Perturbation Analysis of Quantum Reset Models

Géraldine Haack, Alain Joye

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Géraldine Haack* & Alain Joye †

Abstract

This paper is devoted to the analysis of Lindblad operators describing the effective dynamics of tri-partite quantum systems derived from repeated-collision models, known as Quantum Reset Models. We consider a chain of three independent subsystems, coupled by an Hamiltonian term. The two subsystems at each end of the chain are driven, independently from each other, by a reset Lindbladian, while the center system is driven by an Hamiltonian. Under generic assumptions on the coupling term, we prove the existence of a unique steady state for the perturbed reset Lindbladian, analytic in the coupling constant. We further analyze the large times dynamics of the corresponding CPTP Markov semigroup that describes the approach to the steady state. We illustrate these results with concrete exemples corresponding to realistic open quantum systems.

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1 Introduction

A major challenge when investigating small quantum systems is to assess their dynamics in out-of-equilibrium situations, i.e. when those systems are interacting with environments. An Hamiltonian approach using perturbation theory is probably the most standard way to derive an effective evolution equation for the reduced quantum system that takes a Lindblad form (CPTP map for the density operator) [13]. This derivation is possible under several approximations, the Born-Markov assumptions, valid for memoryless environments and sufficiently weak system-bath couplings [5, 24], see the review [11] for an account of mathematical results on such derivations and references. In this context, reset models, known for describing stochastic processes [6,10,12,15,21,23], have been extended to the quantum regime as a way to provide a valid Markovian approximation to assess complex dynamics of open quantum systems. Well-known examples are quantum reset models (QRM for short in the following), derived from a repeated-collision approach to describe dissipation of a quantum system coupled to a thermal bath [3, 18, 25, 27]. Assuming that the bath is a collection of qubits in a thermal state, a linear map describing the effective evolution of the reduced quantum system is obtained by tracing out the bath degrees of freedom after a global unitary evolution of the system-bath. Such models often appear under the name repeated interaction quantum systems in the mathematical physics literature, and many aspects of the Markovian dynamics they give rise to have been analysed rigorously in various settings or regimes [1, 7–9, 14, 17]. On the physics side, QRM have been successfully exploited to investigate thermodynamical properties of these systems [2, 20] and to demonstrate steady-state bipartite and multipartite entanglement [4,28,29]. In this paper, we do not to discuss the validity of QRM, but instead consider them as established models whose properties we examine in some perturbative regime.

Quantum reset models, thanks to their structural simplicity, raise the question to which extent the general properties of the dynamics they generate can be analyzed mathematically. A first step in that direction is performed in the recent work [22] where a single system driven by a Lindbladian subject to a reset process is considered. The spectral properties of the total Lindbladian perturbed by the reset processes are established, under the assumption that the unperturbed Lindbadian possesses a unique stationary state. In the present paper, we consider QRM describing the dynamics of more complex structures that are therefore intrinsically degenerate and not amenable to the cases dealt with above. We show that those degenerate QRM nevertheless allow for a complete mathematical treatment revealing a rich structure.

More precisely, we consider a tripartite structure, $A-C-B$, where A and B are the two quantum systems subject to reset processes, and C is a central system with its own free evolution. The three subsystems are weakly interacting through an Hamiltonian. This model is for instance suitable to describe a chain of small quantum systems, with

each end coupled to its own reservoir. We first prove that QRM are always characterised by Lindblad generators, the dissipators of which we make explicit. Then we analyse the spectral properties of the resulting Lindbladians and the dynamics of the tri-partite system they generate, under generic hypotheses on the coupling term. We conduct this analysis first in absence of interaction between the $A - C - B$ parts of the Hilbert space they are defined on, which gives rise to an uncoupled Lindbladian displaying large degeneracies, *i.e.* a large subspace of invariant states. Then, we introduce a generic interaction between these different parts and perform a perturbative analysis in the coupling constant. We prove uniqueness of an invariant steady state under the coupled dynamics, analytic in the coupling constant, and provide a description of the converging power series of this non-equilibrium steady state that develops in the small system. Building up on our spectral analysis, we elucidate the long time properties of the dynamics of the tri-partite system and its approach to the steady state. Finally, we focus on the case where the uncoupled system has no Hamiltonian drive and we describe in particular the emergence of a natural classical Markov process in the description of the large time behaviour of the coupled system. The paper closes with the study of two examples illustrating the key features of this analysis: the systems A and B are two qubits while the central system C is of arbitrary dimension N and the uncoupled dynamics has no Hamiltonian drive. For a rather general choice of QRM coupled dynamics, we compute the leading order of the steady state for N arbitrary and, for $N = 2$ – when C is another qubit – we determine the steady state up to order three in the coupling constant as well as the associated classical Markov process.

2 Mathematical framework

2.1 Simple Hilbert space setup

As a warmup, we consider a single quantum system of finite dimension characterized by its Hamiltonian H defined on its Hilbert space $\mathcal{H}$ which is coupled to M reservoirs. QRMs assume the state of the quantum system to be reset to a given state τ_l with probablity $\gamma_l\, dt$ within each time interval dt. The QRM-type evolution equation is given by:

$$\dot{\rho}(t) = -i[H, \rho] + \sum_{l=1}^{M} \gamma_l(\tau_l \operatorname{tr}(\rho) - \rho)\,. \tag{2.1}$$

The operator ρ is the reduced density operator of the system defined on $\mathcal{H}$, and γ_l characterizes the coupling rate to the reservoir l, $l = 1, \ldots, M$.

For the sake of comparison with our main concern –tri-partite systems– and to set the notation, we discuss the dynamics of QRM defined in this simple setup. We provide a full description of its generic features, under the following assumptions.

Gen:
Let $\mathcal{H}$ be a Hilbert space, with $\dim \mathcal{H} = N < \infty$. The dissipative part of the generator is characterised by

- $\{\tau_l\}_{1\leq l\leq M}$ a collection of density matrices on $\mathcal{H}$, *i.e.* $\tau_l \in \mathcal{B}(\mathcal{H})$, with $\tau_l \geq 0$ and $\operatorname{tr}(\tau_l) = 1$, for all $l \in 1, \ldots, M$,
- $\gamma_l > 0$, $l \in 1, \ldots, M$, the collection of associated non-zero rates for the coupling to the M baths.

The Hamiltonian part of the generator, $H = H^* \in \mathcal{B}(\mathcal{H})$, is generic in the spectral sense

- $\sigma(H) = \{e_1, e_2, \dots, e_N\}$, consists of simple eigenvalues with associated normalised eigenvectors denoted by $\{\varphi_j\}_{1\leq j\leq N}$, *i.e.* $H\varphi_j = e_j\varphi_j$, $j \in \{1, \cdots, N\}$,
- The differences (Bohr frequencies) $\{e_j - e_k\}_{j\neq k}$ are all distinct.

The generator of QRM is thus the (super-)operator $\mathcal{L} \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ defined by

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_{l=1}^{M} \gamma_l(\tau_l \,\mathrm{tr}(\rho) - \rho), \tag{2.2}$$

where ρ here is arbitrary in $\mathcal{B}(\mathcal{H})$, such that the dynamics of the QRM reads

$$\dot\rho(t) = \mathcal{L}(\rho(t)), \quad t \in (0, \infty), \quad \rho(0) = \rho_0 \in \mathcal{B}(\mathcal{H}). \tag{2.3}$$

In case $\rho \in \mathcal{DM}(\mathcal{H})$, the set of density matrices $\mathcal{DM}(\mathcal{H}) = \{\rho \in \mathcal{B}(\mathcal{H}) \,|\, \rho \geq 0, \mathrm{tr}(\rho) = 1\}$, the trace factor in (2.2) disappears. Indeed, we will see below in a more general framework that the operator $\mathcal{L}$ enjoys further symmetries, being a Lindblad operator, see Proposition 3.2; in particular if $\rho_0 \in \mathcal{DM}(\mathcal{H})$, $\rho(t) \in \mathcal{DM}(\mathcal{H})$, for all $t \in [0, \infty)$.

However, we perform the full spectral analysis of $\mathcal{L}$ as an operator on $\mathcal{B}(\mathcal{H})$ and, accordingly, solve the equation (2.3) without resorting to these symmetries.

We first combine the density matrices τ_l with corresponding rates γ_l into a single density matrix T with corresponding rate Γ: Setting

$$\Gamma = \sum_{l=1}^{M} \gamma_l > 0, \quad T = \frac{1}{\Gamma} \sum \gamma_l\tau_l \in \mathcal{DM}(\mathcal{H}), \tag{2.4}$$

we get that (2.2) writes

$$\mathcal{L}(\rho) = -i[H, \rho] + \Gamma(T\mathrm{tr}(\rho) - \rho). \tag{2.5}$$

In the sequel, we denote the matrix elements of any $A \in \mathcal{B}(\mathcal{H})$ in the basis $\{\varphi_j\}_{1\leq j\leq N}$ by $A_{jk} = \langle\varphi_j|A\varphi_k\rangle$, and the operator $|\varphi\rangle\langle\psi| \in \mathcal{B}(\mathcal{H})$, for $\varphi, \psi \in \mathcal{H}$, is defined by $|\varphi\rangle\langle\psi| : \eta \mapsto \varphi\langle\psi|\eta\rangle$.

Lemma 2.1 *Under our assumptions* **Gen**, *the operator* $\mathcal{L} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *defined by (2.5) is diagonalisable with spectrum given by*

$$\sigma(\mathcal{L}) = \{0, -\Gamma\} \cup \{-i(e_j - e_k) - \Gamma\}_{j\neq k}\}. \tag{2.6}$$

All eigenvalues are simple, except $-\Gamma$ *which has multiplicity* $N - 1$.
Moreover, the solution to (2.3) reads

$$\rho(t) = e^{-t(i[H,\cdot]+\Gamma)}\big(\rho_0 - \mathrm{tr}(\rho_0)\Gamma\big(i[H, \cdot] + \Gamma\big)^{-1}(T)\big) + \mathrm{tr}(\rho_0)\Gamma\big(i[H, \cdot] + \Gamma\big)^{-1}(T). \tag{2.7}$$

Expressed in the eigenbasis of H, *this means that, with* $\lambda_{jk} = i(e_j - e_k) + \Gamma$,

$$\rho_{jk}(t) = e^{-t\lambda_{jk}}\rho_{0jk} + \mathrm{tr}(\rho_0)\Gamma\frac{T_{jk}}{\lambda_{jk}}\big(1 - e^{-t\lambda_{jk}}\big), \quad \textit{for all} \;\; 1 \leq j, k \leq N. \tag{2.8}$$

Remark 2.2 *i) In the limit* $t \to \infty$ *the steady state is independent of the initial condition and reads*

$$\rho^{SS} \equiv \lim_{t\to\infty} \rho(t) = \Gamma\big(i[H, \cdot] + \Gamma\big)^{-1}(T) \tag{2.9}$$

ii) In particular, for $\rho_0 \in \mathcal{DM}(\mathcal{H})$, *all populations decay to* T_{jj} *at the same exponential rate without oscillations* $\rho_{jj}(t) = e^{-t\Gamma}\rho_{0jj} + T_{jj}\big(1 - e^{-t\Gamma}\big)$.

Proof. We first deal with the dynamical aspects and note that $\mathcal{L}(\cdot) = -(i[H,\cdot] + \Gamma\cdot) + \Gamma T \mathrm{tr}(\cdot)$, with tr $T = 1$ implies $\mathrm{tr}\mathcal{L}(\rho) = 0$ for any $\rho \in \mathcal{B}(\mathcal{H})$, so that the trace is conserved by (2.3). Hence, considering the jk matrix element of the differential equation (2.3) we get

$$\dot{\rho}_{jk} = -\lambda_{jk}\rho_{jk} + \Gamma T_{jk}\mathrm{tr}(\rho_0) \quad \text{where} \quad \lambda_{jk} \neq 0, \tag{2.10}$$

which yields (2.8). The basis independent formulation (2.7) follows by the decomposition $\rho = \sum_{1\leq j,k\leq N} \rho_{jk}|\varphi_j\rangle\langle\varphi_k|$ and the observation

$$i[H, |\varphi_j\rangle\langle\varphi_k|] + \Gamma|\varphi_j\rangle\langle\varphi_k|) = \lambda_{jk}|\varphi_j\rangle\langle\varphi_k|, \tag{2.11}$$

which yields $\big(i[H,\cdot]+\Gamma\big)^{-1}(T)_{jk} = T_{jk}/\lambda_{jk}$.

On the spectral side, the observation above immediately yields $\mathcal{L}(|\varphi_j\rangle\langle\varphi_k|) = -\lambda_{jk}|\varphi_j\rangle\langle\varphi_k|$ for $j \neq k$, showing $\{-\lambda_{jk}\}_{j\neq k}$ are simple eigenvalues by our genericity assumption. To compute the other nonzero eigenvalues of $\mathcal{L}$, we note that if ρ is an eigenvector of $\mathcal{L}$ associated with an eigenvalue λ, then $\lambda \mathrm{tr}\rho = 0$. Hence $\lambda \neq 0$ implies $\mathrm{tr}\rho = 0$. Thus, considering the $N-1$ dimensional subspace of diagonal traceless matrices in the eigenbasis of H, $\{\rho = \sum_{1\leq j\leq N} r_j|\varphi_j\rangle\langle\varphi_j| \mid \sum_{1\leq j\leq N} r_j = 0\}$, and making use of the identity $\mathcal{L}(|\varphi_j\rangle\langle\varphi_j|) = \Gamma(T - |\varphi_j\rangle\langle\varphi_j|)$, for any j, we see that it coincides with $\mathrm{Ker}\,(\mathcal{L}+\Gamma\mathbb{I})$. Finally, the one-dimensional kernel of $\mathcal{L}$ is spanned by $\Gamma\big(i[H,\cdot]+\Gamma\big)^{-1}(T)$: the inverse is well defined thanks to (2.11), it has matrix elements $\Gamma T_{jk}/\lambda_{jk}$, and trace one. Thus

$$\mathcal{L}(\Gamma\big(i[H,\cdot]+\Gamma\big)^{-1}(T)) = -\Gamma\big(i[H,\cdot]+\Gamma\cdot\big)\big(\big(i[H,\cdot]+\Gamma\big)^{-1}(T)\big) + \Gamma T = 0. \tag{2.12}$$

□

2.2 Tri-partite Hilbert spaces

We define here the tri-partite systems whose dynamical properties are studied in this paper.

Consider $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B$, where $\mathcal{H}_\#$ are Hilbert spaces, with dimensions noted $n_\# < \infty$, where $\# \in \{A, B, C\}$. Let $\tau_A \in \mathcal{DM}(\mathcal{H}_A)$, $\tau_B \in \mathcal{DM}(\mathcal{H}_B)$ be two density matrices on their respective Hilbert space and $\gamma_A, \gamma_B > 0$ two positive rates. Consider three Hamiltonians H_A, H_B, H_C on their respective Hilbert space that further satisfy

$$[H_A, \tau_A] = 0, \quad \text{and} \quad [H_B, \tau_B] = 0, \tag{2.13}$$

while H_C is arbitrary at this point. In applications, the reset state $\tau_\#$ will typically be defined as a Gibbs state at some inverse temperature $\beta_\#$ associated to $H_\#$; *i.e.* $\tau_\# = e^{-\beta_\# H_\#}/Z_\#$ which satisfies (2.13), where $Z_\#$ is the corresponding partition function. In Sec.3, we perform the analysis of the uncoupled case (system $A - C - B$ is non-interacting), and in Sec.4, we make use of analytic perturbation theory to treat the case where a weak interaction is added to the system $A - C - B$.

3 The non-interacting tripartite QRM

We define the *uncoupled* QRM by the generator

$$\begin{aligned}\mathcal{L}(\rho) = &- i[H_A \otimes \mathbb{I}_C \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_C \otimes \mathbb{I}_B + \mathbb{I}_A \otimes \mathbb{I}_C \otimes H_B, \rho] \\ &+ \gamma_A(\tau_A \otimes \mathrm{tr}_A(\rho) - \rho) + \gamma_B(\mathrm{tr}_B(\rho) \otimes \tau_B - \rho),\end{aligned} \tag{3.1}$$

where $\mathbb{I}_{\#}$ denotes the identity operator on $\mathcal{H}_{\#}$ and $\mathrm{tr}_{\#}$ denotes the operator on the tensor product of Hilbert spaces with indices different from #, obtained by taking the partial trace over $\mathcal{H}_{\#}$. For later purposes, $\mathrm{tr}_{\#\#'}$ denotes the operator on the Hilbert space with index different from # and #′ obtained by taking the partial trace over $\mathcal{H}_{\#} \otimes \mathcal{H}_{\#'}$. For example,

$$\mathrm{tr}_A : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_B), \quad \mathrm{tr}_{AB} : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_C) \tag{3.2}$$

will be viewed as linear maps. We shall abuse notations and write $H_{\#}$ for the Hamiltonian both on $\mathcal{H}_{\#}$ and $\mathcal{H}$, the context making it clear what we mean. Also, we shall denote the non-Hamiltonian part of the generator by

$$\mathcal{D}(\rho) = \gamma_A(\tau_A \otimes \mathrm{tr}_A(\rho) - \rho) + \gamma_B(\mathrm{tr}_B(\rho) \otimes \tau_B - \rho), \tag{3.3}$$

so that $\mathcal{L}(\rho) = -i\big[H_A + H_C + H_B, \rho\big] + \mathcal{D}(\rho)$.

Remark 3.1 *If $n_B = 1$, $\mathcal{H}_B \simeq \mathbb{C}$ and the last tensor product is trivial. Hence the QRM reduces to $\mathcal{L}(\rho) = -i\big[H_A + H_C, \rho\big] + \gamma_A(\tau_A \otimes \mathrm{tr}_A(\rho) - \rho)$ on $\mathcal{H}_A \otimes \mathcal{H}_C$, while keeping $\gamma_B > 0$.*

Let us start by a structural result showing that the QRM at time t, $e^{t\mathcal{L}}(\rho_0)$, with ρ_0 a state, is a CPTP map, by proving that its generator takes the form of a Lindblad operator. More precisely, one needs to show that the non-Hamiltonian part of their generator (3.1) takes the form of a dissipator, *i.e.*

$$\sum_j A_j \rho A_j^* - \frac{1}{2}\{A_j^* A_j, \rho\} = \sum_j \frac{1}{2}\Big\{[A_j\rho, A_j^*] + [A_j, \rho A_j^*]\Big\}, \quad \text{for} \quad A_j \in \mathcal{B}(\mathcal{H}). \tag{3.4}$$

Given (3.1), it is enough to consider $\tau_A \otimes \mathrm{tr}_A(\rho) - \rho$ defined on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_C$.

Proposition 3.2 *Let $\tau_A = \sum_k t_k |\varphi_k\rangle\langle\varphi_k|$ be the spectral decomposition of τ_A, where $\{\varphi_k\}_k$ is a complete orthonormal basis of $\mathcal{H}_A$. Then*

$$\tau_A \otimes \mathrm{tr}_A(\rho) - \rho = \sum_{j,k} \big(A_{jk}\rho A_{jk}^* - \frac{1}{2}\{A_{jk}^* A_{jk}, \rho\}\big), \quad \textit{where} \quad A_{jk} = \sqrt{t_j}|\varphi_j\rangle\langle\varphi_k| \otimes \mathbb{I}_C. \tag{3.5}$$

Remark 3.3 *i) This result applies to the non-Hamiltonian part of the generator of QRM defined on a simple Hilbert space as well, by considering $\mathcal{H}_C = \mathbb{C}$, in which case tr_A reduces to the scalar valued trace.*
ii) The operators A_{jk} can be replaced by $\sqrt{t_j}|\varphi_j\rangle\langle\psi_k| \otimes \mathbb{I}_C$, where $\{\psi_k\}_k$ is any orthonormal basis of $\mathcal{H}_A$ without altering the result.

Proof. One computes $A_{jk}^* A_{jk} = t_j |\varphi_k\rangle\langle\varphi_k| \otimes \mathbb{I}_C$ so that $\sum_{j,k} A_{jk}^* A_{jk} = \mathbb{I}$ on $\mathcal{H}$, using $\sum_j t_j = 1 = \mathrm{tr}(\tau_A)$. Hence the anticommutator term yields ρ. Finally,

$$\begin{aligned} \sum_{jk} A_{jk}\rho A_{jk}^* &= \sum_{j,k} t_j \big(|\varphi_j\rangle\langle\varphi_k| \otimes \mathbb{I}_C\big)\rho\big(|\varphi_k\rangle\langle\varphi_j| \otimes \mathbb{I}_C\big) \\ &= \sum_j t_j |\varphi_j\rangle\langle\varphi_j| \otimes \sum_k \big(\langle\varphi_k| \otimes \mathbb{I}_C\big)\rho\big(|\varphi_k\rangle \otimes \mathbb{I}_C\big) = \tau_A \otimes \mathrm{tr}_A(\rho). \end{aligned} \tag{3.6}$$

□

3.1 Spectrum of the uncoupled QRM

We proceed by analysing the spectrum of the uncoupled QRM $\mathcal{L}$ (3.1) in the tri-partite case, making use of the fact that, by construction, the Hamiltonian part of the decoupled QRM commutes with the dissipator as we quickly check:

$$[H_A, \cdot] \circ (\tau_A \otimes \mathrm{tr}_A(\cdot))(\rho) = [H_A, \tau_A \otimes \mathrm{tr}_A(\rho)] = [H_A, \tau_A] \otimes \mathrm{tr}_A(\rho) = 0, \tag{3.7}$$

since τ_A and H_A commute, while

$$(\tau_A \otimes \mathrm{tr}_A(\cdot)) \circ [H_A, \cdot](\rho) = \tau_A \otimes (\mathrm{tr}_A(H_A\rho) - \mathrm{tr}_A(\rho H_A)) = 0, \tag{3.8}$$

using $\mathrm{tr}_A(\cdot) = \sum_j \langle\varphi_j^A| \otimes \mathbb{I} \ \cdot \ |\varphi_j^A\rangle \otimes \mathbb{I}$ with $\{\varphi_j^A\}_{1\leq j\leq n_A}$ an orthonormal basis of eigenvectors of H_A. Now, replacing H_A by H_B (or H_C for that matter) yields

$$\begin{aligned} &[H_B, \cdot] \circ (\tau_A \otimes \mathrm{tr}_A(\cdot))(\rho) = \tau_A \otimes [H_B, \mathrm{tr}_A(\rho)], \text{ and} \\ &(\tau_A \otimes \mathrm{tr}_A(\cdot)) \circ [H_B, \cdot](\rho) = \tau_A \otimes (\mathrm{tr}_A(H_B\rho) - \mathrm{tr}_A(\rho H_B)) = \tau_A \otimes [H_B, \mathrm{tr}_A(\rho)], \end{aligned} \tag{3.9}$$

since H_B commutes with $\langle\varphi_j^A| \otimes \mathbb{I}$ and $|\varphi_j^A\rangle \otimes \mathbb{I}$. Altogether, the dissipator and the Hamiltonian parts of $\mathcal{L}$ admit a common basis of eigenvectors that we now determine.

Let us start with the dissipator and its spectral properties.

Proposition 3.4 *The dissipator, as an operator on $\mathcal{B}(\mathcal{H})$, admit the following spectral decomposition*

$$\begin{aligned} &\sigma(\gamma_A(\tau_A \otimes \mathrm{tr}_A(\cdot) - \mathbb{I}) + \gamma_B(\mathrm{tr}_B(\cdot) \otimes \tau_B - \mathbb{I})) = \{0, -\gamma_A, -\gamma_B, -(\gamma_A + \gamma_B)\} \\ &\gamma_A(\tau_A \otimes \mathrm{tr}_A(\cdot) - \mathbb{I}) + \gamma_B(\mathrm{tr}_B(\cdot) \otimes \tau_B - \mathbb{I}) = 0Q_0 - \gamma_A Q_A - \gamma_B Q_B - (\gamma_A + \gamma_B)Q_{AB}, \end{aligned}$$

where the spectral projectors $Q_\#$, $\# \in \{0, A, B, AB\}$ are given by

$$\begin{aligned} &Q_0(\rho) = \tau_A \otimes \mathrm{tr}_{AB}(\rho) \otimes \tau_B, \quad Q_{AB}(\rho) = \rho - \mathrm{tr}_B(\rho) \otimes \tau_B - \tau_A \otimes \mathrm{tr}_A(\rho) + \tau_A \otimes \mathrm{tr}_{AB}(\rho) \otimes \tau_B, \\ &Q_A(\rho) = \big(\mathrm{tr}_B(\rho) - \tau_A \otimes \mathrm{tr}_{AB}(\rho)\big) \otimes \tau_B, \quad Q_B(\rho) = \tau_A \otimes \big(\mathrm{tr}_A(\rho) - \mathrm{tr}_{AB}(\rho) \otimes \tau_B\big). \end{aligned}$$

Moreover, the different spectral subspaces in $\mathcal{B}(\mathcal{H})$ are

$$\begin{aligned} &\mathrm{Ran}\,\mathrm{Q}_0 = \mathrm{span}\{\tau_\mathrm{A} \otimes \rho_\mathrm{C} \otimes \tau_\mathrm{B}\}_{\rho_\mathrm{C} \in \mathcal{B}(\mathcal{H}_\mathrm{C})}, && \text{s.t. } \dim(\mathrm{Q}_0) = \mathrm{n}_\mathrm{C}^2 \\ &\mathrm{Ran}\,\mathrm{Q}_\mathrm{A} = \mathrm{span}\{\rho_\mathrm{AC} \otimes \tau_\mathrm{B} \,|\, \mathrm{tr}_\mathrm{A}(\rho_\mathrm{AC}) = 0\}_{\rho_\mathrm{AC} \in \mathcal{B}(\mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{C})}, && \text{s.t. } \dim(\mathrm{Q}_\mathrm{A}) = (\mathrm{n}_\mathrm{A}^2 - 1)\mathrm{n}_\mathrm{C}^2 \\ &\mathrm{Ran}\,\mathrm{Q}_\mathrm{B} = \mathrm{span}\{\tau_\mathrm{A} \otimes \rho_\mathrm{CB} \,|\, \mathrm{tr}_\mathrm{B}(\rho_\mathrm{CB}) = 0\}_{\rho_\mathrm{CB} \in \mathcal{B}(\mathcal{H}_\mathrm{C} \otimes \mathcal{H}_\mathrm{B})}, && \text{s.t. } \dim(\mathrm{Q}_\mathrm{B}) = (\mathrm{n}_\mathrm{B}^2 - 1)\mathrm{n}_\mathrm{C}^2 \\ &\mathrm{Ran}\,\mathrm{Q}_\mathrm{AB} = \mathrm{span}\{\mathrm{tr}_\mathrm{A}(\rho) = 0, \mathrm{tr}_\mathrm{B}(\rho) = 0\}_{\rho \in \mathcal{B}(\mathcal{H})}, && \text{s.t. } \dim(\mathrm{Q}_\mathrm{AB}) = (\mathrm{n}_\mathrm{A}^2 - 1)(\mathrm{n}_\mathrm{B}^2 - 1)\mathrm{n}_\mathrm{C}^2. \end{aligned}$$

Remark 3.5 *i) In case $\gamma_A = \gamma_B$, there are only three distinct eigenvalues and the corresponding spectral projector is $Q_A + Q_B$.*
ii) Being spectral projectors, the $Q_\#$'s, $\# \in \{0, A, B, AB\}$ satisfy $Q_0 + Q_A + Q_B + Q_{AB} = \mathbb{I}$ and $Q_\# Q_{\#'} = \delta_{\#,\#'} Q_\#$
iii) The dimensions referred to correspond to complex dimensions for $\mathcal{B}(\mathcal{H})$.
iv) The result essentially follows from the observation that $\tau_A \otimes \mathrm{tr}_A(\cdot)$ and $\mathrm{tr}_B(\cdot) \otimes \tau_B$ are commuting projectors.

Proof. We start with point iv) of Remark 3.5. For any ρ in $\mathcal{B}(\mathcal{H})$,

$$(\tau_A \otimes \mathrm{tr}_A(\cdot) \circ \mathrm{tr}_B(\cdot) \otimes \tau_B)(\rho) = \tau_A \otimes \mathrm{tr}_{AB}(\rho) \otimes \tau_B = (\mathrm{tr}_B(\cdot) \otimes \tau_B \circ \tau_A \otimes \mathrm{tr}_A(\cdot))(\rho) \tag{3.10}$$

while $\tau_A \otimes \mathrm{tr}_A(\cdot) \circ \tau_A \otimes \mathrm{tr}_A(\cdot) = \tau_A \otimes \mathrm{tr}_A(\cdot)$, and similarly for $\mathrm{tr}_B(\cdot) \otimes \tau_B$. Hence the dissipator is a linear combination of two commuting projectors to which we can apply the next Lemma.

Lemma 3.6 *Let $P, Q \in \mathcal{B}(\mathcal{H})$ such that $P^2 = P, Q^2 = Q$, and $[P, Q] = 0$. Then, for any $\alpha, \beta \in \mathbb{C}$, the identity*

$$\alpha P + \beta Q = 0(\mathbb{I} - P)(\mathbb{I} - Q) + \alpha P(\mathbb{I} - Q) + \beta Q(\mathbb{I} - P) + (\alpha + \beta)PQ, \tag{3.11}$$

provides the spectral decomposition of $\alpha P + \beta Q$, so that $\sigma(\alpha P + \beta Q) = \{0, \alpha, \beta, \alpha + \beta\}$, with respective spectral projectors $(\mathbb{I} - P)(\mathbb{I} - Q), P(\mathbb{I} - Q), Q(\mathbb{I} - P), PQ$, and no eigennilpotent.

The proof of the Lemma is immediate, and in case some eigenvalues coincide, the corresponding spectral projector is simply the sum of the individual projectors.

The identifications $P = \mathbb{I} - \tau_A \otimes \mathrm{tr}_A(\cdot)$, $Q = \mathbb{I} - \mathrm{tr}_B(\cdot) \otimes \tau_B$, $\alpha = -\gamma_A$, $\beta = -\gamma_B$ yield the announced spectral decomposition of the dissipator, together with the explicit spectral projectors. A direct verification then gives the corresponding spectral subspaces. □

The eigenvectors of the Hamiltonian part of $\mathcal{L}$ are readily computed. For $\# \in \{A, B, C\}$, let $\{\varphi_j^\#\}_{1 \le j \le n_\#}$ be an orthonormal basis of $\mathcal{H}_\#$ of eigenvectors of $H_\#$, with associated eigenvalues $e_j^\#$, $1 \le j \le n_\#$. The eigenvalues need not to be distinct at that point. We denote by $P_{j,k}^\# \in \mathcal{B}(\mathcal{H}_\#)$, $j, k \in \{1, \cdots, n_\#\}$, the operators $P_{j,k}^\# = |\varphi_j^\#\rangle\langle\varphi_k^\#|$ that yield a basis of eigenvectors of the Hamiltonian part of (3.1) of $\mathcal{B}(\mathcal{H})$):

$$\begin{aligned} -i\big[H_A + H_C + H_B, & P_{j,k}^A \otimes P_{j',k'}^C \otimes P_{j'',k''}^B\big] \\ &= -i(e_j^A - e_k^A + e_{j'}^C - e_{k'}^C + e_{j''}^B - e_{k''}^B) P_{j,k}^A \otimes P_{j',k'}^C \otimes P_{j'',k''}^B. \end{aligned} \tag{3.12}$$

It remains to take into account the role of the trace in the spectral subspaces of the dissipator to get the sought for common basis of eigenvectors of (3.3). To do so, we introduce the $n_\# - 1$ dimensional basis of diagonal (w.r.t. to the eigenbasis of $H_\#$) traceless matrices

$$\Delta_j^\# = |\varphi_j^\#\rangle\langle\varphi_j^\#| - |\varphi_{j+1}^\#\rangle\langle\varphi_{j+1}^\#|, \quad j = 1, 2, \ldots, n_\# - 1, \tag{3.13}$$

such that $[H_\#, \Delta_j^\#] = 0$. Together with $\tau_\#$, the $\Delta_j^\#$'s form a basis of diagonal matrices. Proposition 3.4 then provides the full spectral analysis of the uncoupled QRM .

Proposition 3.7 *The vectors listed below form a basis of $\mathcal{B}(\mathcal{H})$ consisting in eigenvectors associated with the mentioned eigenvalue of the uncoupled QRM*

$\mathcal{L}(\cdot) = -i\big[H_A + H_C + H_B, \cdot\big] + \mathcal{D}(\cdot)$ *defined on* $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B$ *by (3.1):*

$$\tau_A \otimes P^C_{j',k'} \otimes \tau_B \;\leftrightarrow\; -i(e^C_{j'} - e^C_{k'}), \qquad 1 \le j', k' \le n_C$$

$$\Delta^A_j \otimes P^C_{j',k'} \otimes \tau_B \;\leftrightarrow\; -\gamma_A - i(e^C_{j'} - e^C_{k'}), \qquad 1 \le j \le n_A - 1,\ 1 \le j', k' \le n_C$$

$$P^A_{j,k} \otimes P^C_{j',k'} \otimes \tau_B \;\leftrightarrow\; -\gamma_A - i(e^A_j - e^A_k + e^C_{j'} - e^C_{k'}), \qquad 1 \le j \ne k \le n_A,\ 1 \le j', k' \le n_C$$

$$\tau_A \otimes P^C_{j',k'} \otimes \Delta^B_{j''} \;\leftrightarrow\; -\gamma_B - i(e^C_{j'} - e^C_{k'}), \qquad 1 \le j'' \le n_B - 1,\ 1 \le j', k' \le n_C$$

$$\tau_A \otimes P^C_{j',k'} \otimes P^B_{j'',k''} \;\leftrightarrow\; -\gamma_B - i(e^C_{j'} - e^C_{k'} + e^B_{j''} - e^B_{k''}), \qquad 1 \le j'' \ne k'' \le n_B,\ 1 \le j', k' \le n_C$$

$$\Delta^A_j \otimes P^C_{j',k'} \otimes \Delta^B_{j''} \;\leftrightarrow\; -(\gamma_A + \gamma_B) - i(e^C_{j'} - e^C_{k'}), \qquad 1 \le j \le n_A - 1,\ 1 \le j'' \le n_B - 1,\ 1 \le j', k' \le n_C$$

$$\Delta^A_j \otimes P^C_{j',k'} \otimes P^B_{j'',k''} \;\leftrightarrow\; -(\gamma_A + \gamma_B) - i(e^C_{j'} - e^C_{k'} + e^B_{j''} - e^B_{k''}), \qquad 1 \le j \le n_A - 1,\ 1 \le j'' \ne k'' \le n_B,\ 1 \le j', k' \le n_C$$

$$P^A_{j,k} \otimes P^C_{j',k'} \otimes \Delta^B_{j''} \;\leftrightarrow\; -(\gamma_A + \gamma_B) - i(e^A_j - e^A_k + e^C_{j'} - e^C_{k'}), \qquad 1 \le j \ne k \le n_A,\ 1 \le j'' \ne n_B - 1,\ 1 \le j', k' \le n_C$$

$$P^A_{j,k} \otimes P^C_{j',k'} \otimes P^B_{j'',k''} \;\leftrightarrow\; -(\gamma_A + \gamma_B) - i(e^A_j - e^A_k + e^C_{j'} - e^C_{k'} + e^B_{j''} - e^B_{k''}), \qquad 1 \le j \ne k \le n_A,\ 1 \le j'' \ne k'' \le n_B,\ 1 \le j', k' \le n_C$$

Remark 3.8 *0) The Hamiltonians* $\mathcal{H}_\# \in \mathcal{B}(\mathcal{H}_\#)$ *are arbitrary at that point.*
i) The uncoupled reset model Lindbladian $\mathcal{L}$ *is thus diagonalisable, with eigenvalues located on the (generically) four vertical lines* $\Re z = 0$, $\Re z = -\gamma_A$, $\Re z = -\gamma_B$, $\Re z = -(\gamma_A + \gamma_B)$ *in the complex plane, symmetrically with respect to the real axis.*
ii) In particular, the kernel of $\mathcal{L}$ *is degenerate, since* $\dim \mathrm{Ker}\, \mathcal{L}(\cdot) \ge n_C$.

The spectral projectors of $\mathcal{L}$ can be constructed explicitely, making use of the next Lemma:

Lemma 3.9 *Consider a Hilbert space* $\mathcal{H}$ *and* $\tau \in \mathcal{B}(\mathcal{H})$ *a density matrix. Let* $\{\varphi_j\}_{1 \le j \le n}$ *be an orthonormal basis of eigenvectors of* τ *for* $\mathcal{H}$. *Consider the basis of* $\mathcal{B}(\mathcal{H})$ *given by*

$$P_{jk} = |\varphi_j\rangle\langle\varphi_k|,\ 1 \le j \ne k \le n,\ \Delta_j = |\varphi_j\rangle\langle\varphi_j| - |\varphi_{j+1}\rangle\langle\varphi_{j+1}|,\ 1 \le j \le n-1,\ \textit{and}\ \ \tau. \tag{3.14}$$

Set $\sigma_j = \sum_{k=1}^{j} |\varphi_k\rangle\langle\varphi_k|$, $1 \le j \le n$. *Then the operators on* $\mathcal{B}(\mathcal{H})$ *defined by*

$$\begin{aligned} &Q_{jk}(\cdot) = P_{jk}\mathrm{tr}(P^*_{jk}\,\cdot),\ 1 \le j \ne k \le n, \\ &Q_j(\cdot) = \Delta_j \mathrm{tr}(\sigma_j(\,\cdot - \tau\mathrm{tr}(\cdot))),\ 1 \le j \le n-1,\ \textit{and}\ \ Q_0(\cdot) = \tau\mathrm{tr}(\mathbb{I}\,\cdot) \end{aligned} \tag{3.15}$$

yield a complete set of rank one projectors onto the span of the corresponding basis vectors of (3.14) so that the composition of any two of them equals zero.

Remark 3.10 *The spectral projectors of* $\mathcal{L}$ *corresponding to Proposition 3.7 are then given by the appropriate tensor products of projectors (3.15).*

The solution to $\dot{\rho} = \mathcal{L}(\rho)$, $\rho(0) = \rho_0$ follows immediately by expanding ρ_0 along those eigenvectors. In particular, one gets for this uncoupled QRM model

$$\rho(t) = \tau_A \otimes (e^{-i[H_C,\cdot]t}\mathrm{tr}_{AB}(\rho_0)) \otimes \tau_B + \mathcal{O}(e^{-t\min\{\gamma_A,\gamma_B\}}), \quad t \geq 0, \tag{3.16}$$

where $e^{-i[H_C,\cdot]t}\mathrm{tr}_{AB}(\rho_0)$ satisfies the Hamiltonian evolution equation $\dot{\rho}_C = -i[H_C, \rho_C]$, $\rho_C(0) = \mathrm{tr}_{AB}(\rho_0)$ on $\mathcal{H}_C$, as expected in this uncoupled context.

4 The weakly-interacting tripartite QRM

We consider now the coupled QRM defined by the Lindblad generator on $\mathcal{B}(\mathcal{H})$, with $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B$,

$$\mathcal{L}_g(\rho) = \mathcal{L}(\rho) - ig[H, \rho] \equiv \mathcal{L}_0(\rho) + g\mathcal{L}_1(\rho) \tag{4.1}$$

where $H = H^* \in \mathcal{B}(\mathcal{H})$ is a Hamiltonian that effectively couples the different Hilbert spaces $\mathcal{H}_\#$, while $g \in \mathbb{R}$ is a coupling constant. We focus on the determination of the kernel of $\mathcal{L}_g$, as $g \to 0$, which describes the asymptotic state of the system driven by $\mathcal{L}_g$, under generic hypotheses. Then we turn to the consequences for the dynamics generated by $\mathcal{L}_g$.

4.1 Leading order analytic perturbation theory

When $g = 0$, Proposition 3.7 shows that

$$\mathrm{Ker}\,\mathcal{L}_0 \supset \mathrm{span}\big\{\tau_{\mathrm{A}} \otimes |\varphi_{\mathrm{j}}^{\mathrm{C}}\rangle\langle\varphi_{\mathrm{j}}^{\mathrm{C}}| \otimes \tau_{\mathrm{B}}\big\}_{1\leq \mathrm{j} \leq \mathrm{n_C}}, \tag{4.2}$$

whatever the properties of the Hamiltonian H_C. In case the Hamiltonian H_C is trivial,

$$H_C = 0 \;\Rightarrow\; \mathrm{Ker}\,\mathcal{L}_0 = \mathrm{span}\big\{\tau_{\mathrm{A}} \otimes \rho_{\mathrm{C}} \otimes \tau_{\mathrm{B}}\big\}_{\rho_{\mathrm{C}} \in \mathcal{B}(\mathcal{H}_{\mathrm{C}})} \tag{4.3}$$

has dimension n_C^2, and the corresponding spectral projector coincides with Q_0, the spectral projector on $\mathrm{Ker}\,\mathcal{D}$, see Proposition 3.4. We shall consider below both cases, which give rise to different results. In order to avoid accidental degeneracies when $H_C \neq 0$, we will assume H_C satisfies the spectral hypothesis

Spec(H_C):
The spectrum of $H_C \in \mathcal{B}(\mathcal{H}_C)$ is simple and the Bohr frequencies $\{e_j^C - e_k^C\}_{j\neq k}$ are distinct.

Under this assumption, we have

$$\mathrm{Ker}\,\mathcal{L}_0 = \mathrm{span}\big\{\tau_{\mathrm{A}} \otimes \rho_{\mathrm{C}} \otimes \tau_{\mathrm{B}},\ \text{s.t. } [\rho_{\mathrm{C}}, \mathrm{H_C}] = 0\big\}, \tag{4.4}$$

which is of dimension n_C. The corresponding spectral projector acts as follows

$$Q_0(\rho) = \tau_A \otimes \mathrm{Diag}_C(\mathrm{tr}_{AB}(\rho)) \otimes \tau_B, \tag{4.5}$$

where the projector $\mathrm{Diag}_C : \mathcal{B}(\mathcal{H}_C) \to \mathcal{B}(\mathcal{H}_C)$ defined by

$$\mathrm{Diag}_C(\cdot) = \sum_{j=1}^{n_C} |\varphi_j^C\rangle\langle\varphi_j^C| \ \cdot\ |\varphi_j^C\rangle\langle\varphi_j^C| \tag{4.6}$$

extracts the diagonal part of ρ_C within the normalised eigenbasis of H_C. Observe that $\mathrm{Offdiag}_C : \mathcal{B}(\mathcal{H}_C) \to \mathcal{B}(\mathcal{H}_C)$, extracting the offdiagonal part of ρ_C within the same basis, yields the complementary projector

$$\mathrm{Offdiag}_C = \mathbb{I} - \mathrm{Diag}_C. \tag{4.7}$$

We also note, for later reference, that Q_0 on $\mathcal{B}(\mathcal{H})$ is trace preserving, so that $\mathrm{Ran}\,(\mathbb{I} - \mathrm{Q}_0) \subset \{\rho \,|\, \mathrm{tr}\rho = 0\}$.

Analytic perturbation theory, see e.g. Chapter II §2 [16], allows us to compute the splitting of the degenerate eigenvalue zero of $\mathcal{L}_0$ by the perturbation $g\mathcal{L}_1$. Recall here that $\mathcal{L}_g$ being a Lindblad operator (Proposition 3.2), the following structural constraints hold:

$$0 \in \sigma(\mathcal{L}_g) = \overline{\sigma(\mathcal{L}_g)} \subset \{z \in \mathbb{C} \,|\Re z \leq 0\}, \quad \forall\, g \in \mathbb{R}. \tag{4.8}$$

Moreover, the eigenvalue 0 is semisimple, that is there is no eigennilpotent (Jordan block) corresponding to that eigenvalue in the spectral decomposition of $\mathcal{L}_g$. The same is actually true for all eigenvalues sitting on the imaginary axis.

Let $\{\lambda_j(g)\}_{1\leq j\leq m}$ be the set of eigenvalues of $\mathcal{L}_g$ that stem from the eigenvalue 0 of $\mathcal{L}_0$, with $m = n_C^2$ if $H_C = 0$ or $m = n_C$ if $H_C \neq 0$. They form the so-called $\lambda-$group for $\lambda = 0$, and for $g \in \mathbb{C} \setminus \{0\}$ with $|g|$ is small enough, $\{\lambda_j(g)\}_{1\leq j\leq m}$ are analytic functions of a (fractional) power of g that tend to zero as $g \to 0$. These eigenvalues may be permanently degenerate. For the structural reasons recalled above, one of these eigenvalues, denoted by $\lambda_0(g)$, is identically equal to zero, $\lambda_0(g) \equiv 0$, $\forall g \in \mathbb{C} \setminus 0$, and in case $\lambda_0(g)$ is degenerate, it is semisimple.

We show that under generic hypotheses, $\lambda_0(g) \equiv 0$ is a simple eigenvalue, see Theorem 4.3, and we determine the corresponding eigenvector $\rho_0(g)$, normalized to be a state, *i.e.* $\rho_0(g) \geq 0$ and $\mathrm{tr}\rho_0(g) = 1$.

Let us denote by $Q_0(g)$ the analytic spectral projector of $\mathcal{L}_g$ corresponding to the set of eigenvalues in the 0-group . It writes

$$Q_0(g) = \frac{-1}{2i\pi}\int_{\Gamma_0} (\mathcal{L}_g - z)^{-1} dz = Q_0 + gQ_1 + g^2Q_2 + \mathcal{O}(g^3), \tag{4.9}$$

for $|g|$ is small, where Γ_0 is a circle of small radius centered at the origin. Also, since 0 is a semisimple eigenvalue of $\mathcal{L}_0$,

$$Q_1 = -Q_0\mathcal{L}_1S_0 - S_0\mathcal{L}_1Q_0 = Q_0Q_1(\mathbb{I} - Q_0) + (\mathbb{I} - Q_0)Q_1Q_0, \tag{4.10}$$

where S_0 is the reduced resolvent of $\mathcal{L}_0$ at 0, satisfying $S_0Q_0 = Q_0S_0 = 0$ and $S_0\mathcal{L}_0 = \mathcal{L}_0S_0 = \mathbb{I} - Q_0$. In other words, $S_0 = \mathcal{L}_0^{-1}(\mathbb{I} - Q_0)$, that we shall sometimes abusively write $S_0 = \mathcal{L}_0^{-1}$, with the understanding that it acts on $(\mathbb{I} - Q_0)\mathcal{B}(\mathcal{H})$. The analytic reduced operator in the corresponding subspace which describes the splitting reads

$$\begin{aligned} Q_0(g)\mathcal{L}_gQ_0(g) &= (Q_0 + gQ_1 + g^2Q_2 + \mathcal{O}(g^3))(\mathcal{L}_0 + g\mathcal{L}_1)(Q_0 + gQ_1 + g^2Q_2 + \mathcal{O}(g^3)) \\ &= gQ_0\mathcal{L}_1Q_0 + g^2(Q_1\mathcal{L}_0Q_1 + Q_1\mathcal{L}_1Q_0 + Q_0\mathcal{L}_1Q_1) + \mathcal{O}(g^3), \end{aligned} \tag{4.11}$$

where we used $\mathcal{L}_0Q_0 = Q_0\mathcal{L}_0 = 0$.

Lemma 4.1 *Under assumption* $\mathbf{Spec}(H_C)$ *when* $H_C \neq 0$*, we have*

$$Q_0\mathcal{L}_1Q_0(\rho) = \begin{cases} 0 & \text{if } H_C \neq 0 \\ -i\tau_A \otimes [\overline{H}^{\,\tau}, \mathrm{tr}_{AB}(\rho)] \otimes \tau_B & \text{if } H_C = 0 \end{cases} \tag{4.12}$$

where

$$\begin{aligned}\overline{H}^{\tau} :&= \mathrm{tr}_{AB}(\tau_A^{1/2}\otimes\mathbb{I}_C\otimes\tau_B^{1/2}\,H\,\tau_A^{1/2}\otimes\mathbb{I}_C\otimes\tau_B^{1/2}) \\ &= \mathrm{tr}_{AB}(H\,\tau_A\otimes\mathbb{I}_C\otimes\tau_B) = \mathrm{tr}_{AB}(\tau_A\otimes\mathbb{I}_C\otimes\tau_B\,H)\in\mathcal{B}(\mathcal{H}_C).\end{aligned} \tag{4.13}$$

Explicitly, with $\tau_\# = \sum_{1\le j\le n_\#} t_j^\# |\varphi_j^\#\rangle\langle\varphi_j^\#|$,

$$\overline{H}^{\tau} = \sum_{\substack{1\le j\le n_A\\ 1\le k\le n_B}} t_j^A t_k^B (\langle\varphi_j^A|\otimes\mathbb{I}_C\otimes\langle\varphi_k^B|)\; H\; (|\varphi_j^A\rangle\otimes\mathbb{I}_C\otimes|\varphi_k^B\rangle). \tag{4.14}$$

As a consequence, when $H_C\neq 0$ the splitting is generically described by the order g^2 correction, while in case $H_C = 0$, the non-zero first order correction imposes that the elements of the kernel of $Q_0(g)$ commute with $\overline{H}^{\tau}$ which, generically, decreases the degeneracy from n_C^2 to n_C. In both cases, the eigenvalue zero of $Q_0\mathcal{L}_1Q_0$ is semisimple.

Proof. We first compute for any $\rho_C\in\mathcal{B}(\mathcal{H}_C)$, using (4.13),

$$\mathrm{tr}_{AB}([H,\tau_A\otimes\rho_C\otimes\tau_B]) = [\overline{H}^{\tau},\rho_C]. \tag{4.15}$$

One gets the explicit expression for $\overline{H}^{\tau}$ by expressing the partial trace within the eigenbases of $\tau_\#$. Therefore

$$Q_0\mathcal{L}_1Q_0(\rho) = -i\tau_A\otimes(\mathrm{tr}_{AB}([H,\tau_A\otimes\mathrm{tr}_{AB}(\rho)\otimes\tau_B]))\otimes\tau_B = -i\tau_A\otimes[\overline{H}^{\tau},\mathrm{tr}_{AB}(\rho)]\otimes\tau_B. \tag{4.16}$$

The fact that $H_C\neq 0$ implies $Q_0\mathcal{L}_1Q_0 = 0$ then follows from

$$Q_0\mathcal{L}_1Q_0(\rho) = -i\tau_A\otimes\mathrm{Diag}_C(\mathrm{tr}_{AB}([H,\tau_A\otimes\mathrm{Diag}_C(\mathrm{tr}_{AB}(\rho))\otimes\tau_B]))\otimes\tau_B, \tag{4.17}$$

and the identity

$$\mathrm{Diag}_C(\mathrm{tr}_{AB}([H,\tau_A\otimes\mathrm{Diag}_C(\rho_C)\otimes\tau_B])) = \mathrm{Diag}_C[\overline{H}^{\tau},\mathrm{Diag}_C(\rho_C)] = 0. \tag{4.18}$$

□

Let us investigate the next order correction in order to analyse the splitting from the eigenvalue zero. Following [16] we consider the analytic matrix

$$\begin{aligned}\widetilde{\mathcal{L}}_g &= \frac{1}{g}Q_0(g)\mathcal{L}_gQ_0(g)) \\ &= Q_0\mathcal{L}_1Q_0 + g(Q_1\mathcal{L}_0Q_1 + Q_1\mathcal{L}_1Q_0 + Q_0\mathcal{L}_1Q_1) + \mathcal{O}(g^2) \\ &\equiv \widetilde{\mathcal{L}}_0 + g\widetilde{\mathcal{L}}_1 + \mathcal{O}(g^2),\end{aligned} \tag{4.19}$$

where we observe with (4.10) that

$$\widetilde{\mathcal{L}}_1 = -Q_0\mathcal{L}_1S_0\mathcal{L}_1Q_0 - S_0\mathcal{L}_1Q_0\mathcal{L}_1Q_0 - Q_0\mathcal{L}_1Q_0\mathcal{L}_1S_0. \tag{4.20}$$

Let $\widetilde{Q}_0$ be the eigenprojector onto $\mathrm{Ker}\,\widetilde{\mathcal{L}}_0$. Then the spectrum of $\widetilde{Q}_0\widetilde{\mathcal{L}}_1\widetilde{Q}_0$ describes the splitting to order g^2, see [16], Thm 5.11: for $\tilde{\lambda}_j^{(1)}\in\sigma(\widetilde{Q}_0\widetilde{\mathcal{L}}_1\widetilde{Q}_0)$ of multiplicity $m_j^{(1)}$, there exist exactly $m_j^{(1)}$ eigenvalue of $\mathcal{L}_g$ of the form

$$\lambda_j(g) = g^2\tilde{\lambda}_j^{(1)} + \mathcal{O}(g^3). \tag{4.21}$$

Notice that $\widetilde{Q}_0\widetilde{\mathcal{L}}_1\widetilde{Q}_0$ is viewed as an operator on $Q_0\mathcal{B}(\mathcal{H})$ here.

We observe that $\widetilde{Q}_0 = \widetilde{Q}_0 Q_0 = Q_0 \widetilde{Q}_0$, hence

$$\widetilde{Q}_0 \widetilde{\mathcal{L}}_1 \widetilde{Q}_0 = -\widetilde{Q}_0 (Q_0 \mathcal{L}_1 S_0 \mathcal{L}_1 Q_0) \widetilde{Q}_0 = -\widetilde{Q}_0 \mathcal{L}_1 \mathcal{L}_0^{-1} \mathcal{L}_1 \widetilde{Q}_0, \tag{4.22}$$

since $\mathcal{L}_1 \widetilde{Q}_0 = (\mathbb{I} - Q_0) \mathcal{L}_1 \widetilde{Q}_0$.

In order to proceed, we shall also assume in the sequel that the operator $\overline{H}^{\tau}$ appearing in Lemma 4.1 has generic spectral properties.

Spec($\overline{H}^{\tau}$):
The spectrum of $\overline{H}^{\tau} \in \mathcal{B}(\mathcal{H}_C)$ is simple and the corresponding Bohr frequencies are distinct. We denote the normalised eigenvectors and eigenvalues of $\overline{H}^{\tau}$ by φ_j^{τ} and e_j^{τ}, $1 \leq j \leq n_C$.

Under **Spec($\overline{H}^{\tau}$)**, we get from (4.5) and Lemma 4.1

$$\widetilde{Q}_0(\rho) = \begin{cases} \tau_A \otimes \mathrm{Diag}_C \mathrm{tr}_{AB}(\rho) \otimes \tau_B & \text{if } H_C \neq 0 \\ \tau_A \otimes \mathrm{Diag}_\tau \mathrm{tr}_{AB}(\rho) \otimes \tau_B & \text{if } H_C = 0, \end{cases} \tag{4.23}$$

where Diag_τ is the projector that extracts the diagonal part of the matrices expressed in the orthonormal eigenbasis $\{\varphi_j^{\tau}\}$. Therefore

$$\widetilde{Q}_0 \widetilde{\mathcal{L}}_1 \widetilde{Q}_0 = \tau_A \otimes \mathrm{Diag}\, \mathrm{tr}_{AB}\Big(\big[H, \mathcal{L}_0^{-1}([H, \tau_A \otimes \mathrm{Diag}\, \mathrm{tr}_{AB}(\cdot) \otimes \tau_B])\big] \Big) \otimes \tau_B, \tag{4.24}$$

where Diag stands here for Diag_C (resp. Diag_τ) if $H_C \neq 0$ (resp. $H_C = 0$). Equivalently, $\widetilde{Q}_0 \widetilde{\mathcal{L}}_1 \widetilde{Q}_0$ is fully characterised by the following linear map. Set

$$\Phi(\cdot) := \mathrm{tr}_{AB}\big(\big[H, \mathcal{L}_0^{-1}([H, \tau_A \otimes \mathrm{Diag}(\,\cdot\,) \otimes \tau_B])\big] \big) : \mathcal{B}(\mathcal{H}_C) \to \mathcal{B}(\mathcal{H}_C) \cap \{\rho_C \,|\, \mathrm{tr}\rho_C = 0\}. \tag{4.25}$$

Note that Φ is well defined and takes the form $\Phi(\rho) = \mathrm{tr}_{AB}([H, M(\rho)])$, for $M(\rho) \in \mathcal{B}(\mathcal{H})$, hence $\mathrm{tr}\,\Phi(\rho) = \mathrm{tr}([H, M(\rho)]) = 0$, for any ρ. Then, the restriction of Φ to $\mathrm{Diag}\,\mathcal{B}(\mathcal{H}_C)$, which has dimension n_C, satisfies

$$\Phi_D := \mathrm{Diag}\,\Phi\,|_{\mathrm{Diag}\,\mathcal{B}(\mathcal{H}_C)} \quad \text{and} \quad \widetilde{Q}_0 \widetilde{\mathcal{L}}_1 \widetilde{Q}_0(\cdot) = \tau_A \otimes \Phi_D(\cdot) \otimes \tau_B \circ \mathrm{tr}_{AB}(\cdot). \tag{4.26}$$

We shall abuse notations in the sequel and simply write

$$\widetilde{Q}_0 \widetilde{\mathcal{L}}_1 \widetilde{Q}_0(\cdot) = \tau_A \otimes \Phi_D(\cdot) \otimes \tau_B, \tag{4.27}$$

identifying operators defined on $\widetilde{Q}_0 \mathcal{B}(\mathcal{H})$ and $\mathrm{Diag}\,\mathcal{B}(\mathcal{H}_C)$. Hence

$$\sigma(\widetilde{Q}_0 \widetilde{\mathcal{L}}_1 \widetilde{Q}_0) = \sigma(\Phi_D). \tag{4.28}$$

Note that $\dim \mathrm{Ker}\, \Phi_D \geq 1$, since $\mathrm{Ran}\, \Phi_{\mathrm{D}} \subset \mathrm{Diag}\,\mathcal{B}(\mathcal{H}_{\mathrm{C}}) \cap \{\rho_{\mathrm{C}} \,|\, \mathrm{tr}\rho_{\mathrm{C}} = 0\}$, a subspace of dimension $n_C - 1$, in keeping with the fact that $\mathrm{Ker}\,\mathcal{L}_g$ is never trivial. Hence, for the zero eigenvalue of $\mathcal{L}_g$ to be non-degenerate at second order perturbation in g, we assume the coupling satisfies the assumption.

Coup:
The linear map

$$\Phi_D(\cdot) = \mathrm{Diag}\, \mathrm{tr}_{AB}\big(\big[H, \mathcal{L}_0^{-1}([H, \tau_A \otimes \mathrm{Diag}\,(\,\cdot\,) \otimes \tau_B])\big] \big) \text{ defined on } \mathrm{Diag}\,\mathcal{B}(\mathcal{H}_C), \tag{4.29}$$

where Diag stands here Diag_C (resp. Diag_τ) if $H_C \neq 0$ (resp. $H_C = 0$), is such that $\dim \mathrm{Ker}\, \Phi_D = 1$.

Remark 4.2 *Assumption* **Coup** *is equivalent to the statement*

Φ_D^{-1} exists on the n_c-1 dimension subspace $\mathrm{Diag}\,\mathcal{B}(\mathcal{H}_C)\cap\{\rho_C\,|\,\mathrm{tr}\rho_C=0\}=\mathrm{Ran}\,\Phi_{\mathrm{D}}$.

Indeed, both statements entail $\dim\mathrm{Ker}\,\Phi_D=1$*, and the corresponding spectral projector onto* $\mathrm{Ker}\,\Phi_D$*, say* Π_0*, is such that* $\mathrm{Ran}\,\Phi_{\mathrm{D}}=(\mathbb{I}-\Pi_0)\mathrm{Ran}\,\Phi_{\mathrm{D}}$*, and* $\mathrm{Ker}\,\Phi_D\cap\mathrm{Ran}\,\Phi_{\mathrm{D}}=\{0\}$.

As a consequence,

Theorem 4.3 *Consider the coupled QRM* $\mathcal{L}_g(\rho)$ *defined on* $\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_C\otimes\mathcal{H}_B)$ *by*

$$\mathcal{L}_g(\rho)=-i\big[H_A+H_C+H_B+gH,\rho\big]+\gamma_A(\tau_A\otimes\mathrm{tr}_A(\rho)-\rho)+\gamma_B(\mathrm{tr}_B(\rho)\otimes\tau_B-\rho) \tag{4.30}$$

and assume $\mathbf{Spec}(H_C)$ *if* $H_C\neq 0$ *or* $\mathbf{Spec}(\overline{H}^{\,\tau})$ *if* $H_C=0$*. Then for* $g\in\mathbb{C}\backslash_{\{0\}}$*,* $|g|$ *small enough,* $\dim\mathrm{Ker}\,\mathcal{L}_g=1$ *if* **Coup** *holds.*

Remark 4.4 *Under assumption* $\mathbf{Spec}(\overline{H}^{\,\tau})$*, the non-zero eigenvalues of* $\widetilde{\mathcal{L}}_0$ *are all simple, of the form* $\lambda_{jk}=-i(e_j^\tau-e_k^\tau)$ *with associated eigenvector* $\tau_A\otimes|\varphi_j^\tau\rangle\langle\varphi_k^\tau|\otimes\tau_B$*,* $j\neq k$*, and corresponding eigenprojector*

$$\widetilde{Q}_{\lambda_{jk}}(\rho)=\tau_A\otimes|\varphi_j^\tau\rangle\langle\varphi_k^\tau|\otimes\tau_B\ \mathrm{tr}(\tau_A\otimes|\varphi_k^\tau\rangle\langle\varphi_j^\tau|\otimes\tau_B\,\rho). \tag{4.31}$$

The next order correction, given by the eigenvalue of the operator $-\widetilde{Q}_{\lambda_{jk}}\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_1\widetilde{Q}_{\lambda_{jk}}$*, reads*

$$\tilde{\lambda}_{jk}^{(1)}=\mathrm{tr}\Big\{\mathbb{I}_A\otimes|\varphi_k^\tau\rangle\langle\varphi_j^\tau|\otimes\mathbb{I}_B)\big[H,\mathcal{L}_0^{-1}([H,\tau_A\otimes|\varphi_j^\tau\rangle\langle\varphi_k^\tau|\otimes\tau_B])\big]\Big\}. \tag{4.32}$$

4.2 Dynamics

We push here the spectral analysis a bit further in order to get sufficient information to analyse the behaviour of the dynamics of the coupled QRM $\mathcal{L}_g(\cdot)$, as $g\to 0$. We first discuss the richer case $H_C=0$ and then describe the modifications required for the case $H_C\neq 0$.

Let $Q_0(g)$ be the spectral projector of $\mathcal{L}_g$ given by (4.9), and $Q_0^\perp(g)=\mathbb{I}-Q_0(g)$. We have accordingly

$$e^{t\mathcal{L}_g}=e^{t\mathcal{L}_g^0}Q_0(g)+e^{t\mathcal{L}_g^\perp}Q_0^\perp(g), \tag{4.33}$$

where $\mathcal{L}_g^0=Q_0(g)\mathcal{L}_gQ_0(g)$, and $\mathcal{L}_g^\perp=Q_0^\perp(g)\mathcal{L}_gQ_0^\perp(g)$. Since the spectrum of $\mathcal{L}_g^\perp$ is a positive distance away from the imaginary axis, uniformly in g small enough, functional calculus yields the existence of $\Gamma>0$, independent of g, such that

$$e^{t\mathcal{L}_g}=e^{t\mathcal{L}_g^0}Q_0(g)+\mathcal{O}(e^{-t\Gamma}), \tag{4.34}$$

where $\mathcal{O}$ is uniform in g, since $Q_0(g)$ is analytic in g. Now, by (4.19)

$$\mathcal{L}_g^0=g\widetilde{\mathcal{L}}_g=g(\widetilde{\mathcal{L}}_0+g\widetilde{\mathcal{L}}_1+\mathcal{O}(g^2)), \tag{4.35}$$

where, for $H_C=0$ under assumption $\mathbf{Spec}(\overline{H}^{\,\tau})$,

$$\widetilde{\mathcal{L}}_0=0\,\widetilde{Q}_0+\sum_{j\neq k}\lambda_{jk}\widetilde{Q}_{\lambda_{jk}},\ \text{where}\ \ \lambda_{jk}=-i(e_j^\tau-e_k^\tau), \tag{4.36}$$

with simple non zero eigenvalues, see Remark 4.4. In case $H_C\neq 0$ under hypothesis $\mathbf{Spec}(H_C)$, $\widetilde{\mathcal{L}}_0=0$ by Lemma 4.1, so that (4.36) holds with $\widetilde{Q}_0=Q_0$ and $\widetilde{Q}_{\lambda_{jk}}=0$.

Since $\mathcal{L}_g^0 = \mathcal{O}(g)$ (and even $\mathcal{L}_g^0 = \mathcal{O}(g^2)$ in case $H_C \neq 0$), the long time behaviour of $e^{t\mathcal{L}_g}$ is controlled by the first term in (4.34) when g is small. This requires addressing the behaviour of the non self-adjoint spectral projectors associated to eigenvalues of $\mathcal{L}_g$ that vanish as g goes to zero.

Proposition 4.5 *Assuming $H_C = 0$,* **Spec($\overline{H}^{\tau}$)** *and* **Coup**, *there exists $g_0 > 0$ such that for all $|g| < g_0$, $\mathcal{L}_g$ admits analytic spectral projector $\widetilde{Q}_0(g)$ and $\widetilde{Q}_{\lambda_{jk}}(g)$ and analytic simple eigenvalues $\lambda_{jk}(g)$ such that*

$$\mathcal{L}_g^0 = g\widetilde{Q}_0(g)\widetilde{\mathcal{L}}_g\widetilde{Q}_0(g) + \sum_{j\neq k}\lambda_{jk}(g)\widetilde{Q}_{\lambda_{jk}}(g). \tag{4.37}$$

Here $\widetilde{Q}_0(g) = \widetilde{Q}_0 + \mathcal{O}(g)$, $\widetilde{Q}_{\lambda_{jk}}(g) = \widetilde{Q}_{\lambda_{jk}} + \mathcal{O}(g)$ and $\lambda_{jk}(g) = -ig(e_j^\tau - e_k^\tau) + g^2\tilde{\lambda}_{jk}^{(1)} + \mathcal{O}(g^3)$, see (4.32).

Assuming $H_C \neq 0$, **Spec(H_C)** *and* **Coup**, *the same statement holds with $\widetilde{Q}_0(g) = Q_0 + \mathcal{O}(g)$ and $\widetilde{Q}_{\lambda_{jk}}(g) \equiv 0$, $\lambda_{jk}(g) \equiv 0$ in (4.37).*

Moreover, assuming **Coup** *and* **Spec($\overline{H}^{\tau}$)**, *(respectively* **Spec(H_C)**), *if $H_C = 0$, (respectively $H_C \neq 0$), we have* $\dim\mathrm{Ker}\,\widetilde{Q}_0(g)\widetilde{\mathcal{L}}_g\widetilde{Q}_0(g) \equiv 1$ *and the corresponding spectral projector $\widetilde{Q}_0^S(g)$ is analytic for $|g| < g_0$, and satisfies*

$$\widetilde{Q}_0^S(g)\mathcal{L}_g = \mathcal{L}_g\widetilde{Q}_0^S(g) = 0. \tag{4.38}$$

Here

$$\widetilde{Q}_0(g)\widetilde{\mathcal{L}}_g\widetilde{Q}_0(g) = g\widetilde{Q}_0\widetilde{\mathcal{L}}_1\widetilde{Q}_0 + \mathcal{O}(g^2) = g\tau_A\otimes\Phi_D\otimes\tau_B + \mathcal{O}(g^2) \tag{4.39}$$

and $\widetilde{Q}_0^S(g) = \widetilde{Q}_0^S + \mathcal{O}(g)$ where $\widetilde{Q}_0^S$ is the projector on $\mathrm{Ker}\,\widetilde{Q}_0\widetilde{\mathcal{L}}_1\widetilde{Q}_0$.

Remark 4.6 *The spectral constraints on Lindblad operators imply,*

$$\Re\,\sigma(\widetilde{Q}_0\widetilde{\mathcal{L}}_1\widetilde{Q}_0)\setminus\{0\} \leq 0, \quad \textit{and} \quad \Re\tilde{\lambda}_{jk}^{(1)} \leq 0. \tag{4.40}$$

We give conditions ensuring $\Re\tilde{\lambda}_{jk}^{(1)} < 0$ in case the model has no leading order Hamiltonian drive, $\mathcal{L}_0 = \mathcal{D}$, that we analyse in more details in Section 6.

Proof. We consider $H_C = 0$ only, the other case being similar. Thanks to (4.35) and (4.36), perturbation theory applies to $\widetilde{\mathcal{L}}_g$ and yields the analytic projectors $\widetilde{Q}_0(g)$ and $\widetilde{Q}_{\lambda_{jk}}(g)$ converging to $\widetilde{Q}_0$ and $\widetilde{Q}_{\lambda_{jk}}$ respectively, and the analytic simple eigenvalues $\lambda_{jk}(g)$, such that (4.37) holds. Expanding the first term using $\widetilde{Q}_0\widetilde{\mathcal{L}}_0 = \widetilde{\mathcal{L}}_0\widetilde{Q}_0 = 0$, one gets thanks to (4.26)

$$\widetilde{Q}_0(g)\widetilde{\mathcal{L}}_g\widetilde{Q}_0(g) = g\widetilde{Q}_0\widetilde{\mathcal{L}}_1\widetilde{Q}_0 + \mathcal{O}(g^2) = g\,\tau_A\otimes\Phi_D\otimes\tau_B + \mathcal{O}(g^2). \tag{4.41}$$

Assumption **Coup** implies that $\widetilde{Q}_0\widetilde{\mathcal{L}}_1\widetilde{Q}_0$ has one dimensional kernel, with associated spectral projector we write $\widetilde{Q}_0^S$. Hence, perturbation theory again ensures the existence of an analytic one dimensional spectral projector $\widetilde{Q}_0^S(g)$ of $\widetilde{Q}_0(g)\widetilde{\mathcal{L}}_g\widetilde{Q}_0(g)$ corresponding to the simple zero eigenvalue of $\widetilde{Q}_0\widetilde{\mathcal{L}}_1\widetilde{Q}_0$ at $g = 0$. Necessarily, $\widetilde{Q}_0^S(g)$ coincides with the spectral projector onto the nontrivial kernel of $\mathcal{L}_g$ for all g small enough, which proves (4.38). □

Let us turn to the dynamical implications.

Corollary 4.7 *Under the hypotheses for $H_C = 0$ above, the following holds for g real small enough:*

$$e^{t\mathcal{L}_g} = e^{tg^2(\widetilde{Q}_0\widetilde{\mathcal{L}}_1\widetilde{Q}_0+\mathcal{O}(g))}\widetilde{Q}_0(g) + \sum_{j\neq k} e^{t\lambda_{jk}(g)}\widetilde{Q}_{\lambda_{jk}}(g) + \mathcal{O}(e^{-t\Gamma}). \tag{4.42}$$

Further assuming $\max_{j\neq k}\{\Re\tilde{\lambda}^{(1)}_{jk}\} < 0$, there exists $\delta > 0$ such that for all $t \geq 0$,

$$e^{t\mathcal{L}_g} = \widetilde{Q}^S_0(g) + \mathcal{O}(e^{-\delta g^2 t}), \tag{4.43}$$

where the constant in the $\mathcal{O}$ is uniform in $t \geq 0$ and g small.

Setting $\eta = \min_{j\neq k}\{|\Re\tilde{\lambda}^{(1)}_{jk}|\} > 0$, we have

$$\begin{aligned} e^{t\mathcal{L}_g} &= e^{tg^2\widetilde{Q}_0\widetilde{\mathcal{L}}_1\widetilde{Q}_0}\widetilde{Q}_0 + \mathcal{O}(e^{-tg^2\eta}) + \mathcal{O}(g) + \mathcal{O}(g^3t) \\ &= \tau_A \otimes e^{tg^2\Phi_D}\mathrm{Diag}_\tau \mathrm{tr}_{AB} \otimes \tau_A + \mathcal{O}(e^{-tg^2\eta}) + \mathcal{O}(g) + \mathcal{O}(g^3t) \end{aligned} \tag{4.44}$$

where the constants in all $\mathcal{O}$ are uniform in $t \geq 0$, g small.

Under the hypotheses for $H_C \neq 0$ above, for g real small enough,

$$e^{t\mathcal{L}_g} = e^{tg^2(Q_0\widetilde{\mathcal{L}}_1Q_0+\mathcal{O}(g))}\widetilde{Q}_0(g) + \mathcal{O}(e^{-t\Gamma}), \tag{4.45}$$

and there exists $\delta > 0$ such that for all $t \geq 0$,

$$e^{t\mathcal{L}_g} = \widetilde{Q}^S_0(g) + \mathcal{O}(e^{-\delta g^2 t}), \tag{4.46}$$

where the constant in the $\mathcal{O}$ is uniform in $t \geq 0$ and g small. Moreover,

$$\begin{aligned} e^{t\mathcal{L}_g} &= e^{tg^2Q_0\widetilde{\mathcal{L}}_1Q_0}Q_0 + \mathcal{O}(e^{-t\Gamma}) + \mathcal{O}(g) + \mathcal{O}(g^3t) \\ &= \tau_A \otimes e^{tg^2\Phi_D}\mathrm{Diag}_C \mathrm{tr}_{AB} \otimes \tau_A + \mathcal{O}(e^{-t\Gamma}) + \mathcal{O}(g) + \mathcal{O}(g^3t) \end{aligned} \tag{4.47}$$

where the constants in all $\mathcal{O}$ are uniform in $t \geq 0$, g small.

Remark 4.8 *0) The identical statements (4.43) and (4.46) show that $1/g^2$ is the time scale of the approach to the asymptotic state, as expected.*
i) The full evolution can be approximated by the restriction of $e^{tg^2\tau_A\otimes\Phi_D\otimes\tau_B}$ to $\mathrm{Ran}\,\widetilde{\mathrm{Q}}_0$, (provided η is larger than the absolute value of the real part of the eigenvalues of $\tau_A \otimes \Phi_D \otimes \tau_B$ in case $H_C = 0$).
ii) In case $\mathcal{L}_0 = \mathcal{D}$, we provide in Section 6 an interpretation of the approximate evolution $e^{tg^2\tau_A\otimes\Phi_D\otimes\tau_B}$ as a classical continuous time Markov process.
iii) Set $F = \max\{|\Re\lambda| \ \lambda \in \sigma(\Phi_D)\}$. When $H_C = 0$, the leading term is meaningful if $F < \eta$, and for times which satisfy $1/g^2 < t < \frac{1}{\epsilon+F}|\ln(g)|/g^2$, as $g \to 0$, for any $\epsilon > 0$. When $H_C \neq 0$, the same is true, without constraint on F.

Proof. Again we prove the statements for $H_C = 0$ only, the other case being similar. The first two statements follow from functional calcul, and Proposition 4.5, taking into account the analyticity of the spectral data involved. To get the last statement, we observe that since the CPTP map $e^{t\mathcal{L}_g}$ has a norm which is uniformly bounded in $t \geq 0$ and g small enough, the norm of

$$e^{tg^2(\widetilde{Q}_0\widetilde{\mathcal{L}}_1\widetilde{Q}_0+\mathcal{O}(g))}\widetilde{Q}_0(g) = e^{t\mathcal{L}_g} - \sum_{j\neq k} e^{t\lambda_{jk}(g)}\widetilde{Q}_{\lambda_{jk}}(g) + \mathcal{O}(e^{-t\Gamma}) \tag{4.48}$$

is bounded above by a constant $C > 0$ which uniform in $t \geq 0$ and g small enough. Thus, by Duhamel formula

$$e^{\tau(A+B)} = e^{\tau A} + \int_0^\tau e^{\tau'(A+B)} B e^{(\tau-\tau')A} d\tau' \tag{4.49}$$

applied to $A = \widetilde{Q}_0 \widetilde{\mathcal{L}}_1 \widetilde{Q}_0$ subject to (4.40), $B = \mathcal{O}(g)$, $\tau = g^2 t$ we get

$$e^{tg^2(\widetilde{Q}_0 \widetilde{\mathcal{L}}_1 \widetilde{Q}_0 + \mathcal{O}(g))} \widetilde{Q}_0(g) = e^{tg^2 \widetilde{Q}_0 \widetilde{\mathcal{L}}_1 \widetilde{Q}_0} \widetilde{Q}_0(g) + \mathcal{O}(g^3 t). \tag{4.50}$$

Moreover, $\eta = \min_{j \neq k}\{|\Re \tilde{\lambda}_{jk}^{(1)}|\} > 0$ immediately implies upon expanding $\widetilde{Q}_0(g)$,

$$e^{t\mathcal{L}_g} = e^{tg^2 \widetilde{Q}_0 \widetilde{\mathcal{L}}_1 \widetilde{Q}_0} \widetilde{Q}_0 + \mathcal{O}(e^{-tg^2 \eta}) + \mathcal{O}(g) + \mathcal{O}(g^3 t), \tag{4.51}$$

where the constants in all $\mathcal{O}$ are uniform in $t \geq 0$ and g small. Finally, $\widetilde{Q}_0 \widetilde{\mathcal{L}}_1 \widetilde{Q}_0 = \tau_A \otimes \Phi_D \otimes \tau_B$ allows us to express the exponential in terms of that of Φ_D. □

5 Construction of the asymptotic state

We now turn to the determination of the state $\rho_0(g) \in \operatorname{Ker} \mathcal{L}_g$ where $\mathcal{L}_g = \mathcal{L}_0 + g\mathcal{L}_1 \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ given by a power series in g

$$\rho_0(g) = \rho_0 + g\rho_1 + g^2 \rho_2 + \cdots, \tag{5.1}$$

where $\operatorname{tr}(\rho_0) = 1$ and $\operatorname{tr}(\rho_j) = 0$, $\forall j > 0$. Expanding $\mathcal{L}_0(\rho_0(g)) + g\mathcal{L}_1(\rho_0(g)) \equiv 0$, and equating like powers of g we get

$$\begin{aligned}
&\mathcal{L}_0(\rho_0) = 0 \\
&\mathcal{L}_0(\rho_1) + \mathcal{L}_1(\rho_0) = 0 \\
&\mathcal{L}_0(\rho_2) + \mathcal{L}_1(\rho_1) = 0 \\
&\qquad \vdots \\
&\mathcal{L}_0(\rho_j) + \mathcal{L}_1(\rho_{j-1}) = 0 \quad \forall\ j \geq 1.
\end{aligned} \tag{5.2}$$

The way to solve this set of equations, in principle, is as follows. Note that the spectral decomposition of $\mathcal{L}_0$ yields

$$\operatorname{Ker} \mathcal{L}_0 = \operatorname{Ran} \mathrm{Q}_0 \quad \text{and} \quad \operatorname{Ran} \mathcal{L}_0 = \operatorname{Ker} \mathrm{Q}_0. \tag{5.3}$$

The first equation is solved by picking a trace one element R_0 in $\operatorname{Ker} \mathcal{L}_0 = Q_0(\mathcal{B}(\mathcal{H}))$, described in Proposition 3.7. The addition of any traceless vector $r_0 \in \operatorname{Ker} \mathcal{L}_0$ yields an equally good solution for $\rho_0 := R_0 + r_0$ at that order. The next equation amounts to solve $\mathcal{L}_0(R_1) = -\mathcal{L}_1(R_0 + r_0)$ for R_1, a traceless matrix. This requires $\mathcal{L}_1(R_0 + r_0) \in \operatorname{Ran} \mathcal{L}_0$. Since $\operatorname{Ran} \mathcal{L}_0 = \operatorname{Ker} \mathrm{Q}_0$, this is equivalent to $Q_0 \mathcal{L}_1 Q_0 r_0 = -Q_0 \mathcal{L}_1 R_0$, which determines $r_0 = Q_0 r_0$ up to the addition of an element of $\operatorname{Ker} Q_0 \mathcal{L}_1 Q_0$ ($Q_0 \mathcal{L}_1 Q_0$ viewed as an operator on $Q_0(\mathcal{B}(\mathcal{H}))$). Let us assume for the discussion here that $Q_0 \mathcal{L}_1 Q_0 \neq 0$, *i.e.* $H_C = 0$. This yields $R_1 = -\mathcal{L}_0^{-1}(\mathcal{L}_1(R_0 + r_0))$. Again, the addition of any traceless vector $r_1 = Q_0 r_1 \in \operatorname{Ker} \mathcal{L}_0$ to that R_1 yields an equally good solution $\rho_1 := R_1 + r_1$ to that equation. The next order requires $\mathcal{L}_1(r_1 - \mathcal{L}_0^{-1}(\mathcal{L}_1(R_0 + r_0))) \in \operatorname{Ran} \mathcal{L}_0$, which is equivalent to $Q_0 \mathcal{L}_1 Q_0 r_1 = Q_0 \mathcal{L}_1 \mathcal{L}_0^{-1} \mathcal{L}_1 (R_0 + r_0)$. This equation will then determine r_0 completely, under generic hypotheses, as we shall see. Then we proceed by induction.

The case $H_C \neq 0$ is slightly different, see Lemma 4.1, but is approached in the same spirit. We start by working out the first few steps and then give the general statements about this construction in Theorem 5.2 for $H_C = 0$ and Theorem 5.4 for $H_C \neq 0$.

Again, the inverse of $\mathcal{L}_0$ on its range is the reduced resolvent $S_0 = \mathcal{L}_0^{-1}(\mathbb{I} - Q_0) = \mathcal{L}_0^{-1}|_{(\mathbb{I}-Q_0)\mathcal{B}(\mathcal{H})}$. To express S_0, it is enough to consider the spectral decomposition $\mathcal{L}_0 = \sum_{k>0} \lambda_k Q_k$, where $\lambda_k \neq 0$ and Q_k are the spectral projectors corresponding to Proposition 3.7, while $\lambda_0 = 0$ corresponds to the projector Q_0.

5.1 $H_C = 0$

We consider here that $H_C = 0$ and work under the spectral assumption $\mathbf{Spec}(\overline{H}^{\tau})$ on the self-adjoint operator defined by (4.13). We first work out the orders g^0 and g^1 terms, *i.e.* ρ_0 and ρ_1, and then state an abstract result on the full perturbation series in Theorem 5.2.

The first equation yields $R_0 = \tau_A \otimes \rho_C \otimes \tau_B$ where ρ_C is a state. We choose $\rho_C = \frac{1}{n_C}\mathbb{I}_C$, and $r_0 = \tau_A \otimes r_C^{(0)} \otimes \tau_B$ with any traceless $r_C^{(0)}$ can be added to that choice so that

$$\rho_0 = \tau_A \otimes \rho_C^{(0)} \otimes \tau_B, \text{ with } \quad \rho_C^{(0)} = \frac{1}{n_C}\mathbb{I}_C + r_C^{(0)}. \tag{5.4}$$

Then we compute $Q_0\mathcal{L}_1(R_0 + r_0)$:

$$Q_0(-i[H, R_0 + r_0]) = -i\tau_A \otimes [\overline{H}^{\tau}, \rho_C^{(0)}] \otimes \tau_B = -i\tau_A \otimes [\overline{H}^{\tau}, r_C^{(0)}] \otimes \tau_B. \tag{5.5}$$

The condition to solve the equation for R_1 requires $r_C^{(0)} = \mathrm{Diag}_{\tau}(r_C^{(0)})$, where $\mathrm{Diag}_{\tau}(\cdot)$ extracts the diagonal part of $r_C^{(0)}$ in the normalised eigenbasis of $\overline{H}^{\tau}$. Thanks to our assumption, we set

$$R_1 := i\mathcal{L}_0^{-1}([H, R_0 + r_0]) = i\sum_{k>0} \lambda_k^{-1} Q_k([H, R_0 + r_0]), \tag{5.6}$$

which is traceless, since $R_1 = (\mathbb{I} - Q_0)R_1$, and self-adjoint if $r_C^{(0)}$ is. Next we look for R_2, which requires $Q_0(\mathcal{L}_1(R_1 + r_1)) = 0$, where $r_1 = Q_0(r_1) = \tau_A \otimes r_C^{(1)} \otimes \tau_B$:

$$Q_0([H, \{\mathcal{L}_0^{-1}(i[H, \tau_A \otimes \mathrm{Diag}_{\tau}(\rho_C^{(0)}) \otimes \tau_B]) + \tau_A \otimes r_C^{(1)} \otimes \tau_B\}]) = 0. \tag{5.7}$$

This is equivalent to the equation on $\mathcal{B}(\mathcal{H}_C)$

$$i\mathrm{tr}_{AB}\big([H, \mathcal{L}_0^{-1}([H, \tau_A \otimes \mathrm{Diag}_{\tau}(\rho_C^{(0)}) \otimes \tau_B])]\big) + [\overline{H}^{\tau}, r_C^{(1)}] = 0, \tag{5.8}$$

where we note that $\mathrm{Diag}_{\tau}(r_C^{(1)})$ is arbitrary. Our hypotheses on $\overline{H}^{\tau}$ imply that

$$\mathrm{Ker}\,[\overline{H}^{\tau}, \cdot] = \{\rho_C \,|\, \rho_C = \mathrm{Diag}_{\tau}\rho_C\}, \tag{5.9}$$

$$\mathrm{Ran}\,[\overline{\mathrm{H}}^{\tau}, \cdot] = \{\rho_C \,|\, \rho_C = \mathrm{Offdiag}_{\tau}\rho_C\}. \tag{5.10}$$

Now, assumption **Coup** on H ensures (5.8) determines $\mathrm{Diag}_{\tau} r_C^{(0)}$ and $\mathrm{Offdiag}_{\tau} r_C^{(1)}$: Separating the diagonal from the offdiagonal parts, we have for the former

$$\Phi_D(\rho_C^{(0)}) = 0, \tag{5.11}$$

which determines $\rho_C^{(0)} = \mathbb{I}_C/n_C + \mathrm{Diag}_\tau r_C^{(0)} = \mathrm{Diag}_\tau(\rho_C^{(0)})$ fully since $\dim \mathrm{Ker}\,\Phi_D = 1$, and thus R_1 as well. The offdiagonal part yields

$$\begin{aligned}\mathrm{Offdiag}_\tau r_C^{(1)} &= -i[\overline{H}^{\,\tau},\cdot]^{-1}\Big(\mathrm{Offdiag}_\tau \mathrm{tr}_{AB}\big([H,\mathcal{L}_0^{-1}([H,\tau_A\otimes\rho_C^{(0)}\otimes\tau_B])]\big)\Big)\\ &= -i[\overline{H}^{\,\tau},\cdot]^{-1}\Big(\mathrm{Offdiag}_\tau\Phi(\rho_C^{(0)}))\Big)\end{aligned} \tag{5.12}$$

which fixes $\mathrm{Offdiag}_\tau r_C^{(1)}$ and leaves $\mathrm{Diag}_\tau r_C^{(1)}$ open for now.

At this point, the formula which defines R_2 makes sense,

$$R_2 = i\mathcal{L}_0^{-1}([H,R_1+r_1]) = i\sum_{k>0}\lambda_k^{-1}Q_k([H,R_1+r_1]), \tag{5.13}$$

where R_2 depends parametrically on $\mathrm{Diag}_\tau r_C^{(1)}$. At order two, the contribution is R_2+r_2, where $r_2 = Q_0(r_2) = \tau_A\otimes r_C^{(2)}\otimes\tau_B$ is arbitrary. The term $\mathrm{Diag}_\tau r_C^{(1)}$ is determined by the requirement that $Q_0(\mathcal{L}_1(R_2+r_2)) = 0$ necessary to solve for R_3, *i.e.*

$$\begin{aligned}&\mathrm{tr}_{AB}([H,\{\mathcal{L}_0^{-1}(i[H,R_1+\tau_A\otimes r_C^{(1)}\otimes\tau_B])+\tau_A\otimes r_C^{(2)}\otimes\tau_B\}])\\ &\quad= \mathrm{tr}_{AB}([H,\mathcal{L}_0^{-1}(i[H,R_1+\tau_A\otimes r_C^{(1)}\otimes\tau_B])+[\overline{H}^{\,\tau},r_C^{(2)}]=0.\end{aligned} \tag{5.14}$$

Splitting this equation into its diagonal and offdiagonal parts, we get, making use of (4.29),

$$\mathrm{Diag}_\tau\mathrm{tr}_{AB}([H,\mathcal{L}_0^{-1}(i[H,R_1+\tau_A\otimes\mathrm{Offdiag}_\tau r_C^{(1)}\otimes\tau_B])]+\Phi_D(\mathrm{Diag}_\tau r_C^{(1)})=0, \tag{5.15}$$

$$\mathrm{Offdiag}_\tau\mathrm{tr}_{AB}([H,\mathcal{L}_0^{-1}(i[H,R_1+\tau_A\otimes r_C^{(1)}\otimes\tau_B])]+[\overline{H}^{\,\tau},r_C^{(2)}]=0. \tag{5.16}$$

Using assumption **Coup** under the form: Φ_D is invertible on the subspace $\mathrm{Ran}\,\Phi_\mathrm{D} = \mathrm{Diag}_\tau\mathcal{B}(\mathcal{H}_\mathrm{C})\cap\{\rho_\mathrm{C}\,|\,\mathrm{tr}\rho_\mathrm{C} = 0\}$, the first equation determines

$$\mathrm{Diag}_\tau r_C^{(1)} = -\Phi_D^{-1}(\mathrm{Diag}_\tau\mathrm{tr}_{AB}([H,\mathcal{L}_0^{-1}(i[H,R_1+\tau_A\otimes\mathrm{Offdiag}_\tau r_C^{(1)}\otimes\tau_B])]), \tag{5.17}$$

so that $r_C^{(1)}$ is determined and therefore the second equation yields

$$\mathrm{Offdiag}_\tau r_C^{(2)} = -i[\overline{H}^{\,\tau},\cdot]^{-1}\Big(\mathrm{Offdiag}_\tau\mathrm{tr}_{AB}\big([H,\mathcal{L}_0^{-1}([H,R_1+\tau_A\otimes r_C^{(1)}\otimes\tau_B])]\big)\Big). \tag{5.18}$$

Consequently, we can set

$$R_3 = i\mathcal{L}_0^{-1}([H,R_2+r_2]) = (\mathbb{I}-Q_0)R_3. \tag{5.19}$$

At this point, $\rho_0 = R_0+r_0$, $\rho_1 = R_1+r_1$ are known, as well as R_2, $\mathrm{Offdiag}_\tau r_C^{(2)}$ and R_3.

Remark 5.1 *The fact that $\rho_C^{(0)}\in\mathrm{Ker}\,\Phi_D$ implies $\mathrm{tr}\rho_C^{(0)}\neq 0$, so the assumption that ρ_C is a state in the initial step amounts to set a normalisation.*

Let us formulate a general result that summarises the foregoing and guarantees the process can be pursued:

Theorem 5.2 *Consider the QRM Lindbladian $\mathcal{L}_g$ (4.30) with $H_C = 0$ under the assumptions* **Spec**$(\overline{H}^{\,\tau})$ *and* **Coup**. *Then there exists $g_0>0$ such that $\rho_0(g)$, the unique invariant state of $\mathcal{L}_g$, admits a convergent expansion*

$$\rho_0(g) = \rho_0 + g\rho_1 + g^2\rho_2 + \cdots, \tag{5.20}$$

for all $g \in \mathbb{C}$ with $|g| < g_0$. We have,

$$\rho_0 = \tau_A \otimes \rho_C^{(0)} \otimes \tau_B, \text{ where } \rho_C^{(0)} \in \operatorname{Ker} \Phi_D \tag{5.21}$$

see (4.29) and (5.4), and

$$\rho_j = R_j + \tau_A \otimes r_C^{(j)} \otimes \tau_B \tag{5.22}$$

for all $j \geq 1$. Moreover, there exists a linear map $\mathcal{R} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \cap \{\rho_C \,|\, \mathrm{tr}\rho_C = 0\}$ such that $\rho_j = \mathcal{R}(\rho_{j-1})$, where

$$R_j = i\mathcal{L}_0^{-1}([H, \rho_{j-1}]), \tag{5.23}$$

$$\mathrm{Offdiag}_\tau r_C^{(j)} = -i[\overline{H}^{\,\tau}, \cdot]^{-1}\Big(\mathrm{Offdiag}_\tau \mathrm{tr}_{AB}\big([H, \mathcal{L}_0^{-1}([H, \rho_{j-1}])]\big)\Big), \tag{5.24}$$

$$\mathrm{Diag}_\tau r_C^{(j)} = -\Phi_D^{-1}\big(\mathrm{Diag}_\tau \mathrm{tr}_{AB}([H, \mathcal{L}_0^{-1}(i[H, R_j + \tau_A \otimes \mathrm{Offdiag}_\tau r_C^{(j)} \otimes \tau_B])])\big). \tag{5.25}$$

Consequently, for $|g| < g_0$,

$$\rho_0(g) = (\mathbb{I} - g\mathcal{R})^{-1}(\rho_0). \tag{5.26}$$

Remark 5.3 *0) Replacing R_j and $\mathrm{Offdiag}_\tau r_C^{(j)}$ by their expression into (5.25) shows $\mathrm{Diag}_\tau r_C^{(j)}$ is linear in ρ_{j-1} as well an yields the map $\mathcal{R}$.*
i) Eq. (5.26) is equivalent to

$$\rho_0(g) = \sum_{k=1}^{N} \left(\frac{M_k}{1 - g\mu_k} + \sum_{l=1}^{m_k - 1} \frac{g^l N_k^l}{(1 - g\mu_k)^{l+1}} \right)(\rho_0), \tag{5.27}$$

where μ_k, M_k, N_k and m_k are the eigenvalues, eigenprojectors, eigennilpotents and algebraic multiplicities appearing in the spectral decomposition of $\mathcal{R} = \sum_{k=1}^{N} \mu_k M_k + N_k$. Hence the radius of convergence is $g_0 = 1/\max_{1 \leq k \leq N}(|\mu_k|)$.
ii) In case $\sigma(\mathcal{R}) \cap \mathbb{R}_+^* = \emptyset$, the steady state $\rho_0(g)$ is well defined for all $g \in \mathbb{R}_+^*$.
iii) The iteration terminates if and only if $\mathcal{R}$ has a zero eigenvalue and ρ_0 belongs to the corresponding eigenspace; see Section 8 for examples.
iv) The restriction of the invariant state to $\mathcal{H}_C$ is given by $\mathrm{tr}_{AB}(\rho_0(g)) = \rho_C^{(0)} + \sum_{j \geq 1} g^j r_C^{(j)}$.
v) We provide necessary and sufficient conditions in Proposition 6.1 for **Coup** *to be satisfied in case $\mathcal{L}_0 = \mathcal{D}$ and $H_C = 0$.*

Proof. Recall that $\dim \operatorname{Ker} \mathcal{L}_g = 1$ is proven in Theorem 4.3.
We solve the higher orders equations for $\rho_j = R_j + r_j$ of (5.2) with

$$R_j = (\mathbb{I} - Q_0)R_j, \; r_j = Q_0 r_j = \tau_A \otimes r_C^{(j)} \otimes \tau_B, \tag{5.28}$$

for all j by induction. Let $j \geq 2$ and assume R_k, r_k are given traceless matrices satisfying (5.28) for $1 \leq k \leq j-1$ as well as

$$R_j = i\mathcal{L}_0^{-1}([H, R_{j-1} + r_{j-1}]), \; \tau_A \otimes \mathrm{Offdiag}_\tau r_C^{(j)} \otimes \tau_B \text{ and } R_{j+1} = i\mathcal{L}_0^{-1}([H, R_j + r_j]). \tag{5.29}$$

This is the situation we arrived at for $j = 2$. Consider $Q_0(\mathcal{L}_1(R_{j+1} + r_{j+1})) = 0$, a necessary condition to compute R_{j+2}, which yields

$$\mathrm{tr}_{AB}\big([H, \mathcal{L}_0^{-1}(i[H, R_j + \tau_A \otimes r_C^{(j)} \otimes \tau_B])\big) + [\overline{H}^{\,\tau}, r_C^{(j)}] = 0. \tag{5.30}$$

Splitting the equation into its diagonal and offdiagonal parts gives

$$\mathrm{Diag}_\tau \mathrm{tr}_{AB}\big([H,\mathcal{L}_0^{-1}(i[H,R_j+\tau_A\otimes \mathrm{Offdiag}_\tau r_C^{(j)}\otimes\tau_B]\big)+\Phi_D(\mathrm{Diag}_\tau r_C^{(j)})=0, \quad (5.31)$$

$$\mathrm{Offdiag}_\tau \mathrm{tr}_{AB}\big([H,\mathcal{L}_0^{-1}(i[H,R_j+\tau_A\otimes r_C^{(j)}\otimes\tau_B]\big)+[\overline{H}^\tau,r_C^{(j+1)}]=0. \quad (5.32)$$

The first equation determines

$$\mathrm{Diag}_\tau r_C^{(j)}=-\Phi_D^{-1}\big(\mathrm{Diag}_\tau \mathrm{tr}_{AB}([H,\mathcal{L}_0^{-1}(i[H,R_j+\tau_A\otimes \mathrm{Offdiag}_\tau r_C^{(j)}\otimes\tau_B])])\big), \quad (5.33)$$

so that $r_C^{(j)}$ is fully determined and therefore the second equation yields

$$\mathrm{Offdiag}_\tau r_C^{(j+1)}=-i[\overline{H}^\tau,\cdot]^{-1}\Big(\mathrm{Offdiag}_\tau \mathrm{tr}_{AB}\big(\big[H,\mathcal{L}_0^{-1}([H,R_j+\tau_A\otimes r_C^{(j)}\otimes\tau_B])\big]\big)\Big). \quad (5.34)$$

Consequently we can define

$$R_{j+2}=i\mathcal{L}_0^{-1}([H,R_{j+1}+r_{j+1}]), \quad (5.35)$$

where $\mathrm{Diag}_\tau r_C^{(j+1)}$ remains free, while r_j is determined. This finishes the proof of the induction. □

5.2 $H_C \neq 0$

We consider here $H_C \neq 0$ and the necessary modifications to compute the series (5.1) due to the identities

$$Q_0(\cdot)=\tau_A\otimes \mathrm{Diag}_C(\mathrm{tr}_{AB}(\cdot))\otimes\tau_B \ \text{ and } \ Q_0\mathcal{L}_1Q_0\equiv 0. \quad (5.36)$$

The first equation in (5.2) yields $\rho_0=Q_0\rho_0=\tau_A\otimes\rho_C^{(0)}\otimes\tau_B$, where $\rho_C^{(0)}\in \mathrm{Diag}_C\mathcal{B}(\mathcal{H}_C)$ is free. The condition to solve the second equation is $Q_0\mathcal{L}_1(\rho_0)=Q_0\mathcal{L}_1Q_0(\rho_0)=0$ which is trivially satisfied. Thus, writing $\rho_1=R_1+r_1$ with $R_1=(\mathbb{I}-Q_0)\rho_1$ and $r_1=Q_0\rho_1$, we can solve partially the equation setting

$$R_1=-\mathcal{L}_0^{-1}\mathcal{L}_1(\rho_0). \quad (5.37)$$

The next equation $\mathcal{L}_0(\rho_2)=-\mathcal{L}_1(\rho_1)$ requires $Q_0\mathcal{L}_1(R_1)+Q_0\mathcal{L}_1(r_1)=0$. Thanks to $r_1=Q_0r_1$ and the identity (5.36), this equation reduces to

$$Q_0\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_1Q_0(\rho_0)=0, \quad (5.38)$$

where we used the expression for R_1 and $\rho_0=Q_0\rho_0$. Thanks to assumption **Coup** for $H_C\neq 0$, this determines $\rho_0=\tau_A\otimes\mathrm{Diag}_C\rho_C^{(0)}\otimes\tau_B$ since (5.38) is equivalent to

$$\rho_C^{(0)}\in\mathrm{Ker}\,\Phi_D, \ \text{ where } \ \dim\mathrm{Ker}\,\Phi_D=1. \quad (5.39)$$

Thus R_1 is now determined, while the traceless part $r_1=\tau_A\otimes\mathrm{Diag}_C r_C^{(1)}\otimes\tau_B$ is not. With the familiar decomposition $\rho_2=R_2+r_2$ with respect to the projector Q_0, we set

$$R_2=-\mathcal{L}_0^{-1}\mathcal{L}_1(R_1+r_1) \quad (5.40)$$

and turn to the equation for $\rho_3=R_3+r_3$: $\mathcal{L}_0(\rho_3)=\mathcal{L}_0(R_3)=-\mathcal{L}_1(\rho_2)$. It requires $Q_0\mathcal{L}_1(R_2+r_2)=Q_0\mathcal{L}_1(R_2)=0$, where we used (5.36) and $r_2=Q_0r_2$. With (5.40), this is equivalent to

$$Q_0\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_1Q_0(r_1)=-Q_0\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_1(R_1)=-\tau_A\otimes\mathrm{Diag}_C\mathrm{tr}_{AB}(\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_1(R_1))\otimes\tau_B, \quad (5.41)$$

where $\mathrm{tr}\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_1(R_1) = 0$, since $\mathcal{L}_1(\cdot) = -i[H,\cdot]$. Thanks to **Coup**, we can thus determine $r_1 = \tau_A \otimes \mathrm{Diag}_C r_C^{(1)} \otimes \tau_B$ uniquely in terms of Φ_D

$$r_C^{(1)} = \Phi_D^{-1}\Big(\mathrm{Diag}_C \mathrm{tr}_{AB}\big\{\big[H, \mathcal{L}_0^{-1}([H,R_1]\big]\big\}\Big). \tag{5.42}$$

In turn R_2 is fully determined while $r_2 = \tau_A \otimes \mathrm{Diag}_C r_C^{(2)} \otimes \tau_B$ remains to be computed, and

$$R_3 = -\mathcal{L}_0^{-1}\mathcal{L}_1(R_2 + r_2). \tag{5.43}$$

From there on we can iterate the process to get the equivalent of Theorem 5.2 in the case $H_C \neq 0$. The proof being similar and simpler, we omit it.

Theorem 5.4 *Consider the QRM Lindbladian $\mathcal{L}_g$ (4.30) with $H_C \neq 0$ under the assumptions* **Spec($\overline{H}^{\,\tau}$)** *and* **Coup**. *Then there exists $g_0 > 0$ such that $\rho_0(g)$, the unique invariant state of $\mathcal{L}_g$, admits a convergent expansion*

$$\rho_0(g) = \rho_0 + g\rho_1 + g^2\rho_2 + \cdots, \tag{5.44}$$

for all $g \in \mathbb{C}$ with $|g| < g_0$. We have,

$$\rho_0 = \tau_A \otimes \rho_C^{(0)} \otimes \tau_B, \ \textit{where}\ \rho_C^{(0)} \in \mathrm{Ker}\,\Phi_D \tag{5.45}$$

see (4.29) and (5.39), and $\rho_j = R_j + \tau_A \otimes r_C^{(j)} \otimes \tau_B$ for all $j \geq 1$, with $r_C^{(j)} = \mathrm{Diag}_C(r_C^{(j)})$. Moreover, there exists a linear map $\mathcal{R} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \cap \{\rho_C \,|\, \mathrm{tr}\rho_C = 0\}$ such that $\rho_j = \mathcal{R}(\rho_{j-1})$, where

$$R_j = i\mathcal{L}_0^{-1}([H,\rho_{j-1}]), \tag{5.46}$$

$$\begin{aligned} r_C^{(j)} &= \Phi_D^{-1}\big(\mathrm{Diag}_C \mathrm{tr}_{AB}([H,\mathcal{L}_0^{-1}([H,R_j])])\big) \\ &= i\Phi_D^{-1}\big(\mathrm{Diag}_C \mathrm{tr}_{AB}([H,\mathcal{L}_0^{-1}([H,\mathcal{L}_0^{-1}([H,\rho_{j-1}])])])\big). \end{aligned} \tag{5.47}$$

Consequently, for $|g| < g_0$,

$$\rho_0(g) = (\mathbb{I} - g\mathcal{R})^{-1}(\rho_0). \tag{5.48}$$

Remark 5.5 *0) Remarks i), ii, iii) below Theorem 5.2 remain in force here.*
i) The map $\mathcal{R}$ can be expressed as

$$R_j = -\mathcal{L}_0^{-1}\mathcal{L}_1(\rho_{j-1}), \tag{5.49}$$

$$r_C^{(j)} = -\Phi_D^{-1}\big(\mathrm{tr}_{AB}\big\{Q_0\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_1(R_j)\big\}\big) = \Phi_D^{-1}\big(\mathrm{tr}_{AB}\big\{Q_0\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_1(\rho_{j-1})\big\}\big)$$

so that

$$\rho_j = \Big(-\mathcal{L}_0^{-1}\mathcal{L}_1(\cdot) + \tau_A \otimes \Phi_D^{-1}\big(\mathrm{tr}_{AB}\big\{Q_0\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_1(\cdot)\big\}\big) \otimes \tau_B\Big)(\rho_{j-1}). \tag{5.50}$$

6 No leading order Hamiltonian drive

We consider here the case where $H_A = H_B = H_C = 0$ on their respective spaces, so that $\mathcal{L}_0 = \mathcal{D}$ with τ_A and τ_B arbitrary, while $\mathcal{L}_1 = -i[H,\cdot\,]$ with H arbitrary as well. This allows us to keep things relatively simple, while retaining a certain level of generality, since the dimensions of the different Hilbert spaces are arbitrary as well.

Let us consider the hypothesis **Coup** in this simplified setup, assuming **Spec($\overline{H}^{\,\tau}$)** holds. Recall that $\{\varphi_j^\tau\}_{1\leq j\leq n_C}$ denotes the normalized eigenbasis of $\overline{H}^{\,\tau}$ with respect

to which the projectors Diag_τ and $\mathrm{Offdiag}_\tau$ are defined, and set $P_j^\tau = |\varphi_j^\tau\rangle\langle\varphi_j^\tau|$. Given the definition (4.29) of Φ_D, we need to compute for all $j,k \in \{1,\dots,n_C\}$

$$(\Phi_D)_{jk} := \mathrm{tr}\big\{(\mathbb{I}_A \otimes P_j^\tau \otimes \mathbb{I}_B)\big([H, \mathcal{L}_0^{-1}([H, \tau_A \otimes P_k^\tau \otimes \tau_B])\big)\big\}. \tag{6.1}$$

Thanks to Proposition 3.4, we can express $\mathcal{L}_0^{-1} = \mathcal{D}^{-1}$ in a compact way. Let $\tilde\rho_0 \in \mathcal{B}(\mathcal{H})$ such that $\mathrm{tr}_{AB}(\tilde\rho_0) = 0$, so that $Q_0(\tilde\rho_0) = 0$. Thus

$$\mathcal{D}^{-1}(\tilde\rho_0) = \frac{-1}{\gamma_A+\gamma_B}\left\{\tilde\rho_0 + \frac{\gamma_A}{\gamma_B}\tau_A \otimes \mathrm{tr}_A(\tilde\rho_0) + \frac{\gamma_B}{\gamma_A}\mathrm{tr}_B(\tilde\rho_0)\otimes\tau_B\right\} \tag{6.2}$$

Therefore, introducing

$$\overline{H}^{\tau_A} = \mathrm{tr}_A(H(\tau_A\otimes\mathbb{I}_C\otimes\mathbb{I}_B)) = \mathrm{tr}_A((\tau_A\otimes\mathbb{I}_C\otimes\mathbb{I}_B)H) \in \mathcal{B}(\mathcal{H}_C\otimes\mathcal{H}_B), \tag{6.3}$$

$$\overline{H}^{\tau_B} = \mathrm{tr}_B(H(\mathbb{I}_A\otimes\mathbb{I}_C\otimes\tau_B)) = \mathrm{tr}_B((\mathbb{I}_A\otimes\mathbb{I}_C\otimes\tau_B)H) \in \mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_C) \tag{6.4}$$

and making use of $\mathrm{tr}_{AB}[H, \tau_A\otimes P_k^\tau\otimes\tau_B] = 0$, a straightforward computation yields

$$\begin{aligned}[H, \mathcal{L}_0^{-1}([H,&\tau_A\otimes P_k^\tau\otimes\tau_B])] = -\frac{1}{\gamma_A+\gamma_B}[H,[H,\tau_A\otimes P_k^\tau\otimes\tau_B]] \\ &- \frac{\gamma_A/\gamma_B}{\gamma_A+\gamma_B}[H,\tau_A\otimes[\overline{H}^{\tau_A},P_k^\tau\otimes\tau_B]] - \frac{\gamma_B/\gamma_A}{\gamma_A+\gamma_B}[H,[\overline{H}^{\tau_B},\tau_A\otimes P_k^\tau]\otimes\tau_B].\end{aligned} \tag{6.5}$$

Then we note using the cyclicity of the trace that

$$\begin{aligned}&\mathrm{tr}\big\{(\mathbb{I}_A\otimes P_j^\tau\otimes\mathbb{I}_B)\big([H,[H,\tau_A\otimes P_k^\tau\otimes\tau_B]]\big)\big\} \\ &\quad= 2\big(\delta_{jk}\mathrm{tr}(H(\tau_A\otimes P_k^\tau\otimes\tau_B)H) - \mathrm{tr}((\mathbb{I}_A\otimes P_j^\tau\otimes\mathbb{I}_B)H(\tau_A\otimes P_k^\tau\otimes\tau_B)H)\big)\end{aligned} \tag{6.6}$$

where the operator in the first trace reads

$$\big((\tau_A^{1/2}\otimes P_k^\tau\otimes\tau_B^{1/2})H\big)^*(\tau_A^{1/2}\otimes P_k^\tau\otimes\tau_B^{1/2})H \geq 0, \tag{6.7}$$

while the second trace yields the jj element of its partial tr_{AB}. Hence,

$$\begin{aligned}&\mathrm{tr}\big\{(\mathbb{I}_A\otimes P_j^\tau\otimes\mathbb{I}_B)\big([H,&&[H,\tau_A\otimes P_k^\tau\otimes\tau_B]]\big)\big\} \\ &&&= 2\begin{cases} -\mathrm{tr}_{AB}(H(\tau_A\otimes P_k^\tau\otimes\tau_B)H)_{jj} \leq 0 & \text{if } j\neq k \\ \sum_{l\neq k}\mathrm{tr}_{AB}(H(\tau_A\otimes P_k^\tau\otimes\tau_B)H)_{ll} \geq 0 & \text{if } j = k\end{cases}.\end{aligned} \tag{6.8}$$

Similar considerations can be made for the traces of the other two operators in (6.5):

$$\begin{aligned}\mathrm{tr}\big\{(\mathbb{I}_A\otimes P_j^\tau\otimes\mathbb{I}_B)\big([H,[\overline{H}^{\tau_B},\tau_A\otimes P_k^\tau]\otimes\tau_B]\big)\big\} &= \mathrm{tr}\big\{(\mathbb{I}_A\otimes P_j^\tau)\big([\overline{H}^{\tau_B},[\overline{H}^{\tau_B},\tau_A\otimes P_k^\tau]]\big)\big\} \\ &= 2\big(\delta_{jk}\mathrm{tr}(\overline{H}^{\tau_B}(\tau_A\otimes P_k^\tau)\overline{H}^{\tau_B}) - \mathrm{tr}((\mathbb{I}_A\otimes P_j^\tau)\overline{H}^{\tau_B}(\tau_A\otimes P_k^\tau)\overline{H}^{\tau_B})\big) \\ &= 2\begin{cases} -\mathrm{tr}_A(\overline{H}^{\tau_B}(\tau_A\otimes P_k^\tau)\overline{H}^{\tau_B})_{jj} \leq 0 & \text{if } j\neq k \\ \sum_{l\neq k}\mathrm{tr}_A(\overline{H}^{\tau_B}(\tau_A\otimes P_k^\tau)\overline{H}^{\tau_B})_{ll} \geq 0 & \text{if } j = k\end{cases},\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}\mathrm{tr}\big\{(\mathbb{I}_A\otimes P_j^\tau\otimes\mathbb{I}_B)\big([H,\tau_A\otimes[\overline{H}^{\tau_A},P_k^\tau\otimes\tau_B]]\big)\big\} &= \mathrm{tr}\big\{(P_j^\tau\otimes\mathbb{I}_B)\big([\overline{H}^{\tau_A},[\overline{H}^{\tau_A},P_k^\tau\otimes\tau_B]]\big)\big\} \\ &= 2\big(\delta_{jk}\mathrm{tr}(\overline{H}^{\tau_A}(P_k^\tau\otimes\tau_B)\overline{H}^{\tau_A}) - \mathrm{tr}((P_j^\tau\otimes\mathbb{I}_A)\overline{H}^{\tau_A}(P_k^\tau\otimes\tau_B)\overline{H}^{\tau_A})\big) \\ &= 2\begin{cases} -\mathrm{tr}_B(\overline{H}^{\tau_A}(P_k^\tau\otimes\tau_B)\overline{H}^{\tau_A})_{jj} \leq 0 & \text{if } j\neq k \\ \sum_{l\neq k}\mathrm{tr}_B(\overline{H}^{\tau_B}(P_k^\tau\otimes\tau_B)\overline{H}^{\tau_B})_{ll} \geq 0 & \text{if } j = k\end{cases}.\end{aligned} \tag{6.10}$$

Defining for $1 \leq k \leq n_C$ the non negative operator $h(k) \in \mathcal{B}(\mathcal{H}_C)$ by

$$h(k) = \frac{2}{\gamma_A + \gamma_B} \mathrm{tr}_{AB}(H(\tau_A \otimes P_k^\tau \otimes \tau_B)H) \tag{6.11}$$
$$+ \frac{2\gamma_A/\gamma_B}{\gamma_A + \gamma_B} \mathrm{tr}_B(\overline{H}^{\tau_A}(P_k^\tau \otimes \tau_B)\overline{H}^{\tau_A}) + \frac{2\gamma_B/\gamma_A}{\gamma_A + \gamma_B} \mathrm{tr}_A(\overline{H}^{\tau_B}(\tau_A \otimes P_k^\tau)\overline{H}^{\tau_B}),$$

we eventually obtain

$$(\Phi_D)_{jk} = \begin{cases} -h(k)_{jj} & \geq 0 \quad \text{if } j \neq k \\ +\sum_{l \neq k} h(k)_{ll} & \neq 0 \quad \text{if } j = k \end{cases}, \tag{6.12}$$

where Φ_D is viewed as a matrix on $\mathbb{C}^{n_C}$, and any diagonal matrix $r = \sum_{k=1}^{n_C} r_k P_k^\tau \in \mathrm{Diag}_\tau \mathcal{B}(\mathcal{H}_C)$ is viewed as a vector $\begin{pmatrix} r_1 & r_2 & \cdots & r_{n_C} \end{pmatrix}^t$ of $\mathbb{C}^{n_C}$.

We provide a necessary and sufficient condition on the coupling Hamiltonian H in terms of the diagonal matrix elements of $h(k)$, $1 \leq k \leq n_C$ for assumption **Coup** to hold, *i.e.* that Φ_D restricted to diagonal traceless matrices is invertible.

Proposition 6.1 *Assume $\mathcal{L}_0 = \mathcal{D}$, $\mathcal{L}_1 = -i[H, \cdot\,]$ and consider the non negative operators $\{h(k)\}_{1 \leq k \leq n_C}$ defined by (6.11). Assumption* **Coup** *holds if and only if there exists $j \in \{1, \ldots, n_C\}$ such that $h_{jj}(k) > 0$ for all $1 \leq k \neq j \leq n_C$.*

Remark 6.2 *i) Since $h(k)$ is a sum of non negative operators, it is sufficient to check the condition on any of its constituants.*
ii) Explicit computations show that for $\dim \mathcal{H}_C = 2$, assumption **Coup** *holds as soon as $\Phi_D \neq 0$, while for $\dim \mathcal{H}_C = 3$ it is true if $h_r(j)h_s(k) > 0$ for some $1 \leq j \neq k \leq 3$, $r \neq j$, $s \neq k$ and $(r, s) \neq (k, j)$.*

Proof. Within the framework introduced above we identify Φ_D with its matrix $(\Phi_D)_{jk}$. We need to show it admits zero as a simple eigenvalue, which amounts to showing that $\mathrm{Rank}\, \Phi_D = n_C - 1$.

We use the short hand notations $h_j(k) = h(k)_{jj} \geq 0$ for $j \neq k$ and $h_j(j) = \sum_{k \neq j} h_k(j) \geq 0$ to express the matrix elements of $-\Phi_D$. The proof follows once we establish the following Lemma

Lemma 6.3 *Consider $\mathfrak{h} \in M_n(\mathbb{R})$ given by*

$$\mathfrak{h} = \begin{pmatrix} h_1(1) & -h_1(2) & \cdots & -h_1(n) \\ -h_2(1) & h_2(2) & & -h_2(n) \\ \vdots & & \ddots & \vdots \\ -h_n(1) & -h_n(2) & \cdots & h_n(n) \end{pmatrix}, \text{ where } \begin{cases} h_j(k) \geq 0 & \text{for } j \neq k \\ h_j(j) = \sum_{k \neq j} h_k(j) \geq 0 \end{cases}. \tag{6.13}$$

Then, $\mathrm{Rank}\, \mathfrak{h} = n - 1$ *if and only if* $\exists\, 1 \leq j \leq n$ *such that $h_j(k) > 0$, $\forall\, 1 \leq k \neq j \leq n$.*

Remark 6.4 *It is possible that* $\mathrm{Rank}\, \mathfrak{h} = n - 1$ *and one diagonal element $h_j(j) = 0$, in which case $\mathfrak{h} e_j = 0$, where e_j is the j^{th} canonical basis vector of $\mathbb{C}^n$.*
We can associate to $\mathfrak{h}$ a stochastic matrix $\mathfrak{p}$ the elements of which are

$$\mathfrak{p}_{jk} = \begin{cases} \frac{h_k(j)}{2h_j(j)} & \text{if } \ h_j(j) > 0 \\ \delta_{jk} & \text{if } \ h_j(j) = 0 \end{cases}, \tag{6.14}$$

such that $x \in \mathbb{C}^n$ satisfies $\mathfrak{h}x = 0$ iff $\mathfrak{p}^t y = y$, where $y = \mathrm{Diag}(\mathfrak{h})x \in \mathbb{C}^n$, if $h_k(k) > 0$ for all k. Hence, if $\mathrm{Rank}\, \mathfrak{h} = n - 1$*, the components of x can all be chosen to be non*

negative, by Perron Frobenius theorem.

However $\mathfrak{p}$ *is not necessarily irreducible as one sees from the example* $\mathfrak{h} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ *with* $\sigma(\mathfrak{h}) = \{0, 1, 2\}$ *that admits the non strictly positive eigenvector* $\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T$ *in its kernel*

Proof. We know $0 \in \sigma(\mathfrak{h})$ and by Jacobi's formula,

$$\frac{d}{dz} \det(\mathfrak{h} - z)|_{z=0} = \text{tr } \text{com}^t(\mathfrak{h}) = \sum_{j=1}^{n} \det \hat{\mathfrak{h}}_{jj}, \tag{6.15}$$

where $\text{com}(A)$ is the comatrix of A and $\hat{A}_{jk}$ is obtained by deleting the j^{th} row and k^{th} column of A. In our case

$$\hat{\mathfrak{h}}_{jj} = \begin{pmatrix} h_1(1) & \cdots & -h_1(j-1) & -h_1(j+1) & \cdots & -h_1(n) \\ \vdots & \ddots & & \vdots & & \vdots \\ -h_{j-1}(1) & \cdots & h_{j-1}(j-1) & -h_{j-1}(j+1) & & -h_{j-1}(n) \\ -h_{j+1}(1) & & -h_{j+1}(j-1) & h_{j+1}(j+1) & & -h_{j+1}(n) \\ \vdots & & \vdots & & \ddots & \vdots \\ -h_n(1) & \cdots & -h_n(j-1) & -h_n(j+1) & \cdots & h_n(n) \end{pmatrix} \tag{6.16}$$

is real valued so that $\sigma(\hat{\mathfrak{h}}_{jj}) = \overline{\sigma(\hat{\mathfrak{h}}_{jj})}$. Moreover, by definition, for all $k \neq j$

$$h_k(k) = \sum_{l \neq k} h_l(k) \geq \sum_{\substack{l \neq k \\ l \neq j}} h_l(k), \tag{6.17}$$

so that by Gershgorin Theorem

$$\sigma(\hat{\mathfrak{h}}_{jj}) \subset \bigcup_{k \neq j} \Big\{ z \in \mathbb{C} \,|\, |z - h_k(k)| \leq \sum_{\substack{l \neq k \\ l \neq j}} h_l(k) \Big\} \equiv \bigcup_{k \neq j} G_k \tag{6.18}$$

where the circle G_k centered at $h_k(k)$ of radius $\sum_{\substack{l \neq k \\ l \neq j}} h_l(k)$ intersects the imaginary axis if and only if $h_j(k) = 0$, in which case the intersection reduces to the origin. Since the determinant of $\hat{\mathfrak{h}}_{jj}$ is the product of its complex conjugate eigenvalues, (6.18) yields

$$\det \hat{\mathfrak{h}}_{jj} \geq 0, \text{ with equality iff } \exists\ k \neq j \text{ s.t. } h_j(k) = 0. \tag{6.19}$$

Therefore

$$\sum_{j=1}^{n} \det \hat{\mathfrak{h}}_{jj} \geq 0, \text{ with equality iff } \forall\ 1 \leq j \leq n,\ \exists\ k \neq j \text{ s.t. } h_j(k) = 0. \tag{6.20}$$

□

This ends the proof of the Proposition. □

6.1 Emergence of a classical Markov process

Coming back to Corollary 4.7, we know that for times s.t. $1/g^2 \leq t \leq \frac{1}{F+\epsilon} |\ln(g)|/g^2$, the evolution semigroup $e^{t(\mathcal{D}(\cdot) - ig[H,\cdot])}$ can be approximated by

$$e^{tg^2 \Phi_D} : \text{Diag}_\tau \mathcal{B}(\mathcal{H}_C) \to \text{Diag}_\tau \mathcal{B}(\mathcal{H}_C). \tag{6.21}$$

In the case at hand, Φ_D is expressed in the orthonormal basis $\{|\varphi_j^\tau\rangle\langle\varphi_j^\tau|\}_{1\leq j\leq n_C}$ as the matrix (6.12) denoted by $\mathfrak{h}$ in Lemma 6.3. The negative of the transpose $\mathfrak{h}^T$ of $\mathfrak{h}$ is thus a *transition rate matrix* or *Q-matrix*, associated to a classical continuous time Markov chain with finitely many states, see [19]. Therefore we can associate to our quantum problem $\dot\rho = \mathcal{D}(\rho) - ig[H,\rho]$ a classical continuous time Markov chain $(X_t)_{t\geq 0}$ on the state space $\{|\varphi_j^\tau\rangle\langle\varphi_j^\tau|\}_{1\leq j\leq n_C}$ identified with $\{1,2,\ldots,n\}$ with $n = n_C$, as follows.

Let us recall the general framework. The Markov process $(X_t)_{t\geq 0}$ is characterised by the probability to find the process in state j at time $t \geq 0$, given the process at time 0 is in state i, is denoted by

$$p_{ij}(t) = \mathbb{P}(X_t = j | X_0 = i), \quad i,j \in \{1,2,\ldots,n\}. \tag{6.22}$$

These transition probabilities $P(t) = (p_{ij}(t))_{1\leq i,j\leq n}$ are solutions to the matrix form forward and backward equations

$$P'(t) = P(t)Q, \quad P(0) = \mathbb{I} \quad \Leftrightarrow \quad P'(t)^T = Q^T P^T(t), \quad P(0) = \mathbb{I}, \tag{6.23}$$

where $Q = (q_{ij})_{1\leq i,j\leq n}$ is a transition rate matrix such that $q_{ii} \leq 0$, $q_{ij} \geq 0$ and $\sum_{j=1}^n q_{ij} = 0$. Hence, with the identification $Q = -\mathfrak{h}^T$ we get the following interpretation

Theorem 6.5 *Consider $\mathcal{L}_g(\cdot) = \mathcal{D}(\cdot) - ig[H,\cdot]$ under assumptions* **Spec($\overline{H}^\tau$)** *and* **Coup**. *Then, the operator $e^{tg^2\Phi_D}$ arising in the approximation of $e^{t\mathcal{L}_g}$ provided in (4.44), describes a (rescaled) continuous time Markov process $(X_t)_{t\geq 0}$ on the state space $\{|\varphi_j^\tau\rangle\langle\varphi_j^\tau|\}_{1\leq j\leq n_C} \equiv \{1,\ldots,n\}$ such that for all $\tau \geq 0$,*

$$\mathbb{P}(X_s = j | X_0 = i) = \mathrm{tr}_C\big\{|\varphi_i^\tau\rangle\langle\varphi_i^\tau| e^{s\Phi_D}(|\varphi_j^\tau\rangle\langle\varphi_j^\tau|)\big\}. \tag{6.24}$$

Remark 6.6 *Therefore, for any $s \geq 0$, the transpose of $e^{s\Phi_D}$ is a stochastic matrix.*

Finally, let us address the computation of the order g^2 corrections (4.32) of the simple eigenvalues $\lambda_{jk}(g)$ of $\mathcal{L}_g(\cdot) = \mathcal{D}(\cdot) - ig[H,\cdot]$ given by

$$\tilde\lambda_{jk}^{(1)} = \mathrm{tr}\Big\{(\mathbb{I}_A \otimes |\varphi_k^\tau\rangle\langle\varphi_j^\tau| \otimes \mathbb{I}_B)\big[H, \mathcal{L}_0^{-1}([H, \tau_A \otimes |\varphi_j^\tau\rangle\langle\varphi_k^\tau| \otimes \tau_B])\big]\Big\}. \tag{6.25}$$

We prove in Appendix that

Proposition 6.7 *Consider $\mathcal{L}_g(\cdot) = \mathcal{D}(\cdot) - ig[H,\cdot]$ under assumptions* **Spec($\overline{H}^\tau$)** *and* **Coup**. *Then, the eigenvalues $\lambda_{jk}(g)$ of $\mathcal{L}_g$, see Proposition 4.5, satisfy*

$$\Re\lambda_{jk}(g) \leq -g^2 \frac{\gamma_A^2 + \gamma_A\gamma_B + \gamma_B^2}{\gamma_A\gamma_B(\gamma_A+\gamma_B)}(e_j^\tau - e_k^\tau)^2 + \mathcal{O}(g^3) \tag{6.26}$$

Remark 6.8 *Actually, we show that $\Re\tilde\lambda_{jk}^{(1)}$ is upper bounded by a sum of non positive explicit contributions. Hence one can decrease the contributions stemming from these eigenvalues in the approximations of the dynamics shown in Corollary 4.7 by assuming the coupling Hamiltonian H makes the lower bounds of Lemma 9.1 below large enough.*

7 Example on $\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^2$

We present here an example where the two parts of the Hilbert space on which the dissipator acts non trivially are both $\mathbb{C}^2 = \mathcal{H}_A = \mathcal{H}_B$, while the central part $\mathcal{H}_C = \mathbb{C}^N$, with N arbitrary. The orthonormal bases of $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$ are denoted respectively

by $\{|g\rangle, |e\rangle\}$, $\{|\downarrow\rangle, |\uparrow\rangle\}$ and $\{\varphi_j\}_{j=1}^N$. The reset states associated with rates $\gamma_A, \gamma_B > 0$ are

$$\tau_A = t_A|g\rangle\langle g| + (1-t_A)|e\rangle\langle e|, \quad \tau_B = t_B|\downarrow\rangle\langle\downarrow| + (1-t_B)|\uparrow\rangle\langle\uparrow|, \tag{7.1}$$

where $0 < t_A, t_B < 1$. We consider again a case without leading order Hamiltonian drive, that is $H_A = H_B = H_C = 0$, while the order g Hamiltonian reads

$$H = H_\alpha \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_\beta, \text{ where} \tag{7.2}$$

$$H_\alpha = \sum_{j=1}^N a_j^{(g)}|g\otimes\varphi_j\rangle\langle g\otimes\varphi_j| + a_j^{(e)}|e\otimes\varphi_j\rangle\langle e\otimes\varphi_j| + \sum_{k=1}^N \alpha_k|g\otimes\varphi_1\rangle\langle e\otimes\varphi_k| + \text{h.c.}$$

$$H_\beta = \sum_{j=1}^N b_j^{(\downarrow)}|\varphi_j\otimes\downarrow\rangle\langle\varphi_j\otimes\downarrow| + b_j^{(\uparrow)}|\varphi_j\otimes\uparrow\rangle\langle\varphi_j\otimes\uparrow| + \sum_{k=1}^N \beta_k|\varphi_N\otimes\downarrow\rangle\langle\varphi_k\otimes\uparrow| + \text{h.c.}$$

In other words,

$$H_\alpha = |g\rangle\langle g|\otimes H_a^{(g)} + |e\rangle\langle e|\otimes H_a^{(e)} + |g\rangle\langle e|\otimes|\varphi_1\rangle\langle\Phi_\alpha| + |e\rangle\langle g|\otimes|\Phi_\alpha\rangle\langle\varphi_1| \tag{7.3}$$

$$H_\beta = |\downarrow\rangle\langle\downarrow|\otimes H_b^{(\downarrow)} + |\uparrow\rangle\langle\uparrow|\otimes H_b^{(\uparrow)} + |\downarrow\rangle\langle\uparrow|\otimes|\varphi_N\rangle\langle\Phi_\beta| + |\uparrow\rangle\langle\downarrow|\otimes|\Phi_\beta\rangle\langle\varphi_N| \tag{7.4}$$

with $H_a^{(\#)} = \sum_{j=1}^N a_j^{(\#)}|\varphi_j\rangle\langle\varphi_j|$, $\# \in \{g, e\}$, $\Phi_\alpha = \sum_{k=1}^N \alpha_k\varphi_k$, and similarly for H_β, introducing $H_b^{(\#)} = \sum_{j=1}^N b_j^{(\#)}|\varphi_j\rangle\langle\varphi_j|$ $\# \in \{\downarrow, \uparrow\}$, and $\Phi_\beta = \sum_{k=1}^N \beta_k\varphi_k$.

With these definitions we compute

$$\begin{aligned}\overline{H}^\tau &= t_A H_a^{(g)} + (1-t_A)H_a^{(e)} + t_B H_b^{(\downarrow)} + (1-t_B)H_b^{(\uparrow)} \\ &= \sum_{j=1}^N \big(t_A a_j^{(g)} + (1-t_A)a_j^{(e)} + t_B b_j^{(\downarrow)} + (1-t_B)b_j^{(\uparrow)}\big)|\varphi_j\rangle\langle\varphi_j|,\end{aligned} \tag{7.5}$$

which yields

$$\varphi_j^\tau = \varphi_j \quad \text{and} \quad e_j^\tau = \big(t_A a_j^{(g)} + (1-t_A)a_j^{(e)} + t_B b_j^{(\downarrow)} + (1-t_B)b_j^{(\uparrow)}\big). \tag{7.6}$$

We can choose the real parameters $a_j^{(g)}, a_j^{(e)}, b_j^{(\downarrow)}, b_j^{(\uparrow)}$ so that the generic assumption **Spec** $\overline{H}^\tau$ holds for any choice of $0 < t_A, t_B < 1$.

7.1 Leading order term

The next step consists in determining the diagonal elements of the nonnegative operators $h(k)$ defined in (6.11), $1 \le k \le N$; more precisely $h_j(k) := \langle\varphi_j|h(k)\varphi_j\rangle$, for $j \neq k$. We first compute

$$\overline{H}^{\tau_A} = H_\beta + (t_A H_a^{(g)} + (1-t_A)H_a^{(e)}) \otimes \mathbb{I}_B \tag{7.7}$$

$$\overline{H}^{\tau_B} = H_\alpha + \mathbb{I}_A \otimes (t_B H_b^{(\downarrow)} + (1-t_B)H_b^{(\uparrow)}). \tag{7.8}$$

Since we do not need the elements $\langle\varphi_k|h(k)\varphi_k\rangle$, we do not make explicit their contribution, that we generically denote below by $c_i(k)P_k$, where $c_i(k) \ge 0$, $i = 1, 2, 3, 4$. With this convention, we get for the different elements $h(k)$ is made of

$$\begin{aligned}\mathrm{tr}_{AB}(H(\tau_A\otimes P_k^\tau\otimes\tau_B)H) &= c_1(k)P_k + (1-t_A)|\alpha_k|^2|\varphi_1\rangle\langle\varphi_1| + \delta_{k,1}t_A|\Phi_\alpha\rangle\langle\Phi_\alpha| \\ &\quad + (1-t_B)|\beta_k|^2|\varphi_N\rangle\langle\varphi_N| + \delta_{k,N}t_B|\Phi_\beta\rangle\langle\Phi_\beta| \\ \mathrm{tr}_A(\overline{H}^{\tau_B}(\tau_A\otimes P_k^\tau)\overline{H}^{\tau_B})) &= c_2(k)P_k + (1-t_A)|\alpha_k|^2|\varphi_1\rangle\langle\varphi_1| + \delta_{k,1}t_A|\Phi_\alpha\rangle\langle\Phi_\alpha| \\ \mathrm{tr}_B(\overline{H}^{\tau_A}(P_k^\tau\otimes\tau_B)\overline{H}^{\tau_A})) &= c_3(k)P_k + (1-t_B)|\beta_k|^2|\varphi_N\rangle\langle\varphi_N| + \delta_{k,N}t_B|\Phi_\beta\rangle\langle\Phi_\beta|.\end{aligned} \tag{7.9}$$

Eventually,

$$h(k) = \frac{2}{\gamma_A\gamma_B}\Big\{(1-t_A)|\alpha_k|^2\gamma_B|\varphi_1\rangle\langle\varphi_1| + \delta_{k,1}t_A\gamma_B|\Phi_\alpha\rangle\langle\Phi_\alpha| + \delta_{k,N}t_B\gamma_A|\Phi_\beta\rangle\langle\Phi_\beta| + (1-t_B)|\beta_k|^2\gamma_A|\varphi_N\rangle\langle\varphi_N|\Big\} + c_4(k)P_k\,, \tag{7.10}$$

The offdiagonal elements $h_j(k)$, $j \neq k$, of the matrix $-\Phi_D$ immediately follow: let

$$\begin{aligned} &S_k = \gamma_B(1-t_A)|\alpha_k|^2, && U_k = \gamma_B t_A|\alpha_k|^2, && \text{for } 2 \le k \le N \\ &T_k = \gamma_A(1-t_B)|\beta_k|^2, && V_k = \gamma_A t_B|\beta_k|^2, && \text{for } 1 \le j \le N-1. \end{aligned} \tag{7.11}$$

Therefore the matrix form (8.17) of the operator Φ_D reads

$$\Phi_D = \frac{-2}{\gamma_A\gamma_B} \times \begin{pmatrix} \tilde{h}_1(1) & -S_2 & -S_3 & -S_4 & \cdots & -S_{N-1} & -S_N - V_1 \\ -U_2 & \tilde{h}_2(2) & 0 & 0 & \dots & 0 & -V_2 \\ -U_3 & 0 & \tilde{h}_3(3) & 0 & \dots & 0 & -V_3 \\ \vdots & & \vdots & \ddots & & 0 & \vdots \\ -U_{N-1} & 0 & 0 & \dots & 0 & \tilde{h}_{N-1}(N-1) & -V_{N-1} \\ -U_N - T_1 & -T_2 & -T_3 & -T_4 & \cdots & -T_{N-1} & \tilde{h}_N(N) \end{pmatrix}, \tag{7.12}$$

where the diagonal elements $\tilde{h}_j(j) = T_j + S_j$, $0 < j < N$, $\tilde{h}_1(1) = \sum_{j=2}^N U_j + T_N$ and $\tilde{h}_N(N) = \sum_{j=2}^N V_j + S_N$.

Note that $\alpha_j \neq 0 \Leftrightarrow S_j \neq 0$ and $U_j \neq 0$, while $\beta_j \neq 0 \Leftrightarrow T_j \neq 0$ and $V_j \neq 0$. Hence, looking at the first row of (7.12), one sees that **Coup** holds for this model when

$$\alpha_2\alpha_3\dots\alpha_{N-1} \neq 0 \ \text{ and } \ |\beta_1|^2 + |\alpha_N|^2 \neq 0, \tag{7.13}$$

or, looking at the last row, when

$$\beta_2\beta_3\dots\beta_{N-1} \neq 0 \ \text{ and } \ |\beta_1|^2 + |\alpha_N|^2 \neq 0. \tag{7.14}$$

In either cases, this validates the conclusions of Theorem 5.2 on the invariant state and the way to compute it. From now on, we assume that either (7.13) or (7.14) holds.

The leading term of the invariant state is determined by the one dimensional kernel of Φ_D which turns out to be computable explicitly. We have, noting that $S_j + T_j > 0$ for $2 \le j \le N-1$,

$$\operatorname{Ker}\Phi_D = \mathbb{C}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}, \quad \text{where} \quad \begin{cases} x_1 = S_N + \sum_{j=2}^{N-1} \frac{V_j S_j}{S_j+T_j} + V_1 \\ x_N = U_N + \sum_{j=2}^{N-1} \frac{U_j T_j}{S_j+T_j} + T_1 \\ x_j = \frac{U_j x_1 + V_j x_N}{S_j+T_j},\ 2 \le j \le N-1 \end{cases}. \tag{7.15}$$

The corresponding faithful leading order of the invariant state of the QRM thus reads

$$\rho_0 = \frac{1}{Z}\tau_A \otimes \sum_{j=1}^N x_j|\varphi_j\rangle\langle\varphi_j| \otimes \tau_B, \quad \text{where} \ \ Z = \sum_{k=1}^N x_k. \tag{7.16}$$

Actually, the following more explicit expressions are true. With

$$y(N) = \sum_{j=2}^{N-1} \frac{\gamma_A \gamma_B |\alpha_j \beta_j|^2}{(1-t_A)\gamma_B |\alpha_j|^2 + (1-t_B)\gamma_A |\beta_j|^2}, \tag{7.17}$$

we can write

$$x_1 = (1-t_A)|\alpha_N|^2 \gamma_B + y(N) t_B (1-t_A) + t_B |\beta_1|^2 \gamma_A \tag{7.18}$$

$$x_N = t_A |\alpha_N|^2 \gamma_B + y(N) t_A (1-t_B) + (1-t_B)|\beta_1|^2 \gamma_A \tag{7.19}$$

$$\begin{aligned} x_j = {} & t_A |\alpha_N|^2 \gamma_B + t_B |\beta_1|^2 \gamma_A + y(N) t_A t_B \\ & + \frac{\gamma_A \gamma_B (|\alpha_N|^2 t_A (2t_B - 1)|\beta_j|^2 + |\beta_1|^2 t_B (2t_A - 1)|\alpha_j|^2)}{(1-t_A)\gamma_B |\alpha_j|^2 + (1-t_B)\gamma_A |\beta_j|^2}, \end{aligned} \tag{7.20}$$

for $2 \leq j \leq N$.

Note in particular the generic nontrivial dependence on j of the populations of (the reduced) leading order ρ_0 of the invariant state. Further remarks are in order:

- For non zero coefficients α_j and β_j, x_j is independent of $2 \leq j \leq N-1$ if

$$\frac{(2t_A - 1)t_B}{(1-t_A)} \gamma_A |\beta_1|^2 = \frac{(2t_B - 1)t_A}{(1-t_B)} \gamma_B |\alpha_N|^2. \tag{7.21}$$

- In case we consider thermal states for $\tau_\#$ on $\mathbb{C}^2$, $\# \in \{A, B\}$, such that $t_\# = \frac{1}{1+e^{-\beta_\# E_\#}}$, with excitation energy $E_\# > 0$. We get that $t_\# \to 1$ when $\beta_\# \to \infty$, while $t_\# \to 1/2$ when $\beta_\# \to 0$, which shows that at high temperature, the populations tend to be constant.

8 Example on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

With the previous example considering $H_C \in \mathbb{C}^N$, we could derive the exact expressions of the map Φ_D and of the leading order solution. However, going to first order correction and beyond requires considerable effort and would not be enlightening for the reader. This motivates this second example, where we restrict H_C to be in $\mathbb{C}^2$ and consider an interaction Hamiltonian H that is appropriate to describe effective physical systems. The goal of this section is twofold. First, we derive explicitely higher order corrections illustrating the theorems of Sec. 5, showing that we can capture the main features of the dynamics with relatively little effort as compared to the complexity of the system. Second, we make a clear connection between a tri-partite quantum reset model and models suitable to describe realistic physical systems.

8.1 Model

Explicitly, we consider here a chain of three qubits characterized by their bare energies e_A, e_B, e_C entering H_0. They are interacting through H. The two Hamiltonians are given by

$$H_0 = e_A |1\rangle\langle 1| \otimes \mathbb{I}_C \otimes \mathbb{I}_B + \mathbb{I}_A \otimes e_C |1\rangle\langle 1| \otimes \mathbb{I}_B + \mathbb{I}_A \otimes \mathbb{I}_C \otimes e_B |1\rangle\langle 1|, \tag{8.1}$$

$$\begin{aligned} H = {} & U \left(|11\rangle_{AC}\langle 11| \otimes \mathbb{I}_B + \mathbb{I}_A \otimes (|11\rangle_{CB}\langle 11|) \right. \\ & \left. + (J_\alpha |01\rangle_{AC}\langle 10| \otimes \mathbb{I}_B + \mathbb{I}_A \otimes J_\beta |01\rangle_{CB}\langle 10| + h.c) \right). \end{aligned} \tag{8.2}$$

Without loss of generality, we assume the interaction strengths U, J_α, J_β to be real. This model could be effective for instance for three qubits subject to nearest-neighbour interactions: a Coulomb interaction (set by U) whenever two adjacent qubits are occupied and to a flip-flop interaction term of the form $|01\rangle\langle 10| + h.c.$ (set by J_α, J_β with $J_\alpha \neq J_\beta$). In the ordered computational basis of the three qubits

$$\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}, \tag{8.3}$$

the total Hamiltonian $H_{tot} = H_0 + gH$ reads

$$H_{tot} = \tag{8.4}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_B & gJ_\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & gJ_\beta & e_C & 0 & gJ_\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & e_B + e_C + gU & 0 & gJ_\alpha & 0 & 0 \\ 0 & 0 & gJ_\alpha & 0 & e_A & 0 & 0 & 0 \\ 0 & 0 & 0 & gJ_\alpha & 0 & e_A + e_B & gJ_\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & gJ_\beta & e_A + e_C + gU & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_A + e_B + e_C + 2gU \end{pmatrix}.$$

This model corresponds exactly to the previous example with $N = 2$ and setting:

$$\alpha_1 = \beta_2 = a_2^{(e)} = a_1^{(g)} = a_2^{(g)} = b_2^{(\downarrow)} = b_2^{(\uparrow)} = b_1^{(\uparrow)} = 0 \tag{8.5}$$
$$a_1^{(e)} = b_1^{(\downarrow)} = U,\ \alpha_2 = J_\alpha,\ \beta_1 = J_\beta, \tag{8.6}$$

The ground state for the three qubits is now simply given by $|000\rangle$ and corresponds to $|g \otimes \varphi_2 \otimes \uparrow\rangle$ in the previous example with $N = 2$. For clarity, we provide the expression of H_{tot} in the form introduced in (7.2)

$$\begin{aligned} H_\alpha &= U|e \otimes \varphi_1\rangle\langle e \otimes \varphi_1| + J_\alpha |g \otimes \varphi_1\rangle\langle e \otimes \varphi_2| + \text{h.c.} \\ H_\beta &= U|\varphi_1 \otimes \downarrow\rangle\langle \varphi_1 \otimes \downarrow| + J_\beta |\varphi_2 \otimes \downarrow\rangle\langle \varphi_1 \otimes \uparrow| + \text{h.c.} \end{aligned} \tag{8.7}$$

The two ends (A and B) of the 3-qubit chain are weakly coupled to their own thermal baths at inverse temperatures β_A and β_B with coupling strengths γ_A and γ_B respectively. Dissipation takes place following QRM . The reset states are assumed to be thermal states defined by the Maxwell-Boltzmann distribution with their respective inverse temperature $\beta_\# = 1/T_\#$ ($k_B = 1$ in the following) in the basis $\{|0\rangle, |1\rangle\}$:

$$\tau_\# = \frac{1}{Z_\#} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\beta_\# e_\#} \end{pmatrix} = \begin{pmatrix} t_\# & 0 \\ 0 & 1 - t_\# \end{pmatrix}, \quad \# \in \{A, B\}. \tag{8.8}$$

Note that since the ground state $|0\rangle$ in the C part of the Hilbert space corresponds to $|\uparrow\rangle$, the substitution $t_B \to (1 - t_B)$ is in order to use the results of Section 7.

Let us remark that this model for a tri-partite open quantum system differs from previous works on reset models in the context of quantum thermodynamics, studiying in particular quantum absorption refrigerators and entanglement engines, Refs. [4, 26, 28]. These models consist of a chain of 2, 3 or N qubits, each of them being coupled to its own thermal bath. Dissipation due to the presence of environments is captured through QRM . In Ref. [26], the steady-state solution for 3 qubits with three different environments is derived analytically, whereas the case of two qubits is fully solved in Ref. [4]. In contrast, in this work, we derive the steady-state solution considering an arbitrary system C only coupled to the two ends A and B of the chain, as long as H_C satisfies generic assumptions.

8.2 Generic assumptions

We first check the assumptions for H_C and H. The condition $\mathbf{Spec}(H_C)$, is trivially satisfied in this case as the spectrum $\sigma(H_C) = \{0, e_C\}$ is simple with $e_C \neq 0$. We can then verify $\mathbf{Spec}(\overline{H}^\tau)$ with

$$\overline{H}^\tau = \mathrm{tr}_{AB}(H\tau_A \otimes \mathbb{I}_C \otimes \tau_B) = U(2 - t_A - t_B)|1\rangle\langle 1|\,, \tag{8.9}$$

as defined by Eq. (4.13) . The spectrum $\sigma(\bar{H}^\tau) = \{0, U(2 - t_A - t_B)\}$ with associated eigenvectors $\{|0\rangle, |1\rangle\}$ is simple whenever $U \neq 0$ and $t_A + t_B \neq 2$ where t_A, t_B stand for the ground state populations of the reset states τ_A, τ_B. The identity $t_A + t_B = 2$ is only satisfied for zero temperature reservoirs, $t_A = t_B = 1$. Hence we stay in the generic case, $t_A, t_B < 1$. The condition $U \neq 0$ also tells us that a flip-flop interaction Hamiltonian of the form $(|01\rangle\langle 10| + h.c.)$ is not sufficient to ensure the required non-degeneracy conditions in the 0-subspace of $\mathcal{L}_0$. We easily verify that the kernel of $\mathcal{L}_0$ has dimension $n_C^2 = 4$ if $H_C = 0$ and $n_C = 2$ if $H_C \neq 0$.

In the following, we will restrict the derivation of the steady-state solution up to the second order correction assuming no drive, *i.e.* $H_A = H_B = H_C = 0$. Let us note that in two dimensions, there is no loss of generality to consider the reset states τ_A and τ_B defined as thermal states with respect to H_A and H_B.

8.3 Leading order solution, no drive

Under $\mathbf{Spec}(\bar{H}^\tau)$ and Lemma 4.1, the first-order-correction projector $\tilde{Q}_0\tilde{\mathcal{L}}_1\tilde{Q}_0$ in the 0-eigenvalue subspace is fully characterized by the map Φ acting onto $\mathcal{H}_C$, see Eq. (4.25) and Theorem 4.3

$$\Phi(\cdot) := \mathrm{tr}_{AB}\big(\big[H, \mathcal{L}_0^{-1}([H, \tau_A \otimes \mathrm{Diag}(\,\cdot\,) \otimes \tau_B])\big]\big) : \mathcal{B}(\mathcal{H}_C) \to \mathcal{B}(\mathcal{H}_C) \cap \{\rho_C \,|\, \mathrm{tr}\rho_C = 0\}. \tag{8.10}$$

In contrast to the previous example, we can compute explicitely here the map Φ and not only Φ_D. To this end, we consider ρ_C to be initially in an arbitrary diagonal state (with respect to the eigenbasis of $\overline{H}^\tau$)

$$\rho_C = \begin{pmatrix} r_C^0 & 0 \\ 0 & r_C^1 \end{pmatrix}. \tag{8.11}$$

Defining the linear form on $\mathbb{C}^2$

$$X(r_C^0, r_C^1) = -r_C^0(J_\beta^2(1 - t_B)\gamma_A + J_\alpha^2(1 - t_A)\gamma_B) + r_C^1(J_\beta^2 t_B\gamma_A + J_\alpha^2 t_A\gamma_B)\,, \tag{8.12}$$

we find the matrix $\Phi(\rho_C) \in \mathcal{B}(\mathcal{H}_C)$ to be given by (with respect to the eigenbasis of $\overline{H}^\tau$)

$$\Phi(\rho_C) = \frac{2}{\gamma_A\gamma_B}\begin{pmatrix} X(r_C^0, r_C^1) & 0 \\ 0 & -X(r_C^0, r_C^1) \end{pmatrix}. \tag{8.13}$$

Note that $\Phi(\rho_C)$ is diagonal, so that for this example we have $\Phi(\cdot) = \Phi_D(\cdot)$. In particular

$$\mathrm{Ker}\,\Phi_D(\cdot) = \mathbb{C}\begin{pmatrix} \gamma_A t_B J_\beta^2 + \gamma_B t_A J_\alpha^2 & 0 \\ 0 & \gamma_A(1 - t_B)J_\beta^2 + \gamma_B(1 - t_A)J_\alpha^2 \end{pmatrix} \tag{8.14}$$

is one dimensional, so that Assumption **Coup** is satisfied. Then $\mathrm{Ker}\,\Phi_D$ provides the leading order steady-state solution $\rho_0 = \tau_A \otimes \rho_C^{(0)} \otimes \tau_B$ with

$$\rho_C^{(0)} = \begin{pmatrix} \frac{\gamma_A t_B J_\beta^2 + \gamma_B t_A J_\alpha^2}{\gamma_A J_\beta^2 + \gamma_B J_\alpha^2} & 0 \\ 0 & \frac{\gamma_A(1-t_B)J_\beta^2 + \gamma_B(1-t_A)J_\alpha^2}{\gamma_A J_\beta^2 + \gamma_B J_\alpha^2} \end{pmatrix}. \tag{8.15}$$

Interestingly, the zeroth order solution is the exact solution in the equilibrium situation, i.e. when $\tau_A = \tau_B = \tau$, the state $\rho_0 = \tau \otimes \tau \otimes \tau$ satisfies for any $g \in \mathbb{R}$ (or $\mathbb{C}$) $\mathcal{L}_g(\rho_0) = 0$, an instance of Remark ii) 5.3.

Remark 8.1 *In this example, the matrix Φ_D can also be derived directly from the previous example with $N = 2$, starting from the positive operator $h(k)$:*

$$h(k) = \frac{2}{\gamma_A \gamma_B} \quad \Big(\ |\alpha_k|^2 (1 - t_A)\gamma_B |1\rangle\langle 1| + |\beta_k|^2 t_B \gamma_A |0\rangle\langle 0| \\ + t_A \gamma_B |0\rangle\langle 0| + (1 - t_B)\gamma_A |1\rangle\langle 1| \ \Big) + c_4(k) P(k) \, . \tag{8.16}$$

In the basis $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, given (8.5), the substitution $t_B \to 1 - t_B$, and according to (7.12), the matrix Φ_D reads

$$\Phi_D = \frac{-2}{\gamma_A \gamma_B} \begin{pmatrix} \gamma_A J_\beta^2 (1 - t_B) + \gamma_B J_\alpha^2 (1 - t_A) & -\gamma_A J_\beta^2 t_B - \gamma_B J_\alpha^2 t_A \\ -\gamma_A J_\beta^2 (1 - t_B) - \gamma_B J_\alpha^2 (1 - t_A) & \gamma_A J_\beta^2 t_B + \gamma_B J_\alpha^2 t_A \end{pmatrix} , \tag{8.17}$$

whose kernel in this same basis is generated by the two-dimensional vector

$$\mathrm{Ker}\, \Phi_D = \mathbb{C}(\gamma_A J_\beta^2 t_B + \gamma_B J_\alpha^2 t_A, \gamma_A J_\beta^2 (1 - t_B) + \gamma_B J_\alpha^2 (1 - t_A))^T \, . \tag{8.18}$$

Let us note that Φ_D, when written as a superoperator acting onto diagonal matrices, takes a diagonal form, see Eq. (8.13)*.*

8.4 Underlying Markov process

We have enough information here to determine the natural two-state classical continuous Markov process associated to the model, according to Theorem 6.5. The state space is denoted by $\{0, 1\} \equiv \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, and by (6.24) we need to compute $e^{s\Phi_D}$ to determine the transition probabilities of the process

$$\mathbb{P}(X_s = j | X_0 = k) = \mathrm{tr}\big\{ |k\rangle\langle k| e^{s\Phi_D}(|j\rangle\langle j|) \big\} \equiv (e^{s\Phi_D})_{k,j}, \quad 0 \leq j, k \leq 1 \tag{8.19}$$

The spectral decomposition of Φ_D in the matrix form (8.17) is easily obtained. Introducing

$$\varphi_+ = \gamma_A J_\beta^2 t_B + \gamma_B J_\alpha^2 t_A, \quad \varphi_- = \gamma_A J_\beta^2 (1 - t_B) + \gamma_B J_\alpha^2 (1 - t_A), \tag{8.20}$$

we have

$$\sigma(\Phi_D) = \{0, -2(\varphi_+ + \varphi_-)/(\gamma_A \gamma_B)\} = \{0, -2(\gamma_A J_\beta^2 + \gamma_B J_\alpha^2)/(\gamma_A \gamma_B)\}, \tag{8.21}$$

with eigenvector associated to the non zero eigenvalue proportional to $\begin{pmatrix} 1 & -1 \end{pmatrix}^T$. Hence,

$$\Phi_D = \frac{-2(\varphi_+ + \varphi_-)}{\gamma_A \gamma_B} Q_+ + 0 \, Q_0, \tag{8.22}$$

with spectral projectors

$$Q_0 = \frac{\left| \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|}{\varphi_- + \varphi_+}, \quad Q_+ = \frac{\left| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} \varphi_- \\ -\varphi_+ \end{pmatrix} \right|}{\varphi_- + \varphi_+}. \tag{8.23}$$

Therefore, with $\tilde{s} = \frac{2s(\varphi_+ + \varphi_-)}{\gamma_A \gamma_B}$,

$$e^{s\Phi_D} = e^{-\tilde{s}} Q_+ + Q_0 = \frac{1}{\varphi_- + \varphi_+} \begin{pmatrix} \varphi_+ + e^{-\tilde{s}}\varphi_- & \varphi_+ - e^{-\tilde{s}}\varphi_+ \\ \varphi_- - e^{-\tilde{s}}\varphi_- & \varphi_- + e^{-\tilde{s}}\varphi_+ \end{pmatrix}. \tag{8.24}$$

In turn this eventually yields the sought for transition probabilities

$$\begin{aligned} &\mathbb{P}(X_s = 0 | X_0 = 0) = \frac{\varphi_+ + e^{-\tilde{s}}\varphi_-}{\varphi_- + \varphi_+}, \quad \mathbb{P}(X_s = 1 | X_0 = 1) = \frac{\varphi_- + e^{-\tilde{s}}\varphi_+}{\varphi_- + \varphi_+}, \\ &\mathbb{P}(X_s = 1 | X_0 = 0) = \frac{\varphi_-(1 - e^{-\tilde{s}})}{\varphi_- + \varphi_+}, \quad \mathbb{P}(X_s = 0 | X_0 = 1) = \frac{\varphi_+(1 - e^{-\tilde{s}})}{\varphi_- + \varphi_+}. \end{aligned} \tag{8.25}$$

The appearance of a classical Markov process on the eigenstates of the driving Hamiltonian within the derivation of Lindblad generators for open quantum systems is well known. By contrast, the state space of the Markov process into play in teh present context is determined by the eigenstates of $\overline{H}^{\tau}$, that takes into account the effects of the reset matrices.

8.5 Higer-order corrections, no drive

We now illustrate Theorem 5.2 by deriving the converging expansion of the unique invariant state of $\mathcal{L}_g$

$$\rho_0(g) = \rho_0 + g\,\rho_1 + g^2\,\rho_2 + \dots \quad \text{with,} \quad \rho_0 = \tau_A \otimes \rho_C^{(0)} \otimes \tau_B\,, \tag{8.26}$$

and

$$\rho_j = R_j + \tau_A \otimes r_C^{(j)} \otimes \tau_B \quad \forall j \geq 1\,. \tag{8.27}$$

We recall the definitions for convenience

$$\begin{aligned} R_j &= i\mathcal{L}_0^{-1}([H, \rho_{j-1}]), \\ \mathrm{Offdiag}_\tau r_C^{(j)} &= -i[\overline{H}^{\,\tau}, \cdot]^{-1}\Big(\mathrm{Offdiag}_\tau \mathrm{tr}_{AB}\big([H, \mathcal{L}_0^{-1}([H, \rho_{j-1}])]\big)\Big), \\ \mathrm{Diag}_\tau r_C^{(j)} &= -\Phi_D^{-1}\big(\mathrm{Diag}_\tau \mathrm{tr}_{AB}([H, \mathcal{L}_0^{-1}(i[H, R_j + \tau_A \otimes \mathrm{Offdiag}_\tau r_C^{(j)} \otimes \tau_B])])\big)\,. \end{aligned}$$

For the first-order correction, we start computing $R_1 = i\mathcal{L}_0^{-1}([H, \tau_A \otimes \rho_C^{(0)} \otimes \tau_B])$ which can be expressed with $F_1 = i(|01\rangle\langle 10| - |10\rangle\langle 01|)$ (acting on $\mathcal{H}_A \otimes \mathcal{H}_C$ or $\mathcal{H}_C \otimes \mathcal{H}_B$ depending on the context) as

$$\begin{aligned} R_1 =& \frac{(t_A - t_B)J_\alpha J_\beta}{\gamma_A J_\beta^2 + \gamma_B J_\alpha^2}\Big(J_\beta F \otimes \tau_B + J_\alpha \tau_A \otimes F\Big) \\ =& \frac{(t_A - t_B)J_\alpha J_\beta}{\gamma_A J_\beta^2 + \gamma_B J_\alpha^2}\Big(J_\beta\, i(|01\rangle\langle 10| - |10\rangle\langle 01|) \otimes \tau_B + J_\alpha \tau_A \otimes\, i(|01\rangle\langle 10| - |10\rangle\langle 01|)\Big). \end{aligned} \tag{8.28}$$

We first note that since $\Phi = \Phi_D$, the expression for $\mathrm{Offdiag}_\tau r_C^{(1)}$ reduces to zero:

$$\mathrm{Offdiag}_\tau r_C^{(1)} = -i[\overline{H}^{\,\tau}, \cdot]^{-1}\Big(\mathrm{Offdiag}_\tau \Phi(\rho_C^{(0)})\Big) \equiv 0. \tag{8.29}$$

Then, it remains to determine $\mathrm{Diag}_\tau r_C^{(1)}$ to get the first order correction in g. Thanks to (8.29) and using (8.28) for R_1, we compute

$$\mathrm{Diag}_\tau r_C^{(1)} = -\Phi_D^{-1}\big(\mathrm{Diag}_\tau \mathrm{tr}_{AB}([H, \mathcal{L}_0^{-1}(i[H, R_1])])\big) = 0. \tag{8.30}$$

Hence, the first order correction is simply given by R_1, $\rho_1 = R_1$ and we obtain

$$\rho_0(g) = \tau_A \otimes \rho_C^{(0)} \otimes \tau_B + gR_1 + \mathcal{O}(g^2)\,. \tag{8.31}$$

We proceed with the second-order correction and compute $R_2 = i\mathcal{L}_0^{-1}\big([H, R_1]\big)$. The matrix R_2 is rather complex and we provide the expressions for its diagonal and off-diagonal elements separately. Its 8 diagonal elements in the ordered basis (8.3) are proportional to by

$$\begin{aligned}\mathrm{Diag}(R_2) \;=\; & \frac{J_\alpha J_\beta}{\gamma_A\gamma_B}\Big(t_A t_B(\gamma_A-\gamma_B), -t_A(t_B\gamma_A+\gamma_B(1-t_B)),\\ & t_A\gamma_A(1-t_B)-t_B\gamma_B(1-t_A), (1-t_B)(t_A\gamma_A-\gamma_B(1-t_A)),\\ & t_B(\gamma_B t_A+\gamma_A(1-t_A)), \gamma_A t_A(1-t_B)-\gamma_A t_B(1-t_A),\\ & (1-t_A)(\gamma_B t_B+\gamma_A(1-t_B)), (1-t_A)(1-t_B)(\gamma_B-\gamma_A)\Big).\end{aligned} \tag{8.32}$$

For its off-diagonal elements, we introduce $F_2 = |01\rangle\langle 10| + |10\rangle\langle 01|$ and the coefficient matrices

$$\Gamma_A = \begin{pmatrix} \gamma_A & 0 \\ 0 & \gamma_A+\gamma_B/(1-t_A)\end{pmatrix} \quad ; \quad \Gamma_B = \begin{pmatrix} \gamma_B & 0 \\ 0 & \gamma_B+\gamma_A/(1-t_B)\end{pmatrix}. \tag{8.33}$$

The matrix R_2 can then be written in a compact form

$$\begin{aligned} R_2 \;=\; & \frac{2J_\alpha J_\beta(t_A-t_B)}{J_\beta^2\gamma_A+J_\alpha^2\gamma_B}\Big\{\mathrm{Diag}(R_2)+\frac{1}{\gamma_A+\gamma_B}\\ & \times\Big(\frac{-J_\alpha U(1-t_A)}{2\gamma_B}\tau_A\Gamma_A\otimes F_2+\frac{J_\beta U(1-t_B)}{2\gamma_A}F_2\otimes\tau_B\Gamma_B\\ & -\frac{1}{2}(J_\alpha^2 t_A-J_\beta^2 t_B)|001\rangle\langle 100|+\frac{1}{2}(J_\alpha^2(1-t_A)-J_\beta^2(1-t_B))|110\rangle\langle 011|\Big)\Big\}.\end{aligned} \tag{8.34}$$

For Offdiag$r_C^{(2)}$, we find that it is equal to 0. This leads us to:

$$\mathrm{Diag}\, r_C^{(2)} = \begin{pmatrix} X^{(2)} & 0 \\ 0 & -X^{(2)}\end{pmatrix} \tag{8.35}$$

with

$$\begin{aligned} X^{(2)} \;=\; & \frac{2iJ_\alpha^2J_\beta^2(t_A-t_B)}{\gamma_A^2\gamma_B^2(\gamma_A+\gamma_B)(J_\beta^2\gamma_A+J_\alpha^2\gamma_B)}\\ & \times\Big\{(\gamma_A+\gamma_B)(J_\beta^2\gamma_A(2\gamma_A-\gamma_B)-J_\alpha^2\gamma_B(2\gamma_B-\gamma_A))\\ & +U^2((1-t_A)\gamma_A^2(\gamma_B+(1-t_A)\gamma_A)-(1-t_B)\gamma_B^2(\gamma_A-(1-t_B)\gamma_B))\Big\}.\end{aligned} \tag{8.36}$$

The solution up to the second-order correction is then given by

$$\rho_0(g) = \tau_A\otimes(\rho_C^{(0)}+g^2r_C^{(2)})\otimes\tau_B + g\,R_1 + g^2R_2+\mathcal{O}(g^3)\,. \tag{8.37}$$

We note that coulomb-interaction term like in U starts playing a role when considering the second-order correction.

9 Appendix

We provide here the proof of Proposition 6.7.

Proof. By computations similar to those performed in the determination of Φ_D, we have with $P^\tau_{jk} = |\varphi^\tau_j\rangle\langle\varphi^\tau_k|$,

$$\begin{aligned}[H, \mathcal{L}_0^{-1}([H, \tau_A \otimes P^\tau_{jk} \otimes \tau_B])] = &-\frac{1}{\gamma_A+\gamma_B}[H,[H,\tau_A\otimes P^\tau_{jk}\otimes\tau_B]] \\ &- \frac{\gamma_B/\gamma_A}{\gamma_A+\gamma_B}[H,[\overline{H}^{\,\tau_B},\tau_A\otimes P^\tau_{jk}]\otimes\tau_B] - \frac{\gamma_A/\gamma_B}{\gamma_A+\gamma_B}[H,\tau_A\otimes[\overline{H}^{\,\tau_A},P^\tau_{jk}\otimes\tau_B]] \\ &- \frac{\gamma_A^2+\gamma_A\gamma_B+\gamma_B^2}{\gamma_A\gamma_B(\gamma_A+\gamma_B)}(e^\tau_j - e^\tau_k)[H,\tau_A\otimes P^\tau_{jk}\otimes\tau_B].\end{aligned} \tag{9.1}$$

The last term in (9.1) yields the following contribution to $\tilde\lambda^{(1)}_{jk}$, using cyclicity of the trace and $\overline{H}^{\,\tau}\varphi^\tau_j = e^\tau_j\varphi^\tau_j$,

$$-\frac{\gamma_A^2+\gamma_A\gamma_B+\gamma_B^2}{\gamma_A\gamma_B(\gamma_A+\gamma_B)}(e^\tau_j-e^\tau_k)^2 < 0. \tag{9.2}$$

Then, using cyclicity of the trace and $P^\tau_{kj}P^\tau_{jk} = P^\tau_{kk}$, we have

$$\begin{aligned}\mathrm{tr}(\mathbb{I}_A\otimes P^\tau_{kj}\otimes\mathbb{I}_B[H,[H,\tau_A\otimes P^\tau_{jk}\otimes\tau_B]]) &= -2\mathrm{tr}(\mathbb{I}_A\otimes P^\tau_{kj}\otimes\mathbb{I}_B H\tau_A\otimes P^\tau_{jk}\otimes\tau_B H) \\ &\quad + \mathrm{tr}(H(\tau_A\otimes P^\tau_{jj}\otimes\tau_B)H) + \mathrm{tr}(H(\tau_A\otimes P^\tau_{kk}\otimes\tau_B)H) \\ = \mathrm{tr}(H(\tau_A\otimes P^\tau_{jj}\otimes\tau_B)H) + \mathrm{tr}(H(\tau_A\otimes P^\tau_{kk}\otimes\tau_B)H) &- 2(\mathrm{tr}_{AB}(H(\tau_A\otimes P^\tau_{jk}\otimes\tau_B)H))_{jk}.\end{aligned} \tag{9.3}$$

Here A_{jk} denotes the jk of the matrix $A \in \mathcal{B}(\mathcal{H}_C)$ with respect to the basis $\{\varphi^\tau_j\}$. Note that the operators in the full traces are non negative, whereas the last term is *a priori* complex valued.

Similarly,

$$\begin{aligned}&\mathrm{tr}(\mathbb{I}_A\otimes P^\tau_{kj}\otimes\mathbb{I}_B[H,\tau_A\otimes[\overline{H}^{\,\tau_A},P^\tau_{jk}\otimes\tau_B]]) = \\ &\mathrm{tr}_{AC}(\overline{H}^{\,\tau_A}(P^\tau_{jj}\otimes\tau_B)\overline{H}^{\,\tau_A}) + \mathrm{tr}_{AC}(\overline{H}^{\,\tau_A}(P^\tau_{kk}\otimes\tau_B)\overline{H}^{\,\tau_A}) - 2(\mathrm{tr}_B(\overline{H}^{\,\tau_A}P^\tau_{jk}\otimes\tau_B\overline{H}^{\,\tau_A}))_{jk},\end{aligned} \tag{9.4}$$

and the analogous formula holds for the term involving $\overline{H}^{\,\tau_B}$. These expressions allow us to bound below their real part by a non negative quantity, as the next lemma shows.

Lemma 9.1 *Under the hypotheses above, we compute*

$$\begin{aligned}&\Re\,\mathrm{tr}\Big\{(\mathbb{I}_A\otimes P^\tau_{kj}\otimes\mathbb{I}_B)\big[H,[H,\tau_A\otimes P^\tau_{jk}\otimes\tau_B]\big]\Big\} \\ &\quad\geq \mathrm{tr}\Big\{(\mathbb{I}_A\otimes(\mathbb{I}_C-P^\tau_{jj})\otimes\mathbb{I}_B)H(\tau_A\otimes P^\tau_{jj}\otimes\tau_B)H(\mathbb{I}_A\otimes(\mathbb{I}_C-P^\tau_{jj})\otimes\mathbb{I}_B)\Big\} \\ &\qquad + \text{same with } k\leftrightarrow j.\end{aligned} \tag{9.5}$$

$$\begin{aligned}&\Re\,\mathrm{tr}\Big\{\mathbb{I}_A\otimes P^\tau_{kj}\otimes\mathbb{I}_B[H,\tau_A\otimes[\overline{H}^{\,\tau_A},P^\tau_{jk}\otimes\tau_B]]\Big\} \\ &\quad\geq \mathrm{tr}\Big\{((\mathbb{I}_C-P^\tau_{jj})\otimes\mathbb{I}_B)\overline{H}^{\,\tau_A}(P^\tau_{jj}\otimes\tau_B)\overline{H}^{\,\tau_A}((\mathbb{I}_C-P^\tau_{jj})\otimes\mathbb{I}_B)\Big\} \\ &\qquad + \text{same with } k\leftrightarrow j.\end{aligned} \tag{9.6}$$

$$\begin{aligned}&\Re\,\mathrm{tr}\Big\{\mathbb{I}_A\otimes P^\tau_{kj}\otimes\mathbb{I}_B)\big[H,[\overline{H}^{\,\tau_B},\tau_A\otimes P^\tau_{jk}]\otimes\tau_B\big]\Big\} \\ &\quad\geq \mathrm{tr}\Big\{(\mathbb{I}_A\otimes(\mathbb{I}_C-P^\tau_{jj}))\overline{H}^{\,\tau_B}(\tau_A\otimes P^\tau_{jj})\overline{H}^{\,\tau_B}(\mathbb{I}_A\otimes(\mathbb{I}_C-P^\tau_{jj}))\Big\} \\ &\qquad + \text{same with } k\leftrightarrow j.\end{aligned} \tag{9.7}$$

Remark 9.2 *Since*

$$(\mathbb{I}_A \otimes (\mathbb{I}_C - P^\tau_{jj}) \otimes \mathbb{I}_B)H(\tau_A \otimes P^\tau_{jj} \otimes \tau_B)H(\mathbb{I}_A \otimes (\mathbb{I}_C - P^\tau_{jj}) \otimes \mathbb{I}_B) = \tag{9.8}$$
$$((\tau_A^{1/2} \otimes P^\tau_{jj} \otimes \tau_B^{1/2})H(\mathbb{I}_A \otimes (\mathbb{I}_C - P^\tau_{jj}) \otimes \mathbb{I}_B))^* \times ((\tau_A^{1/2} \otimes P^\tau_{jj} \otimes \tau_B^{1/2})H(\mathbb{I}_A \otimes (\mathbb{I}_C - P^\tau_{jj}) \otimes \mathbb{I}_B))$$

is a non negative operator, we get from (9.1), (9.2) and the Lemma that

$$\Re\tilde{\lambda}^{(1)}_{jk} \leq -\frac{\gamma_A^2 + \gamma_A\gamma_B + \gamma_B^2}{\gamma_A\gamma_B(\gamma_A + \gamma_B)}(e^\tau_j - e^\tau_k)^2 < 0, \tag{9.9}$$

which proves Proposition 6.7.

Proof. We prove the first inequality, the others are similar. Let $G = H(\tau_A^{1/2} \otimes \mathbb{I}_C \otimes \tau_B^{1/2})$, so that the real part we need to consider reads, see (9.3),

$$\mathrm{tr}(G(\mathbb{I}_A \otimes P^\tau_{jj} \otimes \mathbb{I}_B)G^*) + \mathrm{tr}(G(\mathbb{I}_A \otimes P^\tau_{kk} \otimes \mathbb{I}_B)G^*) - 2\mathrm{tr}((\mathbb{I}_A \otimes P^\tau_{jk} \otimes \mathbb{I}_B)G(\mathbb{I}_A \otimes P^\tau_{kj} \otimes \mathbb{I}_B)G^*). \tag{9.10}$$

Spelling out the traces we get

$$\mathrm{tr}(G(\mathbb{I}_A \otimes P^\tau_{jj} \otimes \mathbb{I}_B)G^*) = \sum_{n,m,r,s}\sum_{l} \left|\langle \varphi^A_r \otimes \varphi^\tau_l \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_j \otimes \varphi^B_m\rangle\right|^2 \tag{9.11}$$

$$\mathrm{tr}((\mathbb{I}_A \otimes P^\tau_{jk} \otimes \mathbb{I}_B)G(\mathbb{I}_A \otimes P^\tau_{kj} \otimes \mathbb{I}_B)G^*) \tag{9.12}$$
$$= \sum_{n,m,r,s} \langle \varphi^A_r \otimes \varphi^\tau_k \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_k \otimes \varphi^B_m\rangle\langle \varphi^A_n \otimes \varphi^\tau_j \otimes \varphi^B_m | G\,\varphi^A_r \otimes \varphi^\tau_j \otimes \varphi^B_s\rangle,$$

and we observe that the complex conjugate of (9.12) is obtained by exchanging j and k. Hence we can express the real part of (9.10) as

$$\begin{aligned}
&\sum_{n,m,r,s} \left|\langle \varphi^A_r \otimes \varphi^\tau_j \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_j \otimes \varphi^B_m\rangle\right|^2 + \left|\langle \varphi^A_r \otimes \varphi^\tau_k \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_k \otimes \varphi^B_m\rangle\right|^2 \\
&\quad - \langle \varphi^A_r \otimes \varphi^\tau_k \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_k \otimes \varphi^B_m\rangle\langle \varphi^A_n \otimes \varphi^\tau_j \otimes \varphi^B_m | G\,\varphi^A_r \otimes \varphi^\tau_j \otimes \varphi^B_s\rangle \\
&\quad - \langle \varphi^A_r \otimes \varphi^\tau_j \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_j \otimes \varphi^B_m\rangle\langle \varphi^A_n \otimes \varphi^\tau_k \otimes \varphi^B_m | G\,\varphi^A_r \otimes \varphi^\tau_k \otimes \varphi^B_s\rangle \\
&\quad + \left|\langle \varphi^A_r \otimes \varphi^\tau_k \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_j \otimes \varphi^B_m\rangle\right|^2 + \left|\langle \varphi^A_r \otimes \varphi^\tau_j \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_k \otimes \varphi^B_m\rangle\right|^2 \\
&+ \sum_{n,m,r,s}\sum_{l\notin\{j,k\}} \left|\langle \varphi^A_r \otimes \varphi^\tau_l \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_j \otimes \varphi^B_m\rangle\right|^2 + \left|\langle \varphi^A_r \otimes \varphi^\tau_l \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_k \otimes \varphi^B_m\rangle\right|^2.
\end{aligned} \tag{9.13}$$

With $a = \langle \varphi^A_r \otimes \varphi^\tau_j \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_j \otimes \varphi^B_m\rangle$ and $b = \langle \varphi^A_r \otimes \varphi^\tau_k \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_k \otimes \varphi^B_m\rangle$, we rewrite the first four terms of the summand as

$$|a|^2 + |b|^2 - b\bar{a} - a\bar{b} = \left\langle \begin{pmatrix} a \\ b \end{pmatrix} \middle| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle \geq 0, \tag{9.14}$$

since $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \geq 0$. The remaining terms can reorganised as follows,

$$\begin{aligned}
&\sum_{n,m,r,s} \left|\langle \varphi^A_r \otimes \varphi^\tau_j \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_k \otimes \varphi^B_m\rangle\right|^2 + \sum_{l\notin\{j,k\}} \left|\langle \varphi^A_r \otimes \varphi^\tau_l \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_j \otimes \varphi^B_m\rangle\right|^2 \\
&= \sum_{n,m,r,s}\sum_{l\neq j} \left|\langle \varphi^A_r \otimes \varphi^\tau_l \otimes \varphi^B_s | G\,\varphi^A_n \otimes \varphi^\tau_j \otimes \varphi^B_m\rangle\right|^2 \\
&= \mathrm{tr}\Big\{(\mathbb{I} - \mathbb{I}_A \otimes P^\tau_{jj} \otimes \mathbb{I}_B)G(\mathbb{I}_A \otimes P^\tau_{jj} \otimes \mathbb{I}_B)G^*(\mathbb{I} - \mathbb{I}_A \otimes P^\tau_{jj} \otimes \mathbb{I}_B)\Big\},
\end{aligned} \tag{9.15}$$

and similarly for the terms with second index equal to k, which yields the result. □
And the proof of Proposition 6.7 is finished.

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