Averaged dynamics and control for heat equations with random diffusion.

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Abstract: This paper deals with the averaged dynamics for heat equations in the degenerate case where the diffusivity coefficient, assumed to be constant, is allowed to take the null value. First we prove that the averaged dynamics is analytic. This allows to show that, most often, the averaged dynamics enjoys the property of unique continuation and is approximately controllable. We then determine if the averaged dynamics is actually null controllable or not depending on how the density of averaging behaves when the diffusivity vanishes. In the critical density threshold the dynamics of the average is similar to the $\frac{1}{2}$-fractional Laplacian, which is well-known to be critical in the context of the controllability of fractional diffusion processes. Null controllability then fails (resp. holds) when the density weights more (resp. less) in the null diffusivity regime than in this critical regime.

Key words: Averaged controllability, averaged observability, observability, random heat equation

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Abbreviated title: Averaged controls for the heat equation

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1 Introduction

We analyze the problem of controlling the averaged value of the heat equation. This problem is relevant in applications in which the control has to be chosen independently of the random value, in a robust way. This problem has been studied in the literature in bounded domains and with diffusivities which are random variables independent of the space and time variables and have a strictly positive minimum common to almost every realization. Notably, in [Zua16] and [LZ16] the authors consider diffusivities which follow the uniform and exponential probability distributions respectively, whereas a more general study is done in [CGM19]. In those papers it is shown that, under these assumptions, the averaged dynamics inherits many properties from the dynamics of the heat equation (regularity, controllability, observability, etc.), with the only notable exception of the semi-group property. This is done by considering the Fourier representation of the averaged solutions.

In this paper we pursue that study to diffusivities which are allowed to take any positive value. In this scenario the averaged dynamics is still analytic (see Proposition 4.1 below), and so, we prove that the averaged dynamics is approximately controllable provided that we have a hierarchic decay in the time variable of the different frequencies. However, the averaged dynamics may acquire a fractional nature, or an even less diffusive one, so it may not be null controllable. What determines if we can control it is how fast the density of averaging decays when the diffusivity vanishes. In the critical threshold, which is given by random variables whose density functions decay like $e^{-C\alpha^{-1}}$, the dynamics of the average is similar to the $\frac{1}{2}$-fractional Laplacian, which is well-known to be critical in the context of controllability of fractional diffusion processes.

1.1 The mathematical model and main results

In this paper we treat the random heat equation described by the following system:

$$\begin{cases} y_t - \alpha \Delta y = g, & \text{in } (0,T) \times G, \\ y = h, & \text{on } (0,T) \times \partial G, \\ y(0,\cdot) = y^0, & \text{on } G, \end{cases} \tag{1.1}$$

for G a domain, g a source term, h the Dirichlet boundary conditions, y^0 the initial configuration and α the diffusivity coefficient, which is a positive random variable with density function ρ (the regime in which α is allowed to take negative values is studied in Section 8). We have that the averaged solution of (1.1) is given by:

$$\tilde{y}(t,x;y^0,g,h) := \int_0^{+\infty} y(t,x;\alpha,y^0,g,h)\rho(\alpha)d\alpha.$$

Moreover, we can model a control f located in $G_0 \subset G$ or on $\Gamma \subset \partial G$ by posing $g = f 1_{G_0}$ or $h = f 1_{\Gamma}$.

In order to study (1.1) we rely on being the eigenfunctions of $-\alpha\Delta$ independent of α, as this allows us to work with the Fourier representation of the averaged solution. This necessity prevents us from using the same techniques in more general heat equations, like:

$$y_t - \operatorname{div}(\sigma(x,\alpha)\nabla y) + A(x,\alpha)\cdot\nabla y + a(x,\alpha)y = 0. \tag{1.2}$$

However, by studying the dynamics and controllability of (1.1) we highlight some of the most fundamental phenomena involving (1.2). In addition, the same techniques work for heat equations of the type:

$$y_t - \alpha\mathcal{L}y = 0,$$

for $\mathcal{L}$ a self-adjoint elliptic operator of order 2.

Since studying controllability with internal or boundary controls is almost equivalent, this paper is mainly devoted to controllability with an internal control and the few differences are explained in Section 8. In addition, to study the controllability properties of (1.1) we follow the classical duality approach (see Section 7.1 for further details) and focus on the observability properties of its adjoint system, which is given by:

$$\begin{cases} -\varphi_t - \alpha\Delta\varphi = 0, & \text{in } (0,T)\times G, \\ \varphi = 0, & \text{on } (0,T)\times \partial G, \\ \varphi(T,\cdot) = \phi, & \text{on } G. \end{cases} \tag{1.3}$$

To lighten the notation, as usual, we work in its time-reversed system, which is given by:

$$\begin{cases} u_t - \alpha\Delta u = 0, & \text{in } (0,T)\times G, \\ u = 0, & \text{on } (0,T)\times \partial G, \\ u(0,\cdot) = \phi, & \text{on } G. \end{cases} \tag{1.4}$$

The first property of the averaged solutions of (1.4) that we prove is their analyticity in the time variable from $(0,+\infty)$ to $L^2(G)$. Next, using this together with a hierarchic decay in the time variable of the different frequencies, we obtain some unique continuation results for (1.4). Finally we determine when the averaged dynamics of (1.4) is null observable by combining the Fourier representation of the solutions of (1.4) and the monotonicity of the solutions of (1.1) with respect to the boundary conditions.

In order to illustrate the effect of averaging in the dynamics, let us study the dynamics of (1.4) when $G = \mathbb{R}^d$. As averaging and the Fourier transform commute, we work on the Fourier transform of the fundamental solution of the heat equation, which is given by:

$$\exp(-\alpha|\xi|^2 t).$$

Consequently, the Fourier transform of the average of the fundamental solutions is given by:

$$\int_0^{+\infty} \exp(-\alpha|\xi|^2 t)\rho(\alpha)d\alpha;$$

i.e. the Laplace transform of ρ evaluated in $|\xi|^2 t$. In particular, for $r \in (0,1)$ if $\rho(\alpha) \sim_{0^+} e^{-C\alpha^{-\frac{r}{1-r}}}$ we have that:

$$\int \exp(-\alpha|\xi|^2 t)\rho(\alpha)d\alpha \sim \exp(-C|\xi|^{2r} t^r) \tag{1.5}$$

when $|\xi|^2 t \to +\infty$ as shown in (2.6) below. Thus, for those density functions the averaged dynamics in $\mathbb{R}^d$ has a fractional nature. As we are going to prove, for G bounded this is also true and we have the usual controllability and observability results of fractional dynamics (see, for example, [FR71, MZ06, Mil06b, BWZ17, BHS19]); that is, (1.5) implies that the averaged unique continuation is preserved, but (1.5) preserves the null averaged observability if and only if $r > 1/2$, being the threshold density functions those which satisfy:

$$\rho(\alpha) \sim_{0^+} e^{-C\alpha^{-1}}. \tag{1.6}$$

2 Quantification of the main results

In this section we introduce the precise definition of the previously introduced observability notions, we quantify the main results and we give some specific examples.

To start with, we recall the definitions of the introduced observability notions:

Definition 2.1. Let $G \subset \mathbb{R}^d$ be a domain and $G_0 \subset G$ be a subdomain. System (1.4) is *null averaged observable* or *null observable in average* in G_0 if for all $T > 0$ there is a constant $C > 0$ such that for any $\phi \in L^2(G)$:

$$\|\tilde{u}(T, \cdot; \phi)\|_{L^2(G)} \leq C\|\tilde{u}(\cdot; \phi)\|_{L^2((0,T)\times G_0)}. \tag{2.1}$$

If (1.4) is null averaged observable, we define the *cost* of the null averaged observability as:

$$K(G, G_0, \rho, T) = \sup_{\phi \in L^2(G)\setminus\{0\}} \frac{\|\tilde{u}(T, \cdot; \phi)\|_{L^2(G)}}{\|\tilde{u}(\cdot; \phi)\|_{L^2((0,T)\times G_0)}}. \tag{2.2}$$

Definition 2.2. Let $G \subset \mathbb{R}^d$ be a domain and $G_0 \subset G$ be a subdomain. System (1.4) satisfies the *averaged unique continuation property* in G_0 if for all $T > 0$ the equality $\tilde{u} = 0$ in $(0, T) \times G_0$ implies that $\phi = 0$.

To continue with, we state the precise hypotheses on ρ. For that, we focus on the Laplace transform of ρ, which also appears naturally when G is a bounded domain (see (3.1) below):

- To have the unique continuation we need for some $r > 0$ that:

$$-\frac{d}{ds}\ln\left(\int_0^{+\infty} e^{-s\alpha}\rho(\alpha)d\alpha\right) = \frac{\int_0^{+\infty} e^{-s\alpha}\alpha\rho(\alpha)d\alpha}{\int_0^{+\infty} e^{-s\alpha}\rho(\alpha)d\alpha} \gtrsim s^{r-1}. \tag{2.3}$$

- To have the null observability we need (2.3) for some $r > \frac{1}{2}$.
- To prove the lack of null observability we need for some $C > 0$ and $r \in \left[0, \frac{1}{2}\right)$ that:

$$\int_0^{+\infty} e^{-s\alpha}\rho(\alpha)d\alpha \gtrsim e^{-Cs^r}. \tag{2.4}$$

Example 2.3. In a general way, the density functions which satisfy (2.3) are those which decay fast when the diffusivity vanishes. Similarly, the density functions which satisfy (2.4) are those which do not decay too fast (including those which do not decay at all) when the diffusivity vanishes. Meaningful examples include:

1. Any density function ρ supported in $\mathbb{R}^+$ satisfy (2.3) for $r = 1$.
2. If $k \in (0, +\infty)$, the density function

$$\rho(\alpha) = \frac{e^{-\alpha^{-k}}1_{(0,1)}(\alpha)}{\int_0^1 e^{-s^{-k}}ds} \tag{2.5}$$

   satisfies (2.3) for $r = \frac{k}{k+1}$.
3. If $k \in (0, 1)$, the density function given by (2.5) satisfies (2.4) for $r = \frac{k}{k+1}$.
4. The density functions $\rho(\alpha) = 1_{(0,1)}(\alpha)$ (that is, when α is a random variable with uniform distribution in $(0, 1)$) and $\rho(\alpha) = e^{-\alpha}1_{(0,+\infty)}(\alpha)$ (that is, when α is a random variable with exponential distribution in $(0, +\infty)$) satisfy (2.4) for all $r > 0$. Indeed, any continuous density function ρ such that $\rho(0) > 0$ does so.

The proofs of items 1 and 4 are straightforward. As for items 2 and 3, we can prove them by considering the asymptotic results:

$$\int_0^1 e^{-s\alpha-\alpha^{-k}}d\alpha \sim s^{-\frac{2+k}{2+2k}}e^{-c_k s^{\frac{k}{k+1}}} \quad \text{and} \quad \int_0^1 \alpha e^{-s\alpha-\alpha^{-k}}d\alpha \sim s^{-\frac{4+k}{2+2k}}e^{-c_k s^{\frac{k}{k+1}}}, \tag{2.6}$$

for some $c_k > 0$ when $s \to +\infty$. These asymptotic similarities can be proved with the Laplace method (see, for instance, [BA78, (6.4.35) and Example 6.4.9]).

Let us now state the main results of this paper:

- The first main result of this paper is that in many cases we have the unique continuation property:

  Theorem 2.4. *Let $G \subset \mathbb{R}^d$ be a Lipschitz domain, $G_0 \subset G$ be a subdomain, and $\rho = 1_{(0,1)}$ or ρ be a density function which satisfies* (2.3) *for some $r \in (0,1]$. Then, system* (1.4) *satisfies the averaged unique continuation property in G_0.*

  The proof of Theorem 2.4 is given in Section 4. For the uniform distribution it relies on explicit computations of the averaged solutions, whereas for the more diffusive case it relies on the analyticity of the averaged dynamics from $t \in (0, +\infty)$ to $L^2(G)$ (see Proposition 4.1 below) and on the fact that there is some hierarchy in how the frequencies decay, a technique dating back to [Bor97].

- The second main result of this paper concerns some cases in which we do not have averaged observability:

  Theorem 2.5. *Let $G \subset \mathbb{R}^d$ be a Lipschitz domain, $G_0 \subset G$ be a subdomain such that $G_0 \neq G$ and ρ a density function which satisfies* (2.4) *for some $C > 0$ and $r \in \left[0, \frac{1}{2}\right)$. Then, system* (1.4) *is not null observable in average in G_0.*

  We know from Theorem 2.4 that the lack of observability is not caused by a lack of unique continuation. In fact, we prove Theorem 2.5 in Section 5 by giving a sequence $\phi_N \in L^2(G)$ such that:

$$\lim_{N \to \infty} \frac{\|\tilde{u}(T, \cdot; \phi_N)\|_{L^2(G)}}{\left(\int_0^T \int_{G_0} |\tilde{u}(t, x; \phi_N)|^2 dx dt\right)^{1/2}} = +\infty. \tag{2.7}$$

  This sequence is constructed with functions supported in G/G_0, orthogonal to some low frequencies and, at the same time, not too concentrated on high frequencies. Estimate (2.4) ensures us that the mid frequencies do not decay too fast. The fact that the proof works for all $d \in \mathbb{N}$ and $r \in [0, 1/2)$ is a step forwards with respect to the literature, as in analogous situations with fractional dynamics the lack of controllability for $d \geq 2$ and $r \in [0, 1/2)$ is still unproved.

  Remark 2.6. If $G = G_0$ system (1.4) has the averaged unique continuation property and is null observable in average. Both properties are immediate consequence of the fact that $t \mapsto \|\tilde{u}(t, \cdot; \phi)\|_{L^2(G)}$ is a decreasing function (see Remark 3.7).

- The last main result of the paper concerns some cases in which we have averaged observability:

  Theorem 2.7. *Let $G \subset \mathbb{R}^d$ be a Lipschitz locally star-shaped domain, $G_0 \subset G$ be a subdomain, $T > 0$ and ρ a density function which satisfies* (2.3) *for some $r \in \left(\frac{1}{2}, 1\right]$. Then, system* (1.4) *is null observable in average. In addition, there are $T_0, C > 0$ such that for all $T \in (0, T_0]$ we have that:*

$$K(G, G_0, \rho, T) \leq C e^{C T^{(2r-1)^{-1}}}. \tag{2.8}$$

  We recall that the locally star-shaped domains are defined in [AEWZ14, Section 3] and include all the C^2 domains. We prove Theorem 2.7 in Section 6 by adapting the ideas of [Mil10]; that is, we use the Fourier representation and the decay properties of the averaged dynamics.

  Remark 2.8. The estimate (2.8) is an upper estimate for short-time horizons. Ideally, it would also be good to have a lower bound and to precise the constant of the exponential by some geometric terms as in the heat equation (see, for instance, [Mil04], [Mil06a], [TT07], [EZ11] and [LL18]), though this problem is outside the reach of this work.

The rest of the paper is organized as follows: in Section 3 we present some basic results, in Section 4 we prove Theorem 2.4, in Section 5 we prove Theorem 2.5, in Section 6 we prove Theorem 2.7, in Section 7 we resume the controllability problem, in Section 8 we comment some possible extensions, and in Appendix A we prove a technical result.

3 Preliminaries

In this section we introduce some basic facts and notation that we use later on. In particular, we study the spectral properties of the Dirichlet Laplacian, the size of the solutions of the heat equation and the decay implied by (2.3).

3.1 Some results about the spectral decomposition of the Dirichlet Laplacian

As usual, e_i denotes (starting in $i = 0$) the eigenfunctions of the Dirichlet Laplacian, λ_i their respective eigenvalues and $\Lambda_\lambda := \{i : \lambda_i \leq \lambda\}$. In addition, for any $\lambda > 0$, $\mathcal{P}_\lambda$ denotes the orthogonal projection of $L^2(G)$ into $\langle e_i \rangle_{i \in \Lambda_\lambda}$ and $\mathcal{P}_\lambda^\perp$ the orthogonal projection of $L^2(G)$ into $\langle e_i \rangle_{i \in \Lambda_\lambda}^\perp$.

To begin with, we recall that, as shown in [Zua16], the Fourier representation of the averaged solution is:

$$\tilde{u}(t, x; \phi) := \int_0^{+\infty} u(t, x; \alpha, \phi)\rho(\alpha) d\alpha = \sum_{i \in \mathbb{N}} \int_0^{+\infty} e^{-\alpha \lambda_i t} \rho(\alpha) d\alpha \langle \phi, e_i \rangle_{L^2(G)} e_i(x). \tag{3.1}$$

Next, we recall that the eigenvalues have a growth limited by Weyl's law:

Lemma 3.1 (Weyl's law)**.** *Let $G \subset \mathbb{R}^d$ be a Lipschitz domain. We have:*

$$\lim_{\lambda \to \infty} \frac{|\Lambda_\lambda|}{\lambda^{d/2}} = \frac{\operatorname{Vol}(B(0,1)) \operatorname{Vol}(G)}{(2\pi)^d},$$

In particular, there is $C > 0$ such that for all $\lambda \geq \lambda_0$:

$$|\Lambda_\lambda| \leq C \lambda^{d/2}. \tag{3.2}$$

Weyl's law is proved for instance in [Ivr16].

Finally, we recall the following elliptic result proved in [AEWZ14, Theorem 3]:

Lemma 3.2 ([AEWZ14])**.** *Let G be a locally star-shaped domain and $G_0 \subset G$ a subdomain. There exists a constant $C > 0$ such that for all $\lambda > 0$ and $\{c_i\} \subset \mathbb{R}$:*

$$\left(\sum_{i \in \Lambda_\lambda} |c_i|^2 \right)^{1/2} \leq C e^{C\sqrt{\lambda}} \left\| \sum_{i \in \Lambda_\lambda} c_i e_i \right\|_{L^2(G_0)}. \tag{3.3}$$

Their result is a refinement of [Lü13, Theorem 1.2], which was a refinement of the results proved in [LR95].

3.2 Some result on the heat equation

In this subsection we state some properties of the solutions of the heat equation. We first recall that their time derivative can be estimated by using the analyticity and contraction of the semigroup of the heat equation (see [Paz83, Sections 2.5 and 5.6]) and Cauchy's integration formula:

Lemma 3.3. *Let G a bounded domain. Then, there is $C>0$ such that for all $k\in\mathbb{N}$, $s\in\mathbb{R}^+$ and $\phi\in L^2(G)$ we have that:*

$$\|\partial_s^k v(s,\cdot;\phi)\|_{L^2(G)}\leq \frac{C^k k!}{s^k}\|\phi\|_{L^2(G)}, \tag{3.4}$$

for v the solution of:

$$\begin{cases} v_t-\Delta v=0, & \text{in } (0,T)\times G,\\ v=0, & \text{on } (0,T)\times\partial G,\\ v(0,\cdot)=\phi, & \text{on } G.\end{cases} \tag{3.5}$$

It is interesting to consider the solutions of (3.5) because of the identity:

$$u(t,x;\alpha,\phi)=v(t\alpha,x;\phi), \tag{3.6}$$

for u the solution of (1.4).

Another result that we need is that the propagation of the mass when the initial value in some subdomain is null is exponentially slow:

Lemma 3.4. *Let G be a bounded domain and let $\hat{G},G_0\subset G$ be Lipschitz domains satisfying $\hat{G}\subset\subset G\setminus G_0$. Then, there are $c,C>0$ such that for all ϕ satisfying $\operatorname{supp}(\phi)\subset\hat{G}$ and all $T,\alpha>0$ we have that:*

$$\|u(\cdot;\alpha,\phi)\|_{C^0([0,T];L^2(G_0))}\leq Ce^{-\frac{c}{\alpha T}}\|\phi\|_{L^2(G)}, \tag{3.7}$$

for u the solution of (1.4).

Lemma 3.4, whose originality we do not claim, is a consequence of the comparison principle. Indeed, following for example the ideas of [CG05, Lemma 4], we obtain Lemma 3.4 by comparing the solutions of (1.4) with initial value $\pm\phi$ to the solution of the heat equation in $\mathbb{R}^d$ with initial value $|\phi|1_G$, a solution which can be estimated by using its representation with the kernel of the heat equation.

3.3 The decay properties implied by (2.3)

In this subsection we see that if the density function ρ satisfies (2.3), the averaged solutions of (1.4) have a decay similar to that of the solutions of the fractional heat equation. In particular, we prove the following result:

Lemma 3.5. *Let ρ a density function which satisfies (2.3) for some $r\in(1/2,1]$. Then, there is $c>0$ such that for all $\lambda\geq\lambda_0$ and $t_1,t_2\in[0,1)$ satisfying $t_1<t_2$ we have that:*

$$\int_0^{+\infty}e^{-t_2\lambda\alpha}\rho(\alpha)d\alpha\leq e^{-c\lambda^r(t_2-t_1)}\int_0^{+\infty}e^{-t_1\lambda\alpha}\rho(\alpha)d\alpha. \tag{3.8}$$

We recall that λ_0 is the first eigenvalue of the Laplacian.

Proof of Lemma 3.5. In order to prove Lemma 3.5 we first remark that for all $s \geq 0$ we have that:

$$-\frac{d}{ds}\ln\left(\int_0^{+\infty} e^{-s\alpha}\rho(\alpha)d\alpha\right) = \frac{\int_0^{+\infty} e^{-s\alpha}\alpha\rho(\alpha)d\alpha}{\int_0^{+\infty} e^{-s\alpha}\rho(\alpha)d\alpha} \geq c\min\{s^{r-1}, 1\}. \tag{3.9}$$

Indeed, (3.9) follows from (2.3) and from the fact that the second term on the left-hand side of (3.9) is continuous in $[0, +\infty)$. Thus, from (3.9) we obtain for all $s_1, s_2 \geq 0$ with $s_1 < s_2$ the estimate:

$$\int_0^{+\infty} e^{-s_2\alpha}\rho(\alpha)d\alpha \leq \exp\left(-c\int_{s_1}^{s_2}\min\{s^{r-1}, 1\}ds\right)\int_0^{+\infty} e^{-s_1\alpha}\rho(\alpha)d\alpha. \tag{3.10}$$

Next, we fix t_1 and t_2 and use different approaches depending on the value of λ:

- If $\lambda \in [\lambda_0, t_2^{-1}]$, from (3.10) taking $s_1 = \lambda t_1$ and $s_2 = \lambda t_2$ we obtain that:

$$\int_0^{+\infty} e^{-t_2\lambda\alpha}\rho(\alpha)d\alpha \leq e^{-c\lambda(t_2-t_1)}\int_0^{+\infty} e^{-t_1\lambda\alpha}\rho(\alpha)d\alpha. \tag{3.11}$$

  Indeed, $[s_1\lambda, s_2\lambda] \subset (0, 1]$. In addition, since $\lambda \geq \lambda_0$ we have that:

$$-\lambda \leq -\lambda_0^{1-r}\lambda^r. \tag{3.12}$$

  Thus, from (3.11) and (3.12) we obtain (3.8) for some $c > 0$ and all $\lambda \in [\lambda_0, t_2^{-1}]$.

- If $\lambda \in [t_1^{-1}, +\infty)$, from (3.10) taking $s_1 = \lambda t_1$ and $s_2 = \lambda t_2$ we obtain that:

$$\int_0^{+\infty} e^{-t_2\lambda\alpha}\rho(\alpha)d\alpha \leq e^{-c\lambda^r(t_2^r-t_1^r)}\int_0^{+\infty} e^{-t_1\lambda\alpha}\rho(\alpha)d\alpha. \tag{3.13}$$

  Indeed, $[s_1\lambda, s_2\lambda] \subset [1, +\infty)$. Moreover, we consider that:

$$-(t_2^r - t_1^r) = \frac{(-t_2^r + t_1^r)(t_2^{1-r} + t_1^{1-r})}{t_2^{1-r} + t_1^{1-r}} = \frac{t_1 - t_2 - t_2^r t_1^{1-r} + t_2^{1-r}t_1^r}{t_2^{1-r} + t_1^{1-r}} \leq \frac{t_1 - t_2}{t_2^{1-r} + t_1^{1-r}} \leq -\frac{t_2 - t_1}{2}. \tag{3.14}$$

  We have used in the first inequality of (3.14) that $t_1 < t_2$ and $r \in (1/2, 1]$, and in the second one that $t_1 - t_2 < 0$ and $t_1, t_2 \in (0, 1]$. Thus, from (3.13) and (3.14) we obtain (3.8) for some $c > 0$ and all $\lambda \in [t_1^{-1}, +\infty)$. We can suppose that c is the same constant as in the previous case by just taking the minimum.

- If $\lambda \in (t_2^{-1}, t_1^{-1})$, we can use the results of the two previous cases as:

$$(t_1, t_2) = (t_1, \lambda^{-1}] \cup (\lambda^{-1}, t_2).$$

  Indeed, from the previous results we find that:

$$\int_0^{+\infty} e^{-t_2\lambda\alpha}\rho(\alpha)d\alpha \leq e^{-c\lambda^r(t_2-\lambda^{-1})}\int_0^{+\infty} e^{-\alpha}\rho(\alpha)d\alpha \leq e^{-c\lambda^r(t_2-t_1)}\int_0^{+\infty} e^{-t_1\lambda\alpha}\rho(\alpha)d\alpha. \tag{3.15}$$

  Thus, we also get (3.8) for the constant c of the previous cases and all $\lambda \in (t_2^{-1}, t_1^{-1}]$.

□

In a similar way, we can also prove the following result:

Lemma 3.6. *Let ρ a density function satisfies (2.3) for some $r \in (0,1]$. Then, there is $c > 0$ such that for all $\lambda, \tilde{\lambda}$ such that $\tilde{\lambda} > \lambda \geq \lambda_0$ and $t \in [1, +\infty]$ we have that:*

$$\int_0^{+\infty} e^{-t\tilde{\lambda}\alpha}\rho(\alpha)d\alpha \leq e^{-ct^r((\tilde{\lambda})^r-\lambda^r)} \int_0^{+\infty} e^{-t\lambda\alpha}\rho(\alpha)d\alpha. \tag{3.16}$$

Indeed, we obtain easily from (2.3) that for all $s \geq \lambda_0$:

$$-\frac{d}{ds}\ln\left(\int_0^{+\infty} e^{-s\alpha}\rho(\alpha)d\alpha\right) = \frac{\int_0^{+\infty} e^{-s\alpha}\alpha\rho(\alpha)d\alpha}{\int_0^{+\infty} e^{-s\alpha}\rho(\alpha)d\alpha} \geq cs^{r-1}. \tag{3.17}$$

Thus, integrating both sides of (3.17) in $(\tilde{\lambda}t, \lambda t)$ we find (3.16).

Finally, we underline the following result:

Remark 3.7. A consequence of (3.1) is that $t \mapsto \|\tilde{u}(t,\cdot;\phi)\|_{L^2(G)}$ is a decreasing function. Indeed, we have that:

$$\|\tilde{u}(t,\cdot;\phi)\|^2_{L^2(G)} = \sum_{i\in\mathbb{N}}\left(\int_0^{+\infty} e^{-\alpha\lambda_i t}\rho(\alpha)d\alpha\right)^2 |\langle\phi, e_i\rangle|^2,$$

which is a series of decreasing functions.

4 The unique continuation property for averaged solutions

In this section we prove the unique continuation property for averaged solutions (Theorem 2.4). We first study the uniform distribution and then the density functions which satisfy (2.3).

4.1 Proof of Theorem 2.4 for the uniform distribution

Let us compute the averaged solutions of (1.4) when α has the uniform distribution in $(0,1)$. For that, as in [Zua16, Section 3] and [LZ16, Section 3], we present $\tilde{u}$ as the difference of two terms of known nature:

$$\begin{aligned}\tilde{u}(t,x;\phi) = \sum_{i\in\mathbb{N}}\int_0^1 e^{-\lambda_i\alpha t}\langle\phi, e_i\rangle e_i(x)d\alpha &= \frac{1}{t}\left(\sum_{n\in\mathbb{N}}\frac{1}{\lambda_i}\langle\phi, e_i\rangle e_i(x) - \sum_{n\in\mathbb{N}}\frac{e^{-\lambda_i t}}{\lambda_i}\langle\phi, e_i\rangle e_i(x)\right)\\ &= \frac{1}{t}\left(-\Delta^{-1}\phi + \sum_{i\in\mathbb{N}} e^{-\lambda_i t}\langle\Delta^{-1}\phi, e_i\rangle e_i(x)\right).\end{aligned} \tag{4.1}$$

Consequently, from $\int_0^T\int_{G_0}|\tilde{u}(t,x)|^2 = 0$ and (4.1) we find that:

$$-\Delta^{-1}\phi + \sum_{i\in\mathbb{N}} e^{-\lambda_i t}\langle\Delta^{-1}\phi, e_i\rangle e_i(x) = 0 \text{ in } (0,T)\times G_0,$$

which differentiating in time implies that:

$$\sum_{i\in\mathbb{N}} e^{-\lambda_i t}\langle\phi, e_i\rangle e_i(x) = 0 \text{ in } (0,T)\times G_0. \tag{4.2}$$

Hence, using the analyticity of the solutions of the heat equation we have that (4.2) implies that $\phi = 0$, and thus we have the averaged unique continuation property.

4.2 Proof of Theorem 2.4 for density functions which satisfy (2.3)

The proof consists on several steps. First, we show that assuming (2.3) the averaged dynamic are real-analytic and then use this to prove the unique continuation. To begin with, we prove the analyticity:

Proposition 4.1. *Let G be a Lipschitz domain, α any random variable and $\phi \in L^2(G)$. Then, the function:*

$$U : t \in (0, \infty) \to \tilde{u}(t, \cdot; \phi) \in L^2(G)$$

is analytic.

Proof. In order to prove Proposition 4.1, we prove that $U \in C^\infty$ and that:

$$\forall a_1, a_2 \in (0, \infty)\ \exists C > 0 : \sup_{t \in [a_1, a_2]} \|U^{(k)}(t)\|_{L^2(G)} \leq C^k k! \quad \forall k \in \mathbb{N}, \tag{4.3}$$

which is a characterization of analyticity in $\mathbb{R}^+$. Since

$$U(t) = \int_0^{+\infty} v(\alpha t, \cdot; \phi) \rho(\alpha) d\alpha,$$

for v the solution of (3.5), we can easily see that:

$$U^{(k)}(t) = \int_0^\infty \alpha^k \left(\partial_t^k v\right)(\alpha t, \cdot; \phi) \rho(\alpha) d\alpha, \tag{4.4}$$

and thus $U \in C^\infty$. Moreover, (4.3) follows from (4.4), the triangular inequality and (3.4). □

Now we are ready to prove Theorem 2.7:

End of the proof of Theorem 2.7. Let $\phi \in L^2(G)$ such that $\tilde{u}(t, x; \phi) = 0$ in $(0, T) \times G_0$. By Proposition 4.1 we have that $\tilde{u}(t, x; \phi) = 0$ in $(0, +\infty) \times G_0$. Let us show that the first frequency of ϕ is null by contradiction. If the first frequency is not null, we obtain from (3.1) and (3.16) that:

$$\begin{aligned} \tilde{u}(t, \cdot; \phi) &= \int_0^{+\infty} e^{-\alpha\lambda_0 t} \rho(\alpha) d\alpha \langle \phi, e_0 \rangle_{L^2(G)} e_0 + \sum_{i \in \mathbb{N}_*} \int_0^{+\infty} e^{-\alpha\lambda_i t} \rho(\alpha) d\alpha \langle \phi, e_i \rangle_{L^2(G)} e_i \\ &= \left(\int_0^{+\infty} e^{-\alpha\lambda_0 t} \rho(\alpha) d\alpha \right) \left[\langle \phi, e_0 \rangle_{L^2(G)} e_0 + O\left(e^{-(\lambda_1^r - \lambda_0^r) t^r} \|\phi\|_{L^2(G)} \right) \right]. \end{aligned} \tag{4.5}$$

Thus, by considering (4.5) for large values of t we obtain that $\langle \phi, e_0 \rangle_{L^2(G)} e_0 = 0$ in G_0, which by Lemma 3.2 implies that $\langle \phi, e_0 \rangle_{L^2(G)} e_0 = 0$, arriving at a contradiction.

To continue with, we can prove in an similar way that if ϕ is null up to the N-th frequency, then $\tilde{u}(t, x; \phi) = 0$ in $(0, T) \times G_0$ implies that the $(N + 1)$-th frequency is also null. Consequently, we obtain by induction that $\tilde{u}(t, x; \phi) = 0$ in $(0, T) \times G_0$ implies $\phi = 0$. □

5 The cases in which we do not have averaged null observability

In this section we prove Theorem 2.5. As for the notation used in this section, C (resp. c) denotes a sufficiently large (resp. small) strictly positive constant that may be different each time it appears and which just depends on G, G_0, T and ρ. In particular, it does not depend on the index N that we are going to introduce.

In order to prove Theorem 2.5 we construct a sequence ϕ_N satisfying (2.7). For that purpose, we first state the properties which allow us to have (2.7) and explain why these properties prevent us from having averaged observability:

- The first property is that:
$$\bigcup_{N\geq N_0} \mathrm{supp}(\phi_N) \subset\subset G \setminus \overline{G_0}. \tag{5.1}$$
This requirement is very natural as $G \setminus \overline{G_0}$ is the part of the domain that cannot be observed of system (1.4) when $\alpha = 0$. We use it in addition to Lemma 3.4 to obtain that $u(t, x; \alpha, \phi_N)$ decays exponentially in $\{(x, t, \alpha) : x \in G_0,\ \alpha t < N^{-1/2}\}$.

- The second property is that:
$$\phi_N \in \langle e_i\rangle^{\perp}_{i\in\Lambda_N}. \tag{5.2}$$
The benefit of (5.2) is that $u(t, x; \alpha, \phi_N)$ decays exponentially in $\{(x, t, \alpha) : x \in G,\ \alpha t \geq N^{-1/2}\}$, which follows from (3.1).

- The third property is that there exists $C > 0$ such that for all $N \in \mathbb{N}$:
$$\|\mathcal{P}_{CN}\phi_N\|_{L^2(G)} \geq \sqrt{3}\|\phi_N\|_{L^2(G)}/2. \tag{5.3}$$
This estimate is needed to make sure that $\|\tilde{u}_N(T, \cdot; \phi_N)\|_{L^2(G)}/\|\phi_N\|_{L^2(G)}$ does not decrease too fast.

Let us construct the sequence ϕ_N. For that, we inspire in [FCZ00, Section 6] and consider more or less a linear combination of Dirac masses; that is,
$$\phi_N \approx \sum_{i_1,\dots,i_N=1}^{C\sqrt{N}} c_{i_1,\dots,i_N,N}\delta^0_{x_{i_1,\dots,i_N,N}}. \tag{5.4}$$
The property (5.1) is trivial. As for (5.2), we can obtain it by taking the right linear combination. Indeed, we just have to solve an homogeneous linear system which, for $C > 0$ large enough, by Weyl's law (see Lemma 3.1) has more unknowns than equations. Finally, we can obtain (5.3) by choosing the right approximation with functions whose support has a diameter proportional to $N^{-1/2}$. In particular, we can prove that:

Proposition 5.1. *Let $G \subset \mathbb{R}^d$ be a Lipschitz domain and $G_0 \subset G$. Then, there is a sequence $(\phi_N)_{N\geq N_0}$ satisfying* (5.1)*,* (5.2) *and* (5.3)*.*

The rigorous proof of Proposition 5.1 is a bit technical, and thus is postponed to Section A in the Appendix.

Remark 5.2. Since (5.1), (5.2) and (5.3) just depend on G and G_0, so does the sequence ϕ_N.

Example 5.3. In Figure 1 we illustrate the solutions of the heat equation given by the proof of Proposition 5.1 and we appreciate that the mass of $\tilde{u}(t, \cdot; \phi_N)$ in G_0 is very small compared to the total mass of $\tilde{u}$. For doing these graphs we have taken $G = (0, \pi)$, $G_0 = (0, \pi/2)$, $\rho = 1_{(0,1)}$ and:
$$\varsigma(x) = \exp\left(\frac{-1}{10(x-1)^2(x+1)^2}\right) 1_{(-1,1)}(x). \tag{5.5}$$
We recall that in $(0, \pi)$ we have that $e_i(x) = \sin(ix)$ and $\lambda_i = i^2$.

Let us now prove rigorously Theorem 2.5:

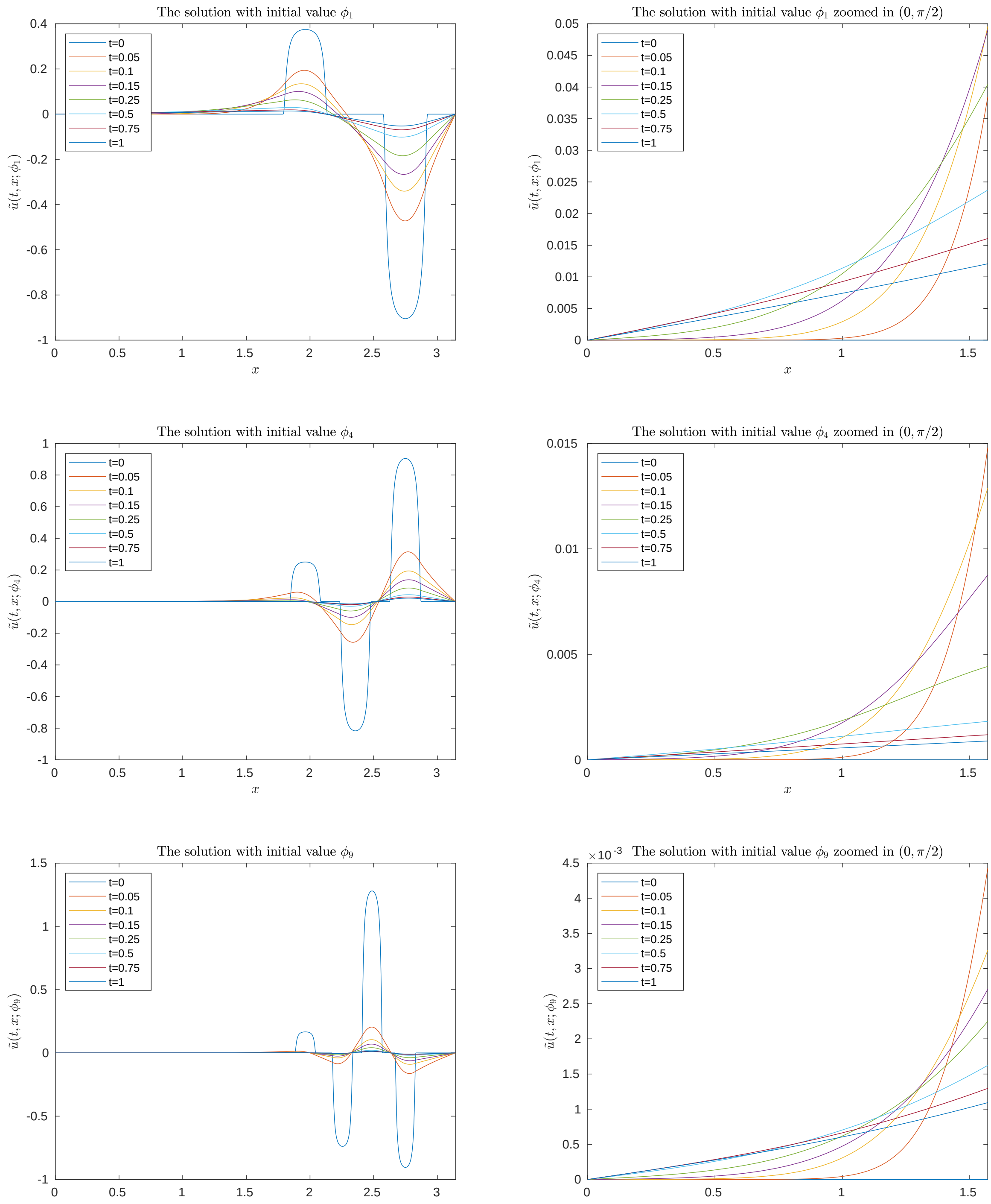

Figure 1: The averaged solutions of the heat equation when $G = (0, \pi)$, $G_0 = (0, \pi/2)$, $\rho = 1_{(0,1)}$ and with initial values of the sequence given in the proof of Proposition (see (A.1)) and with ς given in (5.5).

Proof of Theorem 2.5. We consider ϕ_N given by Proposition 5.1 (for N large enough). We easily find that:

$$\int_0^T \int_{G_0} \left| \int_0^{+\infty} u(t,x;\alpha,\phi_N)\rho(\alpha)d\alpha \right|^2 dxdt \leq \int_0^T \int_{G_0} \int_0^{+\infty} |u(t,x;\alpha,\phi_N)|^2 \rho(\alpha)d\alpha dxdt$$
$$= \int_0^T \int_{G_0} \int_0^{+\infty} |u(t,x;\alpha,\phi_N)|^2 1_{\alpha t \leq N^{-1/2}}(t,\alpha)\rho(\alpha)d\alpha dxdt$$
$$+ \int_0^T \int_{G_0} \int_0^{+\infty} |u(t,x;\alpha,\phi_N)|^2 1_{\alpha t > N^{-1/2}}(t,\alpha)\rho(\alpha)d\alpha dxdt \leq C\left(e^{-c\sqrt{N}} + e^{-\sqrt{N}}\right) \|\phi_N\|^2_{L^2(G)}. \quad (5.6)$$

Indeed, for the first inequality of (5.6) we have used that the L^1 norm in a probabilistic space can be estimated by the L^2 norm. As for the second inequality of (5.6), we have used (5.1) and (3.7) for bounding the first integral, whereas we have used (5.2) and (3.1) for the second one.

To continue with, using (3.1), (2.4) and (5.3) we obtain that:

$$\|\tilde{u}(T,\cdot)\|^2_{L^2(G)} = \sum_{i\in\mathbb{N}} \left(\int_0^\infty e^{-\lambda_i \alpha T}\rho(\alpha)d\alpha\right)^2 |\langle \phi_N, e_i\rangle|^2 \geq c\sum_{i\in\mathbb{N}} e^{-C(\lambda_i T)^r} |\langle \phi_N, e_i\rangle|^2$$
$$\geq ce^{-CN^r} \sum_{i\in\Lambda_{CN}} |\langle \phi_N, e_i\rangle|^2 = ce^{-CN^r}\|\mathcal{P}_{CN}\phi_N\|^2_{L^2(G)} \geq ce^{-CN^r}\|\phi_N\|^2_{L^2(G)}. \quad (5.7)$$

Hence, recalling that $r \in [0,1/2)$ we easily obtain (2.7) from (5.6) and (5.7). □

6 The cases in which we have averaged null observability

In this section we prove Theorem 2.7. As for the notation, C (resp. c) denotes a sufficiently large (resp. small) strictly positive constant that may be different each time it appears and which just depends on G, G_0 and ρ, but which is independent of $T \in (0, T_0)$, for $T_0(G, G_0, \rho)$ small enough.

In order to prove Theorem 2.7 we use the approach introduced in [Mil10, Section 2]. We cannot directly apply its results as the dynamics of the averaged solution just satisfies a decay property and not a semigroup property.

First, we reformulate [Mil10, Lemma 2.1]:

Lemma 6.1. *Let $G \subset \mathbb{R}^d$ a domain, G_0 a subdomain, $T_0 > 0$, $q \in (0,1)$ and f a positive function such that $f(t) \to 0$ as $t \to 0^+$. Suppose that we have for all $\phi \in L^2(G)$ and all $t_1, t_2 \in (0, T_0]$ satisfying $t_1 < t_2$ that:*

$$f(t_2 - t_1)\|\tilde{u}(t_2,\cdot;\phi)\|^2_{L^2(G)} - f(q(t_2-t_1))\|\tilde{u}(t_1,\cdot;\phi)\|^2_{L^2(G)} \leq \int_{t_1}^{t_2}\int_{G_0} |\tilde{u}(\tau,x;\phi)|^2 dxd\tau. \quad (6.1)$$

Then, we have for all $\phi \in L^2(G)$ and $T \in (0, T_0]$ that:

$$\|\tilde{u}(T,\cdot;\phi)\|_{L^2(G)} \leq \sqrt{f((1-q)T)}\|\tilde{u}(\cdot;\phi)\|_{L^2((0,T)\times G_0)}.$$

The proof of Lemma 6.1 is very similar to that of [Mil10, Lemma 2.1]: a telescopic sum considering $t_{2,i} = Tq^i$ and $t_{1,i} = Tq^{i+1}$ for $i \in \mathbb{N}$.

As in [Mil10], we do not prove (6.1) directly, but we prove a similar version, which is the analogue of [Mil10, Lemma 2.3]:

Lemma 6.2. *Let $G \subset \mathbb{R}^d$ a domain, G_0 a subdomain, $T_0, \beta, \gamma_1, \gamma_2, f_0, g_0 > 0$ satisfying $\gamma_1 < \gamma_2$. Suppose that we have for all $\phi \in L^2(G)$ and all $t_1, t_2 \in (0, T_0]$ satisfying $t_1 < t_2$ the inequality:*

$$f(t_2 - t_1)\|\tilde{u}(t_2, \cdot; \phi)\|^2_{L^2(G)} - g(t_2 - t_1)\|\tilde{u}(t_1, \cdot; \phi)\|^2_{L^2(G)} \leq \int_{t_1}^{t_2} \int_{G_0} |\tilde{u}(\tau, x; \phi)|^2 dx d\tau, \tag{6.2}$$

for $f(s) = f_0 \exp(-2/(\gamma_2 s)^\beta)$ and $g(s) = g_0 \exp(-2/(\gamma_1 s)^\beta)$. Then, for any $\gamma \in (0, \gamma_2 - \gamma_1)$ there is $T' \in (0, T_0]$ such that for all $T \in (0, T']$ and $\phi \in L^2(G)$:

$$\|\tilde{u}(T, \cdot; \phi)\|_{L^2(G)} \leq \sqrt{f_0^{-1}} \exp(1/(\gamma T)^\beta) \|\tilde{u}(\cdot; \phi)\|_{L^2((0,T)\times G_0)}.$$

Moreover, if $g_0 < f_0$, we can take $\gamma = \gamma_2 - \gamma_1$ and $T' = T_0$.

The proof of Lemma 6.2 is the same as [Mil10, Lemma 2.3]: bounding superiorly $\frac{g(s)}{f(qs)}$ and using Lemma 6.1.

Now we are ready to prove Theorem 2.7. We do it by following the strategy of [Mil10, Section 2]:

Proof of Theorem 2.7. Let $t_1, t_2 \in [0, 1)$ such that $t_1 < t_2$ and $\phi \in L^2(G)$. First, considering Remark 3.7 we have that:

$$\|\tilde{u}(t_2, \cdot; \phi)\|^2_{L^2(G)} \leq \frac{2}{t_2 - t_1} \int_{(t_1+t_2)/2}^{t_2} \int_G |\tilde{u}(\tau, x; \phi)|^2 dx d\tau. \tag{6.3}$$

We now define:

$$\lambda(t_2, t_1) = \mathfrak{C}(t_2 - t_1)^{-(r-1/2)^{-1}}, \tag{6.4}$$

for $\mathfrak{C}$ a large positive constant to be fixed later. From Lemma 3.2 and (6.3) we obtain that:

$$\|\tilde{u}(t_2, \cdot; \mathcal{P}_\lambda \phi)\|^2_{L^2(G)} \leq Ce^{C((t_2-t_1)^{-1/4}+\sqrt{\lambda})} \int_{(t_1+t_2)/2}^{t_2} \int_{G_0} |\tilde{u}(\tau, x; \mathcal{P}_\lambda \phi)|^2 dx d\tau. \tag{6.5}$$

In addition, considering Remark 3.7 and the Cauchy-Schwarz inequality we find that:

$$\int_{(t_1+t_2)/2}^{t_2} \int_{G_0} |\tilde{u}(\tau, x; \mathcal{P}_\lambda \phi)|^2 dx d\tau \leq 2 \int_{(t_1+t_2)/2}^{t_2} \int_{G_0} |\tilde{u}(\tau, x; \phi)|^2 dx d\tau + (t_2 - t_1)\|\tilde{u}((t_1 + t_2)/2, \cdot; \mathcal{P}_\lambda^\perp \phi)\|^2_{L^2(G)}. \tag{6.6}$$

Moreover, from (3.8) we have that:

$$e^{-C((t_2-t_1)^{-1/4}+\sqrt{\lambda})}\|\tilde{u}(t_2, \cdot; \mathcal{P}_\lambda^\perp \phi)\|^2_{L^2(G)} \leq \left\|\tilde{u}\left((t_2 + t_1)/2, \cdot; \mathcal{P}_\lambda^\perp \phi\right)\right\|^2_{L^2(G)} \leq Ce^{-c\lambda^r(t_2-t_1)}\|\tilde{u}(t_1, \cdot; \mathcal{P}_\lambda^\perp \phi)\|^2_{L^2(G)}. \tag{6.7}$$

Thus, from (6.5)-(6.7), (3.8) and increasing the integration domain in the right-hand side of (6.6) we obtain that:

$$ce^{C((t_2-t_1)^{-1/4}+\sqrt{\lambda})}\|\tilde{u}(t_2, \cdot; \phi)\|^2_{L^2(G)} \leq \int_{t_1}^{t_2} \int_{G_0} |\tilde{u}(\tau, x; \phi)|^2 dx d\tau + Ce^{-c\lambda^r(t_2-t_1)}\|\tilde{u}(t_1, \cdot; \mathcal{P}_\lambda^\perp \phi)\|^2_{L^2(G)}. \tag{6.8}$$

Consequently, we get from (6.8) that:

$$ce^{-C((t_2-t_1)^{-1/4}+\sqrt{\lambda})}\|\tilde{u}(t_2, \cdot; \phi)\|^2_{L^2(G)} - Ce^{-c\lambda^r(t_2-t_1)}\|\tilde{u}(t_1, \cdot; \phi)\|^2_{L^2(G)} \leq \int_{t_1}^{t_2} \int_{G_0} |\tilde{u}(\tau, x; \phi)|^2 dx d\tau.$$

In conclusion, if we substitute the value of λ by the one given in (6.4), we obtain (6.2) for the functions:

$$f(s) = c\exp\left(-C\left(s^{-1/4} + \mathfrak{C}^{1/2}s^{-(2r-1)^{-1}}\right)\right), \quad g(s) = C\exp\left(-c\mathfrak{C}^r s^{-(2r-1)^{-1}}\right).$$

Since $r > 1/2$, we have for $\mathfrak{C}$ sufficiently large and all $s \in (0,1)$ that:

$$c\mathfrak{C}^r s^{-(2r-1)^{-1}} > C\left(s^{-1/4} + \mathfrak{C}^{1/2}s^{-(2r-1)^{-1}}\right),$$

which implies that the functions f and g satisfy the hypothesis of Lemma 6.1 for some (γ_1, γ_2) and for $\beta = (2r-1)^{-1}$, and so we end the proof by using Lemma 6.2. □

7 The controllability problem

In this section we first resume the theoretical study of the controllability problem and then perform some numerical simulations.

7.1 A theoretical study

As stated in the introduction, the observability results that we have obtained in this paper have some implications on the controllability of (1.1). Let us consider the controllability problem given by:

$$\begin{cases} y_t - \alpha\Delta y = f1_{G_0}, & \text{in } (0,T)\times G, \\ y = 0, & \text{on } (0,T)\times \partial G, \\ y(0,\cdot) = y^0, & \text{on } G. \end{cases} \tag{7.1}$$

In particular, we focus on the following notions of controllability, which are introduced in [Zua14]:

Definition 7.1. System (7.1) is *null averaged controllable* or *null controllable in average* if for all $T > 0$ there is $C > 0$ such that for any initial value $y^0 \in L^2(G)$ there is $f \in L^2((0,T)\times G_0)$ satisfying:

$$\|f\|_{L^2((0,T)\times G_0)} \le C\|y^0\|_{L^2(G)},$$

and $\tilde{y}(T,\cdot;y^0,f) = 0$. If (7.1) is null averaged controllable, the *cost* of the null averaged controllability is defined by:

$$\tilde{K}(G, G_0, \rho, T) = \sup_{y^0 \in L^2(G)\setminus\{0\}} \inf_{f:\tilde{y}(T,\cdot;y^0,f)=0} \frac{\|f\|_{L^2((0,T)\times G_0)}}{\|y^0\|_{L^2(G)}}. \tag{7.2}$$

Definition 7.2. System (7.1) is *approximately averaged controllable* or *approximately controllable in average* if for all $T > 0$, $\varepsilon > 0$ and $y^0, y^1 \in L^2(G)$, there exists a control f^ε such that:

$$\left\|\tilde{y}(T,\cdot;y^0,f^\varepsilon) - y^1\right\|_{L^2(G)} < \varepsilon.$$

We now recall the duality result between observability and controllability:

Theorem 7.3 ([LZ16]). *Let $G \subset \mathbb{R}^d$ be a domain and $G_0 \subset G$ be a subdomain. System* (7.1) *is null controllable in average if and only if system* (1.4) *is null observable in average in G_0. In that case $K = \tilde{K}$ (see* (2.2) *and* (7.2)*). Similarly, system* (7.1) *is approximately averaged controllable if and only if system* (1.4) *satisfies the unique continuation property in G_0.*

The proof of Theorem 7.3 can be found in [LZ16, Appendix A]. As an immediate consequence we obtain that Theorems 2.5, 2.4 and 2.7 and Remark 2.6 imply the following controllability results for system (7.1):

Corollary 7.4. *Let $G \subset \mathbb{R}^d$ be a domain, $G_0 \subset G$ be a subdomain and $T > 0$. Then:*

- *Under the hypotheses of Theorem 2.4 system* (7.1) *is approximately controllable in average.*
- *Under the hypotheses of Theorem 2.5 system* (7.1) *is not null controllable in average.*
- *If $G_0 = G$, system* (7.1) *is null and approximately controllable in average for any random variable α.*
- *Under the hypotheses of Theorem 2.7 system* (7.1) *is null controllable in average and there are $C, T_0 > 0$ such that for all $T \in (0, T_0]$ we have the bound:*

$$\tilde{K}(G, G_0, \rho, T) \leq Ce^{CT^{(2r-1)^{-1}}}.$$

As shown in [LZ16, Section A.2], the controls can be obtained (when the system is controlable) by minimizing a quadratic functional, so they can be obtained numerically with usual methods like the gradient descent. Indeed, in the following section we show some numerical simulations of the optimal control made in both the null controllability and the non-null controllability regimes.

7.2 Some numerical experiments

In this section we illustrate experimentally the controllability results obtained in Corollary 7.4. For that, we recall that the optimal control is given by $\varphi(t, x; \phi)1_{G_0}$, for φ the solution of (1.3) and ϕ the state which minimizes the functional:

$$J(\phi) = \frac{1}{2}\int_0^T \int_{G_0} \left| \int_0^{+\infty} \varphi(t, x; \alpha, \phi)\rho(\alpha)d\alpha \right|^2 dxdt + \left\langle y^0, \int_0^{+\infty} \varphi(0; \alpha, \phi)\rho(\alpha)d\alpha \right\rangle.$$

Due to the hardness of the numerical computations in higher dimensions and to get better illustrations we work in $d = 1$, and in particular in $G = (0, \pi)$. We also consider $G_0 = (1, 2)$, $T = 1$ and $y^0 = \frac{1}{2}$. Moreover, to illustrate the difference between diffusivities inside and outside the null controllability regime, we consider $\rho = 1_{(1,2)}$, which is inside, and $\rho = 1_{(0,1)}$, which is outside.

In order to numerically implement this problem, we approximate it by minimizing J in $V_M := \langle e_i \rangle_{i=1}^M$ for $M = 40$, $M = 50$ and $M = 60$. Since V_M is a finite dimensional space, computing the minimum of J is equivalent to solving numerically a linear system, which can be easily done by using any numerical computing environment (in our case with MATLAB). We have the following illustrations:

- We illustrate in Figure 2 (resp. in Figure 3) the controls induced by the minimum of J for $\rho = 1_{(1,2)}$ (resp. for $\rho = 1_{(0,1)}$). For $\rho = 1_{(1,2)}$ the sequence of controls converges, which is something that can be seen in an even more clear way when $t \in [0, 1/2]$. Of course, the closer the time is to 1, the more slowly the punctual values of the control converges pointwise with M (and in $t = 1$ it diverges), but this is a well-known behaviour when controlling a parabolic dynamics (see, for instance, [GL94, MZ10, FCM13]). However, for $\rho = 1_{(0,1)}$ the sequence of controls diverges, which is something that we can appreciate in a more detailed way when $t \in [0, 1/2]$.
- We show in Figure 4 the canonical prolongation of the previously obtained controls to $t = 0$. Again, for $\rho = 1_{(1,2)}$ we have a clear convergence, whereas for $\rho = 1_{(0,1)}$ it diverges.

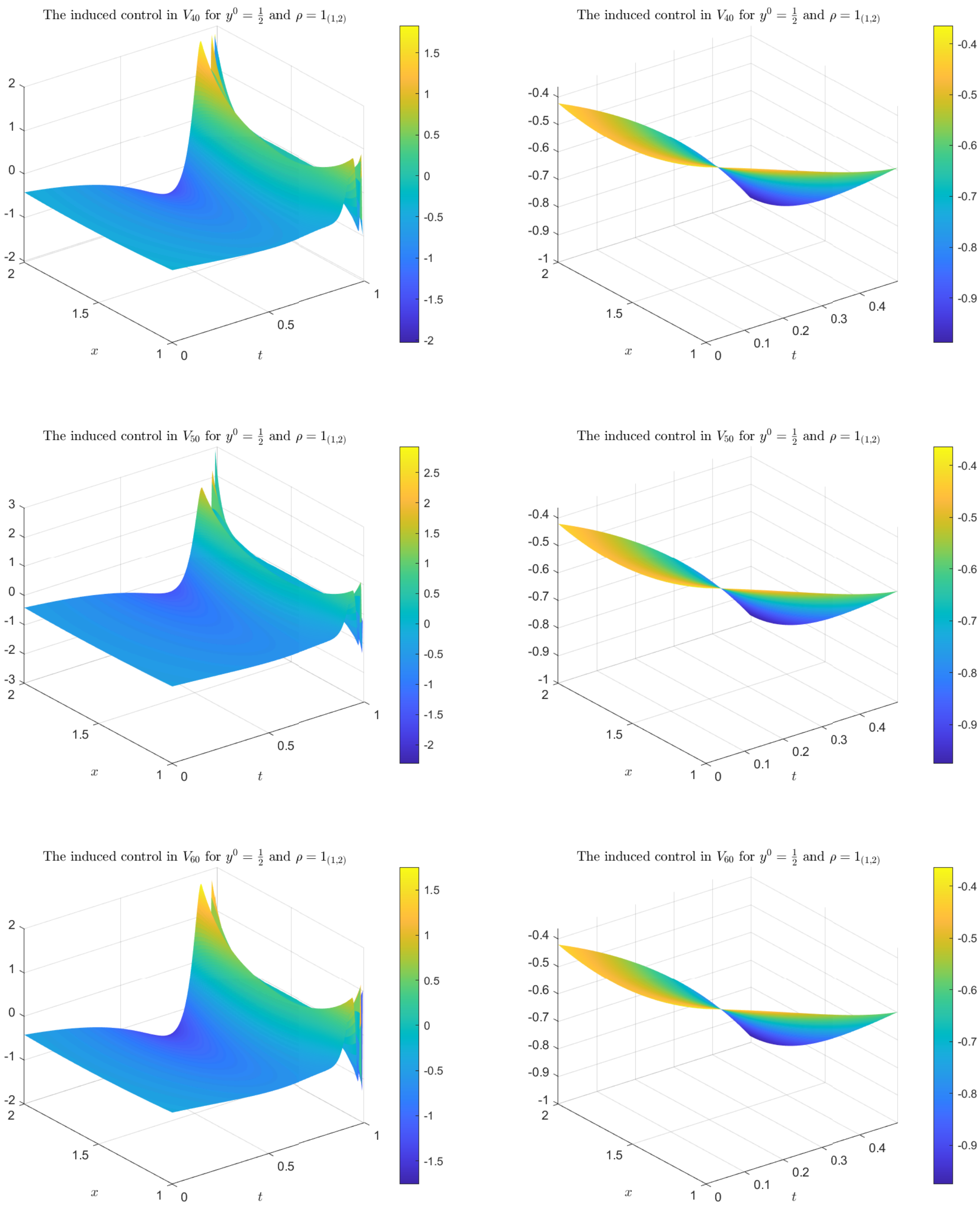

Figure 2: The optimal control for $\rho = 1_{(1,2)}$ and $y^0 = \frac{1}{2}$ induced by the minimum of the functional J in V_{40}, V_{50} and V_{60}. In the left column we illustrate the whole controls, whereas in the right column we illustrate the controls with the time variable zoomed in $[0, 1/2]$.

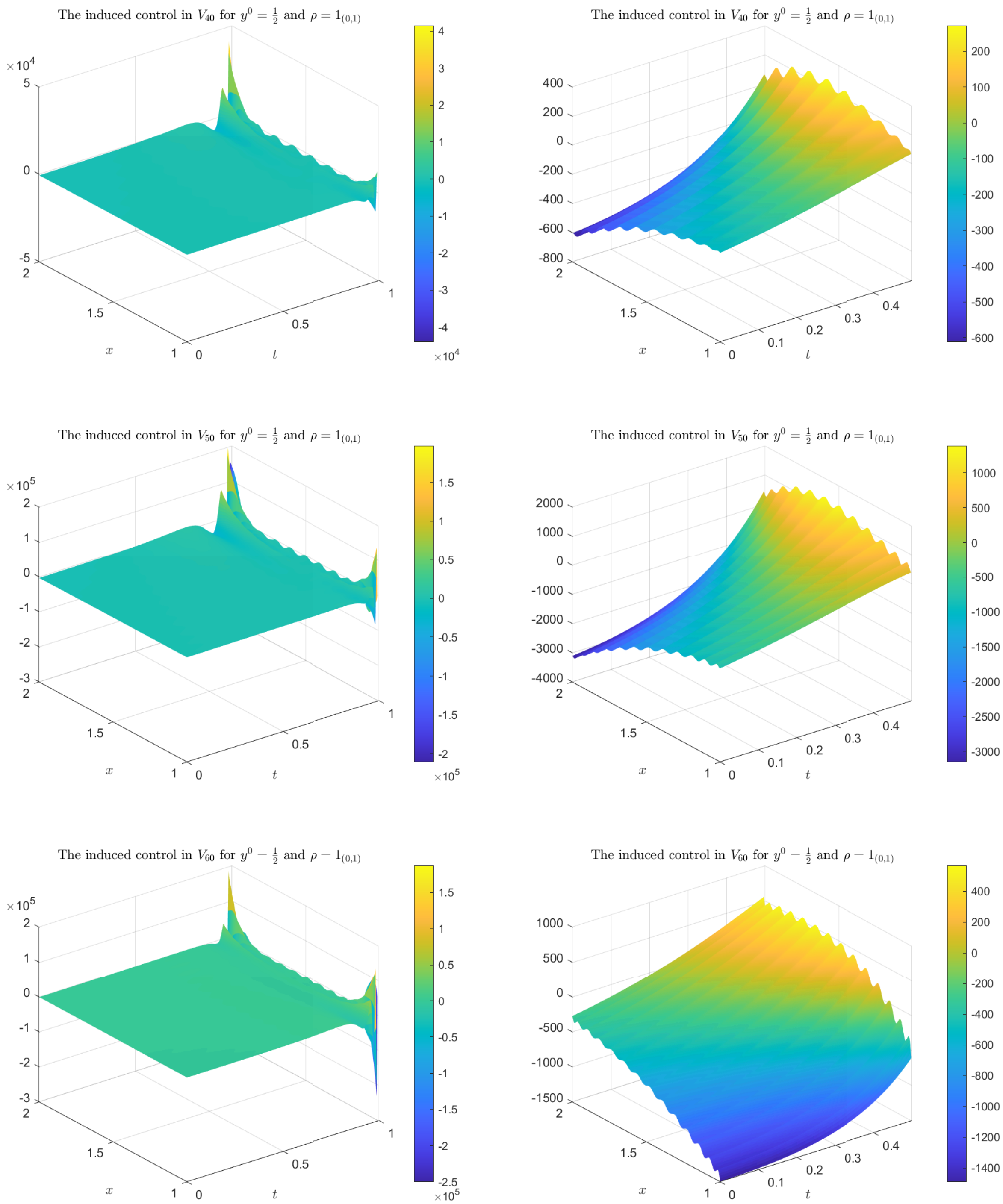

Figure 3: The optimal control for $\rho = 1_{(0,1)}$ and $y^0 = \frac{1}{2}$ induced by the minimum of the functional J in V_{40}, V_{50} and V_{60}. In the left column we illustrate the whole controls, whereas in the right column we illustrate the controls with the time variable zoomed in $[0, 1/2]$.

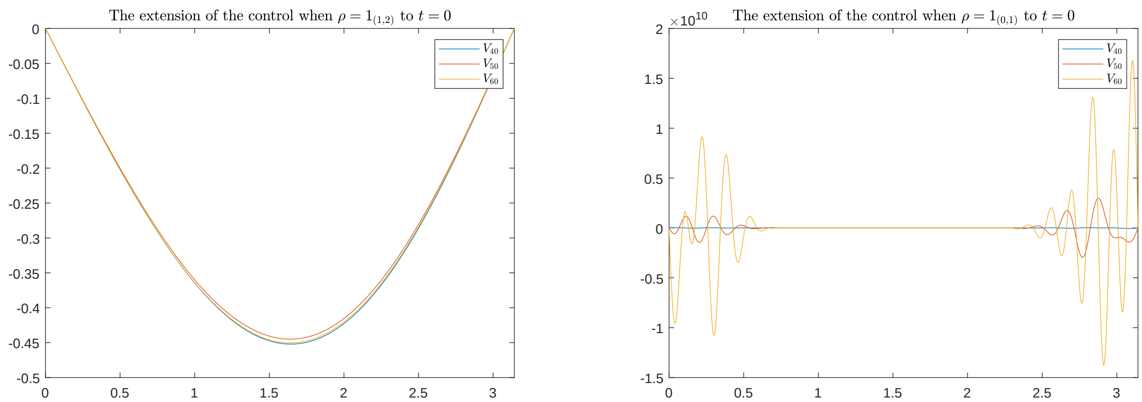

Figure 4: The natural extensions to $t = 0$ of the controls induced by the minimum of the functional J in V_{40}, V_{50} and V_{60} with $y^0 = \frac{1}{2}$. In the left figure we have considered $\rho = 1_{(1,2)}$ and in the right one $\rho = 1_{(0,1)}$.

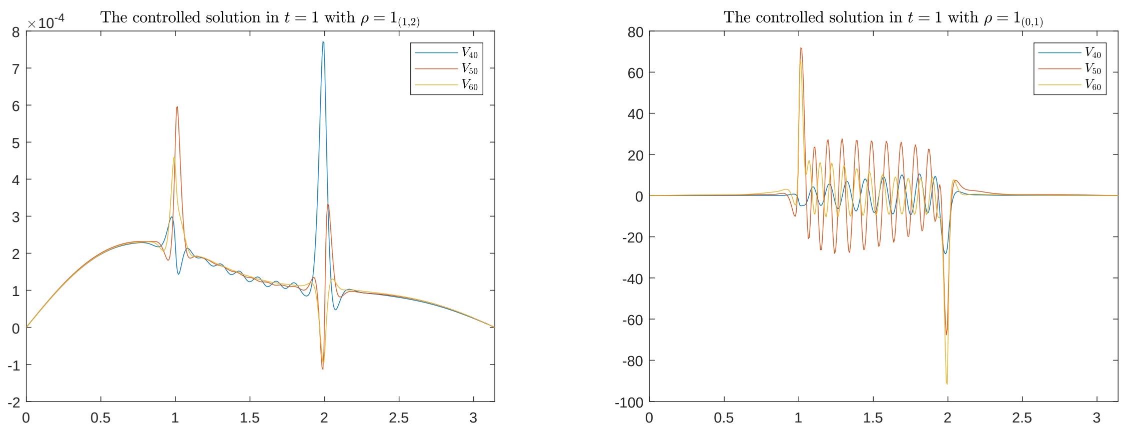

Figure 5: The state in time $t = 1$ of the averaged solutions of the heat equation after applying the control induced by the minimum of J in V_{40}, V_{50} and V_{60} with $y^0 = \frac{1}{2}$. In the left figure we have considered $\rho = 1_{(1,2)}$ and in the right one $\rho = 1_{(0,1)}$.

- We illustrate in Figure 5 the state at $t = 1$ of the respective solutions of the averaged heat equation with the previously obtained controls. For $\rho = 1_{(1,2)}$ the solution converges smoothly to 0, whereas for $\rho = 1_{(0,1)}$ the solution diverges.

8 Further comments and open problems

In this section we underline some extensions of our results to analogous situations and also comment some interesting open problems:

- When dealing with the averaged observability problem, the analogous result of Theorem 2.7 is clearly true as we can repeat the proof step by step. Similarly, the analogous results of Theorems 2.4 and 2.7 are true, which can be proved by using the technique of extensions of domains, the averaged null controllability in an internal domain and an analogous version of Theorem 7.3.

- We have analogous results of Theorems 2.4 and 2.7 for the controllability of the averaged solutions of the heat equation with random diffusion and Neumann boundary conditions. Indeed, we can repeat the proof step by step of those theorems since (3.3) is also true for Neumann boundary conditions (see [Lü13, Theorem 2]). However, whether the analogous of Theorem 2.5 is true remains an open question since we do not have an analogous of Lemma 3.5 for Neumann boundary conditions.

- Even if all the results in this paper have been stated for random variables with a density function, they are true for any random variable whose law satisfies the analogous inequalities of (2.3) and (2.4). Indeed, the proofs can be replicated step by step.

- Even if we have obtained all the results in this paper for the final state in $L^2(G)$ and we have made the observation in $L^2((0,T)\times G_0)$, analogous results are valid for final states in $H^{s_1}(G)$ and the observation in $H^{s_2}((0,T)\times G_0)$ (for any $s_1, s_2 \in \mathbb{R}$). Indeed, the proofs are very similar with the only differences of some polynomial factors of N or λ.

- If (α, G) satisfy the hypotheses of Theorem 2.7, we can easily prove with [EMZ17, Theorem 1] that the free averaged solutions of the heat equation preserve the analyticity with respect to the spatial variable in the interior of the domain.

- Regarding the cases where the diffusion take strictly negative values we do not have null averaged controllability. Indeed, under that hypothesis one can easily prove that:

$$\frac{\|\tilde{u}(T,\cdot;e_n)\|_{L^2(G)}}{\|\tilde{u}(\cdot;e_n)\|_{L^2((0,T)\times G)}} \to +\infty.$$

- There are some edge-case density functions which satisfy neither (2.3) nor (2.4) and whose (non-)observability properties are still unproved, for instance, those satisfying (1.6).

- It would be interesting to have a proof of the unique continuation property for the averaged dynamics of any random variable α, even when it takes negative values. Indeed, there are some random variables whose density functions do not satisfy (2.3) for any $r > 0$ (for instance, $\rho(\alpha) = 2\alpha 1_{(0,1)}(\alpha)$), and thus their unique continuation is still unproved. The probable answer is that the unique continuation is always preserved, though a priori if ρ is too irregular it might not be preserved.

- The observability properties proved in Theorem 2.7 can be extended to sets of the type $E \times G_0$, for E a measurable set. Indeed, we can use the approach of [PW13, AEWZ14, LZ16, CGM19], which complement the ideas of [Mil10] with some results from Measure Theory.

- An interesting problem that remains open is the study of the averaged observability properties of the random heat equation when the lower terms are also random terms, as:

$$y_t - \operatorname{div}(\sigma(x,\alpha)\nabla y) + A(x,\alpha)\cdot\nabla y + a(x,\alpha)y = 0.$$

  In particular, this is interesting when the averaged convection operator and averaged diffusion operator do not commute. Unfortunately, the techniques presented in this paper do not help in that direction since they rely on the fact that the eigenfunctions associated to the elliptic operator are independent of α. Even having an in-depth numerical study would be of high interest.

- There are many other interesting questions involving random PDEs such as Schrödinger, wave or Stokes equations:

  - The Schrödinger equations with random diffusions satisfying the uniform, exponential, Laplace, normal, Chi-squared and Cauchy distributions were studied in [LZ16]. There, the authors show that the averaged dynamics may be conservative or diffusive depending on the probability density, which leads to averaged controllability properties of very different kind. They consider the uniform distribution in any segment of $\mathbb{R}$ and the exponential distribution in $[1,+\infty)$, though their proof is valid in any segment of the type $[K,+\infty)$ for any $K\in\mathbb{R}$. However, the problem of determining the dynamics and controllability properties of the averaged Schrödinger equations with arbitrary distributions is still open.

- The wave equation with random discrete diffusion was studied in [LZ14], whereas understanding the general case is still an open challenge.
- The Stokes equation with random diffusion has not been studied in the literature. However, we can get analogous versions of Theorems 2.4 and 2.7 for the Stokes equation with random diffusion as of the heat equation by considering [CSL16, Theorem 3.1]. Nonetheless, determining if the analogous of Theorem 2.5 is true remains an open problem because of a lack of a comparison theorem prevent us from using analogous arguments.

A Proof of Proposition 5.1

As in Section 5, C denotes a sufficiently large positive constant that may be different each time it appears and which just depends on G, G_0, T and ρ. In particular, it does not depend on the index N. Similarly, $\tilde{C}$ and $\overline{C}$ are a constant that just depends on G, G_0, T and ρ, but they take the same value each time they appear. Finally, i denotes the multi-index $(i_1,\dots,i_d)$ and $[\cdot]$ denotes the floor function of a real number.

Let us fix $q=(q_1,\dots,q_d)$ and $\ell>0$ such that

$$K:=[q_1,q_1+\ell]\times\cdots\times[q_d,q_d+\ell]\subset\subset G\setminus G_0.$$

We also fix a positive non-trivial function $\varsigma\in\mathcal{D}(B_{\mathbb{R}^d}(0,1))$. We define:

$$p(\gamma_1,\dots,\gamma_d):=q+\ell(\gamma_1,\dots,\gamma_d),$$

which is a parametrization of K. With this in mind, we define the functions:

$$\phi_N(x):=\sum_{i_1,\dots,i_d=0}^{[\tilde{C}\sqrt{N}]}c_{i,N}\varsigma_{i,N}(x),\quad \text{for } \varsigma_{i,N}(x):=\varsigma\left(3\sqrt{N}\frac{x-p\left(\frac{i}{[\tilde{C}\sqrt{N}]}\right)}{\tilde{C}\ell}\right),\tag{A.1}$$

for $c_{i,N}$ and $\tilde{C}$ to be defined later on (see Figure 6 for an illustration on how K and the support of ϕ_1 may look like). Let us check that for some $\tilde{C}$ and $c_{i,N}$ the sequence ϕ_N given by (A.1) satisfies (5.1)-(5.3):

- We have that:
$$\operatorname{supp}(\phi_N)\subset\left\{x:d(x,K)<\frac{\tilde{C}\ell}{3\sqrt{N}}\right\}.\tag{A.2}$$
Since the right-hand side of (A.2) is a decreasing sequence of sets and since $K\subset\subset G\setminus G_0$ we can easily prove (5.1).
- In order to have (5.2) we just need to find a non-trivial solution of the system:
$$\{\langle\phi_N,e_i\rangle_{L^2(G)}=0,\quad\forall i\in\Lambda_N.\tag{A.3}$$

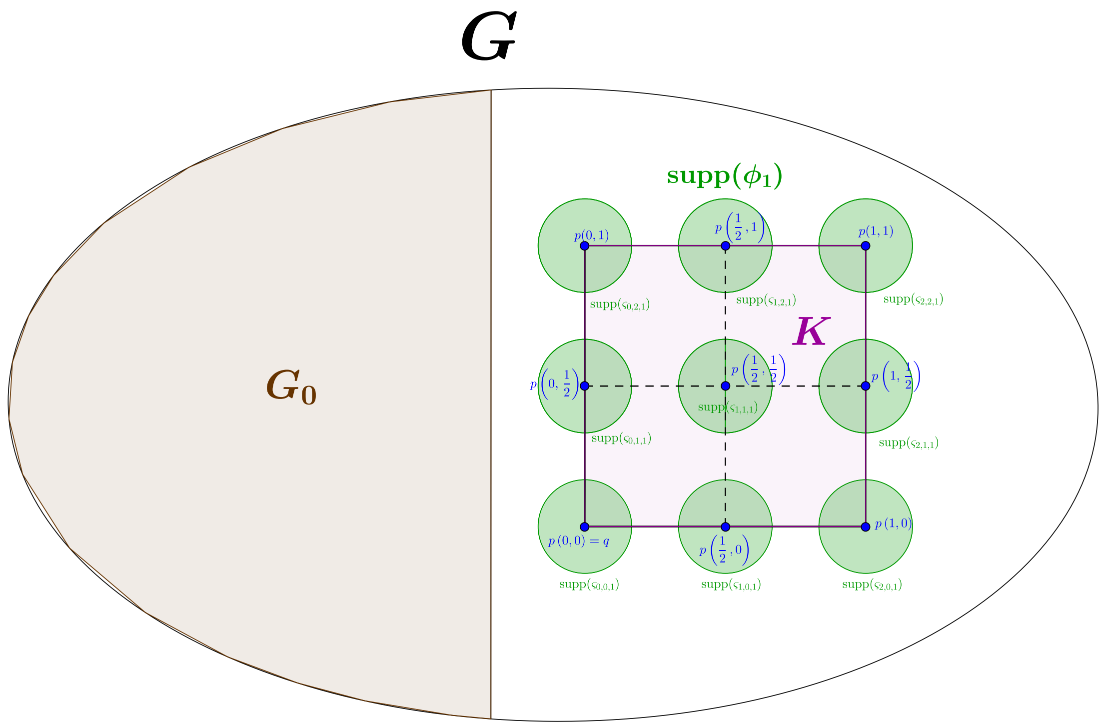

Figure 6: An illustration of the support ϕ_1 in a domain belonging to $\mathbb{R}^2$.

We remark that the system (A.3) is a linear homogeneous system with $\left[\tilde{C}\sqrt{N}\right]^d$ unknowns and $|\Lambda_N|$ equations, so from Weyl's law (see Lemma 3.1) and by taking $\tilde{C} > 0$ large enough we obtain that there are more unknowns than equations, which implies that (A.3) has a non-trivial solution. In particular, we can fix $(c_{i,N})_i$ a non-null tuple such that ϕ_N is a solution of (A.3).

- In order to prove (5.3) it suffices to prove that there is $\overline{C} > 0$ such that:

$$\|\Delta\phi_N\|_{L^2(G)} \leq \frac{\overline{C}N}{2}\|\phi_N\|_{L^2(G)}. \tag{A.4}$$

Indeed, from (A.4) we obtain that:

$$\|\phi_N\|_{L^2(G)} \geq \frac{2\|\Delta\phi_N\|_{L^2(G)}}{\overline{C}N} \geq \frac{2\|\mathcal{P}^{\perp}_{\overline{C}N}\Delta\phi_N\|_{L^2(G)}}{\overline{C}N} \geq 2\|\mathcal{P}^{\perp}_{\overline{C}N}\phi_N\|_{L^2(G)},$$

so we find that:

$$\|\mathcal{P}_{\overline{C}N}\phi_N\|^2_{L^2(G)} = \|\phi_N\|^2_{L^2(G)} - \|\mathcal{P}^{\perp}_{\overline{C}N}\phi_N\|^2_{L^2(G)} \geq \frac{3}{4}\|\phi_N\|^2_{L^2(G)},$$

which is (5.3) squared. So, let us prove (A.4). We clearly have for all $i, \tilde{i} \in \left\{0, \ldots, \left[\tilde{C}\sqrt{N}\right]\right\}^d$ satisfying

$i \neq \tilde{i}$ that $\mathrm{supp}(\varsigma_{i,N}) \cap \mathrm{supp}(\varsigma_{\tilde{i},N}) = \emptyset$. Thus, we have that:

$$\begin{aligned}
\|\Delta \phi_N\|^2_{L^2(G)} &= \sum_{i_1,\ldots,i_d=0}^{[\tilde{C}\sqrt{N}]} c^2_{i,N} \left(\frac{3\sqrt{N}}{\tilde{C}\ell}\right)^4 \int_G |\Delta \varsigma|^2 \left(3\sqrt{N}\frac{x - p\left(\frac{i}{[\tilde{C}\sqrt{N}]}\right)}{\tilde{C}\ell}\right) dx \\
&= \left(\sum_{i_1,\ldots,i_d=0}^{[\tilde{C}\sqrt{N}]} c^2_{i,N}\right) \left(\frac{3\sqrt{N}}{\tilde{C}\ell}\right)^3 \|\Delta\varsigma\|^2_{L^2(B(0,1))} \leq C \left(\sum_{i_1,\ldots,i_d=0}^{[\tilde{C}\sqrt{N}]} c^2_{i,N}\right) \left(\frac{3\sqrt{N}}{\tilde{C}\ell}\right)^3 \|\varsigma\|^2_{L^2(B(0,1))} \\
&= C \sum_{i_1,\ldots,i_d=0}^{[\tilde{C}\sqrt{N}]} c^2_{i,N} \left(\frac{3\sqrt{N}}{\tilde{C}\ell}\right)^4 \int_G |\varsigma|^2 \left(3\sqrt{N}\frac{x - p\left(\frac{i}{[\tilde{C}\sqrt{N}]}\right)}{\tilde{C}\ell}\right) dx \\
&\leq CN^2 \sum_{i_1,\ldots,i_d=0}^{[\tilde{C}\sqrt{N}]} c^2_{i,N} \int_G |\varsigma|^2 \left(3\sqrt{N}\frac{x - p\left(\frac{i}{[\tilde{C}\sqrt{N}]}\right)}{\tilde{C}\ell}\right) dx = CN^2 \|\phi_N\|^2_{L^2(G)}. \qquad \text{(A.5)}
\end{aligned}$$

Consequently, the sequence ϕ_N satisfies (A.4) for $\overline{C}$ large enough, and hence it also satisfies (5.3).

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