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Hilbert space-valued fractionally integrated autoregressive moving average processes with long memory operators

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Abstract

Fractionally integrated autoregressive moving average (FIARMA) processes have been widely and successfully used to model and predict univariate time series exhibiting long range dependence. Vector and functional extensions of these processes have also been considered more recently. Here we study these processes by relying on a spectral domain approach in the case where the processes are valued in a separable Hilbert space \mathcal{H}_0 . In this framework, the usual univariate long memory parameter d is replaced by a long memory *operator* D acting on \mathcal{H}_0 , leading to a class of \mathcal{H}_0 -valued FIARMA(D, p, q) processes, where p and q are the degrees of the AR and MA polynomials. When D is a normal operator, we provide a necessary and sufficient condition for the D -fractional integration of an \mathcal{H}_0 -valued ARMA(p, q) process to be well defined. Then, we derive the best predictor for a class of causal FIARMA processes and study how this best predictor can be consistently estimated from a finite sample of the process. To this end, we provide a general result on quadratic functionals of the periodogram, which incidentally yields a result of independent interest. Namely, for any ergodic stationary process valued in \mathcal{H}_0 with a finite second moment, the empirical autocovariance operator converges, in trace-norm, to the true autocovariance operator almost surely at each lag.

Keywords: ARFIMA processes. Long memory. Spectral representation of random processes.

Functional time series. Hilbert space

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1. Introduction

Over the past several decades, the study of weakly stationary time series valued in a separable Hilbert space has been an active field of research. For example, functional ARMA processes were discussed in [3, 33, 22], a spectral theory was detailed in [27, 26, 34] and several estimation methods were studied in [18, 19, 17, 20, 23, 2, 24, 9, 1, 32]. However, these references mainly focus on short memory processes. The study of long memory processes valued in a separable Hilbert space is a more recent topic as seen in [29, 6, 5, 14, 25]. More specifically, in [25, Section 4], the fractionally integrated autoregressive moving average (often abbreviated as ARFIMA, but we prefer to use FIARMA for reasons that will be made explicit in Remark 1) processes are generalized to the case of *curve*, or *functional*, time series. In short, the authors consider the functional case in which the Hilbert space is an L^2 space of real valued functions defined on a compact subset of \mathbb{R} , say $[0, 1]$, and they introduce the time series $(X_t)_{t \in \mathbb{Z}}$ valued in this Hilbert space defined by

$$X_t(v) = Y_t + \sum_{k=1}^{\infty} \frac{\prod_{j=0}^{k-1} (d+j)}{k!} Y_{t-k}(v), \quad t \in \mathbb{Z}, v \in [0, 1], \quad (1.1)$$

where $-1/2 < d < 1/2$ and Y_t is a functional ARMA process. As pointed out in [25, Remark 9], taking the same d for all $v \in [0, 1]$ in (1.1) is highly restrictive compared to other long memory processes recently introduced. For instance in [6, 5], they consider long memory processes of the form

$$X_t(v) = \sum_{k=0}^{\infty} (1+k)^{-n(v)} \epsilon_{t-k}(v), \quad t \in \mathbb{Z}, v \in \mathcal{V},$$

where $(\mathcal{V}, \mathcal{V}, \xi)$ is a σ -finite measure space, and $(\epsilon_t)_{t \in \mathbb{Z}}$ is a white noise valued in $L^2(\mathcal{V}, \mathcal{V}, \xi)$. Since the ratio in (1.1) is asymptotically equivalent to $(1+k)^{-1+d}$ as $k \rightarrow \infty$, this new process is, in fact, close to the previous one in the case where $n(v) = d - 1$ for all $v \in \mathcal{V}$. A formulation that is not restricted to an L^2 space was proposed in [14] where the author considers long memory processes of the form

$$X_t = \sum_{k=0}^{\infty} (1+k)^{-N} \epsilon_{t-k}, \quad t \in \mathbb{Z}. \quad (1.2)$$

Here, $(\epsilon_t)_{t \in \mathbb{Z}}$ is a white noise valued in a separable Hilbert space \mathcal{H}_0 and N is a bounded normal operator on \mathcal{H}_0 . This suggests defining FIARMA processes in (1.1) with d replaced by a function

50 $d(v)$, or in the case where it is valued in an arbitrary separable Hilbert space \mathcal{H}_0 , by a bounded normal operator D acting on this space.

In this paper, we fill this gap by providing a definition of FIARMA processes valued in a separable Hilbert space \mathcal{H}_0 with a long memory operator D , taken as a bounded linear operator on \mathcal{H}_0 . If the spectrum of D is included in $(-\infty, 1/2)$, the FIARMA process is well defined. However, 55 such a condition is not necessary: if D is normal, then we rely on its spectral decomposition and find necessary and sufficient conditions for the \mathcal{H}_0 -valued FIARMA process with long memory operator D to be well defined, which show that the spectrum of D may in fact reach the value $1/2$. This allows us to compare FIARMA processes with the processes defined by (1.2) as in [14].

The definition of FARMA processes relies on linear filtering in the spectral domain. It is a 60 well known fact that linear filtering of real valued time series in the time domain is equivalent to pointwise multiplication by a transfer function in the frequency domain. This duality also applies to Hilbert space valued time series using a proper spectral representation for them. In this context, pointwise multiplication becomes a pointwise application of an operator-valued transfer function defined on the set of frequencies. A complete account is provided in [13]. Here, we rely 65 on the spectral approach to define a D -fractional integration filter acting on a weakly stationary process X valued in \mathcal{H}_0 . We provide necessary and sufficient conditions for this filter to apply to an \mathcal{H}_0 -valued ARMA process X . When the ARMA process is causal, we derive the best predictor of X_t given its past $(X_s)_{s < t}$. It is thus of interest to investigate whether this best predictor can be consistently estimated from a finite sample X_1, \dots, X_n . We provide a positive answer to this 70 question when the long memory parameter operator D has a positive definite real part, under mild additional conditions. To this end, we study quadratic functionals based on the periodogram of X_1, \dots, X_n . A very simple example of such quadratic functional is the empirical covariance operator at a given lag, for which we obtain the following result of independent interest.

Theorem 1. *Let \mathcal{H}_0 be a separable Hilbert space and let $(X_t)_t$ be an \mathcal{H}_0 -valued ergodic stationary 75 process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and satisfying $\mathbb{E} \left[\|X_0\|_{\mathcal{H}_0}^2 \right] < \infty$. Let us define, for all $n \geq 1$ and $1 \leq k \leq n$,*

$$X_k^{(c,n)} = X_k - \frac{1}{n} \sum_{j=1}^n X_j. \quad (1.3)$$

Then, we have, for all $h \in \mathbb{Z}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\substack{1 \leq k, k' \leq n \\ k - k' = h}} \left(X_k^{(c,n)} \right) \otimes \left(X_{k'}^{(c,n)} \right) = \text{Cov} (X_h, X_0) \quad \text{in } \mathcal{S}_1(\mathcal{H}_0), \quad \mathbb{P}\text{-a.s.}, \quad (1.4)$$

where, for all $x, y \in \mathcal{H}_0$, $x \otimes y$ denotes the usual rank one tensor operator and $\mathcal{S}_1(\mathcal{H}_0)$ denotes the space of trace-class operators endowed with the trace-norm.

80 What makes Theorem 1 of particular interest and novel based on our up-to-date-knowledge, is that the a.s. convergence holds in $\mathcal{S}_1(\mathcal{H}_0)$ in (1.4).

This paper is organized as follows. We first recall in Section 2 the necessary definitions and facts on operator theory and linear filtering needed for our purpose. Then, the construction of FIARMA processes is introduced and discussed in Section 3 with a focus on the case where the 85 long memory operator is normal. In Section 4, the prediction of FIARMA processes is studied. To this end, in Section 4.1, we provide general results for parametric contrast estimation in the spectral domain, based on a finite sample. Then, in Section 4.2, we show how to apply these results for FIARMA prediction. Finally, proofs are provided in Section 5. In particular, Theorem 1 is proven in Section 5.3.

90 2. Preliminaries and useful notation

2.1. Operators, measurability and integrals

Throughout this paper, we denote by $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ the set of continuous linear operators defined on the separable (complex) Hilbert space \mathcal{H}_0 onto the separable (complex) Hilbert space \mathcal{G}_0 .

The operator norm on $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ is denoted by $\|\cdot\|_\infty$. We denote by $\mathcal{S}_\infty(\mathcal{H}_0, \mathcal{G}_0)$ its subset of compact operators, and by $\mathcal{S}_1(\mathcal{H}_0, \mathcal{G}_0)$ and $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$, its subsets of *trace-class* and *Hilbert-Schmidt* operators, respectively, with their respective norms denoted by $\|\cdot\|_1$ and $\|\cdot\|_2$. We follow the usual convention of omitting \mathcal{G}_0 in the notation of operator spaces when $\mathcal{G}_0 = \mathcal{H}_0$. We use the notation P^H for the Hermitian adjoint of an operator $P \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$. We say that $P \in \mathcal{L}_b(\mathcal{H}_0)$ is invertible if P admits an inverse in the \mathbb{C} -algebra $\mathcal{L}_b(\mathcal{H}_0)$. We denote by $\sigma(P)$ the spectrum of an operator $P \in \mathcal{L}_b(\mathcal{H}_0)$, defined as the set $\{z \in \mathbb{C} : z \text{Id}_{\mathcal{H}_0} - P \text{ is not invertible}\}$, where $\text{Id}_{\mathcal{H}_0}$ denotes the identity operator on \mathcal{H}_0 . An operator $P \in \mathcal{L}_b(\mathcal{H}_0)$ is said to be *normal* if $PP^H = P^HP$ and we denote by $\mathcal{N}(\mathcal{H}_0)$ the set of normal bounded operators. We further denote by $\mathcal{L}_b^+(\mathcal{H}_0)$, $\mathcal{S}_1^+(\mathcal{H}_0)$ and $\mathcal{S}_2^+(\mathcal{H}_0)$ the sets of positive, positive trace-class and positive Hilbert-Schmidt operators. Here *positive* refers to *positive-semidefinite*, that is, P is positive if $\langle Px, x \rangle_{\mathcal{H}_0} \geq 0$ for all x . For a positive operator P , the operator $P^{1/2}$ is the unique positive operator satisfying $(P^{1/2})^2 = P$. A general and detailed presentation of operator theory can be found in [35].

We will make extensive use of integrals of functions valued in a Banach space (see [11, Chapter 1] for details). Given a measure space $(\Lambda, \mathcal{A}, \mu)$, a Banach space $(E, \|\cdot\|_E)$ and $p \in [1, \infty]$, we denote by $\mathcal{L}^p(\Lambda, \mathcal{A}, E, \mu)$ the space of functions $f : \Lambda \rightarrow E$ which are Borel-measurable such that $\int \|f\|_E^p d\mu$ (or μ -essup $\|f\|_E$ for $p = \infty$) is finite. Its quotient space for the μ -a.e. equality is denoted by $L^p(\Lambda, \mathcal{A}, E, \mu)$. We use the same notation for $E = \mathcal{L}_b^+(\mathcal{H}_0)$, $\mathcal{S}_1^+(\mathcal{H}_0)$ or $\mathcal{S}_2^+(\mathcal{H}_0)$, in which case $L^p(\Lambda, \mathcal{A}, E, \mu)$ is a cone subset of the corresponding L^p space.

In the particular case where $E = \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ for two separable Hilbert spaces $\mathcal{H}_0, \mathcal{G}_0$, we also use a weaker notion of measurability. Namely, we say that a function $\Phi : \Lambda \rightarrow \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ is *simply measurable* if for all $x \in \mathcal{H}_0$, $\lambda \mapsto \Phi(\lambda)x$ is measurable as a \mathcal{G}_0 -valued function. The set of simple measurable functions from (Λ, \mathcal{A}) to $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ is denoted by $\mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$ where, again, we omit \mathcal{G}_0 if $\mathcal{H}_0 = \mathcal{G}_0$. A mapping $\Phi : \Lambda \rightarrow E$ with $E = \mathcal{S}_1(\mathcal{H}_0, \mathcal{G}_0)$ or $E = \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ is simply measurable if and only if it is Borel measurable (see Lemma 4.1 in [13]). A useful consequence is that, if $\Phi \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1^+(\mathcal{H}_0), \mu)$, then the function $\Phi^{1/2} : \lambda \mapsto \Phi(\lambda)^{1/2}$ is in $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2^+(\mathcal{H}_0), \mu)$.

2.2. Linear filtering of Hilbert space-valued time series in the spectral domain

This section gathers the spectral theory used for linear filtering of times series valued in a separable Hilbert space. We refer the reader to [13, Section 3] for details. In the following, we denote by \mathbb{T} the set $\mathbb{R}/2\pi\mathbb{Z}$, which can be represented by an interval such as $[-\pi, \pi)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and \mathcal{H}_0 a separable Hilbert space. We recall that the expectation of $X \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is the unique vector $\mathbb{E}[X] \in \mathcal{H}_0$ satisfying

$$\langle \mathbb{E}[X], x \rangle_{\mathcal{H}_0} = \mathbb{E}[\langle X, x \rangle_{\mathcal{H}_0}], \quad \text{for all } x \in \mathcal{H}_0.$$

The covariance operator between $X, Y \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is the unique linear operator $\text{Cov}(X, Y) \in \mathcal{L}_b(\mathcal{H}_0)$ satisfying

$$\langle \text{Cov}(X, Y) y, x \rangle_{\mathcal{H}_0} = \text{Cov}(\langle X, x \rangle_{\mathcal{H}_0}, \langle Y, y \rangle_{\mathcal{H}_0}), \quad \text{for all } x, y \in \mathcal{H}_0.$$

A process $X := (X_t)_{t \in \mathbb{Z}}$ is said to be an \mathcal{H}_0 -valued, weakly stationary process if

- (i) For all $t \in \mathbb{Z}$, $X_t \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$.
- (ii) For all $t \in \mathbb{Z}$, $\mathbb{E}[X_t] = \mathbb{E}[X_0]$. We say that X is centered if $\mathbb{E}[X_0] = 0$.
- (iii) For all $t, h \in \mathbb{Z}$, $\text{Cov}(X_{t+h}, X_t) = \text{Cov}(X_h, X_0)$.

We denote by $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ the space of all centered random variables in $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$. As explained in [13, Example 1.8], this space is a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$ -module whose Gramian is defined as the covariance operator. Let $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and $X = (X_t)_{t \in \mathbb{Z}} \in \mathcal{H}^{\mathbb{Z}}$ be a centered, weakly stationary, \mathcal{H}_0 -valued time series. Following [13, Section 3], a spectral representation for X amounts to define a random Gramian-orthogonally scattered measure \hat{X} on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ such that

$$X_t = \int e^{i\lambda t} \hat{X}(d\lambda) \quad \text{for all } t \in \mathbb{Z}. \quad (2.1)$$

The intensity measure $\nu_X : \mathcal{B}(\mathbb{T}) \rightarrow \mathcal{S}_1^+(\mathcal{H}_0)$ of \hat{X} is called the *spectral operator measure* and is characterized by the identity

$$\text{Cov}(X_h, X_0) = \int e^{ih\lambda} \nu_X(d\lambda), \quad \text{for all } h \in \mathbb{Z}.$$

130 The spectral operator measure is a trace-class Positive Operator-Valued Measure (p.o.v.m.) in the sense that it is a mapping from $\mathcal{B}(\mathbb{T})$ to $\mathcal{S}_1^+(\mathcal{H}_0)$ which is σ -additive for the $\|\cdot\|_1$ -norm. Note that, in this case, the mapping $\|\nu_X\|_1 : A \mapsto \|\nu_X(A)\|_1$ is a finite non-negative measure. Throughout this paper, we use the Radon-Nikodym property of the trace-class p.o.v.m. ν_X which is a consequence of Theorem 1 in [10, Chapter III, Section 3]. Namely, for any σ -finite non-negative measure μ on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, which dominates $\|\nu_X\|_1$, there exists a unique $g_X \in L^1(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{S}_1^+(\mathcal{H}_0), \mu)$ such that, 135 for all $A \in \mathcal{B}(\mathbb{T})$, $\nu_X(A) = \int_A g_X d\mu$. In this case, we say that g_X is the *spectral operator density* of X with respect to μ , and we write $d\nu_X = g d\mu$. In the following, when we say that g is the spectral operator density of X with respect to a σ -finite non-negative measure μ , it is implicitly assumed that μ dominates $\|\nu_X\|_1$.

Let us now briefly introduce the linear filtering in the spectral domain. We only state the facts that will be useful in the following and refer the reader to [13] for further details. The spectral representation (2.1) can be extended to define a Gramian-isometric mapping from the modular *spectral domain* $\hat{\mathcal{H}}^X$ to the modular *time domain* \mathcal{H}^X , also denoted as an integral with respect to \hat{X} , namely,

$$Y = \int \Phi(\lambda) \hat{X}(d\lambda), \quad Y \in \mathcal{H}^X, \Phi \in \hat{\mathcal{H}}^X.$$

Here, \mathcal{H}^X is the smallest closed linear subspace of $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$, which contains $\{X_t : t \in \mathbb{Z}\}$ and is stable through the left multiplication by any operator of $\mathcal{L}_b(\mathcal{H}_0)$. The space $\hat{\mathcal{H}}^X$ is its spectral counterpart, a space of operator-valued functions defined on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ which only depends on ν_X and is stable through the same module action, namely, through left multiplication by an operator of $\mathcal{L}_b(\mathcal{H}_0)$. Conversely, given an operator-valued function Φ defined on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, we denote by $\mathcal{S}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ the class of all centered weakly stationary processes X such that $\Phi \in \hat{\mathcal{H}}^X$. Then, the time-shift invariant linear filter with *transfer operator function* Φ is the mapping defined on $\mathcal{S}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ by mapping a centered weakly stationary process X (the input) to the centered weakly stationary process Y (the output) defined by

$$Y_t = \int e^{it\lambda} \Phi(\lambda) \hat{X}(d\lambda), \quad t \in \mathbb{Z},$$

140 which we also write

$$Y = F_\Phi(X) \quad \text{or} \quad \hat{Y}(d\lambda) = \Phi(\lambda) \hat{X}(d\lambda). \quad (2.2)$$

We will use the following result, where we characterize the domain of definition of a filter F_Φ in the case where Φ is valued in $\mathcal{L}_b(\mathcal{H}_0)$. It follows by applying [13, Proposition 3.15] with $\mathbb{G} = \mathbb{Z}$ and $\mathcal{G}_0 = \mathcal{H}_0$.

Proposition 1. *Let \mathcal{H}_0 be a separable Hilbert space, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\Phi \in \mathbb{F}_s(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{H}_0)$. Let X be an \mathcal{H}_0 -valued centered weakly stationary process admitting g_X as a spectral operator density with respect to a σ -finite non-negative measure μ on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$. Then, the mapping $\|\Phi g_X \Phi^H\|_1$ is measurable from $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and we have $X \in \mathcal{S}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ if and only if $\int \|\Phi g_X \Phi^H\|_1 d\mu < \infty$.* 145

3. Hilbert space-valued FIARMA processes

150 In this section, we propose a definition of FIARMA processes valued in a separable Hilbert space thus extending the definition of [25, Section 4] to a long memory operator D . This definition is introduced in Section 3.1 where we also recall known results on the existence of \mathcal{H}_0 -valued ARMA processes. We then state the main results, namely 1) Theorem 2 (resp. Corollary 1),

155 where a sufficient condition is given for a weakly stationary \mathcal{H}_0 -valued process (resp. a \mathcal{H}_0 -valued ARMA process) X to belong the domain of definition of the fractional integration operator filter, 2) Theorem 3 where necessary and sufficient conditions are given for a weakly stationary \mathcal{H}_0 -valued process X to belong the domain of definition of the fractional integration operator filter when D is a normal operator, 3) Corollary 2 where we specify these conditions to the case where X is an ARMA process, thus ensuring the existence of FIARMA processes, and 4) Proposition 4
 160 where we compare the obtained FIARMA processes to the processes introduced in [14]. The main definitions are in Section 3.1, Point 1) is treated in Section 3.2, Points 2) and 3) in Section 3.3 and Point 4) in Section 3.4.

3.1. Definition of FIARMA processes

Let \mathcal{H}_0 be a separable Hilbert space. In the following, for all $D \in \mathcal{L}_b(\mathcal{H}_0)$ and $z \in \mathbb{C} \setminus [1, \infty)$, we will use

$$(1 - z)^D = \exp(D \ln(1 - z)) = \sum_{k=0}^{\infty} \frac{1}{k!} (D \ln(1 - z))^k,$$

165 where \ln denotes the principal complex logarithm, so that $z \mapsto \ln(1 - z)$ is holomorphic on $\mathbb{C} \setminus [1, \infty)$, and so is $z \mapsto (1 - z)^D$, as a $\mathcal{L}_b(\mathcal{H}_0)$ -valued function (see [15, Chapter 1] for an overview on the subject). Let us now introduce the fractional integration operator transfer function. It is an extension of the one-dimensional case introduced in [21].

Definition 1 (Fractional integration operator transfer function \mathbf{I}_D). *Let \mathcal{H}_0 be a separable Hilbert space and $D \in \mathcal{L}_b(\mathcal{H}_0)$. We define the D -order fractional integration operator transfer function \mathbf{I}_D by*

$$\mathbf{I}_D(\lambda) = \begin{cases} (1 - e^{-i\lambda})^{-D} & \text{if } \lambda \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Using the properties of $z \mapsto (1 - z)^D$ recalled previously, we see that \mathbf{I}_D is a mapping from \mathbb{T} to $\mathcal{L}_b(\mathcal{H}_0)$, continuous on $\mathbb{T} \setminus \{0\}$. Then, we have $\mathbf{I}_D \in \mathbb{F}_s(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{H}_0)$ and we can define the filter
 170 $F_{\mathbf{I}_D}$ acting on centered \mathcal{H}_0 -valued weakly stationary processes as in (2.2) of which the domain of definition $\mathcal{S}_{\mathbf{I}_D}(\Omega, \mathcal{F}, \mathbb{P})$ can thus be characterized by Proposition 1. Since \mathbf{I}_D has a singularity at the null frequency, the domain $\mathcal{S}_{\mathbf{I}_D}(\Omega, \mathcal{F}, \mathbb{P})$ is not obvious. For instance, in the scalar case, it is well known that if X has a positive and continuous spectral density at the null frequency, then $F_{\mathbf{I}_d}(X)$ is well defined if and only if $d < 1/2$. Based on Proposition 1, we show that a similar
 175 sufficient condition holds in the Hilbert space valued case in Section 3.2, and provide a complete description of $\mathcal{S}_{\mathbf{I}_D}(\Omega, \mathcal{F}, \mathbb{P})$ in Section 3.3 when D is a normal operator.

A *fractionally integrated autoregressive moving average* (FIARMA) process is simply the output of the filter in the case where X is an \mathcal{H}_0 -valued autoregressive moving average (ARMA) process. Let us first recall a basic result on the existence of weakly stationary ARMA processes (see [33,
 180 Corollary 2.2]).

Proposition 2. *Let \mathcal{H}_0 be a separable Hilbert space and p, q be two positive integers. Let $A_1, \dots, A_p \in \mathcal{L}_b(\mathcal{H}_0)$, $B_1, \dots, B_q \in \mathcal{L}_b(\mathcal{H}_0)$ and $Z = (Z_t)_{t \in \mathbb{Z}}$ be an \mathcal{H}_0 -valued white noise (i.e. a centered weakly stationary \mathcal{H}_0 -valued process with constant spectral density operator). Suppose that*

$$\mathbb{P}(z) := \text{Id}_{\mathcal{H}_0} - \sum_{k=1}^p A_k z^k \quad \text{is invertible for all } z \in \mathbb{U}, \quad (3.1)$$

185 where $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ is the complex unit circle. Then,

$$X_t - \sum_{k=1}^p A_k X_{t-k} = Z_t + \sum_{k=1}^q B_k Z_{t-k}, \quad t \in \mathbb{Z}, \quad (3.2)$$

admits a unique weakly stationary solution. This solution is called an \mathcal{H}_0 -valued ARMA(p, q) process.

Explicit constructions of the solution in the time domain can be found in [3, 33, 22], under various assumptions. Using a spectral approach, with ϕ as in (3.1) and $\theta(z) := \text{Id}_{\mathcal{H}_0} + \sum_{k=1}^p B_k z^k$, the solution is more directly given by

$$\hat{X}(d\lambda) = [\phi(e^{-i\lambda})]^{-1} \theta(e^{-i\lambda}) \hat{Z}(d\lambda),$$

using the notation introduced in (2.2). In the following, for any integer $d \in \mathbb{N}$, $\mathcal{P}_d(\mathcal{H}_0)$ denotes the set of polynomials \mathfrak{p} of degree d with coefficients in $\mathcal{L}_b(\mathcal{H}_0)$, such that $\mathfrak{p}(0) = \text{Id}_{\mathcal{H}_0}$ and $\mathcal{P}_d^*(\mathcal{H}_0)$ denotes the subset of all $\mathfrak{p} \in \mathcal{P}_d(\mathcal{H}_0)$, which are invertible on \mathbb{U} . In particular, (3.1) is equivalent to saying that $\phi \in \mathcal{P}_d^*(\mathcal{H}_0)$. Time domain approaches for defining ARMA processes are easier to derive when Condition (3.1) is extended on the closed unit disk, that is,

$$\phi(z) = \text{Id}_{\mathcal{H}_0} - \sum_{k=1}^p A_k z^k \quad \text{is invertible for all } z \in \overline{\mathbb{D}}, \quad (3.3)$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ denote the open and closed complex unit discs of \mathbb{C} , respectively. We do not need Condition (3.3) for defining FIARMA processes. However, we will assume that ϕ and θ satisfy such a condition to derive predictors (see Section 4), as in the well known case of univariate FIARMA processes.

We can now define FIARMA processes as follows.

Definition 2 (Hilbert space-valued FIARMA processes). *Let \mathcal{H}_0 be a separable Hilbert space and p, q be two non-negative integers. Let $D \in \mathcal{L}_b(\mathcal{H}_0)$, $\theta \in \mathcal{P}_q(\mathcal{H}_0)$, $\phi \in \mathcal{P}_p^*(\mathcal{H}_0)$ and Z be an \mathcal{H}_0 -valued centered white noise. Let X be the ARMA(p, q) process defined by $\hat{X}(d\lambda) = [\phi(e^{-i\lambda})]^{-1} \theta(e^{-i\lambda}) \hat{Z}(d\lambda)$ and suppose that $X \in \mathcal{S}_{\mathbf{I}_D}(\Omega, \mathcal{F}, \mathbb{P})$. Then, the process defined by $Y = F_{\mathbf{I}_D}(X)$, that is, with spectral representation given by*

$$\hat{Y}(d\lambda) = \mathbf{I}_D(\lambda) [\phi(e^{-i\lambda})]^{-1} \theta(e^{-i\lambda}) \hat{Z}(d\lambda), \quad (3.4)$$

is called a FIARMA process of order (p, q) with long memory operator D , abbreviated as FIARMA(D, p, q).

Remark 1. *Definition 2 extends the usual definition of univariate (\mathbb{C} or \mathbb{R} -valued) ARFIMA(p, d, q) processes to the Hilbert space-valued case. In the general case, we use the acronym FIARMA to indicate the order of the operators in the definition (3.4), where the fractional integration operator appears on the left of the autoregressive operator, which then is on the left of the moving average operator. We also respected this order in the list of parameters (D, p, q) . Following this convention, an ARFIMA(p, D, q) process is, in turn, defined as the solution of (3.2), with Z defined as a FIARMA($0, D, q$) process. Having this convention in mind is important since the ARFIMA(p, D, q) process does not coincide with the FIARMA(D, p, q) process, except in highly unique instances such as the univariate case where all operators commute.*

Definition 2 extends the definition of ARFIMA curve time series proposed in [25] where it is restricted to the case where D is a scalar operator, that is $D : f \mapsto d \times f$ for a constant $-1/2 < d < 1/2$. In this particular case, it is rather straightforward to show that $X \in \mathcal{S}_{\mathbf{I}_D}(\Omega, \mathcal{F}, \mathbb{P})$ for any ARMA process X by directly making use of Proposition 1, However, in Remark 2(6), it will be obtained as a special case of Corollary 2.

3.2. Existence : a sufficient condition

In this section, we provide a simple sufficient condition for the existence of FIARMA processes of Definition 2, namely when the real parts of all elements of $\sigma(D)$ are less than $1/2$. We first derive a sufficient condition for the fractional integration of a general weakly stationary process X and then specify this condition to the case where X is an \mathcal{H}_0 -valued ARMA process.

Theorem 2. Let \mathcal{H}_0 be a separable Hilbert space, $D \in \mathcal{L}_b(\mathcal{H}_0)$ and $X = (X_t)_{t \in \mathbb{Z}}$ be a centered \mathcal{H}_0 -valued weakly stationary time series defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with spectral operator measure ν_X . Let $\delta \geq 0$ and suppose that $\Re(z) < \delta$ for all $z \in \sigma(D)$ and that

$$\int |\lambda|^{-2\delta} \|\nu_X\|_1(d\lambda) < \infty. \quad (3.5)$$

Then we have $X \in \mathcal{S}_{\mathbf{I}_D}(\Omega, \mathcal{F}, \mathbb{P})$, where \mathbf{I}_D is as in Definition 1.

If, in the previous result, X is an \mathcal{H}_0 -valued ARMA process, then its spectral measure admits a spectral density operator function, with respect to the Lebesgue measure, and this spectral density operator function is continuous (hence bounded) from \mathbb{T} to $\mathcal{S}_1(\mathcal{H}_0)$. It follows that (3.5) holds for any $\delta < 1/2$. Since the spectrum of a bounded operator is a compact subset of \mathbb{C} , we immediately obtain the following result.

Corollary 1. Let \mathcal{H}_0 be a separable Hilbert space and $D \in \mathcal{L}_b(\mathcal{H}_0)$. Suppose that $\Re(z) < 1/2$ for all $z \in \sigma(D)$. Then the FIARMA process defined by (3.4) is well defined for all $\theta \in \mathcal{P}_q(\mathcal{H}_0)$, $\psi \in \mathcal{P}_p^*(\mathcal{H}_0)$ and any \mathcal{H}_0 -valued centered white noise Z .

Note that the condition of Corollary 1 corresponds to the usual condition for \mathbb{R} -valued FIARMA processes imposing the long memory parameter to be less than $1/2$. In this case, this condition is also necessary as soon as the MA polynomial does not vanish at 1. However, in the functional case, it is more involved to derive a necessary and sufficient condition on the operator D for the FIARMA process to be well defined. We address this question in the next section only in the case where D is a normal operator.

3.3. A necessary and sufficient condition when D is normal

In this section, we provide a necessary and sufficient condition for the existence of FIARMA processes as defined in Definition 2 in the case where D is a normal operator. In this case, we can rely on the spectral theorem, a consequence of which is that D is unitarily equivalent to a multiplication by a bounded function on a L^2 space (see [7, Theorem 9.4.6, Proposition 9.4.7]). Namely, if $D \in \mathcal{N}(\mathcal{H}_0)$, then there exists a σ -finite measure space $(\mathbb{V}, \mathcal{V}, \xi)$, a unitary operator $U : \mathcal{H}_0 \rightarrow L^2(\mathbb{V}, \mathcal{V}, \xi)$ and $d \in L^\infty(\mathbb{V}, \mathcal{V}, \xi)$, such that

$$UDU^H = M_d, \quad (3.6)$$

where M_d denotes the pointwise multiplicative operator on $L^2(\mathbb{V}, \mathcal{V}, \xi)$ associated to d , that is $M_d : f \mapsto d \times f$. Using this representation, we can rely on the process $UX = (UX_t)_{t \in \mathbb{Z}}$ valued in $\mathcal{G}_0 := L^2(\mathbb{V}, \mathcal{V}, \xi)$. Note that \mathcal{G}_0 is separable because it is isometrically isomorphic to \mathcal{H}_0 through the unitary operator U . It is straightforward to check that, if g_X is the spectral operator density of X with respect to a non-negative measure μ on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, then the function $g_{UX} \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{S}_1^+(\mathcal{G}_0), \mu)$ defined by $g_{UX}(\lambda) = Ug_X(\lambda)U^H$, for all $\lambda \in \mathbb{T}$, is the spectral operator density of UX with respect to μ . Note that we can always find a function $h \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{S}_2(\mathcal{G}_0), \mu)$ such that $g_{UX}(\lambda) = h(\lambda)[h(\lambda)]^H$ for μ -a.e. $\lambda \in \mathbb{T}$ (take e.g. $h = g_{UX}^{1/2}$). Then, [35, Theorem 6.11] gives that, for all $\lambda \in \mathbb{T}$, the operator $h(\lambda)$ can be written as an integral operator with a kernel $\mathcal{K}(\cdot, \cdot; \lambda)$ in $L^2(\mathbb{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$. In the following, we need the measurability of \mathcal{K} on $(\mathbb{V}^2 \times \mathbb{T}, \mathcal{V}^{\otimes 2} \otimes \mathcal{B}(\mathbb{T}))$ and therefore introduce the notion of joint kernel in the following lemma and definition.

Lemma 1. Let $(\mathbb{V}, \mathcal{V}, \xi)$ be a σ -finite measure space and suppose that the Hilbert space $\mathcal{G}_0 = L^2(\mathbb{V}, \mathcal{V}, \xi)$ is separable. Let K be a measurable function from (Λ, \mathcal{A}) to $(\mathcal{S}_2(\mathcal{G}_0), \mathcal{B}(\mathcal{S}_2(\mathcal{G}_0)))$. Then, there exists a function $\mathcal{K} : (\mathbb{V}, \mathcal{V}, \xi) \mapsto \mathcal{K}(\mathbb{V}, \mathbb{V}; \lambda)$ measurable from $(\mathbb{V}^2 \times \Lambda, \mathcal{V}^{\otimes 2} \otimes \mathcal{A})$ to $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that, for all $\lambda \in \Lambda$, $f \in \mathcal{H}_0$ and $v \in \mathbb{V}$,

$$[K(\lambda)f](v) = \int \mathcal{K}(v, v'; \lambda) f(v') \xi(dv'). \quad (3.7)$$

Moreover, if $K \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{G}_0), \mu)$ for some non-negative measure μ on (Λ, \mathcal{A}) , then $\mathcal{K} \in \mathcal{L}^2(\mathbb{V}^2 \times \Lambda, \mathcal{V}^{\otimes 2} \otimes \mathcal{A}, \xi^{\otimes 2} \otimes \mu)$.

Based on this lemma, for all $\lambda \in \Lambda$, the identity (3.7) defines $(v, v') \mapsto \mathcal{K}(v, v'; \lambda)$ uniquely over \mathbb{V}^2 up to a $\xi^{\otimes 2}$ -null set. This allows us to introduce the following definition.

Definition 3 (Joint kernel of \mathcal{S}_2 -valued functions). *Under the assumptions of Lemma 1, we call \mathcal{K} the Λ -joint kernel of K .*

270 Assuming that D is normal allows us to characterize the domain of definition of the D -order fractional integration operator filter, as shown in the following result, which may be of independent interest.

Theorem 3. *Let \mathcal{H}_0 be a separable Hilbert space and $X = (X_t)_{t \in \mathbb{Z}}$ be a centered \mathcal{H}_0 -valued weakly stationary time series defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with spectral operator density g_X with respect to a non-negative measure μ on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$. Let D be in $\mathcal{N}(\mathcal{H}_0)$ satisfying the representation (3.6) with $U : \mathcal{H}_0 \rightarrow \mathcal{G}_0 := L^2(\mathbb{V}, \mathcal{V}, \xi)$ unitary and $d \in L^\infty(\mathbb{V}, \mathcal{V}, \xi)$. Let $h \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{S}_2(\mathcal{G}_0), \mu)$ be such that $\lambda \mapsto h(\lambda)[h(\lambda)]^H$ is the spectral operator density of $UX = (UX_t)_{t \in \mathbb{Z}}$ with respect to μ , that is, $h(\lambda)[h(\lambda)]^H = U g_X(\lambda) U^H$ for μ -a.e. $\lambda \in \mathbb{T}$. Let \mathcal{K} denote the \mathbb{T} -joint kernel function of h . Then, the following assertions are equivalent.*

280 (i) We have $X \in \mathcal{S}_{\mathbf{I}_D}(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) There exists $\eta \in (0, \pi)$ such that

$$\int_{\mathbb{V}^2 \times ((-\eta, \eta) \setminus \{0\})} |\lambda|^{-2\Re(d(v))} |\mathcal{K}(v, v'; \lambda)|^2 \xi(dv)\xi(dv')\mu(d\lambda) < \infty,$$

where, for all $z \in \mathbb{C}$, $\Re(z) = (z + \bar{z})/2$ denotes the real part of z .

(iii) We have

$$\int_{\mathbb{V}^2 \times ((-\pi, \pi) \setminus \{0\})} |\lambda|^{-2\Re_+(d(v))} |\mathcal{K}(v, v'; \lambda)|^2 \xi(dv)\xi(dv')\mu(d\lambda) < \infty, \quad (3.8)$$

where, for all $z \in \mathbb{C}$, $\Re_+(z) = \max(0, (z + \bar{z})/2)$ denotes the non-negative real part of z .

285 In the following corollary, we specify Theorem 3 in the case where X is an ARMA process, as in [25]. However, unlike in [25], we let D be any normal operator and not necessarily a scalar one. Our necessary and sufficient condition relies on the following definition:

$$P_n(\Phi, \Theta) = \{ [[\Phi]^{-1} \Theta] \circ \exp \}^{(n)}(0), \quad n \in \mathbb{N}, \Phi \in \mathcal{P}_p^*, \Theta \in \mathcal{P}_q, \quad (3.9)$$

where the exponent $^{(n)}$ here denotes the n -th derivative of the mapping $z \mapsto [\Phi(e^z)]^{-1} \Theta(e^z)$, which is infinitely differentiable in a neighborhood of $z = 0$ as a $\mathcal{L}_b(\mathcal{H}_0)$ -valued function, since $\Phi \in \mathcal{P}_p^*(\mathcal{H}_0)$. In fact we have

$$P_0(\Phi, \Theta) = [\Phi(1)]^{-1} \Theta(1) \quad \text{and} \quad P_n(\Phi, \Theta) = \sum_{k=1}^n b_{n,k} [[\Phi]^{-1} \Theta]^{(k)}(1), \quad n \geq 1,$$

where $b_{n,k}$ are known positive rational coefficients obtained by taking the exponential Bell n -order Bell polynomial at $(1, \dots, 1)$. We can now state our main result on FIARMA processes with normal long memory operator.

290 **Corollary 2.** *In addition to the setting of Theorem 3, let us assume that X is the \mathcal{H}_0 -valued ARMA(p, q) process defined by $\hat{X}(d\lambda) = [\Phi(e^{-i\lambda})]^{-1} \Theta(e^{-i\lambda}) \hat{Z}(d\lambda)$ with $\Theta \in \mathcal{P}_q(\mathcal{H}_0)$, $\Phi \in \mathcal{P}_p^*(\mathcal{H}_0)$ and Z a white noise with covariance operator Σ . Let $\sigma_n : v \mapsto \left(\mathbb{E} [|W_n(v, \cdot)|^2] \right)^{1/2}$, where $W_n = U P_n(\Phi, \Theta) Z_0$ is seen as a \mathbb{C} -valued function defined on $\mathbb{V} \times \Omega$. Then, the two following assertions are equivalent:*

295 (i) We have $X \in \mathcal{S}_{\mathbf{I}_D}(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) For all $n \in \mathbb{N}$, we have $\Re(d(v)) < n + 1/2$, for ξ -a.e. $v \in \{\sigma_n > 0\}$ and

$$\int_{\{v: \Re(d(v)) < n+1/2\}} \frac{\sigma_n^2(v)}{1 + 2n - 2\Re(d(v))} \xi(dv) < \infty. \quad (3.10)$$

Note that there is a slight abuse of notation in the definition of σ_n since the definition of measurability for $L^2(\mathbf{V}, \mathcal{V}, \xi)$ -valued random variables does not necessarily ensure measurability of $W_n(v, \cdot)$ as a \mathbb{C} -valued random variable. This abuse of notation is common in the literature on functional data analysis and is harmless because we can always find a version of W which is jointly measurable on $(\mathbf{V} \times \Omega, \mathcal{V} \otimes \mathcal{F})$, see Proposition 8 in Appendix B.

Remark 2. Let us briefly comment on Corollary 2.

- (1) First, by definition of W_n and σ_n , we have $\sigma_n \in L^2(\mathbf{V}, \mathcal{V}, \xi)$ and thus Assertion (ii) always holds for n large enough, namely, for all $n > \xi\text{-esssup}_{v \in \mathbf{V}}(\Re(d(v))) - \frac{1}{2} = (\|D + D^H\|_\infty - 1)/2$.
- 305 (2) If $\Theta = \Phi$ is the unit polynomial, the ARMA process X equals the white noise process Z , $P_n(\Phi, \Theta) = \text{Id}_{\mathcal{H}_0}$, and $\sigma_n = \sigma_0$ for all $n \in \mathbb{N}$, so that it is sufficient to check Condition (ii) only for $n = 0$. Moreover, in this case, we have $\sigma_0^2(v) = \mathbb{E} [|UZ_0(v, \cdot)|^2]$, that is the variance of a sample of the white noise variable Z_0 mapped to the function space $L^2(\mathbf{V}, \mathcal{V}, \xi)$.
- 310 (3) In the case where $\Theta = \Phi$ is the unit polynomial as in the previous point, we can easily exhibit the reason why the sufficient condition in Corollary 1 is not necessary. Suppose for simplicity that the white noise satisfies $\sigma_0^2(v) = 1$ for all $v \in \mathbf{V}$ with $\mathbf{V} = [0, 1]$ and ξ equal to the Lebesgue measure. Then, the necessary and sufficient condition (3.10) (with $n = 0$) allows $\sigma(D)$ to contain an element with real part $1/2$. Take for instance $d(v) = 1/2 - \sqrt{v}$, in which case (3.10) holds for $n = 0$ and $\sigma(D) = [-1/2, 1/2]$.
- 315 (4) In the N -dimensional case with n finite, we have $\mathbf{V} = \{1, \dots, N\}$, ξ is the counting measure on \mathbf{V} , and U can be interpreted as a $n \times n$ unitary matrix, and d and σ_n as N -dimensional vectors. Condition (ii) then says that $\Re(d(k)) < n + 1/2$ for all $n \in \mathbb{N}$ and $k \in \{1, \dots, N\}$ such that $\sigma_n(k) > 0$.
- 320 (5) For the real univariate case ($N = 1$, $D = d \in \mathbb{R}$ in (4)), Condition (ii) says that $d < n_0 + 1/2$, where n_0 is the smallest n such that $\sigma_n > 0$. Ruling out the case where Z is the null process (in which case $\Sigma = 0$ and $\sigma_n = 0$ for all $n \in \mathbb{N}$), one can see that n_0 equals 0 if $\Theta(1) \neq 0$ and n_0 equals the order of multiplicity of 1 as a root of Θ otherwise (in other words, it corresponds to the difference operator of largest order contained in the MA operator). In particular, we find the usual $d < 1/2$ condition for the existence of a weakly stationary ARFIMA(p, d, q) model in the case where the underlying ARMA(p, q) process is canonical (Φ and Θ do not vanish on the unit disk). If $n_0 \geq 1$, the usual convention is to include the difference operator as a negative exponent of the fractional integration operator hence leading to an ARFIMA($p, d - n_0, q - n_0$) with $d - n_0 < 1/2$.
- 325 (6) We already mentioned in (1.1) the case treated in [25, Section 4]. In the setting of Corollary 2, it corresponds to the case where $D = d \text{Id}_{\mathcal{H}_0}$ is a scalar operator on $\mathcal{H}_0 = \mathcal{G}_0 = L^2(\mathbf{V}, \mathcal{B}(\mathbf{V}), \xi)$ for a compact subset \mathbf{V} of \mathbb{R} , ξ being the Lebesgue measure on \mathbf{V} and $-1/2 < d < 1/2$ (thus $d(v) \equiv d$ and $U = \text{Id}_{\mathcal{H}_0}$). Under this assumption, Condition (4) trivially holds since $1 + 2n - 2d > 0$ and $\sigma_n \in L^2(\mathbf{V}, \mathcal{B}(\mathbf{V}), \xi)$ for all $n \in \mathbb{N}$.

3.4. Other long memory processes

335 Several non-equivalent definitions of long rang dependence, or long memory, are available in the literature for time series. Some approaches focus on the behavior of the autocovariance function at large lags, others on the spectral density at low frequencies (see [28, Section 2.1] and the references

therein). Separating short range from long range dependence is often made more natural within a particular class of models. For instance, for a Hilbert space-valued process $Y = (Y_t)_{t \in \mathbb{Z}}$, one may rely on a causal linear representation, namely

$$Y_t = \sum_{k=0}^{\infty} P_k \epsilon_{t-k}, \quad t \in \mathbb{Z}, \quad (3.11)$$

where $\epsilon = (\epsilon_t)_{t \in \mathbb{Z}}$ is a centered white noise valued in the separable Hilbert space \mathcal{H}_0 and $(P_k)_{k \in \mathbb{Z}}$ is a sequence of $\mathcal{L}_b(\mathcal{H}_0)$ operators. In (3.11), to make Y well defined and weakly stationary, the convergence is meant in $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ or in its subspace \mathcal{H}^ϵ , the time domain of ϵ . Obviously, a sufficient condition for convergence is $\sum_{k=0}^{\infty} \|P_k\|_\infty < \infty$, and this assumption is referred to as the *short range dependence* (or short memory) case, in contrast to *long range dependence* (long memory) case, for which the same series is infinite. The case where $P_k = (k+1)^{-N}$ is investigated in [14] and exhibit long memory for a well chosen operator N . For convenience, we define the same process using the spectral domain, based on the filter operator function

$$\mathbf{J}_N(\lambda) = \sum_{k=0}^{\infty} (k+1)^{-N} e^{-i\lambda k} \quad (3.12)$$

Using the arguments of the proof of [14, Lemma A.1], we immediately have the following result,

Proposition 3. *Let $\epsilon = (\epsilon_t)_{t \in \mathbb{Z}}$ be a centered white noise valued in the separable Hilbert space \mathcal{H}_0 and $N \in \mathcal{N}(\mathcal{H}_0)$. Let $UNU^H = M_n$ be a unitarily equivalent representation of N , where $U : \mathcal{H}_0 \rightarrow \mathcal{G}_0 := L^2(\mathcal{V}, \mathcal{V}, \xi)$ is unitary, $n \in L^\infty(\mathcal{V}, \mathcal{V}, \xi)$ and M_n is the pointwise n -multiplication operator on $L^2(\mathcal{V}, \mathcal{V}, \xi)$. Define $h : v \mapsto \Re(n(v))$ and $\sigma_W^2 : s \mapsto \mathbb{E} [|W(v, \cdot)|^2]$ with $W = U\epsilon_0$ and assume that*

$$h > \frac{1}{2} \quad \xi\text{-a.e.} \quad \text{and} \quad \int_{\mathcal{V}} \frac{\sigma_W^2(v)}{2h(v) - 1} \xi(dv) < \infty. \quad (3.13)$$

Then we have $\epsilon \in \mathcal{S}_{\mathbf{J}_N}(\Omega, \mathcal{F}, \mathbb{P})$ and the process $Y = F_{\mathbf{J}_N}(\epsilon)$ is equivalently defined by

$$Y_t = \sum_{k=0}^{\infty} (k+1)^{-N} \epsilon_{t-k}, \quad t \in \mathbb{Z}. \quad (3.14)$$

In [14, Theorem 2.1], the limit behavior of the partial sums of this process is provided when ϵ is independent and identically distributed. In short, if h is valued in $(1/2, 1)$, the exhibited limit behavior naturally extends the usual behavior observed for univariate long memory processes. The following result shows that the process (3.14) is closely related to a FIARMA process up to a bounded operator C and to an additive short memory process.

Proposition 4. *Let N be as in Proposition 3 and let $D = \text{Id}_{\mathcal{H}_0} - N$. Then we have $\epsilon \in \mathcal{S}_{\mathbf{I}_D}(\Omega, \mathcal{F}, \mathbb{P})$ and there exist $C \in \mathcal{L}_b(\mathcal{H}_0)$ and $(P_k^*)_{k \in \mathbb{N}} \in \mathcal{L}_b(\mathcal{H}_0)^{\mathbb{N}}$ such that*

$$\sum_{k=0}^{\infty} \|P_k^*\|_\infty < \infty \quad \text{and} \quad F_{\mathbf{I}_D}(\epsilon) = C F_{\mathbf{J}_N}(\epsilon) + Z,$$

where Z is the short memory process defined, for all $t \in \mathbb{Z}$, by $Z_t = \sum_{k=0}^{\infty} P_k^* \epsilon_{t-k}$.

Other classes of long memory functional time series have been recently considered. In [1], the long memory part of the functional time series has a finite rank in the sense that it lives in a finite dimensional linear subspace. In [32], a class of long memory processes is introduced by imposing the spectral density operator to be of the form $\lambda^A M(\lambda)$ as the frequency $\lambda \rightarrow 0$, with

($A, M(\lambda)_{\lambda \in (-\pi, \pi)}$) a family of self-adjoint operators sharing the same decomposition operator (hence commuting), see Assumptions II and III in [32] (where ω refers to the frequency while λ refers to an element of $\sigma(D)$). In opposition to these works, the class of FIARMA processes introduced above neither impose a finite rank long memory, nor a particular commuting structure between a power law frequency behavior and a multiplicative bounded operator valued function. In particular [32, Example 1] is a very specific subclass of FIARMA processes where the long memory operator and all the AR and MA operators are assumed to be self-adjoint and compact with the same eigenvectors, see [32, Eq. (3.16) and (3.17)]. In this reference, the author further studies the bias of the periodogram in Hilbert-Schmidt norm and the spectral estimation of A (within a parametric class $(A_\theta)_{\theta \in \Theta}$) under the assumption that $M(\lambda)$ is known and the common decomposition operator of $(A_\theta)_{\theta \in \Theta}$ and $(M(\lambda))_{\lambda \in (-\pi, \pi)}$ is also known. In the next section we study the consistent estimation and prediction of a parametric class of FIARMA processes with unknown AR, MA and long memory operators.

4. Prediction and estimation

4.1. Main assumptions and preliminary result

We denote by $\mu_{\mathbb{T}}$ the Lebesgue measure on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ divided by 2π , so that for any locally integrable 2π -periodic function g ,

$$\int g \, d\mu_{\mathbb{T}} = (2\pi)^{-1} \int_{\mathbb{T}} g(x) \, dx = (2\pi)^{-1} \int_{-\pi}^{\pi} g(x) \, dx .$$

Let \mathcal{H}_0 be a separable Hilbert space, $X = (X_t)_{t \in \mathbb{Z}}$ be a process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and valued in \mathcal{H}_0 , and consider the following assumptions.

(A-1) The process X is stationary and ergodic.

(A-2) The process X is weakly stationary.

Under (A-2), we always denote by ν_X the spectral operator measure of X . Denote the discrete Fourier coefficients of X_1, \dots, X_n by

$$d_n^X(\lambda) = \frac{1}{\sqrt{n}} \sum_{k=1}^n X_k e^{-i\lambda k} , \quad \lambda \in \mathbb{T} , \quad (4.1)$$

and the periodogram by

$$I_n^X(\lambda) = d_n^X(\lambda) \otimes d_n^X(\lambda) , \quad \lambda \in \mathbb{T} .$$

If X is not a centered process, one can use the empirical mean to center it, that is, in (4.1), replace X_k by $X_k^{(c,n)}$ as defined in (1.3), in which case we denote the corresponding discrete Fourier coefficients and the corresponding periodogram by $d_n^{X^{(c,n)}}$ and $I_n^{X^{(c,n)}}$, respectively.

The periodogram is related to the empirical covariance estimators through the following identity, for all $s, t \in \mathbb{Z}$,

$$\hat{\Gamma}_n(s-t) = \int I_n^{X^{(c,n)}}(\lambda) e^{i(s-t)\lambda} \mu_{\mathbb{T}}(d\lambda) = \frac{1}{n} \sum_{\substack{1 \leq k, k' \leq n \\ k-k'=(s-t)}} X_k^{(c,n)} \otimes X_{k'}^{(c,n)} . \quad (4.2)$$

The integral in this equation can be interpreted as a sesquilinear functional $\mathcal{Q}_{I_n^{X^{(c,n)}}}$ applied to exponential functions $\lambda \mapsto e^{is\lambda}$ on the left and $\lambda \mapsto e^{it\lambda}$ on the right, where, for any operator functions L, g and R defined on \mathbb{T} , we set

$$\mathcal{Q}_g(L, R) = \int L g R^H \, d\mu_{\mathbb{T}} . \quad (4.3)$$

395 Similarly, if ν is a trace-class p.o.v.m. defined on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ and valued in $\mathcal{S}_1^+(\mathcal{H}_0)$, we set

$$\mathcal{Q}_\nu(L, R) = \int L d\nu R^H. \quad (4.4)$$

To ensure that these integrals are well defined, we assume that L and R are measurable bounded functions valued in $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, with \mathcal{G}_0 an additional separable Hilbert space. Namely, for any Banach space $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$, we further denote, by $\mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{E})$ the set of bounded measurable functions from $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ to $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$, and we endow $\mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{E})$ with the sup norm, which, for all $L \in \mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{E})$, we denote by

$$\sup(L) = \sup_{\lambda \in \mathbb{T}} \|L(\lambda)\|_{\mathcal{E}}.$$

Then, for all g valued in $\mathcal{S}_1(\mathcal{H}_0)$ and $L, R \in \mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$, we have $\mathcal{Q}_g(L, R) \in \mathcal{S}_1(\mathcal{G}_0)$ with

$$\|\mathcal{Q}_g(L, R)\|_1 \leq \sup(L) \sup(R) \int \|g\|_1 d\mu_{\mathbb{T}}, \quad (4.5)$$

and similarly, for a trace-class p.o.v.m. ν valued in $\mathcal{S}_1^+(\mathcal{H}_0)$, we have $\mathcal{Q}_\nu(L, R) \in \mathcal{S}_1(\mathcal{G}_0)$ with

$$\|\mathcal{Q}_\nu(L, R)\|_1 \leq \sup(L) \sup(R) \|\nu\|_1(\mathbb{T}). \quad (4.6)$$

We denote by $\mathbb{F}_{b,b}(\mathcal{H}_0, \mathcal{G}_0)$ the product vector space $\mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)) \times \mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$, endowed with the max norm defined by

$$\|(L, R)\|_{b,b} := \max(\sup(L), \sup(R)) \quad \text{for all } (L, R) \in \mathbb{F}_{b,b}(\mathcal{H}_0, \mathcal{G}_0).$$

For any two metric spaces (\mathcal{E}_1, d_1) and (\mathcal{E}_2, d_2) , $\mathcal{C}(\mathcal{E}_1, \mathcal{E}_2)$ denotes the space of continuous functions from \mathcal{E}_1 to \mathcal{E}_2 . If F and F_n are in $\mathcal{C}(\mathcal{E}_1, \mathcal{E}_2)$ for all $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathcal{E}_1} d_2(F_n(x), F(x)) = 0,$$

we say that $(F_n)_{n \in \mathbb{N}}$ converges to F uniformly in $\mathcal{C}(\mathcal{E}_1, \mathcal{E}_2)$.

400 Using these definitions, immediate properties of the quadratic functionals $\mathcal{Q}_{I_n^{X^{(c,n)}}}$ and \mathcal{Q}_{ν_X} are summarized in the following proposition.

Proposition 5. *For any X_1, \dots, X_n in \mathcal{H}_0 , the mapping $\mathcal{Q}_{I_n^{X^{(c,n)}}}$ is well defined and belongs to $\mathcal{C}(\mathbb{F}_{b,b}(\mathcal{H}_0, \mathcal{G}_0), \mathcal{S}_1(\mathcal{G}_0))$. If (A-2) holds, we also have $\mathcal{Q}_{\nu_X} \in \mathcal{C}(\mathbb{F}_{b,b}(\mathcal{H}_0, \mathcal{G}_0), \mathcal{S}_1(\mathcal{G}_0))$.*

405 *Proof.* Observe that, under the given assumptions, $X_1^{(c,n)}, \dots, X_n^{(c,n)}$ all are in \mathcal{H}_0 , and so is $d_n^{X^{(c,n)}}(\lambda)$. Moreover, $\left\| d_n^{X^{(c,n)}}(\lambda) \right\|_{\mathcal{H}_0}$ is bounded independently of λ . Consequently, $I_n^{X^{(c,n)}}$ is valued in $\mathcal{S}^1(\mathcal{H}_0)$ and its trace-norm is integrable over \mathbb{T} . The result on $\mathcal{Q}_{I_n^{X^{(c,n)}}}$ thus follows from (4.5). Under (A-2), ν_X is a trace-class p.o.v.m. valued in $\mathcal{S}_1^+(\mathcal{H}_0)$, with $\|\nu_X\|_1(\mathbb{T}) = \mathbb{E} \left[\|X_0\|_{\mathcal{H}_0}^2 \right] < \infty$. The result on \mathcal{Q}_{ν_X} thus follows from (4.6). \square

410 Our next result only exploits (A-1) and (A-2), and is thus of independent interest. It is a uniform convergence result for integral quadratic functionals based on the periodogram. It applies to a parameterized pair of bounded operators functions defined on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ and valued in $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ as in the following.

415 **Definition 4.** *Let (Θ, Δ) be a metric space and L and R be in $\mathcal{C}(\Theta \times \mathbb{T}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$. For all $n \geq 1$, we define $\tilde{\mathcal{Q}}_{I_n^{X^{(c,n)}}}^{(L,R)} : \theta \mapsto \mathcal{Q}_{I_n^{X^{(c,n)}}}(L(\theta, \cdot), R(\theta, \cdot))$ and $\tilde{\mathcal{Q}}_{\nu_X}^{(L,R)} : \theta \mapsto \mathcal{Q}_{\nu_X}(L(\theta, \cdot), R(\theta, \cdot))$, where $\mathcal{Q}_{I_n^{X^{(c,n)}}}$ and \mathcal{Q}_{ν_X} are defined in (4.3) and (4.4), respectively.*

Note that, by Proposition 5, $\tilde{Q}_{I_n^{X^{(c,n)}}}^{(L,R)}$ and $\tilde{Q}_{\nu_X}^{(L,R)}$ are in $\mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{G}_0))$ in Definition 4. We can now state a first result on the convergence of the periodogram quadratic functional, in the case where the left and right operator functions' arrival spaces are finite-dimensional.

Theorem 4. *Let \mathcal{H}_0 be a separable Hilbert space and \mathcal{G}_0 be a finite dimensional space. Let $X = (X_t)_{t \in \mathbb{Z}}$ be a process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and valued in \mathcal{H}_0 satisfying (A-1) and (A-2) and let (Θ, Δ) be a compact metric space. Let L and R be in $\mathcal{C}(\Theta \times \mathbb{T}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$, and set $\tilde{Q}_{I_n^{X^{(c,n)}}}^{(L,R)}$ and $\tilde{Q}_{\nu_X}^{(L,R)}$ as in Definition 4. Then, we have*

$$\lim_{n \rightarrow \infty} \tilde{Q}_{I_n^{X^{(c,n)}}}^{(L,R)} = \tilde{Q}_{\nu_X}^{(L,R)} \quad \text{uniformly in } \mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{G}_0)), \quad \mathbb{P}\text{-a.s.} \quad (4.7)$$

We now consider the case where $\mathcal{G}_0 = \mathcal{H}_0$ and \mathcal{H}_0 is an infinite-dimensional separable Hilbert space. In order to obtain the convergence in $\mathcal{S}_1(\mathcal{H}_0)$, we will rely on an additional assumption in this case. To this end, for any sequence $s = (s_k)_{k \in \mathbb{N}} \in [1, \infty)^\mathbb{N}$ and any orthonormal sequence $(\phi_k)_{k \in \mathbb{N}}$ of \mathcal{H}_0 , we set

$$\mathcal{H}_0^s = \left\{ x \in \overline{\text{Span}}^{\mathcal{H}_0}(\phi_k, k \in \mathbb{N}) : \sum_{k \in \mathbb{N}} s_k^2 |\langle x, \phi_k \rangle_{\mathcal{H}_0}|^2 < \infty \right\}. \quad (4.8)$$

A typical example of such spaces are the Sobolev spaces with index $\alpha > 0$ where $(\phi_k)_{k \in \mathbb{N}}$ is a well chosen Hilbert basis (i.e. orthonormal and complete in \mathcal{H}_0) and $s_k = (1 + k)^\alpha$. The space \mathcal{H}_0^s is a subspace of \mathcal{H}_0 and is itself a separable Hilbert space endowed with the inner product

$$\langle x, y \rangle_{\mathcal{H}_0^s} = \sum_{k \in \mathbb{N}} s_k^2 \langle x, \phi_k \rangle_{\mathcal{H}_0} \overline{\langle y, \phi_k \rangle_{\mathcal{H}_0}}. \quad (4.9)$$

Setting $\xi_k = s_k^{-1} \phi_k$ for all $k \in \mathbb{N}$, we note that $(\xi_k)_{k \in \mathbb{N}}$ is a Hilbert basis of \mathcal{H}_0^s .

Using the space \mathcal{H}_0^s that we just introduced, we have the following result for the infinite-dimensional case.

Theorem 5. *Let \mathcal{H}_0 be an infinite-dimensional separable Hilbert space. Let $X = (X_t)_{t \in \mathbb{Z}}$ be a process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and valued in \mathcal{H}_0 satisfying (A-1) and (A-2) and let (Θ, Δ) be a compact metric space. Let L and R in $\mathcal{C}(\Theta \times \mathbb{T}, \mathcal{L}_b(\mathcal{H}_0))$. Let $s = (s_k)_{k \in \mathbb{N}} \in [1, \infty)^\mathbb{N}$ and $(\phi_k)_{k \in \mathbb{N}}$ be an orthonormal sequence of \mathcal{H}_0 . Define the Hilbert space \mathcal{H}_0^s by (4.8) and (4.9). We suppose that the three following assertions hold.*

- (i) *The sequence $s = (s_k)_{k \in \mathbb{N}} \in [1, \infty)^\mathbb{N}$ is non-decreasing and goes to ∞ .*
- (ii) *We have $X_0 \in \mathcal{H}_0^s$ \mathbb{P} -a.s. with $\mathbb{E}[\|X_0\|_{\mathcal{H}_0^s}^2] < \infty$.*
- (iii) *Defining L_s and R_s by $L_s(\theta, \lambda) = L(\theta, \lambda)|_{\mathcal{H}_0^s}$ and $R_s(\theta, \lambda) = R(\theta, \lambda)|_{\mathcal{H}_0^s}$ for all $(\theta, \lambda) \in \Theta \times \mathbb{T}$, we have L_s and R_s in $\mathcal{C}(\Theta \times \mathbb{T}, \mathcal{L}_b(\mathcal{H}_0^s))$.*

Then, the following convergence holds.

$$\lim_{n \rightarrow \infty} \tilde{Q}_{I_n^{X^{(c,n)}}}^{(L,R)} = \tilde{Q}_{\nu_X}^{(L,R)} \quad \text{uniformly in } \mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{H}_0)), \quad \mathbb{P}\text{-a.s.} \quad (4.10)$$

In fact, as shown by Lemma 12 in Section 5.2, Assumptions (A-1) and (A-2) imply Conditions (i) and (ii) of Theorem 5 for a well chosen s . This fact is useful to prove Theorem 1, by applying Theorem 5 for a specific choice of L and R , for which (iii) holds for any sequence s , see the proof of Theorem 1 in Section 5.3.

4.2. FIARMA prediction and estimation

A common tool for M -estimation for finite dimensional time series is the Whittle contrast, which relies on a Gaussian approximation of d_n^X as $n \rightarrow \infty$, hence suggesting to use a Gaussian likelihood contrast for d_n^X , based on its asymptotic covariance operator. We still have such an approximation for time series valued in a Hilbert space, see [4, Theorem 1]. However, using the Whittle approach directly in infinite dimension does not seem to be directly applicable. Indeed, Gaussian distributions are generally singular to each other in infinite dimension. In particular, the log determinant term of the noise covariance matrix appearing in the Whittle contrast (for example, see the first term in \bar{L}_N on Page 344 of [12]) is not well defined when this matrix is replaced by an infinite dimensional covariance operator.

There are two possible ways of circumventing this issue. The first one is to project the data on some finite-dimensional subspace for statistical inference and then study the behavior of estimators as the dimension of this subspace diverges. The second one is to work on a least square criterion, which does not include the optimization of the noise covariance operator. Here we investigate this second approach.

Recall that, for any integer $d \in \mathbb{N}$, $\mathcal{P}_d(\mathcal{H}_0)$ denotes the set of polynomials \mathfrak{p} of degree d with coefficients in $\mathcal{L}_b(\mathcal{H}_0)$, and such that $\mathfrak{p}(0) = \text{Id}_{\mathcal{H}_0}$. In the following, we further denote by $\mathcal{P}_d^\dagger(\mathcal{H}_0)$, the set of all $\mathfrak{p} \in \mathcal{P}_d(\mathcal{H}_0)$, which are invertible on the closed unit disk $\bar{\mathbb{D}}$. We now derive the best one-step ahead predictor of a FIARMA process (see Theorem 6). We then show that, under some condition, such a predictor can be estimated from the data (see Theorem 7).

Theorem 6. *Let \mathcal{H}_0 be a separable Hilbert space and p, q be two non-negative integers. Let Y be an \mathcal{H}_0 -valued FIARMA process, as in Definition 2, with long memory operator $D \in \mathcal{L}_b(\mathcal{H}_0)$, MA polynomial $\theta \in \mathcal{P}_q^\dagger(\mathcal{H}_0)$, AR polynomial $\phi \in \mathcal{P}_p^\dagger(\mathcal{H}_0)$ and an \mathcal{H}_0 -valued centered white noise Z . Assume that $D \in \mathcal{N}(\mathcal{H}_0)$ or that $\Re(z) < 1/2$ for all $z \in \sigma(D)$. Define $\Phi_{\theta, \phi, D}^\dagger : \mathbb{C} \rightarrow \mathcal{L}_b(\mathcal{H}_0)$ by*

$$\Phi_{\theta, \phi, D}^\dagger(z) = \text{Id}_{\mathcal{H}_0} - \{\theta(z)\}^{-1} \phi(z) (1 - z)^D. \quad (4.11)$$

Then, $\lambda \mapsto \Phi_{\theta, \phi, D}^\dagger(e^{-i\lambda})$ belongs to $\hat{\mathcal{H}}^Y$, and, for all $t \in \mathbb{Z}$, we have

$$\text{proj}(Y_t | \mathcal{H}_{t-1}^Y) = \int_{\mathbb{T}} e^{i\lambda t} \Phi_{\theta, \phi, D}^\dagger(e^{-i\lambda}) \hat{Y}(d\lambda), \quad (4.12)$$

where $\text{proj}(Y_t | \mathcal{H}_{t-1}^Y)$ denotes the best linear predictor of Y_t given its past $\{Y_s : s \leq t-1\}$, that is, the orthogonal projection of Y_t onto the closed space

$$\mathcal{H}_{t-1}^Y = \overline{\text{Span}}^{\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})}(\text{P} Y_s, s = t-1, t-2, \dots, \text{P} \in \mathcal{L}_b(\mathcal{H}_0)).$$

We now derive the best predictor among a collection of FIARMA predictors from a finite sample X_1, \dots, X_n . We will consider ARMA predictors or positive long memory FIARMA predictors. More precisely, Define

$$\mathcal{L}^\dagger(\mathcal{H}_0) := \{D \in \mathcal{L}_b(\mathcal{H}_0) : \text{for all } s \in \sigma(D), \Re(s) > 0\}, \quad (4.13)$$

As a subset of $\mathcal{L}_b(\mathcal{H}_0)$, we endow $\mathcal{L}^\dagger(\mathcal{H}_0)$ with the topology inherited from the operator norm $\|\cdot\|_\infty$. We consider a collection \aleph of FIARMA parameters satisfying the following assumption.

(A-3) Let $\aleph = (D_\theta, \phi_\theta, \theta_\theta)_{\theta \in \Theta}$ be valued in $\mathcal{L}_b(\mathcal{H}_0) \times \mathcal{P}_p^\dagger(\mathcal{H}_0) \times \mathcal{P}_q^\dagger(\mathcal{H}_0)$ for some $p, q \in \mathbb{N}^2$ and indexed by a compact metric space (Θ, Δ) . Moreover, the mappings $(\theta, \lambda) \mapsto \phi_\theta(e^{-i\lambda})$ and $(\theta, \lambda) \mapsto \theta_\theta(e^{-i\lambda})$ belong to $\mathcal{C}(\Theta \times \mathbb{T}, \mathcal{L}_b(\mathcal{H}_0))$ and one of the following two assertions hold.

- (i) For all $\theta \in \Theta$, $D_\theta = 0$.
- (ii) The mapping $\theta \mapsto D_\theta$ belongs to $\mathcal{C}(\Theta, \mathcal{L}^\dagger(\mathcal{H}_0))$.

In **(A-3)**, Condition **(i)** corresponds to using an ARMA predictor. Therefore, we will call \aleph an ARMA predictor model in this case. Condition **(ii)** corresponds to using a FIARMA predictor with *positive* long memory. Therefore, we will call \aleph a positive FIARMA predictor model in this case. We say that the model \aleph of **(A-3)** is *well-specified* for Y when Y is indeed a FIARMA process with a FIARMA parameter (θ, ϕ, D) among the collection \aleph .

Our goal is now to derive, based on a finite sample X_1, \dots, X_n , an approximation of the best possible \aleph -prediction of a weakly stationary process X taken among the ARMA or FIARMA predictors associated to \aleph . We first precise what we mean by this best prediction in the following result, for a centered weakly stationary process Y , and show that, in the well specified case, it corresponds to the best linear predictor of Theorem 6.

Proposition 6 (Definition of \aleph -best prediction). *Let \mathcal{H}_0 be an infinite-dimensional separable Hilbert space and p, q be two non-negative integers. Let $Y = (Y_t)_{t \in \mathbb{Z}}$ be a centered weakly stationary process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and valued in \mathcal{H}_0 and let \aleph be a model satisfying **(A-3)**. Then, we have the following facts and definitions.*

(i) For all $\theta \in \Theta$, there exists an absolutely summable $\mathcal{L}_b(\mathcal{H}_0)$ -valued sequence $(P_k^\dagger(\theta))_{k \geq 1}$ such that, for all $\lambda \in \mathbb{T} \setminus \{0\}$,

$$F^\dagger(\theta, \lambda) := \text{Id}_{\mathcal{H}_0} - \sum_{k=1}^{\infty} P_k^\dagger(\theta) e^{-i\lambda k} = [\theta_\theta(e^{-i\lambda})]^{-1} \phi_\theta(e^{-i\lambda}) (1 - e^{-i\lambda})^{D_\theta}. \quad (4.14)$$

(ii) For all $t \in \mathbb{Z}$ and $\theta \in \Theta$, we can define

$$\hat{Y}_t(\theta) := \sum_{k=1}^{\infty} P_k^\dagger(\theta) Y_{t-k} \in \mathcal{H}_{t-1}^Y. \quad (4.15)$$

(iii) The best \aleph -prediction quadratic risk of Y is defined as

$$\mathbb{E}^2(Y, \aleph) = \inf_{\theta \in \Theta} \mathbb{E} \left[\left\| Y_t - \hat{Y}_t(\theta) \right\|_{\mathcal{H}_0}^2 \right]. \quad (4.16)$$

which does not depend on t by weak stationarity of Y .

(iv) The inf in (4.16) is attained in Θ (hence is a minimum) and we call the argmin set the set of best \aleph -predictors for Y , denoted by

$$\Theta_Y^* := \left\{ \theta \in \Theta : \mathbb{E} \left[\left\| Y_t - \hat{Y}_t(\theta) \right\|_{\mathcal{H}_0}^2 \right] = \mathbb{E}^2(Y, \aleph) \right\}. \quad (4.17)$$

Then, Θ_Y^* is a compact subset of Θ .

(v) If there exists $\hat{Y}_t^* \in \mathcal{H}_{t-1}^Y$ such that the subset $\{\hat{Y}_t(\theta) : \theta \in \Theta_Y^*\}$ of \mathcal{H}_{t-1}^Y reduces to the singleton $\{\hat{Y}_t^*\}$, we call \hat{Y}_t^* the best \aleph -predictor of Y_t . Otherwise we say that the best \aleph -predictor of Y_t is not well defined.

(vi) When the best \aleph -predictor of Y_t is well defined for one t it is well defined for all t . Moreover, in this case, there exists a set of probability one on which, for all $t \in \mathbb{Z}$ and $\theta \in \Theta_Y^*$, $\hat{Y}_t^* = \hat{Y}_t(\theta)$.

(vii) Consider the well-specified case, where we assume that Y is a FIARMA process with parameter (D, θ, ϕ) among \aleph , and suppose that $D \in \mathcal{N}(\mathcal{H}_0)$ or that $\Re(z) < 1/2$ for all $z \in \sigma(D)$. Then, the best \aleph -predictor \hat{Y}_t^* is always well defined and coincides with the best predictor in \mathcal{H}_{t-1}^Y , that is,

$$\mathbb{E}^2(Y, \aleph) = \inf_{V \in \mathcal{H}_{t-1}^Y} \mathbb{E} \left[\left\| Y_t - V \right\|_{\mathcal{H}_0}^2 \right], \quad (4.18)$$

$$\hat{Y}_t^* = \text{proj}(Y_t | \mathcal{H}_{t-1}^Y). \quad (4.19)$$

The next result shows how to estimate a predictor which converges to the best predictor that we have just introduced.

We now introduce our estimation procedure. Using F_θ^\dagger defined by (4.14) and the periodogram $I_n^{X^{(c,n)}}$ defined in Section 4.1, we consider a sequence of estimators $(\hat{\theta}_n)_{n \in \mathbb{N}}$ satisfying

$$\limsup_{n \rightarrow \infty} \left(\Lambda_n(\hat{\theta}_n) - \inf_{\theta \in \Theta} \Lambda_n(\theta) \right) = 0, \quad (4.20)$$

where, for all $n \in \mathbb{N}$ and $\theta \in \Theta$,

$$\Lambda_n(\theta) := \text{Tr} \left(\tilde{Q}_{I_n^{X^{(c,n)}}}^{(F^\dagger, F^\dagger)}(\theta) \right) = \text{Tr} \left(\int F^\dagger(\theta, \lambda) I_n^{X^{(c,n)}}(\lambda) (F^\dagger(\theta, \lambda))^H d\mu_{\mathbb{T}} \right). \quad (4.21)$$

Let $X^{(c)}$ be the centered version of X , $X_t^{(c)} = X_t - \mathbb{E}[X_0]$. Using that $I_n^{X^{(c,n)}}(\lambda) \mu_{\mathbb{T}}(d\lambda)$ approximates $\nu_X(d\lambda) = \nu_{X^{(c)}}(d\lambda)$ with (4.14) and (4.15), $\Lambda_n(\theta)$ can be seen as an approximation of $\mathbb{E} \left[\left\| X_t^{(c)} - \widehat{X}_t^{(c)}(\theta) \right\|_{\mathcal{H}_0}^2 \right]$, and $\hat{\theta}_n$ as an attempt to minimize this risk in θ , mimicking what is done in (4.16). Then, to take into account the unknown mean of X and since we can only use the observations X_1, \dots, X_n to predict X_{n+1} , we truncate the series defining the predictor in (4.15) to keep its n first terms only, apply it to the empirically centered observations $(X_{n+1-k}^{(c,n)})_{1 \leq k \leq n}$ and add the empirical mean to approximate $\mathbb{E}[X_0]$. This leads us to define the predictor of X_{n+1} from the sample X_1, \dots, X_n associated to the estimator $\hat{\theta}_n$ by

$$\hat{X}_{n+1,n} = \frac{1}{n} \sum_{k=1}^n X_k + \sum_{k=1}^n P_k^\dagger(\hat{\theta}_n) X_{n+1-k}^{(c,n)}, \quad (4.22)$$

where $P_k^\dagger(\theta)$ is defined in Proposition 6. Note that the predictor $\hat{X}_{n+1,n}$ can be written as

$$m + \sum_{k=1}^n P_k^\dagger(\theta) (X_{n+1-k} - m) \quad (4.23)$$

by taking $m \in \mathcal{H}_0$ and $\theta \in \Theta$ equal to $\frac{1}{n} \sum_{k=1}^n X_k$ and $\hat{\theta}_n$, respectively. In the following theorem, defining, for all $n \geq 1$, $m \in \mathcal{H}_0$ and $\theta \in \Theta$, the quadratic prediction risk of a predictor of this form by

$$\mathbb{E}_{X,n}^2(m, \theta) = \mathbb{E} \left[\left\| X_{n+1} - \left(m + \sum_{k=1}^n P_k^\dagger(\theta) (X_{n+1-k} - m) \right) \right\|_{\mathcal{H}_0}^2 \right], \quad (4.24)$$

we show that, as a predictor of X_{n+1} , $\hat{X}_{n+1,n}$ asymptotically achieves the same prediction risk as the optimal risk for predicting the centered process $X^{(c)} = X - \mathbb{E}[X_0]$ from its past.

Theorem 7. *Let \mathcal{H}_0 be an infinite-dimensional separable Hilbert space and p, q be two non-negative integers. Let $X = (X_t)_{t \in \mathbb{Z}}$ be a process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and valued in \mathcal{H}_0 satisfying (A-1) and (A-2). Let \mathfrak{N} be a model satisfying (A-3) with compact parameter metric space (Θ, Δ) . Let $s = (s_k)_{k \in \mathbb{N}} \in [1, \infty)^\mathbb{N}$ and $(\phi_k)_{k \in \mathbb{N}}$ be an orthonormal sequence of \mathcal{H}_0 . Define the Hilbert space \mathcal{H}_0^s by (4.8) and (4.9). We suppose that (i) and (ii) of Theorem 5 hold as well as the following condition.*

(iii) *Defining, for all $z \in \mathbb{C}$, $\Phi_{\theta,s}(z) = \Phi_\theta(z)|_{\mathcal{H}_0^s}$ and $\Theta_{\theta,s}(z) = \Theta_\theta(z)|_{\mathcal{H}_0^s}$, we have that $(\theta, \lambda) \mapsto \Phi_{\theta,s}(e^{-i\lambda})$ and $(\theta, \lambda) \mapsto \Theta_{\theta,s}(e^{-i\lambda})$ belong to $\mathcal{D}(\Theta \times \mathbb{T}, \mathcal{L}_b(\mathcal{H}_0^s))$. Under (A-3)(ii), defining $D_{\theta,s} = D_{\theta}|_{\mathcal{H}_0^s}$, assume, in addition, that $\theta \mapsto D_{\theta,s}$ belongs to $\mathcal{C}(\Theta, \mathcal{L}^\dagger(\mathcal{H}_0^s))$.*

Finally, let $(\hat{\theta}_n)_{n \in \mathbb{N}}$ be a sequence of estimators satisfying (4.20). Then, we have

$$\lim_{n \rightarrow \infty} \Delta \left(\hat{\theta}_n, \Theta_{X^{(c)}}^* \right) = 0, \quad \mathbb{P}\text{-a.s.}, \quad (4.25)$$

$$\lim_{n \rightarrow \infty} \mathbb{E}_{X,n}^2 \left(\frac{1}{n} \sum_{k=1}^n X_k, \hat{\theta}_n \right) = \mathbb{E}^2 \left(X^{(c)}, \aleph \right), \quad \mathbb{P}\text{-a.s.} \quad (4.26)$$

where $X^{(c)} = (X_t^{(c)})_{t \in \mathbb{Z}}$ denotes the centered process defined by $X_t^{(c)} = X_t - \mathbb{E}[X_t]$, $\Theta_{X^{(c)}}^*$ is defined in (4.17), $\mathbb{E}_{X,n}^2$ in (4.24), and $\mathbb{E}^2(X^{(c)}, \aleph)$ in (4.16).

Moreover, if $\left(\widehat{X}_t^{(c)} \right)^*$ is well defined (as in Proposition 6 (v)), we further have

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left[\left\| X_{n+1} - \hat{X}_{n+1,n} \right\|_{\mathcal{H}_0}^2 \right] \leq \mathbb{E}^2 \left(X^{(c)}, \aleph \right), \quad (4.27)$$

where $\hat{X}_{n+1,n}$ is defined by (4.22).

Let us briefly comment on the conclusions of Theorem 7. Equation (4.25) says that $\hat{\theta}_n$ is consistent for estimating the optimal θ up to the equivalence relationship $\theta \sim \theta'$ defined by $\mathbb{E} \left[\left\| X_t^{(c)} - \widehat{X}_t^{(c)}(\theta) \right\|_{\mathcal{H}_0}^2 \right] = \mathbb{E} \left[\left\| X_t^{(c)} - \widehat{X}_t^{(c)}(\theta') \right\|_{\mathcal{H}_0}^2 \right]$. Equation (4.26) says that the risk of an estimator of the form (4.23) for predicting X_{n+1} is asymptotically minimal with m and θ replaced by the empirical mean and $\hat{\theta}_n$. Finally, while (4.25) and (4.26) hold in the \mathbb{P} -a.s. sense, Equation (4.27) says that the prediction risk directly defined with the predictor $\hat{X}_{n+1,n}$ is indeed asymptotically optimal. In contrast to (4.26), where the left-hand side is random as a function of the empirical mean and $\hat{\theta}_n$, the left-hand side of (4.27) is not random as the expectation applies to the prediction error square norm.

5. Postponed proofs

5.1. Proofs of Section 3

Many results of Section 3 require to control the behavior of some particular operator-valued functions that depend on the long memory operator D . We obtain some general bounds of such function that exploit assumptions on the spectrum of this operator in Section 5.1.1. Then we proceed with the proofs of the results of Section 3.

5.1.1. Preliminary results

For $\varsigma > 0$ and $\varrho \in \mathbb{R}$, we define the ϱ -above part of the open disk with radius ς by $\overline{\mathbb{D}}_{\varrho, \varsigma}$ and its closure by $\mathbb{D}_{\varrho, \varsigma}$, that is,

$$\mathbb{D}_{\varrho, \varsigma} = \{z \in \mathbb{C} : |z| < \varsigma, \Re(z) > \varrho\} \quad \text{and} \quad \overline{\mathbb{D}}_{\varrho, \varsigma} = \{z \in \mathbb{C} : |z| \leq \varsigma, \Re(z) \geq \varrho\}. \quad (5.1)$$

If $\varrho \leq -\varsigma$, then $\mathbb{D}_{\varrho, \varsigma}$ is the open disk of radius ς and $\overline{\mathbb{D}}_{\varrho, \varsigma}$ its closure, and, in this case, we also use the simplified notation \mathbb{D}_{ς} , respectively $\overline{\mathbb{D}}_{\varsigma}$. We consider some special functions (all analytic on \mathbb{C}), which will be used all along this section, respectively defined as follows for some $k \in \mathbb{N}$, $\rho \in [0, 1]$ and $\lambda \in (-\pi/3, \pi/3) \setminus \{0\}$:

$$f_k(z) = (k+1)^{-z}, \quad (5.2)$$

$$g_k(z) = (k+1)^{-z} - (k+2)^{-z}, \quad (5.3)$$

$$h_{\rho, \lambda}(z) = (1 - \rho e^{-i\lambda})^z, \quad (5.4)$$

$$j_{\rho, \lambda}(z) = (1 - \rho e^{-i\lambda})^{-z} (1 - e^{-i\lambda})^z. \quad (5.5)$$

We first derive the following bounds where these functions are applied on \mathbb{C} .

570 **Lemma 2.** Let $\varsigma > 0$ and $\varrho \in [-\varsigma, \varsigma]$. There exist $C_2, \dots, C_4 > 0$ only depending on ς, ϱ such that, for all $k \in \mathbb{N}$, $\rho \in [0, 1]$ and $\lambda \in (-\pi/3, \pi/3) \setminus \{0\}$,

$$\sup_{z \in \overline{\mathbb{D}}_{\varrho, \varsigma}} |f_k(z)| \leq (k+1)^{-\varrho}, \quad (5.6)$$

$$\sup_{z \in \overline{\mathbb{D}}_{\varrho, \varsigma}} |g_k(z)| \leq C_2 (k+1)^{-\varrho} \ln \left(1 + \frac{1}{k+1} \right), \quad (5.7)$$

$$\sup_{z \in \overline{\mathbb{D}}_{\varrho, \varsigma}} |h_{\rho, \lambda}(z)| \leq C_3 |\lambda|^{-\varrho_-}, \quad (5.8)$$

$$\sup_{z \in \overline{\mathbb{D}}_{\varrho, \varsigma}} |j_{\rho, \lambda}(z)| \leq C_4 |\lambda|^{-\varrho_-}, \quad (5.9)$$

where $\varrho_- = \max(0, -\varrho)$ denotes the negative part of ϱ .

Proof. Let $z \in \overline{\mathbb{D}}_{\varrho, \varsigma}$. Then $|f_k(z)| = (k+1)^{-\Re(z)} \leq (k+1)^{-\varrho}$, so (5.6) holds. To obtain (5.7), observe that

$$\begin{aligned} |g_k(z)| &= (k+1)^{-\Re(z)} \left| 1 - \exp \left(-\ln \left(1 + \frac{1}{k+1} \right) z \right) \right| \\ &\leq \varsigma \alpha(\varsigma \ln(2)) (k+1)^{-\varrho} \ln \left(1 + \frac{1}{k+1} \right), \end{aligned}$$

where, here, we set, for any $r > 0$, $\alpha(r) := \sup \left\{ \left| \frac{1-e^{-z}}{z} \right| : z \in \mathbb{C}, 0 < |z| \leq r \right\}$. To obtain (5.8), we apply Lemma 22 and using that $\Re_-(z) \leq \varsigma$, $|\Im(z)| \leq \varsigma$ and $\Re_-(z) \leq \varrho_-$, we get that

$$|h_{\rho, \lambda}(z)| \leq \left(2\pi/(3\sqrt{3}) \right)^\varsigma e^{\pi\varsigma/2} |\lambda|^{-\varrho_-}.$$

Finally, to obtain (5.9), we apply Lemmas 21 and 22 and get that

$$|j_{\rho, \lambda}(z)| \leq \left(2\pi/(3\sqrt{3}) \right)^{\Re_+(z)} |\lambda|^{-\Re_+(z)} e^{\pi|\Im(z)|/2} (\pi/2)^{\Re_-(z)} |\lambda|^{\Re(z)} e^{\pi|\Im(z)|/2}.$$

575 Reassembling the terms in this upper bound conveniently, and using that $|\Re(z)| \leq \varsigma$, $|\Im(z)| \leq \varsigma$ and $\Re_-(z) \leq \varrho_-$, we get that $|j_{\rho, \lambda}(z)| \leq \left(2\pi/(3\sqrt{3}) \right)^\varsigma (\pi/2)^\varsigma e^{\pi\varsigma} |\lambda|^{-\varrho_-}$. \square

The bounds in Lemma 2 are easily extended to the case where these functions apply to a normal operator whose spectrum is included in $\overline{\mathbb{D}}_{\varrho, \varsigma}$.

580 **Lemma 3.** Let $\varsigma > 0$ and $\varrho \in [-\varsigma, \varsigma]$. Let \mathcal{H}_0 be a separable Hilbert space and let $\mathcal{N}_{\varrho, \varsigma}$ denote the set of all $P \in \mathcal{N}(\mathcal{H}_0)$ such that $\sigma(P) \subseteq \overline{\mathbb{D}}_{\varrho, \varsigma}$. Then, there exist $C_2, \dots, C_4 > 0$ only depending on ς, ϱ such that, for all $k \in \mathbb{N}$, $\rho \in [0, 1]$ and $\lambda \in (-\pi/3, \pi/3) \setminus \{0\}$,

$$\sup_{P \in \mathcal{N}_{\varrho, \varsigma}} \|f_k(P)\|_\infty \leq (k+1)^{-\varrho}, \quad (5.10)$$

$$\sup_{P \in \mathcal{N}_{\varrho, \varsigma}} \|g_k(P)\|_\infty \leq C_2 (k+1)^{-\varrho} \ln \left(1 + \frac{1}{k+1} \right), \quad (5.11)$$

$$\sup_{P \in \mathcal{N}_{\varrho, \varsigma}} \|h_{\rho, \lambda}(P)\|_\infty \leq C_3 |\lambda|^{-\varrho_-}, \quad (5.12)$$

$$\sup_{P \in \mathcal{N}_{\varrho, \varsigma}} \|j_{\rho, \lambda}(P)\|_\infty \leq C_4 |\lambda|^{-\varrho_-}. \quad (5.13)$$

Proof. Let $P \in \mathcal{N}(\mathcal{H}_0)$ such that $\sigma(P) \subseteq \overline{\mathbb{D}}_{\varrho, \varsigma}$. Let $UPU^H = M_p$ be a unitarily equivalent representation of P , where $U : \mathcal{H}_0 \rightarrow \mathcal{G}_0 := L^2(\mathcal{V}, \mathcal{V}, \xi)$ is unitary, $p \in L^\infty(\mathcal{V}, \mathcal{V}, \xi)$ and M_p is the pointwise p -multiplication operator on $L^2(\mathcal{V}, \mathcal{V}, \xi)$ (see [7, Theorem 9.4.6]). Let f be a power function with infinite radius of convergence, we have

$$\|f(P)\|_\infty = \|f(M_p)\|_\infty = \|M_{f \circ p}\|_\infty = \xi\text{-esssup}(f \circ p).$$

Since $\xi(\{P \notin \sigma(P)\}) = 0$, we get that $\|f_k(P)\|_\infty \leq \sup_{z \in \sigma(P)} |f(z)|$. All the functions $f_k, g_k, h_{\rho,\lambda}$ and $j_{\rho,\lambda}$ are analytic on \mathbb{C} , so the bounds directly follow from Lemma 2. \square

We finally derive similar bounds that apply to bounded operators P and not only normal ones. There is a price to pay for this extension, namely the spectrum of P must now be included in the open set $\mathbb{D}_{\varrho,\varsigma}$. We rely on the following lemma.

Lemma 4. *Let G be an open subset of \mathbb{C} and D be a compact subset of G . Let \mathcal{H}_0 be a separable Hilbert space. Then there exists a continuous function $C_{D,G} : \{P \in \mathcal{L}_b(\mathcal{H}_0) : \sigma(P) \subseteq D\} \rightarrow \mathbb{R}_+$ only depending on D and G such that, for all analytic functions $f : G \rightarrow \mathbb{C}$ and all $P \in \mathcal{L}_b(\mathcal{H}_0)$ such that $\sigma(P) \subseteq D$,*

$$\|f(P)\|_\infty \leq C_{D,G}(P) \sup_{z \in G} |f(z)|, \quad (5.14)$$

where $f(P)$ is defined as in [7, VII.4.5].

Proof. By [7, Proposition VII.4.4] and [7, VII.4.5], we can find n functions $\gamma_1, \dots, \gamma_n$ infinitely differentiable from $[0, 1]$ to $G \setminus D$, such that for all $P \in \mathcal{L}_D := \{P \in \mathcal{L}_b(\mathcal{H}_0) : \sigma(P) \subseteq D\}$,

$$f(P) = \frac{1}{2\pi i} \sum_{k=1}^n \int_0^1 f \circ \gamma_k(t) (\gamma_k(t) \text{Id}_{\mathcal{H}_0} - P)^{-1} \gamma'_k(t) dt.$$

The bound (5.14) follows by setting

$$C_{D,G}(P) := \sum_{k=1}^n \frac{\int |\gamma'_k|}{2\pi} \sup_{t \in [0,1]} \left(\left\| (\gamma_k(t) \text{Id}_{\mathcal{H}_0} - P)^{-1} \right\|_\infty \right).$$

It remains to show that $C_{D,G}$ is continuous on \mathcal{L}_D which is implied by continuity of $P \mapsto \sup_{t \in [0,1]} \left\| (\gamma_k(t) \text{Id}_{\mathcal{H}_0} - P)^{-1} \right\|_\infty$ on \mathcal{L}_D for all $1 \leq k \leq n$. This last continuity holds because $(t, P) \mapsto \left\| (\gamma_k(t) \text{Id}_{\mathcal{H}_0} - P)^{-1} \right\|_\infty$ is continuous on $[0, 1] \times \mathcal{L}_D$ and $[0, 1]$ is compact. \square

We can now apply Lemma 4 using the functions defined in (5.2)–(5.5) which are analytic on \mathbb{C} . In contrast to Lemma 3, in the following lemma, the sup is taken on a compact subset of $\mathcal{L}_b(\mathcal{H}_0)$.

Corollary 3. *Let \mathcal{H}_0 be a separable Hilbert space and let K be a compact subset of $\mathcal{L}_b(\mathcal{H}_0)$. Then $\{\|P\|_\infty : P \in K\}$ and $\{\text{Re}(z) : P \in K, z \in \sigma(P)\}$ are compact subsets of \mathbb{R} . Let us denote*

$$\bar{\varsigma} = \max_{P \in K} (\|P\|_\infty) \quad \text{and} \quad \underline{\varrho} = \min_{P \in K, z \in \sigma(P)} (\text{Re}(z)).$$

Then, for all $\varsigma > \bar{\varsigma}$ and $\varrho \in [-\varsigma, \underline{\varrho}]$ there exist $C'_1, \dots, C'_4 > 0$ only depending on ς, ϱ and K such that, for all $k \in \mathbb{N}$, $\rho \in [0, 1]$ and $\lambda \in (-\pi/3, \pi/3) \setminus \{0\}$,

$$\sup_{P \in K} \|f_k(P)\|_\infty \leq C'_1 (k+1)^{-\varrho}, \quad (5.15)$$

$$\sup_{P \in K} \|g_k(P)\|_\infty \leq C'_2 (k+1)^{-\varrho} \ln \left(1 + \frac{1}{k+1} \right), \quad (5.16)$$

$$\sup_{P \in K} \|h_{\rho,\lambda}(P)\|_\infty \leq C'_3 |\lambda|^{-\varrho-}, \quad (5.17)$$

$$\sup_{P \in K} \|j_{\rho,\lambda}(P)\|_\infty \leq C'_4 |\lambda|^{-\varrho-}. \quad (5.18)$$

Proof. Since K is compact and $P \mapsto \|P\|_\infty$ is continuous on K , $\{\|P\|_\infty : P \in K\}$ is compact and $\bar{\varsigma}$ is well defined. We next show that $\{\text{Re}(z) : P \in K, z \in \sigma(P)\}$ is compact. Let $g : K \times \overline{\mathbb{D}_{\bar{\varsigma}}}$ $\rightarrow \mathcal{L}_b(\mathcal{H}_0)$ denote the continuous mapping $(P, z) \mapsto z \text{Id}_{\mathcal{H}_0} - P$. Since the set of non-invertible operators is closed in $\mathcal{L}_b(\mathcal{H}_0)$, its reciprocal set by g , that is, $\tilde{K} = \{(P, z) \in K \times \mathbb{C} : z \in \sigma(P) \cap \overline{\mathbb{D}_{\bar{\varsigma}}}\}$,

is closed, thus compact, as a subset of the compact set $K \times \overline{\mathbb{D}_{\overline{\zeta}}}$. By definition of $\overline{\zeta}$, we have $\widetilde{K} = \{(P, z) \in K \times \mathbb{C} : z \in \sigma(P)\}$. Hence $\{\operatorname{Re}(z) : P \in K, z \in \sigma(P)\}$, as the image of \widetilde{K} by the continuous mapping $(P, z) \mapsto \Re(z)$, is also compact.

Let now $\varsigma > \overline{\zeta}$ and $\varrho \in [-\varsigma, \underline{\varrho}]$. Then, for all $P \in K$, $\sigma(P) \subset \mathbb{D}_{\varrho, \varsigma}$ and we can apply Lemma 4 with $D := \overline{\mathbb{D}_{\varrho, \overline{\zeta}}} \subset G := \mathbb{D}_{\varrho, \varsigma}$ and f equal to one of the functions defined by (5.2)–(5.5). Since $C_{D,G}$ is continuous, it is bounded on K . Then (5.15)–(5.18) follow from (5.14) and (5.6)–(5.9) in Lemma 2. \square

610 5.1.2. Proof of Theorem 2

Let g_X be the spectral operator density of X with respect to $\|\nu_X\|_1$. By Proposition 1, we have $X \in \mathcal{S}_{\mathbf{I}_D}(\Omega, \mathcal{F}, \mathbb{P})$ if and only if $\int \|\mathbf{I}_D g_X \mathbf{I}_D^H\|_1 d\|\nu_X\|_1 < \infty$. Now observe that

$$\begin{aligned} \int \|\mathbf{I}_D g_X \mathbf{I}_D^H\|_1 d\|\nu_X\|_1 &= \int_{\mathbb{T} \setminus \{0\}} \left\| (1 - e^{-i\lambda})^{-D} (g_X(\lambda))^{1/2} \right\|_2^2 \|\nu_X\|_1(d\lambda) \\ &\leq \int_{\mathbb{T} \setminus \{0\}} \left\| (1 - e^{-i\lambda})^{-D} \right\|_\infty^2 \|\nu_X\|_1(d\lambda), \end{aligned}$$

where, in the last inequality, we have used that $\|g_X(\lambda)\|_1 = 1$ for $\|\nu_X\|_1$ -a.e. $\lambda \in \mathbb{T}$. To show that this bound is finite, we first observe that for all $\lambda \in \mathbb{T} \setminus \{0\}$, we have the bound $\left\| (1 - e^{-i\lambda})^{-D} \right\|_\infty \leq e^{\|D\|_\infty |\ln(1 - e^{-i\lambda})|}$. This bound is useless for λ close to 0 but we can use it away from 0. For instance, for $\lambda \in \mathbb{T} \setminus (-\pi/3, \pi/3)$, $|\ln(1 - e^{-i\lambda})|$ remains bounded away from 0 and ∞ and thus $\left\| (1 - e^{-i\lambda})^{-D} \right\|_\infty$ is bounded on this set of frequencies. Since $\int \|\nu_X\|_1(d\lambda) = \mathbb{E} \left[\|X\|_{\mathcal{H}_0}^2 \right] < \infty$, it only remains to show that

$$\int_{(-\pi/3, \pi/3) \setminus \{0\}} \left\| (1 - e^{-i\lambda})^{-D} \right\|_\infty^2 \|\nu_X\|_1(d\lambda) < \infty. \quad (5.19)$$

We apply Corollary 3 with $K = \{-D\}$, which gives $\underline{\varrho} > -\delta$. Then (5.17) with $\varrho = -\delta$ and $\rho = 1$ gives us that, there exists $C > 0$ such that for all $\lambda \in (-\pi/3, \pi/3) \setminus \{0\}$, $\left\| (1 - e^{-i\lambda})^{-D} \right\|_\infty^2 \leq C |\lambda|^{-2\delta}$. This implies (5.19) and the proof is concluded.

5.1.3. Proofs of Lemma 1 and Theorem 3

Lemma 1 is used to show the existence of the joint kernels in Definition 3.

Proof of Lemma 1. Let $(\phi_i)_{0 \leq i < N}$ denote a Hilbert basis of $L^2(\mathbf{V}, \mathcal{V}, \xi)$, assumed to be of dimension $N \in \{1, 2, \dots, \infty\}$. Define $\mathcal{K}_n : (v, v'; \lambda) \mapsto \sum_{0 \leq i, j \leq n} \phi_i^H K(\lambda) \phi_j \phi_i(v) \bar{\phi}_j(v')$ on $\mathbf{V}^2 \times \mathbb{T}$ and, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{T}$, let us denote by $N_n(\lambda)$ the smallest integer in

$$\left\{ 0 \leq k < N : \sum_{i \text{ or } j > k} |\phi_i^H K(\lambda) \phi_j|^2 \leq 2^{-n} \right\}. \quad \text{Note that this set is not empty since, if } N \text{ is finite,}$$

it contains $N - 1$ and, if $N = \infty$, we have $\sum_{i, j} |\phi_i^H K(\lambda) \phi_j|^2 = \|K(\lambda)\|_2 < \infty$. Now let us define, for all $v, v' \in \mathbf{V}$ and $\lambda \in \mathbb{T}$, $\mathcal{K}(v, v'; \lambda) := \lim_{n \rightarrow \infty} \mathcal{K}_{N_n(\lambda)}(v, v'; \lambda)$ whenever this limit exists in \mathbb{C} and set $\mathcal{K}(v, v'; \lambda) = 0$ otherwise. Since $(\phi_k \otimes \bar{\phi}_{k'})_{0 \leq k, k' < N}$ is a Hilbert basis of $L^2(\mathbf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$, we immediately have that, for any $\lambda \in \Lambda$, $\mathcal{K}_{N_n(\lambda)}(\cdot, \cdot; \lambda)$ converges in the sense of this L^2 space to $\sum_{0 \leq i, j < N} \phi_i^H K(\lambda) \phi_j \phi_i \otimes \bar{\phi}_j$, and thus, this limit must be equal to $\mathcal{K}(\cdot, \cdot; \lambda)$, $\xi^{\otimes 2}$ -a.e. It follows that, that for any $\lambda \in \Lambda$, for all $i, j \in \mathbb{N}$, $\int \mathcal{K}(v, v'; \lambda) \bar{\phi}_i(v) \phi_j(v') \xi(dv) \xi(dv') = \phi_i^H K(\lambda) \phi_j$, which gives that $K(\lambda)$ is an integral operator associated with the kernel $\mathcal{K}(\cdot, \cdot; \lambda)$. Since $(v, v', \lambda) \mapsto \mathcal{K}(v, v'; \lambda)$ is measurable by definition, this concludes the proof of the existence of the Λ -joint kernel of K . If, moreover, $K \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \mu)$, then \mathcal{K}_n converges in $L^2(\mathbf{V}^2 \times \Lambda, \mathcal{V}^{\otimes 2} \otimes \mathcal{A}, \xi^{\otimes 2} \otimes \mu)$ and the limit must be equal to \mathcal{K} , $\xi^{\otimes 2} \otimes \mu$ -a.e. since for each $\lambda \in \Lambda$, $\mathcal{K}_n(\cdot, \cdot; \lambda)$ converges to $\mathcal{K}(\cdot, \cdot; \lambda)$ in $L^2(\mathbf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$. Hence, we get that $\mathcal{K} \in L^2(\mathbf{V}^2 \times \Lambda, \mathcal{V}^{\otimes 2} \otimes \mathcal{A}, \xi^{\otimes 2} \otimes \mu)$. \square

We now prove Theorem 3.

640 **Proof of Theorem 3.** We assume without loss of generality that $\mu(\{0\}) = 0$ (since it affects none of the given assertions). The proof is organized as follows. We first show that Assertion (i) is equivalent to

$$\int_{\mathcal{V}^2 \times (-\pi, \pi]} |\lambda|^{-2\Re(d(v))} |\mathcal{R}(v, v'; \lambda)|^2 \xi(dv)\xi(dv')\mu(d\lambda) < \infty. \quad (5.20)$$

Then we successively check that Condition (5.20) is equivalent to Assertions (iii) and (ii).

Step 1:(i) \Leftrightarrow (5.20). By Proposition 1, Assertion (i) is equivalent to

$$\int_{\mathbb{T}} \left\| (1 - e^{-i\lambda})^{-D} g_X(\lambda) [(1 - e^{-i\lambda})^{-D}]^H \right\|_1 \mu(d\lambda) < \infty. \quad (5.21)$$

645 Using the unitarily equivalent representation (3.6), and since U is unitary from \mathcal{H}_0 to $L^2(\mathcal{V}, \mathcal{V}, \xi)$, we get that, for all $\lambda \in \mathbb{T} \setminus \{0\}$,

$$\begin{aligned} \left\| (1 - e^{-i\lambda})^{-D} g_X(\lambda) [(1 - e^{-i\lambda})^{-D}]^H \right\|_1 &= \left\| U^H M_{(1-e^{-i\lambda})^{-d}} U g_X(\lambda) U^H M_{(1-e^{-i\lambda})^{-d}}^H U \right\|_1 \\ &= \left\| M_{(1-e^{-i\lambda})^{-d}} h(\lambda) \right\|_2^2. \end{aligned}$$

Hence, using the \mathbb{T} -joint kernel \mathcal{R} of h , the left-hand side of (5.21) reads $\int \left| (1 - e^{-i\lambda})^{-d(v)} \mathcal{R}(v, v'; \lambda) \right|^2 \xi(dv)\xi(dv') \mu(d\lambda)$. Applying Lemma 21 to $z = -d(v)$, since d is ξ -essentially bounded, we get that (5.21), thus Assertion (i), is equivalent to (5.20).

Step 2:(5.20) \Leftrightarrow (iii). Using the identity $|\lambda|^{-2\Re(d(v))} = |\lambda|^{-2\Re_+(d(v))} + |\lambda|^{2\Re_-(d(v))} - 1$, that $|\lambda|^{2\Re_-(d(v))}$ is bounded independently of v and λ and that

$$\int |\mathcal{R}(v, v'; \lambda)|^2 \xi(dv)\xi(dv')\mu(d\lambda) = \int \|h(\lambda)\|_2^2 \mu(d\lambda) < \infty,$$

650 we get that (5.20) is equivalent to Assertion (iii).

Step 3:(5.20) \Leftrightarrow (ii). This follows similarly by observing that, for any $\eta \in (0, \pi)$, $|\lambda|^{-2\Re(d(v))}$ is bounded independently of v on $\lambda \in (-\pi, \pi] \setminus (-\eta, \eta)$. \square

5.1.4. Proof of Corollary 2

655 We start with a result providing a decomposition of the square root of the spectral operator density of an Hilbert space valued ARMA process. It is required for the proof of Corollary 2.

Lemma 5. Let \mathcal{H}_0 be a separable Hilbert space and X be an ARMA(p, q) process defined by $\hat{X}(d\lambda) = [\hat{\Phi}(e^{-i\lambda})]^{-1} \hat{\Theta}(e^{-i\lambda}) \hat{Z}(d\lambda)$ with $\hat{\Theta} \in \mathcal{P}_q(\mathcal{H}_0)$, $\hat{\Phi} \in \mathcal{P}_p^*(\mathcal{H}_0)$ and Z an \mathcal{H}_0 -valued white noise with covariance operator Σ . Then, there exists $\eta \in (0, \pi)$ such that

$$\sum_{n \in \mathbb{N}} \frac{\eta^n}{n!} \|P_n(\hat{\Phi}, \hat{\Theta})\|_\infty < \infty, \quad (5.22)$$

where $P_n(\hat{\Phi}, \hat{\Theta})$ is defined in (3.9). Moreover, for μ -a.e. $\lambda \in (-\eta, \eta)$, we have

$$g_X(\lambda) = h(\lambda) [h(\lambda)]^H \quad \text{with} \quad h(\lambda) = \sum_{n \in \mathbb{N}} \frac{(-i\lambda)^n}{n!} P_n(\hat{\Phi}, \hat{\Theta}) \Sigma^{1/2}, \quad (5.23)$$

660 where h can be seen as a power series valued in $\mathcal{S}_2(\mathcal{H}_0)$ with a convergence radius at least equal to η .

Proof. Since $z \mapsto [\Phi(z)]^{-1} \Theta(z)$ is holomorphic in an open ring containing \mathbb{U} and the exponential function is holomorphic on \mathbb{C} , by [15, Theorem 1.8.5], there exists $\eta > 0$ such that (5.22) holds and $[\Phi(e^z)]^{-1} \Theta(e^z)$ coincides with the $\mathcal{L}_b(\mathcal{H}_0)$ -valued power series $\sum_{n=0}^{\infty} P_n(\Phi, \Theta) z^n / n!$ on the set $\{z \in \mathbb{C} : |z| \leq \eta\}$. 665

Finally, we observe that $g_X = hh^H$ with $h(\lambda) = [\Phi(e^{-i\lambda})]^{-1} \Theta(e^{-i\lambda}) \Sigma^{1/2}$ and the given expression of h in (5.23) follows from (5.22) and the usual bound

$$\left\| P_n(\Phi, \Theta) \Sigma^{1/2} \right\|_2 \leq \|P_n(\Phi, \Theta)\|_{\infty} \left\| \Sigma^{1/2} \right\|_2 = \|P_n(\Phi, \Theta)\|_{\infty} \|\Sigma\|_1^{1/2}.$$

This concludes the proof. □

Proof of Corollary 2. Before proving the claimed implications, we start with some preliminary facts that follow from Lemma 5, Lemma 24 and Theorem 3. First observe that the process $UX = (UX_t)_{t \in \mathbb{Z}}$ is the \mathcal{G}_0 -valued ARMA(p, q) process defined by $\widehat{UX}(d\lambda) = [\tilde{\Phi}(e^{-i\lambda})]^{-1} \tilde{\Theta}(e^{-i\lambda}) \widehat{UZ}(d\lambda)$, where $\tilde{\Theta} := U\Theta U^H \in \mathcal{P}_q(\mathcal{G}_0)$ and $\tilde{\Phi} := U\Phi U^H \in \mathcal{P}_p^*(\mathcal{G}_0)$, and $UZ = (UZ_t)_{t \in \mathbb{Z}}$ is a \mathcal{G}_0 -valued white noise. Then, applying Lemma 5 with μ as the Lebesgue measure, we get that, for some $\eta > 0$, ν_{UX} has density $h(\lambda)[h(\lambda)]^H$ on $(-\eta, \eta)$ with h a power series valued in $\mathcal{S}_2(\mathcal{G}_0)$ with radius of convergence at least $\eta > 0$, 670

$$h(\lambda) = \sum_{n \in \mathbb{N}} \frac{(-i\lambda)^n}{n!} U P_n(\Phi, \Theta) \Sigma^{1/2} U^H. \quad (5.24)$$

Now, define, for any $\eta' \in (0, \eta)$,

$$I(\eta') := \int_{\mathcal{V}^2 \times (-\eta', \eta')} |\lambda|^{-2\Re(d(v))} |\mathcal{K}(v, v'; \lambda)|^2 \xi(dv) \xi(dv') \frac{d\lambda}{2\pi},$$

where \mathcal{K} is the \mathbb{T} -joint kernel of h in (5.24). By Theorem 3, Assertion (i) holds if, and only if, there exists $\eta' \in (0, \eta)$ such that $I(\eta') < \infty$ which is itself equivalent to having $I(\eta') < \infty$ for all $\eta' \in (0, \eta)$. Using (5.24), we have 675

$$I(\eta') = \int_{\mathcal{V}^2 \times (-\eta', \eta')} |\lambda|^{-2\Re(d(v))} \left| \sum_{n \in \mathbb{N}} \frac{(-i\lambda)^n}{n!} \mathcal{K}_n(v, v') \right|^2 \xi(dv) \xi(dv') \frac{d\lambda}{2\pi}, \quad (5.25)$$

where \mathcal{K}_n denotes the kernel of $U P_n(\Phi, \Theta) \Sigma^{1/2} U^H \in \mathcal{S}_2(\mathcal{G}_0)$. In particular we have by Lemma 24 that

$$\sigma_n(v) = \left(\mathbb{E} \left[|W_n(v, \cdot)|^2 \right] \right)^{1/2} = \|\mathcal{K}_n(v, \cdot)\|_{\mathcal{G}_0}. \quad (5.26)$$

Denote

$$I_n(\eta') := \int_{\mathcal{V}^2 \times (-\eta', \eta')} |\lambda|^{2n-2\Re(d(v))} |\mathcal{K}_n(v, v')|^2 \xi(dv) \xi(dv') \frac{d\lambda}{2\pi}, \quad (5.27)$$

$$\bar{d} = \xi\text{-essup}_{v \in \mathcal{V}}(\Re(d(v))) \quad \text{and} \quad \bar{m} := \inf \{ m \in \mathbb{N} : m > \bar{d} - 1/2 \}.$$

Note that \bar{d} and \bar{m} are finite since $d \in L^\infty(\mathcal{V}, \mathcal{V}, \xi)$. Defining, moreover, 680

$$\bar{I}(\eta') := \int_{\mathcal{V}^2 \times (-\eta', \eta')} |\lambda|^{-2\Re(d(v))} \left| \sum_{0 \leq n < \bar{m}} \frac{(-i\lambda)^n}{n!} \mathcal{K}_n(v, v') \right|^2 \xi(dv) \xi(dv') \frac{d\lambda}{2\pi}$$

$$\text{and} \quad R(\eta') := \sum_{n=\bar{m}}^{\infty} \frac{I_n^{1/2}(\eta')}{n!},$$

and applying the Minkowski inequality in (5.25), for any $\eta' \in (0, \eta)$, we have that, if $R(\eta') < \infty$,

$$I(\eta') < \infty \Leftrightarrow \bar{I}(\eta') < \infty.$$

Let us pick $\eta' \in (0, 1 \wedge \eta)$. Then, for all $\lambda \in (-\eta', \eta')$ and $n \in \mathbb{N}$, we have $|\lambda|^{2n-2\Re(d(v))} \leq |\lambda|^{2n-2\bar{d}}$ and thus, for all $n \geq \bar{m}$,

$$\begin{aligned} I_n(\eta') &\leq \frac{\eta'^{(1+2n-2\bar{d})}}{\pi(1+2n-2\bar{d})} \int_{\mathbb{V}^2} |\mathcal{K}_n(v, v')|^2 \xi(dv)\xi(dv') \\ &= \frac{\eta'^{(1+2n-2\bar{d})}}{\pi(1+2n-2\bar{d})} \left\| \mathbb{P}_n(\Phi, \Theta) \Sigma^{1/2} \right\|_2^2. \end{aligned}$$

Using that $\left\| \mathbb{P}_n(\Phi, \Theta) \Sigma^{1/2} \right\|_2 \leq \left\| \mathbb{P}_n(\Phi, \Theta) \right\|_\infty \left\| \Sigma \right\|_1^{1/2}$ and the bound (5.22) of Lemma 5, we get that $R(\eta') < \infty$. We thus conclude that Assertion (i) is equivalent to have, for some $\eta' \in (0, 1 \wedge \eta)$,

$$\bar{I}(\eta') < \infty. \quad (5.28)$$

685 Next, we show that, for any $\eta' \in (0, 1)$, Condition (ii) is, in fact, equivalent to have

$$I_n(\eta') < \infty \quad \text{for all } n \in \mathbb{N}. \quad (5.29)$$

Indeed, integrating w.r.t. v' and λ in the definition of $I_n(\eta')$ in (5.27), we have that, for all $n \in \mathbb{N}$,

$$I_n(\eta') = \int_{\{v: \Re(d(v)) < n+1/2\}} \frac{\eta'^{(1+2n-2\Re(d(v)))} \sigma_n^2(v)}{1+2n-2\Re(d(v))} \frac{\xi(dv)}{\pi},$$

if $\Re(d(v)) < n+1/2$ for ξ -a.e. $v \in \{\sigma_n > 0\}$, and is equal to ∞ otherwise. Since d is bounded on \mathbb{V} , so is $\eta'^{(1+2n-2\Re(d(v)))}$ on $v \in \mathbb{V}$ and we conclude that Condition (ii) is equivalent to (5.29).

We are now ready to prove each implication of the claimed equivalence successively.

690 **Proof of (i) \Leftarrow (ii).** This is now trivial, since applying the Minkowski inequality in the integral defining \bar{I} in (5.28) and the definition of I_n in (5.27), we immediately see that Condition (5.28) is implied by (5.29).

Proof of (i) \Rightarrow (ii). The proof of this implication is a bit more complex. The first step is to prove that Assertion (i) implies

$$\text{for all } n \in \mathbb{N} \text{ and } \xi\text{-a.e. } v \in \{\sigma_n > 0\}, \Re(d(v)) < n+1/2. \quad (5.30)$$

695 Then, we show that it must also imply (3.10) for all $n \in \mathbb{N}$ in a second and last step. For convenience, from now on, for any $c \in \mathbb{R}$, we simply write $\{\Re(d) < c\}$ for $\{v : \Re(d(v)) < c\}$.

Step 1. Suppose that (5.30) does not hold; let us show that Assertion (i) cannot hold. Since it is equivalent to (5.28), it is sufficient to show that $\bar{I}(\eta') = \infty$ for any arbitrary $\eta' > 0$. Let \underline{m} be the smallest $n \in \mathbb{N}$ for which $\xi(\{\Re(d) \geq n+1/2\} \cap \{\sigma_n > 0\}) > 0$. Note that by their mere definitions, we have $\underline{m} < \bar{m}$. In addition, for all $0 \leq n < \underline{m}$, we have

$$\xi(\{\Re(d) \geq \underline{m}+1/2\} \cap \{\sigma_n > 0\}) \leq \xi(\{\Re(d) \geq n+1/2\} \cap \{\sigma_n > 0\}) = 0.$$

Hence, $\mathcal{K}_n(v, v') = 0$ for $\xi^{\otimes 2}$ -a.e. $(v, v') \in \{\Re(d) \geq \underline{m}+1/2\} \times \mathbb{V}$. Now we have, using the definition of $\bar{I}(\eta')$ in (5.28) and what we just deduced,

$$\begin{aligned} \bar{I}(\eta') &\geq \int_{\{\Re(d) \geq \underline{m}+1/2\} \times \mathbb{V} \times (-\eta', \eta')} |\lambda|^{-2\Re(d(v))} \left| \sum_{0 \leq n < \bar{m}} \frac{(-i\lambda)^n}{n!} \mathcal{K}_n(v, v') \right|^2 \xi(dv)\xi(dv') \frac{d\lambda}{2\pi} \\ &= \int_{\{\Re(d) \geq \underline{m}+1/2\} \times \mathbb{V} \times (-\eta', \eta')} |\lambda|^{-2\Re(d(v))} \left| \sum_{\underline{m} \leq n < \bar{m}} \frac{(-i\lambda)^n}{n!} \mathcal{K}_n(v, v') \right|^2 \xi(dv)\xi(dv') \frac{d\lambda}{2\pi}. \end{aligned}$$

Note that, for all $(v, v') \in \mathbb{V}^2$ and $\lambda \in \mathbb{R}$, we have

$$|\lambda|^{-2\Re(d(v))} \left| \sum_{\underline{m} \leq n < \bar{m}} \frac{(-i\lambda)^n}{n!} \mathcal{K}_n(v, v') \right|^2 = |\lambda|^{2\underline{m}-2\Re(d(v))} \left(\frac{|\mathcal{K}_{\underline{m}}(v, v')|^2}{(\underline{m}!)^2} + o(1) \right),$$

where σ -term tends to 0 as $\lambda \rightarrow 0$. We get that, for all $(v, v') \in (\{\Re(d) \geq \underline{m} + 1/2\} \times \mathbb{V}) \cap \{|\mathcal{K}_{\underline{m}}|^2 > 0\}$ and $\eta' > 0$,

$$\int_{(-\eta', \eta')} |\lambda|^{-2\Re(d(v))} \left| \sum_{\underline{m} \leq n < \overline{m}} \frac{(-i\lambda)^n}{n!} \mathcal{K}_n(v, v') \right|^2 \frac{d\lambda}{2\pi} = \infty.$$

With the previous lower bound on $\bar{I}(\eta')$, we deduce that $\bar{I}(\eta') = \infty$ if

$$\xi^{\otimes 2} \left((\{\Re(d) \geq \underline{m} + 1/2\} \times \mathbb{V}) \cap \{|\mathcal{K}_{\underline{m}}|^2 > 0\} \right) > 0.$$

By definition of σ_n in (5.26), we have, for all $v \in \mathbb{V}$,

$$g(v) := \int_{\mathbb{V}} \mathbb{1}_{\{|\mathcal{K}_{\underline{m}}(v, v')|^2 > 0\}} \xi(dv') > 0 \Leftrightarrow \sigma_{\underline{m}}(v) > 0.$$

Hence,

$$\xi^{\otimes 2} \left((\{\Re(d) \geq \underline{m} + 1/2\} \times \mathbb{V}) \cap \{|\mathcal{K}_{\underline{m}}|^2 > 0\} \right) = \int_{\{\Re(d) \geq \underline{m} + 1/2\}} g \, d\xi$$

is positive if and only if $\xi(\{\Re(d) \geq \underline{m} + 1/2\} \cap \{\sigma_{\underline{m}} > 0\}) > 0$, which is true by definition of \underline{m} . This concludes the first step.

Step 2. Suppose now that (5.30) does hold but (3.10) does not hold for all $n \in \mathbb{N}$ and let us show again that Assertion (i) cannot hold. Let us define \tilde{m} as the smallest $n \in \mathbb{N}$ such that (3.10) does not hold. Again by definition of \overline{m} , we must have $\tilde{m} < \overline{m}$, since (3.10) holds for $n = \overline{m}$ by definition of \overline{m} . Take now an arbitrary $\eta' \in (0, 1 \wedge \eta)$. We have shown in the preamble of the proof that if (5.30) is satisfied, then (3.10) is equivalent to $I_n(\eta') < \infty$. Hence, we can also see \tilde{m} as the smallest $n \in \mathbb{N}$ such that $I_n(\eta') = \infty$. Thus, we have $I_k(\eta') < \infty$ for all $0 \leq k < \tilde{m}$ and $I_{\tilde{m}}(\eta') = \infty$. It follows that Assertion (i) is not only equivalent to having $\bar{I}(\eta') < \infty$ as in (5.28) but also to the condition

$$\tilde{I}(\eta') := \int_{\mathbb{V}^2 \times (-\eta', \eta')} |\lambda|^{-2\Re(d(v))} \left| \sum_{\tilde{m} \leq n < \overline{m}} \frac{(-i\lambda)^n}{n!} \mathcal{K}_n(v, v') \right|^2 \xi(dv) \xi(dv') \frac{d\lambda}{2\pi} < \infty. \quad (5.31)$$

Now, we observe that

$$\tilde{I}(\eta') \geq \int_{\{\Re(d) < \tilde{m} + 1/2\} \times \mathbb{V} \times (-\eta', \eta')} |\lambda|^{-2\Re(d(v))} \left| \sum_{\tilde{m} \leq n < \overline{m}} \frac{(-i\lambda)^n}{n!} \mathcal{K}_n(v, v') \right|^2 \xi(dv) \xi(dv') \frac{d\lambda}{2\pi}.$$

Therefore, to conclude that $\tilde{I}(\eta') = \infty$ (implying that Assertion (i) does not hold), by the Minkowski inequality, it is sufficient to show that

$$\tilde{I}_{\tilde{m}}(\eta') = \infty \quad \text{and, for all } n > \tilde{m}, \quad \tilde{I}_n(\eta') < \infty, \quad (5.32)$$

where, for all $n \in \mathbb{N}$, we denoted

$$\tilde{I}_n(\eta') := \int_{\{\Re(d) < \tilde{m} + 1/2\} \times \mathbb{V} \times (-\eta', \eta')} |\lambda|^{2n - 2\Re(d(v))} |\mathcal{K}_n(v, v')|^2 \xi(dv) \xi(dv') \frac{d\lambda}{2\pi}.$$

For all $n \geq \tilde{m}$ we have, as in the previous computation of I_n that

$$\tilde{I}_n(\eta') < \infty \Leftrightarrow \int_{\{\Re(d) < \tilde{m} + 1/2\}} \frac{\sigma_n^2(v)}{1 + 2n - 2\Re(d(v))} \xi(dv) < \infty.$$

For an integer $n > \tilde{m}$, we have $1 + 2n - 2\Re(d(v)) \geq 2$ on $\{\Re(d) < \tilde{m} + 1/2\}$, hence the right-hand side of (5.32) follows as a consequence of $\int \sigma_n^2 \, d\xi < \infty$. For $n = \tilde{m}$, the left-hand side of (5.32) follows as a consequence of (3.10) not being satisfied for $n = \tilde{m}$ by definition of \tilde{m} . \square

5.1.5. Proof of Proposition 4

715 The proof of Proposition 4 relies on the following two lemmas where we recall that the open and closed complex unit discs of \mathbb{C} are denoted by \mathbb{D} and $\overline{\mathbb{D}}$, respectively. The first lemma involves Banach-space-valued series and is used in the next one.

Lemma 6. *Let E be a Banach space and $(a_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$ such that $\|a_n\|_E \xrightarrow{n \rightarrow \infty} 0$ and the series $\sum \|a_n - a_{n+1}\|_E$ converges. Then for all $z_0 \in \overline{\mathbb{D}} \setminus \{1\}$, the series $\sum_{n=0}^{\infty} a_n z_0^n$ converges in E and the mapping $z \mapsto \sum_{n=0}^{\infty} a_n z^n$ is uniformly continuous on $[0, z_0]$.*

Proof. By assumption on (a_n) , $\sum a_n z^n$ is a power series valued in E with a convergence radius at least equal to 1, and hence is uniformly continuous on any compact subset of \mathbb{D} . When $|z_0| = 1$, the result follows using Abel's transform. \square

725 The following lemma provides a useful identity between operator-valued functions defined on convenient domains of \mathbb{C} , whose restrictions to the unit circle are the transfer functions \mathbf{I}_D and \mathbf{J}_N . It is needed in the proofs of Proposition 4 and Lemma 18.

Lemma 7. *Let \mathcal{H}_0 be a separable Hilbert space and $\varsigma > 0$. Then there exist $C_1, C_2 > 0$ only depending on ς such that, for all $N \in \mathcal{L}_b(\mathcal{H}_0)$ satisfying $\|N\|_{\infty} \leq \varsigma$, there exist $Q \in \mathcal{L}_b(\mathcal{H}_0)$ and $(P_k^*)_{k \in \mathbb{N}} \in \mathcal{L}_b(\mathcal{H}_0)^{\mathbb{N}}$ satisfying the four following assertions.*

730 (i) We have $\|Q\|_{\infty} \leq C_1$.

(ii) We have, for all $k \geq 0$, $\|P_k^*\|_{\infty} \leq C_2 (k+1)^{-1} \|(k+1)^{-N}\|_{\infty}$.

(iii) We have, for all $z \in \mathbb{D}$, $(1-z)^{N - \text{Id}_{\mathcal{H}_0}} = Q \left(\sum_{k=0}^{\infty} (k+1)^{-N} z^k \right) + \sum_{k=0}^{\infty} P_k^* z^k$, where the two infinite sums converge in $\mathcal{L}_b(\mathcal{H}_0)$.

(iv) If $\Re(z) > 0$ for all $z \in \sigma(N)$, then Assertion (iii) can be extended to all $z \in \overline{\mathbb{D}} \setminus \{1\}$ and we have $\sum_{k=0}^{\infty} \|P_k^*\|_{\infty} < \infty$.

Proof. Let $z \in \mathbb{D}$, then

$$(1-z)^{N - \text{Id}_{\mathcal{H}_0}} = \text{Id}_{\mathcal{H}_0} + \sum_{k \geq 1} N_k z^k \quad \text{with} \quad N_k = \prod_{j=1}^k \left(\text{Id}_{\mathcal{H}_0} - \frac{N}{j} \right), \quad \text{for all } k \geq 1. \quad (5.33)$$

The proof is now three steps. First, by separating the above sum in two parts $k < k_0$ and $k \geq k_0$ and by decomposing N_k conveniently for $k \geq k_0$, we obtain Assertion (iii) for well defined Q and P_k^* , $k \geq 0$. Then we show that Assertions (i) and (iii) hold. Finally we prove Assertion (iv).

740 **Step 1.** Define the integer $k_0 \geq 1$ by the condition $\varsigma < k_0 \leq \varsigma + 1$. Then, for all $j \geq k_0$, $\text{Id}_{\mathcal{H}_0} - \frac{N}{j} = \exp \left(\ln \left(\text{Id}_{\mathcal{H}_0} - \frac{N}{j} \right) \right) = \exp \left(- \sum_{\ell \geq 1} \frac{N^{\ell}}{\ell j^{\ell}} \right)$ and therefore, for all $k \geq k_0$,

$$\begin{aligned} N_k &= \prod_{j=1}^{k_0-1} \left(\text{Id}_{\mathcal{H}_0} - \frac{N}{j} \right) \exp \left(- \sum_{\ell \geq 1} \frac{N^{\ell}}{\ell} \sum_{j=k_0}^k \frac{1}{j^{\ell}} \right) \\ &= \prod_{j=1}^{k_0-1} \left(\text{Id}_{\mathcal{H}_0} - \frac{N}{j} \right) \exp \left(-N \sum_{j=k_0}^k \frac{1}{j} \right) \exp \left(- \sum_{\ell \geq 2} \frac{N^{\ell}}{\ell} \sum_{j=k_0}^k \frac{1}{j^{\ell}} \right). \end{aligned} \quad (5.34)$$

Moreover, we have the following asymptotic expansions. For all $k \geq k_0$,

$$\begin{aligned} \sum_{j=k_0}^k \frac{1}{j} &= \sum_{j=1}^k \frac{1}{j} - \sum_{j=1}^{k_0-1} \frac{1}{j} = \ln(k+1) + \gamma_e - \sum_{j=1}^{k_0-1} \frac{1}{j} + \frac{\alpha_k}{k}, \\ \sum_{j=k_0}^k \frac{1}{j^\ell} &= \sum_{j=k_0}^{\infty} \frac{1}{j^\ell} - \sum_{j=k+1}^{\infty} \frac{1}{j^\ell} = \frac{\beta_\ell}{k_0^\ell} + \frac{\eta_{k,\ell}}{(\ell-1)k^{\ell-1}}, \text{ for all } \ell \geq 2, \end{aligned}$$

where γ_e is Euler's constant, $\beta_\ell := \sum_{k=k_0}^{\infty} \left(\frac{k_0}{k}\right)^\ell$, and $(\alpha_k)_{k \geq 1}$ and $(\eta_{k,\ell})_{k \geq 1, \ell \geq 2}$ are some universal constants satisfying

$$\sup_{k \geq 1} |\alpha_k| < \infty \quad \text{and} \quad \sup_{k \geq 1, \ell \geq 2} |\eta_{k,\ell}| < \infty. \quad (5.35)$$

745 Also note that

$$\sup_{\ell \geq 2} \beta_\ell = \beta_2 < \infty. \quad (5.36)$$

Using these definitions in (5.34), we obtain, for all $k \geq k_0$,

$$N_k = Q(k+1)^{-N} \exp\left(-N \frac{\alpha_k}{k} - \sum_{\ell \geq 2} \frac{N^\ell \eta_{k,\ell}}{(\ell-1)k^{\ell-1}}\right)$$

where

$$Q = \prod_{j=1}^{k_0-1} \left(\text{Id}_{\mathcal{H}_0} - \frac{N}{j}\right) \exp\left(-N \left(\gamma_e - \sum_{t=1}^{k_0-1} \frac{1}{t}\right)\right) \exp\left(-\sum_{\ell \geq 2} \left(\frac{N}{k_0}\right)^\ell \frac{\beta_\ell}{\ell}\right). \quad (5.37)$$

Using the previous equations in (5.33), for all $z \in \mathbb{D}$, we can write $(1-z)^{N-\text{Id}_{\mathcal{H}_0}}$ as

$$\text{Id}_{\mathcal{H}_0} + \sum_{k=1}^{k_0-1} \prod_{j=1}^k \left(\text{Id}_{\mathcal{H}_0} - \frac{N}{j}\right) z^k + Q \sum_{k \geq k_0} (k+1)^{-N} \exp\left(-N \frac{\alpha_k}{k} - \sum_{\ell \geq 2} \frac{N^\ell \eta_{k,\ell}}{(\ell-1)k^{\ell-1}}\right) z^k.$$

Thus, Assertion (iii) immediately follows by setting

$$P_0^* := \text{Id}_{\mathcal{H}_0} - Q, \quad (5.38)$$

$$P_k^* := \prod_{j=1}^k \left(\text{Id}_{\mathcal{H}_0} - \frac{N}{j}\right) - Q(k+1)^{-N}, \quad \text{for } 1 \leq k \leq k_0 - 1, \quad (5.39)$$

$$P_k^* := Q(k+1)^{-N} \left[\exp\left(-N \frac{\alpha_k}{k} - \sum_{\ell \geq 2} \frac{N^\ell \eta_{k,\ell}}{(\ell-1)k^{\ell-1}}\right) - \text{Id}_{\mathcal{H}_0} \right] \quad \text{for } k \geq k_0. \quad (5.40)$$

Step 2. We now prove Assertion (i) and Assertion (ii). Since $\|N\|_\infty \leq \varsigma$ and k_0 only depends on ς , using (5.37) and (5.36), we immediately get Assertion (i) for $C_1 > 0$ well chosen only depending on ς . By (5.38) and (5.39), for $k < k_0 \leq \varsigma + 1$ we can similarly bound $\|P_k^*\|_\infty$ by a constant only depending on ς . Now let $k \geq k_0$ and define

$$\Phi_k := -N \frac{\alpha_k}{k} - \sum_{\ell \geq 2} \frac{N^\ell \eta_{k,\ell}}{(\ell-1)k^{\ell-1}}.$$

Relation (5.40) yields

$$\|P_k^*\|_\infty \leq \|Q\|_\infty \|(k+1)^{-N}\|_\infty \sum_{t \geq 1} \frac{\|\Phi_k\|_\infty^t}{t!}.$$

Using that $\|N\|_\infty \leq \varsigma$, we have

$$\begin{aligned} \|\Phi_k\|_\infty &\leq \varsigma \frac{|\alpha_k|}{k} + \sum_{\ell \geq 2} \frac{\varsigma^\ell \eta_{k,\ell}}{(\ell-1)k^{\ell-1}} \\ &= \frac{\varsigma}{k} \left(|\alpha_k| + \sum_{\ell \geq 1} \frac{\varsigma^\ell}{\ell} \eta_{k,\ell+1} k^{1-\ell} \right) \\ &\leq \frac{\varsigma}{k} \left(|\alpha_k| + \left(\sup_{k \geq 1, \ell \geq 2} |\eta_{k,\ell}| \right) \varsigma \left(1 - \frac{\varsigma}{k_0}\right)^{-1} \right). \end{aligned}$$

750 The two previous bounds, with Assertion (i) and (5.35), yield Assertion (iii).

Step 3. We conclude with the proof of Assertion (iv) and therefore assume that $\Re(z) > 0$ for all $z \in \sigma(N)$. Since $\sigma(N)$ is a compact subset of \mathbb{C} , there exists $\varsigma > 0$ and $\varrho \in (0, \varsigma)$ such that $\sigma(N) \subset \mathbb{D}_{\varrho, \varsigma}$ with $\mathbb{D}_{\varrho, \varsigma}$ defined as in (5.1). Applying Corollary 3 with $K = \{N\}$, the bounds (5.15) and (5.16) gives us that

$$\|(k+1)^{-N}\|_\infty = O((k+1)^{-\varrho}), \quad (5.41)$$

$$\|(k+1)^{-N} - (k+2)^{-N}\|_\infty = O((k+1)^{-\varrho-1}). \quad (5.42)$$

We can now extend Assertion (iii) to $\overline{\mathbb{D}} \setminus \{1\}$, that is, to the case $z = e^{-i\lambda}$ for some $\lambda \in \mathbb{T} \setminus \{0\}$. For such a λ , as a consequence of Assertion (iii), we already have, for all $0 < a < 1$,

$$(1 - ae^{-i\lambda})^{N - \text{Id}_{\mathcal{H}_0}} = \mathbb{Q} \sum_{k \geq 0} (k+1)^{-N} a^k e^{-i\lambda k} + \sum_{k \geq 0} P_k^* a^k e^{-i\lambda k}.$$

755 Moreover, $(1 - e^{-i\lambda})^{N - \text{Id}_{\mathcal{H}_0}} = \lim_{a \uparrow 1} (1 - ae^{-i\lambda})^{N - \text{Id}_{\mathcal{H}_0}}$ by continuity of $z \mapsto (1 - z)^{N - \text{Id}_{\mathcal{H}_0}}$ in $\mathbb{C} \setminus [1, \infty)$. It remains to show that, for $\lambda \in \mathbb{T} \setminus \{0\}$, $\sum_{k \geq 0} (k+1)^{-N} a^k e^{-i\lambda k}$ converges to $\sum_{k \geq 0} (k+1)^{-N} e^{-i\lambda k}$ and $\sum_{k \geq 0} P_k^* a^k e^{-i\lambda k}$ converges to $\sum_{k \geq 0} P_k^* e^{-i\lambda k}$ and as $a \uparrow 1$ and that these convergences hold in $\mathcal{L}_b(\mathcal{H}_0)$. The first convergence follows by applying Lemma 6 with $a_k = (k+1)^{-N}$ using (5.41) and (5.42). The second convergence follows from Assertion (ii) and (5.41), which concludes the proof. \square

We can now prove Proposition 4.

Proof of Proposition 4. Let us first show that $\epsilon \in \mathcal{S}_{\text{Id}_D}(\Omega, \mathcal{F}, \mathbb{P})$. Since ϵ is a white noise, as explained in Remark 2 (2), Assertion (ii) of Corollary 2 needs to be checked only for $n = 0$. The result follows since this case precisely corresponds to the conditions in (3.13) with $D = \text{Id}_{\mathcal{H}_0} - N$.

765 We apply Lemma 7 with $\varsigma = \|N\|_\infty$. Let h be as in Proposition 3, we have $\Re(z) \geq \varrho := \xi\text{-essinf}(h)$ for all $z \in \sigma(N)$. The first condition in (3.13) implies that $\varrho \geq 1/2$. Then, Assertion (iv) of Lemma 7 gives us that, for all $\lambda \in \mathbb{T} \setminus \{0\}$, we have $(1 - e^{-i\lambda})^{N - \text{Id}_{\mathcal{H}_0}} = \mathbb{Q} \sum_{k=0}^\infty (k+1)^{-N} e^{-i\lambda k} + \sum_{k=0}^\infty P_k^* e^{-i\lambda k}$ in $\mathcal{L}_b(\mathcal{H}_0)$ and $\sum_{k=0}^\infty \|P_k^*\|_\infty < \infty$. Thus $C F_{\mathbf{J}_N}(\epsilon) + Z$ and $F_{\text{Id}_D}(\epsilon)$ coincide in the spectral domain of ϵ , which concludes the proof. \square

770 5.2. Proofs of Section 4.1

5.2.1. Preliminary results

In the following, for two separable Hilbert spaces \mathcal{H}_0 and \mathcal{I}_0 and a finite non-negative measure μ on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, we denote by $\|\cdot\|_{1,1}$ the natural norm of the Bochner space $L^{1,1}(\mathcal{H}_0, \mathcal{I}_0, \mu) := L^1(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{S}_1(\mathcal{H}_0, \mathcal{I}_0), \mu)$, that is,

$$\|g\|_{1,1} := \int \|g(\lambda)\|_1 d\mu.$$

We use the notation

$$B_{1,1}(r, \mathcal{H}_0, \mathcal{I}_0, \mu) = \left\{ g \in L^1(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{S}_1(\mathcal{H}_0, \mathcal{I}_0), \mu) : \|g\|_{1,1} \leq r \right\},$$

for the ball of radius r in the Banach space $L^{1,1}(\mathcal{H}_0, \mathcal{I}_0, \mu)$. As usual, if $\mathcal{I}_0 = \mathcal{H}_0$, we drop \mathcal{I}_0 in the notation, thus writing $L^{1,1}(\mathcal{H}_0, \mu)$ and $B_{1,1}(r, \mathcal{H}_0, \mu)$ in this case. Also if $\mu = \mu_{\mathbb{T}}$ we drop the measure in the notation, thus writing $L^{1,1}(\mathcal{H}_0, \mathcal{I}_0)$ and $B_{1,1}(r, \mathcal{H}_0, \mathcal{I}_0)$, or $L^{1,1}(\mathcal{H}_0)$ and $B_{1,1}(r, \mathcal{H}_0)$ if $\mathcal{I}_0 = \mathcal{G}_0$, in this case.

These definitions and those introduced in Section 4.1 (such as \mathcal{Q} and $\tilde{\mathcal{Q}}$) will be useful in the following. Recall, in particular, that $\mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{E})$ denotes the set of bounded measurable functions from $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ to $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$. We will also need to define $\mathbb{F}_{b,b}((\mathcal{H}_0, \mathcal{G}_0), (\mathcal{I}_0, \mathcal{J}_0))$ as the product vector space $\mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{I}_0, \mathcal{J}_0)) \times \mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$, endowed with the max norm

$$\|(L, R)\|_{b,b} := \max(\sup(L), \sup(R)) \quad \text{for all } (L, R) \in \mathbb{F}_{b,b}((\mathcal{H}_0, \mathcal{G}_0), (\mathcal{I}_0, \mathcal{J}_0)).$$

This simply extends the definition of $\mathbb{F}_{b,b}(\mathcal{H}_0, \mathcal{G}_0)$ already introduced in Section 4.1, which can be seen as a short-hand notation for $\mathbb{F}_{b,b}((\mathcal{H}_0, \mathcal{G}_0), (\mathcal{H}_0, \mathcal{G}_0))$.

We now derive a series of useful lemmas.

Lemma 8. *Let μ be a finite non-negative measure on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ and $\mathcal{H}_0, \mathcal{G}_0, \mathcal{I}_0$ and \mathcal{J}_0 be four separable Hilbert spaces and $g \in L^{1,1}(\mathcal{H}_0, \mathcal{I}_0, \mu)$. Then the mapping $\mathcal{Q}_{g,\mu}$ defined by*

$$\mathcal{Q}_{g,\mu}(L, R) = \int L g R^{\text{H}} d\mu \tag{5.43}$$

is a continuous sesquilinear mapping from $\mathbb{F}_{b,b}((\mathcal{H}_0, \mathcal{G}_0), (\mathcal{I}_0, \mathcal{J}_0))$ to $\mathcal{S}_1(\mathcal{G}_0, \mathcal{J}_0)$ satisfying

$$\sup \left\{ \|\mathcal{Q}_{g,\mu}(L, R)\|_1 : \|(L, R)\|_{b,b} \leq 1 \right\} \leq \|g\|_{1,1}.$$

Consequently, for any positive radius r , the set $\{\mathcal{Q}_{g,\mu} : g \in B_{1,1}(r, \mathcal{H}_0, \mathcal{I}_0, \mu)\}$ is equicontinuous in $\mathcal{C}(\mathbb{F}_{b,b}((\mathcal{H}_0, \mathcal{G}_0), (\mathcal{I}_0, \mathcal{J}_0)), \mathcal{S}_1(\mathcal{G}_0, \mathcal{I}_0))$.

Proof. For all $(L, R) \in \mathbb{F}_{b,b}((\mathcal{H}_0, \mathcal{G}_0), (\mathcal{I}_0, \mathcal{J}_0))$, $g \in L^{1,1}(\mathcal{H}_0, \mathcal{I}_0)$ and $\lambda \in \mathbb{T}$, we have

$$\|L(\lambda) g(\lambda) R(\lambda)^{\text{H}}\|_1 \leq \|L(\lambda)\|_{\infty} \|g(\lambda)\|_1 \|R(\lambda)^{\text{H}}\|_{\infty},$$

which is thus integrable with respect to μ . Moreover, we obtain that

$$\|\mathcal{Q}_{g,\mu}(L, R)\|_1 \leq \|g\|_{1,1} \sup(L) \sup(R).$$

The given claim immediately follows as well as its consequence. \square

We also derive the following lemma, which will be useful in the following.

Lemma 9. *Let $\mathcal{H}_0, \mathcal{G}_0$ and \mathcal{I}_0 be three separable Hilbert spaces, Θ be a compact metric space, and let L and R be two continuous mappings from $\Theta \times \mathbb{T}$ into $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and $\mathcal{L}_b(\mathcal{H}_0, \mathcal{I}_0)$, respectively. Then, for any positive radius r , the set $\mathcal{R}(r) := \left\{ \tilde{\mathcal{Q}}_g^{(L,R)} : g \in B_{1,1}(r, \mathcal{H}_0) \right\}$ is equicontinuous in $\mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{I}_0, \mathcal{G}_0))$.*

Proof. Let $r > 0$. Since $\Theta \times \mathbb{T}$ is compact, R and L are uniformly continuous on $\Theta \times \mathbb{T}$ and we get that $\theta \mapsto (L(\theta, \cdot), R(\theta, \cdot)) \in \mathcal{C}(\Theta, \mathbb{F}_{b,b}((\mathcal{H}_0, \mathcal{G}_0), (\mathcal{H}_0, \mathcal{I}_0)))$. By Lemma 8, we get that $\mathcal{R}(r)$ is equicontinuous in $\mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{I}_0, \mathcal{G}_0))$. \square

We next provide four last preliminary lemmas, one about the non-centered periodogram, and two dealing with the centering term.

Lemma 10. Let \mathcal{H}_0 and \mathcal{I}_0 be two separable Hilbert spaces such that \mathcal{I}_0 is continuously embedded in \mathcal{H}_0 . Assume **(A-1)** and suppose moreover that $X_0 \in \mathcal{I}_0$ \mathbb{P} -a.s. with $\mathbb{E} \left[\|X_0\|_{\mathcal{I}_0}^2 \right] < \infty$. Then, for all $n \geq 1$, $I_n^X \in L^{1,1}(\mathcal{H}_0, \mathcal{I}_0)$ \mathbb{P} -a.s. and we have

$$\sup_{n \geq 1} \|I_n^X\|_{1,1} < \infty \quad \mathbb{P}\text{-a.s.}$$

Proof. Note that, for all $n \geq 1$, we have, using the definition of I_n^X , and then that of d_n^X ,

$$\begin{aligned} \|I_n^X\|_{1,1} &= \int \|I_n^X\|_1 \, d\mu_{\mathbb{T}} \\ &\leq \int \|d_n^X(\lambda)\|_{\mathcal{I}_0} \|d_n^X(\lambda)\|_{\mathcal{H}_0} \, d\mu_{\mathbb{T}} \\ &\leq \left(\int \|d_n^X(\lambda)\|_{\mathcal{I}_0}^2 \, d\mu_{\mathbb{T}} \right)^{1/2} \left(\int \|d_n^X(\lambda)\|_{\mathcal{H}_0}^2 \, d\mu_{\mathbb{T}} \right)^{1/2} \\ &= \left(\frac{1}{n} \sum_{k=1}^n \|X_k\|_{\mathcal{I}_0}^2 \right)^{1/2} \left(\frac{1}{n} \sum_{k=1}^n \|X_k\|_{\mathcal{H}_0}^2 \right)^{1/2}, \end{aligned}$$

795 where in the right-hand side of the first line $\|\cdot\|_1$ denotes the $\mathcal{S}_1(\mathcal{H}_0, \mathcal{I}_0)$ -norm. By **(A-1)**, with the Birkhoff ergodic theorem, we get that the right-hand of the previous bound converges \mathbb{P} -a.s. The claim \mathbb{P} -a.s. uniform bound of $\left(\|I_n^X\|_{1,1} \right)_{n \geq 1}$ follows. \square

Lemma 11. Recall that I_n^X and $I_n^{X^{(c,n)}}$ denote the periodograms respectively computed from X_1, \dots, X_n and from $X_1^{(c,n)}, \dots, X_n^{(c,n)}$, as defined in **(1.3)**. Suppose that $X_1, \dots, X_n \in \mathcal{H}_0$. Then, I_n^X and $I_n^{X^{(c,n)}}$ belong to $\mathcal{S}_1(\mathcal{H}_0)$ and we have, for all $\lambda \in \mathbb{T}$,

$$\left\| I_n^X(\lambda) - I_n^{X^{(c,n)}}(\lambda) \right\|_1 \leq \left\| \frac{1}{n} \sum_{j=1}^n X_j \right\|_{\mathcal{H}_0}^2 F_n(\lambda) + 2 \left\| \frac{1}{n} \sum_{j=1}^n X_j \right\|_{\mathcal{H}_0} \|d_n^X(\lambda)\|_{\mathcal{H}_0} (F_n(\lambda))^{1/2},$$

where F_n denotes the Fejér kernel defined by

$$F_n(\lambda) = \frac{1}{n} \left| \sum_{k=1}^n e^{-i\lambda k} \right|^2. \quad (5.44)$$

Proof. By **(1.3)** and **(4.1)**, we have, for all $\lambda \in \mathbb{T}$,

$$d_n^{X^{(c,n)}}(\lambda) = d_n^X(\lambda) - \left(\frac{1}{n} \sum_{j=1}^n X_j \right) \left(\frac{1}{\sqrt{n}} \sum_{k=1}^n e^{-i\lambda k} \right).$$

800 Computing $I_n^{X^{(c,n)}}(\lambda) = d_n^{X^{(c,n)}}(\lambda) \otimes d_n^{X^{(c,n)}}(\lambda)$ and using that $\|x \otimes y\|_1 = \|x\|_{\mathcal{H}_0} \|y\|_{\mathcal{H}_0}$ for all $x, y \in \mathcal{H}_0$, we easily get the result. \square

We have the following result on the process X .

Lemma 12. Let \mathcal{H}_0 be a separable Hilbert space. Assume **(A-1)** and **(A-2)**. Then, the process X is valued in a finite-dimensional space $\mathcal{G}_0 \subset \mathcal{H}_0$ or, if it is not the case, there always exists an orthonormal sequence $(\phi_k)_{k \in \mathbb{N}}$ of \mathcal{H}_0 and a sequence $s = (s_k)_{k \in \mathbb{N}} \in [1, \infty)^{\mathbb{N}}$ such that Conditions **(i)** and **(ii)** in Theorem 5 hold. 805

Proof. Define $\Sigma = \mathbb{E}[X_0 \otimes X_0]$. Since $\Sigma \in \mathcal{S}_1^+(\mathcal{H}_0)$, there exists a finite or countable non-increasing sequence $(\sigma_k)_{0 \leq k < K} \in (0, \infty)^\mathbb{N}$ and an orthonormal sequence $(\phi_k)_{0 \leq k < K} \in \mathcal{H}_0^\mathbb{N}$ such that

$$\Sigma = \sum_{0 \leq k < K} \sigma_k^2 \phi_k \otimes \phi_k \quad \text{and} \quad \sum_{0 \leq k < K} \sigma_k^2 < \infty .$$

In particular, we have that \mathbb{P} -a.s., X_0 is valued in $\overline{\text{Span}}^{\mathcal{H}_0}(\phi_k, 0 \leq k < K)$. If K is finite, then X_0 is valued in the finite-dimensional space $\mathcal{G}_0 = \text{Span}(\phi_k, 0 \leq k < K)$.

From now on, we take $K = \infty$. By Lemma 23, we can find $s = (s_k)_{k \in \mathbb{N}} \in [1, \infty)^\mathbb{N}$, non-decreasing and going to ∞ (hence, satisfying Condition (i) in Theorem 5), such that $\sum_{k \in \mathbb{N}} s_k^2 \sigma_k^2 < \infty$. Defining \mathcal{H}_0^s by (4.8) and its inner product by (4.9), we get that

$$\mathbb{E} \left[\|X_0\|_{\mathcal{H}_0^s}^2 \right] = \sum_{k \in \mathbb{N}} s_k^2 \sigma_k^2 < \infty .$$

We thus have Condition (ii) in Theorem 5. \square

The following lemma is used to treat the centering term in the next result, and also to prove 810 Theorem 1.

Lemma 13. *Let \mathcal{H}_0 be a separable Hilbert space. Assume (A-1) and (A-2). Then,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n X_j = \mathbb{E}[X_0] \quad \text{in } \mathcal{H}_0, \quad \mathbb{P}\text{-a.s.} \quad (5.45)$$

Proof. If we can find a finite-dimensional space $\mathcal{G}_0 \subset \mathcal{H}_0$ such that X is valued in \mathcal{G}_0 , then (5.45) follows straightforwardly from the Birkhoff ergodic theorem. If not, by Lemma 12, we can find an orthonormal sequence $(\phi_k)_{k \in \mathbb{N}}$ of \mathcal{H}_0 such that Conditions (i) and (ii) in Theorem 5 hold. As a consequence, by the Birkhoff ergodic theorem, we get that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \|X_j\|_{\mathcal{H}_0^s} = \mathbb{E} \left[\|X_0\|_{\mathcal{H}_0^s} \right] \quad \mathbb{P}\text{-a.s.}$$

Since $\left\| \frac{1}{n} \sum_{j=1}^n X_j \right\|_{\mathcal{H}_0^s} \leq \frac{1}{n} \sum_{j=1}^n \|X_j\|_{\mathcal{H}_0^s}$ for all $n \geq 1$, we get that, \mathbb{P} -a.s., there exists $r > 0$ such that $\sup_{n \geq 1} \left\| \frac{1}{n} \sum_{j=1}^n X_j \right\|_{\mathcal{H}_0^s} \leq r$. Using that $(s_k)_{k \in \mathbb{N}}$ is going to infinity, we get that the operator $\sum_{k \in \mathbb{N}} s_k^{-1} \phi_k \otimes \phi_k$ belongs to $\mathcal{S}_\infty(\mathcal{H}_0)$. Since the image by this operator of the unit \mathcal{H}_0 -ball is the unit \mathcal{H}_0^s -ball, we get that all \mathcal{H}_0^s -balls are compact in \mathcal{H}_0 . In particular, we have that $\left(\frac{1}{n} \sum_{j=1}^n X_j \right)_{n \geq 1}$ is \mathbb{P} -a.s. valued in a compact subset of \mathcal{H}_0 . Therefore, it only remains to show that, \mathbb{P} -a.s., the only possible accumulation point of this sequence is $\mathbb{E}[X_0]$. This fact follows by using the Birkhoff ergodic theorem again, which gives us, for all $x \in \mathcal{H}_0$,

$$\lim_{n \rightarrow \infty} \left\langle \frac{1}{n} \sum_{j=1}^n X_j, x \right\rangle_{\mathcal{H}_0} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \langle X_j, x \rangle_{\mathcal{H}_0} = \mathbb{E} [\langle X_0, x \rangle_{\mathcal{H}_0}] = \langle \mathbb{E}[X_0], x \rangle_{\mathcal{H}_0} \quad \mathbb{P}\text{-a.s.}$$

Hence we get the convergence (5.45). \square

We can now state a result which applies both in the context of Theorem 4 and Theorem 5.

Theorem 8. *Let \mathcal{H}_0 and \mathcal{G}_0 be two separable Hilbert spaces. Assume (A-1) and (A-2), and suppose that (Θ, Δ) is a compact metric space. Let L and R in $\mathcal{C}(\Theta \times \mathbb{T}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$. Moreover, suppose that, \mathbb{P} -a.s., there exists, for all $\theta \in \Theta$, a compact subset $B \subset \mathcal{S}_1(\mathcal{G}_0)$ such that, for all $n \geq 1$ $\tilde{\mathcal{Q}}_{I_n^X}^{(L,R)}(\theta) \in B$. Then, we have*

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{Q}}_{I_n^{X^{(c,n)}}}^{(L,R)} = \tilde{\mathcal{Q}}_{\nu_X}^{(L,R)} \quad \text{uniformly in } \mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{G}_0)), \quad \mathbb{P}\text{-a.s.} \quad (5.46)$$

Proof. Thanks to the centering (1.3), we can replace all X_k 's by $X_k - \mathbb{E}[X_0]$ without modifying $X_k^{(c,n)}$ for any $n, k \geq 1$. Hence, without loss of generality, from now on in this proof, we assume that $\mathbb{E}[X_0]$ is zero, that is, the process X is centered.

By Lemma 11 and using that $\int F_n \, d\mu_{\mathbb{T}} = 1$, first note that for all $\theta \in \Theta$ we have, for all $n \geq 1$,

$$\left\| \tilde{\mathcal{Q}}_{I_n^{X^{(c,n)}}}(\theta) - \tilde{\mathcal{Q}}_{I_n^X}(\theta) \right\|_1 \leq A_n (A_n + B_n) \sup_{\substack{\theta \in \Theta \\ \lambda \in \mathbb{T}}} \|L(\theta, \lambda)\|_{\infty} \sup_{\substack{\theta \in \Theta \\ \lambda \in \mathbb{T}}} \|R(\theta, \lambda)\|_{\infty},$$

where we set

$$A_n := \left\| \frac{1}{n} \sum_{j=1}^n X_j \right\|_{\mathcal{H}_0} \quad \text{and} \quad B_n = 2 \int \|d_n^X\|_{\mathcal{H}_0} (F_n)^{1/2} \, d\mu_{\mathbb{T}}.$$

By the Cauchy-Schwartz inequality and then the Parseval identity, we have

$$B_n \leq 2 \left(\int \|d_n^X\|_{\mathcal{H}_0}^2 \, d\mu_{\mathbb{T}} \right)^{1/2} = 2 \left(\frac{1}{n} \sum_{k=1}^n \|X_k\|_{\mathcal{H}_0}^2 \right)^{1/2},$$

which, by the Birkhoff ergodic theorem and (A-2), converges \mathbb{P} -a.s.. On the other hand, using Lemma 13, we have that, \mathbb{P} -a.s., $\lim_{n \rightarrow \infty} A_n = 0$. Hence, we finally get that

$$\lim_{n \rightarrow \infty} \sup_{\theta \in \Theta} \left\| \tilde{\mathcal{Q}}_{I_n^{X^{(c,n)}}}(\theta) - \tilde{\mathcal{Q}}_{I_n^X}(\theta) \right\|_1 = 0 \quad \mathbb{P}\text{-a.s.}$$

Therefore, to prove (5.46), we can replace $I_n^{X^{(c,n)}}$ by I_n^X , that is, it only remains to prove that

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{Q}}_{I_n^X}^{(L,R)} = \tilde{\mathcal{Q}}_{\nu_X}^{(L,R)} \quad \text{uniformly in } \mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{G}_0)), \quad \mathbb{P}\text{-a.s.} \quad (5.47)$$

We immediately have from Proposition 5 that $\tilde{\mathcal{Q}}_{I_n^X}^{(L,R)}$ and $\tilde{\mathcal{Q}}_{\nu_X}^{(L,R)}$ belong to $\mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{G}_0))$. The rest of the proof is now in three steps. First, we show that, \mathbb{P} -a.s., every sequence valued in $\left\{ \tilde{\mathcal{Q}}_{I_n^X}^{(L,R)} : n \geq 1 \right\}$ admits a subsequence which converges uniformly in $\mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{G}_0))$. Second, we show that, for all $x, y \in \mathcal{G}_0$ and $\theta \in \Theta$, we have

$$\lim_{n \rightarrow \infty} x^H \tilde{\mathcal{Q}}_{I_n^X}^{(L,R)}(\theta) y = x^H \tilde{\mathcal{Q}}_{\nu_X}^{(L,R)}(\theta) y, \quad \mathbb{P}\text{-a.s.} \quad (5.48)$$

We then conclude in Step 3 from these two results.

Step 1. By (A-2), we have $\mathbb{E} \left[\|X_0\|_{\mathcal{H}_0}^2 \right] < \infty$, and we can apply Lemma 10 with $\mathcal{I}_0 = \mathcal{H}_0$ and get that, \mathbb{P} -a.s., there exists C such that, for all $n \in \mathbb{N}$, $\|I_n^X\|_{1,1} \leq C$, where here $\|\cdot\|_{1,1}$ denotes the norm in $L^{1,1}(\mathcal{H}_0)$. We conclude with Lemma 9 that, \mathbb{P} -a.s., $\left\{ \tilde{\mathcal{Q}}_{I_n^X}^{(L,R)} : n \geq 1 \right\}$ is equicontinuous in $\mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{G}_0))$. Using the last assumption of the theorem, we also have that, \mathbb{P} -a.s., for all $\theta \in \Theta$, there is a compact subset B of $\mathcal{S}_1(\mathcal{G}_0)$ such that $\left\{ \tilde{\mathcal{Q}}_{I_n^X}^{(L,R)}(\theta) : n \geq 1 \right\}$ is included in B . Thus, \mathbb{P} -a.s., the Ascoli-Arzelà theorem (see [31, Section 7.10]) applies and any sequence in $\left\{ \tilde{\mathcal{Q}}_{I_n^X}^{(L,R)} : n \geq 1 \right\}$ admits a uniformly convergent subsequence in $\mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{G}_0))$, which concludes the proof of Step 1.

Step 2. This step is similar to the scalar case and we follow the ideas of [16]. Let $x, y \in \mathcal{G}_0$, $\theta \in \Theta$, and denote $\mathbf{x} = x^H L(\theta, \cdot)$ and $\mathbf{y} = y^H R(\theta, \cdot)$, so that \mathbf{x} and \mathbf{y} are continuous functions from \mathbb{T} to $\mathcal{H}_0^H = \mathcal{L}_b(\mathcal{H}_0, \mathbb{C})$, and (5.48) can be written as

$$\lim_{n \rightarrow \infty} \mathcal{Q}_{I_n^X}(\mathbf{x}, \mathbf{y}) = \mathcal{Q}_{\nu_X}(\mathbf{x}, \mathbf{y}), \quad \mathbb{P}\text{-a.s.} \quad (5.49)$$

We can write, for any $N \in \mathbb{N}^*$,

$$\mathbf{x} = F_N \star \mathbf{x} + (\mathbf{x} - F_N \star \mathbf{x}),$$

where \star denotes the convolution of locally integrable 2π -periodic functions,

$$f \star g(\lambda) = \int f(\lambda') g(\lambda - \lambda') \mu_{\mathbb{T}}(d\lambda'), \quad \lambda \in \mathbb{T},$$

and F_N is the Fejér kernel defined by (5.44). Using standard properties of Fejér's kernel and the fact that $\lambda \mapsto \mathbf{x}(\lambda)$ is continuous on \mathbb{R} , we have, denoting $\mathbf{e}_\ell(\lambda) = e^{i\lambda\ell}$ so that $(\mathbf{e}_\ell)_{\ell \in \mathbb{Z}}$ is a Hilbert basis of $\mathcal{L}_0 := L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu_{\mathbb{T}})$,

$$F_N \star \mathbf{x} = \sum_{\ell=-N}^N \alpha_\ell(N) c_\ell(\mathbf{x}) \mathbf{e}_\ell, \quad (5.50)$$

$$\text{with } \alpha_\ell(N) = 1 - \frac{|\ell|}{N} \quad \text{and} \quad c_\ell(\mathbf{x}) = \int \mathbf{x}(\lambda) e^{-i\lambda\ell} \mu_{\mathbb{T}}(d\lambda),$$

$$\lim_{N \rightarrow \infty} \sup_{\lambda \in \mathbb{R}} \|\mathbf{x}(\lambda) - F_N \star \mathbf{x}(\lambda)\|_\infty = 0. \quad (5.51)$$

Eq (5.51) can be interpreted as saying that $F_N \star \mathbf{x}$ converges to \mathbf{x} in $\mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{H}_0, \mathbb{C}))$. The same holds with \mathbf{y} replacing \mathbf{x} and, applying Lemma 8 with $\mathcal{I}_0 = \mathcal{H}_0$ and $\mathcal{G}_0 = \mathcal{J}_0 = \mathbb{C}$ and $\mu = \mu_{\mathbb{T}}$, since by Step 1, $\{I_n^X : n \geq 1\}$ remains in a ball of $L^{1,1}$ \mathbb{P} -a.s., we have, \mathbb{P} -a.s.,

$$\lim_{N \rightarrow \infty} \sup_{g \in \{I_n^X : n \geq 1\}} |\mathcal{Q}_g(F_N \star \mathbf{x}, F_N \star \mathbf{y}) - \mathcal{Q}_g(\mathbf{x}, \mathbf{y})|. \quad (5.52)$$

Similarly by the continuity of $\mathcal{Q}_{\nu_X} = \mathcal{Q}_{g_X, \mu}$ (with g_X the density of ν_X with respect to $\mu = \|\nu_X\|_1$) established in Lemma 8, we have

$$\lim_{N \rightarrow \infty} \mathcal{Q}_{\nu_X}(F_N \star \mathbf{x}, F_N \star \mathbf{y}) = \mathcal{Q}_{\nu_X}(\mathbf{x}, \mathbf{y}). \quad (5.53)$$

Next, using (5.50), we have

$$\begin{aligned} \mathcal{Q}_{I_n^X}(F_N \star \mathbf{x}, F_N \star \mathbf{y}) &= \sum_{\ell, \ell'=-N}^N \alpha_\ell(N) \alpha_{\ell'}(N) c_\ell(\mathbf{x}) \mathcal{Q}_{I_n^X}(\mathbf{e}_\ell, \mathbf{e}_{\ell'}) c_{\ell'}(\mathbf{y})^H \\ &= \sum_{\ell, \ell'=-N}^N \alpha_\ell(N) \alpha_{\ell'}(N) c_\ell(\mathbf{x}) \tilde{\Gamma}_n(\ell - \ell') c_{\ell'}(\mathbf{y})^H, \end{aligned} \quad (5.54)$$

where $\tilde{\Gamma}_n$ denotes the empirical covariance defined as in (4.2), but with $X^{(c,n)}$ replaced by X , that is,

$$\tilde{\Gamma}_n(s-t) = \int I_n^X(\lambda) e^{i(s-t)\lambda} \mu_{\mathbb{T}}(d\lambda) = \frac{1}{n} \sum_{\substack{1 \leq k, k' \leq n \\ k-k'=(s-t)}} X_k \otimes X_{k'}.$$

In particular, we have for any $\ell, \ell' \in \{-N, \dots, N\}$,

$$c_\ell(\mathbf{x}) \tilde{\Gamma}_n(\ell - \ell') c_{\ell'}(\mathbf{y})^H = \frac{1}{n} \sum_{1 \vee (1+\ell-\ell') \leq k \leq n \wedge (n+\ell-\ell')} c_\ell(\mathbf{x}) X_k X_{k-(\ell-\ell')}^H c_{\ell'}(\mathbf{y})^H$$

By (A-1), with the Birkhoff ergodic theorem, we get, for any $\ell, \ell' \in \{-N, \dots, N\}$, \mathbb{P} -a.s.,

$$\begin{aligned} \lim_{n \rightarrow \infty} c_\ell(\mathbf{x}) \tilde{\Gamma}_n(\ell - \ell') c_{\ell'}(\mathbf{y})^H &= \mathbb{E} [c_\ell(\mathbf{x}) X_{\ell-\ell'} X_0^H c_{\ell'}(\mathbf{y})^H] \\ &= c_\ell(\mathbf{x}) \text{Cov}(X_{\ell-\ell'}, X_0) c_{\ell'}(\mathbf{y})^H \\ &= c_\ell(\mathbf{x}) \mathcal{Q}_{\nu_X}(\mathbf{e}_\ell, \mathbf{e}_{\ell'}) c_{\ell'}(\mathbf{y})^H. \end{aligned}$$

where we used that X is centered and that ν_X is the spectral operator measure of X . From (5.54), using (5.50) again, we get that, \mathbb{P} -a.s., for any $N \geq 1$,

$$\lim_{n \rightarrow \infty} \mathcal{Q}_{I_n^X}(F_N \star \mathbf{x}, F_N \star \mathbf{y}) = \mathcal{Q}_{\nu_X}(F_N \star \mathbf{x}, F_N \star \mathbf{y}).$$

This, with (5.52) and (5.53), concludes Step 2.

Step 3. From Step 1, there exists $\Omega' \in \mathcal{F}$ with probability 1 such that on Ω' , any sequence valued in $\{\tilde{\mathcal{Q}}_{I_n^x}^{(L,R)} : n \geq 1\}$ admits a subsequence uniformly converging in $\mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{H}_0))$. To obtain (5.47), we will exhibit $\Omega'' \subset \Omega'$ with probability one such that, on Ω'' , $\tilde{\mathcal{Q}}_{\nu_x}^{(L,R)}$ is the only possible accumulation point of the sequence $\left(\tilde{\mathcal{Q}}_{I_n^x}^{(L,R)}\right)_{n \geq 1}$. Let E_0 be a countable linearly dense subset of \mathcal{G}_0 and let $(\theta_j)_{j \in \mathbb{N}}$ be a dense sequence in Θ , which exists since Θ is compact. Then, from Step 2, we have, \mathbb{P} -a.s.,

$$\forall j \in \mathbb{N}, \forall x, y \in E_0, \lim_{n \rightarrow \mathbb{N}} x^H \tilde{\mathcal{Q}}_{I_n^x}^{(L,R)} y = x^H \tilde{\mathcal{Q}}_{\nu_x}^{(L,R)} y .$$

We can thus take $\Omega'' \subset \Omega'$ with probability one, on which the previous display holds. Let $\omega \in \Omega''$ and take an accumulation point $\tilde{\mathcal{Q}}_\infty$ of $\left(\tilde{\mathcal{Q}}_{I_n^{x(\omega)}}^{(L,R)}\right)_{n \geq 1}$ in $\mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{G}_0))$. Then, for all $j \in \mathbb{N}$, using the previous display and the fact that $\tilde{\mathcal{Q}}_\infty$ must also be an accumulation point for the weak operator topology, we get that, for all $x, y \in E_0$, $x^H \tilde{\mathcal{Q}}_\infty(\theta_j)y = x^H \tilde{\mathcal{Q}}_{\nu_x}^{(L,R)}(\theta_j)y$, which implies $\tilde{\mathcal{Q}}_\infty(\theta_j) = \tilde{\mathcal{Q}}_{\nu_x}^{(L,R)}(\theta_j)$. Since $\tilde{\mathcal{Q}}_\infty$ and $\tilde{\mathcal{Q}}_{\nu_x}^{(L,R)}$ are continuous on Θ , and $(\theta_j)_{j \in \mathbb{N}}$ is dense in Θ , we get that $\tilde{\mathcal{Q}}_\infty$ and $\tilde{\mathcal{Q}}_{\nu_x}^{(L,R)}$ coincide, which concludes the proof. \square

5.2.2. Proof of Theorem 4

We can now prove Theorem 4 as a direct application of Theorem 8.

Proof of Theorem 4. Theorem 4 directly follows from Theorem 8, if we can prove that, \mathbb{P} -a.s., there exists B , a compact subset of $\mathcal{S}_1(\mathcal{G}_0)$, such that $\tilde{\mathcal{Q}}_{I_n^x}^{(L,R)}(\theta) \in B$ for all $n \geq 1$ and $\theta \in \Theta$. Because \mathcal{G}_0 is finite-dimensional, so is $\mathcal{S}_1(\mathcal{G}_0)$, and we only need to show that, \mathbb{P} -a.s., there exists $C > 0$ such that $\left\|\tilde{\mathcal{Q}}_{I_n^x}^{(L,R)}(\theta)\right\|_1 \leq C$ for all $n \geq 1$ and $\theta \in \Theta$. By Lemma 8, this follows from the fact that, \mathbb{P} -a.s., there exists $r > 0$ such that $\left\|I_n^x\right\|_1 \leq r$ for all $n \geq 1$, which has already been used in the proof of Theorem 8 and is a consequence of Lemma 10 with (A-2) in the case $\mathcal{H}_0 = \mathcal{I}_0$. This concludes the proof. \square

5.2.3. Proof of Theorem 5

The proof of Theorem 5 essentially follows the same path as that of Theorem 4. However, in the infinite-dimensional case, we will need an additional result (Proposition 7) to prove the assumption involving the set B in Theorem 8. The result relies on the space \mathcal{H}_0^s introduced in Section 4.1. In this section, we will make extensive use of partial isometries as (see [8, Definition 3.8]). We recall that a partial isometry U on the Hilbert space \mathcal{H}_0 onto another Hilbert space \mathcal{G}_0 is a bounded operator which is an isometry on $(\ker(U))^\perp$. The subspaces $(\ker(U))^\perp$ and $\text{Im}(U)$ are respectively called the *initial space* and *final space* of U . We recall that, if U is a partial isometry, then $U^H U$ and $U U^H$ are the orthogonal projections onto the initial and the final space of U respectively.

Let us start with the following lemma, whose proof is straightforward, but which contains some important definitions

Lemma 14. Let \mathcal{H}_0 be a separable Hilbert space and let $(\phi_k)_{k \in \mathbb{N}}$ be an orthonormal sequence in \mathcal{H}_0 . Let $s = (s_k)_{k \in \mathbb{N}} \in [1, \infty)^\mathbb{N}$ and define $(\mathcal{H}_0^s, \langle \cdot, \cdot \rangle_{\mathcal{H}_0^s})$ by (4.8) and (4.9). Denote by \mathbb{J}_s the continuous $\mathcal{H}_0^s \hookrightarrow \mathcal{H}_0$ -inclusion map defined on \mathcal{H}_0^s onto \mathcal{H}_0 by $x \mapsto x$. Further denote by U_s the partial isometry with initial space $\overline{\text{Span}}^{\mathcal{H}_0}(\phi_k, k \in \mathbb{N})$ and final space \mathcal{H}_0^s such that, for all $k \in \mathbb{N}$, $U_s \phi_k = s_k^{-1} \phi_k$. Finally, let us set $\mathbb{J}_s = \mathbb{J}_s U_s \in \mathcal{L}_b(\mathcal{H}_0)$. Then, for all $x \in \mathcal{H}_0$, we have

$$\mathbb{J}_s = \sum_{k \in \mathbb{N}} s_k^{-1} \phi_k \otimes \phi_k . \quad (5.55)$$

Moreover, suppose that s is non-decreasing and going to ∞ . Then, we have $\mathbb{J}_s \in \mathcal{S}_\infty(\mathcal{H}_0)$.

We now have the following result.

Proposition 7. *Let \mathcal{H}_0 be a separable Hilbert space and let $(\phi_k)_{k \in \mathbb{N}}$ be an orthonormal sequence in \mathcal{H}_0 . Define \mathbb{J}_s as in Lemma 14 for $s = (s_k)_{k \in \mathbb{N}} \in [1, \infty)^\mathbb{N}$ non-decreasing and going to ∞ . Define $\mathbb{B}_1 = \left\{ P \in \mathcal{S}_1(\mathcal{H}_0) : \|P\|_1 \leq 1 \text{ and } (\mathbb{J}_s P)^H = \mathbb{J}_s P \right\}$. Then the set $B_1 = \{\mathbb{J}_s P : P \in \mathbb{B}_1\}$ is compact in $\mathcal{S}_1(\mathcal{H}_0)$.*

Proof. Let $(P_n)_{n \in \mathbb{N}}$ be a sequence valued in B_1 and let us prove that it admits a subsequence which converges in B_1 . By definition, we can write, for all $n \in \mathbb{N}$, $P_n = \mathbb{J}_s \tilde{P}_n$ with $\tilde{P}_n \in \mathbb{B}_1$. We use that $\mathcal{S}_1(\mathcal{H}_0)$ is isometric to the dual of the space $\mathcal{S}_\infty(\mathcal{H}_0)$ (see [8, Theorem 19.1]). Then, by the Banach-Alaoglu Theorem (see Theorem 3.1 in [7, Chapter V]), we get that the unit ball of $\mathcal{S}_1(\mathcal{H}_0)$ is compact for the weak-star topology, that is the topology generated by the family of semi-norms $\{P \mapsto |\text{Tr}(CP)| : C \in \mathcal{S}_\infty(\mathcal{H}_0)\}$. This implies that $(\tilde{P}_n)_{n \in \mathbb{N}}$ admits a subsequence $(\tilde{P}_{a_n})_{n \in \mathbb{N}}$ converging to an element \tilde{Q} in the unit ball of $\mathcal{S}_1(\mathcal{H}_0)$ in the sense of the weak-star topology, that is, for all $C \in \mathcal{S}_\infty(\mathcal{H}_0)$, we have

$$\lim_{n \rightarrow \infty} \text{Tr} \left(C \tilde{P}_{a_n} \right) = \text{Tr} \left(C \tilde{Q} \right).$$

Observe that for all $x, y \in \mathcal{H}_0$, the operator $C = x \otimes y \mathbb{J}_s$ is a rank-one (hence compact) linear operator on \mathcal{H}_0 onto \mathcal{H}_0 . The last display thus gives that \tilde{P}_{a_n} converges to \tilde{Q} in weak operator topology (that is, for all $x, y \in \mathcal{H}_0$, $\langle \tilde{P}_{a_n} x, y \rangle_{\mathcal{H}_0}$ converges to $\langle \tilde{Q} x, y \rangle_{\mathcal{H}_0}$). Since $\mathbb{J}_s \tilde{P}_{a_n}$ is hermitian for all n , we get that $\mathbb{J}_s \tilde{Q}$ is hermitian as well and we finally get that \tilde{Q} must be in \mathbb{B}_1 . (In fact we have shown that \mathbb{B}_1 is compact for the weak-star topology).

Let us set $Q = \mathbb{J}_s \tilde{Q} \in B_1$ and for all $n \in \mathbb{N}$, $\Delta_n = P_{a_n} - Q = \mathbb{J}_s \tilde{\Delta}_n$ with $\tilde{\Delta}_n = \tilde{P}_{a_n} - \tilde{Q}$, and let us summarize our findings so far. We already know that P_{a_n} and Q are in B_1 (hence are hermitian and so is Δ_n), that $\|\tilde{\Delta}_n\|_1 \leq 2$ and that $(\tilde{\Delta}_n)_{n \rightarrow \infty}$ converges to zero in $\mathcal{S}_1(\mathcal{H}_0)$ for the weak-star topology, which also implies the convergence in weak operator topology. To conclude, we now proceed in two steps. First, we show that $(\Delta_n)_{n \in \mathbb{N}}$ converges to 0 for the strong operator topology. Second, we use the first step to show that $\lim_{n \rightarrow +\infty} \|\Delta_n\|_1 = 0$.

Step 1. Let $x \in \mathcal{H}_0$, then, for all $n \in \mathbb{N}$, using (5.55), we have

$$\|\Delta_n x\|_{\mathcal{H}_0}^2 = \sum_{k \in \mathbb{N}} |\langle \Delta_n x, \phi_k \rangle_{\mathcal{H}_0}|^2 = \sum_{k \in \mathbb{N}} s_k^{-2} \left| \langle \tilde{\Delta}_n x, \phi_k \rangle_{\mathcal{H}_0} \right|^2.$$

Since $(\tilde{\Delta}_n)_{n \in \mathbb{N}}$ converges to 0 for the weak operator topology, we have, for all $m \geq 1$,

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{m-1} s_k^{-2} \left| \langle \tilde{\Delta}_n x, \phi_k \rangle_{\mathcal{H}_0} \right|^2 = 0.$$

On the other hand, for all $m, n \geq 1$, using the fact that $\|\tilde{\Delta}_n\|_\infty \leq \|\tilde{\Delta}_n\|_1 \leq 2$ and that s is non-decreasing, we get that

$$\begin{aligned} \sum_{k=m}^{\infty} s_k^{-2} \left| \langle \tilde{\Delta}_n x, \phi_k \rangle_{\mathcal{H}_0} \right|^2 &\leq s_m^{-2} \sum_{k=0}^{\infty} \left| \langle \tilde{\Delta}_n x, \phi_k \rangle_{\mathcal{H}_0} \right|^2 \\ &= s_m^{-2} \|\tilde{\Delta}_n x\|_{\mathcal{H}_0}^2 \leq 4 s_m^{-2} \|x\|_{\mathcal{H}_0}^2, \end{aligned}$$

hence converges to 0 independently of n as $m \rightarrow \infty$, by assumption on s . With the two previous displays, we conclude that $(\Delta_n x)_{n \in \mathbb{N}}$ converges to 0 in \mathcal{H}_0 . Hence, $(\Delta_n)_{n \in \mathbb{N}}$ converges to 0 for the strong operator topology.

Step 2. Let $n \in \mathbb{N}$. Since $\tilde{\Delta}_n \in \mathcal{S}_1(\mathcal{H}_0)$ and $\mathbb{J}_s \in \mathcal{L}_b(\mathcal{H}_0)$, we have $\Delta_n \in \mathcal{S}_1(\mathcal{H}_0)$. Consider the polar decomposition of Δ_n , that is $\Delta_n = V_n |\Delta_n|$ where V_n is a partial isometry with initial space

905 $\ker(\Delta_n)^\perp = \overline{\text{Im}}(|\Delta_n|)$ and final space $\overline{\text{Im}}(\Delta_n)$ (see §3.9 in [8]). Since Δ_n is autoadjoint we have $\ker(\Delta_n)^\perp = \overline{\text{Im}}(\Delta_n)$ and we get that $\overline{\text{Im}}(|\Delta_n|) = \overline{\text{Im}}(\Delta_n) \subset \overline{\text{Im}}(\mathbb{J}_s) = \overline{\text{Span}}(\phi_k, k \in \mathbb{N})$. Hence

$$\|\Delta_n\| = \text{Tr}(|\Delta_n|) = \sum_{k \in \mathbb{N}} \langle |\Delta_n| \phi_k, \phi_k \rangle_{\mathcal{H}_0} = \sum_{k \in \mathbb{N}} \langle \Delta_n \phi_k, V_n \phi_k \rangle_{\mathcal{H}_0}, \quad (5.56)$$

where the last equality comes from the fact that $|\Delta_n| = V_n^H \Delta_n$.

Now, note that for all $m \geq 1$,

$$\sum_{k=0}^{m-1} |\langle \Delta_n \phi_k, V_n \phi_k \rangle_{\mathcal{H}_0}| \leq \sum_{k=0}^{m-1} \|\Delta_n \phi_k\|_{\mathcal{H}_0},$$

which converges to zero by Step 1. Thus,

$$\text{for all } m \geq 1, \quad \lim_{n \rightarrow \infty} \sum_{k=0}^{m-1} \langle \Delta_n \phi_k, V_n \phi_k \rangle_{\mathcal{H}_0} = 0. \quad (5.57)$$

On the other hand, using the fact that Δ_n is hermitian and (5.55), we have, for all $n, k \in \mathbb{N}$,

$$\begin{aligned} \langle \Delta_n \phi_k, V_n \phi_k \rangle_{\mathcal{H}_0} &= \langle \phi_k, \Delta_n V_n \phi_k \rangle_{\mathcal{H}_0} = \left\langle \phi_k, \mathbb{J}_s \tilde{\Delta}_n V_n \phi_k \right\rangle_{\mathcal{H}_0} = s_k^{-1} \left\langle \phi_k, \tilde{\Delta}_n V_n \phi_k \right\rangle_{\mathcal{H}_0} \\ &= s_k^{-1} \left\langle \tilde{\Delta}_n \phi_k, V_n \phi_k \right\rangle_{\mathcal{H}_0}. \end{aligned}$$

It follows that, for all $m \geq 1$,

$$\sum_{k \geq m} |\langle \Delta_n \phi_k, V_n \phi_k \rangle_{\mathcal{H}_0}| \leq s_m^{-1} \sum_{k \in \mathbb{N}} \left| \left\langle \tilde{\Delta}_n \phi_k, V_n \phi_k \right\rangle_{\mathcal{H}_0} \right| \leq s_m^{-1} \left\| \tilde{\Delta}_n \right\|_1,$$

where we used [8, Corollary 18.12]. Since $\left\| \tilde{\Delta}_n \right\|_1 \leq 1$ and s_m^{-1} converges to 0, we obtain that

$$\lim_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} \sum_{k \geq m} |\langle \Delta_n \phi_k, V_n \phi_k \rangle_{\mathcal{H}_0}| = 0.$$

910 This with (5.56) and (5.57) concludes the second and final step. \square

We can now prove Theorem 5.

Proof of Theorem 5. By the polarization formula we can write $\tilde{\mathcal{Q}}_{I_n^{X(c,n)}}^{(L,R)}$ as a linear combination of $\tilde{\mathcal{Q}}_{I_n^{X(c,n)}}^{(W,W)}$ with W in $\{L+R, L-R, L+iR, L-iR\}$. The same formula holds for expressing $\tilde{\mathcal{Q}}_{\nu^X}^{(L,R)}$ using $\tilde{\mathcal{Q}}_{\nu^X}^{(W,W)}$ with the same W 's. Hence, to obtain the claimed result, it suffices to show that, for all $W \in \{L+R, L-R, L+iR, L-iR\}$, we have

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{Q}}_{I_n^{X(c,n)}}^{(W,W)} = \tilde{\mathcal{Q}}_{\nu^X}^{(W,W)} \quad \text{uniformly in } \mathcal{C}(\Theta, \mathcal{S}_1(\mathcal{H}_0)), \quad \mathbb{P}\text{-a.s.} \quad (5.58)$$

So, take $W \in \{L+R, L-R, L+iR, L-iR\}$, and let us show (5.58).

By assumption on L and R , we have $W \in \mathcal{C}(\Theta \times \mathbb{T}, \mathcal{L}_b(\mathcal{H}_0))$ and, using Condition (iii), $W_s \in \mathcal{C}(\Theta \times \mathbb{T}, \mathcal{L}_b(\mathcal{H}_0^s))$ where $W_s(\theta, \lambda) = W(\theta, \lambda)|_{\mathcal{H}_0^s}$ for all $(\theta, \lambda) \in \Theta \times \mathbb{T}$. By Condition (ii), we can apply Lemma 10 with $\mathcal{I}_0 = \mathcal{H}_0^s$, and obtain that, \mathbb{P} -a.s.,

$$\text{there exists } r_1 > 0 \text{ such that } \left\{ I_n^{X(c,n)} : n \geq 1 \right\} \subset B_{1,1}(r_1, \mathcal{H}_0, \mathcal{H}_0^s). \quad (5.59)$$

Now let us define J_s , U_s and \mathbb{J}_s as in Lemma 14. Applying these definitions carefully and using the fact that UU^H is the orthogonal projection onto $\text{Im}(U_s) = \mathcal{H}_0^s$, it straightforwardly yields that, for all $(\theta, \lambda) \in \Theta \times \mathbb{T}$, and $x \in \mathcal{H}_0^s$,

$$\mathbb{J}_s U_s^H W_s(\theta, \lambda) x = J_s W_s(\theta, \lambda) x = W(\theta, \lambda) x.$$

920 Thus, for all $n \in \mathbb{N}$, \mathbb{P} -a.s., $I_n^X \in L^{1,1}(\mathcal{H}_0, \mathcal{H}_0^s)$, and for all $\theta \in \Theta$,

$$\tilde{\mathcal{Q}}_{I_n^X}^{(W,W)}(\theta) = \mathbb{J}_s \tilde{\mathcal{Q}}_{I_n^X}^{(U_s^H W_s, W)}(\theta). \quad (5.60)$$

Observe that $W_s \in \mathcal{C}(\Theta \times \mathbb{T}, \mathcal{L}_b(\mathcal{H}_0^s))$ immediately implies that $U_s^H W_s(\theta, \cdot) \in \mathbb{F}_b(\mathcal{H}_0^s, \mathcal{H}_0)$. Thus, for all θ , we can apply Lemma 8 with $\mathcal{I}_0 = \mathcal{H}_0^s$ and $\mathcal{G}_0 = \mathcal{J}_0 = \mathcal{H}_0$, $L = U_s^H W_s(\theta, \cdot)$, $R = W$, $g = I_n^X$ and $\mu = \mu_{\mathbb{T}}$, which, with (5.59), gives us that, \mathbb{P} -a.s.: for all $\theta \in \Theta$, there exists $r > 0$ such that $\left\| \tilde{\mathcal{Q}}_{I_n^X}^{(U_s^H W_s, W)}(\theta) \right\|_1 \leq r$, where, here, $\|\cdot\|_1$ denotes the trace-class norm in $\mathcal{S}_1(\mathcal{H}_0)$. With (5.60) and the fact that $\tilde{\mathcal{Q}}_{I_n^X}^{(W,W)}(\theta)$ is an hermitian operator for all θ , we get that, \mathbb{P} -a.s.: for all $\theta \in \Theta$, there exists $r > 0$ such that $\tilde{\mathcal{Q}}_{I_n^X}^{(W,W)}(\theta)$ belongs to the set $B = rB_1$, with B_1 defined as in Proposition 7. Since B is compact by Proposition 7, we can apply Theorem 8 with $L = R := W$ and we obtain (5.58), which concludes the proof. \square

5.3. Proof of Theorem 1

930 Hereafter we prove (1.4) by relying on Theorem 5. Of course the uniform convergence in θ does not matter in this case and it could be proven by relying directly on Proposition 7, by mimicking the argument used in the proof of Theorem 5 to show that $\left(\hat{\Gamma}_n(h) \right)_{n \in \mathbb{N}}$ remains in a compact subset of $\mathcal{S}_1(\mathcal{H}_0)$, \mathbb{P} -a.s. (since the convergence of $\hat{\Gamma}_n(h)$ to $\Gamma(h) = \text{Cov}(X_h, X_0)$ in weak operator topology is obvious under the assumptions of Theorem 1).

935 **Proof of Theorem 1.** Let h be a given lag in \mathbb{Z} . The claimed result follows from Theorem 5 with $L(\theta, \lambda) = e^{i h \lambda} \text{Id}_{\mathcal{H}_0}$ and $R(\theta, \lambda) = \text{Id}_{\mathcal{H}_0}$, with θ being an arbitrary point and Θ the corresponding singleton. Indeed, with these definitions, the convergence (1.4) can be rewritten as (5.46). Thus, we only need to show that $(X_t)_{t \in \mathbb{Z}}$ and the above defined L and R satisfy the assumptions of Theorem 5. Obviously (A-1) and (A-2) are satisfied. As for Conditions (i) and (ii), they follow from (A-1) and (A-2) by Lemma 12, for well chosen s and $(\phi_k)_{k \in \mathbb{N}}$. Now, defining L_s and R_s as in Theorem 5 with L and R as above, we get $L_s(\theta, \lambda) = e^{i h \lambda} \text{Id}_{\mathcal{H}_0^s}$ and $R_s(\theta, \lambda) = \text{Id}_{\mathcal{H}_0^s}$, and L, R, L_s and R_s obviously satisfy the assumptions of the theorem (including Condition (iii)). Hence Theorem 5 applies and the proof is finished. \square

5.4. Proofs of Section 4.2

945 5.4.1. Preliminary results

The following simple lemma gathers basic properties related to the function $\Phi_{\theta, \phi, D}^\dagger$ and is used repeatedly in the following.

950 **Lemma 15.** *Let \mathcal{H}_0 be a separable Hilbert space and p, q be two non-negative integers. Let $D \in \mathcal{L}_b(\mathcal{H}_0)$, $\theta \in \mathcal{P}_q^\dagger(\mathcal{H}_0)$ and $\phi \in \mathcal{P}_p(\mathcal{H}_0)$. Define $\Phi_{\theta, \phi, D}^\dagger$ by (4.11). Then $\theta^{-1} : z \mapsto [\theta(z)]^{-1}$ and $\Phi_{\theta, \phi, D}^\dagger$ are holomorphic functions on \mathbb{D} onto $\mathcal{L}_b(\mathcal{H}_0)$. Moreover θ^{-1} is continuous on $\overline{\mathbb{D}}$ and $(z, \theta, \phi, D) \mapsto \Phi_{\theta, \phi, D}^\dagger(z)$ is continuous from $(\overline{\mathbb{D}} \setminus \{1\}) \times \mathcal{P}_q^\dagger(\mathcal{H}_0) \times \mathcal{P}_p(\mathcal{H}_0) \times \mathcal{L}_b(\mathcal{H}_0)$ to $\mathcal{L}_b(\mathcal{H}_0)$.*

Proof. By holomorphic in Lemma 15 we mean the same as in [15, Definition 1.1.1]. Since ϕ and θ are polynomials they are holomorphic in \mathbb{C} . For $\theta \in \mathcal{P}_q^\dagger(\mathcal{H}_0)$, we further have that $z \mapsto [\theta(z)]^{-1}$ is holomorphic on \mathbb{D} and continuous on $\overline{\mathbb{D}}$. Since the principal logarithmic function is holomorphic on $\mathbb{C} \setminus [1, \infty)$, and the exponential function $P \mapsto \exp(P)$ is a power series with infinite radius of convergence, $z \mapsto (1 - z)^D$ is holomorphic as a $\mathcal{L}_b(\mathcal{H}_0)$ -valued function (see [15, Chapter 1]). Moreover, $(z, D) \mapsto (1 - z)^D$ is continuous on $(\mathbb{C} \setminus [1, \infty)) \times \mathcal{L}_b(\mathcal{H}_0)$. The remaining claims follow from these facts and the definition of $\Phi_{\theta, \phi, D}^\dagger$ in (4.11). \square

The following lemma is useful in the proof of Theorem 6.

960 **Lemma 16.** Let \mathcal{H}_0 be a separable Hilbert space and p, q be two non-negative integers. Let $D \in \mathcal{L}_b(\mathcal{H}_0)$, $\theta \in \mathcal{P}_q^\dagger(\mathcal{H}_0)$ and $\phi \in \mathcal{P}_p(\mathcal{H}_0)$ and define $\Phi_{\theta, \phi, D}^\dagger$ by (4.11). Let moreover $X \in \mathcal{S}_{\mathbf{I}D}(\Omega, \mathcal{F}, \mathbb{P})$ with spectral operator density g_X with respect to μ . We assume that one of the two following condition holds.

(i) There exists $\delta \geq 0$ such that $\Re(z) < \delta$ for all $z \in \sigma(D)$ and $\int_{\mathbb{T} \setminus \{0\}} |\lambda|^{-2\delta} g_X(\lambda) \mu(d\lambda) < \infty$.

965 (ii) We have $D \in \mathcal{N}(\mathcal{H}_0)$.

Then, we have

$$\int_{(-\pi/3, \pi/3) \setminus \{0\}} \sup_{0 \leq \rho \leq 1} \|\Psi(\rho, \lambda) g_X(\lambda) \Psi^H(\rho, \lambda)\|_1 \mu(d\lambda) < \infty, \quad (5.61)$$

in the two following cases:

$$\text{if } \Psi(\rho, \lambda) := (1 - \rho e^{-i\lambda})^{-D} - (1 - e^{-i\lambda})^{-D}. \quad (5.62)$$

$$\text{or if } \Psi(\rho, \lambda) := \left(\Phi_{\theta, \phi, D}^\dagger(\rho e^{-i\lambda}) - \Phi_{\theta, \phi, D}^\dagger(e^{-i\lambda}) \right) (1 - e^{-i\lambda})^{-D}. \quad (5.63)$$

Proof. The proof is three steps. First, we first find sufficient conditions for (5.61) that hold respectively under (5.62) and (5.63). Then we show that these conditions are satisfied under Condition (i). Finally we show that they are also satisfied under Condition (ii).

970 **Step 1.** Consider the two conditions:

$$\int_{(-\pi/3, \pi/3) \setminus \{0\}} \sup_{0 \leq \rho \leq 1} \left\| (1 - \rho e^{-i\lambda})^{-D} (g_X(\lambda))^{1/2} \right\|_2^2 \mu(d\lambda) < \infty, \quad (5.64)$$

$$\int_{(-\pi/3, \pi/3) \setminus \{0\}} \sup_{0 \leq \rho \leq 1} \left\| (1 - \rho e^{-i\lambda})^D (1 - e^{-i\lambda})^{-D} (g_X(\lambda))^{1/2} \right\|_2^2 \mu(d\lambda) < \infty. \quad (5.65)$$

In this step we show that under (5.62), the bound (5.61) is implied by (5.64) and under (5.63), the bound (5.61) is implied by (5.65). Indeed, for $\Psi(\rho, \lambda)$ as in (5.62), we have, for all $0 \leq \rho \leq 1$ and $\lambda \neq 0$,

$$\|\Psi(\rho, \lambda) g_X(\lambda) \Psi^H(\rho, \lambda)\|_1 = \left\| \Psi(\rho, \lambda) (g_X(\lambda))^{1/2} \right\|_2^2 \leq 4 \sup_{0 \leq \rho \leq 1} \left\| (1 - \rho e^{-i\lambda})^{-D} (g_X(\lambda))^{1/2} \right\|_2^2.$$

Hence we get that (5.61) follows from (5.64). Let us now take $\Psi(\rho, \lambda)$ as in (5.63). By (4.11), for all $\lambda \in \mathbb{T} \setminus \{0\}$ and $0 \leq \rho \leq 1$, the difference $\Phi_{\theta, \phi, D}^\dagger(\rho e^{-i\lambda}) - \Phi_{\theta, \phi, D}^\dagger(e^{-i\lambda})$ can be written as

$$[\theta(e^{-i\lambda})]^{-1} \phi(e^{-i\lambda}) (1 - e^{-i\lambda})^D - [\theta(\rho e^{-i\lambda})]^{-1} \phi(\rho e^{-i\lambda}) (1 - \rho e^{-i\lambda})^D.$$

By Lemma 15, θ^{-1} is continuous on $\overline{\mathbb{D}}$, hence θ^{-1} and the polynomial ϕ have bounded operator norms over $\overline{\mathbb{D}}$. Thus, for some positive constants C_1, C_2 , we have, for all $0 \leq \rho \leq 1$ and $\lambda \neq 0$,

$$\begin{aligned} \left\| \Psi(\rho, \lambda) (g_X(\lambda))^{1/2} \right\|_2 &\leq C_1 \left\| (1 - e^{-i\lambda})^D (1 - e^{-i\lambda})^{-D} (g_X(\lambda))^{1/2} \right\|_2 \\ &\quad + C_2 \left\| (1 - \rho e^{-i\lambda})^D (1 - e^{-i\lambda})^{-D} (g_X(\lambda))^{1/2} \right\|_2 \\ &\leq (C_1 + C_2) \sup_{\rho \in [0, 1]} \left\| (1 - \rho e^{-i\lambda})^D (1 - e^{-i\lambda})^{-D} (g_X(\lambda))^{1/2} \right\|_2. \end{aligned}$$

We get that (5.65) implies (5.61), which concludes this step.

Step 2. We now show that Condition (i) implies (5.64) and (5.65). We apply Corollary 3 with $K = \{-D\}$, which gives $\underline{\varrho} > -\delta$ by the first part of Condition (i). Then (5.17) with $\varrho = -\delta$ gives us that there exists $C > 0$ such that for all $\lambda \in (-\pi/3, \pi/3) \setminus \{0\}$,

$$\sup_{0 \leq \rho \leq 1} \left\| (1 - \rho e^{-i\lambda})^{-D} (g_X(\lambda))^{1/2} \right\|_2^2 \leq \sup_{0 \leq \rho \leq 1} \left\| (1 - \rho e^{-i\lambda})^{-D} \right\|_\infty^2 \|g_X(\lambda)\|_1 \leq C |\lambda|^{-2\delta} \|g_X(\lambda)\|_1.$$

975 We thus get (5.64) by using the second part of Condition (i). The proof of (5.65) is similar, but based on (5.18).

Step 3. We now assume that Condition (ii) holds. In this case, we can apply Theorem 3. This gives that $X \in \mathcal{S}_{\mathbf{I}_D}(\Omega, \mathcal{F}, \mathbb{P})$ implies

$$\int_{\mathbb{V}^2 \times ((-\pi, \pi) \setminus \{0\})} |\lambda|^{-2\Re_+(\mathbf{d}(v))} |\mathcal{K}(v, v'; \lambda)|^2 \xi(\mathbf{d}v) \xi(\mathbf{d}v') \mu(\mathbf{d}\lambda) < \infty, \quad (5.66)$$

980 where \mathcal{K} denote the \mathbb{T} -joint kernel function of h such that $h(\lambda)[h(\lambda)]^{\mathbf{H}} = U g_X(\lambda) U^{\mathbf{H}}$ for μ -a.e. $\lambda \in \mathbb{T}$. Using the same unitarily equivalent representation (3.6), Conditions (5.64) and (5.65) are respectively implied by

$$\begin{aligned} & \int_{\mathbb{V}^2 \times ((-\pi/3, \pi/3) \setminus \{0\})} \sup_{0 \leq \rho \leq 1} \left| (1 - \rho e^{-i\lambda})^{-\mathbf{d}(v)} \mathcal{K}(v, v'; \lambda) \right|^2 \xi(\mathbf{d}v) \xi(\mathbf{d}v') \mu(\mathbf{d}\lambda) < \infty, \\ & \int_{\mathbb{V}^2 \times ((-\pi/3, \pi/3) \setminus \{0\})} \sup_{0 \leq \rho \leq 1} \left| \left(\frac{1 - \rho e^{-i\lambda}}{1 - e^{-i\lambda}} \right)^{\mathbf{d}(v)} \mathcal{K}(v, v'; \lambda) \right|^2 \xi(\mathbf{d}v) \xi(\mathbf{d}v') \mu(\mathbf{d}\lambda) < \infty. \end{aligned}$$

Now, for each $v \in \mathbb{V}$, we apply Lemma 2 with $\varrho = -\Re(\mathbf{d}(v))$ and $\varsigma = |\mathbf{d}(v)|$. Then $\varrho_- = \Re_+(\mathbf{d}(v))$ and the two above conditions follow from (5.8), (5.9), and (5.66). \square

The following remark will be useful.

985 **Remark 3.** We note that if X is an ARMA process then $\|\nu_X\|_1$ admits a bounded density with respect to the Lebesgue measure and we thus have $\int |\lambda|^{-2\delta} g_X(\lambda) \mu_{\mathbb{T}}(\mathbf{d}\lambda) < \infty$ for all $\delta \in [0, 1/2)$. Thus, if $\Re(z) < 1/2$ for all $z \in \sigma(D)$ then Condition (i) holds since, by compactness of $\sigma(D)$, there would exists $\delta \in [0, 1/2)$ such that $\Re(z) < \delta$ for all $z \in \sigma(D)$.

In the following, for any positive integer p , we endow the set of polynomials of degree less than or equal to p (or any of its subsets $\mathcal{P}_p(\mathcal{H}_0)$, $\mathcal{P}_p^\dagger(\mathcal{H}_0)$ or $\mathcal{P}_p^*(\mathcal{H}_0)$) with the max of the $\|\cdot\|_\infty$ -norms of its $\mathcal{L}_b(\mathcal{H}_0)$ coefficients. For instance if $\Psi(z) := \sum_{k=0}^p A_k z^k$ we denote

$$\|\Psi\| = \max \{ \|A_k\|_\infty : k = 0, \dots, p \}.$$

990 It is straightforward to show that the convergence of a $\mathcal{P}_p(\mathcal{H}_0)$ -valued sequence in the obtained Banach space is equivalent to the uniform convergence of this sequence in $\mathcal{C}(\mathbb{U}, \mathcal{L}_b(\mathcal{H}_0))$. In particular, if $(\theta, \lambda) \mapsto \Phi_\theta(e^{-i\lambda})$ and $(\theta, \lambda) \mapsto \Theta_\theta(e^{-i\lambda})$ are continuous on $\Theta \times \mathbb{T}$ onto $\mathcal{L}_b(\mathcal{H}_0)$ as assumed in (A-3), then $\theta \mapsto \Phi_\theta$ and $\theta \mapsto \Theta_\theta$ are continuous on Θ onto $\mathcal{P}_p(\mathcal{H}_0)$.

The following lemma is needed in the following.

995 **Lemma 17.** Let K be a compact subset of $\mathcal{P}_q^\dagger(\mathcal{H}_0)$. Then there exists $r > 1$ such that for all $z \in \mathbb{C}$ with $|z| < r$ and all $\theta \in K$, $\theta(z)$ is invertible in $\mathcal{L}_b(\mathcal{H}_0)$.

1000 *Proof.* We proceed by contradiction. Suppose that the conclusion of the lemma is not true. Then there exists a sequence (z_n) valued in $\{z \in \mathbb{C} : 1 < |z| \leq 2\}$ and a sequence $(P_n) \in K^{\mathbb{N}}$ such that $P_n(z_n)$ is not invertible and $\lim |z_n| = 1$. Since K is compact, we can extract a subsequence (z'_n, P'_n) of (z_n, P_n) that converges to an element (z^*, P^*) of $\overline{\mathbb{D}} \times K$ and since the set of invertible operators is open in $\mathcal{L}_b(\mathcal{H}_0)$ and $(z, \theta) \mapsto \theta(z)$ is continuous from $\mathbb{C} \times \mathcal{P}_q(\mathcal{H}_0)$ to $\mathcal{L}_b(\mathcal{H}_0)$ we get that $P^*(z^*)$ is not invertible. But this contradicts the assumption on K . \square

We can now state a lemma gathering all the important properties of the power series expansion of $\Phi_{\theta, \Phi, D}^\dagger$, that are repeatedly used in the proofs of the main results.

1005 **Lemma 18.** Let \mathcal{H}_0 be a separable Hilbert space and p, q be two non-negative integers. Let $\Phi_{\theta, \Phi, D}^\dagger$ be defined by (4.11). Then, for all $(\theta, \Phi, D) \in \mathcal{P}_q^\dagger(\mathcal{H}_0) \times \mathcal{P}_p(\mathcal{H}_0) \times \mathcal{L}_b(\mathcal{H}_0)$ and all $k \in \mathbb{N}$,

$$P_k^\dagger(\theta, \Phi, D) := \frac{1}{2i\pi} \int_{z \in \mathbb{C}, |z|=\rho} \Phi_{\theta, \Phi, D}^\dagger(z) z^{-k-1} dz \quad (5.67)$$

is well defined as $\mathcal{L}_b(\mathcal{H}_0)$ -valued Bochner integral for any $\rho \in (0, 1)$ and does not depend on ρ . Moreover, the following assertions hold.

(i) For all $(\theta, \phi, D) \in \mathcal{P}_q^\dagger(\mathcal{H}_0) \times \mathcal{P}_p(\mathcal{H}_0) \times \mathcal{L}_b(\mathcal{H}_0)$ and $z \in \mathbb{D}$,

$$\Phi_{\theta, \phi, D}^\dagger(z) = \sum_{k=1}^{\infty} P_k^\dagger(\theta, \phi, D) z^k . \quad (5.68)$$

(ii) For any $k \geq 1$, $(\theta, \phi, D) \mapsto P_k^\dagger(\theta, \phi, D)$ is continuous on $\mathcal{P}_q^\dagger(\mathcal{H}_0) \times \mathcal{P}_p(\mathcal{H}_0) \times \mathcal{L}_b(\mathcal{H}_0)$.

1010 (iii) For any compact subset $\bar{K} \subset \mathcal{P}_q^\dagger(\mathcal{H}_0) \times \mathcal{P}_p(\mathcal{H}_0) \times (\{0\} \cup \mathcal{L}^\dagger(\mathcal{H}_0))$, we have

$$\sum_{k=1}^{\infty} \sup_{(\theta, \phi, D) \in \bar{K}} \|P_k^\dagger(\theta, \phi, D)\|_\infty < \infty . \quad (5.69)$$

(iv) For all $(\theta, \phi, D) \in \mathcal{P}_q^\dagger(\mathcal{H}_0) \times \mathcal{P}_p(\mathcal{H}_0) \times \mathcal{L}^\dagger(\mathcal{H}_0)$ and $z \in \bar{\mathbb{D}} \setminus \{1\}$,

$$\Phi_{\theta, \phi, D}^\dagger(z) = \sum_{k=1}^{\infty} P_k^\dagger(\theta, \phi, D) z^k . \quad (5.70)$$

Proof. By Lemma 15, $\Phi_{\theta, \phi, D}^\dagger$ is continuous on \mathbb{D} for all $(\theta, \phi, D) \in \mathcal{P}_q^\dagger(\mathcal{H}_0) \times \mathcal{P}_p(\mathcal{H}_0) \times \mathcal{L}_b(\mathcal{H}_0)$. Thus the integral (5.67) is well defined. We now successively prove Assertions (i)-(iv).

Proof of (i). Using Lemma 15 and [15, Theorem 1.8.5], we can expand $\Phi_{\theta, \phi, D}^\dagger$ as a power series on the open unit disk \mathbb{D} , that is (5.68) holds with $P_k^\dagger(\theta, \phi, D)$ defined by (5.67) for any $\rho \in (0, 1)$. Note that the sum in the right-hand side of (5.68) starts at $k = 1$ because $P_0^\dagger(\theta, \phi, D) = 0$ since, by the Cauchy Formula (see [15, Theorem 1.5.1]), we have $P_0^\dagger(\theta, \phi, D) = \Phi_{\theta, \phi, D}^\dagger(0)$, which is the null operator following (4.11) and $\phi(0) = \phi(0) = 1^D = \text{Id}_{\mathcal{H}_0}$.

1020 **Proof of (ii).** The mapping $(z, \theta, \phi, D) \mapsto \Phi_{\theta, \phi, D}^\dagger(z)$ is continuous as a $\mathbb{D} \times \mathcal{P}_q^\dagger(\mathcal{H}_0) \times \mathcal{P}_p(\mathcal{H}_0) \times \mathcal{L}_b(\mathcal{H}_0) \rightarrow \mathcal{L}_b(\mathcal{H}_0)$ function by Lemma 15. Hence Assertion (ii) follows from (5.67) by dominated convergence.

Proof of (iii). Let \bar{K} be a compact subset of $\mathcal{P}_q^\dagger(\mathcal{H}_0) \times \mathcal{P}_p(\mathcal{H}_0) \times \{0\}$ or of $\mathcal{P}_q^\dagger(\mathcal{H}_0) \times \mathcal{P}_p(\mathcal{H}_0) \times \mathcal{L}^\dagger(\mathcal{H}_0)$. Then $\{\phi : (\theta, \phi, D) \in \bar{K}\}$ is a compact subset of $\mathcal{P}_q^\dagger(\mathcal{H}_0)$ and applying Lemma 17, there exists $r > 1$ such that for all $(\theta, \phi, D) \in \bar{K}$, θ does not vanish over the open disk of radius r . It follows that for all $(\theta, \phi, D) \in \bar{K}$, $z \mapsto [\theta(z)]^{-1} \phi(z)$ is a power series with a radius of convergence at least equal to r and that, for all $(z, \theta, \phi, D) \in \bar{\mathbb{D}} \times \bar{K}$,

$$[\theta(z)]^{-1} \phi(z) = \text{Id}_{\mathcal{H}_0} + \sum_{k=1}^{\infty} C_k(\theta, \phi) z^k , \quad (5.71)$$

where, for all $k \geq 1$ and all $\rho \in (0, r)$, $C_k(\theta, \phi) = \frac{1}{2i\pi} \int_{z \in \mathbb{C}, |z|=\rho} [\theta(z)]^{-1} \phi(z) z^{-k-1} dz$. As a result, picking any $\rho_1 \in (r^{-1}, 1)$, there exists $c_1 > 0$ such that,

$$\text{for all } k \geq 1 \quad \sup_{(\theta, \phi, D) \in \bar{K}} \|C_k(\theta, \phi)\|_\infty \leq c_1 \rho_1^k \quad (5.72)$$

If $\bar{K} \subset \mathcal{P}_q^\dagger(\mathcal{H}_0) \times \mathcal{P}_p(\mathcal{H}_0) \times \{0\}$, we have $P_k^\dagger(\theta, \phi, D) = -C_k(\theta, \phi)$ and (5.69) follows.

1030 We now consider the case where $\bar{K} \subset \mathcal{P}_q^\dagger(\mathcal{H}_0) \times \mathcal{P}_p(\mathcal{H}_0) \times \mathcal{L}^\dagger(\mathcal{H}_0)$. For all $D \in \mathcal{L}^\dagger(\mathcal{H}_0)$, applying Lemma 7 with $N = D + \text{Id}_{\mathcal{H}_0}$ and any $\varsigma \geq \|D + \text{Id}_{\mathcal{H}_0}\|_\infty$, we get, for all $z \in \mathbb{D}$, $(1-z)^D = \sum_{k=0}^{\infty} (Q(D) (k+1)^{-\text{Id}_{\mathcal{H}_0}-D} + P_k^*(D)) z^k$ where $Q(D), P_k^*(D) \in \mathcal{L}_b(\mathcal{H}_0)$ satisfy

$$\|Q(D) (k+1)^{-N} + P_k^*(D)\|_\infty \leq C \|(k+1)^{-\text{Id}_{\mathcal{H}_0}-D}\|_\infty , \quad (5.73)$$

for some $C > 0$ only depending on ς . By (4.11) and (5.71), setting $C_0(\theta, \phi) = \text{Id}_{\mathcal{H}_0}$, we obtain, for all $z \in \mathbb{D}$ and $(\theta, \phi, D) \in \bar{K}$,

$$\Phi_{\theta, \phi, D}^\dagger(z) = - \sum_{k=1}^{\infty} \left(\sum_{\ell=0}^k C_{k-\ell}(\theta, \phi) (Q(D) (\ell+1)^{-\text{Id}_{\mathcal{H}_0}-D} + P_\ell^*(D)) \right) z^k.$$

By (5.68), we thus have, for all $k \geq 1$,

$$P_k^\dagger(\theta, \phi, D) = - \left(\sum_{\ell=0}^k C_{k-\ell}(\theta, \phi) (Q(D) (\ell+1)^{-\text{Id}_{\mathcal{H}_0}-D} + P_\ell^*(D)) \right),$$

Let $K = \{D + \text{Id}_{\mathcal{H}_0} : (\theta, \phi, D) \in \bar{K}\}$. Then it is a compact subset of $\mathcal{L}_b(\mathcal{H}_0)$ and $\Re(z) > 1$ for all $P \in K$ and $z \in \sigma(P)$. Thus $\underline{\rho}$ as defined in Corollary 3 is in $(1, \infty)$ and (5.15) in this corollary gives us that we can find $\varsigma > 1$ and $\underline{\rho} \in (1, \varsigma)$ such that, for all $(\theta, \phi, D) \in \bar{K}$, $N = D + \text{Id}_{\mathcal{H}_0}$ satisfies $\|(\ell+1)^{-N}\|_\infty \leq C' (\ell+1)^{-\underline{\rho}}$ for some C' only depending on $(\varsigma, \underline{\rho})$. With the previous display, (5.72) and (5.73), this implies that

$$\text{for all } k \in \mathbb{N}, \quad \sup_{(\theta, \phi, D) \in \bar{K}} \|P_k^\dagger(\theta, \phi, D)\|_\infty \leq c_1 C C' \sum_{\ell=0}^k \rho_1^{k-\ell} (\ell+1)^{-\underline{\rho}}, \quad (5.74)$$

from which (5.69) is straightforward.

Proof of (iv). Assertions (ii) and (iii) imply that the right-hand side of (5.70) is continuous on $\bar{\mathbb{D}}$. Since, by Lemma 15, the left-hand side is continuous on $\bar{\mathbb{D}} \setminus \{1\}$, with Assertion (i), we conclude that Assertion (iv) holds. \square

Finally, the following lemma will be useful.

Lemma 19. *Let \mathcal{H}_0 be a separable Hilbert space and $X = (X_t)_{t \in \mathbb{Z}}$ be an ergodic stationary process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ valued in \mathcal{H}_0 such that $\mathbb{E}[\|X\|_{\mathcal{H}_0}^2] < \infty$. Then we have*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{k=1}^n X_k - \mathbb{E}[X] \right\|_{\mathcal{H}_0}^2 \right] = 0.$$

Proof. The assumptions imply that X is weakly stationary. Moreover, the space of shift-invariant elements in \mathcal{H}^X (where the shift is defined by $X_t \mapsto X_{t-1}$) is the null set, otherwise X would not be ergodic : take V shift-invariant in the sense of \mathcal{H}^X , then, for all $x \in \mathcal{H}_0$, $\langle V, x \rangle_{\mathcal{H}_0}$ is shift-invariant in the \mathbb{P}^X -a.s. sense. Therefore, the result simply follows from the von Neumann ergodic theorem (see [30, Theorem II.11]). \square

In the previous proof, we quickly explained why the assumptions of the von Neumann ergodic theorem hold. To conclude this section, we next state a more precise and detailed result for the sake of completeness.

Lemma 20. *Let \mathcal{H}_0 -valued be a separable Hilbert space and $X := (X_t)_{t \in \mathbb{Z}}$ be a centered \mathcal{H}_0 -valued weakly stationary process. Denote by U^X the shift operator defined on the modular time domain \mathcal{H}^X by $U^X : X_t \mapsto X_{t+1}$ and let ν_X be the spectral operator measure of X . Then the two following assertions are equivalent and they hold if X is an ergodic stationary process.*

- (i) For all $Y \in \mathcal{H}^X$, we have $U^X Y = Y$ if and only if $Y = 0$.
- (ii) We have $\nu_X(\{0\}) = 0$.

Proof. By the Kolmogorov isomorphism theorem (see [13, Theorem 3.8]), we can represent any $Y \in \mathcal{H}^X$ as $Y = \int \Phi \, d\hat{X}$ with $\Phi \in \hat{\mathcal{H}}^X$, Assertion (i) is thus equivalent to saying that for all $\Phi \in \hat{\mathcal{H}}^X$, we have $\int_{\mathbb{T}} |1 - e^{i\lambda}|^2 \left\| \Phi g_X^{1/2} \right\|_2^2 d\|\nu_X\|_1 = 0$ if and only if $\int_{\mathbb{T}} \left\| \Phi g_X^{1/2} \right\|_2^2 d\|\nu_X\|_1 = 0$, where $g_X = \frac{d\nu_X}{d\|\nu_X\|_1}$. Since $\|g_X\|_1 = 1$ $\|\nu_X\|_1$ -a.e., $\nu_X(\{0\}) \neq 0$ is equivalent to have $\|\nu_X\|_1(\{0\}) > 0$ and we clearly obtain that Assertions (i) and (ii) are equivalent.

Suppose now that X is an ergodic stationary process and let us show that Assertion (i) holds. The ergodicity of X means that $(\mathcal{H}_0^{\mathbb{Z}}, \mathcal{B}(\mathcal{H}_0)^{\otimes \mathbb{Z}}, \mathbb{P}^X, T)$ is an ergodic measure preserving dynamical system, where \mathbb{P}^X is the distribution of $X = (X_t)_{t \in \mathbb{Z}}$ defined on the canonical space $(\mathcal{H}_0^{\mathbb{Z}}, \mathcal{B}(\mathcal{H}_0)^{\otimes \mathbb{Z}})$ and T is the shift operator on $\mathcal{H}_0^{\mathbb{Z}}$ defined by $(x_t)_{t \in \mathbb{Z}} \mapsto (x_{t+1})_{t \in \mathbb{Z}}$. Now take $Y \in \mathcal{H}^X$. Setting $\Omega = \mathcal{H}_0^{\mathbb{Z}}$ and $\mathcal{F} = \mathcal{B}(\mathcal{H}_0)^{\otimes \mathbb{Z}}$, Y can be seen as the equivalence class in $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P}^X)$ of a measurable function $h : \Omega \rightarrow \mathcal{H}_0$. Then $h \circ T$ belongs to the equivalence class $U^X Y$. To prove Assertion (i), let us suppose that $Y = U^X Y$ (as elements of \mathcal{H}^X) and show that $Y = 0$ (since the reverse implication is obvious). From what precedes, $Y = U^X Y$ implies $h \circ T = h$, \mathbb{P}^X -a.s. Since $(\mathcal{H}_0^{\mathbb{Z}}, \mathcal{B}(\mathcal{H}_0)^{\otimes \mathbb{Z}}, \mathbb{P}^X, T)$ is ergodic, this implies that h is constant, \mathbb{P}^X -a.s., which in turn implies $Y = 0$, since all elements in \mathcal{H}^X have mean zero. The proof is concluded. \square

5.4.2. Proof of main results

Proof of Theorem 6. By Definition 2, denoting by Σ the covarariance operator of Z , Y admits the spectral density

$$g_Y(\lambda) = (1 - e^{-i\lambda})^{-D} [\Phi(e^{-i\lambda})]^{-1} \Theta(e^{-i\lambda}) \Sigma \left((1 - e^{-i\lambda})^{-D} [\Phi(e^{-i\lambda})]^{-1} \Theta(e^{-i\lambda}) \right)^H$$

with respect to the normalized Lebesgue measure $\mu_{\mathbb{T}}$. Let $t \in \mathbb{Z}$. We first show that the right-hand side of (4.12) is well defined, that is, that $\lambda \mapsto e^{i\lambda t} \Phi_{\Theta, \Phi, D}^{\dagger}(e^{-i\lambda})$ belongs to $\hat{\mathcal{H}}^Y$. Since this mapping is continuous on $\mathbb{T} \setminus \{0\}$ onto $\mathcal{L}_b(\mathcal{H}_0)$, by Proposition 1, this is equivalent to have

$$\int \left\| \Phi_{\Theta, \Phi, D}^{\dagger}(e^{-i\lambda}) g_Y(\lambda) \Phi_{\Theta, \Phi, D}^{\dagger}(e^{-i\lambda})^H \right\|_1 \mu_{\mathbb{T}}(d\lambda) < \infty .$$

By definition of g_Y , we thus have to show that

$$\int \left\| \Phi_{\Theta, \Phi, D}^{\dagger}(e^{-i\lambda}) (1 - e^{-i\lambda})^{-D} [\Phi(e^{-i\lambda})]^{-1} \Theta(e^{-i\lambda}) \Sigma^{1/2} \right\|_2^2 \mu_{\mathbb{T}}(d\lambda) < \infty . \quad (5.75)$$

By definition of $\Phi_{\Theta, \Phi, D}^{\dagger}$ in (4.11), we have, for all $\lambda \in \mathbb{T} \setminus \{0\}$,

$$\left(\text{Id}_{\mathcal{H}_0} - \Phi_{\Theta, \Phi, D}^{\dagger}(e^{-i\lambda}) \right) (1 - e^{-i\lambda})^{-D} [\Phi(e^{-i\lambda})]^{-1} \Theta(e^{-i\lambda}) = \text{Id}_{\mathcal{H}_0} .$$

We thus get that, for all $\lambda \in \mathbb{T} \setminus \{0\}$, $\Phi_{\Theta, \Phi, D}^{\dagger}(e^{-i\lambda}) (1 - e^{-i\lambda})^{-D} [\Phi(e^{-i\lambda})]^{-1} \Theta(e^{-i\lambda}) \Sigma^{1/2}$ can be expressed as

$$(1 - e^{-i\lambda})^{-D} [\Phi(e^{-i\lambda})]^{-1} \Theta(e^{-i\lambda}) \Sigma^{1/2} - \Sigma^{1/2} .$$

Thus, since $\|\Sigma^{1/2}\|_2^2 = \|\Sigma\|_1 < \infty$, Condition (5.75) is implied by

$$\int \left\| (1 - e^{-i\lambda})^{-D} [\Phi(e^{-i\lambda})]^{-1} \Theta(e^{-i\lambda}) \Sigma^{1/2} \right\|_2^2 \mu_{\mathbb{T}}(d\lambda) < \infty .$$

On the other hand, the square \mathcal{S}_2 -norm inside the previous integral is equal to $\|g_Y\|_1$ which is $\mu_{\mathbb{T}}$ -integrable as a spectral density. We thus get that the right-hand side of (4.12) is well defined and, in the following, we denote

$$\hat{Y}_t = \int e^{i\lambda t} \Phi_{\Theta, \Phi, D}^{\dagger}(e^{-i\lambda}) \hat{Y}(d\lambda) .$$

It only remains to show the two following assertions.

1075 (i) We have $\hat{Y}_t \in \mathcal{H}_{t-1}^Y$.

(ii) We have $Y_t - \hat{Y}_t \perp \mathcal{H}_{t-1}^Y$.

Let us first prove Assertion (i). By Assertion (i) in Lemma 18, we immediately have that, for all $\rho \in (0, 1)$,

$$\hat{Y}_t^{(\rho)} = \int e^{i\lambda t} \Phi_{\Theta, \Phi, D}^\dagger(\rho e^{-i\lambda}) \hat{Y}(d\lambda) \in \mathcal{H}_{t-1}^Y .$$

To conclude Assertion (i), it is thus sufficient to show that

$$\lim_{\rho \uparrow 1} \hat{Y}_t^{(\rho)} = Y_t \quad \text{in } \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P}). \quad (5.76)$$

By the Kolmogorov Gramian isometric theorem (see [13, Theorem 3.8]), setting

$$\Psi(\rho, \lambda) := \Phi_{\Theta, \Phi, D}^\dagger(\rho e^{-i\lambda}) - \Phi_{\Theta, \Phi, D}^\dagger(e^{-i\lambda}) ,$$

we can write, for any $\eta \in (0, \pi)$ and $\rho \in (0, 1)$,

$$\begin{aligned} \mathbb{E} \left[\left\| \hat{Y}_t^{(\rho)} - \hat{Y}_t \right\|_{\mathcal{H}_0}^2 \right] &= \int \left\| \Psi(\rho, \lambda) g_Y(\lambda) \Psi^H(\rho, \lambda) \right\|_1 \mu_{\mathbb{T}}(d\lambda) \\ &\leq \left(\int \|g_Y\|_1 d\mu_{\mathbb{T}} \right) \sup_{\lambda \in \mathbb{T} \setminus [-\eta, \eta]} \|\Psi(\rho, \lambda)\|_\infty^2 \\ &\quad + \int_{-\eta}^{\eta} \sup_{0 \leq \rho \leq 1} \left\| \Psi(\rho, \lambda) g_Y(\lambda) \Psi^H(\rho, \lambda) \right\|_1 \mu_{\mathbb{T}}(d\lambda) . \end{aligned}$$

1080 By Lemma 15 the first term of this bound tends to zero as $\rho \uparrow 1$ for all $\eta \in (0, \pi)$. It thus only remain to check that the second term can be made arbitrarily small as $\eta \downarrow 0$, which follows if there exists $\eta > 0$ such that

$$\int_{-\eta}^{\eta} \sup_{0 \leq \rho \leq 1} \left\| \Psi(\rho, \lambda) g_Y(\lambda) \Psi^H(\rho, \lambda) \right\|_1 \mu_{\mathbb{T}}(d\lambda) < \infty .$$

By definition of g_Y , setting X as the ARMA(p, q) process defined by

$$\hat{X}(d\lambda) = [\Phi(e^{-i\lambda})]^{-1} \Theta(e^{-i\lambda}) \hat{Z}(d\lambda) , \quad (5.77)$$

denoting by g_X the density of X with respect to $\mu_{\mathbb{T}}$, the previous condition is equivalent to

$$\int_{-\eta}^{\eta} \sup_{0 \leq \rho \leq 1} \left\| \tilde{\Psi}(\rho, \lambda) g_X(\lambda) \tilde{\Psi}^H(\rho, \lambda) \right\|_1 \mu_{\mathbb{T}}(d\lambda) < \infty , \quad (5.78)$$

$$\text{where } \tilde{\Psi}(\rho, \lambda) := \Psi(\rho, \lambda) (1 - e^{-i\lambda})^{-D} = \left(\Phi_{\Theta, \Phi, D}^\dagger(\rho e^{-i\lambda}) - \Phi_{\Theta, \Phi, D}^\dagger(e^{-i\lambda}) \right) (1 - e^{-i\lambda})^{-D} .$$

1085 Since X here is an ARMA process and we assume in Theorem 6 that the real parts of all elements of $\sigma(D)$ is less than 1/2 or that $D \in \mathcal{N}(\mathcal{H}_0)$, Condition (i) (see Remark 3) or Condition (ii) of Lemma 16 hold, and we can apply this lemma. This gives us that the Condition (5.78) holds, which concludes the proof of (5.76) and thus of Assertion (i).

We now prove Assertion (ii). By definition of \hat{Y}_t , (4.11) and (3.4), we have

$$\begin{aligned} Y_t - \hat{Y}_t &= \int e^{i\lambda t} \left(\text{Id}_{\mathcal{H}_0} - \Phi_{\Theta, \Phi, D}^\dagger(e^{-i\lambda}) \right) \hat{Y}(d\lambda) \\ &= \int e^{i\lambda t} [\Theta(e^{-i\lambda})]^{-1} \Phi(e^{-i\lambda}) (1 - e^{-i\lambda})^D \hat{Y}(d\lambda) \\ &= \int e^{i\lambda t} \hat{Z}(d\lambda) = Z_t . \end{aligned}$$

Since Z is a white noise we have $Z_t \perp \mathcal{H}_{t-1}^Z$. To prove Assertion (ii), it thus only remains to show that \mathcal{H}_s^Y is included in \mathcal{H}_s^Z for all $s \in \mathbb{Z}$. Since we assumed $\mathbb{f} \in \mathcal{P}_p^\dagger(\mathcal{H}_0)$, we have that \mathbb{f}^{-1} is holomorphic in an open domain that includes $\overline{\mathbb{D}}$ and thus can be written as a power series on the unit circle. It follows that the ARMA process X defined by (5.77) satisfies $\mathcal{H}_s^X \subset \mathcal{H}_s^Z$ for all $s \in \mathbb{Z}$. To conclude, we now prove that $\mathcal{H}_s^Y \subset \mathcal{H}_s^X$ for all $s \in \mathbb{Z}$. Observe that, for all $s \in \mathbb{Z}$,

$$Y_s = \int e^{i\lambda s} (1 - e^{-i\lambda})^{-D} \hat{X}(d\lambda).$$

Using that $z \mapsto (1 - z)^{-D}$ is holomorphic on $\mathbb{C} \setminus [1, \infty)$, it can be expanded as a power series on \mathbb{D} and it follows that, for all $s \in \mathbb{Z}$ and $\rho \in (0, 1)$,

$$Y_s^{(\rho)} = \int e^{i\lambda s} (1 - \rho e^{-i\lambda})^{-D} \hat{X}(d\lambda) \in \mathcal{H}_s^X.$$

Using the same trick as for the proof of Assertion (i), we write, for any $\eta \in (0, \pi)$ and $\rho \in (0, 1)$,

$$\begin{aligned} \mathbb{E} \left[\left\| Y_s - Y_s^{(\rho)} \right\|_{\mathcal{H}_0}^2 \right] &\leq \left(\int \|g_X\|_1 d\mu_{\mathbb{T}} \right) \sup_{\lambda \in \mathbb{T} \setminus [-\eta, \eta]} \left\| (1 - \rho e^{-i\lambda})^{-D} - (1 - e^{-i\lambda})^{-D} \right\|_{\infty}^2 \\ &\quad + \int_{-\eta}^{\eta} \sup_{0 \leq \rho \leq 1} \left\| \Psi(\rho, \lambda) g_X(\lambda) \Psi^H(\rho, \lambda) \right\|_1 \mu_{\mathbb{T}}(d\lambda), \end{aligned}$$

$$\text{where, here, } \Psi(\rho, \lambda) := (1 - \rho e^{-i\lambda})^{-D} - (1 - e^{-i\lambda})^{-D}.$$

1090 By continuity of $z \mapsto (1 - z)^{-D}$ on $\mathbb{C} \setminus [1, \infty)$, the first term in the upper bound tends to zero for all η , while the second bound can be made arbitrarily small as $\eta \downarrow 0$ by using again Lemma 16. We thus get the claim $\mathcal{H}_s^Y \subset \mathcal{H}_s^X$ and the proof is concluded. \square

Proof of Proposition 6. All the assertions follow straightforwardly from Lemma 18 and other previous results. Observe indeed that (4.14) immediately follows from the fact that $F^\dagger(\theta, \lambda) = \text{Id}_{\mathcal{H}_0} - \Phi_{D_\theta, \mathbb{f}_\theta, \Theta_\theta}^\dagger(e^{-i\lambda})$. As for the other claimed facts, here are some details.

1095 Moreover, the continuity of $(D, \mathbb{f}, \Theta) \mapsto P_k^\dagger(\Theta, \mathbb{f}, D)$ (and thus with (A-3), that of $\theta \mapsto P_k^\dagger(\theta)$, and the bound (5.69) gives us that $\theta \mapsto \hat{Y}_t(\theta)$ is continuous on Θ onto \mathcal{H}_{t-1}^Y , hence the expectation in the right-hand side of (4.16) is continuous in θ , which shows that the inf is attained on a compact subset of Θ .

1100 When the best predictor \hat{Y}_t^* is well defined for one $t \in \mathbb{Z}$, by weak stationarity of Y it must be well defined for all t . Then, for all $t \in \mathbb{Z}$ and all $\theta \in \Theta_Y^*$, we have $\hat{Y}_t^* = \hat{Y}_t(\theta)$, \mathbb{P} -a.s.. Of course, since \mathbb{Z} is countable, we can exchange the \mathbb{P} -a.s. with the “for all $t \in \mathbb{Z}$ ”. We can also exchange the \mathbb{P} -a.s. with the “for all $\theta \in \Theta_Y^*$ ” as claimed in Assertion (vi) of the proposition because the arguments above also give that, \mathbb{P} -a.s., $\theta \mapsto \hat{Y}_t(\theta)$ is continuous on Θ onto \mathcal{H}_0 , and consequently, 1105 is uniquely defined by its value on a dense countable subset of (the compact set) Θ_Y^* .

Finally, in the well-specified case, we have that Y is a FIARMA process with parameter (D, θ, \mathbb{f}) such that $(D, \theta, \mathbb{f}) = (D_\theta, \mathbb{f}_\theta, \Theta_\theta)$ for some $\theta^* \in \Theta$. Since we suppose that $D \in \mathcal{N}(\mathcal{H}_0)$ or that $\Re(z) < 1/2$ for all $z \in \sigma(D)$ in Assertion (vii), we can apply Theorem 6. Observe that, by Assertion (iv) of Lemma 18, $\hat{Y}_t(\theta^*)$ defined as in (4.15) with $\theta = \theta^*$ is the right-hand side of (4.12). This gives us that $\text{proj}(Y_t | \mathcal{H}_{t-1}^Y) = \hat{Y}_t(\theta^*)$. Since the projection is uniquely defined and $\hat{Y}_t(\theta) \in \mathcal{H}_{t-1}^Y$ for all $\theta \in \Theta$, we conclude that the inf in (4.16) is achieved with $\theta = \theta^*$, and from (4.17) that $\hat{Y}_t(\theta) = \text{proj}(Y_t | \mathcal{H}_{t-1}^Y)$ for all $\theta \in \Theta_Y^*$. Hence \hat{Y}_t^* of Assertion (v) is well defined and coincides with $\text{proj}(Y_t | \mathcal{H}_{t-1}^Y)$. This concludes the proof of Assertion (vii). \square

Proof of Theorem 7. By Lemma 18, we have that $F^\dagger \in \mathcal{C}(\Theta \times \mathbb{T}, \mathcal{L}_b(\mathcal{H}_0))$ with F^\dagger defined in (4.14). Similarly, using Condition (iii) in Theorem 7, we also have $F_s^\dagger \in \mathcal{C}(\Theta \times \mathbb{T}, \mathcal{L}_b(\mathcal{H}_0^s))$, with $F_s^\dagger(\theta, \lambda) := F^\dagger(\theta, \lambda)|_{\mathcal{H}_0^s}$. Hence Assertions (i), (ii) and (iii) of Theorem 5 hold with $L = R = F^\dagger$. Applying this theorem, with the fact that the trace is continuous on $\mathcal{S}_1(\mathcal{H}_0)$, we obtain that, \mathbb{P} -a.s.,

$$\lim_{n \rightarrow \infty} \Lambda_n = \text{Tr} \left(\tilde{Q}_{\nu_X}^{(F^\dagger, F^\dagger)} \right) \quad \text{uniformly in } \mathcal{C}(\Theta, \mathbb{R}).$$

Now, applying Proposition 6 to $Y = X^{(c)}$, (4.14) and (4.15) yield, for all $\theta \in \Theta$,

$$\mathrm{Tr} \left(\tilde{Q}_{\nu_X}^{(\mathbb{F}^\dagger, \mathbb{F}^\dagger)}(\theta) \right) = \mathbb{E} \left[\left\| X_0^{(c)} - \widehat{X_0^{(c)}}(\theta) \right\|_{\mathcal{H}_0}^2 \right].$$

Therefore, with (4.20) and (4.16), we can write that, \mathbb{P} -a.s.,

$$\limsup_{n \rightarrow \infty} \Lambda_n(\hat{\theta}_n) \leq \limsup_{n \rightarrow \infty} \inf_{\theta \in \Theta} \Lambda_n(\theta) = \mathbb{E}^2 \left(X^{(c)}, \aleph \right).$$

Thus, \mathbb{P} -a.s., all accumulation points θ of the Θ -valued sequence $(\hat{\theta}_n)_{n \geq 1}$ satisfy

$$\mathbb{E} \left[\left\| X_0^{(c)} - \widehat{X_0^{(c)}}(\theta) \right\|_{\mathcal{H}_0}^2 \right] \leq \mathbb{E}^2 \left(X^{(c)}, \aleph \right), \quad \text{which implies } \theta \in \Theta_{X^{(c)}}^*.$$

1115 Since Θ is compact, we obtain (4.25).

We now prove (4.26). By Assertion (iii) in Lemma 18, we have

$$\sup_{n \in \mathbb{N}} \sup_{\theta \in \Theta} \left\| \mathrm{Id}_{\mathcal{H}_0} - \sum_{k=1}^n \mathbb{P}_k^\dagger(\theta) \right\|_\infty \leq 1 + \sum_{k=1}^\infty \sup_{\theta \in \Theta} \left\| \mathbb{P}_k^\dagger(\theta) \right\|_\infty < \infty. \quad (5.79)$$

Let us denote, for all $\theta \in \Theta$,

$$\mathbb{E}_\infty^2(\theta) = \mathbb{E} \left[\left\| X_0^{(c)} - \left(\sum_{k=1}^\infty \mathbb{P}_k^\dagger(\theta) X_{-k}^{(c)} \right) \right\|_{\mathcal{H}_0}^2 \right]. \quad (5.80)$$

By (4.24) and Minkowski's inequality, we have, for all $(m, \theta) \in \mathcal{H}_0 \times \Theta$ and $n \geq 1$,

$$|\mathbb{E}_\infty(\theta) - \mathbb{E}_{X,n}(m, \theta)| \leq |\mathbb{E}_\infty(\theta) - \mathbb{E}_{X^{(c)},n}(0, \theta)| + \left\| \left(\mathrm{Id}_{\mathcal{H}_0} - \sum_{k=1}^\infty \mathbb{P}_k^\dagger(\theta) \right) (\mathbb{E}[X_0] - m) \right\|_{\mathcal{H}_0}.$$

Using this bound and (5.79), to obtain (4.26), it is sufficient to prove the three following facts:

$$\lim_{n \rightarrow \infty} \mathbb{E}_\infty^2(\hat{\theta}_n) = \mathbb{E}^2 \left(X^{(c)}, \aleph \right) \quad \mathbb{P}\text{-a.s.}, \quad (5.81)$$

$$\lim_{n \rightarrow \infty} \sup_{\theta \in \Theta} |\mathbb{E}_\infty(\theta) - \mathbb{E}_{X^{(c)},n}(0, \theta)| = 0, \quad (5.82)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_k = \mathbb{E}[X_0] \quad \mathbb{P}\text{-a.s.} \quad (5.83)$$

1120 We prove these fact in reverse order. Lemma 13 directly gives us (5.83). To obtain (5.82), we use that, for all $m \in \mathcal{H}_0$ and $n \geq 1$,

$$\begin{aligned} \sup_{\theta \in \Theta} |\mathbb{E}_\infty(\theta) - \mathbb{E}_{X^{(c)},n}(0, \theta)| &\leq \sup_{\theta \in \Theta} \left(\mathbb{E} \left[\left\| \sum_{k=n+1}^\infty \mathbb{P}_k^\dagger(\theta) X_{-k}^{(c)} \right\|_{\mathcal{H}_0}^2 \right] \right)^{1/2} \\ &\leq \left(\sum_{k=n+1}^\infty \sup_{\theta \in \Theta} \left\| \mathbb{P}_k^\dagger(\theta) \right\|_\infty \right) \left(\mathbb{E} \left[\left\| X_0^{(c)} \right\|_{\mathcal{H}_0}^2 \right] \right)^{1/2}, \end{aligned}$$

which, by (5.79), converges to zero as $n \rightarrow \infty$. We finally prove (5.81). By (4.25), there exists a set Ω^* of probability 1 on which $\Delta(\hat{\theta}_n, \Theta_{X^{(c)}}^*)$ converges to 0. Thus (5.81) holds if we can

show that, for any (deterministic) sequence $(\theta_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} \Delta(\theta_n, \Theta_{X^{(c)}}^*) = 0$, we have

1125 $\lim_{n \rightarrow \infty} \mathbb{E}_\infty^2(\theta_n) = \mathbb{E}^2(X^{(c)}, \aleph)$. Take such a sequence $(\theta_n)_{n \in \mathbb{N}}$ and set $\ell_n := \mathbb{E}_\infty^2(\theta_n)$ for all n . Observe that, by (5.79) and Assertion (ii) of Lemma 18, we have that \mathbb{E}_∞^2 is continuous on Θ onto \mathbb{R}_+ . Since Θ is compact, $(\ell_n)_{n \geq 1}$ is a bounded sequence in \mathbb{R}_+ and it is sufficient to show that all its accumulation points are equal to the constant $\mathbb{E}^2(X^{(c)}, \aleph)$. By compactness of Θ , from any subsequence of $(\ell_n)_{n \geq 1}$ converging to an accumulation point ℓ^* , we can extract a subsubsequence

1130 $(\ell'_n)_{n \geq 1}$ for which the corresponding extraction (θ'_n) from (θ_n) converges in Θ . By the assumption made on (θ_n) , the limit of (θ'_n) must be an element $\theta^* \in \Theta_{X^{(c)}}^*$ (which is closed by Assertion (iv) in Proposition 6). By continuity of \mathbb{E}_∞^2 , we get that $\ell^* = \mathbb{E}_\infty^2(\theta^*)$, thus is equal to $\mathbb{E}^2(X^{(c)}, \aleph)$ by definition of $\Theta_{X^{(c)}}^*$ in (4.17).

Let us show the last assertion of the theorem. To this end, we suppose that $\widehat{X_t^{(c)}}^*$ is well defined and show that (4.27) holds. Since $X_k^{(c,n)}$ is unmodified by translating X to $X^{(c)}$, we have

$$X_{n+1} - \widehat{X}_{n+1,n} = X_{n+1}^{(c)} - \sum_{k=1}^n P_k^\dagger(\widehat{\theta}_n) X_{n+1-k}^{(c)} - \left(\text{Id}_{\mathcal{H}_0} - \sum_{k=1}^n P_k^\dagger(\widehat{\theta}_n) \right) \left(\frac{1}{n} \sum_{k=1}^n X_k^{(c)} \right).$$

Therefore, to obtain (4.27), we only need to show that

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left[\left\| X_{n+1}^{(c)} - \sum_{k=1}^n P_k^\dagger(\widehat{\theta}_n) X_{n+1-k}^{(c)} \right\|_{\mathcal{H}_0}^2 \right] \leq \mathbb{E}^2(X^{(c)}, \aleph), \quad (5.84)$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left\| \left(\text{Id}_{\mathcal{H}_0} - \sum_{k=1}^n P_k^\dagger(\widehat{\theta}_n) \right) \left(\frac{1}{n} \sum_{k=1}^n X_k^{(c)} \right) \right\|_{\mathcal{H}_0}^2 \right] = 0. \quad (5.85)$$

1135 Let us start with (5.85). Then, we get (5.85) by applying Lemma 19.

We now prove (5.84). For all $n \geq 1$, the squared norm in the left-hand side's expectation of (5.84) can be written as

$$\inf_{\theta \in \Theta_{X^{(c)}}^*} \left\| X_{n+1}^{(c)} - \sum_{k=1}^n P_k^\dagger(\theta) X_{n+1-k}^{(c)} + \sum_{k=1}^n \left(P_k^\dagger(\theta) - P_k^\dagger(\widehat{\theta}_n) \right) X_{n+1-k}^{(c)} \right\|_{\mathcal{H}_0}^2.$$

We thus have, for all $n \geq 1$,

$$\mathbb{E} \left[\left\| X_{n+1}^{(c)} - \sum_{k=1}^n P_k^\dagger(\widehat{\theta}_n) X_{n+1-k}^{(c)} \right\|_{\mathcal{H}_0}^2 \right] \leq \mathbb{E} \left[(A_n + B_n)^2 \right], \quad (5.86)$$

where we set

$$A_n := \inf_{\theta \in \Theta_{X^{(c)}}^*} \left\| \sum_{k=1}^n \left(P_k^\dagger(\theta) - P_k^\dagger(\widehat{\theta}_n) \right) X_{n+1-k}^{(c)} \right\|_{\mathcal{H}_0},$$

$$B_n := \sup_{\theta \in \Theta_{X^{(c)}}^*} \left\| X_{n+1}^{(c)} - \sum_{k=1}^n P_k^\dagger(\theta) X_{n+1-k}^{(c)} \right\|_{\mathcal{H}_0}.$$

We are going to show, successively that

$$\lim_{n \rightarrow \infty} \mathbb{E} [A_n^2] = 0, \quad (5.87)$$

$$\lim_{n \rightarrow \infty} \mathbb{E} [B_n^2] = \mathbb{E}^2(X^{(c)}, \aleph). \quad (5.88)$$

1140 These two facts with (5.86) indeed imply (5.84). First observe that Lemma 18 straightforwardly yields

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{\theta \in \Theta} \left\| \sum_{k=n+1}^{\infty} \mathbb{P}_k^\dagger(\theta) X_{n+1-k}^{(c)} \right\|_{\mathcal{H}_0}^2 \right] = 0. \quad (5.89)$$

Thus, to have (5.87) and (5.88), and by stationarity of $X^{(c)}$, we can use

$$A'_n := \inf_{\theta \in \Theta_{X^{(c)}}^*} \left\| \sum_{k=1}^{\infty} \left(\mathbb{P}_k^\dagger(\theta) - \mathbb{P}_k^\dagger(\hat{\theta}_n) \right) X_{-k}^{(c)} \right\|_{\mathcal{H}_0},$$

$$B' := \sup_{\theta \in \Theta_{X^{(c)}}^*} \left\| X_0^{(c)} - \sum_{k=1}^{\infty} \mathbb{P}_k^\dagger(\theta) X_{-k}^{(c)} \right\|_{\mathcal{H}_0},$$

and prove instead

$$\lim_{n \rightarrow \infty} \mathbb{E} [A_n'^2] = 0, \quad (5.90)$$

$$\mathbb{E} [B'^2] = \mathbb{E}^2 \left(X^{(c)}, \aleph \right). \quad (5.91)$$

To get Relation (5.91), we observe that, with the assumption that the best \aleph -predictor is well defined, Assertion (vi) in Proposition 6 and (4.15) give that, \mathbb{P} -a.s., for all $\theta \in \Theta_{X^{(c)}}^*$, $\widehat{X}_0^{(c)}(\theta) =$
 1145 $\sum_{k=1}^{\infty} \mathbb{P}_k^\dagger(\theta) X_{-k}^{(c)} = \widehat{X}_0^{(c)*}$. Hence, \mathbb{P} -a.s., $B' = \left\| X_0^{(c)} - \widehat{X}_0^{(c)*} \right\|$. We thus have (5.91).

We conclude with the proof of (5.90). Note that

$$A'_n \leq 2 \sup_{\theta \in \Theta_{X^{(c)}}^*} \left\| \sum_{k=1}^{\infty} \mathbb{P}_k^\dagger(\theta) X_{-k}^{(c)} \right\|_{\mathcal{H}_0}.$$

Note that the L^2 -norm of this upper bound satisfies

$$\left(\mathbb{E} \left[\sup_{\theta \in \Theta_{X^{(c)}}^*} \left\| \sum_{k=1}^{\infty} \mathbb{P}_k^\dagger(\theta) X_{-k}^{(c)} \right\|_{\mathcal{H}_0}^2 \right] \right)^{1/2} \leq \left(\sum_{k=1}^{\infty} \sup_{\theta \in \Theta_{X^{(c)}}^*} \left\| \mathbb{P}_k^\dagger(\theta) \right\|_{\infty} \right) \left(\mathbb{E} \left[\left\| X_0^{(c)} \right\|_{\mathcal{H}_0}^2 \right] \right)^{1/2},$$

which is finite by Lemma 18. Thus, we can apply the dominated convergence theorem, and (5.90) follows from

$$\lim_{n \rightarrow \infty} A'_n = 0 \quad \mathbb{P}\text{-a.s.},$$

which we now prove by contradiction. Suppose that, with positive probability, we can find $\eta > 0$ and an increasing sequence $(n_j)_{j \in \mathbb{N}}$ of integers such that $A'_{n_j} \geq \eta$ for all j . Then, by (4.25) and since Θ is compact, with positive probability, there also exists a subsequence $(n'_j)_{j \in \mathbb{N}}$ integers such that $A'_{n'_j} \geq \eta$ for all j and $\hat{\theta}_{n'_j}$ converges to some $\theta \in \Theta_{X^{(c)}}^*$ as $j \rightarrow \infty$. By Lemma 18, this latter fact implies that, for this θ ,

$$\lim_{j \rightarrow \infty} \left\| \sum_{k=1}^{\infty} \left(\mathbb{P}_k^\dagger(\theta) - \mathbb{P}_k^\dagger(\hat{\theta}_{n'_j}) \right) X_{-k}^{(c)} \right\|_{\mathcal{H}_0} = 0.$$

But since $\theta \in \Theta_{X^{(c)}}^*$, this contradicts the assumption that yielded $A'_{n'_j} \geq \eta > 0$ for all j . This finishes the proof. \square

Appendix A. Some technical lemmas

We start with two lemmas on some useful complex valued functions.

1150 **Lemma 21.** *For all $z \in \mathbb{C}$ and $\lambda \in [-\pi, \pi] \setminus \{0\}$, we have*

$$(2/\pi)^{2\Re_+(z)} |\lambda|^{2\Re(z)} e^{-\pi|\Im(z)|} \leq |(1 - e^{-i\lambda})^z|^2 \leq (\pi/2)^{2\Re_-(z)} |\lambda|^{2\Re(z)} e^{\pi|\Im(z)|}, \quad (\text{A.1})$$

where $\Re(z) = (z + \bar{z})/2$, $\Re_+(z) = \max(\Re(z), 0)$ and $\Re_-(z) = \max(-\Re(z), 0)$.

Proof. Let $z \in \mathbb{C}$ and $\lambda \in (-\pi, \pi] \setminus \{0\}$. By definition of the principal logarithm, we have

$$\text{for all } y \in \mathbb{C} \setminus \mathbb{R}_-, \quad |y^z|^2 = |\exp(z \ln(y))|^2 = |y|^{2\Re(z)} e^{-2\Im(z)b(y)}, \quad (\text{A.2})$$

where $b(y)$ denotes the argument in the polar form of y in $(-\frac{\pi}{2}, \frac{\pi}{2})$. It follows that $e^{-\pi|\Im(z)|} \leq e^{-2\Im(z)b(y)} \leq e^{\pi|\Im(z)|}$. Applying (A.2) with $y = 1 - e^{-i\lambda}$, using that $\frac{2|\lambda|}{\pi} \leq |2 \sin(\lambda/2)| = |1 - e^{-i\lambda}| \leq |\lambda|$ for all $\lambda \in (-\pi, \pi)$ and separating the cases where $\Re(z) \geq 0$ and where $\Re(z) < 0$, we get (A.1). \square

Lemma 22. *For all $z \in \mathbb{C}$ and $\lambda \in [-\pi/3, \pi/3] \setminus \{0\}$, we have*

$$\sup_{0 \leq \rho \leq 1} |(1 - \rho e^{-i\lambda})^z|^2 \leq \left(2\pi/(3\sqrt{3})\right)^{2\Re_-(z)} |\lambda|^{-2\Re_-(z)} e^{\pi|\Im(z)|}, \quad (\text{A.3})$$

where $\Re(z) = (z + \bar{z})/2$, $\Re_+(z) = \max(\Re(z), 0)$ and $\Re_-(z) = \max(-\Re(z), 0)$.

Proof. Applying (A.2) with $y = 1 - \rho e^{-i\lambda}$ and using that $b(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we get that

$$\sup_{0 \leq \rho \leq 1} |(1 - \rho e^{-i\lambda})^z|^2 \leq \sup_{0 \leq \rho \leq 1} |1 - \rho e^{-i\lambda}|^{2\Re(z)} e^{\pi|\Im(z)|}.$$

Let now $z \in \mathbb{C}$ and $\lambda \in [-\pi/3, \pi/3] \setminus \{0\}$. It is straightforward to show that, in this case,

$$\sin^2(\lambda) = \inf_{0 \leq \rho \leq 1} |(1 - \rho e^{-i\lambda})|^2 \leq \sup_{0 \leq \rho \leq 1} |(1 - \rho e^{-i\lambda})|^2 = 1.$$

1160 Separating the cases where $\Re(z) \geq 0$ and where $\Re(z) < 0$, and, in the latter case, using that $|\sin(\lambda)| \geq 3\sqrt{3}|\lambda|/(2\pi)$ for $|\lambda| \leq \pi/3$, we easily get (A.3). \square

We conclude this appendix with a lemma on non-negative sequences.

Lemma 23. *Let $(u_k)_{k \in \mathbb{N}}$ be a non-negative non-increasing sequence such that $\sum_{k \in \mathbb{N}} u_k < \infty$. Then there exists a non-decreasing sequence $(v_k)_{k \in \mathbb{N}}$, going to ∞ as $k \rightarrow \infty$ such that $\sum_{k \in \mathbb{N}} u_k v_k < \infty$.*

Proof. Let $k_0 = 0$, and for all $n \geq 1$, define by induction

$$k_n = \min \left\{ j > k_{n-1} : \sum_{k=j}^{\infty} u_k \leq 4^{-n} \right\}.$$

Then (k_n) is an increasing sequence of integers going to ∞ as $n \rightarrow \infty$. Define, for all $n \geq 1$, and for all $k_{n-1} \leq k < k_n$, $v_k = 2^n$. Then $(v_k)_{k \in \mathbb{N}}$ is a non-decreasing sequence going to ∞ and we have, by definition of (v_k) , using that (u_k) is non-negative and then, by definition of (k_n) ,

$$\sum_{k \in \mathbb{N}} u_k v_k = \sum_{n=0}^{\infty} 2^n \left(\sum_{k_{n-1} \leq k < k_n} u_k \right) \leq \sum_{n=0}^{\infty} 2^n \left(\sum_{k=k_{n-1}}^{\infty} u_k \right) \leq 4 \sum_{n=0}^{\infty} 2^{-n} < \infty.$$

1165 The proof is concluded. \square

Appendix B. Some details on $L^2(\mathbf{V}, \mathcal{V}, \xi)$ -valued weakly stationary time series

Within this appendix, we set $\mathcal{H}_0 = L^2(\mathbf{V}, \mathcal{V}, \xi)$ for a σ -finite measured space $(\mathbf{V}, \mathcal{V}, \xi)$ and we assume that the Hilbert space \mathcal{H}_0 is separable with dimension $N \in \{1, 2, \dots, \infty\}$. This will allow us to use a Hilbert basis $(\phi_i)_{0 \leq i < N}$ of \mathcal{H}_0 .

1170 We first show that we can always find a version of an \mathcal{H}_0 -valued random variable which is jointly measurable on $\mathbf{V} \times \Omega$.

Proposition 8. *Let $(\mathbf{V}, \mathcal{V}, \xi)$ be a σ -finite measured space. Assume that $\mathcal{H}_0 = L^2(\mathbf{V}, \mathcal{V}, \xi)$ is separable and let Y be an \mathcal{H}_0 -valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then Y admits a version $(v, \omega) \mapsto \tilde{Y}(v, \omega)$ jointly measurable on $(\mathbf{V} \times \Omega, \mathcal{V} \otimes \mathcal{F})$.*

1175 *Proof.* Let us define for all $0 \leq n < N$, $\omega \in \Omega$, $v \in \mathbf{V}$ and $\epsilon > 0$, $S_n^Y(v, \omega) := \sum_{k=0}^n \langle Y(\omega), \phi_k \rangle \phi_k(v)$ and $N_\epsilon^Y(\omega) := \inf \left\{ n < N : \|S_n^Y(\cdot, \omega) - Y(\omega)\|_{\mathcal{H}_0}^2 \leq \epsilon \right\}$. Then it is straightforward to show that, for all $\omega \in \Omega$ and $\epsilon > 0$, $N_\epsilon^Y(\omega)$ is well defined in \mathbb{N} and that $(N_{2^{-n}}^Y(\omega))_n$ is a non-decreasing sequence. We now defined \tilde{Y} on $\mathbf{V} \times \Omega$ by $\tilde{Y}(v, \omega) = \lim_{n \rightarrow \infty} S_{N_{2^{-n}}^Y(\omega)}^Y(v, \omega)$ if the limit exists in \mathbb{C} and 0 otherwise. It follows that, for all $\omega \in \Omega$, $S_{N_{2^{-n}}^Y(\omega)}^Y(\cdot, \omega)$ converges to Y in \mathcal{H}_0 and
1180 $S_{N_{2^{-n}}^Y(\omega)}^Y(v, \omega)$ converges to $\tilde{Y}(v, \omega)$ for ξ -a.e. $v \in \mathbf{V}$ as $n \rightarrow \infty$ and that $\tilde{Y}(\cdot, \omega) = Y(\omega)$ (as elements of \mathcal{H}_0). The result follows since S_n^Y is jointly measurable on $\mathbf{V} \times \Omega$ for all $n \in \mathbb{N}$ and N_ϵ^Y is measurable on Ω for all $\epsilon > 0$. \square

Hence, an \mathcal{H}_0 -valued random variable Y can always be assumed to be represented by a $\mathbf{V} \times \Omega \rightarrow \mathbb{C}$ -measurable function \tilde{Y} . If, moreover, $Y \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$, then, by Fubini's theorem, we can see
1185 \tilde{Y} as an element of $L^2(\mathbf{V} \times \Omega, \mathcal{V} \otimes \mathcal{F}, \xi \otimes \mathbb{P})$, and we can write $\tilde{Y}(v, \omega) = \sum_{0 \leq k < N} \langle Y(\omega), \phi_k \rangle \phi_k(v)$, where the convergence holds in $L^2(\mathbf{V} \times \Omega, \mathcal{V} \otimes \mathcal{F}, \xi \otimes \mathbb{P})$. As expected, in this case, the covariance operator $\text{Cov}(Y)$ is an integral operator with kernel $(v, v') \mapsto \text{Cov}(\tilde{Y}(v, \cdot), \tilde{Y}(v', \cdot))$. It is tempting to write that $\text{Var}(\tilde{Y}(v, \cdot))$ is equal to the kernel of $\text{Cov}(Y)$ on the diagonal $\{v = v' : (v, v') \in \mathbf{V}^2\}$. However, because this diagonal set has null $\xi^{\otimes 2}$ -measure, this ‘‘equality’’ is meaningless. In the
1190 following lemma we make this statement rigorous by relying on a decomposition of the form $\text{Cov}(Y) = KK^H$ for some $K \in \mathcal{S}_2(\mathcal{H}_0)$. In particular, this can be used to give a rigorous definition of σ_W in Corollary 2 or (3.13).

Lemma 24. *Let $(\mathbf{V}, \mathcal{V}, \xi)$ be a σ -finite measured space. Assume that $\mathcal{H}_0 = L^2(\mathbf{V}, \mathcal{V}, \xi)$ is separable and let Y be an \mathcal{H}_0 -valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $K \in \mathcal{S}_2(\mathcal{H}_0)$ and denote by
1195 \mathcal{K} its kernel in $L^2(\mathbf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$. Suppose that $\text{Cov}(Y) = KK^H$. Then, we have, for ξ -a.e. $v \in \mathbf{V}$,*

$$\mathbb{E} \left[\left| \tilde{Y}(v, \cdot) \right|^2 \right] = \int |\mathcal{K}(v, v')|^2 \xi(\mathrm{d}v') = \|\mathcal{K}(v, \cdot)\|_{\mathcal{H}_0}^2, \quad (\text{B.1})$$

where \tilde{Y} is a version of Y in $L^2(\mathbf{V} \times \Omega, \mathcal{V} \otimes \mathcal{F}, \xi \otimes \mathbb{P})$.

Proof. As explained before the lemma, we have that $\tilde{Y}_n : (v, \omega) \mapsto \sum_{0 \leq k < n} \langle Y(\omega), \phi_k \rangle_{\mathcal{H}_0} \phi_k(v)$ converges to \tilde{Y} as $n \rightarrow N$ in $L^2(\mathbf{V} \times \Omega, \mathcal{V} \otimes \mathcal{F}, \xi \otimes \mathbb{P})$. Let us define, for all $v, v' \in \mathbf{V}$ and $0 \leq n \leq N$, $\mathcal{K}_n(v, v') = \sum_{0 \leq k < n} \langle \mathcal{K}(\cdot, v'), \phi_k \rangle_{\mathcal{H}_0} \phi_k(v)$. Then, using that $\mathcal{K} \in L^2(\mathbf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$, it is easy to show that \mathcal{K}_n converges to \mathcal{K} in $L^2(\mathbf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$ as $n \rightarrow N$. By the Cauchy-Schwartz inequality, the mappings $(g, h) \mapsto [v \mapsto \mathbb{E} [g(v, \cdot) \overline{h(v, \cdot)}]]$ and $(g, h) \mapsto [v \mapsto \langle g(v, \cdot), h(v, \cdot) \rangle_{\mathcal{H}_0}]$ sesquilinear continuous from $L^2(\mathbf{V} \times \Omega, \mathcal{V} \otimes \mathcal{F}, \xi \otimes \mathbb{P})$ to $L^1(\mathbf{V}, \mathcal{V}, \xi)$ and from $L^2(\mathbf{V}^2, \mathcal{V}^{\otimes 2}, \xi^{\otimes 2})$ to $L^1(\mathbf{V}, \mathcal{V}, \xi)$, respectively. This, with the two previous convergence results, shows that $v \mapsto \mathbb{E} \left[\left| \tilde{Y}_n(v, \cdot) \right|^2 \right]$ and $v \mapsto \|\mathcal{K}_n(v, \cdot)\|_{\mathcal{H}_0}^2$ both converge
in $L^1(\mathbf{V}, \mathcal{V}, \xi)$, to $\mathbb{E} \left[\left| \tilde{Y}(v, \cdot) \right|^2 \right]$ and $\|\mathcal{K}(v, \cdot)\|_{\mathcal{H}_0}^2$, respectively, that is to the left-hand side and

right-hand side of (B.1). Hence, to conclude, we only have to show that, for all $v \in \mathbb{V}$ and $0 \leq n < N$, $\mathbb{E} \left[\left| \tilde{Y}_n(v, \cdot) \right|^2 \right] = \|\mathcal{K}_n(v, \cdot)\|_{\mathcal{H}_0}^2$. To this end, we write $\mathbb{E} \left[\left| \tilde{Y}_n(v, \cdot) \right|^2 \right] = \mathbb{E} \left[\sum_{0 \leq j, k < n} \langle Y, \phi_j \rangle_{\mathcal{H}_0} \langle \phi_k, Y \rangle_{\mathcal{H}_0} \phi_j(v) \overline{\phi_k(v)} \right] = \sum_{0 \leq j, k < n} \phi_j^H \text{Cov}(Y) \phi_k \phi_j(v) \overline{\phi_k(v)}$. Using the fact that $\text{Cov}(Y) = KK^H$ and Fubini's theorem, we get

$$\phi_j^H \text{Cov}(Y) \phi_k = \int \langle \mathcal{K}(\cdot, v''), \phi_j \rangle_{\mathcal{H}_0} \overline{\langle \mathcal{K}(\cdot, v''), \phi_k \rangle_{\mathcal{H}_0}} \xi(dv'').$$

Inserting this in the previous equation and moving the double sum inside the integral with respect to $\xi(dv'')$, this double sum becomes a product of two conjugate sums. Namely, we get that

$$\mathbb{E} \left[\left| \tilde{Y}_n(v, \cdot) \right|^2 \right] = \int \left| \sum_{0 \leq k < n} \langle \mathcal{K}(\cdot, v''), \phi_k \rangle_{\mathcal{H}_0} \phi_k(v) \right|^2 \xi(dv'') = \|\mathcal{K}_n(v, \cdot)\|_{\mathcal{H}_0}^2,$$

which concludes the proof. \square

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