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ESTIMATION OF THE SPECTRAL MEASURE FROM CONVEX COMBINATIONS OF REGULARLY VARYING RANDOM VECTORS

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The extremal dependence structure of a regularly varying random vector \mathbf{X} is fully described by its limiting spectral measure. In this paper, we investigate how to recover characteristics of the measure, such as extremal coefficients, from the extremal behaviour of convex combinations of components of \mathbf{X} . Our considerations result in a class of new estimators of moments of the corresponding combinations for the spectral vector. We show asymptotic normality by means of a functional limit theorem and, focusing on the estimation of extremal coefficients, we verify that the minimal asymptotic variance can be achieved by a plug-in estimator.

1. Introduction. One of the main challenges in multivariate extreme value analysis is the inference of the dependence among the extremes of a random vector $\mathbf{X} = (X_1, \dots, X_d)^\top \in (0, \infty)^d$ from a sample of independent copies $\mathbf{X}_1, \dots, \mathbf{X}_n$. In risk management applications, particular interest lies in the case that \mathbf{X} possesses asymptotically dependent components, i.e. extremes tend to appear jointly in various components, with heavy tails. Many of these cases are covered by the framework of multivariate regular variation.

It is common practice in extreme value analysis to consider the extremal dependence structure of \mathbf{X} separately from the marginal properties of the components X_1, \dots, X_d . Thus, focus is often put on the joint tail behaviour of some standardized vector \mathbf{X}^* obtained from \mathbf{X} via marginal transformations. More precisely, the margins are standardized such that the tails are regularly varying with index 1 and asymptotically identical in the sense that

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(X_i^* > x)}{\mathbb{P}(X_j^* > x)} = 1 \quad \text{for all } i, j \in \{1, \dots, d\}.$$

By such a transformation, the property of multivariate regular variation is inherited from \mathbf{X} to \mathbf{X}^* . Thus, extremal dependence of \mathbf{X}^* can be fully described via its so-called spectral measure, that is, the limit distribution of the angular component $\mathbf{X}^*/\|\mathbf{X}^*\|_\infty$ conditional on $\|\mathbf{X}^*\|_\infty > u$ as $u \rightarrow \infty$.

In the past decades, the estimation of the spectral measure and related concepts has attracted a lot of attention, starting from the bivariate setting ($d = 2$), where estimation of the spectral measure is equivalent to inference for the so-called Pickands' dependence function. Here, seminal estimation procedures come from Pickands [1975] with refinements such as Capéraà et al. [1997] using a copula approach assuming known margins. These procedures have been extended by Genest and Segers [2009] to the setting with unknown margins of \mathbf{X} using the rank transform. All the above mentioned works were assuming that the observations are max-stable, i.e. that the asymptotic extreme regime is achieved, a typical assumption for observations that correspond to maxima over certain temporal blocks such as annual maxima.

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The pioneer works of Einmahl et al. [1993] and Einmahl et al. [1997] deal with the estimation of the cumulative distribution function of the spectral measure with observations in the domain of attraction of max-stable distribution, including the case of regularly varying distributions. The two-step procedure in Einmahl et al. [1997] for dealing with unknown margins is replaced by a more efficient rank transform in Einmahl et al. [2001]. An alternative approach due to Drees and Huang [1998] focuses on the estimation of the stable tail dependence function

$$L(\mathbf{x}) = \mathbb{E} \left[\max_{i=1, \dots, d} x_i \Theta_i \right], \quad \mathbf{x} = (x_1, \dots, x_d) \in (0, \infty)^d,$$

using the rank transform for $d = 2$. This work was extended to the multivariate case $d > 2$ in Einmahl et al. [2012]. We refer to Chapter 7 of the monograph De Haan and Ferreira [2006] for a general overview of non-parametric approaches.

When d is large, any inference of the complete spectral distribution suffers from the curse of dimensionality. Thus, it is more reasonable in practice to focus on the joint tail behaviour of the components of lower-dimensional subvectors $\mathbf{X}_I^* = (X_i^*)_{i \in I}$ for sufficiently many “directions” I that are non-empty subsets of $\{1, \dots, d\}$. In this framework, one often makes use of summary statistics such as the so-called extremal coefficients [cf. Smith, 1990, Schlather and Tawn, 2003] given by

$$(1) \quad \tau_I = \lim_{u \rightarrow \infty} \frac{\mathbb{P}(\max_{i \in I} X_i^* > u)}{\mathbb{P}(X_j^* > u)}, \quad j \in I,$$

a number between 1 and $|I|$. Such parameter has attracted a lot of attention as it quantifies the effective number of asymptotically independent components among I . Some properties, such as self-consistency for sets of extremal coefficients, and various procedures for estimating extremal coefficients were introduced by Schlather and Tawn [2003]. For max-stable random fields, Cooley et al. [2006] showed that inference can also be based on the madogram, a graphical tool originally developed in geostatistics.

In this paper, we also consider the extremal dependence structure of subvectors \mathbf{X}_I^* described in terms of the corresponding spectral vector Θ^I , that is, the limit in distribution of $\mathbf{X}_I^* / \|\mathbf{X}_I^*\|_\infty$ conditional on $\ell_I(\mathbf{X}^*) := \|\mathbf{X}_I^*\|_\infty > u$ as $u \rightarrow \infty$. While existing works typically focus on a direct estimation of the distribution of Θ^I , i.e. the spectral measure, we focus on moment estimators of Θ^I . Our approach further differs from previous works since we impose the assumption that the original observations \mathbf{X} are standard regularly varying with a unique tail index $\alpha > 0$ and tail equivalent margins. This restriction allows us to consider a two-step procedure in the spirit of Einmahl et al. [1997] with a simpler first standardization step relying on the componentwise normalization via upper order statistics and the estimation of a unique coefficient $\alpha > 0$ via a Hill type estimator. Our second step is then based on the estimation of moments of convex combinations of the spectral vector, i.e. estimators of $\mathbb{E}[(\mathbf{v}^\top \Theta^I)^p]$ with $p \geq 1$ and \mathbf{v} on the simplex of the direction I . We prove that the collection of such moments $\mathbb{E}[(\mathbf{v}^\top \Theta^I)^p]$, $p \geq 1$, is characterizing the spectral measure of \mathbf{X}_I^* . Perceiving the resulting estimators as functions of the weight vector \mathbf{v} on the simplex, we obtain functional central limit theorems with precise asymptotic (co-)variances. Here, we consider both the case that the margins of \mathbf{X} are known, i.e. we can directly work with exactly standardized observations \mathbf{X}^* , and the case that the margins are unknown and a preliminary empirical standardization step has to be included.

In accordance to the above-mentioned paradigm that, in large dimensions d , emphasis is often put on simpler summary statistics, we suggest to mainly focus on small p for which our approach provides simple functions that contain important information on the dependence among extremes. For instance, by tail equivalence of the marginals of \mathbf{X}^* we have that

$\mathbb{E}[\Theta_i^I] = \mathbb{E}[\Theta_j^I] = 1/\tau_I$ for any $i, j \in I$ and, consequently, for any positive weights \mathbf{v} such that $\sum_{i \in I} v_i = 1$ we have that $\mathbb{E}[\mathbf{v}^T \Theta^I] = 1/\tau_I$. Thus, the moment of order $p = 1$ of convex combinations of Θ^I does not depend on the weight vector \mathbf{v} . This observation motivates an additional third step of our procedure. More precisely, similarly to Mainik and Rüschendorf [2010] who estimate the optimal portfolio of jointly regularly varying assets, we use the functional central limit theorem to estimate the weight vector \mathbf{v}^* that minimizes the asymptotic variance and show that this minimal asymptotic variance can be achieved by a plug-in estimator. We note that the determination of \mathbf{v}^* is interesting on its own and provides important information on the dependence among extremes. The determination of a prominent direction in the data is related to recent developments on extreme PCA in Cooley and Thibaud [2019] and Drees and Sabourin [2019]. Moreover we discuss how the weights \mathbf{v}^* are impacted by the fact that the margins are known or not, even in our setting of standard regular variation where the extremes in different directions are comparable.

The paper is organized as follows. In Section 2, we gather several results on convex combinations of the spectral components of a regularly varying vector. The asymptotic normality of empirical moment estimators is given in Section 3 both for the case of known and the case of unknown margins. These results specified to first order moments are then applied in Section 4 in order to obtain estimators for extremal coefficients with minimal asymptotic variance. Some comparison to benchmark estimators based on the definition of τ_I and some illustration are provided. We conclude with a short discussion on our results in Section 5.

2. Extremes of convex combinations of regularly varying random vectors. Let $\mathbf{X} = (X_1, \dots, X_d)^\top$ be a regularly varying $E = [0, \infty)^d$ -valued random vector, that is, there exists a regularly varying function $a : (0, \infty) \rightarrow (0, \infty)$ such that

$$a(u)\mathbb{P}(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu(\cdot), \quad u \rightarrow \infty,$$

for a measure μ on $E \setminus \{\mathbf{0}\}$ which is homogeneous of order $-\alpha < 0$, the so-called *exponent measure*. In the following, we will assume that the exponent measure is non-degenerate, i.e.

$$(2) \quad r_i := \mu(\{\mathbf{x} \in E : x_i > 1\}) > 0 \quad \text{for all } i = 1, \dots, d.$$

In particular, $a(u)\mathbb{P}(X_i > u) \rightarrow r_i$ as $u \rightarrow \infty$ for any $1 \leq i \leq d$ and thus

$$\frac{\mathbb{P}(X_i > u)}{\mathbb{P}(X_j > u)} \rightarrow \frac{r_i}{r_j} \in (0, \infty), \quad u \rightarrow \infty,$$

for all $i, j \in \{1, \dots, d\}$, i.e. X_1, \dots, X_d are tail equivalent. Assuming that the index $\alpha > 0$ and the scaling factors $r_1, \dots, r_d > 0$ are known, we will directly focus on the transformed vector $\mathbf{X}^* = (r_1^{-1}X_1^\alpha, \dots, r_d^{-1}X_d^\alpha)$ which is necessarily regularly varying with index 1. With $a^*(u) = a(u^{1/\alpha})$, we obtain that

$$(3) \quad a^*(u)\mathbb{P}(u^{-1}\mathbf{X}^* \in \cdot) \xrightarrow{v} \mu^*(\cdot), \quad u \rightarrow \infty,$$

for some normalized exponent measure μ^* which is homogeneous of order -1 and satisfies

$$(4) \quad \mu^*(\{\mathbf{x} \in E : x_i > 1\}) = \lim_{u \rightarrow \infty} a^*(u)\mathbb{P}(X_i^* > u) = 1,$$

i.e. $a^*(u) \sim \mathbb{P}(X_i^* > u)^{-1}$ for all $i = 1, \dots, d$. This verifies that the tail behaviour of all the components of \mathbf{X}^* is asymptotically identical, i.e. its marginal distribution functions F_1^*, \dots, F_d^* satisfy

$$\frac{1 - F_i^*(u)}{1 - F_j^*(u)} \rightarrow 1, \quad u \rightarrow \infty,$$

for all $i, j \in \{1, \dots, d\}$, which also implies

$$(5) \quad \frac{(F_i^*)^{-1}(p)}{(F_j^*)^{-1}(p)} \rightarrow 1, \quad p \rightarrow 1.$$

Equation (3) can be equivalently expressed in terms of a spectral component for an arbitrary norm $\|\cdot\|$ on \mathbb{R}^d . More precisely, we obtain the weak convergence

$$(6) \quad \mathcal{L}(u^{-1} \mathbf{X}^* \mid \|\mathbf{X}^*\| > u) \xrightarrow{w} \mathcal{L}(Y \Theta), \quad u \rightarrow \infty,$$

where Y is a unit Pareto random variable and, independently from Y , Θ is a $[0, \infty)^d$ -valued random vector satisfying $\|\Theta\| = 1$ almost surely, i.e.

$$\mathbb{P}(\Theta \in S_{d-1}^+) = 1, \quad \text{where } S_{d-1}^+ = \{\mathbf{x} \in [0, \infty)^d : \|\mathbf{x}\| = 1\}.$$

In the following, we will focus on the maximum norm $\|\cdot\| = \|\cdot\|_\infty$ which is not a serious restriction as all norms on \mathbb{R}^d are equivalent.

There is a one-to-one correspondence between the normalized exponent measure μ^* and the distribution of Θ , the so-called (normalized) *spectral measure* (w.r.t. the norm $\|\cdot\|$) via the relation

$$(7) \quad \mu^*(\{\mathbf{x} \in E : \|\mathbf{x}\| > r, \mathbf{x}/\|\mathbf{x}\| \in B\}) = \tau r^{-\alpha} \mathbb{P}(\Theta \in B), \quad r > 0, B \subset S_{d-1}^+$$

for some constant $1 \leq \tau \leq d$ given by

$$(8) \quad \tau = \mu^*(\{\mathbf{x} \in E : \|\mathbf{x}\| > 1\}) = \lim_{u \rightarrow \infty} a^*(u) \mathbb{P}(\|\mathbf{X}^*\| > u).$$

Thus, the extremal dependence structure of the components of the normalized random vector \mathbf{X}^* , and, thus, also of the original vector \mathbf{X} , is fully described by the distribution of the spectral vector Θ which we will estimate in the following.

More precisely, we now describe the tail behaviour of convex combinations of X_1^*, \dots, X_d^* in terms of the corresponding spectral measure. To this end, we will make use of the following well-known lemma which is a direct consequence of the weak convergence in (6).

LEMMA 1. *Let \mathbf{X}^* , Y and Θ be as above. Furthermore, let $g : (0, \infty) \times S_{d-1}^+ \rightarrow \mathbb{R}$ be a continuous bounded function. Then, we have*

$$\lim_{u \rightarrow \infty} \mathbb{E} [g(u^{-1} \|\mathbf{X}^*\|, \mathbf{X}^*/\|\mathbf{X}^*\|) \mid \|\mathbf{X}^*\| > u] = \mathbb{E}[g(Y, \Theta)].$$

In particular, the lemma implies that

$$(9) \quad \lim_{u \rightarrow \infty} \mathbb{E}[\tilde{g}(\mathbf{X}^*/\|\mathbf{X}^*\|) \mid \|\mathbf{X}^*\| > u] = \mathbb{E}[\tilde{g}(\Theta)]$$

for any continuous function $\tilde{g} : S_{d-1}^+ \rightarrow \mathbb{R}$.

Furthermore, the lemma extends to indicator functions of sets that are continuous with respect to the measure of (Y, Θ) , i.e. conditional probabilities of events that can be expressed in terms of $u^{-1} \|\mathbf{X}^*\|$ and $\mathbf{X}^*/\|\mathbf{X}^*\|$. For instance, for convex combinations of X_1^*, \dots, X_d^* with coefficient vector $\mathbf{v} \in B_1(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_1 \leq 1\}$, $\mathbf{v} \neq \mathbf{0}$, we obtain

$$(10) \quad \begin{aligned} \lim_{u \rightarrow \infty} \mathbb{P}(\mathbf{v}^\top \mathbf{X}^* > u \mid \|\mathbf{X}^*\| > u) &= \mathbb{P}(Y \mathbf{v}^\top \Theta > 1) \\ &= \mathbb{E}[(\mathbf{v}^\top \Theta)_+ \wedge 1] = \mathbb{E}[(\mathbf{v}^\top \Theta)_+] \end{aligned}$$

using that $\mathbf{v}^\top \Theta \leq \|\mathbf{v}\|_1 \cdot \|\Theta\| = \|\mathbf{v}\|_1 \leq 1$ a.s. It can be shown that the law of Θ is uniquely determined by the set of all the limits in (10) with $\mathbf{v} \in \partial B_1(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_1 = 1\}$, see Basrak et al. [2002], Klüppelberg and Pergamenchtchikov [2007], Boman and Lindskog [2009]. This fact motivates the use of extremes of convex combinations of observations for inference of the spectral measure. In this paper, we modify the above approach by assessing the limiting quantity on the RHS of Equation (10) in an alternative way based on the following result which is a trivial consequence of Lemma 1.

LEMMA 2. *Let \mathbf{X}^* be a regularly varying random vector with standard Pareto margins. Then, for any $\mathbf{v} \in \mathbb{R}^d$ and $p \in \mathbb{N}$, we have*

$$(11) \quad \mathbb{E} \left[\left(\mathbf{v}^\top \frac{\mathbf{X}^*}{\|\mathbf{X}^*\|} \right)_+^p \mid \|\mathbf{X}^*\| > u \right] \longrightarrow \mathbb{E}[(\mathbf{v}^\top \Theta)_+^p]$$

as $u \rightarrow \infty$.

PROOF. The result directly follows by applying Equation (9) to the continuous functions $\tilde{g}_{\mathbf{v},p}(\theta) = (\mathbf{v}^\top \theta)_+^p$ with $\mathbf{v} \in \mathbb{R}^d$ and $p \in \mathbb{N}$. \square

Note that, for $\mathbf{v} \not\geq \mathbf{0}$, the term $\mathbf{v}^\top \Theta$ is negative with positive probability, which hampers the calculation of closed-form expressions for the RHS of (11). In order to avoid these difficulties, we restrict our attention to non-negative vectors $\mathbf{v} \in \partial B_1^+(\mathbf{0}) = \partial B_1(\mathbf{0}) \cap [0, \infty)^d$, i.e. we consider only convex combinations of components of \mathbf{X}^* with positive coefficients, while allowing for arbitrary powers of these combinations. This ensures the identifiability of the spectral measure from the resulting limits as the following proposition shows.

PROPOSITION 3. *Let \mathbf{X}^* be a non-negative regularly varying random vector with index 1. Then, the spectral measure of Θ is uniquely determined by the moments*

$$(12) \quad \left\{ \mathbb{E}[(\mathbf{v}^\top \Theta)_+^p] : \mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in \mathbb{N} \right\} = \left\{ \mathbb{E}[(\mathbf{v}^\top \Theta)^p] : \mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in \mathbb{N} \right\}.$$

PROOF. By a series expansion, for all $\mathbf{s} \in [0, \infty)^d$, the limits in (12) uniquely determine

$$\mathbb{E}[\exp(-\mathbf{s}^\top \Theta)] = \sum_{p=0}^{\infty} (-1)^p \cdot \frac{\|\mathbf{s}\|_1^p}{p!} \cdot \mathbb{E} \left[\left(\frac{\mathbf{s}^\top}{\|\mathbf{s}\|_1} \Theta \right)^p \right], \quad \mathbf{s} \in [0, \infty),$$

i.e. the Laplace transform of the spectral measure. \square

Let us list some properties of the function $\mathbf{v} \mapsto m_p(\mathbf{v}) = \mathbb{E}[(\mathbf{v}^\top \Theta)^p]$ for $p = 1, 2, \dots$ and $\mathbf{v} \in \partial B_1^+(\mathbf{0})$. We identify $m_1(\mathbf{v}) = \sum_{i=1}^d v_i \mathbb{E}[\Theta_i]$, i.e. the function is linear for $p = 1$. As, for all $i = 1, \dots, d$,

$$(13) \quad \begin{aligned} \mathbb{E}[\Theta_i] &= \int_0^1 \mathbb{P}(\Theta_i > x) dx = \int_1^\infty y^{-2} \mathbb{P}(\Theta_i > 1/y) dy \\ &= \mathbb{P}(Y\Theta_i > 1) = \lim_{u \rightarrow \infty} \mathbb{P}(X_i^* > u \mid \|\mathbf{X}^*\| > u) = \frac{1}{a^*(u) \mathbb{P}(\|\mathbf{X}^*\| > u)} = \frac{1}{\tau}, \end{aligned}$$

where τ is defined as in Equation (8), we even obtain $m_1(\mathbf{v}) = \frac{1}{\tau} \sum_{i=1}^d v_i$, which implies that m_1 is constant on $\partial B_1^+(\mathbf{0})$. For $p \geq 2$, the function m_p is strongly convex. Thus, one easily deduces from Jensen's inequality that

$$m_p(\mathbf{v}) \leq \max_{1 \leq i \leq d} \mathbb{E}[\Theta_i^p].$$

For many popular limit models, closed-form expressions for $\mathbb{E}[(\mathbf{v}^\top \Theta)^p]$ are difficult to obtain as the spectral measure w.r.t. the maximum norm $\|\cdot\| = \|\cdot\|_\infty$ is often difficult to handle in high dimensional setting. Instead, spectral vectors are often normalized w.r.t. some less complex functional $\ell_I(\mathbf{x}) = \bigvee_{i \in I} x_i$ over some smaller subset $I \subset \{1, \dots, d\}$. Such functionals ℓ_I play an important role when asserting extremal coefficients, see Section 4. Then, we have that $\partial\{\mathbf{x} \in E : \ell_I(\mathbf{x}) > 1\} \subset \{\mathbf{x} \in E : \ell_I(\mathbf{x}) = 1\}$ is a μ^* -null set and, thus, the limit

$$(14) \quad \tau_I := \mu^*(\{\mathbf{x} \in E : \ell_I(\mathbf{x}) > 1\}) = \lim_{u \rightarrow \infty} a^*(u) \mathbb{P}(\ell_I(\mathbf{X}^*) > u)$$

exists and, by Equation (3), equals $\mu^*(\{\mathbf{x} \in E : \ell_I(\mathbf{x}) > 1\})$. Furthermore, for any $I \neq \emptyset$, it satisfies the relation

$$(15) \quad \tau_I = \mathbb{E}[\ell_I(\Theta)] \cdot \tau \in [1, |I|].$$

Since $\mathbb{E}[\ell_I(\Theta)] > 0$ as soon as $I \neq \emptyset$, we can study the behaviour of the ℓ_I -spectral vector Θ^I whose distribution can be defined from the original spectral measure via the relation

$$(16) \quad \mathbb{P}(\Theta^I \in A) = \frac{1}{\mathbb{E}[\ell_I(\Theta)]} \int_{[0, \infty)^d} \mathbf{1}\{\boldsymbol{\theta}/\ell_I(\boldsymbol{\theta}) \in A\} \ell_I(\boldsymbol{\theta}) \mathbb{P}(\Theta \in d\boldsymbol{\theta}),$$

where $A \subset \{\mathbf{w} \in [0, \infty)^d : \ell_I(\mathbf{w}) = 1\}$. Note that, from Equation (15) and Equation (16), we directly obtain an analogous result to Equation (13), namely

$$(17) \quad \mathbb{E}[\Theta_i^I] = \frac{1}{\tau_I}, \quad i = 1, \dots, d.$$

Analogues to Equation (10) and to Lemma 2 are given in the following proposition.

PROPOSITION 4. *Let \mathbf{X}^* be a normalized regularly varying random vector with index 1. Then, for any $\mathbf{v} \in \mathbb{R}^d$ and $k \in \mathbb{N}$, we have*

$$(18) \quad \mathbb{P}\left(\mathbf{v}^\top \mathbf{X}^* > u \mid \ell_I(\mathbf{X}^*) > u\right) \longrightarrow \mathbb{E}\left[\left(\mathbf{v}^\top \Theta^I\right)_+ \wedge 1\right] = \frac{\mathbb{E}\left[\left(\mathbf{v}^\top \Theta\right)_+ \wedge \ell_I(\Theta)\right]}{\mathbb{E}[\ell_I(\Theta)]}$$

$$(19) \quad \mathbb{E}\left[\left(\mathbf{v}^\top \frac{\mathbf{X}^*}{\ell_I(\mathbf{X}^*)}\right)_+^p \mid \ell_I(\mathbf{X}^*) > u\right] \longrightarrow \mathbb{E}\left[\left(\mathbf{v}^\top \Theta^I\right)_+^p\right] = \frac{\mathbb{E}\left[\left(\mathbf{v}^\top \frac{\Theta}{\ell_I(\Theta)}\right)_+^p \cdot \ell_I(\Theta)\right]}{\mathbb{E}[\ell_I(\Theta)]}$$

as $u \rightarrow \infty$.

PROOF. First, we note that by definition of ℓ_I we $\ell_I(\mathbf{x}) \leq \|\mathbf{x}\|$ on $[0, \infty)^d \setminus \{\mathbf{0}\}$. Thus $\ell_I(\mathbf{X}^*) > u$ implies that $\|\mathbf{X}^*\| > u$ and we have

$$\begin{aligned} \lim_{u \rightarrow \infty} \mathbb{P}\left(\mathbf{v}^\top \mathbf{X}^* > u \mid \ell_I(\mathbf{X}^*) > u\right) &= \lim_{u \rightarrow \infty} \frac{\mathbb{P}\left(\mathbf{v}^\top \mathbf{X}^* > u, \ell_I(\mathbf{X}^*) > u \mid \|\mathbf{X}^*\| > u\right)}{\mathbb{P}\left(\ell_I(\mathbf{X}^*) > u \mid \|\mathbf{X}^*\| > u\right)} \\ &= \frac{\mathbb{P}\left(Y(\mathbf{v}^\top \Theta) > 1, Y \ell_I(\Theta) > 1\right)}{\mathbb{P}\left(Y \ell_I(\Theta) > 1\right)} = \frac{\mathbb{E}\left[\left(\mathbf{v}^\top \Theta\right)_+ \wedge \ell_I(\Theta) \wedge 1\right]}{\mathbb{E}\left[\ell_I(\Theta) \wedge 1\right]} \\ &= \frac{1}{\mathbb{E}[\ell_I(\Theta)]} \cdot \mathbb{E}\left[\left(\mathbf{v}^\top \Theta\right)_+ \wedge \ell_I(\Theta)\right] = \mathbb{E}\left[\left(\mathbf{v}^\top \Theta^I\right)_+ \wedge 1\right], \end{aligned}$$

where we used the fact that $\ell_I(\Theta) \leq \|\Theta\| = 1$ and we apply the change of measure (16). Using the same arguments and applying Lemma 1 to the bounded measurable function

$g(y, \theta) = (\mathbf{v}^\top \theta)_+^p \mathbf{1}\{\ell_I(y\theta) > 1\}$ whose discontinuity set $\{\ell_I(y\theta) = 1\}$ has no limiting mass, i.e. $\mathbb{P}(\ell_I(Y\Theta) = 1) = \mathbb{P}(Y\ell_I(\Theta) = 1) = 0$, we further obtain

$$\begin{aligned} & \lim_{u \rightarrow \infty} \mathbb{E} \left[\left(\mathbf{v}^\top \frac{\mathbf{X}^*}{\ell_I(\mathbf{X}^*)} \right)_+^p \mid \ell_I(\mathbf{X}^*) > u \right] \\ &= \lim_{u \rightarrow \infty} \frac{\mathbb{E} \left[\left(\mathbf{v}^\top \frac{\mathbf{X}^*}{\ell_I(\mathbf{X}^*)} \right)_+^p \mathbf{1}\{\ell_I(\mathbf{X}^*) > u\} \mid \|\mathbf{X}^*\| > u \right]}{\mathbb{P}(\ell_I(\mathbf{X}^*) > u \mid \|\mathbf{X}^*\| > u)} \\ &= \frac{\mathbb{E} \left[\left(\mathbf{v}^\top \frac{\Theta}{\ell_I(\Theta)} \right)_+^p \mathbf{1}\{Y\ell_I(\Theta) > 1\} \right]}{\mathbb{P}(Y\ell_I(\Theta) > 1)} = \frac{\mathbb{E} \left[\left(\mathbf{v}^\top \frac{\Theta}{\ell_I(\Theta)} \right)_+^p \cdot (\ell_I(\Theta) \wedge 1) \right]}{\mathbb{E}[\ell_I(\Theta) \wedge 1]} \\ &= \frac{1}{\mathbb{E}[\ell_I(\Theta)]} \cdot \mathbb{E} \left[\left(\mathbf{v}^\top \frac{\Theta}{\ell_I(\Theta)} \right)_+^p \cdot \ell_I(\Theta) \right] = \mathbb{E} \left[\left(\mathbf{v}^\top \Theta^I \right)_+^p \right]. \end{aligned}$$

□

Similarly to Proposition 3, it can be shown that the distribution of Θ^I is uniquely determined by the limits

$$\left\{ \mathbb{E}[(\mathbf{v}^\top \Theta^I)_+^p]; \mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in \mathbb{N} \right\} = \left\{ \mathbb{E}[(\mathbf{v}^\top \Theta^I)^p]; \mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in \mathbb{N} \right\}.$$

While the ℓ_I -spectral measure, i.e. the distribution of Θ^I , can be written in terms of the original spectral measure as in (16), the original spectral measure can be uniquely recovered from the distribution of Θ^I only if $\mathbb{P}(\ell_I(\Theta) = 0) = 0$. Then, the inverse transformation is given by

$$(20) \quad \mathbb{P}(\Theta \in A) = \frac{1}{\mathbb{E}[\|\Theta^I\|]} \int_{[0, \infty)^d} \mathbf{1}\{\theta / \|\theta\| \in A\} \|\theta\| \mathbb{P}(\Theta^I \in d\theta), \quad A \subset S_{d-1}^+.$$

Note that the discussion above can be easily extended to any positively homogeneous continuous functional $\ell : [0, \infty)^d \rightarrow [0, \infty)$ with corresponding constant τ_ℓ which is positive.

3. Asymptotic normality.

3.1. Asymptotic normality for deterministic thresholds. Assume that we observe independent copies $\mathbf{X}_l^* = (X_{l1}^*, \dots, X_{ld}^*)^\top$, $l = 1, \dots, n$, of the normalized regularly varying random vector \mathbf{X}^* . Consider a non-empty subset $I \subset \{1, \dots, d\}$. One of the usual approaches to estimate the distribution of the ℓ_I -spectral vector Θ^I is based on the conditional exceedance probabilities of convex combinations given in relation (18), i.e.

$$\lim_{u \rightarrow \infty} \mathbb{P}(\mathbf{v}^\top \mathbf{X}^* > u \mid \ell_I(\mathbf{X}^*) > u) = \mathbb{E} \left[(\mathbf{v}^\top \Theta^I)_+ \wedge 1 \right]$$

for $\mathbf{v} \in \partial B_1(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_1 = 1\}$. The associated empirical estimators are of the form

$$(21) \quad \frac{\widehat{P}_{n,u,I}^{(\text{conv})}(\mathbf{v})}{\widehat{P}_{n,u,I}} = \frac{n^{-1} \sum_{l=1}^n \mathbf{1}\{\mathbf{v}^\top \mathbf{X}_l^* > u, \ell_I(\mathbf{X}_l^*) > u\}}{n^{-1} \sum_{l=1}^n \mathbf{1}\{\ell_I(\mathbf{X}_l^*) > u\}}, \quad \mathbf{v} \in \partial B_1(\mathbf{0}).$$

In view of Proposition 3, we promote in this paper the use of the alternative characterization of the distribution of the ℓ_I -spectral vector Θ^I based on the limits of moments of convex combinations provided by Equation (19), i.e.

$$\lim_{u \rightarrow \infty} \mathbb{E} \left[\left(\mathbf{v}^\top \frac{\mathbf{X}^*}{\ell_I(\mathbf{X}^*)} \right)_+^p \mid \ell_I(\mathbf{X}^*) > u \right] = \mathbb{E} \left[(\mathbf{v}^\top \Theta^I)^p \right],$$

for any $\mathbf{v} \in \partial B_1^+(\mathbf{0})$ and $p \in \mathbb{N}_0$. The associated empirical estimators are of the form

$$(22) \quad \frac{\widehat{M}_{n,u,I}(\mathbf{v}, p)}{\widehat{P}_{n,u,I}} = \frac{n^{-1} \sum_{l=1}^n \left(\mathbf{v}^\top \frac{\mathbf{X}_l^*}{\ell_I(\mathbf{X}_l^*)} \right)^p \mathbf{1}\{\ell_I(\mathbf{X}_l^*) > u\}}{n^{-1} \sum_{l=1}^n \mathbf{1}\{\ell_I(\mathbf{X}_l^*) > u\}}, \quad \mathbf{v} \in \partial B_1^+(\mathbf{0}).$$

In the following, we will consider the new estimators $\widehat{M}_{n,u,\ell}(\mathbf{v}, p)/\widehat{P}_{n,u,\ell}$ for any $p \in \mathbb{N}_0$. As the functional ℓ_I puts emphasis on the components $(X_i^*)_{i \in I}$ of \mathbf{X}^* only, we will focus on the same components in the convex combination, i.e. we will consider weight vectors \mathbf{v}_I where \mathbf{v}_I is the vector with components v_i for $i \in I$ and 0 otherwise. Thus, we obtain $\mathbf{v}_I^\top \mathbf{X}^* = \sum_{i \in I} v_i X_i^*$.

The next theorem will provide a functional central limit theorem for the estimators $\widehat{M}_{n,u,I}(\mathbf{v}_I, p)$ noting that $\widehat{P}_{n,u,I} = \widehat{M}_{n,u,I}(\mathbf{v}_I, 0)$ for all $\mathbf{v} \in \partial B_1^+(\mathbf{0})$. Then we can apply the functional delta method to obtain the asymptotic result for the ratio estimator $\widehat{M}_{n,u,I}(\mathbf{v}_I, p)/\widehat{P}_{n,u,I}$.

While, in this subsection, we assume observations from the normalized random vector \mathbf{X}^* , in Subsection 3.2 the more general case of observations from the regularly varying vector \mathbf{X} with unknown index $\alpha > 0$ and unknown marginal scales is considered. To be disposed for this extension, in the next theorem, we do not only prove a functional central limit theorem for $\widehat{M}_{n,u,I}(\mathbf{v}_I, p)$ with $\mathbf{v} \in \partial B_1^+(\mathbf{0})$ and $p \in \mathbb{N}_0$, but already for a more general version.

Indeed, it will turn out to be useful to allow for multiplication of single components of \mathbf{X}^* by a vector $\mathbf{s} \in [0, \infty)^d$ of factors and for raising them up to a power of $1/\beta > 0$. In the following, let $(\mathbf{s} \circ \mathbf{X}^*)^{1/\beta} = (s_i^{1/\beta} X_i^*)_{i=1}^d$. With this notation, we can write

$$\ell_I(\mathbf{X}^*) = \max_{i \in I} |X_i^*| = \|\mathbf{1}_I \circ \mathbf{X}^*\|,$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$, i.e., according to the notation introduced above, $\mathbf{1}_I$ is the vector with the i th component being equal to 1 if $i \in I$ and being equal to 0 otherwise. Furthermore, $\mathbf{v}_I^\top \mathbf{X}^* = \mathbf{v}^\top (\mathbf{1}_I \circ \mathbf{X}^*)$. Thus, we can rewrite

$$(23) \quad \widehat{M}_{n,u,I}(\mathbf{v}_I, p) = \frac{1}{n} \sum_{l=1}^n \left(\mathbf{v}^\top \frac{\mathbf{1}_I \circ \mathbf{X}_l^*}{\|\mathbf{1}_I \circ \mathbf{X}_l^*\|} \right)^p \mathbf{1}\{\|\mathbf{1}_I \circ \mathbf{X}_l^*\| > u\},$$

with the right-hand-side of the equation depends on the components v_i , $i \in I$, of \mathbf{v} only. In our extended version of the functional central limit theorem, we will not restrict our attention to the vector $\mathbf{1}_I \circ \mathbf{X}^*$ being extreme, but a perturbed vector $(\mathbf{s} \circ \mathbf{X}^*)^{1/\beta}$ with \mathbf{s} being close to $\mathbf{1}_I$ and β being close to 1. More precisely, we consider $\mathbf{s} \in A_\delta$ where

$$A_{\delta,I} = \left\{ \mathbf{s} \in (0, \infty)^d : s_i = 0 \text{ for all } i \notin I, (1 + \delta)^{-1} \leq s_i \leq 1 + \delta \text{ for all } i \in I \right\}$$

for some $\delta \geq 0$, i.e. we allow for a perturbation of a factor between $(1 + \delta)^{-1}$ and $1 + \delta$ in each component of \mathbf{s} with index $i \in I$ and β . We also denote the disjoint union

$$A_\delta = \bigcup_{\emptyset \neq I \subset \{1, \dots, d\}} A_{\delta,I}.$$

Replacing $\mathbf{1}_I \circ \mathbf{X}^*$ in Equation (23) by $(\mathbf{s} \circ \mathbf{X}^*)^{1/\beta}$, we obtain the generalized estimator

$$\widehat{M}_{n,u}(\mathbf{v}, \mathbf{s}, \beta, p) = \frac{1}{n} \sum_{l=1}^n \left(\mathbf{v}^\top \frac{(\mathbf{s} \circ \mathbf{X}_l^*)^{1/\beta}}{\|(\mathbf{s} \circ \mathbf{X}_l^*)^{1/\beta}\|} \right)^p \mathbf{1}\{\|\mathbf{s} \circ \mathbf{X}_l^*\| > u\},$$

for $\mathbf{v} \in \partial B_1^+(\mathbf{0})$, $(\mathbf{s}, \beta) \in A'_\delta$ and $p \in \mathbb{N}_0$, where $A'_\delta = A_\delta \times [(1 + \delta)^{-1}, (1 + \delta)^{-1}]$. By construction, $\widehat{M}_{n,u,I}(\mathbf{v}_I, p) = \widehat{M}_{n,u}(\mathbf{v}_I, \mathbf{1}_I, 1, p)$ for every $I \subset \{1, \dots, d\}$. In particular, the corresponding index set I of components being relevant for the maximum can

be read off from the components of \mathbf{s} that are strictly positive. Analogously, we define $\widehat{P}_{n,u}(\mathbf{s}) = \widehat{M}_{n,u}(\mathbf{v}, \mathbf{s}, \beta, 0)$ which depends neither on $\mathbf{v} \in \partial B_1^+(\mathbf{0})$ nor on $\beta > 0$, and satisfies $\widehat{P}_{n,u,I} = \widehat{P}_{n,u}(\mathbf{1}_I)$.

Applying Proposition 4, we obtain

$$\begin{aligned}
& a^*(u) \mathbb{E} \left[\widehat{M}_{n,u}(\mathbf{v}, \mathbf{s}, \beta, p) \right] \\
&= a^*(u) \mathbb{P}(\|\mathbf{s} \circ \mathbf{X}^*\| > u) \cdot \mathbb{E} \left[\left(\mathbf{v}^\top \frac{(\mathbf{s} \circ \mathbf{X}_l^*)^{1/\beta}}{\|(\mathbf{s} \circ \mathbf{X}_l^*)^{1/\beta}\|} \right)^p \middle| \|\mathbf{s} \circ \mathbf{X}^*\| > u \right] \\
(24) \quad & \xrightarrow{u \rightarrow \infty} \tau_\ell \cdot \mathbb{E} \left[\left(\mathbf{v}^\top (\boldsymbol{\Theta}^I)^{1/\beta} \right)^p \right] = \tau \cdot \mathbb{E} \left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \boldsymbol{\Theta}}{\|\mathbf{s} \circ \boldsymbol{\Theta}\|} \right)^{1/\beta} \right)^p \|\mathbf{s} \circ \boldsymbol{\Theta}\| \right],
\end{aligned}$$

where we used Equation (15) and Equation (16). Beyond the limiting behaviour of the expectation, the following theorem establishes asymptotic normality of $\widehat{M}_{n,u}(\mathbf{v}, \mathbf{1}_I, 1, p)$. A detailed proof is given in Appendix A.

THEOREM 5. *Let \mathbf{X}_l^* , $l \in \mathbb{N}$, be independent copies of a regularly varying $[0, \infty)^d$ -valued random vector \mathbf{X}^* with index $\alpha = 1$ satisfying Equation (4) and denote the corresponding spectral vector by $\boldsymbol{\Theta}$. Furthermore, let $u_n \rightarrow \infty$ such that $n/a^*(u_n) \rightarrow \infty$. Then, for any finite set $K_0 \subset \mathbb{N}_0$ and any $\delta \geq 0$, the sequence of processes $(\{G_n(\mathbf{v}, \mathbf{s}, \beta, p); \mathbf{v} \in \partial B_1^+(\mathbf{0}), (\mathbf{s}, \beta) \in A'_\delta, p \in K_0\})_{n \in \mathbb{N}}$ with*

$$(25) \quad G_n(\mathbf{v}, \mathbf{s}, \beta, p) = \sqrt{\frac{n}{a^*(u_n)}} \left[a^*(u_n) \widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, p) - a^*(u_n) \mathbb{E} \left[\widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, p) \right] \right]$$

converges weakly in $\ell^\infty(\partial B_1^+(\mathbf{0}) \times A'_\delta \times K_0)$ to a tight centered Gaussian process G with covariance

$$\begin{aligned}
(26) \quad & \text{Cov}(G(\mathbf{v}, \mathbf{s}, \beta_1, p_1), G(\mathbf{w}, \mathbf{t}, \beta_2, p_2)) \\
&= \tau \mathbb{E} \left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \boldsymbol{\Theta}}{\|\mathbf{s} \circ \boldsymbol{\Theta}\|} \right)^{1/\beta_1} \right)^{p_1} \left(\mathbf{w}^\top \left(\frac{\mathbf{t} \circ \boldsymbol{\Theta}}{\|\mathbf{t} \circ \boldsymbol{\Theta}\|} \right)^{1/\beta_2} \right)^{p_2} (\|\mathbf{s} \circ \boldsymbol{\Theta}\| \wedge \|\mathbf{t} \circ \boldsymbol{\Theta}\|) \right].
\end{aligned}$$

A direct consequence of the weak convergence of G_n is that $a^*(u_n) \widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, p)$ converges in probability to the limit expression in Equation (24). Consequently, the ratio estimator $\widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, p) / \widehat{P}_{n,u_n}(\mathbf{s})$ is consistent:

$$(27) \quad \frac{\widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, p)}{\widehat{P}_{n,u_n}(\mathbf{s})} = \frac{\widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, p)}{\widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, 0)} \rightarrow_p c(\mathbf{v}, \mathbf{s}, \beta, p)$$

for all $\mathbf{v} \in \partial B_1^+(\mathbf{0})$, $(\mathbf{s}, \beta) \in A'_\delta$ and $p \in \mathbb{N}_0$ where

$$c(\mathbf{v}, \mathbf{s}, \beta, p) := \frac{1}{\mathbb{E}[\|\mathbf{s} \circ \boldsymbol{\Theta}\|]} \cdot \mathbb{E} \left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \boldsymbol{\Theta}}{\|\mathbf{s} \circ \boldsymbol{\Theta}\|} \right)^{1/\beta} \right)^p \cdot \|\mathbf{s} \circ \boldsymbol{\Theta}\| \right].$$

Applying Theorem 5 and using the functional Delta method, we can even establish the desired functional limit theorem for the ratio estimator $\widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, p) / \widehat{P}_{n,u_n}(\mathbf{s})$ provided that we can neglect the preasymptotic bias that arises from the fact that Equation (24) yields an asymptotic relation only.

COROLLARY 6. *Let the assumptions of Theorem 5 hold and assume that there exist some $\delta \geq 0$ and some finite subset $K \subset \mathbb{N}$ such that, for all $p \in K \cup \{0\}$,*

$$(28) \quad \sqrt{\frac{n}{a^*(u_n)}} \left| a^*(u_n) \mathbb{E}[\widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, p)] - \tau \mathbb{E}[\|\mathbf{s} \circ \mathbf{X}^*\|] c(\mathbf{v}, \mathbf{s}, \beta, p) \right| \rightarrow 0$$

uniformly in $\mathbf{v} \in \partial B_1^(\mathbf{0})$ and $(\mathbf{s}, \beta) \in A'_\delta$.*

Then, the sequence of processes $(\{\tilde{G}_n(\mathbf{v}, \mathbf{s}, \beta, p); \mathbf{v} \in \partial B_1^+(\mathbf{0}), (\mathbf{s}, \beta) \in A'_\delta, p \in K\})_{n \in \mathbb{N}}$ with

$$\tilde{G}_n(\mathbf{v}, \mathbf{s}, \beta, p) = \sqrt{\frac{n}{a^*(u_n)}} \left(\frac{\widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, p)}{\widehat{P}_{n,u_n}(\mathbf{s})} - c(\mathbf{v}, \mathbf{s}, \beta, p) \right),$$

converges weakly in $\ell^\infty(\partial B_1^+(\mathbf{0}) \times A'_\delta \times K)$ to a tight centered Gaussian process \tilde{G} .

The proof of Corollary 6 is deferred to Appendix B. The expression of the covariance of \tilde{G} is provided there in Equation (51).

REMARK 7. 1. For the original estimator with deterministic threshold as given in (22), it is sufficient to restrict to the case that $\mathbf{s} = \mathbf{t} = \mathbf{1}_I$ and $\beta = 1$ in Corollary 6. Then, we have $\|\mathbf{s} \circ \Theta\|_\infty = \|\mathbf{t} \circ \Theta\|_\infty = \ell_I(\Theta)$ and the covariance of \tilde{G} given in (51) simplifies to

$$\begin{aligned} & \text{Cov} \left(\tilde{G}(\mathbf{v}, \mathbf{1}_I, 1, p_1), \tilde{G}(\mathbf{w}, \mathbf{1}_I, 1, p_2) \right) \\ &= \frac{1}{\tau \mathbb{E} \ell_I(\Theta)} \cdot \left(\frac{1}{\mathbb{E} \ell_I(\Theta)} \mathbb{E} \left[\left(\mathbf{v}^\top \frac{\Theta_I}{\ell_I(\Theta)} \right)^{p_1} \cdot \left(\mathbf{w}^\top \frac{\Theta_I}{\ell_I(\Theta)} \right)^{p_2} \cdot \ell_I(\Theta) \right] \right. \\ & \quad \left. - \frac{1}{(\mathbb{E} \ell_I(\Theta))^2} \mathbb{E} \left[\left(\mathbf{v}^\top \frac{\Theta_I}{\ell_I(\Theta)} \right)^{p_1} \cdot \ell_I(\Theta) \right] \mathbb{E} \left[\left(\mathbf{w}^\top \frac{\Theta_I}{\ell_I(\Theta)} \right)^{p_2} \cdot \ell_I(\Theta) \right] \right) \\ &= \frac{1}{\tau_I} \text{Cov} \left((\mathbf{v}^\top \Theta_I^I)^{p_1}, (\mathbf{w}^\top \Theta_I^I)^{p_2} \right). \end{aligned}$$

2. Using the same techniques as in the proof of Corollary 6, one can easily show the joint weak convergence of $(G_n(\mathbf{v}, \mathbf{s}, \beta_1, p_1), \tilde{G}_n(\mathbf{w}, \mathbf{t}, \beta_2, p_2))_{n \in \mathbb{N}}$ for any $(\mathbf{v}, \mathbf{s}, \beta_1, p_1), (\mathbf{w}, \mathbf{t}, \beta_2, p_2)$ in $\partial B_1^+(\mathbf{0}) \times A'_\delta \times K$ to $(G(\mathbf{v}, \mathbf{s}, \beta_1, p_1), \tilde{G}(\mathbf{w}, \mathbf{t}, \beta_2, p_2))$ which are jointly Gaussian. with covariance

$$\begin{aligned} & \text{Cov}(G(\mathbf{v}, \mathbf{s}, \beta_1, p_1), \tilde{G}(\mathbf{w}, \mathbf{t}, \beta_2, p_2)) \\ &= \frac{\mathbb{E} \left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \Theta}{\|\mathbf{s} \circ \Theta\|} \right)^{1/\beta_1} \right)^{p_1} \left(\mathbf{w}^\top \left(\frac{\mathbf{t} \circ \Theta}{\|\mathbf{t} \circ \Theta\|} \right)^{1/\beta_2} \right)^{p_2} (\|\mathbf{s} \circ \Theta\| \wedge \|\mathbf{t} \circ \Theta\|) \right]}{\mathbb{E}[\|\mathbf{t} \circ \Theta\|]} \\ & \quad - \frac{\mathbb{E} \left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \Theta}{\|\mathbf{s} \circ \Theta\|} \right)^{1/\beta_1} \right)^{p_1} (\|\mathbf{s} \circ \Theta\| \wedge \|\mathbf{t} \circ \Theta\|) \right] \mathbb{E} \left[\left(\mathbf{w}^\top \left(\frac{\mathbf{t} \circ \Theta}{\|\mathbf{t} \circ \Theta\|} \right)^{1/\beta_2} \right)^{p_2} \|\mathbf{t} \circ \Theta\| \right]}{(\mathbb{E}[\|\mathbf{t} \circ \Theta\|])^2}. \end{aligned}$$

3. From the proofs of Theorem 5 and Corollary 6, it follows that, for fixed $p \in \mathbb{N}_0$, the sequences of processes $G_n(\cdot, \cdot, \cdot, p)$ and $\tilde{G}_n(\cdot, \cdot, \cdot, p)$ are asymptotically uniformly equicontinuous on $\partial B_1^+(\mathbf{0}) \times A'_\delta$ w.r.t. the semimetric

$$\rho((\mathbf{v}, \mathbf{s}, \beta_1), (\mathbf{w}, \mathbf{t}, \beta_2)) = \max\{\|\mathbf{v} - \mathbf{w}\|, \|\mathbf{s} - \mathbf{t}\|, |\beta_1 - \beta_2|\}.$$

Consequently, almost all sample paths of the tight limits G and \tilde{G} , respectively, are uniformly ρ -continuous [cf. Addendum 1.5.8 in Van der Vaart and Wellner, 1996]. Since ρ induces the standard topology on $\partial B_1^+(\mathbf{0}) \times A'_\delta$ as subset of $\mathbb{R}^{|\mathcal{I}|} \times \mathbb{R}^{d+1}$, it follows that almost all sample paths of $G(\cdot, \cdot, \cdot, p)$ and $\tilde{G}(\cdot, \cdot, \cdot, p)$ are uniformly continuous.

3.2. *Asymptotic normality for random thresholds via order statistics.* Consider the general case of a random vector \mathbf{X} that is regularly varying and satisfies Equation (2). In practice, the marginal distributions of X_1, \dots, X_d are unknown. In particular, the index α of regular variation and the scaling factors r_1, \dots, r_d as defined in (2) that are needed for the transformation to the normalized vector \mathbf{X}^* cannot be assumed to be given. Furthermore, the normalizing function $a^*(\cdot) \sim \mathbb{P}(X_i^* > \cdot)$ is unknown. To cope with these difficulties, we propose to work with an estimator $\hat{\alpha}$ of α and order statistics of \mathbf{X} . More precisely, let $X_{k:n,i}$ denote the k th upper order statistic of X_{1i}, \dots, X_{ni} for $i = 1, \dots, d$ and let $\mathbf{X}_{k:n} = (X_{k:n,1}, \dots, X_{k:n,d})$. We propose the following estimator

$$\frac{\widetilde{M}_{n,k,I}(\mathbf{v}, p)}{\widetilde{P}_{n,k,I}} = \frac{n^{-1} \sum_{l=1}^n \left(\mathbf{v}_I^\top \left(\frac{\mathbf{X}_l / \mathbf{X}_{k:n}}{\ell_I(\mathbf{X}_l / \mathbf{X}_{k:n})} \right)^{\hat{\alpha}} \right)^p \mathbf{1}\{\ell_I(\mathbf{X}_l / \mathbf{X}_{k:n}) > 1\}}{n^{-1} \sum_{l=1}^n \mathbf{1}\{\ell_I(\mathbf{X}_l / \mathbf{X}_{k:n}) > 1\}}$$

for some $k < n$, where the ratio of vectors $\mathbf{X}_l / \mathbf{X}_{k:n}$ is to be interpreted componentwise. Expanding each of the ratios $\mathbf{X}_l / \mathbf{X}_{k:n}$ by $r_1^{-1}, \dots, r_d^{-1}$, respectively, and taking the power of α , it can be easily seen that

$$\frac{\widetilde{M}_{n,k,I}(\mathbf{v}, p)}{\widetilde{P}_{n,k,I}} = \frac{n^{-1} \sum_{l=1}^n \left(\mathbf{v}_I^\top \left(\frac{\mathbf{X}_l^* / \mathbf{X}_{k:n}^*}{\ell_I(\mathbf{X}_l^* / \mathbf{X}_{k:n}^*)} \right)^{\hat{\alpha}/\alpha} \right)^p \mathbf{1}\{\ell_I(\mathbf{X}_l^* / \mathbf{X}_{k:n}^*) > 1\}}{n^{-1} \sum_{l=1}^n \mathbf{1}\{\ell_I(\mathbf{X}_l^* / \mathbf{X}_{k:n}^*) > 1\}},$$

i.e. we can rewrite

$$(29) \quad \frac{\widetilde{M}_{n,k,I}(\mathbf{v}, p)}{\widetilde{P}_{n,k,I}} = \frac{\widehat{M}_{n,u_n}(\mathbf{v}_I, u_n / \mathbf{X}_{k:n}^* \circ \mathbf{1}_I, \alpha / \hat{\alpha}, p)}{\widehat{P}_{n,u_n}(u_n / \mathbf{X}_{k:n}^* \circ \mathbf{1}_I)}.$$

For an appropriate sequence $\{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ with $k_n \rightarrow \infty$ such that $k_n/n \rightarrow 0$, we want to show that

$$\left(\left\{ \sqrt{k_n} \left(\frac{\widetilde{M}_{n,k_n,I}(\mathbf{v}, p)}{\widetilde{P}_{n,k_n,I}} - \mathbb{E}[(\mathbf{v}_I^\top \Theta_I^I)^p] \right); \mathbf{v} \in \partial B_1^+(\mathbf{0}) \right\} \right)_{n \in \mathbb{N}}$$

converges weakly to a Gaussian process. The functional central limit theorem will essentially be proven by plugging the random scaling factor $\mathbf{s} = u_n / \mathbf{X}_{k:n}^* \circ \mathbf{1}_I$ and the random power $\beta = \alpha / \hat{\alpha}$ into the results of Subsection 3.1 according to Equation (29). Provided that both the random \mathbf{s} and the random β are asymptotically normal, we can use the functional central limit theorem given in Corollary 6 to obtain the desired result.

To this end, we will first specify our estimator $\hat{\alpha}$ of α which will be a Hill type estimator. Similarly to Section 3.1, we will first start with an estimator with a fixed sequence of thresholds $\{u_n\}_{n \in \mathbb{N}}$ and normalized observations $\mathbf{X}_1^*, \dots, \mathbf{X}_n^*$. To mimic the behaviour of the real non-normalized data $\mathbf{X} = (\mathbf{r} \circ \mathbf{X}^*)^{1/\alpha}$ where $\mathbf{r} = (r_1, \dots, r_d)$, we also allow for a scaling factor $\mathbf{s} \in A_\delta$ and take the norm of the scaled vector to the power of $1/\alpha$. Thus, we obtain the estimator

$$(30) \quad \frac{\widehat{L}_{n,u}(\mathbf{s})}{\widehat{P}_{n,u}(\mathbf{s})} = \frac{n^{-1} \sum_{l=1}^n \log \left(\left\| \mathbf{s} \circ \frac{\mathbf{X}_l^*}{u_n} \right\|^{1/\alpha} \right) \mathbf{1}\{\|\mathbf{s} \circ \mathbf{X}_l^*\| > u_n\}}{\widehat{P}_{n,u}(\mathbf{s})}, \quad \mathbf{s} \in A_\delta.$$

Under the assumptions of Theorem 5, it can be shown that

$$\begin{aligned} a^*(u_n) \mathbb{E}[\widehat{L}_{n,u_n}(\mathbf{s})] &\sim \tau \mathbb{E}[\|\mathbf{s} \circ \Theta\|] \cdot \frac{1}{\alpha} \mathbb{E} \left[\log \left(\left\| \mathbf{s} \circ \frac{\mathbf{X}^*}{u_n} \right\| \right) \mid \|\mathbf{s} \circ \mathbf{X}^*\| > u_n \right] \\ &\rightarrow \tau \mathbb{E}[\|\mathbf{s} \circ \Theta\|] \cdot \frac{1}{\alpha} \end{aligned}$$

as $n \rightarrow \infty$. If we assume that the bias is negligible, i.e.

$$(31) \quad \sup_{\mathbf{s} \in A_\delta} \sqrt{\frac{n}{a^*(u_n)}} \left| a^*(u_n) \mathbb{E}[\widehat{L}_{n,u_n}(\mathbf{s})] - \tau \mathbb{E}[\|\mathbf{s} \circ \Theta\|] \frac{1}{\alpha} \right| = 0,$$

we can establish asymptotic normality for our final Hill type estimator $\widehat{\alpha}_{k,n,I}$ defined via order statistics

$$(32) \quad \frac{1}{\widehat{\alpha}_{n,k,I}} = \frac{n^{-1} \sum_{l=1}^n \log(\ell_I(\mathbf{X}_l / \mathbf{X}_{k:n})) \mathbf{1}_{\{\ell_I(\mathbf{X}_l / \mathbf{X}_{k:n}) > 1\}}}{\widetilde{P}_{n,k,I}}$$

in the following theorem. A detailed proof is given in Appendix C.

THEOREM 8. *Let \mathbf{X}_l , $l \in \mathbb{N}$, be independent copies of a regularly varying $[0, \infty)^d$ -valued random vector \mathbf{X} satisfying Equation (2). Denote the corresponding transformed vector by $\mathbf{X}^* = (r_1^{-1} X_1^\alpha, \dots, r_d^{-1} X_d^\alpha)$ and its marginal distribution F_i^* for all $1 \leq i \leq d$. Let $\{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ be a sequence such that $k_n \rightarrow \infty$. If a sequence $u_n \sim (F_i^*)^{-1}(1 - k_n/n)$ (which automatically holds for all $i \in \{1, \dots, d\}$ simultaneously by Equation (5)) satisfies the assumptions of Theorem 5 as well as Equation (31) and Equation (28) for $p = 0$ and some $\delta > 0$. Then we have*

$$\sqrt{k_n} \left(\frac{1}{\widehat{\alpha}_{n,k_n,I}} - \frac{1}{\alpha} \right) \rightarrow_d \widetilde{H}_I, \quad n \rightarrow \infty, \quad I \subset \{1, \dots, d\}, I \neq \emptyset,$$

where \widetilde{H}_I is a centered Gaussian random variable with variance $1/(\tau_I \alpha^2)$.

REMARK 9. We can show the joint convergence of the different estimates to the limit processes G and \widetilde{G} and the variables \widetilde{H}_I , $I \subset \{1, \dots, d\}$, as obtained in Theorem 5, Corollary 6 and Theorem 8, respectively. The covariance between the processes G and \widetilde{G} is given in the second part of Remark 7. Similar calculations as above reveal that, for all $\mathbf{v} \in \partial B_1^+(\mathbf{0})$, $\mathbf{s} \in A_\delta$, $\beta \in [(1 + \delta)^{-1}, 1 + \delta]$ and $p \in \mathbb{N}_0$,

$$(33) \quad \text{Cov}(\widetilde{H}_I, G(\mathbf{v}, \mathbf{s}, \beta, p)) \\ = \frac{\mathbb{E} \left[-\log \left(1 \wedge \frac{\|\mathbf{s} \circ \Theta\|}{\|\mathbf{1}_I \circ \Theta\|} \right) \cdot \left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \Theta}{\|\mathbf{s} \circ \Theta\|} \right)^{1/\beta} \right)^p \cdot (\|\mathbf{1}_I \circ \Theta\| \wedge \|\mathbf{s} \circ \Theta\|) \right]}{\alpha \mathbb{E}[\|\mathbf{1}_I \circ \Theta\|]}.$$

For the covariance between \widetilde{H}_I and $\widetilde{G}(\mathbf{v}, \mathbf{s}, \beta, p)$, we are particularly interested in the case that $\mathbf{s} = \mathbf{1}_I$ for the same index set I . Here, using similar arguments as above, we obtain

$$(34) \quad \text{Cov}(\widetilde{H}_I, \widetilde{G}(\mathbf{v}, \mathbf{1}_I, \beta, p)) = 0$$

for all $\mathbf{v} \in \partial B_1^+(\mathbf{0})$, $\beta \in [(1 + \delta)^{-1}, 1 + \delta]$ and $p \in \mathbb{N}_0$.

Making use of the asymptotic results for the Hill-type estimator in Theorem 8 and for the estimators in the case of deterministic thresholds given in Section 3.1, we will establish a functional central limit theorem for $\widetilde{M}_{n,k,I}(\mathbf{v}_I, p) / \widetilde{P}_{n,k,I}$. Compared to its analogue for deterministic thresholds, the estimator is blurred by the random marginal normalization based on the order statistics. To ensure that the estimator is unbiased, an additional assumption is required which can be interpreted as a second order condition on the marginal distributions. It corresponds to condition (28) for $\mathbf{s} \in A_{\delta, \{i\}}$ and $p = 0$, i.e. it ensures the negligibility of the presasymptotic bias in the convergence of $G_n(\cdot, \mathbf{s}, \cdot, 0)$ to $G(\cdot, \mathbf{s}, \cdot, 0)$ for $\mathbf{s} \in A_{\delta, \{i\}}$. Remind that, as discussed above, for given $\mathbf{s} \in A_\delta$, the processes $G_n(\cdot, \mathbf{s}, \cdot, 0)$ and, consequently, also

the limit process $G(\cdot, \mathbf{s}, \cdot, 0)$ are constant on $\partial B_1^+(\mathbf{0}) \times [(1 + \delta)^{-1}, 1 + \delta]$. Thus, to ease notation, we just write $G^0(\mathbf{s}) = G(\cdot, \mathbf{s}, \cdot, 0)$.

More precisely, we assume that

$$(35) \quad \sup_{(1+\delta)^{-1} \leq s \leq 1+\delta} \sqrt{k_n} |a^*(u_n) \mathbb{P}(sX_i^* > u_n) - s| \rightarrow 0, \quad n \rightarrow \infty.$$

for some $\delta > 0$ and $u_n \sim (F_i^*)^{-1}(k_n/n)$. Imposing some additional conditions on the function c , we obtain the following main result.

THEOREM 10. *Let \mathbf{X}_l , $l \in \mathbb{N}$, be independent copies of a regularly varying $[0, \infty)^d$ -valued random vector \mathbf{X} satisfying Equation (2) such that the corresponding transformed vector $\mathbf{X}^* = (r_1^{-1}X_1^\alpha, \dots, r_d^{-1}X_d^\alpha)$ and the sequence $\{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ satisfy the assumptions of Corollary 6 and Theorem 8 for some $\delta > 0$ and some $K \subset \mathbb{N}$ and Equation (35) holds for all $i \in \{1, \dots, d\}$. Assume that, for all $p \in K$, all the partial derivatives*

$$c_i(\mathbf{v}, \mathbf{s}, \beta, p) = \frac{\partial}{\partial s_i} c(\mathbf{v}, \mathbf{s}, \beta, p), \quad i \in \{1, \dots, d\},$$

$$c_\beta(\mathbf{v}, \mathbf{s}, \beta, p) = \frac{\partial}{\partial \beta} c(\mathbf{v}, \mathbf{s}, \beta, p),$$

exist and are continuous on $\partial B_1^+(\mathbf{0}) \times A'_\delta$. Then, for any non-empty $I \subset \{1, \dots, d\}$,

$$\left\{ \sqrt{k} \left(\frac{\widetilde{M}_{n,k,I}(\mathbf{v}_I, p)}{\widetilde{P}_{n,k,I}} - c(\mathbf{v}_I, \mathbf{1}_I, 1, p) \right); \mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K \right\}$$

$$\longrightarrow \left\{ \widetilde{G}(\mathbf{v}_I, \mathbf{1}_I, 1, p) - \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, p) G^0(\mathbf{1}_{\{i\}}) + c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, p) \alpha \widetilde{H}_I \right\}$$

weakly in $\ell^\infty(\partial B_1^+(\mathbf{0}) \times K)$ as $n \rightarrow \infty$.

The proof of Theorem 10 is deferred to Appendix D.

REMARK 11. Using the second part of Remark 7 and Equation (33) as well as the fact that $\|\mathbf{1}_{\{i\}} \circ \Theta\| = |\Theta_i| \leq \max_{j \in I} |\Theta_j| = \|\mathbf{1}_I \circ \Theta\|$ for all $i \in I$, we obtain that

$$\begin{aligned} & \text{Cov}(G^0(\mathbf{1}_{\{i\}}), \widetilde{G}(\mathbf{v}_I, \mathbf{1}_I, p)) \\ &= \frac{\mathbb{E} \left[\left(\mathbf{v}^\top \frac{\mathbf{1}_I \circ \Theta}{\|\mathbf{1}_I \circ \Theta\|} \right)^p |\Theta_i| \right]}{\mathbb{E}[\|\mathbf{1}_I \circ \Theta\|]} - \frac{\mathbb{E}[|\Theta_i|]}{\mathbb{E}[\|\mathbf{1}_I \circ \Theta\|]} \cdot \frac{\mathbb{E} \left[\left(\mathbf{v}^\top \frac{\mathbf{1}_I \circ \Theta}{\|\mathbf{1}_I \circ \Theta\|} \right)^p \|\mathbf{1}_I \circ \Theta\| \right]}{\mathbb{E}[\|\mathbf{1}_I \circ \Theta\|]} \\ &= \text{Cov} \left(\Theta_i^I, (\mathbf{v}_I^\top \Theta^I)^p \right), \end{aligned}$$

and

$$\text{Cov}(\widetilde{H}_I, G^0(\mathbf{1}_{\{i\}})) = \frac{\mathbb{E} \left[-\log \left(1 \wedge \frac{|\Theta_i|}{\|\mathbf{1}_I \circ \Theta\|} \right) |\Theta_i| \right]}{\alpha \mathbb{E}[\|\mathbf{1}_I \circ \Theta\|]} = \frac{1}{\alpha} \mathbb{E} \left[-\log(\Theta_i^I) \Theta_i^I \right],$$

respectively, while Equation (26) implies that

$$\text{Cov}(G^0(\mathbf{1}_{\{i\}}), G^0(\mathbf{1}_{\{j\}})) = \tau \cdot \mathbb{E}[\Theta_i \wedge \Theta_j] = \tau_I \cdot \mathbb{E}[\Theta_i^I \wedge \Theta_j^I]$$

for all $i, j \in I$. By Equation (34), $\widetilde{H}(\mathbf{1}_I)$ is uncorrelated to all random variables $\widetilde{G}(\mathbf{v}, 1, \mathbf{1}_I, p)$, $\mathbf{v} \in \partial B_1^+(\mathbf{0})$, $p \in K$. Thus, we can calculate the asymptotic variance

$$\widetilde{V}(\mathbf{v}_I, p) = \lim_{n \rightarrow \infty} k \cdot \text{Var} \left(\frac{\widetilde{M}_{n,k,I}(\mathbf{v}_I, p)}{\widetilde{P}_{n,k,I}} \right)$$

and obtain

$$\begin{aligned}\tilde{V}(\mathbf{v}_I, p) &= \frac{1}{\tau_I} \text{Var} \left((\mathbf{v}_I^\top \boldsymbol{\Theta}^I)^p \right) - 2 \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, p) \text{Cov} \left(\Theta_i^I, (\mathbf{v}_I^\top \boldsymbol{\Theta}^I)^p \right) \\ &\quad + \tau_I \cdot \sum_{i \in I} \sum_{j \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, p) c_j(\mathbf{v}_I, \mathbf{1}_I, 1, p) \mathbb{E}[\Theta_i^I \wedge \Theta_j^I] \\ &\quad - 2c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, p) \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, p) \mathbb{E}[-\log(\Theta_i^I) \Theta_i^I] + \frac{1}{\tau_I} c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, p)^2.\end{aligned}$$

4. Application: Estimation of extremal coefficients. One of the most popular summary statistics for extremal dependence in the (sub-)vector $(X_i^*)_{i \in I}$ is the extremal coefficient τ_I given by Equation (14), i.e.

$$\tau_I = \lim_{u \rightarrow \infty} a^*(u) \mathbb{P} \left(\max_{i \in I} X_i^* > u \right) \sim \frac{\mathbb{P}(\max_{i \in I} X_i^* > u)}{\mathbb{P}(X_j^* > u)} \quad \text{for all } j \in I.$$

From this representation and the union bound for $\mathbb{P}(\max_{i \in I} X_i^* > u)$, it can be easily seen that $\tau_I \in [1, |I|]$ where $\tau_I = 1$ corresponds to the case that all the components of $(X_i)_{i \in I}$ are (asymptotically) fully dependent, while they are all asymptotically independent if $\tau_I = |I|$. This gives rise to the common interpretation of the extremal coefficient as the number of asymptotically independent variables among $(X_i^*)_{i \in I}$.

From Equation (17), we obtain that

$$\mathbb{E} \left[\mathbf{v}_I^\top \boldsymbol{\Theta}^I \right] = \mathbb{E} \left[(\mathbf{v}_I^\top \boldsymbol{\Theta}^I)_+ \wedge 1 \right] = \frac{1}{\tau_I}$$

for all $\mathbf{v}_I \in \partial B_1^+(\mathbf{0})$. This motivates the use of the two classes of estimators considered in Equation (21) and Equation (22), respectively. We will distinguish two cases; Section 4.1 collects the asymptotic normality results for the case of known marginal distributions where we can exploit the results for deterministic thresholds given in Section 3.1 with $\mathbf{s} = \mathbf{1}_I$ and $\beta = 1$. The results derived in Section 3.2 can be used when the margins are unknown as we will investigate in Section 4.2.

4.1. Known margins. In this section, we focus on the case where the margins are known, i.e. \mathbf{X}_i^* is observed.

4.1.1. The inefficiency of the benchmark. For this case, as a first class of estimators, we can directly consider the estimators based on relative frequencies of threshold exceedances by convex combinations, see Equation (21), namely

$$\frac{\hat{P}_{n,u,I}^{(\text{conv})}(\mathbf{v}_I)}{P_{n,u,I}} = \frac{n^{-1} \sum_{l=1}^n \mathbf{1}\{\mathbf{v}_I^\top \mathbf{X}_l^* > u\}}{n^{-1} \sum_{l=1}^n \mathbf{1}\{\ell_I(\mathbf{X}_l^*) > u\}}, \quad \mathbf{v}_I \in \partial B_1^+(\mathbf{0}),$$

where we use the fact that $\mathbf{v}_I^\top \mathbf{X}^* > u$ already implies that $\ell_I(\mathbf{X}^*) > u$. This type of estimator will serve as a benchmark. By standard arguments, for a sequence $\{u_n\}_{n \in \mathbb{N}}$ of thresholds such that $u_n \rightarrow \infty$ and $n/a^*(u_n) \rightarrow \infty$ as $n \rightarrow \infty$, we obtain the asymptotic normality

$$\sqrt{\frac{n}{a^*(u_n)}} \begin{pmatrix} a^*(u_n) \hat{P}_{n,u_n,I}^{(\text{conv})}(\mathbf{v}_I) - a^*(u_n) \mathbb{P}(\mathbf{v}_I^\top \mathbf{X}^* > u_n) \\ a^*(u_n) \hat{P}_{n,u_n,I} - a^*(u_n) \mathbb{P}(\ell_I(\mathbf{X}^*) > u_n) \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}),$$

where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \tau_I \mathbb{E}[\mathbf{v}_I^\top \boldsymbol{\Theta}^I] & \tau_I \mathbb{E}[\mathbf{v}_I^\top \boldsymbol{\Theta}^I] \\ \tau_I \mathbb{E}[\mathbf{v}_I^\top \boldsymbol{\Theta}^I] & \tau_I \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \tau_I \end{pmatrix}.$$

Assuming that the bias is negligible, i.e.

$$\sqrt{\frac{n}{a^*(u_n)}} \left(a^*(u_n) \mathbb{P}(\mathbf{v}_I^\top \mathbf{X}^* > u_n) - 1 \right) \rightarrow 0$$

and

$$(36) \quad \sqrt{\frac{n}{a^*(u_n)}} \left(a^*(u_n) \mathbb{P}(\ell_I(\mathbf{X}^*) > u_n) - \tau_I \right) \rightarrow 0,$$

the Delta method yields asymptotic normality of the benchmark estimator for the case of known margins,

$$\frac{1}{\widehat{\tau}_{I,n}^{BK}(\mathbf{v}_I)} = \frac{\widehat{P}_{n,u_n,I}^{(\text{conv})}(\mathbf{v}_I)}{\widehat{P}_{n,u_n,I}}, \quad \mathbf{v} \in \partial B_1^+(\mathbf{0}),$$

i.e.

$$\sqrt{\frac{n}{a^*(u_n)}} \left(\frac{1}{\widehat{\tau}_{I,n}^{BK}(\mathbf{v}_I)} - \frac{1}{\tau_I} \right) \xrightarrow{d} \mathcal{N}(0, \tau_I^{-3}(\tau_I - 1)).$$

Here, it is important to note that the asymptotic variance does not depend on the choice of $\mathbf{v} \in \partial B_1^+(\mathbf{0})$.

Secondly, we consider moment-based estimators as discussed in Section 3.1, i.e. estimators based on empirical means of convex combinations, namely $\widehat{M}_{n,u,I}(\mathbf{v}_I, 1) / \widehat{P}_{n,u,I}$ with weights $\mathbf{v}_I \in \partial B_1^+(\mathbf{0})$ as defined in Equation (22). Provided that the bias is negligible, i.e.

$$\sqrt{\frac{n}{a^*(u_n)}} \left(a^*(u_n) \mathbb{E} \left[\mathbf{v}_I^\top \frac{\mathbf{X}^*}{\ell_I(\mathbf{X}^*)} \cdot \mathbf{1}_{\{\ell_I(\mathbf{X}^*) > u_n\}} \right] - 1 \right) \rightarrow 0$$

and (36), we obtain the asymptotic normality

$$\sqrt{\frac{n}{a^*(u_n)}} \left(\frac{\widehat{M}_{n,u,I}(\mathbf{v}_I)}{\widehat{P}_{n,u,I}} - \frac{1}{\tau_I} \right) \xrightarrow{d} \mathcal{N} \left(0, \tau_I^{-1} \text{Var}(\mathbf{v}_I^\top \Theta^I) \right),$$

cf. the results in Corollary 6 and the first part of Remark 7.

Here, as $\mathbf{v}_I^\top \Theta^I \leq 1$ a.s., we have that

$$\frac{1}{\tau_I} \text{Var}(\mathbf{v}_I^\top \Theta^I) = \frac{1}{\tau_I} \left(\mathbb{E}[(\mathbf{v}_I^\top \Theta^I)^2] - \frac{1}{\tau_I^2} \right) \leq \frac{1}{\tau_I} \left(\mathbb{E}[(\mathbf{v}_I^\top \Theta^I)] - \frac{1}{\tau_I^2} \right) = \frac{\tau_I - 1}{\tau_I^3}$$

and, therefore, the asymptotic variance of the moment-based approach is always less than or equal to the asymptotic variance of the benchmark estimator. Note that the variances of both estimators are equal, i.e. $\mathbb{E}[(\mathbf{v}_I^\top \Theta^I)^2] = \mathbb{E}[\mathbf{v}_I^\top \Theta^I] = 1/\tau_I$, if and only if $\Theta_i^I \equiv 1$ a.s. for all i with $v_i > 0$. This is the case if and only if $\tau_I = 1$ and then both asymptotic variances are equal to 0. Thus, except for this degenerate case, the moment-based approach is always preferable to the benchmark in terms of the asymptotic variance.

Furthermore, in general, the asymptotic variance of the moment-based estimator depends on the vector $\mathbf{v}_I \in \partial B_1^+(\mathbf{0})$. As we will see in Example 14, the minimal variance is typically not achieved for a standard vector of the type $\mathbf{v}_{\{i\}}$ for $i \in I$, i.e. the empirical estimators of $\mathbb{E}[\Theta_i]$, but by a non-trivial mixture of different basis vectors. This improvement can be attributed to the fact that the moment-based estimator is linear in \mathbf{v} . Thus optimizing in $\mathbf{v}_I \in \partial B_1^+(\mathbf{0})$ can be interpreted as aggregating different estimators. It is well-known that the use of a combination of estimators can improve the accuracy. We note that we could also aggregate

the benchmark estimator for different \mathbf{v} . However this strategy is much more intricate to study as the benchmark estimator is not linear in \mathbf{v} .

Based on this discussion, for the remainder of Section 4.1, we focus on convex combinations of moment-based estimators rather than on the benchmark based on indicators. In the following, we optimize the asymptotic variance $\tau_I^{-1} \text{Var}(\mathbf{v}_I^\top \Theta^I)$ w.r.t. the weight vectors \mathbf{v}_I and we find some approximation of this best aggregation. We should notice that the optimal asymptotic variance $\text{Var}(\mathbf{v}_I^\top \Theta^I)$ may be degenerate even when $\tau_I > 1$. In these cases the results hereinafter still hold with a null limit.

REMARK 12. The asymptotic variance of the moment-based estimator is zero if and only if there exists a weight vector \mathbf{v}_I such that $\mathbf{v}_I^\top \Theta^I = 1/\tau_I$ almost surely. This happens in case of an asymptotically independent vector \mathbf{X} with \mathbf{v} satisfying $v_i = \frac{1}{|I|}$ for $i \in I$. It might also happen in more complex cases. For instance, assume that the vector Θ^I takes at only a finite number of different possible values $\theta^{(j)}$, $j \in 1, \dots, m$. Then, \mathbf{v}_I is given as the solution of the linear system $\mathbf{v}_I^\top \theta^{(j)} = 1/\tau_I$, $j \in 1, \dots, m$ provided that the solution is in the simplex.

4.1.2. Variance minimization. We focus on the moment estimators $\widehat{M}_{n,u,I}(\mathbf{v}_I, 1)/\widehat{P}_{n,u,I}$ for different $\mathbf{v}_I \in \partial B_1^+(\mathbf{0})$. As all of these estimators are consistent, it is appealing to choose the estimator that provides the minimal asymptotic variance. Thus, in the following, we will focus on the determination of weight vectors $\mathbf{v}_I^* \in \partial B_1^+(\mathbf{0})$ that minimize the asymptotic variance

$$\tau_I^{-1} \text{Var}(\mathbf{v}_I^\top \Theta^I) = \tau_I^{-3} \left(\tau_I^2 \cdot \mathbb{E}[(\mathbf{v}_I^\top \Theta^I)^2] - 1 \right),$$

or, equivalently, the function

$$f^{(I)} : \mathbf{v}_I \mapsto \mathbb{E}[(\mathbf{v}_I^\top \Theta^I)^2].$$

We will show that, under mild conditions, the minimizer of $f^{(I)}$ in $\partial B_1^+(\mathbf{0})$ is unique.

PROPOSITION 13. *Let Θ be a spectral vector. Then, for any non-empty $I \subset \{1, \dots, d\}$, it holds: If the matrix $\mathbf{V}_I = (\mathbb{E}[\Theta_i^I \Theta_j^I])_{i,j \in I}$ is conditionally positive definite, i.e.*

$$\mathbf{a}^\top \mathbf{V}_I \mathbf{a} > 0$$

for all $\mathbf{a} \in \mathbb{R}^{|I|} \setminus \{\mathbf{0}\}$ such that $\sum_{i \in I} a_i = 0$, then the function $\mathbf{v}_I \mapsto f^{(I)}(\mathbf{v}_I)$ attains its minimum on $\partial B_1^+(\mathbf{0})$ at some unique $\mathbf{v}_I^* \in \partial B_1^+(\mathbf{0})$.

PROOF. First, we note that

$$f^{(I)}(\mathbf{v}_I) = \sum_{i \in I} \sum_{j \in I} v_i v_j \mathbb{E}[\Theta_i^I \Theta_j^I] = (\mathbf{v}_I)_{i \in I}^\top \mathbf{V}_I (\mathbf{v}_I)_{i \in I}.$$

By a slight abuse of notation, henceforth, we will write $f^{(I)}(\mathbf{v}_I) = \mathbf{v}_I^\top \mathbf{V}_I \mathbf{v}_I$. Now, assume that $f^{(I)}$ attains its minimum at two distinct points $\mathbf{v}_I \in \partial B_1^+(\mathbf{0})$ and $\mathbf{w}_I \in \partial B_1^+(\mathbf{0})$. Then, for all $\lambda \in (0, 1)$, we have $\lambda \mathbf{v}_I + (1 - \lambda) \mathbf{w}_I \in \partial B_1^+(\mathbf{0})$, $\sum_{i \in I} (v_i - w_i) = 0$ and, therefore,

$$\begin{aligned} f^{(I)}(\lambda \mathbf{v}_I + (1 - \lambda) \mathbf{w}_I) &= (\mathbf{w}_I + \lambda(\mathbf{v}_I - \mathbf{w}_I))^\top \mathbf{V}_I (\mathbf{v}_I + (1 - \lambda)(\mathbf{w}_I - \mathbf{v}_I)) \\ &= -\lambda(1 - \lambda)(\mathbf{v}_I - \mathbf{w}_I)^\top \mathbf{V}_I (\mathbf{v}_I - \mathbf{w}_I) + \lambda \mathbf{v}_I^\top \mathbf{V}_I \mathbf{v}_I + (1 - \lambda) \mathbf{w}_I^\top \mathbf{V}_I \mathbf{w}_I \\ &< \lambda f^{(I)}(\mathbf{v}_I) + (1 - \lambda) f^{(I)}(\mathbf{w}_I), \end{aligned}$$

which is a contradiction to the choice of \mathbf{v}_I and \mathbf{w}_I as minimizers of $f^{(I)}$. \square

EXAMPLE 14. In the case of a pairwise extremal coefficient τ_I with $I = \{i, j\}$, we obtain an explicit expression for the minimizer $\mathbf{v}_{\{i,j\}}^*$ with $v_i^*, v_j^* \geq 0$ and $v_i^* + v_j^* = 1$ of the function $f^{(I)}(\mathbf{v}_{\{i,j\}}) = \mathbb{E}[(v_i \Theta_i^I + v_j \Theta_j^I)^2] = (v_i, v_j) \mathbf{V}_I (v_i, v_j)^\top$. To this end, we note that the function

$$v_i \mapsto (v_i, 1 - v_i) \mathbf{V}_I \begin{pmatrix} v_i \\ 1 - v_i \end{pmatrix} = v_i^2 \mathbb{E}[(\Theta_i^I)^2] + 2v_i(1 - v_i) \mathbb{E}[\Theta_i^I \Theta_j^I] + (1 - v_i)^2 \mathbb{E}[(\Theta_j^I)^2]$$

is a quadratic function which possesses a unique minimizer $v_i^* \in [0, 1]$ if and only if $\mathbb{E}[(\Theta_i^I)^2] - 2\mathbb{E}[\Theta_i^I \Theta_j^I] + \mathbb{E}[(\Theta_j^I)^2] > 0$, i.e. if and only if \mathbf{V}_I is conditionally positive definite, cf. Proposition 13. From

$$\begin{aligned} \mathbb{E}[\Theta_i^I \cdot \Theta_j^I] &= \frac{1}{\mathbb{E}[\ell_I(\Theta)]} \cdot \mathbb{E}\left[\frac{\Theta_i \Theta_j}{\Theta_i \vee \Theta_j}\right] = \frac{1}{\mathbb{E}[\ell_I(\Theta)]} \cdot \mathbb{E}[\Theta_i \wedge \Theta_j] \\ &= \frac{1}{\mathbb{E}[\ell_I(\Theta)]} \cdot \mathbb{E}[\Theta_i + \Theta_j - \Theta_i \vee \Theta_j] = \frac{\tau}{\tau_I} \cdot \left(\frac{2}{\tau} - \frac{\tau_I}{\tau}\right) = \frac{2}{\tau_I} - 1, \end{aligned}$$

we obtain that

$$(37) \quad \mathbb{E}[(\Theta_i^I)^2] \geq [\mathbb{E}(\Theta_i^I)]^2 = \frac{1}{\tau_I^2} \geq \frac{2}{\tau_I} - 1 = \mathbb{E}[\Theta_i^I \cdot \Theta_j^I]$$

and, analogously,

$$(38) \quad \mathbb{E}[(\Theta_j^I)^2] \geq \mathbb{E}[\Theta_i^I \cdot \Theta_j^I].$$

Note that equality in Equation (37) and in Equation (38) holds if and only if we have $\mathbb{E}[(\Theta_i^I)^2] = \mathbb{E}[\Theta_i^I] = 1$ and $\mathbb{E}[(\Theta_j^I)^2] = \mathbb{E}[\Theta_j^I] = 1$, respectively. Thus, for $\tau_I > 1$, there exists a unique minimizer which can be easily computed:

$$v_i^* = \frac{\mathbb{E}[(\Theta_j^I)^2] - \mathbb{E}[\Theta_i^I \Theta_j^I]}{\mathbb{E}[(\Theta_i^I)^2] - 2\mathbb{E}[\Theta_i^I \Theta_j^I] + \mathbb{E}[(\Theta_j^I)^2]} = \frac{\mathbb{E}[(\Theta_j^I)^2] - \mathbb{E}[\Theta_i^I \Theta_j^I]}{\mathbb{E}[(\Theta_i^I - \Theta_j^I)^2]}.$$

and

$$v_j^* = 1 - v_i^* = \frac{\mathbb{E}[(\Theta_i^I)^2] - \mathbb{E}[\Theta_i^I \Theta_j^I]}{\mathbb{E}[(\Theta_i^I - \Theta_j^I)^2]}.$$

By Equations (37) and (38), we obtain $v_i^*, v_j^* > 0$, i.e. $\mathbf{v}_{\{i,j\}}^* \in \partial B_1^+(\mathbf{0})$. Some further calculations lead to the minimal asymptotic variance $\tau_I^{-3}(\tau_I^2 \cdot f^{(I)}(\mathbf{v}_I) - 1)$ with

$$f^{(I)}(\mathbf{v}_I^*) = \frac{\mathbb{E}[(\Theta_i^I)^2] \cdot \mathbb{E}[(\Theta_j^I)^2] - (\mathbb{E}[\Theta_i^I \Theta_j^I])^2}{\mathbb{E}[(\Theta_i^I - \Theta_j^I)^2]}.$$

4.1.3. *Asymptotic normality of the plug-in estimator.* We use the functional central limit theorems Corollary 6 and Theorem 10 to show that estimators with minimal asymptotic variance can be obtained by plugging-in consistent estimators of the optimal weight vectors.

The variance minimizing vector \mathbf{v}_I^* can be estimated by taking the minimum of the empirical counterpart of $f^{(I)}$:

$$\widehat{\mathbf{v}}_{n,u,I}^* = \operatorname{argmin}_{\mathbf{v}_I \in \partial B_1^+(\mathbf{0})} \frac{\widehat{M}_{n,u,I}(\mathbf{v}_I, 2)}{\widehat{P}_{n,u,I}}.$$

It is important to note that, being constructed as the empirical second moment of a convex combination, $\mathbf{v}_i \mapsto \widehat{M}_{n,u,I}(\mathbf{v}_I, 2)/\widehat{P}_{n,u,I}$ is a positive semi-definite quadratic form. Thus, the

evaluation of $\widehat{M}_{n,u,I}(\mathbf{v}_I, 2)/\widehat{P}_{n,u,I}$ for a finite number of vectors \mathbf{v}_I is sufficient to reconstruct the whole function $\{\widehat{M}_{n,u,I}(\mathbf{v}_I, 2)/\widehat{P}_{n,u,I} : \mathbf{v}_I \in \partial B_1^+(\mathbf{0})\}$ and, thus, to calculate its minimizer.

Here, even though $f^{(I)}$ is conditionally positive definite, i.e. the theoretical minimizer \mathbf{v}_I^* is unique, under the assumptions of Proposition 13, in practice, definiteness of the empirical counterpart $\mathbf{v}_I \mapsto \widehat{M}_{n,u,I}(\mathbf{v}_I, 2)/\widehat{P}_{n,u,I}$ is not guaranteed for a finite sample, i.e. its minimizer $\widehat{\mathbf{v}}_{n,u,I}^*$ might no longer be unique. In such a case, we just choose an arbitrary minimizer. In particular, $\widehat{\mathbf{v}}_{n,u,I}^*$ can be an arbitrary vector of $\partial B_1^+(\mathbf{0})$ if $\ell_I(\mathbf{X}_l^*) \leq u$ for all $l \in \{1, \dots, n\}$. We first show consistency of this estimator.

PROPOSITION 15. *Let $\mathbf{X}_l^* = (X_{l1}^*, \dots, X_{ld}^*)^\top$, $l \in \mathbb{N}$, be independent copies of a d -dimensional random vector \mathbf{X}^* satisfying the assumptions of Proposition 13 for all non-empty $I \subset \{1, \dots, d\}$. Furthermore, let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence such that $u_n \rightarrow \infty$ and $n/a^*(u_n) \rightarrow \infty$ as $n \rightarrow \infty$ and assume that Equation (28) holds for $p = 0, 2$ and $\delta = 0$. Then,*

$$\widehat{\mathbf{v}}_{n,u_n,I}^* \xrightarrow{p} \mathbf{v}_I^*$$

for any sequence of minimizers $\widehat{\mathbf{v}}_{n,u_n,I}^*$ and any $I \subset \{1, \dots, d\}$.

Furthermore, $\widehat{M}_{n,u_n,I}(\widehat{\mathbf{v}}_{n,u_n,I}^*, 2) \rightarrow_p f^{(I)}(\mathbf{v}_I^*)$.

PROOF. We observe that, for any non-empty $I \subset \{1, \dots, d\}$, we have $f^{(I)}(\mathbf{v}_I) = c(\mathbf{v}_I, \mathbf{1}_I, 2)$ for all $\mathbf{v}_I \in \partial B_1^+(\mathbf{0})$ and, thus, Corollary 6 yields

$$(39) \quad \sup_{\mathbf{v}_I \in \partial B_1^+(\mathbf{0})} |\widehat{M}_{n,u_n,I}(\mathbf{v}_I, 2) - f^{(I)}(\mathbf{v}_I)| \xrightarrow{p} 0.$$

As the continuous function $f^{(I)}$ attains its unique minimum on $\partial B_1^+(\mathbf{0})$ at \mathbf{v}_I^* (see Proposition 13), for every $\varepsilon > 0$, there exists some $\eta = \eta(\varepsilon) > 0$ such that $f^{(I)}(\mathbf{v}) - f^{(I)}(\mathbf{v}_I^*) > \eta$ for all \mathbf{v} with $\|\mathbf{v} - \mathbf{v}_I^*\| > \varepsilon$. Consequently, if $\|\widehat{\mathbf{v}}_{n,u_n,I}^* - \mathbf{v}_I^*\| > \varepsilon$ then we have that $f^{(I)}(\widehat{\mathbf{v}}_{n,u_n,I}^*) - f^{(I)}(\mathbf{v}_I^*) > \eta$, which further implies

$$f^{(I)}(\widehat{\mathbf{v}}_{n,u_n,I}^*) - \widehat{M}_{n,u_n,I}(\widehat{\mathbf{v}}_{n,u_n,I}^*, 2) + \widehat{M}_{n,u_n,I}(\mathbf{v}_I^*, 2) - f^{(I)}(\mathbf{v}_I^*) > \eta$$

as $\widehat{M}_{n,u_n,I}(\widehat{\mathbf{v}}_{n,u_n,I}^*, 2) \leq \widehat{M}_{n,u_n,I}(\mathbf{v}_I^*, 2)$ by definition of $\widehat{\mathbf{v}}_{n,u_n,I}^*$. In particular,

$$\sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0})} |\widehat{M}_{n,u_n,I}(\mathbf{v}, 2) - f^{(I)}(\mathbf{v})| > \eta/2.$$

Noticing that $\widehat{\mathbf{v}}_{n,u_n,I}^*$ is well defined only if $\bigvee_{1 \leq l \leq n} \ell_I(\mathbf{X}_l^*) > u_n$, we obtain

$$\mathbb{P}(\|\widehat{\mathbf{v}}_{n,u_n,I}^* - \mathbf{v}_I^*\| > \varepsilon) \leq \mathbb{P}(\ell_I(\mathbf{X}^*) \leq u_n)^n + \mathbb{P}\left(\sup_{\mathbf{v}_I \in \partial B_1^+(\mathbf{0})} |\widehat{M}_{n,u_n,I}(\mathbf{v}_I, 2) - f^{(I)}(\mathbf{v}_I)| > \frac{\eta}{2}\right),$$

which tends to 0 as $n \rightarrow \infty$ by Equation (39).

The statement that

$$\widehat{M}_{n,u_n,I}(\widehat{\mathbf{v}}_{n,u_n,I}^*, 2) \rightarrow_p f^{(I)}(\mathbf{v}_I^*)$$

follows from the facts that $f^{(I)}(\widehat{\mathbf{v}}_{n,u_n,I}^*) - \widehat{M}_{n,u_n,I}(\widehat{\mathbf{v}}_{n,u_n,I}^*, 2) \rightarrow_p 0$ by Equation (39) and $f^{(I)}(\widehat{\mathbf{v}}_{n,u_n,I}^*) - f^{(I)}(\mathbf{v}_I^*) \rightarrow_p 0$ due to $\widehat{\mathbf{v}}_{n,u_n,I}^* \rightarrow_p \mathbf{v}_I^*$ and the continuity of $f^{(I)}$.

The second part can be shown analogously using the results from Theorem 10 instead of Corollary 16. \square

Now, we are ready establish the desired limit theorem for the plug-in estimator, the moment-based estimator for the known margin case,

$$\frac{1}{\widehat{\tau}_{I,n}^{MK}} = \frac{\widehat{M}_{n,u_n,I}(\widehat{\mathbf{v}}_{n,u_n,I}^*)}{\widehat{P}_{n,u_n,I}},$$

by combining the results of Proposition 15 with Corollary 6.

COROLLARY 16. *Let $\mathbf{X}_l^* = (X_{l1}^*, \dots, X_{ld}^*)^\top$, $l \in \mathbb{N}$, be independent copies of a d -dimensional random vector \mathbf{X}^* satisfying the assumptions of Proposition 13 for all non-empty $I \subset \{1, \dots, d\}$. Furthermore, let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence such that $u_n \rightarrow \infty$ and $n/a^*(u_n) \rightarrow \infty$ as $n \rightarrow \infty$ and assume that Equation (28) holds for $p = 0, 1, 2$ and $\delta = 0$. Then, for all non-empty $I \subset \{1, \dots, d\}$,*

$$\sqrt{\frac{n}{a^*(u_n)}} \left(\frac{1}{\widehat{\tau}_{I,n}^{MK}} - \frac{1}{\tau_I} \right) \xrightarrow{d} \mathcal{N} \left(0, \tau_I^{-1} \text{Var}(\mathbf{v}_I^{*\top} \Theta^I) \right).$$

4.2. Unknown margins. Similarly, in case that the marginal distributions of X_1, \dots, X_d are unknown, i.e. Equation (4) is no longer satisfied and the index α of regular variation may be different from 1, we need to replace the deterministic threshold both in the benchmark estimator and in the moment-based estimator by a random one and include a Hill type estimator as described in Subsection 3.2.

4.2.1. The benchmark and the moment-based estimator. In case of the benchmark defined in Equation (21), the transformations above lead to the estimator

$$\frac{\widetilde{P}_{n,k,I}^{(conv)}(\mathbf{v}_I)}{\widetilde{P}_{n,k,I}} = \frac{n^{-1} \sum_{l=1}^n \mathbf{1}\{\mathbf{v}_I^\top (\mathbf{X}_l / \mathbf{X}_{k:n})^{\widehat{\alpha}} > 1\}}{n^{-1} \sum_{l=1}^n \mathbf{1}\{\ell_I(\mathbf{X}_l / \mathbf{X}_{k:n}) > 1\}}, \quad \mathbf{v}_I \in \partial B_1^+(\mathbf{0}).$$

Of particular interest is the case that \mathbf{v}_I is a standard basis vector, i.e. $\mathbf{v}_I = \mathbf{1}_{\{i\}}$ for some $i \in I$, since then the enumerator satisfies $\widetilde{P}_{n,k,I}^{(conv)}(\mathbf{1}_{\{i\}}) = k/n$ a.s., i.e. it possesses variance zero. Thus, the estimator simplifies to

$$\frac{1}{\frac{n}{k} \widetilde{P}_{n,k,I}} = \frac{1}{\widehat{L}_{k,n}(\mathbf{1}_I)}$$

where $\widehat{L}_{k,n}$ is the well-known non-parametric estimator for the stable-tail dependence function [cf. Drees and Huang, 1998]. Provided that the bias is negligible and that the function $\mathbf{s} \mapsto \mathbb{E}[\|\mathbf{s} \circ \Theta^I\|]$ is differentiable at $\mathbf{s} = \mathbf{1}_I$, this estimator is asymptotically normal:

$$\sqrt{k_n} \left(\frac{1}{\widehat{L}_{k_n,n}(\mathbf{1}_I)} - \frac{1}{\tau_I} \right) \xrightarrow{d} \mathcal{N} \left(0, \frac{\sigma_L^2}{\tau_I^4} \right)$$

where the asymptotic variance of the stable dependence function estimator $\widehat{L}_{k_n,n}(\mathbf{1}_I)$ is provided in Chapter 7.4 of De Haan and Ferreira [2006] by the expression

$$\sigma_L^2 = \tau_I^3 \sum_{i \in I} \sum_{j \in I} \frac{\partial}{\partial s_i} \mathbb{E}[\|\mathbf{s} \circ \Theta^I\|] \Big|_{\mathbf{s}=\mathbf{1}_I} \frac{\partial}{\partial s_j} \mathbb{E}[\|\mathbf{s} \circ \Theta^I\|] \Big|_{\mathbf{s}=\mathbf{1}_I} \mathbb{E}[\Theta_i^I \wedge \Theta_j^I] - \tau_I.$$

Using that $\mathbb{E}[\Theta_i^I \wedge \Theta_j^I] \leq \mathbb{E}(\Theta_i^*) = 1/\tau_I$ for all $i \in I$ and $\sum_{i \in I} \frac{\partial}{\partial s_i} \mathbb{E}[\|\mathbf{s} \circ \Theta^I\|]_{\mathbf{s}=\mathbf{1}_I} = 1$ by Euler's homogeneous function theorem due to the positive 1-homogeneity of the function $\mathbf{s} \mapsto \mathbb{E}[\|\mathbf{s} \circ \Theta^I\|]$, we obtain that

$$\frac{\sigma_I^2}{\tau_I^4} \leq \frac{\tau_I^2 - \tau_I}{\tau_I^4} = \frac{\tau_I - 1}{\tau_I^3},$$

i.e. the asymptotic variance of $1/\widehat{L}_{k_n, n}(\mathbf{1}_I)$ is always smaller than the asymptotic variance of the benchmark estimators for known margins, $1/\widehat{\tau}_{I, n}(\mathbf{v}_I)$. In particular, the estimator becomes more efficient by using order statistics even if the marginal distributions are known.

For the moment-based estimators, as already discussed in Subsection 3.2, the marginal transformations yield estimators of the type $\widetilde{M}_{n, k, I}(\mathbf{v}_I, 1)/\widetilde{P}_{n, k, I}$, $\mathbf{v}_I \in \partial B_1^+(\mathbf{0})$. Provided that the assumptions of Theorem 10 hold, we obtain the asymptotic normality

$$\sqrt{k_n} \left(\frac{\widetilde{M}_{n, k, I}(\mathbf{v}_I)}{\widetilde{P}_{n, k, I}} - \frac{1}{\tau_I} \right) \xrightarrow{d} \mathcal{N}(0, \widetilde{V}(\mathbf{v}_I, 1)),$$

with $\widetilde{V}(\mathbf{v}_I, 1)$ being given in Remark 11.

4.2.2. Variance minimization. Analogously to Section 4.1, we aim at choosing the estimator that provides the minimal asymptotic variance, i.e. use the weight vector $\widetilde{\mathbf{v}}_I \in \partial B_1^+(\mathbf{0})$ that minimizes $\widetilde{V}(\cdot, 1)$, respectively. To this end, we first exploit the results given in Remark 11, to see that

$$\begin{aligned} \widetilde{V}(\mathbf{v}_I, 1) &= \frac{1}{\tau_I} \text{Var} \left(\mathbf{v}_I^\top \Theta^I \right) - 2 \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \text{Cov} \left(\Theta_i^I, \mathbf{v}_I^\top \Theta^I \right) \\ &\quad + \tau_I \cdot \sum_{i \in I} \sum_{j \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) c_j(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \mathbb{E}[\Theta_i^I \wedge \Theta_j^I] \\ &\quad - 2c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \mathbb{E}[-\log(\Theta_i^I) \Theta_i^I] + \frac{1}{\tau_I} c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1)^2 \\ &= \frac{1}{\tau_I} \text{Var} \left(\mathbf{v}_I^\top \Theta^I - \tau_I \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \Theta_i^I \right) \\ &\quad + \tau_I \cdot \sum_{i \in I} \sum_{j \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) c_j(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \left(\mathbb{E}[\Theta_i^I \wedge \Theta_j^I] - \mathbb{E}[\Theta_i^I \cdot \Theta_j^I] \right) \\ (40) \quad &\quad - 2c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \mathbb{E}[-\log(\Theta_i^I) \Theta_i^I] + \frac{1}{\tau_I} c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1)^2, \end{aligned}$$

where we used that

$$\left(\sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \mathbb{E}[\Theta_i^I] \right)^2 = \frac{1}{\tau_I^2} \left(\sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \right)^2 = 0,$$

which follows by Euler's homogeneous function theorem as

$$(41) \quad \mathbf{s} \mapsto c(\mathbf{v}_I, \mathbf{s}, 1, 1) = \frac{1}{\tau_I} \frac{\mathbf{v}_I^\top \mathbf{s}}{\mathbb{E}[\|\mathbf{s} \circ \Theta^I\|]}$$

is positively 0-homogeneous. By a straightforward computation, we obtain that

$$(42) \quad \begin{aligned} \tilde{V}(\mathbf{v}_I, 1) &= \frac{1}{\tau_I} \text{Var} \left(\mathbf{v}_I^\top \Theta^I - \tau_I \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \Theta_i^I \right) \\ &\quad + \tau_I \cdot \left(\begin{array}{c} (c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1))_{i \in I} \\ \tau_I^{-1} c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \end{array} \right)^\top \tilde{\mathbf{V}}_I \left(\begin{array}{c} (c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1))_{i \in I} \\ \tau_I^{-1} c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \end{array} \right) \end{aligned}$$

with

$$\tilde{\mathbf{V}}_I = \begin{pmatrix} (\mathbb{E}[\Theta_i^I \wedge \Theta_j^I - \Theta_i^I \Theta_j^I])_{i,j \in I} & (\mathbb{E}[-\log(\Theta_i^I) \Theta_i^I])_{i \in I} \\ (\mathbb{E}[-\log(\Theta_i^I) \Theta_i^I])_{i \in I}^\top & 1 \end{pmatrix}$$

where both the expectation and the integral are interpreted componentwise. From the identity

$$\tilde{\mathbf{V}}_I = \mathbb{E} \left[\int_0^1 \begin{pmatrix} (\mathbf{1}\{\Theta_i^I > u\} - \Theta_i^I)_{i \in I} \\ 1 + \log(u) \end{pmatrix} \begin{pmatrix} (\mathbf{1}\{\Theta_i^I > u\} - \Theta_i^I)_{i \in I} \\ 1 + \log(u) \end{pmatrix}^\top du \right],$$

it can be seen that $\tilde{\mathbf{V}}$ is always positive semi-definite. If it is even strictly positive definite, we obtain an analogous result to Proposition 13.

PROPOSITION 17. *Let Θ be a spectral vector. Assume that all the partial derivatives*

$$\begin{aligned} c_i(\mathbf{v}, \mathbf{s}, \beta, 1) &= \frac{\partial}{\partial s_i} c(\mathbf{v}, \mathbf{s}, \beta, 1), \quad i \in \{1, \dots, d\}, \\ c_\beta(\mathbf{v}, \mathbf{s}, \beta, 1) &= \frac{\partial}{\partial \beta} c(\mathbf{v}, \mathbf{s}, \beta, 1), \end{aligned}$$

exist and are continuous on $\partial B_1^+(\mathbf{0}) \times A'_\delta$ and that all the partial derivatives of the function

$$\mathbf{s} \mapsto \mathbb{E}(\|\mathbf{s} \circ \Theta^I\|)$$

exist and are continuous in A_δ .

Then, for all non-empty $I \subset \{1, \dots, d\}$, it holds: If the matrix $\tilde{\mathbf{V}}_I$ is strictly positive definite, then the function $\mathbf{v}_I \mapsto \tilde{V}(\mathbf{v}_I)$ attains its minimum on $\partial B_1^+(\mathbf{0})$ at some unique $\tilde{\mathbf{v}}_I \in \partial B_1^+(\mathbf{0})$.

PROOF. First, we note that all the derivatives c_i , $i \in \{1, \dots, d\}$, and c_β are linear w.r.t. the first component, i.e. they satisfy

$$\begin{aligned} c_i(\mathbf{v}, \mathbf{s}, 1, 1) &= \sum_{j=1}^d v_j c_i(\mathbf{1}_{\{j\}}, \mathbf{s}, 1, 1), \quad i \in I \setminus \{1, \dots, d\}, \\ c_\beta(\mathbf{v}, \mathbf{s}, 1, 1) &= \sum_{j=1}^d v_j c_\beta(\mathbf{1}_{\{j\}}, \mathbf{s}, 1, 1) \end{aligned}$$

for all $\mathbf{v} \in \mathbb{R}^d$. Thus, $\mathbf{v}_I \mapsto \tilde{V}(\mathbf{v}_I)$ is a quadratic form and, analogously to the proof of Proposition 13, it remains to show that $\tilde{V}(\mathbf{w}_I) \geq 0$ for all $\mathbf{w}_I \in \mathbb{R}^d$ with $\sum_{i \in I} w_i = 0$ and equality holds only if $\mathbf{w} = \mathbf{0}$.

We use the decomposition

$$\tilde{V}(\mathbf{v}_I, 1) = g_1(\mathbf{v}_I) + g_2(\mathbf{v}_I)$$

where

$$\begin{aligned} g_1(\mathbf{v}_I) &= \frac{1}{\tau_I} \text{Var} \left(\mathbf{v}_I^\top \Theta^I - \tau_I \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \Theta_i^I \right), \\ g_2(\mathbf{v}_I) &= \tau_I \cdot \left(\begin{array}{c} (c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1))_{i \in I} \\ \tau_I^{-1} c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \end{array} \right)^\top \tilde{\mathbf{V}}_I \left(\begin{array}{c} (c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1))_{i \in I} \\ \tau_I^{-1} c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \end{array} \right), \end{aligned}$$

which follows directly from (42). By construction, $g_1(\mathbf{w}) \geq 0$ as a variance. By the positive definiteness of \tilde{V}_I , we have that $g_2(\mathbf{w}) \geq 0$ for all $\mathbf{w} \in \mathbb{R}^d$, while equality holds only if $c_i(\mathbf{w}_I, \mathbf{1}_I, 1, 1) = 0$ for all $i \in I$ and $c_\beta(\mathbf{w}_I, \mathbf{1}_I, 1, 1) = 0$. Differentiating (41) w.r.t. the i -th component of \mathbf{s} yields

$$c_i(\mathbf{w}_I, \mathbf{s}, 1, 1) = \frac{1}{\tau_I} \left(\frac{w_i}{\mathbb{E}[\|\mathbf{s} \circ \Theta^I\|]} - \frac{(\mathbf{w}^T \mathbf{s}) \cdot \frac{\partial}{\partial s_i} \mathbb{E}[\|\mathbf{s} \circ \Theta^I\|]}{\mathbb{E}[\|\mathbf{s} \circ \Theta^I\|]^2} \right).$$

In particular, for $\mathbf{s} = \mathbf{1}_I$ and $\sum_{i \in I} w_i = 0$, we obtain $c_i(\mathbf{w}_I, \mathbf{1}_I, 1, 1) = \tau_I^{-1} w_i$. Thus, $g_2(\mathbf{w}) = 0$ only if $w_i = 0$ for all $i \in I$, i.e. $\mathbf{w} = \mathbf{0}$. \square

4.2.3. Asymptotic normality of the plug-in estimator: We note that, in contrast to $f^{(I)}$, there is no direct empirical counterpart of $\tilde{V}(\cdot, 1)$. In order to enable the analytical minimization of that function, we will construct an estimator that retains the quadratic form of $\tilde{V}(\cdot, 1)$. Noting that $\mathbb{E}[\Theta_i^I] = \tau_I^{-1}$ for all $i \in I$,

$$\mathbb{E}[\Theta_i^I \wedge \Theta_j^I] = \mathbb{E}[\Theta_i^I] + \mathbb{E}[\Theta_j^I] - \mathbb{E}[\Theta_i^I \vee \Theta_j^I] = \frac{2}{\tau_I} - \frac{\tau_{\{i,j\}}}{\tau_I}, \quad i, j \in I,$$

and

$$\mathbb{E}[-\log(\Theta_i^I) \Theta_i^I] = c_\beta(\mathbf{1}_{\{i\}}, \mathbf{1}_I, 1, 1), \quad i \in I,$$

we can rewrite

$$\begin{aligned} \tilde{V}(\mathbf{v}_I, 1) &= \frac{1}{\tau_I} \text{Var} \left(\mathbf{v}_I^\top \Theta^I \right) - 2 \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \text{Cov} \left(\Theta_i^I, \mathbf{v}_I^\top \Theta^I \right) \\ &\quad + \tau_I \cdot \sum_{i \in I} \sum_{j \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) c_j(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \mathbb{E}[\Theta_i^I \wedge \Theta_j^I] \\ &\quad - 2c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \mathbb{E}[-\log(\Theta_i^I) \Theta_i^I] + \frac{1}{\tau_I} c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1)^2 \\ &= \frac{1}{\tau_I} \sum_{i \in I} \sum_{j \in I} v_i v_j \left(\mathbb{E}[\Theta_i^I \Theta_j^I] - \frac{1}{\tau_I^2} \right) - 2 \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \sum_{j \in I} v_j \left(\mathbb{E}[\Theta_i^I \Theta_j^I] - \frac{1}{\tau_I^2} \right) \\ &\quad + \sum_{i \in I} \sum_{j \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) c_j(\mathbf{v}_I, \mathbf{1}_I, 1, 1) (2 - \tau_{\{i,j\}}) \\ (43) \quad &\quad - 2c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1) c_\beta(\mathbf{1}_{\{i\}}, \mathbf{1}_I, 1, 1) + \frac{1}{\tau_I} c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1)^2. \end{aligned}$$

We will estimate thus $\tilde{V}(\cdot, 1)$ replacing the unknown quantities τ_I , $\tau_{\{i,j\}}$, $\mathbb{E}[\Theta_i^I \Theta_j^I]$, $c_i(\mathbf{v}_I, \mathbf{1}_I, 1, 1)$ and $c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, 1)$ in the right-hand side of (43) by consistent sequences of estimators $\tilde{\tau}_{I,n}$, $\tilde{\tau}_{\{i,j\},n}$, $\tilde{\mathbb{E}}_n[\Theta_i^I \Theta_j^I]$, $\tilde{c}_{i,n}(\mathbf{v}_I, \mathbf{1}_I, 1, 1)$ and $\tilde{c}_{\beta,n}(\mathbf{v}_I, \mathbf{1}_I, 1, 1)$, respectively. If we define

$$\tilde{c}_{i,n}(\mathbf{v}, \mathbf{1}_I, 1, 1) := \sum_{j=1}^d v_j \tilde{c}_{i,n}(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1), \quad i \in \{1, \dots, d\}$$

and

$$\tilde{c}_{i,n}(\mathbf{v}, \mathbf{1}_I, 1, 1) := \sum_{j=1}^d v_j \tilde{c}_{\beta,n}(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1)$$

as linear estimators and choose estimators $\tilde{\tau}_{\{i,j\},n}$ and $\tilde{\mathbb{E}}_n[\Theta_i^I \Theta_j^I]$ that are symmetric in the sense that $\tilde{\tau}_{\{i,j\},n} = \tilde{\tau}_{\{j,i\},n}$ and $\tilde{\mathbb{E}}_n[\Theta_i^I \Theta_j^I] = \tilde{\mathbb{E}}_n[\Theta_j^I \Theta_i^I]$ for all $i, j \in \{1, \dots, d\}$, the resulting variance estimator

$$\begin{aligned} \tilde{g}_{n,I}(\mathbf{v}_I) &= \frac{1}{\tilde{\tau}_{I,n}} \sum_{i \in I} \sum_{j \in I} v_i v_j \left(\tilde{\mathbb{E}}_n[\Theta_i^I \Theta_j^I] - \frac{1}{\tilde{\tau}_{I,n}^2} \right) - 2 \sum_{i \in I} \tilde{c}_{i,n}(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \sum_{j \in I} v_j \left(\tilde{\mathbb{E}}_n[\Theta_i^I \Theta_j^I] - \frac{1}{\tilde{\tau}_{I,n}^2} \right) \\ &\quad + \sum_{i \in I} \sum_{j \in I} \tilde{c}_{i,n}(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \tilde{c}_{j,n}(\mathbf{v}_I, \mathbf{1}_I, 1, 1) (2 - \tilde{\tau}_{\{i,j\},n}) \\ &\quad - 2 \tilde{c}_{\beta,n}(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \sum_{i \in I} \tilde{c}_{i,n}(\mathbf{v}_I, \mathbf{1}_I, 1, 1) \tilde{c}_{\beta,n}(\mathbf{1}_{\{i\}}, \mathbf{1}_I, 1, 1) + \frac{1}{\tilde{\tau}_{I,n}} \tilde{c}_{\beta,n}(\mathbf{v}_I, \mathbf{1}_I, 1, 1)^2, \end{aligned}$$

is of a quadratic form $\tilde{g}_{n,I}(\mathbf{v}_I) = \mathbf{v}_I^\top \tilde{\mathbf{A}}_n \mathbf{v}_I$ for a sequence of random symmetric matrices $(\tilde{\mathbf{A}}_n)_{n \in \mathbb{N}}$. Thus, a finite number of evaluations is sufficient to analytically calculate

$$\tilde{\mathbf{v}}_{n,I} = \operatorname{argmin}_{\mathbf{v}_I \in \partial B_1^+(\mathbf{0})} \tilde{g}_{n,I}(\mathbf{v}_I).$$

Here, without further conditions on the underlying estimators, no kind of (semi-)definiteness is guaranteed for the matrices $\tilde{\mathbf{A}}_n$. However, being a continuous function, $\mathbf{v}_i \mapsto \tilde{g}_{n,I}(\mathbf{v}_I)$ attains its minimum on the compact set $\partial B_1^+(\mathbf{0})$ at least once. Here, an arbitrary minimizer might be chosen for $\tilde{\mathbf{v}}_{n,I}$ provided that it is not unique.

Analogously to Proposition 15, consistency of the estimator can be shown.

PROPOSITION 18. *Let Θ be a spectral vector satisfying the assumptions of Proposition 17. Then, for all non-empty $I \subset \{1, \dots, d\}$, it holds: If $\tilde{\tau}_{I,n} \rightarrow_p \tau_I$,*

$$\begin{aligned} \tilde{\mathbb{E}}_n(\Theta_i^I \Theta_j^I) &\rightarrow_p \mathbb{E}(\Theta_i^I \Theta_j^I), & i, j \in I, \\ \tilde{c}_{i,n}(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1) &\rightarrow_p c_i(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1), & i, j \in I, \\ \text{and } \tilde{c}_{\beta,n}(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1) &\rightarrow_p c_\beta(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1), & j \in I, \end{aligned}$$

then we have that $\tilde{\mathbf{v}}_{n,I} \rightarrow_p \tilde{\mathbf{v}}_I$ for any sequence of minimizers $\tilde{\mathbf{v}}_{n,I}$.

Furthermore, $\tilde{g}_{n,I}(\tilde{\mathbf{v}}_{n,I}) \rightarrow_p \tilde{V}(\tilde{\mathbf{v}}_I)$.

PROOF. Recall that, by construction, $\tilde{g}_{n,I}(\mathbf{v}_I) = \mathbf{v}_I^\top \tilde{\mathbf{A}}_n \mathbf{v}_I$ for an appropriate sequence of random symmetric matrices $(\tilde{\mathbf{A}}_n)_{n \in \mathbb{N}}$, while $\tilde{V}(\mathbf{v}_I) = \mathbf{v}_I^\top \tilde{\mathbf{A}} \mathbf{v}_I$ for a conditionally positive definite matrix $\tilde{\mathbf{A}}$. The assumptions on the consistency of all the estimators involved in $\tilde{\mathbf{A}}_n$ imply that $\|\tilde{\mathbf{A}}_n - \tilde{\mathbf{A}}\|_\infty \rightarrow_p 0$. Consequently,

$$\sup_{\mathbf{v}_I \in \partial B_1^+(\mathbf{0})} |\tilde{g}_{n,I}(\mathbf{v}_I) - \tilde{V}(\mathbf{v}_I)| \leq d^2 \|\tilde{\mathbf{A}}_n - \tilde{\mathbf{A}}\|_\infty \rightarrow_p 0,$$

which is the analogue to Equation (39). The remainder of the proof runs analogously to the proof of Proposition 15. \square

Combining the results of Proposition 18 and Theorem 10, we obtain the desired limit theorem for the plug-in estimator, the moment based estimator for the case of unknown margins

$$\frac{1}{\tilde{\tau}_{I,n}^{MU}} = \frac{\tilde{M}_{n,k,I}(\tilde{\mathbf{v}}_{n,I}, 1)}{\tilde{P}_{n,k,I}}.$$

COROLLARY 19. Let \mathbf{X}_l , $l \in \mathbb{N}$, be independent copies of a regularly varying $[0, \infty)^d$ -valued random vector \mathbf{X} satisfying Equation (2) such that the corresponding transformed vector $\mathbf{X}^* = (r_1^{-1}X_1^\alpha, \dots, r_d^{-1}X_d^\alpha)$, the spectral vector Θ and the sequence $\{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ satisfy the assumptions of Proposition 18 for all non-empty $I \subset \{1, \dots, d\}$ and the assumptions of Theorem 10 for $K = \{1\}$. Then, for all non-empty $I \subset \{1, \dots, d\}$, the plug-in estimator satisfies

$$\sqrt{k} \left(\frac{1}{\widehat{\tau}_{I,n}^{MU}} - \frac{1}{\tau_I} \right) \xrightarrow{d} \mathcal{N}(0, \widetilde{V}(\widetilde{\mathbf{v}}_I)), \quad n \rightarrow \infty.$$

In Appendix E we provide consistent estimators satisfying the conditions of Proposition 18. The quantities in $\widetilde{g}_{n,I}(\mathbf{v}_I)$ based on moments of the spectral vector, i.e. τ_I , $\tau_{\{i,j\}}$ and $\mathbb{E}[\Theta_i^I \Theta_j^I]$, $i, j \in I$, can be directly estimated via the estimators constructed in Section 3.2. The partial derivatives c_i , $i \in I$, and c_β are approximated via difference quotients in Appendix E.

4.3. *Simulated example.* We consider the following bivariate max-linear model

$$X_1 = \frac{1}{2+p+q} \max\{Z_1, pZ_2, Z_3, qZ_4\}$$

$$X_2 = \frac{1}{2+p+q} \max\{pZ_1, Z_2, qZ_3, Z_4\},$$

where $Z_1, \dots, Z_4 \sim_{i.i.d.} \Phi_1$ with parameters $p, q \in [0, 1]$.

By definition, the vector (X_1, X_2) possesses unit Fréchet marginal distributions with extremal coefficient

$$\tau = \frac{4}{2+p+q},$$

i.e. the model can interpolate between full dependence ($p = q = 1$) and asymptotic independence ($p = q = 0$). The corresponding spectral vector Θ^* has a discrete distribution

$$\Theta^* = \begin{cases} (1, p) & \text{with probability } \frac{1}{4}, \\ (p, 1) & \text{with probability } \frac{1}{4}, \\ (1, q) & \text{with probability } \frac{1}{4}, \\ (q, 1) & \text{with probability } \frac{1}{4}. \end{cases}$$

We compare the four types of estimators for τ , namely the benchmark estimators and the moment-based estimators, both with known and unknown margins as described in Sections 4.1 and 4.2, respectively. More precisely, in the case of known margins where we can work with standardized observations $\mathbf{X}_1^*, \mathbf{X}_2^*, \dots, \mathbf{X}_n^*$ and use

- the benchmark estimator

$$\frac{1}{\widehat{\tau}_{I,n}^{BK}(\mathbf{v})} = \frac{\widehat{P}_{n,u,\{1,2\}}^{(\text{conv})}(\mathbf{v})}{P_{n,u,\{1,2\}}};$$

- the moment-based plug-in estimator

$$\frac{1}{\widehat{\tau}_{I,n}^{MK}} = \frac{\widehat{M}_{n,u_n,\{1,2\}}(\widehat{\mathbf{v}}_{n,u_n,\{1,2\}}^*)}{\widehat{P}_{n,u_n,\{1,2\}}}$$

based on the estimated “optimal” weights $\widehat{\mathbf{v}}_{n,u_n,\{1,2\}}^*$.

Note that, for simulated data from the bivariate max-linear model, no standardization is required as the (unit Fréchet) marginal distributions are already identical and regularly varying with index $\alpha = 1$. From Example 14, it can be seen that the variance minimizing linear combination is unique whenever $p < 1$ or $q < 1$ and is given by

$$\mathbf{v}^* = \left(\frac{1}{2}, \frac{1}{2} \right).$$

As discussed in Section 4.1, the asymptotic distribution of the benchmark estimator does not depend on the choice of $\mathbf{v}_I \in \partial B_1^+(\mathbf{0})$. Here we choose $\widehat{\tau}_{I,n}^{BK} = \widehat{\tau}_{I,n}^{BK}(\mathbf{v}^*)$ as benchmark.

Even though the margins are known, it might sometimes be beneficial to work with the rank transformations designed for the case of unknown margins as discussed in Section 4.2. Thus, we also consider

- the benchmark estimator

$$\widehat{\tau}_{I,n}^{BU} = \frac{1}{\frac{n}{k} \widetilde{P}_{n,k,\{1,2\}}} = \frac{1}{\widehat{L}_{k,n}(\mathbf{1})}$$

taking the role of the benchmark;

- the moment-based plug-in estimator

$$\widehat{\tau}_{I,n}^{MU} = \frac{\widetilde{M}_{n,k,\{1,2\}}(\widetilde{\mathbf{v}}_{n,\{1,2\}}, 1)}{\widetilde{P}_{n,k,\{1,2\}}}$$

based on estimated “optimal” weights $\widetilde{\mathbf{v}}_{n,\{1,2\}}$.

From symmetry arguments, it can be easily seen that the “optimal” linear combination (if unique) is given by

$$\widetilde{\mathbf{v}} = \left(\frac{1}{2}, \frac{1}{2} \right).$$

We first compare these four estimators in terms of their asymptotic variances, that is, $\text{AVar}^{BK} = \tau_I^{-3}(\tau_I - 1)$ (cf. Section 4.1), $\text{AVar}^{MK} = \tau_I^{-1} \text{Var}(\mathbf{v}_I^{*\top} \boldsymbol{\Theta}_I^\ell)$ (cf. Corollary 16), $\text{AVar}^{BU} = \tau_I^{-4} \sigma_L^2$ (cf. Section 4.2) and $\text{AVar}^{MU} = \widetilde{V}(\widetilde{\mathbf{v}}_I)$ (cf. Corollary 19).

The asymptotic standard deviations $\sqrt{\text{AVar}^{BK}}$, $\sqrt{\text{AVar}^{MK}}$, $\sqrt{\text{AVar}^{BU}}$ and $\sqrt{\text{AVar}^{MU}}$ are displayed in Figure 1 as functions of p and q . First, it can be seen that all the estimators have asymptotic variance 0 in the case of full dependence ($p = q = 1$). While, in all the other cases, the benchmark type estimators have a strictly positive asymptotic variance, an asymptotic variance of 0 occurs for the new moment-based estimators also in the case of asymptotic independence and, if the margins are known, in all the cases where $p = q$ as expected since Remark 12 applies in both cases. Complementing the results we obtained in Section 4 in a more general setting, Figure 1 provides a strict ordering of the four estimators in terms of the asymptotic variances:

$$\text{AVar}^{MK} \leq \text{AVar}^{MU} \leq \text{AVar}^{BU} \leq \text{AVar}^{BK} \quad \text{for all } 0 \leq p, q \leq 1,$$

where equalities hold in some degenerate cases with $p, q \in \{0, 1\}$ only. Thus, in our bivariate max-linear model, we expect the new moment-based estimators to always outperform the benchmark estimators, but the differences are larger for the estimators with known margins than for their analogues based on order statistics.

Furthermore, we perform a simulation study to compare the estimators in a setting with finite sample sizes. Here, we consider three different scenarios:

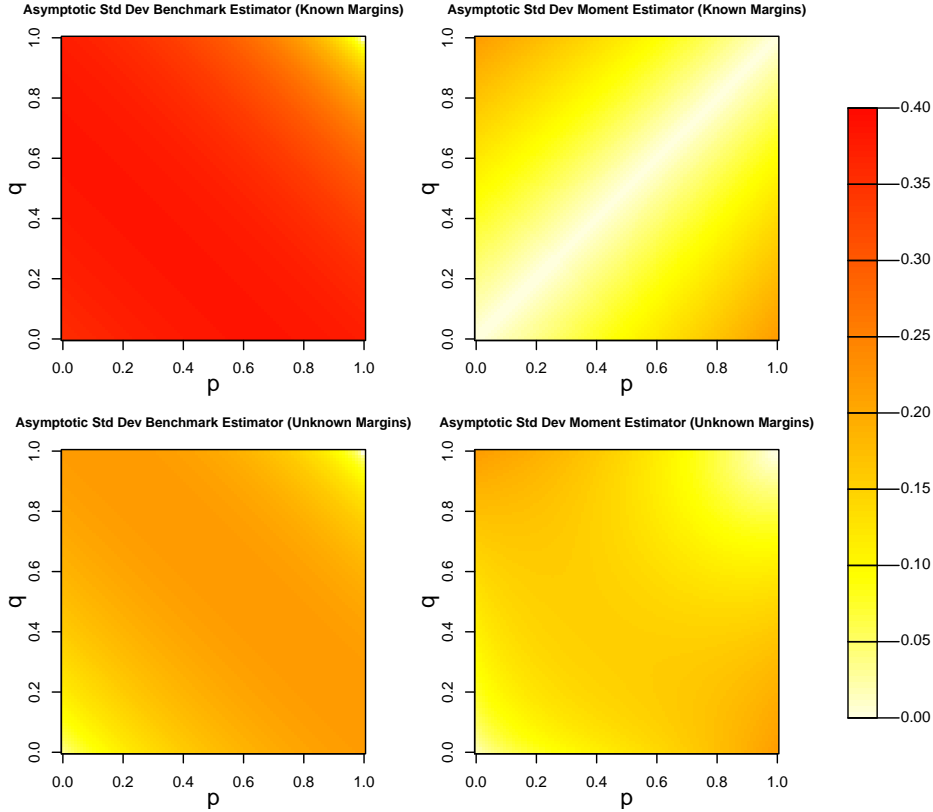


FIG 1. The asymptotic standard deviations of the four estimators, $\sqrt{\text{AVar}^{BK}}$, $\sqrt{\text{AVar}^{MK}}$, $\sqrt{\text{AVar}^{BU}}$ and $\sqrt{\text{AVar}^{MU}}$ (form top left to bottom right) as functions of the model parameters p and q .

- Scenario 1: $p = 0.1, q = 0.2$ (weak dependence)
- Scenario 2: $p = 0.4, q = 0.6$ (medium dependence)
- Scenario 3: $p = 0.8, q = 0.9$ (strong dependence)

In each of these scenarios, we consider $n = 1000$ independent observations of $\mathbf{X} = (X_1, X_2)$ and apply the estimators with a threshold $u_n = \Phi_1^{-1}(0.95)$ and $k/n = 0.05$, respectively. For the estimation of the partial derivatives of c that are needed for the calculation of $\tilde{\mathbf{v}}_{n,\{1,2\}}$, we choose $\varepsilon = k/n$. We determine the biases and standard deviations by 5000 repetitions. For comparison, we also calculate the theoretical standard deviations according to the central limit theorems, that is, we compute the square roots of the four terms $n \cdot \mathbb{P}(X_1 > u_n) \cdot \text{AVar}^{BK}$, $n \cdot \mathbb{P}(X_1 > u_n) \cdot \text{AVar}^{MK}$, $k \cdot \text{AVar}^{BU}$ and $k \cdot \text{AVar}^{MU}$, respectively.

The results are displayed in Table 1. First, it can be seen that, in the weak dependence scenario, all the biases are of a similar order as the standard deviations. In contrast, for the other two scenarios, compared to the standard deviation, the bias seems to be negligible for two of the four estimators, namely $\hat{\tau}_{I,n}^{BK}$ and $\hat{\tau}_{I,n}^{MU}$. The empirical standard deviations confirm the theoretical ordering of the four methods. For three of the four estimators, they are close to their theoretical counterparts. The only exception is the moment-based estimator in case of known margins, $\hat{\tau}_{I,n}^{MK}$, where the empirical standard deviation is significantly larger than the theoretical one. This can be explained that this estimator is in fact a plug-in estimator based on the estimated minimizer $\hat{\mathbf{v}}_{n,u_n,\{1,2\}}^*$ of the asymptotic variance. Even though the theoretical asymptotic variance of the plug-in estimator corresponds to the optimal one, the deviations

$p = 0.1, q = 0.2$ ($\tau = 40/23$, weak dependence)

	$1/\widehat{\tau}_{I,n}^{BK}$	$1/\widehat{\tau}_{I,n}^{MK}$	$1/\widehat{\tau}_{I,n}^{BU}$	$1/\widehat{\tau}_{I,n}^{MU}$
bias	0.026	0.019	0.024	0.013
std. deviation	0.053	0.011	0.021	0.015
theo. std. deviation	0.053	0.003	0.019	0.015

$p = 0.4, q = 0.6$ ($\tau = 4/3$, moderate dependence)

	$1/\widehat{\tau}_{I,n}^{BK}$	$1/\widehat{\tau}_{I,n}^{MK}$	$1/\widehat{\tau}_{I,n}^{BU}$	$1/\widehat{\tau}_{I,n}^{MU}$
bias	0.009	0.013	0.023	0.002
std. deviation	0.054	0.017	0.032	0.022
theo. std. deviation	0.053	0.006	0.031	0.021

$p = 0.8, q = 0.9$ ($\tau = 40/37$, strong dependence)

	$1/\widehat{\tau}_{I,n}^{BK}$	$1/\widehat{\tau}_{I,n}^{MK}$	$1/\widehat{\tau}_{I,n}^{BU}$	$1/\widehat{\tau}_{I,n}^{MU}$
bias	0.002	0.008	0.021	-0.003
std. deviation	0.036	0.008	0.025	0.012
theo. std. deviation	0.036	0.003	0.024	0.010

TABLE 1

Simulation results for the three scenarios specified above with $n = 1000$, $k = 50$ and $u_n = \Phi^{-1}(0.95)$.

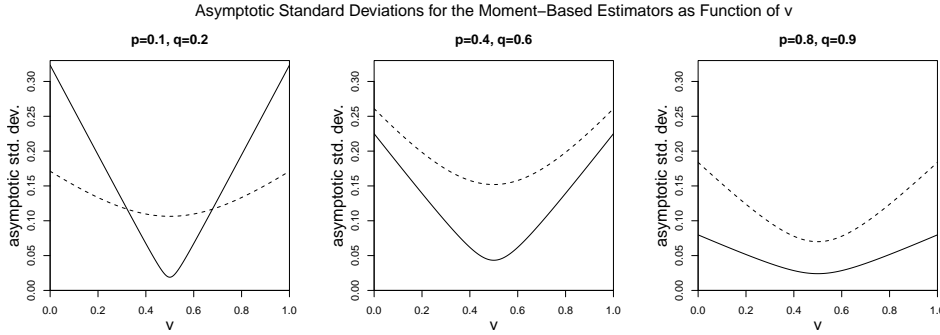


FIG 2. The asymptotic standard deviations $\sqrt{V((v, 1-v))}$ (solid lines) and $\sqrt{\widehat{V}((v, 1-v))}$ (dashed line) of the moment-based estimators with known and unknown margins, respectively, for each of the three scenarios reflecting weak, moderate and strong dependence (from left to right).

between $\widehat{v}_{n, u_n, \{1,2\}}^*$ and the true minimizer v^* , in practice, may lead to a significant increase of the asymptotic variance. This is demonstrated in Figure 2 where the asymptotic variance is plotted as a function of the weight vector v . Compared to the case of known margins, the relative changes of the asymptotic variance for the moment-based estimator for unknown margins are rather small. This is in line with the results of our simulations which show that the empirical standard deviation is close to the theoretical one in that case.

5. Conclusion. We show some benefits of estimating extremal properties with moments of convex combination of spectral components. An analysis of the asymptotic variances yields a natural strategy consisting in plugging in the asymptotically optimal weights. This methodology is shown to efficiently reduce the variance when estimating the extremal coefficient by first order moments. We believe that moments of higher orders $\mathbb{E}[(v_J^T \Theta^J)^p]$, $p \geq 1$, are also meaningful and can be used for asserting other properties of the spectral distribution. They represent interesting alternatives to the stable tail dependence function L as, for small p , such functions of v have nice regularity properties (constant for $p = 1$, quadratic for $p = 2$) that have proven to be very useful for inference purpose. For some simplistic spectral distribution

we actually show that the asymptotic variance can be degenerate to 0 yielding some potential superefficiency considerations for extremal coefficient estimation.

We focus on reducing the variance under ad-hoc assumptions implying bias negligibility. In our numerical illustrations, the biases of our procedures were empirically small. A systematic bias analysis and potential bias reduction for this new class of estimators should be considered in the future. Finally the first two steps of our procedure in the unknown margin settings are specific to tail-equivalence of margins X_i , $1 \leq i \leq d$. It would be interesting for future work to extend spectral moment based approaches to the more general setting of regularly varying vector with different tail indexed margins as the one of Einmahl et al. [1997]. A one-step procedure based on rank transforms similar to the one of Einmahl et al. [2001] could be investigated. However the ratio-type statistics involved in moment estimators have not yet been proven to relate well with rank transforms.

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APPENDIX A: PROOF OF THEOREM 5

In order to prove functional convergence, we apply the functional Central Limit Theorem 2.11.9 in Van der Vaart and Wellner [1996]. Let us first check the convergence of the covariances.

For $\mathbf{v} \in \partial B_1^+(\mathbf{0})$, $(\mathbf{s}, \beta) \in A'_\delta$ and $p \in K_0$, define the function

$$f_{\mathbf{v}, \mathbf{s}, \beta, p} : (0, \infty)^d \rightarrow [0, \infty), \quad f_{\mathbf{v}, \mathbf{s}, \beta, p}(\mathbf{x}) = \left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \mathbf{x}}{\|\mathbf{s} \circ \mathbf{x}\|} \right)^{1/\beta} \right)^p \mathbf{1}_{\{\|\mathbf{s} \circ \mathbf{x}\| > 1\}}.$$

Define the function space $\mathcal{F} = \{f_{\mathbf{v}, \mathbf{s}, \beta, p} : \mathbf{v} \in \partial B_1^+(\mathbf{0}), (\mathbf{s}, \beta) \in A'_\delta, p \in K_0\}$ equipped with the semimetric $\rho(f_{\mathbf{v}, \mathbf{s}, \beta, p_1}, f_{\mathbf{w}, \mathbf{t}, \beta_2, p_2}) = \max\{\|\mathbf{v} - \mathbf{w}\|, \|\mathbf{s} - \mathbf{t}\|, |\beta_1 - \beta_2|, |p_1 - p_2|\}$. Furthermore, for $l = 1, \dots, n$, $n \in \mathbb{N}$, let

$$Z_{nl}(f) = \sqrt{\frac{a^*(u_n)}{n}} f(u_n^{-1} \mathbf{X}_l^*), \quad f \in \mathcal{F}.$$

Then, each element of the sequence $(\{G_n(\mathbf{v}, \mathbf{s}, \beta, p); \mathbf{v} \in \partial B_1^+(\mathbf{0}), (\mathbf{s}, \beta) \in A'_\delta, p \in K_0\})_{n \in \mathbb{N}}$ can be rewritten as

$$\{G_n(\mathbf{v}, \mathbf{s}, \beta, p)\}_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), (\mathbf{s}, \beta) \in A'_\delta, p \in K} = \left\{ \sum_{l=1}^n (Z_{nl}(f) - \mathbb{E}Z_{nl}(f)) \right\}_{f \in \mathcal{F}}, \quad n \in \mathbb{N}.$$

Analogously to the results from Proposition 4, we can calculate

$$\begin{aligned} & \text{Cov} \left(\sum_{l=1}^n Z_{nl}(f_{\mathbf{v}, \mathbf{s}, \beta_1, p_1}), \sum_{l=1}^n Z_{nl}(f_{\mathbf{w}, \mathbf{t}, \beta_2, p_2}) \right) \\ &= a^*(u_n) \mathbb{E} [f_{\mathbf{v}, \mathbf{s}, \beta_1, p_1}(u_n^{-1} \mathbf{X}^*) f_{\mathbf{w}, \mathbf{t}, \beta_2, p_2}(u_n^{-1} \mathbf{X}^*)] \\ & \quad - a^*(u_n) \mathbb{E} [f_{\mathbf{v}, \mathbf{s}, \beta_1, p_1}(u_n^{-1} \mathbf{X}^*)] \mathbb{E} [f_{\mathbf{w}, \mathbf{t}, \beta_2, p_2}(u_n^{-1} \mathbf{X}^*)] \\ &= \mathbb{E} \left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \mathbf{X}^*}{\|\mathbf{s} \circ \mathbf{X}^*\|} \right)^{1/\beta_1} \right)^{p_1} \left(\mathbf{w}^\top \left(\frac{\mathbf{t} \circ \mathbf{X}^*}{\|\mathbf{t} \circ \mathbf{X}^*\|} \right)^{1/\beta_2} \right)^{p_2} \mathbf{1}_{\{\|\mathbf{s} \circ \mathbf{X}^*\| \wedge \|\mathbf{t} \circ \mathbf{X}^*\| > u_n\}} \right] \\ & \quad \cdot a^*(u_n) \mathbb{P}(\|\mathbf{s} \circ \mathbf{X}^*\| \wedge \|\mathbf{t} \circ \mathbf{X}^*\| > u_n) \\ & \quad - \frac{1}{\sqrt{a^*(u_n)}} \mathbb{E} \left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \mathbf{X}^*}{\|\mathbf{s} \circ \mathbf{X}^*\|} \right)^{1/\beta_1} \right)^{p_1} \mathbf{1}_{\{\|\mathbf{s} \circ \mathbf{X}^*\| > u_n\}} \right] \cdot a^*(u_n) \mathbb{P}(\|\mathbf{s} \circ \mathbf{X}^*\| > u_n) \\ & \quad \cdot \frac{1}{\sqrt{a^*(u_n)}} \mathbb{E} \left[\left(\mathbf{w}^\top \left(\frac{\mathbf{t} \circ \mathbf{X}^*}{\|\mathbf{t} \circ \mathbf{X}^*\|} \right)^{1/\beta_2} \right)^{p_2} \mathbf{1}_{\{\|\mathbf{t} \circ \mathbf{X}^*\| > u_n\}} \right] \cdot a^*(u_n) \mathbb{P}(\|\mathbf{t} \circ \mathbf{X}^*\| > u_n). \end{aligned}$$

This sequence of covariances converges to, as $n \rightarrow \infty$,

$$\begin{aligned} & \frac{\mathbb{E} \left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \Theta}{\|\mathbf{s} \circ \Theta\|} \right)^{1/\beta_1} \right)^{p_1} \left(\mathbf{w}^\top \left(\frac{\mathbf{t} \circ \Theta}{\|\mathbf{t} \circ \Theta\|} \right)^{1/\beta_2} \right)^{p_2} (\|\mathbf{s} \circ \Theta\| \wedge \|\mathbf{t} \circ \Theta\|) \right]}{\mathbb{E}[\|\mathbf{s} \circ \Theta\| \wedge \|\mathbf{t} \circ \Theta\|]} \cdot \tau \cdot \mathbb{E}[\|\mathbf{s} \circ \Theta\| \wedge \|\mathbf{t} \circ \Theta\|] \\ &= \tau \cdot \mathbb{E} \left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \Theta}{\|\mathbf{s} \circ \Theta\|} \right)^{1/\beta_1} \right)^{p_1} \left(\mathbf{w}^\top \left(\frac{\mathbf{t} \circ \Theta}{\|\mathbf{t} \circ \Theta\|} \right)^{1/\beta_2} \right)^{p_2} (\|\mathbf{s} \circ \Theta\| \wedge \|\mathbf{t} \circ \Theta\|) \right]. \end{aligned}$$

Furthermore, as $\sup_{\mathbf{x} \in (0, \infty)^d} |f(\mathbf{x})| \leq 1$ for all $f \in \mathcal{F}$, we have that

$$\|Z_{nl}\|_{\mathcal{F}} = \sup_{f \in \mathcal{F}} |Z_{nl}(f)| \leq \sqrt{\frac{a^*(u_n)}{n}} \text{ a.s.}$$

for all $l = 1, \dots, n$ and $n \in \mathbb{N}$.

Consequently, we check the Lindeberg condition: for any $k \in \mathbb{N}$, we have

$$(44) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E}[\|Z_{ni}\|_{\mathcal{F}}^k \mathbf{1}\{\|Z_{ni}\|_{\mathcal{F}} > \eta\}] \leq \lim_{n \rightarrow \infty} n \left(\frac{a^*(u_n)}{n} \right)^{k/2} \mathbf{1}\{\sqrt{a^*(u_n)/n} > \eta\} = 0,$$

where we used that, for any $\eta > 0$, eventually $a^*(u_n)/n < \eta^2$ since $\lim_{n \rightarrow \infty} n/a^*(u_n) = \infty$. Making use of (44) for $k = 2$ one can easily see that, for all $a_1, \dots, a_m \in \mathbb{R}$, $f_1, \dots, f_m \in \mathcal{F}$ and $m \in \mathbb{N}$, the triangular arrays $(\sum_{i=1}^m a_i Z_{ni}(f_i))_{1 \leq l \leq n, n \in \mathbb{N}}$ satisfy a Lindeberg condition and, therefore, G_n converges to G in terms of finite-dimensional distributions. Setting $k = 1$ in (44) gives the Lindeberg type condition of Theorem 2.11.9 in Van der Vaart and Wellner [1996].

We further check the equi-continuity condition, i.e. the second condition of that theorem. To this end, we consider two elements $f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, \beta_1, p} \in \mathcal{F}$ and $f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, \beta_2, p} \in \mathcal{F}$ with distance $\max\{\|\mathbf{v}^{(1)} - \mathbf{v}^{(2)}\|, \|\mathbf{s}^{(1)} - \mathbf{s}^{(2)}\|, |\beta_1 - \beta_2|\} < \eta$ for some $0 < \eta < \delta$. Notice that for sufficiently small η , we have that $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ necessarily belong to $A_{\delta, I}$ for the same non-empty index set I , i.e. that the same components of both vectors are different from zero. Moreover we assume with no loss of generality that there exists some $\mathbf{y} \in A_{\delta, I}$ satisfying $s_i^{(1)}, s_i^{(2)} \in [y_i, y_i + \eta]$ for all $i \in I$.

We distinguish three cases to obtain an upper bound for $|f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, \beta_1, p}(\mathbf{x}) - f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, \beta_2, p}(\mathbf{x})|$:

1. $\|\mathbf{s}^{(1)} \circ \mathbf{x}\| \leq 1$ and $\|\mathbf{s}^{(2)} \circ \mathbf{x}\| \leq 1$:

In this case, we have

$$|f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, \beta_1, p}(\mathbf{x}) - f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, \beta_2, p}(\mathbf{x})| = |0 - 0| = 0.$$

2. $\|\mathbf{s}^{(1)} \circ \mathbf{x}\| \geq 1$ and $\|\mathbf{s}^{(2)} \circ \mathbf{x}\| \geq 1$:

In this case, both the indicator functions in $f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, \beta_1, p}(\mathbf{x})$ and in $f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, \beta_2, p}(\mathbf{x})$ are equal to 1. The fact that $b^p - a^p = \int_a^b p x^{p-1} dx \leq p(b - a)$ for $0 \leq a \leq b \leq 1$ and $p \in \mathbb{N}_0$, leads to the bound

$$|f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, \beta_1, p}(\mathbf{x}) - f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, \beta_2, p}(\mathbf{x})| \leq p \cdot |f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, \beta_1, 1}(\mathbf{x}) - f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, \beta_2, 1}(\mathbf{x})|.$$

Thus, with $\mathbf{v} = \mathbf{v}^{(1)} \wedge \mathbf{v}^{(2)}$, $\mathbf{s} = \mathbf{s}^{(1)} \wedge \mathbf{s}^{(2)}$ denoting componentwise minima and $\beta = \beta_1 \wedge \beta_2$, and the fact that $\mathbf{s} + \eta \mathbf{1}_I \leq (1 + (1 + \delta)\eta)\mathbf{s}$ for $\mathbf{s} \in A_{\delta, I}$, we obtain

$$\begin{aligned} & |f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, \beta_1, p}(\mathbf{x}) - f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, \beta_2, p}(\mathbf{x})| \\ & \leq p(\mathbf{v} + \eta \mathbf{1}_I)^\top \left[\left(\frac{(\mathbf{s} + \eta \mathbf{1}_I) \circ \mathbf{x}}{\|\mathbf{s} \circ \mathbf{x}\|} \right)^{1/(\beta+\eta)} \vee \left(\frac{(\mathbf{s} + \eta \mathbf{1}_I) \circ \mathbf{x}}{\|\mathbf{s} \circ \mathbf{x}\|} \right)^{1/\beta} \right] \\ & \quad - p\mathbf{v}^\top \left[\left(\frac{\mathbf{s} \circ \mathbf{x}}{\|(\mathbf{s} + \eta \mathbf{1}_I) \circ \mathbf{x}\|} \right)^{1/\beta} \wedge \left(\frac{\mathbf{s} \circ \mathbf{x}}{\|(\mathbf{s} + \eta \mathbf{1}_I) \circ \mathbf{x}\|} \right)^{1/(\beta+\eta)} \right] \\ & \leq p(\mathbf{v} + \eta \mathbf{1}_I)^\top \left[\left(\frac{\mathbf{s} \circ \mathbf{x}}{\|\mathbf{s} \circ \mathbf{x}\|} (1 + (1 + \delta)\eta) \right)^{1/(\beta+\eta)} \vee \left(\frac{\mathbf{s} \circ \mathbf{x}}{\|\mathbf{s} \circ \mathbf{x}\|} (1 + (1 + \delta)\eta) \right)^{1/\beta} \right] \\ & \quad - p\mathbf{v}^\top \left[\left(\frac{\mathbf{s} \circ \mathbf{x}}{\|\mathbf{s} \circ \mathbf{x}\|} \frac{1}{1 + (1 + \delta)\eta} \right)^{1/\beta} \wedge \left(\frac{\mathbf{s} \circ \mathbf{x}}{\|\mathbf{s} \circ \mathbf{x}\|} \frac{1}{1 + (1 + \delta)\eta} \right)^{1/(\beta+\eta)} \right] \end{aligned}$$

$$\begin{aligned}
 &\leq p(\mathbf{v} + \eta \mathbf{1})^\top \left(\frac{\mathbf{s} \circ \mathbf{x}}{\|\mathbf{s} \circ \mathbf{x}\|} \right)^{1/(\beta+\eta)} (1 + (1 + \delta)\eta)^{1+\delta} - p\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \mathbf{x}}{\|\mathbf{s} \circ \mathbf{x}\|} \right)^{1/\beta} \frac{1}{(1 + (1 + \delta)\eta)^{1+\delta}} \\
 &\leq p\eta \mathbf{1}^\top \mathbf{1}_I (1 + (1 + \delta)\eta)^{1+\delta} \\
 &\quad + p\mathbf{v}^\top \mathbf{1}_I \max_{i \in I} \left(\left(\frac{s_i x_i}{\|\mathbf{s} \circ \mathbf{x}\|} \right)^{1/(\beta+\eta)} (1 + (1 + \delta)\eta)^{1+\delta} - \left(\frac{s_i x_i}{\|\mathbf{s} \circ \mathbf{x}\|} \right)^{1/\beta} \frac{1}{(1 + (1 + \delta)\eta)^{1+\delta}} \right) \\
 &\leq p\eta d (1 + (1 + \delta)\eta)^{1+\delta} + p \max_{i \in I} \left(\frac{s_i x_i}{\|\mathbf{s} \circ \mathbf{x}\|} \right)^{1/(\beta+\eta)} \cdot \left[(1 + (1 + \delta)\eta)^{1+\delta} - \frac{1}{(1 + (1 + \delta)\eta)^{1+\delta}} \right] \\
 &\quad + p \max_{i \in I} \left(\left(\frac{s_i x_i}{\|\mathbf{s} \circ \mathbf{x}\|} \right)^{1/(\beta+\eta)} - \left(\frac{s_i x_i}{\|\mathbf{s} \circ \mathbf{x}\|} \right)^{1/\beta} \right) \cdot \frac{1}{(1 + (1 + \delta)\eta)^{1+\delta}} \\
 &\leq pd(1 + (1 + \delta)\eta)^{1+\delta} \eta + p \frac{(1 + (1 + \delta)\eta)^{2+2\delta} - 1}{(1 + (1 + \delta)\eta)^{1+\delta}} \\
 &\quad + p \max_{i \in I} \left(\left(\frac{s_i x_i}{\|\mathbf{s} \circ \mathbf{x}\|} \right)^{1/(\beta+\eta)} - \left(\frac{s_i x_i}{\|\mathbf{s} \circ \mathbf{x}\|} \right)^{1/\beta} \right) \\
 &\leq pd(1 + (1 + \delta)\eta)^{1+\delta} \eta + p2(1 + \delta)^2 (1 + (1 + \delta)\eta)^\delta \eta + p(1 + \delta)^2 \left[\sup_{z \in [0,1]} \log(z) z^{1/(1+\delta)} \right] \eta.
 \end{aligned}$$

Bounding p by $p_{\max} = \max_{p \in K_0} p$ and noting that $\sup_{z \in [0,1]} \log(z) z^{1/(1+\delta)} < \infty$, we have an overall bound of the form $C\eta$ some constant $C = C(\delta, p_{\max}, d) > 0$ for sufficiently small η .

This case occurs only if $(1 + \delta)\|\mathbf{x}\| > 1$ as $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ belong to $A_{\delta, I}$.

3. $\min\{\|\mathbf{s}^{(1)} \circ \mathbf{x}\|, \|\mathbf{s}^{(2)} \circ \mathbf{x}\|\} \leq 1 < \max\{\|\mathbf{s}^{(1)} \circ \mathbf{x}\|, \|\mathbf{s}^{(2)} \circ \mathbf{x}\|\}$:

In this case one of the functions $f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, \beta_1, p}(\mathbf{x})$ and $f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, \beta_2, p}(\mathbf{x})$ equals zero, while the other one is bounded by 1. Consequently, noting that this case does not appear if $\mathbf{s}^{(1)} = \mathbf{s}^{(2)}$, we have

$$\begin{aligned}
 &|f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, \beta_1, p}(\mathbf{x}) - f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, \beta_2, p}(\mathbf{x})| \\
 (45) \quad &\leq \mathbf{1} \left\{ \bigcup_{i \in I, s_i^{(1)} \neq s_i^{(2)}} \{u_n / (s_i^{(1)} \vee s_i^{(2)}) \leq x_i < u_n / (s_i^{(1)} \wedge s_i^{(2)})\} \right\}.
 \end{aligned}$$

As, by construction, $y_i \leq \min\{s_i^{(1)}, s_i^{(2)}\} \leq \max\{s_i^{(1)}, s_i^{(2)}\} \leq y_i + \eta$ this case occurs only if $y_i x_i \leq 1 < (y_i + \eta)x_i$ for at least one $i \in I$.

Combining the three cases, we obtain the following uniform bound on $\|\mathbf{v}^{(1)} - \mathbf{v}^{(2)}\| < \eta$ and $s_i^{(1)}, s_i^{(2)} \in [y_i, y_i + \eta]$, $i \in I$, for some $\mathbf{y} \in A_{\delta, I}$:

$$\begin{aligned}
 &(Z_{nl}(f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, \beta_1, p}) - Z_{nl}(f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, \beta_2, p}))^2 \\
 &\leq \frac{a^*(u_n)}{n} C^2 \eta^2 \mathbf{1} \{ \|\mathbf{X}_l^*\| > (1 + \delta)^{-1} u_n \} \\
 &\quad + \frac{a^*(u_n)}{n} \cdot \mathbf{1} \left\{ \bigcup_{i \in I} \{u_n (y_i + \eta)^{-1} \leq X_{li}^* < u_n y_i^{-1}\} \right\} \\
 (46) \quad &=: Q_{nl}^{(1)}(\eta) + Q_{nl}^{(2)}(\eta, \mathbf{y}).
 \end{aligned}$$

In the following, we will consider the two summands separately. From the definition of $Q_{nl}^{(1)}$ and regular variation of the function a^* , it directly follows that

$$\begin{aligned} \sum_{l=1}^n \mathbb{E} \left[Q_{nl}^{(1)}(\eta) \right] &\leq a^*(u_n) C^2 \eta^2 \mathbb{P}(\|\mathbf{X}^*\| > (1+\delta)^{-1} u_n) \\ &\sim \tau \frac{a^*(u_n)}{a^*((1+\delta)^{-1} u_n)} C^2 \eta^2 \sim \tau (1+\delta) C^2 \eta^2, \quad n \rightarrow \infty. \end{aligned}$$

As $\tau \leq d$ we obtain for n sufficiently large

$$(47) \quad \sum_{l=1}^n \mathbb{E} \left[Q_{nl}^{(1)}(\eta) \right] \leq 2d(1+\delta) C^2 \eta^2.$$

For $Q_{nl}^{(2)}$ we have that, as $n \rightarrow \infty$,

$$\begin{aligned} \sum_{l=1}^n \mathbb{E} \left[Q_{nl}^{(2)}(\eta, \mathbf{y}) \right] &\leq a^*(u_n) \sum_{i \in I} \mathbb{P}(u_n(y_i + \eta)^{-1} \leq X_i^* < u_n y_i^{-1}) \\ &\sim \sum_{i \in I} \left(\frac{\mathbb{P}(X_i^* > u_n(y_i + \eta)^{-1})}{\mathbb{P}(X_i^* > u_n)} - \frac{\mathbb{P}(X_i^* > u_n y_i^{-1})}{\mathbb{P}(X_i^* > u_n)} \right). \end{aligned}$$

with $\mathbf{s}^{(1)}, \mathbf{s}^{(2)}$ in $A_{\delta, I}$. As the function $u \mapsto \mathbb{P}(X_i^* > u)$ is regular varying with index -1 for each $i = 1, \dots, d$, the convergence of $\mathbb{P}(X_i^* > u_n y) / \mathbb{P}(X_i^* > u_n) \rightarrow y^{-1}$ is uniform in $y \in [(1+\delta)^{-1}, 1+\delta]$, i.e. there exists a sequence of positive numbers $\{f_n\}_{n \in \mathbb{N}}$ with $f_n \rightarrow 0$ such that

$$(1 - f_n) y^{-1} \leq \frac{\mathbb{P}(X_i^* > u_n y)}{\mathbb{P}(X_i^* > u_n)} \leq (1 + f_n) y^{-1}$$

for all $n \in \mathbb{N}$, $i \in \{1, \dots, d\}$ and $y \in [(1+\delta)^{-1}, 1+\delta]$. Thus, we obtain, as $n \rightarrow \infty$,

$$\begin{aligned} &\sup_{\mathbf{y} \in A_{\delta, I}} \sum_{i \in I} \left(\frac{\mathbb{P}(X_i^* > u_n(y_i + \eta)^{-1})}{\mathbb{P}(X_i^* > u_n)} - \frac{\mathbb{P}(X_i^* > u_n y_i^{-1})}{\mathbb{P}(X_i^* > u_n)} \right) \\ &\leq |I| \cdot (1 + f_n) \cdot \eta + 2f_n \cdot \sup_{\mathbf{y} \in A_{\delta, I}} \sum_{i \in I} y_i \leq 2d\eta + 2df_n(1+\delta) \end{aligned}$$

for sufficiently large n .

Combining the estimates obtained for $Q_{nl}^{(1)}$ and $Q_{nl}^{(2)}$ and denoting $\mathbf{s}^{(1)}, \mathbf{s}^{(2)} \in [\mathbf{y}, \mathbf{y} + \eta]$ for $\mathbf{s}^{(1)}, \mathbf{s}^{(2)} \in A_{\delta, I}$ satisfying $s_i^{(1)}, s_i^{(2)} \in [y_i, y_i + \eta]$ for all $i \in I$, I uniquely determined by $\mathbf{y} \in A_{\delta, I}$ for any $\mathbf{y} \in A_{\delta}$, we obtain for n large enough

$$\begin{aligned} &\sup_{\mathbf{y} \in A_{\delta}} \sum_{l=1}^n \mathbb{E} \left[\sup_{\|\mathbf{v}^{(1)} - \mathbf{v}^{(2)}\| < \eta, (\mathbf{s}^{(1)}, \mathbf{s}^{(2)}) \in [\mathbf{y}, \mathbf{y} + \eta]^2, (\beta_1, \beta_2) \in [\beta, \beta + \eta]} (Z_{nl}(f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, \beta_1, p}) - Z_{nl}(f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, \beta_2, p}))^2 \right] \\ &\leq 2d \left((1+\delta) C^2 \eta^2 + \eta + (1+\delta) f_n \right). \end{aligned}$$

Reminding that the relation $\eta < \delta$ holds, we obtain

$$\sup_{\rho(f, g) < \delta_n} \sum_{l=1}^n \mathbb{E} \left[(Z_{nl}(f) - Z_{nl}(g))^2 \right] \leq 2d \left((1+\delta_n) C^2 \eta^2 + \eta + (1+\delta_n) f_n \right) \rightarrow 0$$

as $\delta_n \rightarrow 0$.

Now, we are checking the entropy condition of Theorem 2.11.9 in Van der Vaart and Wellner [1996] by constructing and counting ε -brackets for $\varepsilon > 0$. To this end, for each $i \in I$, we consider the conditional distribution of $X_i^* \mid X_i^* > (1 + \delta)^{-1}$. Putting all the atoms $\{m_j\}$ of mass larger than $\varepsilon^2/(4(1 + \delta)d)$ separately into singletons (there are at most $1 \leq j \leq \lceil 4(1 + \delta)d/\varepsilon^2 \rceil$ of them), the set $[(1 + \delta)^{-1}, 1 + \delta] \setminus \cup_j \{m_j\}$ can be divided in at most $\lceil 4(1 + \delta)d/\varepsilon^2 \rceil$ sets $J_j^{(i,n)}$, $\lceil 4(1 + \delta)d/\varepsilon^2 \rceil + 1 \leq j \leq 2\lceil 4(1 + \delta)d/\varepsilon^2 \rceil$ such that

$$\mathbb{P}\left(X_i^* \in J_j^{(i,n)} \mid X_i^* > (1 + \delta)^{-1}u_n\right) \leq \frac{\varepsilon^2}{4(1 + \delta)d},$$

for $\lceil 4(1 + \delta)d/\varepsilon^2 \rceil + 1 \leq j \leq 2\lceil 4(1 + \delta)d/\varepsilon^2 \rceil$. Thus, denoting $J_j^{(i,n)} = \{m_j\}$ for any $1 \leq j \leq \lceil 4(1 + \delta)d/\varepsilon^2 \rceil$, the set $A_\delta \subset [0, 1 + \delta]^d$ can be divided into $2^d \lceil 4(1 + \delta)d/\varepsilon^2 \rceil^d$ sets of the form

$$(48) \quad J = J_{j_1, \dots, j_d} = J_{j_1}^{(1,n)} \times J_{j_2}^{(2,n)} \times \dots \times J_{j_d}^{(d,n)}, \quad 1 \leq j_1, \dots, j_d \leq 2\lceil 4(1 + \delta)d/\varepsilon^2 \rceil.$$

Choosing n sufficiently large such that $a^*(u_n) \leq 2(1 + \delta)\mathbb{P}(X_i^* > u_n(1 + \delta)^{-1})^{-1}$ for all $i = 1, \dots, d$, we obtain that

$$\begin{aligned} & \mathbb{E} \left[\sup_{\mathbf{s}^{(1)}, \mathbf{s}^{(2)} \in J} a^*(u_n) \mathbf{1} \left\{ \bigcup_{i \in I, s_i^{(1)} \neq s_i^{(2)}} \{u_n/(s_i^{(1)} \vee s_i^{(2)}) \leq X_i^* < u_n/(s_i^{(1)} \wedge s_i^{(2)})\} \right\} \right] \\ & \leq 2(1 + \delta) \sum_{i \in I, j_i > \lceil 4(1 + \delta)d/\varepsilon^2 \rceil} \mathbb{P}\left(X_i^* \in J_{j_i}^{(i,n)} \mid X_i^* > (1 + \delta)^{-1}u_n\right) \\ & \leq \frac{\varepsilon^2}{2} \end{aligned}$$

Setting $\eta_0 = \varepsilon/(2C\sqrt{d(1 + \delta)})$, Equations (45) and (47) imply that

$$\begin{aligned} & \mathbb{E} \left[\sup_{\|\mathbf{v}^{(1)} - \mathbf{v}^{(2)}\| < \eta_0, \|\mathbf{s}^{(1)} - \mathbf{s}^{(2)}\| < \eta_0, \mathbf{s}^{(1)}, \mathbf{s}^{(2)} \in J, |\beta_1 - \beta_2| < \eta_0} (Z_{nl}(f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, \beta_1, p}) - Z_{nl}(f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, \beta_2, p}))^2 \right] \\ & \leq \sum_{l=1}^n \mathbb{E} \left[Q_{nl}^{(1)}(\eta_0) \right] \\ & + \mathbb{E} \left[\sup_{\mathbf{s}^{(1)}, \mathbf{s}^{(2)} \in J} a^*(u_n) \mathbf{1} \left\{ \bigcup_{i \in I, s_i^{(1)} \neq s_i^{(2)}} \{u_n/(s_i^{(1)} \vee s_i^{(2)}) \leq X_i^* < u_n/(s_i^{(1)} \wedge s_i^{(2)})\} \right\} \right] \\ & \leq \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2. \end{aligned}$$

Now, we note that $\partial B_1^+(\mathbf{0}) \subset [0, 1]^d$ can be covered by $\lceil 1/\eta_0 \rceil^d$ hyperrectangles B_1, B_2, \dots of side length η_0 , while $A'_\delta \subset [0, 1 + \delta]^{d+1}$ can be covered by at most $\lceil (1 + \delta)/\eta_0 \rceil^{d+1}$ hyperrectangles C_1, C_2, \dots of the same side length: Combining each set B_k with each intersection of a set C_j with each set J_{j_1, \dots, j_d} from (48) extended to $J_{j_1, \dots, j_d} \times [(1 + \delta)^{-1}, 1 + \delta]$, we obtain at most

$$\left\lceil \frac{1}{\eta_0} \right\rceil^{d+1} \cdot \left\lceil \frac{1 + \delta}{\eta_0} \right\rceil^d \cdot \left\lceil \frac{4(1 + \delta)}{\varepsilon^2} \right\rceil^d = \left\lceil \frac{2C\sqrt{d(1 + \delta)}}{\varepsilon} \right\rceil^d \cdot \left\lceil \frac{2C\sqrt{d(1 + \delta)^3}}{\varepsilon} \right\rceil^{d+1} \cdot \left\lceil \frac{4(1 + \delta)}{\varepsilon^2} \right\rceil^d$$

sets of the type $B_k \times (C_j \cap J_{j_1, \dots, j_d})$ satisfying

$$\mathbb{E} \left[\sup_{\mathbf{v}^{(1)}, \mathbf{v}^{(2)} \in B_k, \mathbf{s}^{(1)}, \mathbf{s}^{(2)} \in (C_j \cap J_{j_1, \dots, j_d})} (Z_{nl}(f_{\mathbf{v}^{(1)}, \mathbf{s}^{(1)}, p}) - Z_{nl}(f_{\mathbf{v}^{(2)}, \mathbf{s}^{(2)}, p}))^2 \right] \leq \varepsilon^2.$$

As this number are the same for each $p \in K_0$, the function space \mathcal{F} can be partitioned into at most

$$N_{[]}(\varepsilon, \mathcal{F}, L_2) = |K_0| \left[\frac{2C\sqrt{d(1+\delta)}}{\varepsilon} \right]^d \cdot \left[\frac{2C\sqrt{d(1+\delta)^3}}{\varepsilon} \right]^{d+1} \cdot \left[\frac{4(1+\delta)}{\varepsilon^2} \right]^d \in \mathcal{O}(\varepsilon^{-(4d+1)})$$

ε -brackets. Consequently,

$$\int_0^{\varepsilon_n} \sqrt{\log N_{[]}(\varepsilon, \mathcal{F}, L_2)} d\varepsilon \rightarrow 0$$

as $\varepsilon_n \rightarrow 0$. Now, weak convergence follows from Theorem 2.11.9 in Van der Vaart and Wellner [1996].

APPENDIX B: PROOF OF COROLLARY 6

From Theorem 5 and Equation (28), we deduce that

$$(49) \quad \left\{ \sqrt{\frac{n}{a^*(u_n)}} \left(a^*(u_n) \widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, p) - \tau \mathbb{E} \left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \Theta}{\|\mathbf{s} \circ \Theta\|} \right)^{1/\beta} \right)^p \|\mathbf{s} \circ \Theta\| \right] \right) \right\} \rightarrow G$$

weakly in $\ell^\infty(\partial B_1^+(\mathbf{0}) \times A'_\delta \times (K \cup \{0\}))$ as $n \rightarrow \infty$.

For the case $p = 0$ where $\widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, 0) = \widehat{P}_{n,u_n}(\mathbf{s})$, we can use the monotonicity of \widehat{P}_{n,u_n} to conclude that, for all $\mathbf{s} \in A_{\delta,I}$,

$$a^*(u_n) \widehat{P}_{n,u_n,\ell}(\mathbf{s}) \geq a^*(u_n) \widehat{P}_{n,u_n,\ell}((1+\delta)^{-1} \mathbf{1}_I) \rightarrow_p \tau \cdot \mathbb{E}(\|(1+\delta)^{-1} \cdot \Theta\|) = \tau_I (1+\delta)^{-1},$$

due to Equation (14). Thus, for any $\varepsilon \in (0, \tau_I (1+\delta)^{-1})$, we have

$$(50) \quad \begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}(a^*(u_n) \widehat{P}_{n,u_n}(\mathbf{s}) > \varepsilon \text{ for all } \mathbf{s} \in A_{\delta,I}) \\ \geq \lim_{n \rightarrow \infty} \mathbb{P}(a^*(u_n) \widehat{P}_{n,u_n}((1+\delta)^{-1} \mathbf{1}_I) > \varepsilon) = 1. \end{aligned}$$

This can be strengthened to

$$\lim_{n \rightarrow \infty} \mathbb{P}(a^*(u_n) \widehat{P}_{n,u_n}(\mathbf{s}) > \varepsilon \text{ for all } \mathbf{s} \in A_\delta) = 1$$

as they are only finitely many $\emptyset \neq I \subset \{1, \dots, d\}$. This implies that, for $p = 0$ in Equation (49), the term $a^*(u_n) \widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, 0) = a^*(u_n) \widehat{P}_{n,u_n}(\mathbf{s})$ may be replaced by the pointwise maximum $\varepsilon \vee \{a^*(u_n) \widehat{P}_{n,u_n}(\mathbf{s})\}$. As the map

$$\phi : \ell^\infty(\partial B_1^+(\mathbf{0}) \times A_\delta) \times \left(\ell^\infty(\partial B_1^+(\mathbf{0}) \times A_\delta) \cap [\varepsilon, \infty)^{\partial B_1^+(\mathbf{0}) \times A_\delta} \right) \rightarrow \ell^\infty(\partial B_1^+(\mathbf{0}) \times A_\delta),$$

$$\phi(f, g)(\mathbf{v}, \mathbf{s}, \beta) = f(\mathbf{v}, \mathbf{s}, \beta) / g(\mathbf{v}, \mathbf{s}, \beta)$$

is Hadamard-differentiable [cf. Lemma 3.9.25 in Van der Vaart and Wellner, 1996], weak convergence of the process

$$\left(\left\{ \sqrt{\frac{n}{a(u_n)}} \left(\frac{a^*(u_n) \widehat{M}_{n,u_n}(\mathbf{v}, \mathbf{s}, \beta, p)}{\varepsilon \vee \{a^*(u_n) \widehat{P}_{n,u_n}(\mathbf{s})\}} - c(\mathbf{v}, \mathbf{s}, \beta, p) \right); \mathbf{v} \in \partial B_1^+(\mathbf{0}), (\mathbf{s}, \beta) \in A'_\delta, p \in K \right\} \right)_{n \in \mathbb{N}}$$

to \tilde{G} is obtained by the functional delta method [cf. Theorem 3.9.4 in Van der Vaart and Wellner, 1996, for instance] with covariance of the form

(51)

$$\text{Cov}(\tilde{G}(\mathbf{v}, \mathbf{s}, \beta_1, p_1), \tilde{G}(\mathbf{w}, \mathbf{t}, \beta_2, p_2))$$

$$\begin{aligned}
&= \frac{\mathbb{E}\left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \Theta}{\|\mathbf{s} \circ \Theta\|}\right)^{1/\beta_1}\right)^{p_1} \left(\mathbf{w}^\top \left(\frac{\mathbf{t} \circ \Theta}{\|\mathbf{t} \circ \Theta\|}\right)^{1/\beta_2}\right)^{p_2} (\|\mathbf{s} \circ \Theta\| \wedge \|\mathbf{t} \circ \Theta\|)\right]}{\tau \cdot \mathbb{E}[\|\mathbf{s} \circ \Theta\|] \cdot \mathbb{E}[\|\mathbf{t} \circ \Theta\|]} \\
&\quad - \frac{\mathbb{E}\left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \Theta}{\|\mathbf{s} \circ \Theta\|}\right)^{1/\beta_1}\right)^{p_1} \|\mathbf{s} \circ \Theta\|\right] \cdot \mathbb{E}\left[\left(\mathbf{w}^\top \left(\frac{\mathbf{t} \circ \Theta}{\|\mathbf{t} \circ \Theta\|}\right)^{1/\beta_2}\right)^{p_2} (\|\mathbf{s} \circ \Theta\| \wedge \|\mathbf{t} \circ \Theta\|)\right]}{\tau \cdot (\mathbb{E}[\|\mathbf{s} \circ \Theta\|])^2 \cdot \mathbb{E}[\|\mathbf{t} \circ \Theta\|]} \\
&\quad - \frac{\mathbb{E}\left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \Theta}{\|\mathbf{s} \circ \Theta\|}\right)^{1/\beta_1}\right)^{p_1} (\|\mathbf{s} \circ \Theta\| \wedge \|\mathbf{t} \circ \Theta\|)\right] \mathbb{E}\left[\left(\mathbf{w}^\top \left(\frac{\mathbf{t} \circ \Theta}{\|\mathbf{t} \circ \Theta\|}\right)^{1/\beta_2}\right)^{p_2} \|\mathbf{t} \circ \Theta\|\right]}{\tau \cdot \mathbb{E}[\|\mathbf{s} \circ \Theta\|] \cdot (\mathbb{E}[\|\mathbf{t} \circ \Theta\|])^2} \\
&\quad + \frac{\mathbb{E}\left[\left(\mathbf{v}^\top \left(\frac{\mathbf{s} \circ \Theta}{\|\mathbf{s} \circ \Theta\|}\right)^{1/\beta_1}\right)^{p_1} \|\mathbf{s} \circ \Theta\|\right] \cdot \mathbb{E}\left[\left(\mathbf{w}^\top \left(\frac{\mathbf{t} \circ \Theta}{\|\mathbf{t} \circ \Theta\|}\right)^{1/\beta_2}\right)^{p_2} \|\mathbf{t} \circ \Theta\|\right] \cdot \mathbb{E}[\|\mathbf{s} \circ \Theta\| \wedge \|\mathbf{t} \circ \Theta\|]}{\tau \cdot (\mathbb{E}[\|\mathbf{s} \circ \Theta\|])^2 \cdot (\mathbb{E}[\|\mathbf{t} \circ \Theta\|])^2}.
\end{aligned}$$

Using again Equation (50), the assertion follows.

APPENDIX C: PROOF OF THEOREM 8

Analogously to Corollary 6, We first show a functional limit theorem for the estimator $\widehat{L}_{n,u}(\mathbf{s})/\widehat{P}_{n,u_n}(\mathbf{s})$ based on normalized observations \mathbf{X}_i^* .

PROPOSITION 20. *Let the assumptions of Theorem 5 hold and assume that there exists some $\delta > 0$ such that Equation (31) is satisfied and Equation (28) holds for $p = 0$. Then, the sequence of processes $(\{\widehat{H}_n(\mathbf{s}) : \mathbf{s} \in A_\delta\})_{n \in \mathbb{N}}$ with*

$$\widetilde{H}_n(\mathbf{s}) = \sqrt{\frac{n}{a^*(u_n)}} \left(\frac{\widehat{L}_{n,u}(\mathbf{s})}{\widehat{P}_{n,u_n}(\mathbf{s})} - \frac{1}{\alpha} \right)$$

converges weakly in $\ell^\infty(A_\delta)$ to a tight centered Gaussian process \widetilde{H} with covariance

$$(52) \quad \text{Cov}(\widetilde{H}(\mathbf{s}), \widetilde{H}(\mathbf{t})) = \frac{1}{\tau \alpha^2} \cdot \frac{\mathbb{E}[\|\mathbf{s} \circ \Theta\| \wedge \|\mathbf{t} \circ \Theta\|]}{\mathbb{E}[\|\mathbf{s} \circ \Theta\|] \cdot \mathbb{E}[\|\mathbf{t} \circ \Theta\|]}, \quad \mathbf{s}, \mathbf{t} \in A_\delta.$$

PROOF. First, we will show that the sequence of processes $(\{H_n(\mathbf{s}) : \mathbf{s} \in A_\delta\})_{n \in \mathbb{N}}$ with

$$H_n(\mathbf{s}) = \sqrt{\frac{n}{a^*(u_n)}} \left(a^*(u_n) \widehat{L}_{n,u_n}(\mathbf{s}) - a^*(u_n) \mathbb{E}[\widehat{L}_{n,u_n}(\mathbf{s})] \right)$$

converges weakly in $\ell^\infty(A_\delta)$ to a centered Gaussian process H . Analogously to the proof of Theorem 5, we define the function space $\mathcal{F}' = \{f_{\mathbf{s}} : \mathbf{s} \in A_\delta\}$ equipped with the semi-metric $\rho'(f_{\mathbf{s}^{(1)}}, f_{\mathbf{s}^{(2)}}) = \|\mathbf{s}^{(1)} - \mathbf{s}^{(2)}\|$ where

$$f_{\mathbf{s}} : (0, \infty)^d \rightarrow [0, \infty), \quad f_{\mathbf{s}}(\mathbf{x}) = \frac{1}{\alpha} \log(\|\mathbf{s} \circ \mathbf{x}\|) \mathbf{1}\{\|\mathbf{s} \circ \mathbf{x}\| > 1\}.$$

For $l = 1, \dots, n, n \in \mathbb{N}$, we set

$$Z_{nl}(f) = \sqrt{\frac{a^*(u_n)}{n}} f(u_n^{-1} \mathbf{X}_l^*)$$

such that

$$\{H_n(\mathbf{s})\}_{\mathbf{s} \in A_\delta} = \left\{ \sum_{l=1}^n (Z_{nl}(f) - \mathbb{E}Z_{nl}(f)) \right\}_{f \in \mathcal{F}'}$$

By construction, each H_n is a centered process and it can be shown that its covariance function converges pointwise.

Now, using the Potter bound, for every $\varepsilon > 0$, there exists some $C_\varepsilon > 0$ such that

$$\frac{\mathbb{P}(\|\mathbf{X}_I^*\| > u_n t)}{\mathbb{P}(\|\mathbf{X}_I^*\| > u_n)} \leq C_\varepsilon t^{-(1-\varepsilon)}, \quad t \geq 1,$$

for sufficiently large n . Thus, for every $\eta > 0$ and $k \in \{1, 2\}$, we obtain

$$\begin{aligned} & \sum_{l=1}^n \mathbb{E}[\|Z_{n,l}\|_{\mathcal{F}'}^k \mathbf{1}\{\|Z_{n,l}\|_{\mathcal{F}'} > \eta\}] \\ & \leq n \left(\frac{a^*(u_n)}{n}\right)^{k/2} \mathbb{E} \left[\log^k((1+\delta)\|\mathbf{X}_I^*\|/u_n) \mathbf{1}\{\log((1+\delta)\|\mathbf{X}_I^*\|/u_n) > \sqrt{n/a^*(u_n)\eta}\} \right] \\ & = n \left(\frac{a^*(u_n)}{n}\right)^{k/2} \int_0^\infty \mathbb{P} \left(\log^k((1+\delta)\|\mathbf{X}_I^*\|/u_n) \mathbf{1}\{\log((1+\delta)\|\mathbf{X}_I^*\|/u_n) > \sqrt{n/a^*(u_n)\eta}\} > x \right) dx \\ & = n \left(\frac{a^*(u_n)}{n}\right)^{k/2} \left[\mathbb{P} \left(\|\mathbf{X}_I^*\| > \frac{u_n}{1+\delta} \exp \left(\sqrt{\frac{n}{a^*(u_n)\eta}} \right) \right) \cdot \left(\sqrt{\frac{n}{a^*(u_n)\eta}} \right)^k \right. \\ & \quad \left. + \int_{(\sqrt{n/a^*(u_n)\eta})^k}^\infty \mathbb{P} \left(\|\mathbf{X}_I^*\| > \frac{u_n}{1+\delta} \exp(x^{1/k}) \right) dx \right] \\ & = n \left(\frac{n}{a^*(u_n)}\right)^{k/2} \eta^k \cdot \left[\mathbb{P} \left(\|\mathbf{X}_I^*\| > \frac{u_n}{1+\delta} \exp \left(\sqrt{\frac{n}{a^*(u_n)\eta}} \right) \right) \right. \\ & \quad \left. + \int_1^\infty \mathbb{P} \left(\|\mathbf{X}_I^*\| > \frac{u_n}{1+\delta} \exp \left(\sqrt{\frac{n}{a^*(u_n)\eta}} u^{1/k} \right) \right) du \right] \\ & \leq C_\varepsilon n \mathbb{P}(\|\mathbf{X}_I^*\| > u_n) \left(\frac{n}{a^*(u_n)}\right)^{k/2} \frac{\eta^k}{(1+\delta)^{1-\varepsilon}} \\ & \quad \cdot \left[\exp \left(-(1-\varepsilon)\eta \sqrt{\frac{n}{a^*(u_n)}} \right) + \int_1^\infty \exp \left(-(1-\varepsilon)\eta \sqrt{\frac{n}{a^*(u_n)}} u^{1/k} \right) du \right] \\ & \sim C_\varepsilon \tau_I \left(\frac{n}{a^*(u_n)}\right)^{1+k/2} \frac{\eta^k}{(1+\delta)^{1-\varepsilon}} \cdot \left[\exp \left(-(1-\varepsilon)\eta \sqrt{\frac{n}{a^*(u_n)}} \right) \right. \\ & \quad \left. + \int_1^\infty \exp \left(-(1-\varepsilon)\eta \sqrt{\frac{n}{a^*(u_n)}} u^{1/k} \right) du \right] \rightarrow 0 \end{aligned}$$

because of $n/a^*(u_n) \rightarrow 0$ as $n \rightarrow \infty$. For $k = 2$, this results in a Lindeberg condition that guarantees convergence of H_n to a centered Gaussian process H in terms of finite-dimensional distributions.

Analogously to the proof of Theorem 5, we verify the conditions from Theorem 2.11.9 in Van der Vaart and Wellner [1996] in order to ensure functional convergence. Thus, we consider two functions $f_{\mathbf{s}^{(1)}}, f_{\mathbf{s}^{(2)}} \in \mathcal{F}'$ such that $\|\mathbf{s}^{(1)} - \mathbf{s}^{(2)}\| < \eta$ for some $0 < \eta < \delta$ and we assume that $\mathbf{s}^{(1)}, \mathbf{s}^{(2)} \in A_{\delta, I}$ for non-empty same index set I , that is, there exists some $\mathbf{y} \in A_{\delta, I}$ such that $s_i^{(1)}, s_i^{(2)} \in [y_i, y_i + \eta]$ for all $i \in I$. Again, we distinguish three cases:

1. $\|\mathbf{s}^{(1)} \circ \mathbf{x}\| \leq 1$ and $\|\mathbf{s}^{(2)} \circ \mathbf{x}\| \leq 1$:

In this case, we have

$$|f_{\mathbf{s}^{(1)}}(\mathbf{x}) - f_{\mathbf{s}^{(2)}}(\mathbf{x})| = |0 - 0| = 0.$$

2. $\|\mathbf{s}^{(1)} \circ \mathbf{x}\| \geq 1$ and $\|\mathbf{s}^{(2)} \circ \mathbf{x}\| \geq 1$:

In this case, both the indicator functions in $f_{\mathbf{s}^{(1)}}(\mathbf{x})$ and in $f_{\mathbf{s}^{(2)}}(\mathbf{x})$ are equal to 1. Choosing $0 < \beta < 1/2$ and using that $|\log(x) - \log(y)| < \beta^{-1}|x - y|^\beta$, we obtain that

$$|f_{\mathbf{s}^{(1)}}(\mathbf{x}) - f_{\mathbf{s}^{(2)}}(\mathbf{x})| \leq \frac{1}{\beta} \eta^\beta \|\mathbf{x}_I\|^\beta.$$

This case occurs only if $(1 + \delta)\|\mathbf{x}\| > 1$ as $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ belong to $A_{\delta, I}$.

3. $\min\{\|\mathbf{s}^{(1)} \circ \mathbf{x}\|, \|\mathbf{s}^{(2)} \circ \mathbf{x}\|\} \leq 1 < \max\{\|\mathbf{s}^{(1)} \circ \mathbf{x}\|, \|\mathbf{s}^{(2)} \circ \mathbf{x}\|\}$:

In this case one of the functions $f_{\mathbf{s}^{(1)}}(\mathbf{x})$ and $f_{\mathbf{s}^{(2)}}(\mathbf{x})$ equals zero, while the other one is bounded by 1. Here, it is important to use that $\|\mathbf{s} \circ \mathbf{X}\| \leq u_n$ for some $\mathbf{s} \in A_\delta$ implies that $\|u_n^{-1} \mathbf{X}_I\| \leq 1 + \delta$. Thus, we will have $f_{\mathbf{s}^{(i)}}(\mathbf{x}) \leq \alpha^{-1} \log((1 + \delta)^2)$ in this case, and, thus,

$$|f_{\mathbf{s}^{(1)}}(\mathbf{x}) - f_{\mathbf{s}^{(2)}}(\mathbf{x})| \leq \frac{2}{\alpha} \log(1 + \delta) \mathbf{1} \left\{ \bigcup_{i \in I, s_i^{(1)} \neq s_i^{(2)}} \{u_n / (s_i^{(1)} \vee s_i^{(2)}) \leq x_i < u_n / (s_i^{(1)} \wedge s_i^{(2)})\} \right\}.$$

This case occurs only if $y_i x_i \leq 1 < (y_i + \eta)x_i$ for at least one $i \in I$.

Thus, we obtain a uniform bound on $\|\mathbf{v}^{(1)} - \mathbf{v}^{(2)}\| < \eta$ and $s_i^{(1)}, s_i^{(2)} \in [y_i, y_i + \eta]$, $i \in I$, for some $\mathbf{y} \in A_{\delta, I}$

$$\begin{aligned} Z_{nl}(f_{\mathbf{s}^{(1)}}) - Z_{nl}(f_{\mathbf{s}^{(2)}}) &\leq \frac{a^*(u_n)}{n} \frac{1}{\beta^2} \|u_n^{-1} \mathbf{X}_I^*\|^{2\beta} \eta^{2\beta} \mathbf{1}\{\|\mathbf{X}_I^*\| > (1 + \delta)^{-1} u_n\} \\ &\quad + \frac{a^*(u_n)}{n} \frac{4}{\alpha} \log^2(1 + \delta) \cdot \mathbf{1} \left\{ \bigcup_{i \in I} \{u_n (y_i \eta)^{-1} \leq X_{li}^* \leq u_n y_i^{-1}\} \right\} \\ &=: \tilde{Q}_{nl}^{(1)}(\eta) + \frac{4}{\alpha} \log^2(1 + \delta) \cdot Q_{nl}^{(2)}(\eta, \mathbf{y}) \end{aligned}$$

where $Q_{nl}^{(2)}(\eta, \mathbf{y})$ is as in (46).

The definition of $\tilde{Q}_{nl}^{(1)}(\eta)$ and regular variation of the function a^* imply that

$$\begin{aligned} \sum_{l=1}^n \mathbb{E} \left[\tilde{Q}_{nl}^{(1)}(\eta) \right] &\leq a^*(u_n) \frac{1}{\beta^2} \eta^{2\beta} \mathbb{E} \left(\|u_n^{-1} \mathbf{X}_I^*\|^{2\beta} \mathbf{1}\{\|\mathbf{X}_I^*\| > (1 + \delta)^{-1} u_n\} \right) \\ &\sim \tau \frac{a^*(u_n)}{a^*((1 + \delta)^{-1} u_n)} \frac{1}{\beta^2} \eta^{2\beta} \mathbb{E} \left(\|u_n^{-1} \mathbf{X}_I^*\|^{2\beta} \mid \|\mathbf{X}_I^*\| > (1 + \delta)^{-1} u_n \right) \\ &\sim \frac{\tau_I (1 + \delta)^{1-2\beta}}{\beta^2} \mathbb{E}(\|Y \Theta_I^*\|^{2\beta}) \cdot \eta^{2\beta} \leq \frac{\tau_I (1 + \delta)^{1-2\beta}}{(1 - 2\beta)\beta^2} \cdot \eta^{2\beta}, \quad n \rightarrow \infty, \end{aligned}$$

since $\mathbb{E}(Y^{2\beta}) = 1/(1 - 2\beta) < \infty$ for all $\beta < 1/2$. Thus, analogously to (47), for n sufficiently large, we have that

$$\sum_{l=1}^n \mathbb{E} \left[\tilde{Q}_{nl}^{(1)}(\eta) \right] \leq \frac{2d(1 + \delta)^{1-2\beta}}{(1 - 2\beta)\beta^2} \cdot \eta^{2\beta}$$

Combining this bound with the bound with the bounds for $\sum_{l=1}^n \mathbb{E} \left[Q_{nl}^{(2)}(\eta) \right]$ provided in the proof of Theorem 5, the equi-continuity condition and the entropy condition of Theorem 2.11.9 in Van der Vaart and Wellner [1996] can be verified, following the lines of the proof of Theorem 5. Thus, we obtain weak convergence of $\{H_n(\mathbf{s}) : \mathbf{s} \in A_\delta\}$ to H in $\ell^\infty(A_\delta)$. Making use of (31), the bias can be neglected asymptotically, i.e.

$$\left\{ \sqrt{\frac{n}{a^*(u_n)}} \left(a^*(u_n) \hat{L}_{n, u_n}(\mathbf{s}) - \tau \mathbb{E}[\|\mathbf{s} \circ \Theta\|] \frac{1}{\alpha} \right); \mathbf{s} \in A_\delta \right\} \rightarrow \{H(\mathbf{s}); \mathbf{s} \in A_\delta\}$$

weakly in $\ell^\infty(A_\delta)$. By Theorem 5 and (28) for $p = 0$, we also obtain

$$\left\{ \sqrt{\frac{n}{a^*(u_n)}} \left(a^*(u_n) \widehat{P}_{n,u_n}(\mathbf{s}) - \tau \mathbb{E}[\|\mathbf{s} \circ \Theta\|] \right); \mathbf{s} \in A_\delta \right\} \rightarrow \{G(\mathbf{v}, \mathbf{s}, \beta, 0); \mathbf{s} \in A_\delta\}$$

weakly $\ell^\infty(A_\delta)$ for any fixed $\mathbf{v} \in \partial B_1^+(\mathbf{0})$ and $\beta \in [(1 + \delta)^{-1}, 1 + \delta]$ as the RHS does not depend on these variables. Note that both convergences hold jointly. Thus, analogously to the proof of Corollary 6, we can apply the functional Delta method to obtain the desired convergence

$$\left\{ \sqrt{\frac{n}{a(u_n)}} \left(\frac{\widehat{L}_{n,u}(\mathbf{s})}{\widehat{P}_{n,u_n}(\mathbf{s})} - \frac{1}{\alpha} \right); \mathbf{s} \in A_\delta \right\} \rightarrow_w \{\widetilde{H}(\mathbf{s}); \mathbf{s} \in A_\delta\}$$

for some centered Gaussian process \widetilde{H} . The covariance structure of \widetilde{H} follows from classical computation. \square

Using this result, we are ready to prove Theorem 8.

PROOF OF THEOREM 8. First, we note that

$$\begin{aligned} \frac{1}{\widehat{\alpha}_{n,k_n,I}} &= \frac{n^{-1} \sum_{l=1}^n \log(\|\mathbf{1}_I \circ \mathbf{X}_l / \mathbf{X}_{k_n:n}\|) \mathbf{1}_{\{\|\mathbf{1}_I \circ \mathbf{X}_l / \mathbf{X}_{k_n:n}\| > 1\}}}{\widetilde{P}_{n,k_n,I}} \\ &= \frac{n^{-1} \sum_{l=1}^n \log(\|\mathbf{1}_I \circ \mathbf{X}_l^* / \mathbf{X}_{k_n:n}^*\|^{1/\alpha}) \mathbf{1}_{\{\|\mathbf{1}_I \circ \mathbf{X}_l^* / \mathbf{X}_{k_n:n}^*\| > 1\}}}{\widetilde{P}_{n,k_n,I}} \\ &= \frac{\widehat{L}_{n,u_n}(u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I)}{\widehat{P}_{n,u_n}(u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I)}, \end{aligned}$$

cf. Equation (29). Thus, we have to show that $\widetilde{H}_n(u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I) \rightarrow \widetilde{H}(\mathbf{1}_I) =: \widetilde{H}_I$. To this end, we use the consistency of the tail empirical measure to obtain

$$(53) \quad \frac{u_n}{\mathbf{X}_{k_n:n}^*} \sim \left(\frac{(F_i^*)^{-1}(1 - k_n/n)}{X_{k_n:n,i}^*} \right)_{i=1,\dots,d} \rightarrow_p \mathbf{1}, \quad n \rightarrow \infty,$$

cf. Equation (4.17) in Resnick [2007]. By Skohorod's representation theorem, without loss of generality, we may reformulate the statement in Theorem 20 as

$$(54) \quad \sup_{\mathbf{s} \in A_\delta} \left| \widetilde{H}_n(\mathbf{s}) - \widetilde{H}(\mathbf{s}) \right| \rightarrow 0 \quad \text{a.s.}, \quad n \rightarrow \infty.$$

Now, we estimate

$$(55) \quad \left| \widetilde{H}_n(u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I) - \widetilde{H}_I \right| \leq \left| \widetilde{H}_n(u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I) - \widetilde{H}(u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I) \right| + \left| \widetilde{H}(u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I) - \widetilde{H}_I \right|.$$

Equation (53) guarantees that, with probability tending to 1, $\{u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I \in A_\delta\}$, and, in that case, we can bound the right-hand side of Equation (55) by

$$\sup_{\mathbf{s} \in A_\delta} \left| \widetilde{H}_n(\mathbf{s}) - \widetilde{H}(u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{s}) \right| + \left| \widetilde{H}(u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I) - \widetilde{H}_I \right|.$$

Here, the first term vanishes asymptotically by Equation (54) and the second term tends to zero because of Equation (53) and the uniform continuity of the sample paths of the tight process \tilde{H} (cf. Theorem 20). Thus, we obtain the desired convergence

$$\left| \tilde{H}_n(u_n/\mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I) - \tilde{H}_I \right| \rightarrow_p 0.$$

□

APPENDIX D: PROOF OF THEOREM 10

Before showing Theorem 10, we prove to auxiliary results. We start by establishing a first (but biased) version of the functional central limit theorem for the sequence $\{\tilde{M}_{n,k_n,I}(\mathbf{v}, p)/\tilde{P}_{n,k_n,I}; \mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K\}$ of processes.

PROPOSITION 21. *Let \mathbf{X}_l , $l \in \mathbb{N}$, be independent copies of a regularly varying $[0, \infty)^d$ -valued random vector \mathbf{X} satisfying Equation (2). Denote the corresponding transformed vector by $\mathbf{X}^* = (r_1^{-1}X_1^\alpha, \dots, r_d^{-1}X_d^\alpha)$. Let $\{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ be a sequence such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$. If a sequence (u_n) satisfies the assumptions of Corollary 6 and Theorem 8 for some $\delta > 0$ and some $K \subset \mathbb{N}$. Then, for all non-empty $I \subset \{1, \dots, d\}$, we have that*

$$\left\{ \sqrt{k_n} \left(\frac{\tilde{M}_{n,k_n,I}(\mathbf{v}, p)}{\tilde{P}_{n,k_n,I}} - c \left(\mathbf{v}, \frac{u_n}{\mathbf{X}_{k_n:n}^*} \circ \mathbf{1}_I, \frac{\alpha}{\hat{\alpha}_{n,k_n,I}}, p \right) \right); \mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K \right\} \\ \rightarrow \left\{ \tilde{G}(\mathbf{v}, \mathbf{1}_I, 1, p); \mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K \right\}, \quad n \rightarrow \infty,$$

weakly in $\ell^\infty(\partial B_1^+(\mathbf{0}) \times K)$ where \tilde{G} is defined as in Corollary 6.

PROOF. By Equation (29), we have that

$$\sqrt{k_n} \left(\frac{\tilde{M}_{n,k_n,I}(\mathbf{v}, p)}{\tilde{P}_{n,k_n,I}} - c \left(\mathbf{v}, \frac{u_n}{\mathbf{X}_{k_n:n}^*} \circ \mathbf{1}_I, \frac{\alpha}{\hat{\alpha}_{n,k_n,I}}, p \right) \right) = \tilde{G}_n \left(\mathbf{v}, \frac{u_n}{\mathbf{X}_{k_n:n}^*} \circ \mathbf{1}_I, \frac{\alpha}{\hat{\alpha}_{n,k_n,I}}, p \right).$$

Furthermore, Equation (53) and Theorem 8 provide

$$(56) \quad \left(\frac{u_n}{\mathbf{X}_{k_n:n}^*} \circ \mathbf{1}_I, \frac{\alpha}{\hat{\alpha}_{n,k_n,I}} \right) \rightarrow_p (\mathbf{1}_I, 1), \quad n \rightarrow \infty.$$

Again, Skohorod's representation theorem allows us to rewrite the result of Corollary 6 as

$$(57) \quad \sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), (\mathbf{s}, \beta) \in A'_\delta, p \in K} \left| \tilde{G}_n(\mathbf{v}, \mathbf{s}, \beta, p) - \tilde{G}(\mathbf{v}, \mathbf{s}, \beta, p) \right| \rightarrow 0 \quad \text{a.s.}, \quad n \rightarrow \infty.$$

Using Equation (56), with probability tending to 1, we can bound the right-hand side of

$$\sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K} \left| \tilde{G}_n(\mathbf{v}, u_n/\mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I, \alpha/\hat{\alpha}_{n,k_n,I}, p) - \tilde{G}(\mathbf{v}, \mathbf{1}_I, 1, p) \right| \\ \leq \sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K} \left| \tilde{G}_n(\mathbf{v}, u_n/\mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I, \alpha/\hat{\alpha}_{n,k_n,I}, p) - \tilde{G}(\mathbf{v}, u_n/\mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I, \alpha/\hat{\alpha}_{n,k_n,I}, p) \right| \\ + \sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K} \left| \tilde{G}(\mathbf{v}, u_n/\mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I, \alpha/\hat{\alpha}_{n,k_n,I}, p) - \tilde{G}(\mathbf{v}, \mathbf{1}_I, 1, p) \right|.$$

by

$$\begin{aligned} & \sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), (\mathbf{s}, \beta) \in A'_\delta, p \in K} \left| \tilde{G}_n(\mathbf{v}, \mathbf{s}, \beta, p) - \tilde{G}(\mathbf{v}, \mathbf{s}, \beta, p) \right| \\ & + \sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K} \left| \tilde{G}(\mathbf{v}, u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I, \alpha / \hat{\alpha}_{n, k_n, I}, p) - \tilde{G}(\mathbf{v}, \mathbf{1}_I, 1, p) \right|. \end{aligned}$$

While the first term tends to zero by Equation (57), the second term vanishes asymptotically because of Equation (56) and the uniform continuity of $\tilde{G}(\cdot, \cdot, p)$, cf. the third part of Remark 7. Consequently,

$$\sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K} \left| \tilde{G}_n(\mathbf{v}, u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I, \alpha / \hat{\alpha}_{n, k_n, I}, p) - \tilde{G}(\mathbf{v}, \mathbf{1}_I, 1, p) \right| \rightarrow_p 0, \quad n \rightarrow \infty.$$

□

Note that the centering term in Proposition 21, $c(\mathbf{v}_I, u_n / \mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I, \alpha / \hat{\alpha}_{n, k_n, I}, p)$ is random and needs to be replaced by the desired limit $c(\mathbf{v}_I, \mathbf{1}_I, 1, p)$. It turns out that, in general, the difference is not negligible, i.e. the replacement of the centering term will influence the asymptotic distribution. The proof will rely on the following lemma.

LEMMA 22. *Let the assumptions of Proposition 21 hold and assume that there exists some $\delta > 0$ such that Equation (35) holds for all $i \in \{1, \dots, d\}$. Then we have*

$$\left\{ \sqrt{k_n} \left(\frac{u_n}{X_{[sk_n]:n,i}^*} - s \right); (1 + \delta)^{-1} \leq s \leq 1 + \delta \right\} \rightarrow_w \{-G^0(s \mathbf{1}_{\{i\}})\}$$

in $\ell^\infty([(1 + \delta)^{-1}, 1 + \delta])$.

PROOF. Applying Theorem 5 to the case $p = 0$ and $\mathbf{s} = s \mathbf{1}_{\{i\}}$ and making use of Equation (35), we obtain that

$$\left\{ \sqrt{k_n} \left(\frac{1}{k_n} \sum_{l=1}^n \mathbf{1}\{s X_{li}^* > u_n\} - s \right); (1 + \delta)^{-1} \leq s \leq 1 + \delta \right\} \rightarrow_w \{G^0(s \mathbf{1}_{\{i\}})\}$$

in $\ell^\infty([(1 + \delta)^{-1}, 1 + \delta])$ as $n \rightarrow \infty$. Using the notation $\hat{F}_{n,i}$ for the empirical cumulative distribution function of the sample $X_{1i}^*, \dots, X_{ni}^*$ and Skohorod's representation theorem, we may assume that

$$\sup_{(1+\delta)^{-1} \leq s \leq 1+\delta} \left| \sqrt{k_n} \left(\frac{n}{k_n} (1 - \hat{F}_{n,i}(u_n/s)) - s \right) - G^0(s \mathbf{1}_{\{i\}}) \right| \rightarrow 0 \quad \text{a.s.}, \quad n \rightarrow \infty.$$

Applying Vervaat's Lemma [cf. Lemma A.0.2 in De Haan and Ferreira, 2006, for instance] to the nondecreasing functions $x_n(s) = n/k_n(1 - \hat{F}_{n,i}(u_n/s))$, we obtain

$$\sup_{(1+\delta)^{-1} \leq s \leq 1+\delta} \left| \sqrt{k_n} \left(u_n / X_{[sk_n]:n,i}^* - s \right) + G^0(s \mathbf{1}_{\{i\}}) \right| \rightarrow 0 \quad \text{a.s.},$$

where we used that $x_n^{\leftarrow}(s) = u_n / X_{[sk_n]:n,i}^*$. □

Imposing some additional conditions on the function c and using our previous results, we can follow the same lines as the proof of Theorem 4.6 in Einmahl et al. [2012] in order to establish the desired convergence result.

PROOF OF THEOREM 10. By Skohorod's representation theorem, we may assume that the weak convergences in Theorem 8, Proposition 21 and in Lemma 22 take place a.s. Then, the triangular inequality yields the bound

$$\begin{aligned}
& \sup_{\mathbf{v} \in B_1^+(\mathbf{0}), p \in K} \left| \sqrt{k} \left(\frac{\widetilde{M}_{n,k_n,I}(\mathbf{v}_I, p)}{\widetilde{P}_{n,k_n,I}} - c(\mathbf{v}_I, \mathbf{1}_I, p) \right) - \widetilde{G}(\mathbf{v}_I, \mathbf{1}_I, 1, p) \right. \\
& \quad \left. + \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, p) \cdot G^0(\mathbf{1}_{\{i\}}) - c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, p) \cdot \alpha \cdot \widetilde{H}_I \right| \\
& \leq \sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K} \left| \widetilde{G}_n \left(\mathbf{v}_I, \frac{u_n}{\mathbf{X}_{k:n}^*} \circ \mathbf{1}_I, \frac{\alpha}{\widehat{\alpha}_{n,k_n,I}}, p \right) - \widetilde{G}(\mathbf{v}_I, \mathbf{1}_I, 1, p) \right| \\
& \quad + \sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K} \left| \sqrt{k} \left(c \left(\mathbf{v}_I, \frac{u_n}{\mathbf{X}_{k:n}^*} \circ \mathbf{1}_I, \frac{\alpha}{\widehat{\alpha}_{n,k_n,I}}, p \right) - c(\mathbf{v}_I, \mathbf{1}_I, 1, p) \right) \right. \\
& \quad \left. + \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, p) \cdot G^0(\mathbf{1}_{\{i\}}) - c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, p) \cdot \alpha \cdot \widetilde{H}_I \right| \\
& =: D_1 + D_2.
\end{aligned}$$

We will show that $D_1 \rightarrow_p 0$ and $D_2 \rightarrow_p 0$. By Proposition 21, we directly obtain that $D_1 \rightarrow 0$ a.s. In order to show that $D_2 \rightarrow_p 0$, we first define

$$C_n(\mathbf{v}_I, p) := \sqrt{k} \left(c \left(\mathbf{v}_I, \frac{u_n}{\mathbf{X}_{k:n}^*} \circ \mathbf{1}_I, \frac{\alpha}{\widehat{\alpha}_{n,k_n,I}}, p \right) - c(\mathbf{v}_I, \mathbf{1}_I, 1, p) \right), \quad \mathbf{v}_I \in \partial B_1^+(\mathbf{0}), p \in K.$$

The mean value theorem yields

$$(58) \quad \frac{1}{\sqrt{k}} C_n(\mathbf{v}_I, p) = \sum_{i \in I} \left(\frac{u_n}{\mathbf{X}_{k:n,i}^*} - 1 \right) \cdot c_i(\mathbf{v}_I, \boldsymbol{\xi}_n, \psi_n, p) + \left(\frac{\alpha}{\widehat{\alpha}_{n,k_n}} - 1 \right) \cdot c_\beta(\mathbf{v}_I, \boldsymbol{\xi}_n, \psi_n, p)$$

for some $\boldsymbol{\xi}_n = \boldsymbol{\xi}_n(\mathbf{v}_I, p)$ between $u_n/\mathbf{X}_{k:n}^* \circ \mathbf{1}_I$ and $\mathbf{1}_I$ and $\psi_n = \psi_n(\mathbf{v}_I, p)$ between $\alpha/\widehat{\alpha}_{n,k_n}$ and 1. If $(u_n/\mathbf{X}_{k:n}^* \circ \mathbf{1}_I, \alpha/\widehat{\alpha}_{n,k_n}) \in A'_{\delta,I}$, which happens with probability tending to 1 by Equation (56), we obtain from (58) that

$$\begin{aligned}
D_2 &= \sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K} \left| C_n(\mathbf{v}_I, p) + \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, p) G^0(\mathbf{1}_{\{i\}}) - c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, p) \alpha \widetilde{H}_I \right| \\
&\leq \sum_{i \in I} \sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K} \left| c_i(\mathbf{v}_I, \boldsymbol{\xi}_n, \psi_n, p) \sqrt{k} \left(\frac{u_n}{\mathbf{X}_{k:n,i}^*} - 1 \right) + c_i(\mathbf{v}_I, \mathbf{1}_I, 1, p) G^0(\mathbf{1}_{\{i\}}) \right| \\
&\quad + \sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K} \left| c_\beta(\mathbf{v}_I, \boldsymbol{\xi}_n, \psi_n, p) \sqrt{k} \left(\frac{\alpha}{\widehat{\alpha}_{n,k_n,I}} - 1 \right) - c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, p) \alpha \widetilde{H}_I \right| \\
&\leq \sum_{i \in I} \left(\sup_{\mathbf{v} \in B_1^+(\mathbf{0}), (\mathbf{s}, \beta) \in A'_{\delta,I}, p \in K} |c_i(\mathbf{v}_I, \mathbf{s}, \beta, p)| \cdot \left| \sqrt{k} \left(\frac{u_n}{\mathbf{X}_{k:n,i}^*} - 1 \right) + G^0(\mathbf{1}_{\{i\}}) \right| \right. \\
&\quad \left. + \sup_{\mathbf{v} \in B_1^+(\mathbf{0}), p \in K} |c_i(\mathbf{v}_I, \boldsymbol{\xi}_n, p) - c_i(\mathbf{v}_I, \mathbf{1}_I, p)| \cdot |G^0(\mathbf{1}_{\{i\}})| \right) \\
&\quad + \sup_{\mathbf{v} \in B_1^+(\mathbf{0}), (\mathbf{s}, \beta) \in A'_{\delta,I}, p \in K} |c_\beta(\mathbf{v}_I, \mathbf{s}, \beta, p)| \cdot \left| \sqrt{k} \left(\frac{\alpha}{\widehat{\alpha}_{n,k_n,I}} - 1 \right) - \alpha \widetilde{H}_I \right|
\end{aligned}$$

$$\begin{aligned}
 & + \sup_{\mathbf{v} \in B_1^+(\mathbf{0}), p \in K} |c_\beta(\mathbf{v}_I, \boldsymbol{\xi}_n, \psi_n, p) - c_\beta(\mathbf{v}_I, \mathbf{1}_I, \beta, p)| \cdot |\alpha \tilde{H}_I| \\
 & =: \sum_{i \in I} (D_{3,i} \cdot D_{4,i} + D_{5,i} \cdot D_{6,i}) + D_{3,\beta} \cdot D_{4,\beta} + D_{5,\beta} \cdot D_{6,\beta}.
 \end{aligned}$$

Since c_i , $i \in I$, and c_β are continuous on the compact set $\partial B_1^*(\mathbf{0}) \times A'_{\delta,I} \times K$, we have $D_{3,i} < \infty$, $i \in I$, and $D_{3,\beta} < \infty$ while $D_{4,i} \rightarrow_p 0$, $i \in I$ and $D_{4,\beta} \rightarrow_p 0$ by Lemma 22 and Theorem 8, respectively. Furthermore, we have

$$\sup_{\mathbf{v} \in B_1^+(\mathbf{0}), p \in K} \left\| \begin{pmatrix} \boldsymbol{\xi}_n \\ \psi_n \end{pmatrix} - \begin{pmatrix} \mathbf{1}_I \\ 1 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} \frac{u_n}{\mathbf{X}_{k_n:n}^* \circ \mathbf{1}_I} \\ \frac{\alpha}{\tilde{\alpha}_{n,k_n}} \end{pmatrix} - \begin{pmatrix} \mathbf{1}_I \\ 1 \end{pmatrix} \right\| \rightarrow_p 0$$

by (56). Consequently, using the uniform continuity of c_i , $i \in I$, and c_β , we obtain that $D_{5,i} \rightarrow_p 0$ and $D_{5,\beta} \rightarrow_p 0$. The facts that $D_{6,i} < \infty$ a.s., $i \in I$, and $D_{6,\beta} < \infty$ a.s. finally yield

$$D_2 = \sup_{\mathbf{v} \in \partial B_1^+(\mathbf{0}), p \in K} \left| C_n(\mathbf{v}_I, p) + \sum_{i \in I} c_i(\mathbf{v}_I, \mathbf{1}_I, 1, p) G^0(\mathbf{1}_{\{i\}}) - c_\beta(\mathbf{v}_I, \mathbf{1}_I, 1, p) \alpha \tilde{H}_I \right| \rightarrow_p 0.$$

□

APPENDIX E: CHECK OF THE CONDITIONS OF PROPOSITION 18

Indeed there exist consistent sequences of estimators satisfying the conditions of Proposition 18. The quantities in $\tilde{g}_{n,I}(\mathbf{v}_I)$ based on moments of the spectral vector, i.e. τ_I , $\mathbb{E}[\Theta_i^I \Theta_j^I]$, $i, j \in \{1, \dots, d\}$, can be directly estimated via the estimators constructed in Section 3.2. The partial derivatives c_i , $i \in \{1, \dots, d\}$, and c_β are more difficult to estimate. We propose an approximation via difference quotients.

PROPOSITION 23. *Let \mathbf{X}_l , $l \in \mathbb{N}$, be independent copies of a regularly varying $[0, \infty)^d$ -valued random vector \mathbf{X} satisfying Equation (2) such that the corresponding transformed vector $\mathbf{X}^* = (r_1^{-1} X_1^\alpha, \dots, r_d^{-1} X_d^\alpha)$, the spectral vector $\boldsymbol{\Theta}$ and the sequence $\{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ satisfy the assumptions of Theorem 10 for $K = \{1, 2\}$. Then, for all $i, j \in \{1, \dots, d\}$,*

$$\tilde{\mathbb{E}}_n[\Theta_i^I \Theta_j^I] = 2 \frac{\tilde{M}_{n,k_n,\{i,j\}}(\frac{1}{2} \mathbf{1}_{\{i\}} + \frac{1}{2} \mathbf{1}_{\{j\}}, 2)}{\tilde{P}_{n,k_n,\{i,j\}}} - \frac{1}{2} \sum_{m \in \{i,j\}} \frac{\tilde{M}_{n,k_n,\{i,j\}}(\mathbf{1}_{\{m\}}, 2)}{\tilde{P}_{n,k_n,\{i,j\}}} \rightarrow_p \mathbb{E}[\Theta_i^I \Theta_j^I]$$

and for any non-empty $I \subset \{1, \dots, d\}$:

1. $\frac{1}{\tilde{\tau}_{I,n}} = \frac{\tilde{M}_{n,k_n,I}(|I|^{-1} \mathbf{1}_I, 1)}{\tilde{P}_{n,k_n,I}} \rightarrow_p \frac{1}{\tau_I}$
2. For all $i, j \in I$ and any sequence $\varepsilon_n \rightarrow 0$ such that $\varepsilon_n^2 k_n \rightarrow \infty$, we have

$$\begin{aligned}
 & \tilde{c}_{i,n}(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1) \\
 & = \frac{1}{2\varepsilon_n} \left(\frac{\sum_{l=1}^n \mathbf{1}_{\{j\}}^\top \frac{(\mathbf{1}_I + \varepsilon_n \mathbf{1}_{\{i\}}) \circ \mathbf{X}_l^{\hat{\alpha}} / \mathbf{X}_{k_n}^{\hat{\alpha}}}{\|(\mathbf{1}_I + \varepsilon_n \mathbf{1}_{\{i\}}) \circ \mathbf{X}_l^{\hat{\alpha}} / \mathbf{X}_{k_n}^{\hat{\alpha}}\|}}{\sum_{l=1}^n \mathbf{1}\{\|(\mathbf{1}_I + \varepsilon_n \mathbf{1}_{\{i\}}) \circ \mathbf{X}_l^{\hat{\alpha}} / \mathbf{X}_{k_n}^{\hat{\alpha}}\| > 1\}} \mathbf{1}\{\|(\mathbf{1}_I + \varepsilon_n \mathbf{1}_{\{i\}}) \circ \mathbf{X}_l^{\hat{\alpha}} / \mathbf{X}_{k_n}^{\hat{\alpha}}\| > 1\}} \right. \\
 & \quad \left. - \frac{\sum_{l=1}^n \mathbf{1}_{\{j\}}^\top \frac{(\mathbf{1}_I - \varepsilon_n \mathbf{1}_{\{i\}}) \circ \mathbf{X}_l^{\hat{\alpha}} / \mathbf{X}_{k_n}^{\hat{\alpha}}}{\|(\mathbf{1}_I - \varepsilon_n \mathbf{1}_{\{i\}}) \circ \mathbf{X}_l^{\hat{\alpha}} / \mathbf{X}_{k_n}^{\hat{\alpha}}\|}}{\sum_{l=1}^n \mathbf{1}\{\|(\mathbf{1}_I - \varepsilon_n \mathbf{1}_{\{i\}}) \circ \mathbf{X}_l^{\hat{\alpha}} / \mathbf{X}_{k_n}^{\hat{\alpha}}\| > 1\}} \mathbf{1}\{\|(\mathbf{1}_I - \varepsilon_n \mathbf{1}_{\{i\}}) \circ \mathbf{X}_l^{\hat{\alpha}} / \mathbf{X}_{k_n}^{\hat{\alpha}}\| > 1\}} \right) \\
 & \rightarrow_p c_{i,n}(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1)
 \end{aligned}$$

3. For all $j \in I$ and any sequence $\varepsilon_n \rightarrow 0$ such that $\varepsilon_n^2 k_n \rightarrow \infty$, we have

$$\begin{aligned} \tilde{c}_{\beta,n}(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1) &= \frac{1}{2\varepsilon_n} \left(\frac{\sum_{l=1}^n \mathbf{1}_{\{j\}}^\top \left(\frac{\mathbf{X}_l/\mathbf{X}_{k:n}}{\ell_I(\mathbf{X}_l/\mathbf{X}_{k:n})} \right)^{\hat{\alpha}/(1+\varepsilon)} \mathbf{1}\{\ell_I(\mathbf{X}_l/\mathbf{X}_{k:n}) > 1\}}{\sum_{l=1}^n \mathbf{1}\{\ell_I(\mathbf{X}_l/\mathbf{X}_{k:n}) > 1\}} \right. \\ &\quad \left. - \frac{\sum_{l=1}^n \mathbf{1}_{\{j\}}^\top \left(\frac{\mathbf{X}_l/\mathbf{X}_{k:n}}{\ell_I(\mathbf{X}_l/\mathbf{X}_{k:n})} \right)^{\hat{\alpha}/(1-\varepsilon)} \mathbf{1}\{\ell_I(\mathbf{X}_l/\mathbf{X}_{k:n}) > 1\}}{\sum_{l=1}^n \mathbf{1}\{\ell_I(\mathbf{X}_l/\mathbf{X}_{k:n}) > 1\}} \right) \\ &\rightarrow_p c_{i,n}(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1) \end{aligned}$$

PROOF. The consistency of $\tilde{\mathbb{E}}_n[\Theta_i^I \Theta_j^I]$ and $\tilde{\tau}_{I,n}$ directly follow from 10. To prove the consistency of $\tilde{c}_{i,n}(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1)$, we first note that

$$\begin{aligned} &\frac{\sum_{l=1}^n \mathbf{1}_{\{j\}}^\top \frac{(\mathbf{1}_I \pm \varepsilon_n \mathbf{1}_{\{i\}}) \circ \mathbf{X}_l^{\hat{\alpha}} / \mathbf{X}_{k:n}^{\hat{\alpha}}}{\|(\mathbf{1}_I \pm \varepsilon_n \mathbf{1}_{\{i\}}) \circ \mathbf{X}_l^{\hat{\alpha}} / \mathbf{X}_{k:n}^{\hat{\alpha}}\|} \mathbf{1}\{\|(\mathbf{1}_I \pm \varepsilon_n \mathbf{1}_{\{i\}}) \circ \mathbf{X}_l^{\hat{\alpha}} / \mathbf{X}_{k:n}^{\hat{\alpha}}\| > 1\}}}{\sum_{l=1}^n \mathbf{1}\{\|(\mathbf{1}_I + \varepsilon_n \mathbf{1}_{\{i\}}) \circ \mathbf{X}_l^{\hat{\alpha}} / \mathbf{X}_{k:n}^{\hat{\alpha}}\| > 1\}} \\ &= \frac{\widehat{M}_{n,u_n}(\mathbf{1}_{\{j\}}, (\mathbf{1}_I \pm \varepsilon_n \mathbf{1}_{\{i\}})^{\alpha/\hat{\alpha}} \circ u_n / \mathbf{X}_{k:n}, \alpha/\hat{\alpha}, 1)}{\widehat{P}_{n,u_n}((\mathbf{1}_I \pm \varepsilon_n \mathbf{1}_{\{i\}})^{\alpha/\hat{\alpha}} \circ u_n / \mathbf{X}_{k:n})}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\tilde{c}_{i,n}(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1) \\ &= \frac{1}{2\varepsilon_n} \left(c(\mathbf{1}_{\{j\}}, (\mathbf{1}_I + \varepsilon_n \mathbf{1}_{\{i\}})^{\alpha/\hat{\alpha}} \circ \frac{u_n}{\mathbf{X}_{k:n}}, \frac{\alpha}{\hat{\alpha}}, 1) - c(\mathbf{1}_{\{j\}}, (\mathbf{1}_I - \varepsilon_n \mathbf{1}_{\{i\}})^{\alpha/\hat{\alpha}} \circ \frac{u_n}{\mathbf{X}_{k:n}}, \frac{\alpha}{\hat{\alpha}}, 1) \right. \\ &\quad \left. + \frac{1}{\sqrt{k}} \left[\tilde{G}_n(\mathbf{1}_{\{j\}}, (\mathbf{1}_I + \varepsilon_n \mathbf{1}_{\{i\}})^{\alpha/\hat{\alpha}} \circ \frac{u_n}{\mathbf{X}_{k:n}}, \frac{\alpha}{\hat{\alpha}}, 1) - \tilde{G}_n(\mathbf{1}_{\{j\}}, (\mathbf{1}_I - \varepsilon_n \mathbf{1}_{\{i\}})^{\alpha/\hat{\alpha}} \circ \frac{u_n}{\mathbf{X}_{k:n}}, \frac{\alpha}{\hat{\alpha}}, 1) \right] \right) \\ &= \frac{1}{2\varepsilon_n} \left(c(\mathbf{1}_{\{j\}}, \mathbf{1}_I + \varepsilon_n \mathbf{1}_{\{i\}}, 1, 1) - c(\mathbf{1}_{\{j\}}, \mathbf{1}_I - \varepsilon_n \mathbf{1}_{\{i\}}, 1, 1) + O_p(1/\sqrt{k}) \right) \rightarrow_p c_i(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1), \end{aligned}$$

where the O_p term is obtained similarly as in the proof of Theorem 10.

The proof of the consistency of $\tilde{c}_{\beta,n}(\mathbf{1}_{\{j\}}, \mathbf{1}_I, 1, 1)$ runs analogously based on the identity

$$\begin{aligned} &\frac{n^{-1} \sum_{l=1}^n \mathbf{1}_{\{j\}}^\top \left(\frac{\mathbf{X}_l/\mathbf{X}_{k:n}}{\ell_I(\mathbf{X}_l/\mathbf{X}_{k:n})} \right)^{\hat{\alpha}/(1\pm\varepsilon)} \mathbf{1}\{\ell_I(\mathbf{X}_l/\mathbf{X}_{k:n}) > 1\}}{n^{-1} \sum_{l=1}^n \mathbf{1}\{\ell_I(\mathbf{X}_l/\mathbf{X}_{k:n}) > 1\}} \\ &= \frac{\widehat{M}_{n,u_n}(\mathbf{1}_{\{j\}}, u_n / \mathbf{X}_{k:n} \circ \mathbf{1}_I, (1 \pm \varepsilon)\alpha/\hat{\alpha}, 1)}{\widehat{P}_{n,u_n}(u_n / \mathbf{X}_{k:n} \circ \mathbf{1}_I)}. \end{aligned}$$

□