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APPROXIMATING BACKSTOP VIABILITY KERNELS

GUILLAUME DEFFUANT , ISABELLE ALVAREZ , SOPHIE MARTIN , AND PATRICK SAINT-PIERRE

Abstract. This paper defines a specific viability domain called backstop viability kernel and proposes a new algorithm that approximates such sets in a compact constraint set of d dimensions. The algorithm uses a set approximation technique based on a sample chosen in a regular grid of n^d vertices covering the considered constraint set. We show that, in some conditions, the result of the algorithm is the direct approximation of the backstop viability kernel by this approximation technique. This theoretical result is illustrated on examples for which the viability kernel is equal to the backstop viability kernel and can be derived analytically, using two set approximation methods: the nearest neighbour, ensuring a convergence rate in $\mathcal{O}(n^{-1})$, and a recently developed technique, the recursive simplex stars (resistars), ensuring a convergence rate in $\mathcal{O}(n^{-2})$.

Key words. viability kernel, viability domain, resistar.

1. Introduction. Viability theory [2, 4] addresses the problem of maintaining a controlled dynamical system inside a given set of states, generally called the constraint set. This framework is particularly relevant for modelling sustainability problems in which the constraint set is interpreted as an acceptable or desirable property of the system that should be sustained [14, 9, 27, 22, 24]. It appeared also relevant in a variety of engineering problems [30, 18, 23] and in finance and economics [15, 29]. In [21, 8, 13, 26] viability theory is at the core of a mathematical definition of resilience, viewed as the capacity of the system to viably restore the property if it has been lost. Recently, [17] extended this view to a general theory of sustainable management.

One of the main concepts of viability theory is the viability kernel, the set of states from which the system can remain indefinitely in the constraint set. From a state located outside the viability kernel, it is certain that the dynamical system will cross the limits of the constraint set after a finite time, whatever the chosen controls over time. The viability kernel is also important because it is easy to derive from it a variety of control policies keeping the system indefinitely inside the constraint set (and actually also inside the viability kernel itself).

Generally, it is not possible to determine a viability kernel analytically and several methods provide numerical approximations [5, 6, 10, 12, 9, 14, 19, 20, 1]. In the frequent case of dynamics defined with ordinary differential equations in a continuous state space, most algorithms start from a regularly distributed sample of points (vertices of a regular grid) covering the constraint set and another one covering the control space. The first algorithm proposed by Saint-Pierre [28] uses a discrete approximation of the dynamical system on the vertices of a regular mesh. The algorithm computes a sequence of mesh subsets, until reaching a fixed point. The final set provides a discrete approximation of the viability kernel, which converges to the viability kernel when the time step tends to 0 and n , the number of points by grid axis, tends to infinity.

In the particular case of viability kernels defined as epigraphs of a function, [7] showed that the approximation error of the Saint-Pierre algorithm is in $\mathcal{O}(n^{-1})$, when choosing a time step in $\mathcal{O}(n^{-1})$. In the general case, [25] showed that the approximation error (defined with the Hausdorff distance) of the Saint-Pierre algorithm is linear in n^{-1} and in the time step, if the problem satisfies some specific conditions (the shadowing property in particular). As far as we know, there is no other algorithm for which the convergence rate has been established, even in restrictive conditions.

48 In this paper, we focus on approximating the backstop viability kernel, a subset
49 of the viability kernel, with the aim to get a better convergence rate.

50 The first contribution of this paper is theoretical. It defines the backstop viability
51 kernel as the union of all viability domains satisfying a specific property. It introduces
52 the extended discrete time in which the control can change at each time step (as
53 in usual discrete time dynamics) and also when the system is about to leave the
54 constraint set K , and extends the definition of the backstop viability kernel to the
55 extended discrete time. It establishes that, in some conditions, the backstop viability
56 kernels in extended discrete time and in continuous time are equal.

57 The second contribution is an algorithm approximating backstop viability kernels.
58 Approximating backstop viability kernels is easier than approximating viability ker-
59 nels in general, because it boils down to testing long trajectories that change control
60 at most once, starting from points located in a limited subset of the state space. The
61 algorithm uses a set approximation technique which is assumed based on a sample
62 derived from a regular grid of n^d points covering K . In some conditions, the final re-
63 sult is equal to the direct approximation of the exact backstop viability kernel by the
64 set approximation technique. Therefore, the Hausdorff distance between the backstop
65 viability kernel and its approximation by our algorithm when n grows is the same as if
66 the set approximation technique was directly applied to the backstop viability kernel
67 itself.

68 The third contribution is a report on tests of the algorithm on viability problems
69 for which the backstop viability kernel is equal to the viability kernel and can be de-
70 rived analytically. The tests use two approximation techniques: the nearest neighbour
71 and a recently developed method, the resistar surfaces [11]. These methods guaran-
72 tee (if the backstop viability kernel holds some smoothness properties) a Hausdorff
73 approximation error respectively in $\mathcal{O}(n^{-1})$ and $\mathcal{O}(n^{-2})$. The results of the tests are
74 in line with the theory.

75 The remaining of the paper is organised as follows: Section 2 defines the backstop
76 viability kernel in continuous time and extended discrete time control and determines
77 some conditions in which these sets are equal. Section 3 presents the approximation
78 algorithm and the theoretical study of its convergence. Section 4 reports tests of the
79 algorithm convergence on examples. The final section discusses the contributions of
80 the paper.

81 **2. Backstop viability kernels.**

82 **2.1. Continuous time.**

83 **2.1.1. Viability problem and viability kernel.** We consider a controlled
84 dynamical system defined by its state $x(t) \in \mathbb{R}^d$ which can be influenced by a control
85 $u(t)$, chosen in a compact set $U \subset \mathbb{R}^m$, defined by an ordinary differential equation
86 where φ is a continuous function from $\mathbb{R}^d \times \mathbb{R}^m$ to \mathbb{R}^d :

$$87 \quad (2.1) \quad \begin{cases} \dot{x}(t) = \varphi(x(t), u(t)) \\ u(t) \in U. \end{cases}$$

88 A viability problem from point x_0 is to determine a measurable function $u(t) :$
89 $\mathbb{R}_+ \rightarrow U$, such that the trajectory from x_0 when applying $u(\cdot)$ remains for all t in
90 compact set $K \subset \mathbb{R}^n$.

91 Integrating equation 2.1, from initial state x_0 and for a chosen control function
92 $u(\cdot)$, determines the successor of x_0 at time t as follows:

93 (2.2)
$$x_{x_0, u(\cdot)}(t) := x_0 + \int_0^t \varphi(x_{x_0, u(\cdot)}(t'), u(t')) dt'.$$

94 We also denote all the successors (or the trajectory) of x_0 during time interval
 95 $[0, t]$ when applying control function $u(\cdot)$ as follows:

96 (2.3)
$$X_{x_0, u(\cdot)}(t) := \bigcup_{t' \in [0, t]} x_{x_0, u(\cdot)}(t').$$

97 **DEFINITION 2.1.** *The viability kernel $Viab(K)$ of set K under the dynamics de-*
 98 *finied by function φ and control set U is the set of states x_0 , for which there exists a*
 99 *control function $u(\cdot)$ such that all successors of x_0 when applying $u(\cdot)$ are in K :*

100 (2.4)
$$Viab(K) = \{x_0 \in K, \exists u(\cdot) : \mathbb{R}_+ \rightarrow U, \forall t \in \mathbb{R}_+, x_{x_0, u(\cdot)}(t) \in K\},$$

101 where \mathbb{R}_+ denotes the set of positive real numbers.

102 Determining the viability kernel is important because a variety of control func-
 103 tions $u(\cdot)$ keeping the trajectory indefinitely within K can easily be derived from it.
 104 However, the analytical determination of this set is generally impossible and it should
 105 therefore be approximated numerically. In this paper, we define the backstop viability
 106 kernel, a subset of the viability kernel which is easier to approximate.

107 **2.1.2. Backstop viability domain and backstop viability kernel.** We need
 108 a few preliminary notations and definitions:

- 109 • For $(x, y) \in K^2$, $D(x, y)$ denotes the distance between x and y in K , which
 110 is the length of the shortest continuous path connecting x to y in K (K is
 111 supposed to be path-connected). Of course, if K is convex, $D(x, y) = \|x - y\|$,
 112 the usual Euclidean distance.
- 113 • For a set $A \subset K$, the distance from $x \in K$ to A in K is: $D(x, A) =$
 114 $\inf_{y \in A} D(x, y)$. By convention: $D(x, \emptyset) = \infty$.
- 115 • For any two sets (A, B) included in K , the distance from A to B in K is:
 116 $D(A, B) = \inf_{x \in A} D(x, B)$.
- 117 • \mathbb{R}_+^* denotes the set of strictly positive real numbers.
- 118 • For a couple of sets A and B , such that $B \subset A$, $A - B$ denotes the comple-
 119 mentary set of B in A .
- 120 • If $u(t') = u$ for all $t' \in [0, t]$, then $x_{x_0, u(\cdot)}(t)$ and $X_{x_0, u(\cdot)}(t)$ are respectively
 121 denoted $x_{x_0, u}(t)$ and $X_{x_0, u}(t)$.

122 **DEFINITION 2.2.** *The exit time from a set A for an evolution starting from x_0*
 123 *when applying the control function $u(\cdot)$ is:*

124 (2.5)
$$\tau_A^\sharp(x_0, u(\cdot)) = \inf\{t \in \mathbb{R}_+, x_{x_0, u(\cdot)}(t) \notin A\}.$$

126 The exit time from set A and point x_0 obtained when applying a constant control
 127 u is denoted: $\tau_A^\sharp(x_0, u)$.

128 **DEFINITION 2.3.** *For any couple $(u, v) \in U^2$, the constant control from $x_0 \in K$*
 129 *with one possible bounce on the boundary of K , denoted $(u, v)_{x_0, K}(\cdot)$, is the function*

130 from \mathbb{R}_+ to U defined as follows:

$$131 \quad (2.6) \quad (u, v)_{x_0, K}(t) = \begin{cases} u & \text{for } 0 \leq t \leq \tau_K^\#(x_0, u) \\ v & \text{for } t > \tau_K^\#(x_0, u). \end{cases}$$

132

133 The point reached at time t when applying this control function from x_0 is denoted
134 $x_{x_0, (u, v)_K}(t)$ and the corresponding trajectory is denoted $X_{x_0, (u, v)_K}(t)$.

135 We can now define a backstop viability domain.

136 **DEFINITION 2.4.** *The set $V \subset K$ is a backstop viability domain of K under dy-*
137 *namics 2.1 for time threshold $\theta \in \mathbb{R}_+^*$ iff there exists a set $B \subset V$ such that:*

- 138 • $D(B, K - V) > 0$;
- 139 • For all $x_0 \in B$, there exist $(u, v) \in U^2$ such that $X_{x_0, (u, v)}(\theta) \subset V$;
- 140 • For all $x_0 \in V - B$, there exists $(u, v) \in U^2$, such that:
 - 141 – either for all $t \in \mathbb{R}_+$, $x_{x_0, (u, v)_K}(t) \in V$,
 - 142 – or there exists $t \in \mathbb{R}_+^*$, $t \geq \theta$ | $X_{x_0, (u, v)_K}(t) \subset V$ and $x_{x_0, (u, v)_K}(t) \in B$.

143 B is called a backstop in V for time threshold θ , $V - B$ is called the catch of B in
144 V , $D(B, K - V)$ the width of the catch of B in V . By convention, the empty set is a
145 backstop viability domain for any catch width and any time threshold.

146 **PROPOSITION 2.5.** *Let V be a backstop viability domain of K and B a backstop*
147 *in V of catch width $\delta \in \mathbb{R}_+^*$ for time threshold $\theta \in \mathbb{R}_+^*$. Let $B' \subset V$ be such that*
148 *$B \subset B'$ and $D(B', K - V) = \delta'$ with $\delta' \leq \delta$. For any $\theta' \in \mathbb{R}_+^*$ such that $0 < \theta' \leq \theta$, B'*
149 *is a backstop in V for time threshold θ' of catch width δ' .*

150 *Proof.* Consider $x_0 \in V - B'$. Because $B \subset B'$, $x_0 \in V - B$. Therefore, there
151 exist $(u, v) \in U^2$ such that:

- 152 • either $\forall t \in \mathbb{R}_+$, $x_{x_0, (u, v)_K}(t) \in V$,
- 153 • or $\exists t \in \mathbb{R}_+^*$, $t \geq \theta$ such that $X_{x_0, (u, v)_K}(t) \subset V$ and $x_{x_0, (u, v)_K}(t) \in B$. There-
154 fore $t \geq \theta'$ and since $B \subset B'$, $x_{x_0, (u, v)_K}(t) \in B'$.

155 Now consider $x_0 \in B'$. If $x_0 \in B$, there exists $(u, v) \in U^2$ such that $X_{x_0, u}(\theta) \subset V$
156 thus $X_{x_0, u}(\theta') \subset V$. If $x_0 \in B' - B$, then $x_0 \in V - B$ and there exists $t \in \mathbb{R}_+^*$, $t \geq \theta$
157 such that $X_{x_0, (u, v)_K}(t) \subset V$, and $t \geq \theta$ implies $t \geq \theta'$.

158 Therefore, B' satisfies the conditions for being a backstop in V for time threshold
159 θ' and its catch width is δ' by definition. \square

160 **PROPOSITION 2.6.** *For any $(\delta, \theta) \in (\mathbb{R}_+^*)^2$, the union of all backstop viability do-*
161 *maines of K under dynamics 2.1 including a backstop of catch width greater or equal*
162 *to δ for time threshold θ is a backstop viability domain of K including a backstop of*
163 *catch width greater or equal to δ for time threshold θ . It is called the backstop viability*
164 *kernel of catch width δ for time threshold θ and denoted $BSViab(\delta, \theta, K)$.*

165 *Proof.* Let $\mathcal{D}_{Bstop}(\delta, \theta, K)$ be the set of all backstop viability domains in K in-
166 cluding a backstop of catch width at least δ for time threshold θ . Let:

$$167 \quad (2.7) \quad V = \bigcup_{V' \in \mathcal{D}_{Bstop}(\delta, \theta, K)} V' \text{ and } B = \bigcup_{V' \in \mathcal{D}_{Bstop}(\delta, \theta, K)} B_{V'},$$

168 where $B_{V'}$ is a backstop of V' of catch width δ for time threshold θ , for each
169 $V' \in \mathcal{D}_{Bstop}(\delta, \theta, K)$.

170 Consider $x_0 \in B$. There exists $V' \in \mathcal{D}_{Bstop}(\delta, \theta, K)$ such that $x_0 \in B_{V'}$, and
171 $D(x_0, K - V') \geq \delta$. Because $(K - V) \subset (K - V')$, $D(x_0, K - V) \geq \delta$.

172 Consider $x_0 \in V - B$. There exists $V' \in \mathcal{D}_{Bstop}(\delta, \theta, K)$ such that $x_0 \in V'$, and
 173 for all $V'' \in \mathcal{D}_{Bstop}(\delta, \theta, K)$, $x_0 \notin B_{V''}$, which implies $x_0 \in V' - B_{V'}$. Therefore, there
 174 exist $(u, v) \in U^2$ such that:

- 175 • either $\forall t \in \mathbb{R}_+, x_{x_0, (u, v)_K}(t) \in V'$, then $\forall t \in \mathbb{R}_+, x_{x_0, (u, v)_K}(t) \in V$,
- 176 • or $\exists t \in \mathbb{R}_+^*, t \geq \theta$ such that $X_{x_0, (u, v)_K}(t) \subset V'$ then $X_{x_0, (u, v)_K}(t) \subset V$
 177 because $V' \subset V$ and $x_{x_0, (u, v)_K}(t) \in B_{V'}$, then $x_{x_0, (u, v)_K}(t) \in B$, because
 178 $B_{V'} \subset B$.

179 Consider $x_0 \in B$. There exists $V' \in \mathcal{D}_{Bstop}(\delta, \theta, K)$ such that $x_0 \in B_{V'}$. Therefore,
 180 there exists $(u, v) \in U^2$ such that $X_{x_0, (u, v)_K}(\theta) \subset V'$, hence $X_{x_0, (u, v)_K}(\theta) \subset V$. \square

181 **DEFINITION 2.7.** *If there exists $(\delta_0, \theta_0) \in (\mathbb{R}_+^*)^2$ such that:*

$$182 \quad (2.8) \quad BSViab(\delta_0, \theta_0, K) = \bigcup_{(\delta, \theta) \in (\mathbb{R}_+^*)^2} BSViab(\delta, \theta, K),$$

183 *then this set is called the backstop viability kernel of K under dynamics 2.1 and denoted*
 184 *$BSViab(K)$.*

185 Section 4.1 describes examples of viability problems for which the backstop via-
 186 bility kernel exists.

187 2.2. Extended discrete time.

188 **2.2.1. Viability kernel in extended discrete time.** We now define the control
 189 functions in extended discrete time (abbreviated as EDT), namely the control
 190 function for which it is possible to change the control at each clock tick (as usual
 191 in discrete time) and also just before the system leaves the constraint set K . This
 192 extended discrete time will be used in the algorithm approximating backstop viability
 193 kernels.

194 **DEFINITION 2.8.** *The EDT control function set of time step $\theta \in \mathbb{R}_+^*$, for con-*
 195 *straint set K , dynamics 2.1 and starting from $x_0 \in K$ is denoted $\mathcal{E}_\theta(x_0, K, U)$ and*
 196 *is the set of piece-wise constant functions $u(\cdot) : \mathbb{R}_+ \rightarrow U$, defined by the sequence*
 197 *$(u_j, v_j)_{j \in \mathbb{N}}$ of pairs of elements of U such that, for $j \in \mathbb{N}$:*

$$198 \quad (2.9) \quad \forall t \in [j\theta, (j+1)\theta[, u(t) = (u_j, v_j)_{x_j, K}(t - j\theta),$$

199 *where $x_j = x_{x_0, u(\cdot)}(j\theta)$ and $(u_j, v_j)_{x_j, K}(t - j\theta)$ refers to definition 2.3.*

200 **DEFINITION 2.9.** *The viability kernel of K in EDT of time step θ under dynamics*
 201 *2.1, denoted $Viab_{EDT}(\theta, K)$, is the set of states $x_0 \in K$ for which there exists a control*
 202 *function $u(\cdot)$ in $\mathcal{E}_\theta(x_0, U, K)$ such that the trajectory from x_0 applying control function*
 203 *$u(\cdot)$ remains indefinitely in K :*

$$204 \quad (2.10) \quad Viab_{EDT}(\theta, K) = \{x_0 \in K, \exists u(\cdot) \in \mathcal{E}_\theta(x_0, U, K) \mid \forall t \in \mathbb{R}_+, x_{x_0, u(\cdot)}(t) \in K\}.$$

205 We now define a viability domain sewed in K with time step θ under dynamics
 206 2.1. The image of sewing is suggested by considering the trajectory $X_{x_0, (u, v)_K}(\theta)$ as
 207 a thread that should be included in K while the stitch $x_{x_0, (u, v)_K}(\theta)$ should belong to
 208 the viability domain.

209 **DEFINITION 2.10.** *A set $E \subset K$ is a viability domain sewed in K with time step*
 210 *θ under dynamics 2.1, iff:*

$$211 \quad (2.11) \quad \forall x_0 \in E, \exists (u, v) \in U^2 \mid X_{x_0, (u, v)_K}(\theta) \subset K \text{ and } x_{x_0, (u, v)_K}(\theta) \in E.$$

212 PROPOSITION 2.11. *The viability kernel of K in EDT of time step θ is the largest*
 213 *viability domain sewed in K with time step θ .*

214 *Proof.* For all $x_0 \in \text{Viab}_{EDT}(\theta, K)$ there exists $u(\cdot) \in \mathcal{E}_\theta(x_0, U, K)$ such that for
 215 all $t \in \mathbb{R}_+$, $X_{x_0, u(\cdot)}(t) \subset K$. Let $u(\cdot)$ be defined by the infinite series $\{(u_0, v_0), \dots,$
 216 $(u_j, v_j), \dots\}$ with, for $j \in \mathbb{N}$, $(u_j, v_j) \in U^2$. We have thus $X_{x_0, (u_0, v_0)_K}(\theta) \subset K$. Let
 217 $x_1 = x_{x_0, (u_0, v_0)_K}(\theta)$. The function $u'(\cdot)$ defined by $\{(u_1, v_1), \dots, (u_j, v_j), \dots\}$ is such
 218 that for all $t \in \mathbb{R}_+$, $X_{x_1, u'(\cdot)}(t) \subset K$. Therefore, $x_1 \in \text{Viab}_{EDT}(\theta, K)$. Therefore
 219 $\text{Viab}_{EDT}(\theta, K)$ is a viability domain sewed in K with time step θ .

220 Let $E \subset K$ be a viability domain sewed in K with time step θ . Consider
 221 $x_0 \in E$. By definition, there exist $(u_0, v_0) \in U^2$ such that $X_{x_0, (u_0, v_0)_K}(\theta) \subset K$
 222 and $x_{x_0, (u_0, v_0)_K}(\theta) \in E$.

223 Consider now, for $j \in \mathbb{N}$, $\{(u_0, v_0), \dots, (u_{j-1}, v_{j-1})\} \in (U^2)^j$, such that, denoting
 224 $x_{p+1} = x_{x_p, (u_p, v_p)_K}(\theta)$, for $p \in \{0, \dots, j-1\}$, we have: $X_{x_p, (u_p, v_p)_K}(\theta) \subset K$ and
 225 $x_{x_p, (u_p, v_p)_K}(\theta) \in E$. In particular, $x_j = x_{x_{j-1}, (u_{j-1}, v_{j-1})_K}(\theta) \in E$. Therefore there
 226 exist $(u_j, v_j) \in U^2$ such that $x_{j+1} = x_{x_j, (u_j, v_j)_K}(\theta) \in E$ and $X_{x_j, (u_j, v_j)_K}(\theta) \subset K$
 227 because E is a viability domain sewed in K .

228 Therefore, there exists $u(\cdot) \in \mathcal{E}(x_0, U, K)$ such that for all $t \in \mathbb{R}_+$, $X_{x_0, u(\cdot)}(t) \subset K$,
 229 hence $x_0 \in \text{Viab}_{EDT}(\theta, K)$. Therefore $E \subset \text{Viab}_{EDT}(\theta, K)$. \square

230 2.2.2. Backstop viability kernel sewed in K .

231 DEFINITION 2.12. *$V \subset K$ is a backstop viability domain sewed in K with time*
 232 *step θ under dynamics 2.1 iff there exists a set $B \subset V$ such that:*

- 233 • $D(B, K - V) > 0$;
- 234 • for all $x_0 \in B$, there exists $(u, v) \in U^2$ such that $X_{x_0, (u, v)_K}(\theta) \in K$ and
 235 $x_{x_0, (u, v)_K}(\theta) \in V$;
- 236 • For all $x_0 \in V - B$, there exist $(u, v) \in U^2$, such that:
 237 – either $\forall j \in \mathbb{N}$, $X_{x_0, (u, v)_K}(j\theta) \in K$ and $x_{x_0, (u, v)_K}(j\theta) \in V$,
 238 – or $\exists j \in \mathbb{N}^*$, $X_{x_0, (u, v)_K}(j\theta) \subset K$ and $x_{x_0, (u, v)_K}(j\theta) \in B$ and $\forall p < j$,
 239 $x_{x_0, (u, v)_K}(p\theta) \in V$.

240 B is called a backstop of V in EDT time step θ and $D(B, K - V)$ is called the catch
 241 width of the backstop V .

242 It can immediately be seen that a backstop viability domain sewed in K with
 243 time step θ is a viability domain sewed in K with time step θ .

244 PROPOSITION 2.13. *For $(\delta, \theta) \in (\mathbb{R}_+^*)^2$, the union of all backstop viability domains*
 245 *sewed in K with time step θ including a backstop of catch width greater or equal to δ*
 246 *is itself a backstop viability domain sewed in K with time step θ including a backstop*
 247 *of catch width greater or equal to δ . It is called the backstop viability kernel sewed in*
 248 *K with time step θ of catch width δ and denoted $BS\text{Viab}_{EDT}(\delta, \theta, K)$.*

249 *Proof.* The proof is similar to the one of proposition 2.6. \square

250 2.3. Connection between backstop viability kernels in continuous and 251 in extended discrete time.

252 PROPOSITION 2.14. *Let V_θ be a backstop viability domain sewed in K with time*
 253 *step θ under dynamics 2.1, holding set B as a backstop of catch width $\delta \in \mathbb{R}_+^*$. There*
 254 *exists a backstop viability domain V of K holding B as a backstop for time threshold*
 255 *θ such that:*

$$256 \quad (2.12) \quad V_\theta \subset V.$$

257

258 *Proof.* Let B be a backstop of V_θ of catch width δ and let $\mathcal{C}(B)$ be the set of
 259 points $x_0 \in K$ for which there exists $(u, v) \in U^2$ such that:

- 260 • either for all $t \in \mathbb{R}_+$, $X_{x_0, (u, v)_K}(t) \subset K$;
- 261 • or there exists $t \in \mathbb{R}_+^*$, $t \geq \theta$ such that $X_{x_0, (u, v)_K}(t) \subset K$ and $x_{x_0, (u, v)_K}(t) \in$
 262 B ;

263 Let $V = B \cup \mathcal{C}(B)$. For all $x_0 \in V_\theta - B$, $x_0 \in \mathcal{C}(B)$, therefore, $V_\theta \subset V$. Therefore
 264 $\inf_{x_0 \in B} D(x_0, K - V) \geq \delta$. V is thus a backstop viability domain holding B as a
 265 backstop of catch width greater or equal to δ , for time threshold θ . \square

COROLLARY 2.15. *For any $(\delta, \theta) \in (\mathbb{R}_+^*)^2$, we have:*

$$BSViab_{EDT}(\delta, \theta, K) \subset BSViab(\delta, \theta, K).$$

266 *Proof.* The proof comes directly from proposition 2.14 and from the definitions
 267 of $BSViab(\delta, \theta, K)$ and $BSViab_{EDT}(\delta, \theta, K)$. \square

268 PROPOSITION 2.16. *Let V be a backstop viability domain of K under dynamics*
 269 *2.1, holding a backstop B of catch width δ for time threshold θ and let:*

$$270 \quad (2.13) \quad M = \max\{\|\varphi(x, u)\|, x \in K, u \in U\}.$$

271 *For all $\theta' \leq \min(\theta, \frac{\delta}{2M})$, V is a backstop viability domain sewed in K with time step*
 272 *θ' holding a backstop of catch width $\frac{\delta}{2}$.*

273 *Proof.* Consider $\theta' \in \mathbb{R}_+^*$ such that $\theta' \leq \min(\theta, \frac{\delta}{2M})$. Let $B' = \{x \in V, D(x, K -$
 274 $V) \geq \frac{\delta}{2}\}$. For all $x \in B$, $D(x, K - B') \geq \frac{\delta}{2}$. For $x_0 \in B'$, because $\theta' \leq \theta$, there
 275 exists $(u, v) \in U^2$ such that $X_{x_0, (u, v)_K}(\theta') \subset K$ and $x_{x_0, (u, v)_K}(\theta') \in V$. Consider
 276 $x_0 \in V - B'$. Because $B \subset B'$, $x_0 \in V - B$. Therefore, because B is a backstop in V ,
 277 there exists $(u, v) \in U^2$ such that:

- 278 • Either for all $t \in \mathbb{R}_+$, $x_{x_0, (u, v)_K}(t) \in V$, then $\forall j \in \mathbb{N}$, $x_{x_0, (u, v)_K}(j\theta') \in V$, and
 279 $X_{x_0, (u, v)_K}(j\theta') \subset V$, thus $X_{x_0, (u, v)_K}(j\theta') \subset K$,
- 280 • Or there exists $t > 0$, $t \geq \theta$ | $X_{x_0, (u, v)_K}(t) \subset V$ and $y = x_{x_0, (u, v)_K}(t) \in B$.
 281 Then $y \in B'$ because $B \subset B'$. We know that $D(y, V - B') \geq \frac{\delta}{2}$ which implies
 282 that $t > \theta'$ and for all $t' \in [t - \theta', t]$, $x_{x_0, (u, v)_K}(t') \in B'$. Therefore, taking j
 283 as the integer part of $\frac{t}{\theta'}$ ensures $j\theta' \in [t - \theta', t]$ thus $x_{x_0, (u, v)_K}(j\theta') \in B'$.

284 Therefore, V is a backstop viability domain in EDT of time step θ' holding B' as a
 285 backstop of catch width $\frac{\delta}{2}$. \square

COROLLARY 2.17. *Let M be defined by equation 2.13. For any $(\delta, \theta) \in (\mathbb{R}_+^*)^2$, for*
any $(\delta', \theta') \in (\mathbb{R}_+^)^2$ such that $\delta' \leq \frac{\delta}{2}$ and $\theta' \leq \min(\theta, \frac{\delta}{2M})$, we have:*

$$BSViab(\delta, \theta, K) \subset BSViab_{EDT}(\delta', \theta', K).$$

286 *Proof.* The proof comes directly from proposition 2.16 and from the definitions
 287 of $BSViab(\delta, \theta, K)$ and $BSViab_{EDT}(\delta', \theta', K)$. \square

288 PROPOSITION 2.18. *Let M be defined by equation 2.13. If $BSViab(K)$, the back-*
 289 *stop viability kernel of K exists and $BSViab(K) = BSViab(\delta_0, \theta_0, K)$, for $(\delta_0, \theta_0) \in$
 290 $(\mathbb{R}_+^*)^2$, then:*

$$291 \quad (2.14) \quad \forall \delta \leq \frac{\delta_0}{2}, \forall \theta \leq \min(\theta_0, \frac{\delta_0}{2M}), BSViab_{EDT}(\delta, \theta, K) = BSViab(K).$$

292
 293
 294 *Proof.* The proposition is a direct consequence of corollaries 2.15 and 2.17. \square

295 Proposition 2.18 provides a connection between the backstop viability kernel and
 296 backstop viability kernels sewed in K (in EDT) which is important for the approxi-
 297 mation algorithm described in the next section.

298 3. Approximating backstop viability kernels.

299 **3.1. Algorithm and its basic properties.** Algorithm 3.1 takes as input: $n \in$
 300 \mathbb{N} the number of points by axis of the grid, $\delta \in \mathbb{R}_+^*$ a catch width value, $\theta \in \mathbb{R}_+^*$ a
 301 time step value and $j_A \in \mathbb{N}$ a maximum number of tested steps. It uses algorithm \mathcal{A}
 302 approximating sets (or classification functions, this is equivalent) from a finite sample
 303 $\Gamma(n)$ of points in K . This sample is based on a regular grid of n^d points (n points by
 304 axis) covering K for the nearest vertex approximation and a more elaborated sample
 305 drawn on edges of the cubes defined by the grid for resistars (which are used in our
 306 tests of the algorithm reported in section 4). Other set approximation algorithms
 307 such as decision trees or support vector machines could be chosen. Considering a set
 308 $E \subset K$, $\mathcal{A}(\Gamma(n) \cap E)$ denotes the approximation of E derived from sample $\Gamma(n)$ by
 309 algorithm \mathcal{A} .

310 Algorithm 3.1 builds sets $H^i \cap \Gamma(n)$ from which are derived $\hat{H}^i = \mathcal{A}(H^i \cap \Gamma(n))$, the
 311 iterative approximations of the backstop viability kernel, with sets B_{in}^i as the approxi-
 312 mation of their backstops. At each iteration, some points of the sample are eliminated
 313 from the definition of the next approximation if they do not pass `testInBstop` (for
 314 points in B_{in}^i) or `testInCatch` (for points in $H^i - B_{in}^i$). Under some assumptions
 315 (see further), the sets B_{out}^i contain the sets \hat{H}^i , thus the sets $B_{out}^i - B_{in}^i$ are wide
 316 approximations of the catches. Actually, a point passes `testInCatch` if there exists a
 317 constant control with one possible bounce that defines a trajectory sewed in K with
 318 stitches in B_{out}^i during j_A time steps, or reaching B_{in}^i in less than j_A time steps, with-
 319 out getting out from B_{out}^i . As will be shown further, in some conditions, at the last
 320 iteration of the algorithm, this test determines with full accuracy if a point belongs to
 321 the backstop viability kernel, independently from the precision of the approximation
 322 \hat{H}^i . The last iteration is reached when no point of the sample is eliminated. The
 323 algorithm also stops if $H^1 \cap \Gamma(n) = K \cap \Gamma(n)$.

324 The algorithm uses the following definitions:

- 325 • For any set $H \subset K$ and any $\delta \in \mathbb{R}_+$, the sets $(H)_\delta$ and $(H)_{-\delta}$ are respectively
 326 the erosion and the dilatation of size δ in K :

$$327 \quad (H)_\delta := \{x \in H \mid D(x, K - H) \geq \delta\},$$

$$328 \quad (H)_{-\delta} := \{x \in K \mid D(x, H) \leq \delta\}.$$

- 330 • The sets B_{in}^i and B_{out}^i approximate respectively $(\hat{H}^i)_\delta$ and $(\hat{H}^i)_{-\delta}$:

$$331 \quad B_{in}^i := \mathcal{A}((\hat{H}^i)_\delta \cap \Gamma(n)),$$

$$332 \quad B_{out}^i := \mathcal{A}((\hat{H}^i)_{-\delta} \cap \Gamma(n)).$$

- 334 • The sets H^i are recursively defined by:

$$335 \quad H^1 \leftarrow \{x_0 \in K \mid \exists (u, v) \in U^2, X_{x_0, (u, v)_K}(\theta) \subset K\},$$

$$336 \quad H^{i+1} \leftarrow \{x_0 \in B_{in}^i \mid \text{testInBstop}(x_0, B_{in}^i, B_{out}^i)\} \cup$$

$$337 \quad \{x_0 \in H^i \mid x_0 \notin B_{in}^i \text{ and } \text{testInCatch}(x_0, B_{in}^i, B_{out}^i)\}.$$

Algorithm 3.1 BSVApp(n, δ, θ, j_A). Backstop viability kernel approximation.

Input: $n \in \mathbb{N}$ number of points by grid axis, $\delta \in \mathbb{R}_+^*$ width of backstop catch, $\theta \in \mathbb{R}_+^*$ time step, $j_A \in \mathbb{N}$ number of tested steps.

$H^1 \cap \Gamma(n) \leftarrow \{\xi \in \Gamma(n), \exists (u, v) \in U^2, X_{\xi, (u, v)_K}(\theta) \subset K\};$

if $H^1 \cap \Gamma(n) = K \cap \Gamma(n)$ **then**
| **return** K ;

end

$\hat{H}^1 \leftarrow \mathcal{A}(H^1 \cap \Gamma(n));$

$i \leftarrow 0$;

repeat

| $i \leftarrow i + 1$;

| $B_{in}^i \leftarrow \mathcal{A}((\hat{H}^i)_\delta \cap \Gamma(n)); B_{out}^i \leftarrow \mathcal{A}((\hat{H}^i)_{-\delta} \cap \Gamma(n));$

| $H^{i+1} \cap \Gamma(n) \leftarrow \{\xi \in B_{in}^i \cap \Gamma(n) \mid \text{testInBstop}(\xi, B_{in}^i, B_{out}^i)\} \cup$
| $\{\xi \in H^i \cap \Gamma(n) \mid \xi \notin B_{in}^i \text{ and } \text{testInCatch}(\xi, B_{in}^i, B_{out}^i)\}$

| $\hat{H}^{i+1} \leftarrow \mathcal{A}(H^{i+1} \cap \Gamma(n));$

until $H^i \cap \Gamma(n) = H^{i+1} \cap \Gamma(n)$;

return (\hat{H}^i) ;

Algorithm 3.2 testInCatch(x_0, B_{in}^i, B_{out}^i). Test selecting points in $B_{out}^i - B_{in}^i$.

Input: $x_0 \in B_{out}^i - B_{in}^i, B_{in}^i, B_{out}^i$.

$b \leftarrow \exists (u, v) \in U^2 \mid$ either $\forall j \leq j_A, X_{x_0, (u, v)_K}(j\theta) \subset K$ and $x_{x_0, (u, v)_K}(j\theta) \in B_{out}^i$

or $\exists j < j_A \begin{cases} X_{x_0, (u, v)_K}(j\theta) \subset K \text{ and } x_{x_0, (u, v)_K}(j\theta) \in B_{in}^i \\ \text{and } \forall p < j, x_{x_0, (u, v)_K}(p\theta) \in B_{out}^i. \end{cases}$

return (b) ;

Algorithm 3.3 testInBstop(x_0, B_{in}^i, B_{out}^i). Test selecting points in B_{in}^i .

Input: $x_0 \in B_{in}^i, B_{in}^i, B_{out}^i$.

$b \leftarrow \exists (u, v) \in U^2 \mid X_{x_0, (u, v)_K}(\theta) \subset K$ and $y \leftarrow x_{x_0, (u, v)_K}(\theta) \mid$

$\begin{cases} y \in B_{in}^i \text{ or} \\ y \in B_{out}^i - B_{in}^i \text{ and } \text{testInCatch}(y, B_{in}^i, B_{out}^i) \end{cases}$

return (b) ;

339 The basic properties of algorithm 3.1 require the following assumption.

340 ASSUMPTION 1. For any $\epsilon \in \mathbb{R}_+^*$, there exists $n_1(\epsilon) \in \mathbb{N}^*$, such that for all $n >$
341 $n_1(\epsilon)$, at any iteration i of the algorithm, for any set $H \in \{H^i, (\hat{H}^i)_\delta, (\hat{H}^i)_{-\delta}\}$, we
342 have:

$$343 \quad (H)_\epsilon \subset \hat{H} \subset (H)_{-\epsilon} \text{ and } (\hat{H})_\epsilon \subset H \subset (\hat{H})_{-\epsilon},$$

345 where $\hat{H} := \mathcal{A}(H \cap \Gamma(n))$.

346 PROPOSITION 3.1. Under assumption 1, for all $n > n_1(\frac{\delta}{4})$, at each iteration of
347 algorithm 3.1, we have:

$$348 \quad (H^i)_{\frac{3\delta}{2}} \subset B_{in}^i \subset (H^i)_{\frac{\delta}{2}} \text{ and } (H^i)_{-\frac{\delta}{2}} \subset B_{out}^i \subset (H^i)_{-\frac{3\delta}{2}}.$$

350 *Proof.* For $\epsilon \in \mathbb{R}_+^*$, $\epsilon < \frac{\delta}{2}$, for $n > n_1(\epsilon)$, at any iteration i , we have:

$$351 \quad (H^i)_\epsilon \subset \hat{H}^i$$

353 This implies:

$$354 \quad (3.1) \quad ((H^i)_\epsilon)_\delta \subset (\hat{H}^i)_\delta \text{ hence: } (H^i)_{\delta+\epsilon} \subset (\hat{H}^i)_\delta.$$

356 and assumption 1 ensures also:

$$357 \quad (\hat{H}^i)_\epsilon \subset H^i,$$

359 which implies:

$$360 \quad (3.2) \quad ((\hat{H}^i)_\epsilon)_{\delta-\epsilon} \subset (H^i)_{\delta-\epsilon} \text{ hence: } (\hat{H}^i)_\delta \subset (H^i)_{\delta-\epsilon}.$$

362 Therefore:

$$363 \quad (3.3) \quad (H^i)_{\delta+\epsilon} \subset (\hat{H}^i)_\delta \subset (H^i)_{\delta-\epsilon}.$$

365 Applying assumption 1 to set $(\hat{H}^i)_\delta$, and setting $B_{in} = \mathcal{A}((\hat{H}^i)_\delta \cap \Gamma(n))$ we get:

$$366 \quad ((\hat{H}^i)_\delta)_\epsilon \subset B_{in}^i \subset ((\hat{H}^i)_\delta)_{-\epsilon} \text{ hence (eq: 3.3):}$$

$$367 \quad ((H^i)_{\delta+\epsilon})_\epsilon \subset B_{in}^i \subset ((H^i)_{\delta-\epsilon})_{-\epsilon} \text{ hence:}$$

$$368 \quad (H^i)_{\delta+2\epsilon} \subset B_{in}^i \subset (H^i)_{\delta-2\epsilon}.$$

370 Applying the same reasoning to B_{out}^i (*mutatis mutandis*) and taking $\epsilon = \frac{\delta}{4}$ completes
371 the proof. \square

372 **PROPOSITION 3.2.** *Under assumption 1, for $n > n_1(\frac{\delta}{4})$, algorithm 3.1 stops after*
373 *a finite number of iterations.*

374 *Proof.* Proposition 3.1 guarantees that, for $n > n_1(\frac{\delta}{4})$, at each iteration i of
375 algorithm 3.1, we have: $B_{in}^i \subset H^i$ and thus $H^{i+1} \cap \Gamma(n) \subset H^i \cap \Gamma(n)$. Since $\Gamma(n)$
376 is finite, the procedure reaches $H^{i+1} \cap \Gamma(n) = H^i \cap \Gamma(n)$ after a finite number of
377 iterations. \square

378 **PROPOSITION 3.3.** *Under assumption 1, for $n > n_1(\frac{\delta}{4})$, at any step i of algorithm*
379 *3.1 we have:*

$$380 \quad (3.4) \quad BSViab_{EDT}(\frac{3\delta}{2}, \theta, K) \subset H^i.$$

381

382 *Proof.* Let $V = BSViab_{EDT}(\frac{3\delta}{2}, \theta, K)$. Obviously $V \subset H^1$.

383 Assume now $V \subset H^i$ and consider $x_0 \in V$. Two cases occur:

384 • $x_0 \in V - (V)_{\frac{3\delta}{2}}$, then there exists $(u, v) \in U^2$ such that:

385 – either for all $j \in \mathbb{N}$, $X_{x_0, (u, v)_K}(j\theta) \subset K$ and $x_{x_0, (u, v)_K}(j\theta) \in V$, then
386 for all $j \in \mathbb{N}$, $x_{x_0, (u, v)_K}(j\theta) \in B_{out}^i$, because $V \subset H^i$ and because of
387 proposition 3.1, $H^i \subset B_{out}^i$;

388 – or there exists $j \in \mathbb{N}^* \mid X_{x_0, (u, v)_K}(j\theta) \subset K$ and $x_{x_0, (u, v)_K}(j\theta) \in (V)_{\frac{3\delta}{2}}$
389 and for all $p < j$, $x_{x_0, (u, v)_K}(p\theta) \in V$, then $x_{x_0, (u, v)_K}(j\theta) \in B_{in}^i$, (because
390 $V \subset H^i$, $(V)_{\frac{3\delta}{2}} \subset (H^i)_{\frac{3\delta}{2}}$ and because of proposition 3.1, $H_{\frac{3\delta}{2}}^i \subset B_{in}^i$)

391 and for all $p < j$, $x_{x_0, (u, v)_K}(p\theta) \in B_{out}^i$.

392 Therefore $\text{testInCatch}(x_0, B_{in}^i, B_{out}^i) = \text{true}$. If $x_0 \notin B_{in}^i$, then $x_0 \in H^{i+1}$.
393 If $x_0 \in B_{in}^i$, this implies $\text{testInBstop}(x_0, B_{in}^i, B_{out}^i) = \text{true}$, thus $x_0 \in H^{i+1}$.
394 • $x_0 \in (V)_{\frac{3\delta}{2}}$, then $x_0 \in B_{in}^i$ (because of 3.1) and there exists $(u, v) \in U^2$ such
395 that $X_{x_0, (u, v)_K}(\theta) \subset K$ and $y = x_{x_0, (u, v)_K}(\theta) \in V$. Therefore $y \in B_{out}^i$. If
396 $y \in V - (V)_{\frac{3\delta}{2}}$, and as shown in the previous case, $\text{testInCatch}(y, B_{in}^i, B_{out}^i)$
397 $= \text{true}$. Else $y \in (V)_{\frac{3\delta}{2}}$ hence $y \in B_{in}^i$ therefore $\text{testInBstop}(x_0, B_{in}^i, B_{out}^i)$
398 $= \text{true}$.

399 Finally, in all cases, $x_0 \in H^{i+1}$, therefore $V \subset H^{i+1}$. \square

400 **3.2. Inclusion of the algorithm output in viability kernel in EDT.** The
401 next propositions require assumption 2 and a new definition.

402 ASSUMPTION 2. For any $\epsilon \in \mathbb{R}_+^*$, there exists $n_2(\epsilon) \in \mathbb{N}$, such that for all $n >$
403 $n_2(\epsilon)$, at any iteration i of the algorithm, for any set H^i , we have:

$$404 \quad \forall x \in K - H^i, \exists \xi \in (K - H^i) \cap \Gamma(n), D(x, \xi) < \epsilon,$$

406 DEFINITION 3.4. For $j \in \mathbb{N}$, let \mathcal{H}^j be the set of points of K for which the exit
407 time when applying the EDT control of time step θ is greater than $j\theta$:

$$408 \quad (3.5) \quad \mathcal{H}^j := \{x_0 \in K, \exists u(\cdot) \in \mathcal{E}_\theta(x_0, U, K), \tau_K^\#(x_0, u(\cdot)) \geq j\theta\}.$$

410 We start by showing that the final set H^q defined by the algorithm is included
411 in \mathcal{H}^{j_A} . Then we distinguish the case where there exists $j_A \in \mathbb{N}$ such that $\mathcal{H}^{j_A} =$
412 $\text{Viab}_{EDT}(\theta, K)$ from the case where there is none.

PROPOSITION 3.5. Under assumptions 1 and 2, for $n > \max(n_1(\frac{\delta}{4}), n_2(\frac{\delta}{2}))$, at
the last iteration q of algorithm 3.1, we have:

$$H^q \subset \mathcal{H}^{j_A}.$$

413

414 *Proof.* At iteration i of algorithm 3.1, let $\mathcal{F}^i = B_{in}^i \cap (K - \mathcal{H}^{j_A})$. Assume $\mathcal{F}^i \neq \emptyset$
415 and, for all $x_0 \in \mathcal{F}^i$, $\text{testInBstop}(x_0, B_{in}^i, B_{out}^i) = \text{true}$. For each point $x_0 \in \mathcal{F}^i$, we
416 define $U(x_0) \subset U^4$ and $j(x_0) \in \mathbb{N}$ as follows:

- 417 • If there exists $(u, v) \in U^2$ such that $X_{x_0, (u, v)_K}(\theta) \subset K$ and $x_{x_0, (u, v)_K}(\theta) \in$
418 B_{in}^i . Then $U(x_0) := \emptyset$ and $j(x_0) := 0$;
- 419 • Else there exists $(u, v) \in K^2 \mid X_{x_0, (u, v)_K}(\theta) \subset K$ and $y := x_{x_0, (u, v)_K}(\theta) \in$
420 $B_{out}^i - B_{in}^i$, and, because $x_0 \notin \mathcal{H}^{j_A}$, there exists $(u', v') \in U^2$ and $j \in \mathbb{N}$,
421 $j < j_A$ such that $X_{y, (u', v')_K}(j\theta) \subset K$, $x_{y, (u', v')_K}(j\theta) \in B_{in}^i$ and for $p \in \mathbb{N}$,
422 $0 < p < j$, $x_{y, (u', v')_K}(p\theta) \in B_{out}^i$. Then $U(x_0) := (u, v, u', v')$ and $j(x_0) := j$.

423 We define set E as follows:

$$424 \quad E := \mathcal{F}^i \cup$$

$$425 \quad \{x_{x_0, (u, v)_K}(\theta), (u', v')_K}(p\theta), x_0 \in \mathcal{F}^i, (u, v, u', v') = U(x_0) \neq \emptyset, 0 \leq p \leq j(x_0)\}.$$

427 It can easily be verified that E is a viability domain sewed in K with time step θ .
428 Therefore $E \subset \mathcal{H}^{j_A}$. This is impossible because $\mathcal{F}^i \subset E$ and $\mathcal{F}^i \subset (K - \mathcal{H}^{j_A})$, by
429 definition.

430 Therefore, there exists $x_0 \in \mathcal{F}^i$, such that $\text{testInBstop}(x_0, B_{in}^i, B_{out}^i) = \text{false}$,
431 thus $x_0 \in K - H^{i+1}$. Because of assumption 2, there exists $\xi \in (K - H^{i+1}) \cap \Gamma(n)$
432 such that $D(x_0, \xi) < \frac{\delta}{2}$ therefore, $\xi \in H^i$ because $x_0 \in B_{in}^i$ and assumption 1 ensures
433 $B_{in}^i \subset (H^i)_{\frac{\delta}{2}}$, hence $D(x_0, K - H^i) \geq \frac{\delta}{2}$.

434 Consider now the case when $\mathcal{F}^i = \emptyset$, hence $B_{in}^i \subset \mathcal{H}^{j_A}$, and consider a point
435 $x_0 \in H^i$ such that $x_0 \notin \mathcal{H}^{j_A}$. For $(u, v) \in U^2$, let $y = x_{x_0, (u, v)_K}(\theta)$. $y \notin \mathcal{H}^{j_A}$
436 (otherwise we would have $x_0 \in \mathcal{H}^{j_A}$), therefore $y \notin B_{in}^i$. Assume $y \in B_{out}^i - B_{in}^i$.
437 Then, for any $(u', v') \in U^2$, $X_{y, (u', v')_K}(j_A \theta) \subset K$ is impossible because $x_0 \notin \mathcal{H}^{j_A}$.
438 Suppose that there exist $(u', v') \in U^2$ and $j \leq j_A$, such that $X_{y, (u', v')_K}(j\theta) \subset K$ and
439 $x_{y, (u', v')_K}(j\theta) \in B_{in}^i$. $B_{in}^i \subset \mathcal{H}^{j_A}$ implies $x_{y, (u', v')_K}(j\theta) \in \mathcal{H}^{j_A}$ which is impossible
440 because $x_0 \notin \mathcal{H}^{j_A}$. Therefore, $\text{testInBstop}(x_0, B_{in}^i, B_{out}^i) = \text{false}$ and, for all $x_0 \in$
441 H^i such that $x_0 \notin \mathcal{H}^{j_A}$, $x_0 \notin H^{i+1}$. Therefore $H^{i+1} \subset \mathcal{H}^{j_A}$.

442 To summarize, while B_{in}^i is not included in \mathcal{H}^{j_A} , $H^{i+1} \cap \Gamma(n) \neq H^i \cap \Gamma(n)$ hence
443 the algorithm does not stop and at the first iteration i such that $B_{in}^i \subset \mathcal{H}^{j_A}$, we have
444 $H^{i+1} \subset \mathcal{H}^{j_A}$. Assumption 1 ensures that the algorithm stops after a finite number of
445 iterations q and that $H^q \subset H^{i+1} \subset \mathcal{H}^{j_A}$. \square

446 **DEFINITION 3.6.** *The supremum of steps that the system's trajectory remains in*
447 *K when applying the EDT control of time step θ from points of K which are not viable*
448 *for the EDT control of time step θ , is:*

$$449 \quad j_{EDT}^\sharp(\theta, K) := \sup_{x_0 \in K - \text{Viab}_{EDT}(\theta, K)} \{j \in \mathbb{N}, \exists u(\cdot) \in \mathcal{E}_\theta(x_0, U, K), X_{x_0, u(\cdot)}(j\theta) \subset K\}.$$

451 For $n \in \mathbb{N}$, the same supremum restricted to $(K - \text{Viab}_{EDT}(\theta, K)) \cap \Gamma(n)$ is:

$$452 \quad j_{EDT}^\sharp(n, \theta, K) := \sup_{\xi \in (K - \text{Viab}_{EDT}(\theta, K)) \cap \Gamma(n)} \{j \in \mathbb{N}, \exists u(\cdot) \in \mathcal{E}_\theta(\xi, U, K),$$

$$453 \quad X_{\xi, u(\cdot)}(j\theta) \subset K\}.$$

455 **PROPOSITION 3.7.** *Under assumptions 1 and 2, for $n > \max(n_1(\frac{\delta}{4}), n_2(\frac{\delta}{2}))$, and*
456 *if $j_{EDT}^\sharp(\theta, K) < \infty$ and $j_A > j_{EDT}^\sharp(\theta, K)$, at the last step q of algorithm 3.1 we have:*

$$457 \quad (3.6) \quad H^q \subset \text{Viab}_{EDT}(\theta, K).$$

460 *Proof.* The proof comes directly from proposition 3.5 and from the fact that
461 $\mathcal{H}^{j_A} = \text{Viab}_{EDT}(\theta, K)$ because $j_A > j_{EDT}^\sharp(\theta, K)$. \square

462 **PROPOSITION 3.8.** *Assume $j_{EDT}^\sharp(n, \theta, K) < \infty$. Under assumptions 1 and 2,*
463 *for $n > \max(n_1(\frac{\delta}{4}), n_2(\frac{\delta}{2}))$, and if $j_A > j_{EDT}^\sharp(n, \theta, K)$, the last iteration $q \in \mathbb{N}$ of*
464 *algorithm 3.1 defines set H^q such that:*

$$465 \quad (3.7) \quad (H^q \cap \Gamma(n)) \subset (\text{Viab}_{EDT}(\theta, K) \cap \Gamma(n)).$$

467 *Proof.* Proposition 3.5 ensures that for all $\xi \in \Gamma(n)$, if $\xi \in H^q$ then $\xi \in \mathcal{H}^{j_A}$.
468 Because $j_A > j_{EDT}^\sharp(n, \theta, K)$, $\xi \in \mathcal{H}^{j_A}$ implies $\xi \in \text{Viab}_{EDT}(\theta, K)$. \square

469 **3.3. Convergence to the backstop viability kernel.** The following proposi-
470 tion requires a new assumption and a new definition.

471 **ASSUMPTION 3.** *There exists $(\theta_0, \delta_0) \in (\mathbb{R}_+^*)^2$ such that $\text{BSViab}(\delta_0, \theta_0, K) =$*
472 *$\text{BSViab}(K)$ is the backstop viability kernel of K .*

473 **DEFINITION 3.9.** *Under assumption 3, the maximum number of time steps in K*
474 *for constant control with one possible bounce starting from points in $K - \text{BSViab}(K)$,*
475 *denoted $j_C^\sharp(\theta, K)$ is:*

$$476 \quad (3.8) \quad j_C^\sharp(\theta, K) = \sup_{x_0 \in K - \text{BSViab}(K)} \{j \in \mathbb{N}, \exists (u, v) \in U^2, X_{x_0, (u, v)_K}(j\theta) \subset K\}.$$

PROPOSITION 3.10. Let M be defined by equation 2.13. If:

• Assumptions 1, 2 and 3 are satisfied and $n > \max(n_1(\frac{\delta}{4}), n_2(\frac{\delta}{4}))$,

• $j_C^\sharp(\theta, K) < \infty$ and $j_A > j_C^\sharp(\theta, K)$,

then for all $\delta \in \mathbb{R}_+^*$ such that $\frac{3\delta}{2} \leq \frac{\delta_0}{2}$, for all $\theta \leq \min(\theta_0, \frac{\delta_0}{2M})$, at the last step q of algorithm 3.1 we have:

$$H^q = BSViab(K).$$

Proof. The structure of the proof is similar to the one of proposition 3.5. Let $V = BSViab(K)$. Because of proposition 2.18, the choice of δ and θ ensures $V = BSViab_{EDT}(\frac{\delta}{4}, \theta, K)$. Suppose $(H^i)_{\frac{\delta}{4}} \cap (K - V) \neq \emptyset$ and that:

• for all points $x_0 \in (H^i)_{\frac{\delta}{4}} - B_{in}^i$, $\text{testInCatch}(x_0, B_{in}^i, B_{out}^i) = \text{true}$ and

• for all points $x_0 \in B_{in}^i$, $\text{testInBstop}(x_0, B_{in}^i, B_{out}^i) = \text{true}$.

For $x_0 \in (H^i)_{\frac{\delta}{4}}$ we define $V(x_0) \in U^2$ or $U(x_0) \in U^4$ and $j(x_0) \in \mathbb{N}$ as follows:

• If $x_0 \in (H^i)_{\frac{\delta}{4}} - B_{in}^i$, there exists $(u, v) \in U^2$, and $j \in \mathbb{N}$, $j \leq j_C^\sharp(\theta, K) < j_A$, such that $X_{x_0, (u, v)_K}(j\theta) \subset K$, $x_{x_0, (u, v)_K}(j\theta) \in B_{in}^i$ and for all $p < j$, $x_{x_0, (u, v)_K}(p\theta) \in B_{out}^i$. Then, $V(x_0) := (u, v)$ and $j(x_0) := j$,

• If $x_0 \in B_{in}^i$, because $j_C^\sharp(\theta, K) < j_A$, we can define $U(x_0)$ and $j(x_0)$ like we did in the proof of proposition 3.5.

Let E be defined as follows:

$$E := (H^i)_{\frac{\delta}{4}} \cup$$

$$\{x_{x_0, (u, v)_K}(p\theta), x_0 \in (H^i)_{\frac{\delta}{4}} - B_{in}^i, (u, v) = V(x_0), 0 < p \leq j(x_0)\} \cup$$

$$\{x_{x_0, (u, v)_K}(\theta), (u', v')_K(p\theta), x_0 \in B_{in}^i, (u, v, u', v') = U(x_0) \neq \emptyset, 0 \leq p \leq j(x_0)\}.$$

By construction, E is a backstop viability domain sewed in K with time step θ holding B_{in}^i as a backstop of catch width greater or equal to $\frac{\delta}{4}$. E is not included in V because $E \subset (H^i)_{\frac{\delta}{4}}$ and by hypothesis, $(H^i)_{\frac{\delta}{4}} \cap (K - V) \neq \emptyset$. This is impossible because the choice of θ ensures that $V = BSViab_{EDT}(\frac{\delta}{4}, \theta, K)$ and thus includes all backstop viability domains sewed in K of catch width greater or equal to $\frac{\delta}{4}$ with time step θ , by definition.

Therefore, there exists $x_0 \in (H^i)_{\frac{\delta}{4}} - B_{in}^i$, such that $\text{testInCatch}(x_0, B_{in}^i, B_{out}^i) = \text{false}$, or there exists $x_0 \in B_{in}^i$, such that $\text{testInBstop}(x_0, B_{in}^i, B_{out}^i) = \text{false}$. In both cases, there exists a point $x_0 \in (H^i)_{\frac{\delta}{4}}$ thus such that $D(x_0, K - H^i) \geq \frac{\delta}{4}$ and $x_0 \notin H^{i+1}$. Because of assumption 2, there exists $\xi \in \Gamma(n)$ such that $\xi \notin H^{i+1}$ and $D(x_0, \xi) < \frac{\delta}{4}$, therefore $\xi \in H^i$. Therefore $H^i \cap \Gamma(n) \neq H^{i+1} \cap \Gamma(n)$.

Suppose now $(H^i)_{\frac{\delta}{4}} \subset V$. Consider $x_0 \in H^i - V$. $x_0 \in H^i - B_{in}^i$ because $B_{in}^i \subset (H^i)_{\frac{\delta}{4}}$. Suppose $\text{testInCatch}(x_0, B_{in}^i, B_{out}^i) = \text{true}$. Then there exist $(u, v) \in U^2$ and $j \in \mathbb{N}$, $j \leq j_C^\sharp(\theta, K) < j_A$ | $X_{x_0, (u, v)_K}(j\theta) \subset K$ and $x_{x_0, (u, v)_K}(j\theta) \in B_{in}^i$ and for all $p < j$, $x_{x_0, (u, v)_K}(p\theta) \in B_{out}^i$. Then the set $V \cup \{x_{x_0, (u, v)_K}(p\theta), p \in \{0, \dots, j\}\}$ is a backstop viability domain of time threshold θ admitting B_{in}^i as a backstop of catch width at least $\frac{\delta}{4}$ and this backstop viability domain is not included in V . This is impossible. Therefore, $\text{testInCatch}(x_0, B_{in}^i, B_{out}^i) = \text{false}$. This implies: $H^{i+1} \subset V$. Because of proposition 2.18, the choice of δ and θ ensures $V = BSViab_{EDT}(\frac{3\delta}{2}, \theta, K)$. Proposition 3.3 can be applied because assumption 1 is satisfied thus $V \subset H^{i+1}$ and $V \subset H^{i+2}$ therefore $H^{i+1} = H^{i+2} = V$.

519 Overall, while $(H^i)_{\frac{\delta}{4}}$ is not included in V , $H^i \cap \Gamma(n) \neq H^{i+1} \cap \Gamma(n)$ therefore
520 the algorithm does not stop. Once $(H^i)_{\frac{\delta}{4}} \subset V$, the algorithm stops after at most one
521 iteration and $H^{i+1} = BSViab(K)$. \square

522 **3.4. Convergence rate.** We now derive the implications of the previous propo-
523 sitions on the convergence rate of the algorithm output to the backstop viability
524 kernel when n , the number of points by axis of the grid, increases. This requires a
525 new assumption which uses the Hausdorff distance $D_h(A, B)$ between two sets A and
526 B :

$$527 \quad (3.9) \quad D_h(A, B) = \max(D(A, B), D(B, A)).$$

529 ASSUMPTION 4. *There exists a function $\epsilon(n) : \mathbb{N} \rightarrow \mathbb{R}_+$, $\epsilon(n) \rightarrow 0$ when $n \rightarrow \infty$,
530 such that, for any set H satisfying some smoothness properties (see examples in section
531 4), there exists $n_4 \in \mathbb{N}$ such that for $n > n_4$:*

$$532 \quad (3.10) \quad D_h(\mathcal{A}(H \cap \Gamma(n)), H) < \epsilon(n).$$

534 PROPOSITION 3.11. *Assume the conditions of proposition 3.10. Under assump-
535 tion 4 and if $BSViab(K)$ satisfies the required smoothness conditions and n is large
536 enough, at the final iteration q of algorithm 3.1, we have:*

$$537 \quad (3.11) \quad D_h(\hat{H}^q, BSViab(K)) < \epsilon(n).$$

539 *Proof.* Let $V = BSViab(K)$. Proposition 3.10 ensures $H^q = V$ and because of
540 assumption 4, $D_h(\mathcal{A}(V \cap \Gamma(n)), V) < \epsilon(n)$. \square

542 PROPOSITION 3.12. *Assume that $BSViab(K) = BSViab(\delta_0, \theta_0)$ is the backstop
543 viability kernel and is equal to the viability kernel: $BSViab(K) = Viab(K)$. Assume
544 that the conditions of proposition 3.8 are satisfied and $j_A > j_{EDT}^\#(n, \theta, K)$. Under
545 assumption 4 and if $Viab(K)$ satisfies the required smoothness conditions and n is
546 large enough, then for all $\delta \in \mathbb{R}_+^*$ such that $\frac{3\delta}{2} \leq \frac{\delta_0}{2}$, for all $\theta \leq \min(\theta_0, \frac{\delta_0}{2M})$, at the
547 final iteration q of algorithm 3.1, we have:*

$$548 \quad (3.12) \quad D_h(\hat{H}^q, Viab(K)) < \epsilon(n).$$

550 *Proof.* Let $V = Viab(K)$. In the considered conditions, $V = Viab_{EDT}(\theta, K)$
551 because $BSViab(K) \subset Viab_{EDT}(\theta, K)$ and $Viab_{EDT}(\theta, K) \subset V$. Proposition 3.8
552 ensures $H^q \cap \Gamma(n) = V \cap \Gamma(n)$ and because of assumption 4, $D_h(\mathcal{A}(V \cap \Gamma(n)), V) < \epsilon(n)$. \square

554 Note that the case $Viab(K) = BSViab(K)$ is particularly interesting, because the
555 convergence rate $\epsilon(n)$ can be ensured even for $j_{EDT}^\#(\theta, K) = \infty$ and $j_C^\#(\theta, K) = \infty$,
556 whereas if $Viab(K) \neq BSViab(K)$, it requires $j_C^\#(\theta, K) < \infty$. The next section
557 reports tests performed in this case.

558 **4. Tests of the convergence rate when viability kernel and backstop**
559 **viability kernel are equal.** In the tests, the trajectories $X_{x_0, (u, v)_K}(j\theta)$ are approx-
560 imated by the Runge and Kutta method with a time step smaller than θ ensuring
561 that several trajectory points are computed in each grid cube. Hence the approxima-
562 tion error on the trajectory (of the order of n^{-4}) is negligible with respect to the set
563 approximation error (at best of the order of n^{-2}).

564 **4.1. Tested viability problems.** The tests are performed on three viability
565 problems in d dimensions.

566 **4.1.1. Population problem.** In the original 2D version [3], $x_1(t)$ represents
567 the size of a population, which grows or diminishes with the evolution rate $x_2(t)$ that
568 can be modified by the control. The populations should be kept within some bounds.
569 The system is written as follows, $m_1, M_1, m_2, M_2, m_u, M_u$ being positive parameters:

$$570 \quad (4.1) \quad \begin{cases} x_1'(t) = x_1(t)x_2(t) \\ x_2'(t) = u(t), \end{cases} \quad \text{with} \quad \begin{cases} -m_u \leq u(t) \leq +M_u, \\ (x_1(t), x_2(t)) \in K = [m_1, M_1] \times [-m_2, M_2]. \end{cases}$$

571 In the extension to d dimensions, the model includes $d - 1 \geq 2$ populations
572 x_1, x_2, \dots, x_{d-1} and it uses variable $\tilde{x}(t)$:

$$573 \quad (4.2) \quad \tilde{x}(t) = \sum_{i=2}^{d-1} \frac{(M_1 - m_1)^2}{4} - \left(x_i(t) - \frac{M_1 + m_1}{2} \right)^2.$$

574
575 The extended viability problem is:

$$576 \quad (4.3) \quad \begin{cases} x_1'(t) = y(t)(x_1(t) + \alpha\tilde{x}(t)) \\ x_i'(t) = 0, i \in \{2, \dots, d-1\} \\ x_d'(t) = u(t), \end{cases} \quad \text{with} \quad \begin{cases} -m_u \leq u(t) \leq M_u \\ m_1 \leq x_i(t) \leq M_1, i \in \{1, \dots, d-1\}, \\ -m_2 \leq x_d(t) \leq M_2. \end{cases}$$

577 Where α is a parameter. In this system, the dynamics of population x_1 non-
578 linearly depends on the abundance of the other populations which are constant.

579 The analytical definition of the viability kernel of the d -dimensional problem can
580 be directly derived from the one of the 2D case (provided in [3]):

$$581 \quad (4.4) \quad \text{Viab}(K) = \{x \in K, \partial V_-(x) \leq x_d \leq \partial V_+(x)\}, \quad \text{with:}$$

$$582 \quad (4.5) \quad \begin{cases} \partial V_+(x) = \sqrt{2m_u \log \left(\frac{M_x + \alpha\tilde{x}}{x_1 + \alpha\tilde{x}} \right)}, \\ \partial V_-(x) = -\sqrt{2M_u \log \left(\frac{x_1 + \alpha\tilde{x}}{m_x + \alpha\tilde{x}} \right)}. \end{cases}$$

584 The backstop viability kernel of this problem exists and is equal to the viability
585 kernel.

586 **4.1.2. Consumption problem.** In the original 2D problem [3], $x_1(t)$ represents
587 the consumption of a primary good and $x_2(t)$ a critical level of consumption above
588 which the prices can decrease and accelerate consumption and below which, on the
589 contrary the prices increase and decrease the consumption. The critical level x_2 can
590 be modified by a control within some bounds in order to maintain the consumption
591 within some bounds. The system is written as follows (m_u, M_u, m_1, M_1 and M_2 are
592 positive):

$$593 \quad (4.6) \quad \begin{cases} x_1'(t) = x_1(t) - x_2(t) \\ x_2'(t) = u(t), \end{cases} \quad \text{with} \quad \begin{cases} -m_u \leq u(t) \leq M_u, \\ (x_1(t), x_2(t)) \in K = [-m_1, M_1] \times [0, M_2]. \end{cases}$$

594 Like the population model, we extend the consumption model to d dimensions by
 595 considering $d - 1 \geq 2$ consumption variables x_1, x_2, \dots, x_{d-1} . We also define variable
 596 $\tilde{x}(t)$ with equation 4.2. The extended system is:

$$(4.7) \quad \begin{cases} x_1'(t) = (x_1(t) + \alpha\tilde{x}(t)) - x_d(t) \\ x_i'(t) = 0, i = 2, \dots, d - 1 \\ x_d'(t) = u(t), \end{cases} \quad \text{with} \quad \begin{cases} -M_u \leq u(t) \leq M_u, \\ -m_1 \leq x_i(t) \leq M_1, i \in \{1, \dots, d - 1\}, \\ 0 \leq x_d(t) \leq M_2. \end{cases}$$

598 The analytical definition of the viability kernel can easily be derived from its
 599 expression in the 2D problem (provided in [3]):

$$600 \quad (4.8) \quad Viab(K) = \{x \in K, \partial V_-(x) \leq x_1 \leq \partial V_+(x)\} \text{ with:}$$

$$601 \quad (4.9) \quad \begin{cases} \partial V_-(x) = x_d - m_u + m_u \exp\left(\frac{-x_d}{m_u}\right) - \alpha\tilde{x}, \\ \partial V_+(x) = x_d + M_u - M_u \exp\left(\frac{x_d - M_1}{M_u}\right) - \alpha\tilde{x}. \end{cases}$$

603 For this problem also, it can be verified that the backstop viability kernel exists
 604 and equals the viability kernel.

605 **4.1.3. Variants of the models with oblique trajectories.** In both popu-
 606 lation and consumption problems, the fact that $x_i'(t) = 0$ for $i \in \{2, \dots, d - 1\}$ can
 607 be seen as an easy particular case for the resistar approximation because in each 2-
 608 dimensional grid defined by axes x_1 and x_d , the problem to solve is the same as in
 609 2 dimensions. Increasing the dimensionality requires only the interpolation by the
 610 resistars between these 2-dimensional classifications.

611 In order to test how the approach performs on a more difficult problem, instead
 612 of keeping all trajectories in the 2 D spaces parallel to (b_1, b_d) , (b_i , $i \in \{1, \dots, d\}$ being
 613 the canonical basis vectors), for point $x = (x_1, \dots, x_d)$ the trajectory is set in the 2D
 614 space generated by vectors $(z(x), b_d)$, with:

$$615 \quad (4.10) \quad z(x) = b_1 + \sum_{i=2}^{d-1} z_i(x)b_i \text{ with, setting } m(x_1) = \frac{1 - \beta}{2} + \beta x_1,$$

$$616 \quad (4.11) \quad \text{for } i \in \{2, \dots, d - 1\}, z_i(x) = \begin{cases} \beta \frac{x_i}{m(x_1)}, & \text{if } x_i \leq m(x_1), \\ \beta \frac{1 - x_i}{1 - m(x_1)}, & \text{otherwise,} \end{cases}$$

618 where β is a parameter ($0 \leq \beta < 1$) and the problem is rescaled so that $K = [0, 1]^d$.
 619 When x is located on the right line of direction $b_1 + \beta \sum_{i=2}^{d-1} b_i$ which includes point
 620 $(\frac{1-\beta}{2}, \dots, \frac{1-\beta}{2})$, it makes the maximum angle with b_1 . The components $z_i(x)$ equal 0
 621 for $x_i = 0$ or $x_i = 1$.

622 This leads to population or consumption problems in the spaces $(z(x), b_d)$ with
 623 a constraint set equal to $[0, \|z(x)\|] \times [0, 1]$ instead of $[0, 1] \times [0, 1]$. Their viability
 624 kernel in the plane $(z(x), b_d)$ can be derived directly from the viability kernels of the
 625 2D problems.

626 **4.1.4. Spirals problem.** Equation 4.12 defines the spirals problem in d dimen-
 627 sions, with $0 < r_0 < 1$ and $\omega > 0$.

628 (4.12)
$$\begin{cases} r = \sqrt{x_1(t)^2 + x_2(t)^2}, \\ x_1'(t) = -x_2(t) + \omega(r - r_0)x_1(t), \\ x_2'(t) = x_1(t) + \omega(r - r_0)x_2(t), \\ x_i'(t) = -\sigma x_i(t), i \in \{3, \dots, d\}, \end{cases} \quad \text{with } -1 \leq x_i(t) \leq 1, i \in \{1, \dots, d\}.$$

629 Qualitatively, in the plane P generated by canonical basis vectors $b_1 = (1, 0, \dots, 0)$
 630 and $b_2 = (0, 1, 0, \dots, 0)$ and which includes point $c = (0, \dots, 0)$, the dynamics turns
 631 around c in P and if the distance to c is higher than r_0 , then the trajectory is a spiral
 632 which increases its distance to c while the spiral goes towards c when the distance to
 633 c is smaller than r_0 . When the point is exactly at the distance r_0 from c , the radius
 634 is kept constant. The parameter ω rules the increase or decrease of the distance to
 635 c . When x does not belong to P , the dynamics is the combination of the spiral in
 636 the plane generated by (b_1, b_2) which includes x and a translation towards P with
 637 a speed proportional to the distance from x to P (ruled by parameter σ). For sake
 638 of simplicity, there is no control in this problem (in our framework, the control set
 639 includes a single value).

640 In the spirals problem, $j_{EDT}^\sharp(\theta, K)$, the supremum of number of steps in K for
 641 the non-viable points of K , is infinite, while it is finite for the two first problems.
 642 Indeed, consider $x \in K$ such that $\|x - c\| = r_0 + \epsilon$ ($\epsilon > 0$); when ϵ tends to 0, the
 643 trajectory starting from x makes an indefinitely increasing number of turns around c
 644 before exiting from the constraint set. In the 2D example, we set $\omega = 0.01$ and the
 645 system makes a large number of rounds before exiting K even when it is moderately
 646 close to the circle of centre c and radius r_0 .

647 It can be verified that the backstop viability kernel exists and is equal to the
 648 viability kernel. The viability kernel can easily be defined analytically:

649 (4.13)
$$Viab(K) = \{x \in K \mid x_1^2 + x_2^2 \leq r_0^2\}.$$

651 **4.1.5. Parameter values.** Table 1 breaks down the parameter values used in
 652 the tests of the population and consumption models and Table 2 provides the values
 653 used in the tests of the spirals model.

TABLE 1
Parameters of population and consumption models.

Model	m_1	M_1	m_2	M_2	m_u	M_u	α	β	δ	θ	j_A
Popul.2D	0.2	3	2	2	0.5	0.5	0	n.a.	0.1	0.6	50
Popul.dD	0.2	3	2	2	2	0.5	0.15	0.6	0.1	0.3	50
Cons.2D	0	2	0	2.5	0.9	0.9	0	n.a.	0.1	0.4	50
Cons.dD	0	2	0	2.2	0.9	0.9	-0.15	0.6	0.1	0.4	50

TABLE 2
Parameters of spirals model.

	r_0	ω	σ	δ	θ	j_A
2D	0.75	0.01	n.a.	0.1	0.6	10000
dD	0.75	0.1	0.2	0.1	0.6	100

654 In both tables the values are given for problems with modified axis scales so
 655 that $K = [0, 1]^d$. We checked experimentally that these parameters are such that

656 $Viab(K) = BSViab(K) = BSViab_{EDT}(\theta, \delta, K)$. Note that $m_u = 2$ in the Population
657 model in d dimensions in order to get a smoother boundary $\partial V_+(x)$, ensuring that
658 the conditions of the convergence for resistars are satisfied in the considered range of
659 values of n .

660 **4.2. Set approximation algorithms and their convergence rates.** In this
661 subsection, we assume $K = [0, 1]^d$. The tests use two set approximation algorithms.

662 **4.2.1. Nearest vertex.** The nearest vertex approximation $N_n(H)$ of set $H \subset K$
663 is defined from G_n , the set of vertices of the regular grid covering K , as follows:

$$664 \quad (4.14) \quad x \in N_n(H) \iff \exists \xi \in G_n \cap H \mid \|x - \xi\| = \min_{\xi' \in G_n} \|x - \xi'\|.$$

666 PROPOSITION 4.1. *For any $H \subset K$ such that, for all $x \in H$ there exists $\xi \in$
667 $G_n \cap H$ such that $D(x, \xi) \leq \frac{\sqrt{d}}{n}$, we have:*

$$668 \quad (4.15) \quad D_h(H, N_n(H)) = \mathcal{O}(n^{-1}).$$

670

671 *Proof.* By definition, for all point $x \in K$ such that $x \in N_n(H)$ there exists $\xi \in$
672 $(G_n \cap H)$ such that ξ is the point of G_n which is the nearest to x , thus $D(x, \xi) \leq \frac{\sqrt{d}}{2n}$.
673 Therefore, $D(x, H) \leq \frac{\sqrt{d}}{2n}$. Moreover, by hypothesis, for all $x \in H$ there exists $\xi \in G_n$
674 such that $D(x, \xi) < \frac{\sqrt{d}}{n}$. Therefore, $D(x, N_n(H)) \leq \frac{\sqrt{d}}{n}$. We get: $D_h(H, N_n(H)) \leq$
675 $\frac{\sqrt{d}}{n}$. \square

676 **4.2.2. Recursive simplex stars (resistars).** We assume that set $H \subset K$ is a
677 d -dimensional manifold, ∂H , the boundary of H and $\partial_K H = \partial H - (\partial H \cap \partial K)$ are
678 $(d - 1)$ -dimensional manifolds. The resistars are designed for approximating $\partial_K H$ by
679 hypersurfaces made of $(d - 1)$ -dimensional simplices. The first step for deriving resistar
680 surfaces is determining the boundary points $B_H(G_n)$ which are approximations of the
681 intersections between $\partial_K H$ and the edges of the grid. These points are computed on
682 the edges $[v, v']$ of the grid such that one of the vertices is inside H and the other is
683 outside. The estimation of a boundary point is done by successive dichotomies. The
684 simplices defined in a cube share the barycentre of the boundary points located in the
685 cube as a vertex, and their other vertices are defined similarly in the facets and faces
686 of the cube. The complete description is available in [11].

687 THEOREM 4.2 ([11]). *If $\partial_K H$ is a $(d - 1)$ -dimensional manifold in K of reach¹ r
688 such that $r > \sqrt{2}dn^{-1}$, if for all j -dimensional faces F of K , setting $H_F = H \cap F$,
689 $\partial_F H_F$ is a $(j - 1)$ -dimensional manifold of reach $r_F > \sqrt{2}jn^{-1}$, and if all the boundary
690 points are determined with at least $\log_2(n)$ dichotomies, then the Hausdorff distance
691 between H and its resistar approximation decreases like $\mathcal{O}(dn^{-2})$.*

692 **4.3. Evaluating the Hausdorff distance between the viability kernel and**
693 **its approximations.** The procedure evaluates the distance from \hat{H}^q (result of the
694 approximation algorithm) to $Viab(K)$ (known analytically). It assumes that it is close
695 to the distance from $Viab(K)$ to \hat{H}^q . It focuses on the points of the boundary $\partial_K \hat{H}^q$
696 that are likely to be the furthest to $\partial Viab(K)$ in order to limit the computation time.

¹The reach of $\partial_K H$ is the supremum of ρ such that for any point x of K for which $D(x, \partial_K H) = \rho$, there is only one point $y \in \partial_K H$ such that $\|x - y\| = \rho$ [16].

697 **4.3.1. Resistar approximation.** Among the vertices of the simplices defining
698 $\partial_K \hat{H}^q$, the barycentres of all boundary points of a grid cube are likely to be the furthest
699 from $\partial_K \text{Viab}(K)$. The procedure estimates the distance from such vertices x to the
700 projection of x on $\partial_K \text{Viab}(K)$ parallel to the estimated normal vector to the resistar
701 surface at x and it returns the maximum of these distances. It is indeed assumed
702 that the normal vector to $\partial_K \hat{H}^q$ at x is close to the normal vector to $\partial \text{Viab}(K)$ at
703 the nearest point to x of $\partial \text{Viab}(K)$. The procedure estimates the intersection y of
704 $\partial \text{Viab}(K)$ with the segment $[x, x + \nu D_M]$ or with the segment $[x, x - \nu D_M]$ (D_M
705 being a parameter) by performing successive dichotomies. The case when $x + \nu D_M$
706 or $x - \nu D_M$ is located outside K requires a specific management. The estimation
707 of the normal vector to $\partial_K \hat{H}^q$ at point x , is based on the estimation (by successive
708 dichotomies) of $d - 1$ affinely independent points of $\partial_K \hat{H}^q$ at a given distance from
709 x , from which the normal vector can be derived. The method includes a specific
710 treatment for the cases when x is on the border of K or very close to it.

711 **4.3.2. Nearest vertex approximation.** The set \hat{H}^q is now the nearest vertex
712 approximation. The estimation of the Hausdorff distance from \hat{H}^q to $\text{Viab}(K)$ is
713 derived from the method defined for the resistar set approximation. It uses indeed
714 the resistar surface denoted H^* , defined from the boundary points $b = (v+v')/2$ where
715 v and v' define a grid edge $[v, v']$ such that $v \in \hat{H}^q$ and $v' \notin \hat{H}^q$. It estimates the
716 distance from the centres of the cubes containing boundary points of H^* to $\partial \text{Viab}(K)$
717 using the normal to H^* estimated at the barycentre of the boundary points of the
718 cube. Indeed, the centre of the cube is always on the boundary of \hat{H}^q and is likely to
719 be the point from which $\partial \text{Viab}(K)$ is the furthest. The direction normal to $\partial_K H^*$ at
720 the barycentre of the boundary points of the cube is a reasonable approximation of
721 the direction from the centre of the cube to its nearest point in $\partial \text{Viab}(K)$.

722 **4.4. Results.** Figure 1 shows the sets \hat{H}^i for all the iterations of algorithm
723 3.1 applied on the 2D problems. The final result can be visually compared with
724 the theoretical viability kernel (quantified evaluations of the Hausdorff distance are
725 shown on figure 3). Note that the algorithm stops after 4 iterations for the population
726 and consumption problems and after 6 iterations for the spirals problem (even with
727 $\omega = 0.01$). The number of iterations is similar in higher dimensionality. Figure 2
728 shows examples of final results on the 3D problems. The smooth non-linearity along
729 the x_2 axis ruled by parameters α and β appears in the viability kernel approximations
730 of population and consumption problems. Panels (a), (b) and (c) of Figure 4 show the
731 intersection with three chosen hyperplanes of viability kernel resistar approximations
732 of algorithm 3.1 in 6 dimensions with $n = 5$ (grid size: 5^6).

733 On Figure 3, panels (a), (b) and (c) show the estimated Hausdorff distance be-
734 tween the viability kernel and its approximation (y axis) by the nearest vertex and by
735 a resistar surface, in dimensionality 2, 3, 4 and 5, and for different values of the grid
736 size n (x axis). For resistar approximations, the values of n are 7, 9, 13, 17, 25, 33,
737 49, 65, 97, 129, 193, 257 for the 2D problems, 7, 9, 13, 17, 25, 33, 49, 65, 97 for the
738 3D problems, 7, 9, 13, 17, 25, 33, 49 for the 4D problems and 7, 9, 13, 17, 25 for the
739 5D problems. For nearest vertex approximations, the values 7 and 9 are not tested
740 because they are too small for defining properly the sets B_{in}^i . The other values of n
741 are the same. The axes are in a logarithmic scale.

742 Table 3 shows the estimation of the slopes of the logarithm of the Hausdorff
743 distance as a function of the logarithm of n , for the problems in 2 and 3 dimensions.
744 These results are in good agreement with the theoretical prediction of an Hausdorff
745 distance decreasing like n^{-1} for the approximation with the nearest vertex and like

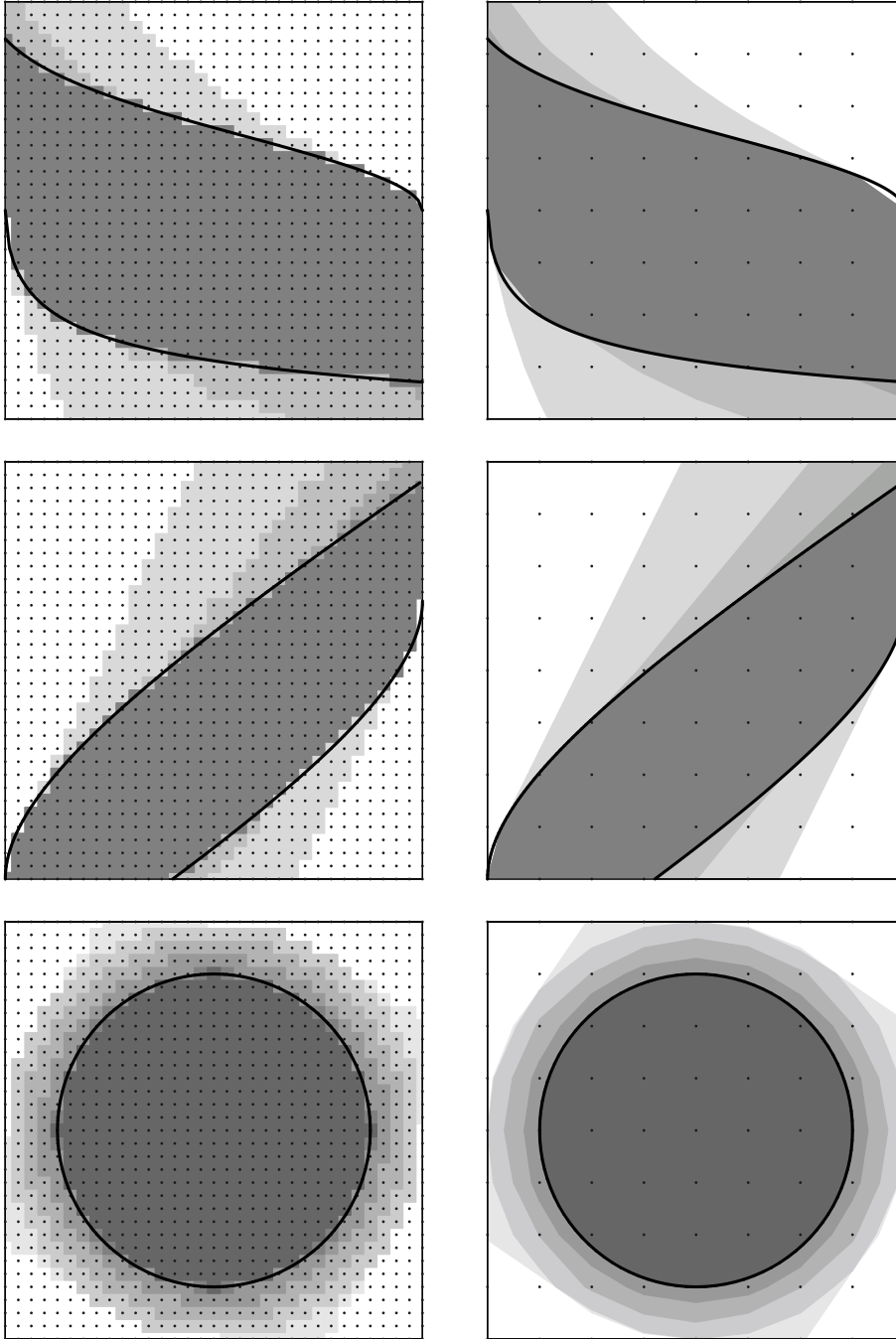


FIGURE 1. Approximations of the viability kernel in 2D for the three problems (first row: population, second row: consumption, third row: spirals). Left column: nearest vertex approximation with $n = 33$ (grid size: 33^2), right column: resistor approximation with $n = 9$ (grid size: 9^2). The black curves are the boundaries of the theoretical viability kernel. The approximations \hat{H}^i are represented in darker and darker grey as i increases. The darkest set is the output of algorithm 3.1.

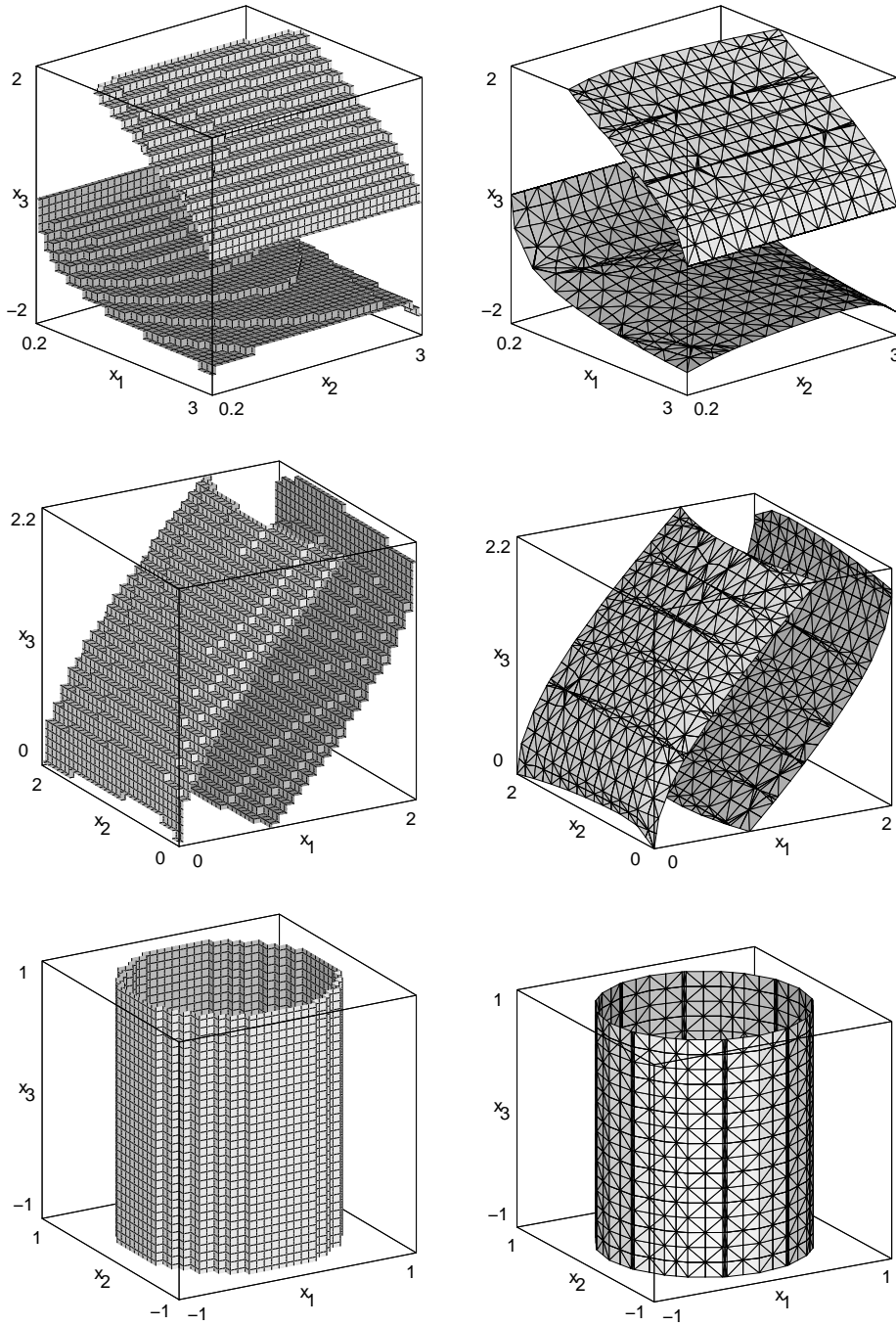


FIGURE 2. Approximations of the viability kernel in 3D for the three problems (first row: population, second row: consumption, third row: spirals). Left column: nearest vertex approximation with $n = 33$ (grid size: 33^3), right column: resistar approximation with $n = 9$ (grid size: 9^3).

	Nearest v.		Resistars	
	2D	3D	2D	3D
Population	-1.00	-0.97	-2.00	-1.99
Consumption	-1.01	-1.03	-2.01	-1.98
Spiral	-1.02	-0.94	-2.02	-2.01

TABLE 3

Slopes of the linear regression of the logarithm of the estimated Hausdorff distance between the viability kernel and its approximation as a function of the logarithm of n . The R^2 values are all superior to 0.98.

746 n^{-2} for the approximation with the resistars.

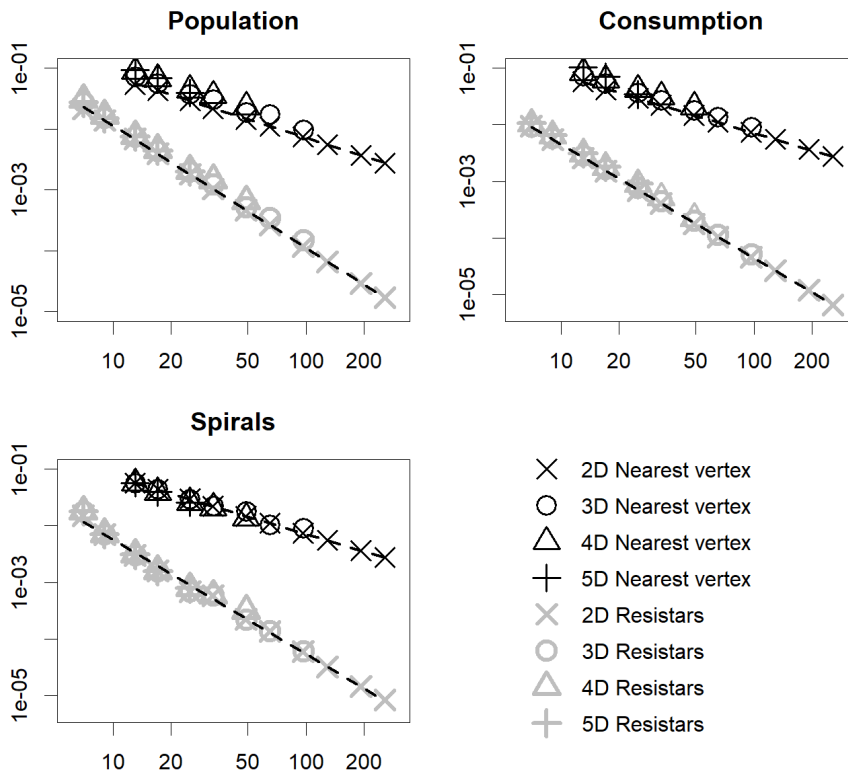


FIGURE 3. Estimation of the Hausdorff distance between the viability kernel and its approximation (y axis) as a function of n defining the grid size as n^d (x axis) for the population, consumption and spirals problems in $d \in \{2, 3, 4, 5\}$ dimensions. The dashed lines are the linear regressions on the 2D values (slopes given in table 3).

747 On figure 3, for a given value of n , the error does not change much when increasing
 748 the dimensionality. This observation is confirmed on Figure 4, panel (d) showing that
 749 the error of the resistar approximation for a grid defined by $n = 5$ does not vary
 750 significantly when the dimensionality d varies from 3 to 9.

5. Discussion - conclusion. In general conditions, there exists a range of parameter values for which $\text{BSViabApp}(n, \delta, \theta, j_A)$, the output of the approximation

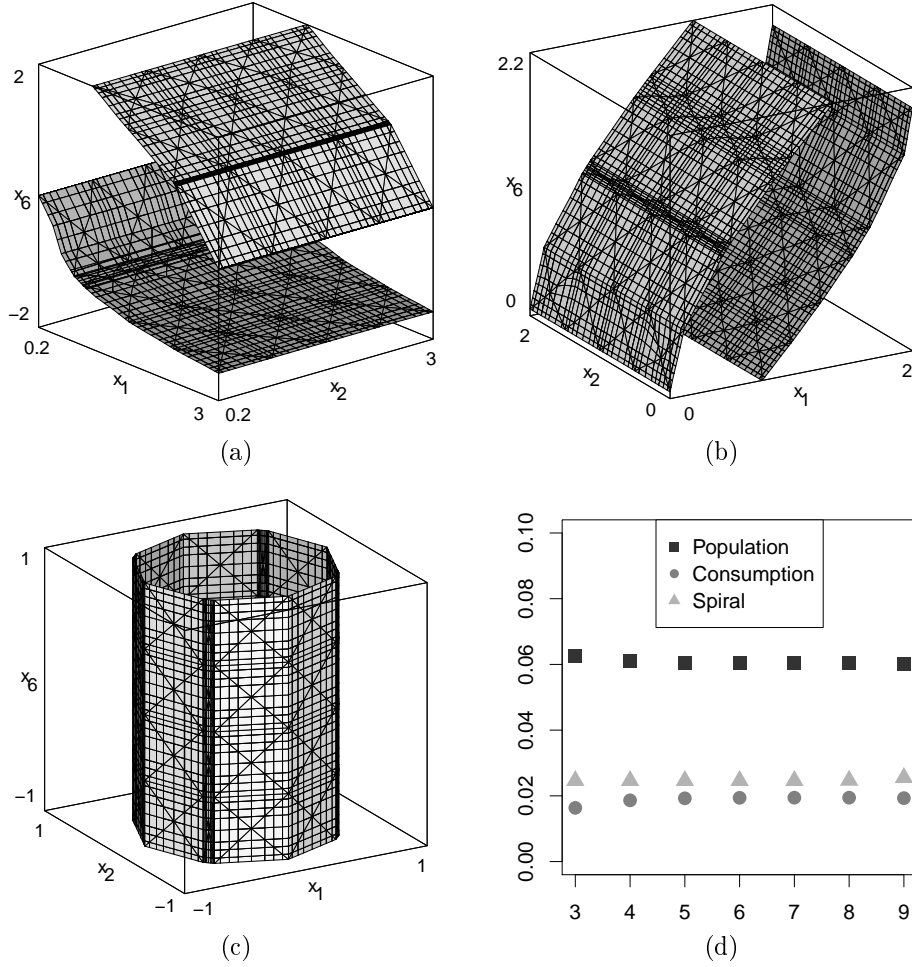


FIGURE 4. Panels (a), (b) and (c): Resistar backstop viability kernel approximations in dimensionality $d = 6$ and $n = 5$ (grid size: 5^6), for respectively the population, consumption and spirals problems. Each panel represents the intersection of the resistar approximation with 3 hyperplanes ($x_3 = 0.02$, $x_4 = 0.05$ and $x_5 = 0.08$ in $K = [0, 1]^6$). Panel (d): Hausdorff distance between viability kernel and its resistar approximation (y axis) for the population, consumption and spirals problems in a grid of size $n = 5$ and dimensionality varying from 3 to 9 (x axis).

algorithm, satisfies:

$$BSViab\left(\frac{3\delta}{2}, \theta, K\right) \cap \Gamma(n) \subset BSViabApp(n, \delta, \theta, j_A) \cap \Gamma(n) \subset Viab_{EDT}(\theta, K) \cap \Gamma(n).$$

The second inclusion implies that all the points of $BSViabApp(n, \delta, \theta, j_A) \cap \Gamma(n)$ are viable. This is generally not guaranteed with the Saint-Pierre algorithm which provides an approximation of a set which contains the viability kernel. This difference is important when the objective is to guarantee the viability of a system. When the backstop viability kernel $BSViab(K)$ exists, under general conditions, there exists a range of parameter values such that:

$$BSViabApp(n, \delta, \theta, j_A) \cap \Gamma(n) = BSViab(K) \cap \Gamma(n).$$

751 Overall, these results lead to a convergence rate of the algorithm to the backstop
752 viability kernel which is the same as the convergence rate of the chosen set approxi-
753 mation technique.

754 When the viability kernel is equal to the backstop viability kernel, the conditions
755 to get this convergence rate are more general. Even when using the nearest vertex
756 approximation, we expect our algorithm to then outperform the current techniques
757 approximating viability kernels for three reasons. Firstly, the convergence to the
758 viability kernel is ensured without decreasing the time step to 0, which is a major
759 difference. Secondly, for a given time step, our algorithm requires a lower number
760 of iterations, especially when the supremum of time steps in K for the non-viable
761 points is infinite (as illustrated on the spiral problem). Thirdly, our algorithm avoids
762 cumulating the error of the successive set approximations taking place in standard
763 algorithms.

764 When using resistars as set approximation technique, if the best conditions are
765 satisfied, the convergence rate of our algorithm is like $\mathcal{O}(n^{-2})$ which significantly
766 increases the advantage over the standard methods, converging at best like $\mathcal{O}(n^{-1})$.
767 Indeed, in order to be as accurate as a resistar approximation using a grid of n^d
768 points, the standard methods need a grid of at least n^{2d} points. For instance, we have
769 shown that it is possible to run resistars approximations in 5 dimensions using a grid
770 of 25^5 (about $8 \cdot 10^6$) points. In order to reach the same accuracy as the one of these
771 approximations, the standard methods would require a grid of at least $(25^2)^5$ (about
772 $9.5 \cdot 10^{13}$) points, which is not manageable by current standard computers.

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REFERENCES

- 774 [1] I. ALVAREZ, R. REUILLON, AND R. D. ALDAMA, *Viabilitree: a kd-tree framework for viability-*
775 *based decision*, archives-ouvertes.fr, (2016).
776 [2] J. AUBIN, *Viability theory*, Birkhäuser, 1991.
777 [3] J. AUBIN, *Elements of viability theory for the analysis of dynamic economics*, Ecole thematique
778 du CNRS ‘Economie Cognitive’, (2002).
779 [4] J.-P. AUBIN, A. BAYEN, AND P. SAINT-PIERRE, *Viability Theory: New Directions*, Springer,
780 2011.
781 [5] O. BOKANOWSKI, S. MARTIN, R. MUNOS, AND H. ZIDANI, *An anti-diffusive scheme for*
782 *viability problems*, Applied Numerical Mathematics, 56 (2006), pp. 1147–1162.
783 [6] N. BONNEUIL, *Computing the viability kernel in large state dimension*, Journal of Mathematical
784 Analysis and Applications, 323 (2006), pp. 1444–1454.
785 [7] P. CARDALIAGUET, M. QUINCAMPOIS, AND P. SAINT-PIERRE, *Numerical schemes for discon-*
786 *tinuous value functions of optimal control*, Set-Valued Analysis, 8 (2000), pp. 11–126.
787 [8] L. CHAPEL, X. CASTELLO, C. BERNARD, G. DEFFUANT, V. EGILUZ, AND S. MARTIN, *Via-*
788 *bility and resilience of languages in competition*, PLOS ONE, 5 (2010).
789 [9] L. CHAPEL, G. DEFFUANT, S. MARTIN, AND C. MULLON, *Defining yield policies in a viability*
790 *approach*, Ecological Modelling, 212 (2008), pp. 10–15.
791 [10] P. COQUELIN, S. MARTIN, AND R. MUNOS, *A dynamic programming approach to viability*
792 *problems*, in IEEE ADPRL Symposium on Approximate Dynamic Programming and Rein-
793 *forcement Learning*, 2007.
794 [11] G. DEFFUANT, *Recursive simplex stars*, arxiv.org/abs/1707.08373, (2018), [http://arxiv.org/](http://arxiv.org/abs/1707.08373)
795 [abs/1707.08373](http://arxiv.org/abs/1707.08373).
796 [12] G. DEFFUANT, L. CHAPEL, AND S. MARTIN, *Approximating viability kernel with support*
797 *vector machines*, IEEE Transactions on Automatic Control, 52 (2007), pp. 933–937.
798 [13] G. DEFFUANT AND N. GILBERT, eds., *Viability and Resilience of Complex Systems: Concepts,*
799 *Methods and Case Studies from Ecology and Society*, Springer, 2011.
800 [14] M. DELARA AND L. DOYEN, *Sustainable Management of Natural Resources. Mathematical*
801 *Models and Methods*, Springer, 2008.
802 [15] L. DOYEN, A. CISSÉ, S. GOURGET, L. MOUYSSSET, P. HARDY, C. B. B. . BLANCHARD,
803 F. JIGUET, J. PERREAU, AND O. THÉBAUD, *Ecological-economic modelling for the sustain-*
804 *able management of biodiversity*, Computational Management Science, 10 (2013), pp. 353–

- 805 364.
- 806 [16] H. FEDERER, *Curvature measures*, Transactions of the American Mathematical Society, 93
807 (1959), pp. 418–491.
- 808 [17] J. HEITZIG, T. KITTEL, J. F. DONGES, AND N. MOLKENTHIN, *Topology of sustainable man-*
809 *agement of dynamical systems with desirable states: from defining planetary boundaries to*
810 *safe operating spaces in the Earth system*, Earth System Dynamics, 7 (2016), pp. 21–50,
811 <https://doi.org/10.5194/esd-7-21-2016>.
- 812 [18] M. KALISIAK AND M. VAN DE PANNE, *Approximate safety enforcement using computed viability*
813 *envelopes*, in Proceedings of IEEE International Conference on Robotics and Automation,
814 2004, pp. 4289–4294.
- 815 [19] J. B. KRAWCZYCK AND A. PHARO, *Viability Kernel Approximation, Analysis and Simulation*
816 *Application - VIKAASA* *Viability Kernel Approximation, Analysis and Simulation Appli-*
817 *cation - VIKAASA*, 2011.
- 818 [20] J. MAIDENS, S. KAYNAMA, I. MITCHELL, M. OISHI, AND G. DUMONT, *Lagrangian methods*
819 *for approximating the viability kernel in high-dimensional systems*, Automatica, 49 (2013),
820 pp. 2017–2029.
- 821 [21] S. MARTIN, *The cost of restoration as a way of defining resilience: a viability approach ap-*
822 *plied to a model of lake eutrophication*, Ecology and Society, 9(2) (2004), [http://www.](http://www.ecologyandsociety.org/vol9/iss2/art8)
823 [ecologyandsociety.org/vol9/iss2/art8](http://www.ecologyandsociety.org/vol9/iss2/art8).
- 824 [22] J. MATHIAS, B. BONTÉ, T. CORDONNIER, AND F. DEMOROGUES, *Using the viability theory*
825 *for assessing flexibility of forest managers under ecological intensification*, Environmental
826 Management, 56 (2015), pp. 1170–1183.
- 827 [23] S. MESMOUDI, I. ALVAREZ, S. MARTIN, R. REULLON, M. SICARD, AND N. PERROT., *Coupling*
828 *geometric analysis and viability theory for system exploration: Application to a living food*
829 *system*, Journal of Process Control, (2014).
- 830 [24] A. OUBRAHAM AND G. ZACCOUR, *A survey of applications of viability theory to the sustainable*
831 *exploitation of renewable resources*, Ecological economics, 145 (2018), pp. 346–367.
- 832 [25] J. RIEGER, *Shadowing and the viability kernel algorithm*, Applied Mathematics and Optimiza-
833 tion, 60 (2009), pp. 429–441.
- 834 [26] C. ROUGÉ, J.-D. MATHIAS, AND G. DEFFUANT, *Extending the viability theory framework*
835 *of resilience to uncertain dynamics, and application to lake eutrophication*, Ecological
836 Indicators, 29 (2013), pp. 420–433.
- 837 [27] R. SABATIER, L. DOYEN, AND M. TICHIT, *Action versus result-oriented schemes in a grassland*
838 *agroecosystem: a dynamic modelling approach*, PLOS ONE, 7 (2012).
- 839 [28] P. SAINT-PIERRE, *Approximation of viability kernel*, App. Math. Optim., 29 (1994), pp. 187–
840 209.
- 841 [29] A. SCHULBAUER AND R. SUMALA, *Economic viability and small-scale fisheries: A review*,
842 Ecological Economics, 124 (2016), pp. 69–75.
- 843 [30] C. TOMLIN, I. MITCHELL, A. BAYEN, AND M. OSISHI, *Computational techniques for the*
844 *verification and control of hybrid systems*, in Proceedings of IEEE 91, 2003.