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The resolvent of the Nelson Hamiltonian improves positivity

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We prove that the resolvent of the renormalised Nelson Hamiltonian at fixed total momentum P improves positivity in the (momentum) Fock-representation, for every P . Our argument is based on an explicit representation of the renormalised operator and its domain using interior boundary conditions.

1 Introduction

An operator on a Hilbert space is said to preserve positivity if it leaves a cone of “positive elements”, e.g. functions that are point-wise non-negative, invariant. It is said to improve positivity if it maps any non-zero positive element to a strictly positive element, meaning that the scalar product with any positive element is strictly positive. This property has important consequences for the spectral theory of the operator. For example, if a self-adjoint bounded operator improves positivity and has a maximal eigenvalue, then this eigenvalue is simple, with a strictly positive eigenfunction, by the Perron-Frobenius-Faris theorem [Far72]. This method has played an important role in the spectral analysis of Hamiltonians from quantum field theory (QFT) for a long time [GJ70, Gro72, Far72, Frö74, BFS98, Mø105, DH19].

An important example is the Nelson model for the interaction of a non-relativistic particle with a bosonic field. At fixed total momentum P this model is described by a self-adjoint Hamiltonian acting on the symmetric Fock space

$$\mathcal{H}_P := \Gamma(L^2(\mathbb{R}^3)) = \bigoplus_{n=0}^{\infty} (L^2(\mathbb{R}^3))^{\otimes_{\text{sym}} n}. \quad (1)$$

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The Hamiltonian has the formal expression

$$H(P) = (P - d\Gamma(k))^2 + d\Gamma(\omega) + a(v) + a^*(v). \quad (2)$$

where $d\Gamma(k)$ is the field momentum, acting on each factor $L^2(\mathbb{R}^3)$ as multiplication by the respective variable, ω denotes operator of multiplication by the dispersion relation $\omega(k) = \sqrt{k^2 + m^2}$, $m \geq 0$ and $v(k) = \omega(k)^{-1/2}$ is the form-factor of the interaction. This expression cannot be interpreted as a sum of densely defined operators on \mathcal{H}_P since $v \notin L^2(\mathbb{R}^3)$. However, it is possible to define a self-adjoint renormalised Hamiltonian corresponding to the formal expression, which was constructed for the translation-invariant model by Nelson [Nel64] and by Cannon [Can71] for the model at fixed momentum (see also [GW18, DH19] for a recent exposition and refinements). This operator is obtained as the limit $\Lambda \rightarrow \infty$ (in norm-resolvent sense) of the operators $H_\Lambda(P) - E_\Lambda \rightarrow H_{\text{ren}}(P)$ with ultraviolet (UV) cutoff, where where v is replaced by $v_\Lambda(k) = v(k)1(|k| \leq \Lambda)$ and $E_\Lambda \in \mathbb{R}$ are appropriately chosen numbers.

Fröhlich [Frö73, Frö74], and Møller [Møl05], showed for the Nelson model with UV cutoff that the resolvent of the Hamiltonian as well as the generated semigroup improve positivity, and used this to prove that the ground state of the Hamiltonian $H_\Lambda(P)$ (which exists for $m > 0$ and small P) is simple. In [Frö73, Frö74] it was also announced that the same positivity property holds for the renormalised operator. However, a complete proof was given only recently by Miyao [Miy19, Miy18], who showed that the semigroup generated by the renormalised Hamiltonian improves positivity for every P , which implies the same property for the resolvent. This was then used by Dam and Hinrichs [DH19], who proved non-existence of the ground state for $m = 0$. Positivity of $e^{-\beta H_{\text{ren}}(0)}$ had been shown earlier by Gross [Gro72], and positivity for the translation-invariant Hamiltonian plus a potential in the path-integral representation by Matte and Møller [MM17]. The difficulty in proving these results is, roughly speaking, that $H_{\text{ren}}(P)$ is defined as a limit, and in this limit positive quantities could converge to zero.

In this article we give a new proof that the resolvent of $H_{\text{ren}}(P)$ improves positivity. We use a representation of the operator and its domain in terms of generalised boundary conditions, called interior boundary conditions, which allows us to work only with the renormalised operator and avoid approximation by operators with cutoff and the difficulties this entails.

1.1 The renormalised Hamiltonian and interior boundary conditions

In this section we review the representation of the renormalised Hamiltonian using interior boundary conditions. This approach to the UV problem was proposed by Teufel and Tumulka [TT16, TT20]. It was applied to the massive ($m > 0$) Nelson model by Schmidt and the author [LS19], and generalised to the massless case ($m = 0$) by Schmidt [Sch20]. The proofs of key lemmas are provided in Appendix A.

For a unitary $U \in \mathcal{B}(L^2(\mathbb{R}^3))$ we denote by $\Gamma(U)$ the induced unitary on \mathcal{H}_P acting on $\mathcal{H}_P^{(n)} := (L^2(\mathbb{R}^3))^{\otimes_{\text{sym}} n}$ as $U^{\otimes n}$. By $d\Gamma(A)$ we denote the self-adjoint generator of $\Gamma(e^{-itA})$. Let

$$L_P := (P - d\Gamma(k))^2 + d\Gamma(\omega) \quad (3)$$

with $\omega(k) = \sqrt{k^2 + m^2}$, $m \geq 0$, $v(k) = g\omega(k)^{-1/2}$, where $g \in \mathbb{R}$ is a coupling constant. This operator leaves the particle-number invariant and acts on an n -particle wavefunction as multiplication by the non-negative function of $K = (k_1, \dots, k_n)$

$$L_P(K) := \left(P - \sum_{j=1}^n k_j \right)^2 + \sum_{j=1}^n \omega(k_j). \quad (4)$$

It is thus self-adjoint on its maximal domain $D(L_P) \subset \mathcal{H}_P$ and non-negative. For $\lambda > 0$ we set

$$G_\lambda^* := -a(v)(L_P + \lambda)^{-1}, \quad (5)$$

where $a(v)$ is the annihilation operator that acts on (a dense subspace of) $\mathcal{H}_P^{(n+1)}$ as

$$\left(a(v)\psi^{(n+1)} \right)(K) = \sqrt{n+1} \int_{\mathbb{R}^3} v(\xi)\psi^{(n+1)}(K, \xi) d\xi. \quad (6)$$

The operator G_λ^* is bounded on \mathcal{H}_P (see Lemma A.1), so its adjoint is also bounded, and $a(v)$ is densely defined on $D(a(v)) = D(L_P)$. The action of the adjoint is given on $\mathcal{H}_P^{(n)}$ by

$$\left(G_\lambda \psi^{(n)} \right)(K) = -\frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} \frac{v(k_j)\psi^{(n)}(\hat{K}_j)}{L_P(K) + \lambda}, \quad (7)$$

where $\hat{K}_j \in \mathbb{R}^{3(n-1)}$ denotes the vector K with the entry k_j removed.

The domain of the Hamiltonian at total momentum P is

$$D(H_P) = \left\{ \psi \in \mathcal{H}_P : (1 - G_\lambda)\psi \in D(L_P) \right\}. \quad (8)$$

The condition $(1 - G_\lambda)\psi \in D(L_P)$ is the interior boundary condition that encodes the behaviour of ψ for $k \rightarrow \infty$. Note that $\text{ran}(G_\lambda - G_\mu) \subset D(L_P)$, so this condition is independent of $\lambda > 0$.

The action of H_P on its domain can be expressed as

$$H_P = (1 - G_\lambda)^*(L_P + \lambda)(1 - G_\lambda) + T_\lambda - \lambda, \quad (9)$$

with the operator $T_\lambda = T_{d,\lambda} + T_{od,\lambda}$, whose “diagonal part” acts on $\mathcal{H}_P^{(n)}$, $n \in \mathbb{N}_0$, as the operator of multiplication by

$$(T_{d,\lambda})(K) = \int_{\mathbb{R}^3} |v(\xi)|^2 \left(\frac{1}{\xi^2 + \omega(\xi)} - \frac{1}{L_P(K, \xi) + \lambda} \right) d\xi \quad (10)$$

and the off-diagonal part acts as the integral operator (for $n > 0$)

$$(T_{od,\lambda}\psi^{(n)})(K) = - \sum_{j=1}^n \int_{\mathbb{R}^3} \frac{\overline{v(\xi)}v(k_j)\psi^{(n)}(\hat{K}_j, \xi)}{L_P(K, \xi) + \lambda} d\xi. \quad (11)$$

To understand the connection of the Hamiltonian (9) and the formal expression (2), we expand the former into a sum of terms that are individually not elements of \mathcal{H}_P , but of $D(L_P^{-1})$ (the completion of \mathcal{H}_P under the norm $\|\psi\| = \|(L_P + 1)^{-1}\psi\|_{\mathcal{H}_P}$).

Let $D(T_\lambda) := D(L_P^\varepsilon + d\Gamma(\omega)^{1/2}) \subset \mathcal{H}_P$ be the domain of T_λ (for appropriate $\varepsilon > 0$, see Equation (17) below). Then

$$\begin{aligned} A : D(L_P) \oplus G_\lambda D(T_\lambda) &\rightarrow \mathcal{H}_P \\ A(\psi + G_\lambda\varphi) &= a(v)\psi + T_\lambda\varphi \end{aligned} \quad (12)$$

extends $a(v)$ to a domain that contains $D(H_P)$ (see the argument below Theorem 1.1). The operator A is well defined and independent of λ since

$$\begin{aligned} A(\psi + G_\lambda\varphi) &= a(v)\psi + T_\lambda\varphi = a(v)\psi + (\lambda - \mu)G_\lambda^*G_\mu\varphi + T_\mu\varphi \\ &= a(v)(\psi + \underbrace{(G_\lambda - G_\mu)\varphi}_{\in D(L_P)}) + AG_\mu\varphi \end{aligned} \quad (13)$$

holds for all $\mu, \lambda > 0$, as one easily checks using the resolvent identity. For $\psi \in D(H_P)$, i.e. with $(1 - G_\lambda)\psi \in D(L_P)$ we then have

$$A\psi = A(1 - G_\lambda)\psi + AG_\lambda\psi = a(v)(1 - G_\lambda)\psi + T_\lambda\psi. \quad (14)$$

This implies that the action of H_P on its domain can be expressed, as a sum in $D(L_P^{-1})$, by

$$\begin{aligned} H_P\psi &= (L_P + \lambda)(1 - G_\lambda)\psi + a(v)(1 - G_\lambda)\psi + T_\lambda\psi - \lambda\psi \\ &= L_P\psi + a^*(v)\psi + A\psi. \end{aligned} \quad (15)$$

This shows that H_P is also independent of λ and essentially acts as the formal expression (2), up to the choice of extension $A \supset a(v)$.

It was proved in [LS19, Sch20] that the translation-invariant Nelson Hamiltonian (which is unitarily equivalent to the direct integral of the fibre-operators H_P , see [Can71]) is self-adjoint, bounded from below and equals the renormalised operator constructed in [Nel64, Can71, GW18]. Formulated for the fibre operator H_P , we have:

Theorem 1.1. *The operator H_P is self-adjoint on $D(H_P)$, bounded from below, and $H_P = H_{\text{ren}}(P)$.*

The proof goes as follows. First, $(1 - G_\lambda)$ has a bounded inverse (see Lemma A.1), so $D(H_P)$ is dense and

$$(1 - G_\lambda^*)(L_P + \lambda)(1 - G_\lambda) \quad (16)$$

is self-adjoint and non-negative on $D(H_P)$. One then shows that T_λ is bounded relative to this operator with infinitesimal bound. The key is that, by Lemma A.2 and Lemma A.3, we have for any $\varepsilon > 0$, $n \in \mathbb{N}_0$ and $\psi^{(n)} \in \mathcal{H}_P^{(n)}$

$$\|T_\lambda \psi\|_{\mathcal{H}_P} \leq C_\varepsilon \left(\|(L_P + \lambda)^\varepsilon \psi\|_{\mathcal{H}_P} + \|\text{d}\Gamma(\omega)^{1/2} \psi\|_{\mathcal{H}_P} \right). \quad (17)$$

This implies that $T_\lambda(1 - G_\lambda)$ is infinitesimally bounded relative to (16), by boundedness of $(1 - G_\lambda^*)^{-1}$. Taking $\varepsilon < 1/4$ it follows from Lemma A.1b) and Lemma A.1c) that $T_\lambda G_\lambda$ is a bounded operator on $D(\text{d}\Gamma(\omega)^{1/2})$. Since $(1 - G_\lambda)$ is an isomorphism on $D(\text{d}\Gamma(\omega)^{1/2})$ by Lemma A.1d), the operator $T_\lambda G_\lambda$ is bounded relative to $(1 - G_\lambda)^* \text{d}\Gamma(\omega)^{1/2} (1 - G_\lambda)$ and thus infinitesimally bounded relative to (16).

This implies self-adjointness and the lower bound. The equality $H_P = H_{\text{ren}}(P)$ follows from the equality of the translation-invariant operators [LS19, Thm.1.4] by unitary transformation (see [Can71, Thm.5.2]).

The method of proof also yields a formula for $(H_P + \mu)^{-1}$ for large enough μ by the proof of the Kato-Rellich theorem (see also [Pos20] for alternative representations of the resolvent). In order to show positivity of the resolvent we will basically follow this proof with some modifications that give control over the sign of certain terms. In particular, the arguments of Section 2.2 reprove self-adjointness of H_P in a slightly different way.

Remark 1.2. The results of [LS19, Sch20] apply to general Nelson-type models in $d \leq 3$ dimensions with v, ω satisfying $|v(k)| \leq |k|^{-\alpha}$, $\omega(k) \geq (m^2 + k^2)^{\beta/2}$ with appropriate conditions on α, β (see also [Sch19] for models with relativistic particles). Our method also works for these more general models under the additional hypothesis that $\beta/3 > d - 2 - 2\alpha$ (which ensures that $T_{d,\lambda} G_\lambda$ is bounded).

2 Positivity of the resolvent of the Nelson Hamiltonian

In this section we formulate and prove our main result.

Definition 2.1. We define the cone of positive elements $\mathcal{C}_+ \subset \mathcal{H}_P$ by

$$\mathcal{C}_+ := \{ \psi \in \mathcal{H}_P \mid \forall n \in \mathbb{N}_0 : \psi^{(n)}(k_1, \dots, k_n) \geq 0 \text{ a.e.} \}. \quad (18)$$

We write that $\psi \geq 0$ iff $\psi \in \mathcal{C}_+$ and $\psi > 0$ iff $\langle \psi, \varphi \rangle > 0$ for all $\varphi \geq 0$. An operator $A \in \mathcal{B}(\mathcal{H}_P)$ preserves positivity if $A\mathcal{C}_+ \subset \mathcal{C}_+$, and improves positivity if $A\psi > 0$ for all $0 \neq \psi \in \mathcal{C}_+$.

We will show that the resolvent of H_P improves positivity if $g < 0$, that is if $v(k)$ is strictly negative.

If instead $g > 0$ and v is positive all our results hold instead with positivity defined by the cone

$$\mathcal{C}_- := \{\psi \in \mathcal{H}_P \mid \forall n \in \mathbb{N}_0 : (-1)^n \psi^{(n)}(k_1, \dots, k_n) \geq 0 \text{ a.e.}\}, \quad (19)$$

since the operators with interactions v and $-v$ are unitarily equivalent via $U = \Gamma(-1)$.

Theorem 2.2. *Let $P \in \mathbb{R}^3$, $m \geq 0$ and $g < 0$. Then for all $\lambda > -\inf \sigma(H_P)$ the resolvent $(H_P + \lambda)^{-1}$ improves positivity with respect to \mathcal{C}_+ .*

To see why this might hold, first note that $(L_P + \lambda)^{-1}$ preserves positivity since it is obtained from a multiplication operator by a positive function. Using the sign of v , one easily sees that $G_\lambda, (1 - G_\lambda)^{-1} = \sum_{j=0}^{\infty} G_\lambda^j$ (for $\lambda \gg 1$) and their adjoints preserve positivity by inspection of the formula (7).

With this, the inverse of

$$(1 - G_\lambda)^*(L_P + \lambda)(1 - G_\lambda) \quad (20)$$

preserves positivity, and it is not difficult to show that it improves positivity (see Lemma 2.7). In the formula (9) for $H_P + \lambda$, this is perturbed by the operator T_λ . The off-diagonal part $T_{\text{od},\lambda}$ is an integral operator with negative kernel, so $-T_{\text{od},\lambda}$ preserves positivity. This can be dealt with by a perturbative argument due to Faris [Far72], the essential point of which is that the Neumann series $(1 + A)^{-1} = \sum_{j=0}^{\infty} (-A)^j$ is positivity-preserving if $-A$ is. However, the diagonal part $T_{\text{d},\lambda}$ is, for fixed particle number, a multiplication operator by a function that does take positive values.

Remark 2.3. If, in deviating from our hypothesis,

$$\int_{\mathbb{R}^3} \frac{|v(k)|^2}{k^2 + \omega(k)} dk < \infty, \quad (21)$$

as is the case for the H. Fröhlich's polaron model where $\omega \equiv 1$ and $v(k) \propto 1/k$, the formal Hamiltonian makes sense as a quadratic form. Self-adjointness can be proved as in [LS19, Sect.2], and the operator T_λ is simply given by

$$T_\lambda = -a(v)(L_P + \lambda)^{-1}a^*(v). \quad (22)$$

Hence $-T_\lambda$ preserves positivity and the proof that the resolvent of H_P improves positivity is rather straightforward starting from there.

2.1 A modified representation of H_P

We will deal with the positive part of $T_{d,\lambda}$ by absorbing it with L_P and modifying the representation of H_P . A similar idea was used in the renormalisation of more singular Hamiltonians of Nelson type [Lam19, Lam20].

For arbitrary $n \in \mathbb{N}_0$ (which we suppress in the notation) and $K \in \mathbb{R}^{3n}$ let

$$\tau_{+,\lambda}(K) := \left(\int_{\mathbb{R}^3} |v(\xi)|^2 \left(\frac{1}{\xi^2 + \omega(\xi)} - \frac{1}{L_P(K, \xi) + \lambda} \right) d\xi \right)_+ \quad (23)$$

denote the positive part of $T_{d,\lambda}(K)$. By scaling (see Lemma A.2 for details) we have

$$\tau_{+,\lambda}(K) \leq C(L_P(K) + \lambda)^\varepsilon \quad (24)$$

for any $\varepsilon > 0$ and some C . Thus for every $\lambda > 0$ and $\varepsilon > 0$, $\tau_{+,\lambda}(K)$ defines a bounded operator from $D(L_P^\varepsilon)$ to \mathcal{H}_P . We denote this operator by $\tau_{+,\lambda}$ and define $\tau_{-,\lambda}$ as

$$\tau_{-,\lambda} := T_{d,\lambda} - \tau_{+,\lambda} \leq 0. \quad (25)$$

For $\lambda > 0$ we now define F_λ , a modification of G_λ , as the adjoint of

$$F_\lambda^* = -a(v)(L_P + \tau_{+,\lambda} + \lambda)^{-1}. \quad (26)$$

Lemma 2.4. *The family of operators F_λ has the following properties:*

- a) F_λ is bounded;
- b) $\text{ran } F_\lambda \subset D(L_P^s)$ for all $0 \leq s < 1/4$, and for all $\lambda_0 > 0$

$$\sup_{\lambda \geq \lambda_0} \|(L_P + \lambda)^s F_\lambda\|_{\mathcal{B}(\mathcal{H}_P)} < \infty.$$

- c) F_λ maps $D(d\Gamma(\omega)^{1/2})$ to itself and there exists $C > 0$ so that for all $\lambda > 0$ and $\psi \in D(d\Gamma(\omega)^{1/2})$

$$\|d\Gamma(\omega)^{1/2} F_\lambda \psi\|_{\mathcal{H}_P} \leq C \lambda^{-1/4} \|d\Gamma(\omega)^{1/2} \psi\|_{\mathcal{H}_P};$$

- d) There exists $\lambda_0 > 0$ so that for $\lambda > \lambda_0$, $1 - F_\lambda$ is invertible on \mathcal{H}_P and $D(d\Gamma(\omega)^{1/2})$ with

$$\sup_{\lambda > \lambda_0} \left(\|(1 - F_\lambda)^{-1}\|_{\mathcal{B}(\mathcal{H}_P)} + \|(1 - F_\lambda)^{-1}\|_{\mathcal{B}(D(d\Gamma(\omega)^{1/2}))} \right) < \infty$$

- e) $D(H_P) = (1 - F_\lambda)^{-1} D(L_P)$.

Proof. Statements a)–c) are proved by reduction to the corresponding properties of G_λ . By (24) we have for $\psi \in \mathcal{H}_P$

$$\left\| \tau_{+,\lambda}(L_P + \tau_{+,\lambda} + \lambda)^{-1}\psi \right\| \leq C \left\| (L_P + \lambda)^{-1+\varepsilon}\psi \right\|. \quad (27)$$

By the resolvent formula this implies that

$$F_\lambda^* = G_\lambda^* \left(1 - \tau_{+,\lambda}(L_P + \tau_{+,\lambda} + \lambda)^{-1} \right) \quad (28)$$

is bounded, by boundedness of G_λ (see Lemma A.1a)). The difference of the adjoints is then

$$G_\lambda - F_\lambda = (L_P + \tau_{+,\lambda} + \lambda)^{-1} \tau_{+,\lambda} G_\lambda. \quad (29)$$

By Lemma A.1b), $(L_P + \lambda)^s G_\lambda$ is bounded, uniformly in λ . Thus by taking $\varepsilon = s < 1/4$ in (24), $\tau_{+,\lambda} G_\lambda$ is bounded. This implies b) and c) since $L_P + \tau_+ \geq L_P$ and thus $(L_P + \lambda)(G_\lambda - F_\lambda)$ is a bounded operator, uniformly in λ .

Statement d) follows from b) and c) by Neumann series, as in Lemma A.1d).

Finally, e) follows from the fact that $\text{ran}(G_\lambda - F_\lambda) \subset D(L_P)$ proved above. \square

Proposition 2.5. *For every $\lambda > 0$ we have the identity*

$$H_P = (1 - F_\lambda)^*(L_P + \tau_{+,\lambda} + \lambda)(1 - F_\lambda) + S_\lambda - \lambda$$

with $S_\lambda = S_{d,\lambda} + S_{od,\lambda}$, $D(S_\lambda) = D(T_\lambda)$, given by

$$\begin{aligned} & \left(S_{d,\lambda} \psi^{(n)} \right) (K) \\ &= \psi^{(n)}(K) \left(\tau_{-,\lambda}(K) + \int_{\mathbb{R}^3} \frac{|v(\xi)|^2 \tau_{+,\lambda}(K, \xi)}{(L_P(K, \xi) + \lambda)(L_P(K, \xi) + \tau_{+,\lambda}(K, \xi) + \lambda)} d\xi \right) \end{aligned}$$

and

$$\left(S_{od,\lambda} \psi^{(n)} \right) (K) = - \sum_{j=1}^n \int_{\mathbb{R}^3} \frac{\overline{v(\xi)} v(k_j) \psi_P^{(n)}(\hat{K}_j, \xi)}{L_P(K, \xi) + \tau_{+,\lambda}(K, \xi) + \lambda} d\xi.$$

Proof. We can rewrite the expression for $S_{od,\lambda}$ as

$$\begin{aligned} & \left(S_{od,\lambda} \psi^{(n)} \right) (K) \\ &= \left(T_{od,\lambda} \psi^{(n)} \right) (K) + \sum_{j=1}^n \int_{\mathbb{R}^3} \frac{\overline{v(\xi)} v(k_j) \tau_{+,\lambda}(K, \xi) \psi_P^{(n)}(\hat{K}_j, \xi)}{(L_P(K, \xi) + \lambda)(L_P(K, \xi) + \tau_{+,\lambda}(K, \xi) + \lambda)} d\xi. \end{aligned} \quad (30)$$

Putting the second term together with the second term of $S_{d,\lambda}$ yields the identity

$$\begin{aligned} S_\lambda &= T_{\text{od},\lambda} + \tau_{-,\lambda} + a(v)(L_P + \tau_{+,\lambda} + \lambda)^{-1} \tau_{+,\lambda} (L_P + \lambda)^{-1} a^*(v) \\ &= T_{\text{od},\lambda} + \tau_{-,\lambda} + F_\lambda^* \tau_{+,\lambda} G_\lambda, \end{aligned} \quad (31)$$

where the last expression defines a bounded operator on \mathcal{H}_P by the argument of Lemma 2.4. Now using this,

$$\begin{aligned} H_P &= (1 - G_\lambda)^*(L_P + \lambda)(1 - G_\lambda) + T_\lambda - \lambda \\ &= (1 - G_\lambda)^*(L_P + \lambda)(1 - G_\lambda) + \tau_{+,\lambda} + S_\lambda - F_\lambda^* \tau_{+,\lambda} G_\lambda - \lambda \\ &= (1 - G_\lambda)^*(L_P + \tau_{+,\lambda} + \lambda)(1 - G_\lambda) + S_\lambda - \lambda \\ &\quad + (1 - G_\lambda^*) \tau_{+,\lambda} G_\lambda + G_\lambda^* \tau_{+,\lambda} - F_\lambda^* \tau_{+,\lambda} G_\lambda. \end{aligned} \quad (32)$$

By (29) we have $\tau_{+,\lambda} G_\lambda = (L_P + \tau_{+,\lambda} + \lambda)(G_\lambda - F_\lambda)$, so we can rewrite the last line as

$$\begin{aligned} &G_\lambda^* \tau_{+,\lambda} + (1 - G_\lambda^*) \tau_{+,\lambda} G_\lambda - F_\lambda^* \tau_{+,\lambda} G_\lambda \\ &= G_\lambda^* \tau_{+,\lambda} (1 - F_\lambda) + (1 - F_\lambda^*) \tau_{+,\lambda} G_\lambda - G_\lambda^* \tau_{+,\lambda} (G_\lambda - F_\lambda) \\ &= (G_\lambda - F_\lambda)^*(L_P + \tau_{+,\lambda} + \lambda)(1 - F_\lambda) + (1 - F_\lambda^*)(L_P + \tau_{+,\lambda} + \lambda)(G_\lambda - F_\lambda) \\ &\quad - (G_\lambda - F_\lambda)^*(L_P + \tau_{+,\lambda} + \lambda)(G_\lambda - F_\lambda) \\ &= (1 - F_\lambda^*)(L_P + \tau_{+,\lambda} + \lambda)(1 - F_\lambda) - (1 - G_\lambda)^*(L_P + \tau_{+,\lambda} + \lambda)(1 - G_\lambda). \end{aligned} \quad (33)$$

This proves the identity as claimed. \square

2.2 Proof of positivity

Lemma 2.6. *For any $\lambda > 0$ the operator $(L_P + \tau_{+,\lambda} + \lambda)^{-1}$ preserves positivity.*

Proof. The operator preserves particle number and its restriction to $\mathcal{H}_P^{(n)}$ is explicitly given as multiplication by a positive function, so it preserves positivity. \square

Lemma 2.7. *Assume that $g < 0$ and let λ_0 as in 2.4d). Then, for all $\lambda > \lambda_0$ the operator*

$$\begin{aligned} R_0(\lambda) &:= \left((1 - F_\lambda)^*(L_P + \tau_{+,\lambda} + \lambda)(1 - F_\lambda) \right)^{-1} \\ &= (1 - F_\lambda)^{-1} (L_P + \tau_{+,\lambda} + \lambda)^{-1} (1 - F_\lambda^*)^{-1} \end{aligned}$$

is positivity-improving with respect to \mathcal{C}_+ .

Proof. We need to show that for all non-zero $\varphi, \psi \geq 0$ we have

$$\left\langle (1 - F_\lambda^*)^{-1}\varphi, (L_P + \tau_{+,\lambda} + \lambda)^{-1}(1 - F_\lambda^*)^{-1}\psi \right\rangle > 0. \quad (34)$$

Let $n \geq 0$ be such that $\varphi^{(n+1)} \neq 0$. From the formula

$$F_\lambda^* \varphi^{(n+1)} = \sqrt{n+1} \int_{\mathbb{R}^3} \frac{-v(\xi)}{L_P(K, \xi) + \tau_{+,\lambda}(K, \xi) + \lambda} \varphi^{(n+1)}(K, \xi) d\xi \quad (35)$$

we see that $F_\lambda^* \varphi^{(n+1)} \neq 0$, since the first factor of the integrand is strictly positive. By induction then $(F_\lambda^*)^{n+1} \varphi^{(n+1)} \neq 0$. Since $(1 - F_\lambda^*)^{-1} = \sum_{j=1}^{\infty} (F_\lambda^*)^j$ (by choice of λ_0), this implies that

$$\left\langle (1 - F_\lambda^*)^{-1}\varphi, \varnothing \right\rangle > 0, \quad (36)$$

where $\varnothing \in \mathcal{H}_P$ denotes the vacuum vector. This proves (34), since the same applies to $(1 - F_\lambda^*)^{-1}\psi$ and the restriction of $(L_P + \tau_{+,\lambda} + \lambda)^{-1}$ to the vacuum sector is a strictly positive number, so both arguments of the scalar product have non-zero overlap with the vacuum. \square

To complete the proof of Theorem 2.2, we will use a perturbative argument for S_λ . Since $S_{\text{od},\lambda}$ is an integral operator with negative kernel, the key point is that now $S_{\text{d},\lambda}$ is essentially negative.

Lemma 2.8. *For all $\lambda_0 > 0$ there exists μ_0 such that for $\mu > \mu_0$ and every $\lambda \geq \lambda_0$ the operator $-(S_\lambda - \mu)$ is positivity-preserving.*

Proof. The operator $S_{\text{od},\lambda}$ is an integral operator with negative kernel, hence $-S_{\text{od},\lambda}$ is positivity-preserving. On the n -particle sector, the operator $S_{\text{d},\lambda}$ is a multiplication operator by the sum of $\tau_{-,\lambda}(K)$, which is non-positive, and a non-negative function. We will need that this function is bounded (uniformly in $\lambda \geq \lambda_0$ and n) and then choose

$$\mu_0 = \sup_{\lambda \geq \lambda_0, n \in \mathbb{N}_0, K \in \mathbb{R}^{dn}} S_{\text{d},\lambda}(K), \quad (37)$$

whence $-S_{\text{d},\lambda}(K) + \mu \geq 0$ a.e.. For this, first use (24) and then the Hardy-Littlewood rearrangement inequality, to obtain for $\lambda \geq \lambda_0$

$$\begin{aligned} S_{\text{d},\lambda}(K) - \tau_{-,\lambda} &= \int_{\mathbb{R}^3} \frac{|v(\xi)|^2 \tau_{+,\lambda}(K, \xi)}{(L_P(K, \xi) + \lambda)(L_P(K, \xi) + \tau_{+,\lambda}(K, \xi) + \lambda)} d\xi \\ &\leq \int_{\mathbb{R}^3} \frac{1}{|\xi|((P - \sum_{j=1}^n k_j - \xi)^2 + \lambda_0)^2} d\xi \\ &\leq \int_{\mathbb{R}^3} \frac{1}{|\xi|(\xi^2 + \lambda_0)^2} d\xi, \end{aligned} \quad (38)$$

which yields the claim. \square

Proof of Theorem 2.2. We start by deriving a formula for the resolvent of H_P for sufficiently large λ . Let $\mu > \mu_0$ as in Lemma 2.8 and $\lambda \geq \max\{\mu, \lambda_0\}$, $R_0(\lambda)$ as in Lemma 2.7. Then, by the representation of Proposition 2.5, we have

$$(H_P + \lambda - \mu)R_0(\lambda) = 1 + (S_\lambda - \mu)R_0(\lambda). \quad (39)$$

We now prove that this is invertible by a Neumann series for sufficiently large λ .

By Lemma A.2 and Lemma 2.8

$$\begin{aligned} & \|(S_{d,\lambda} - \mu)R_0(\lambda)\|_{\mathcal{B}(\mathcal{H}_P)} \\ & \leq \|\tau_{-, \lambda} R_0(\lambda)\|_{\mathcal{B}(\mathcal{H}_P)} + \|(S_{d,\lambda} - \tau_{-, \lambda} - \mu)R_0(\lambda)\|_{\mathcal{B}(\mathcal{H}_P)} \\ & \leq C\|(L_P + \lambda)^\varepsilon R_0(\lambda)\|_{\mathcal{B}(\mathcal{H}_P)} + \mu\|R_0(\lambda)\|_{\mathcal{B}(\mathcal{H}_P)}. \end{aligned} \quad (40)$$

Using that $(1 - F_\lambda)^{-1} = 1 + F_\lambda(1 - F_\lambda)^{-1}$, we obtain for $\varepsilon < 1/4$ by Lemma 2.4

$$\begin{aligned} (40) & \leq C\|(1 - F_\lambda^*)^{-1}\|_{\mathcal{B}(\mathcal{H}_P)} \left(\|(L_P + \lambda)^{-1+\varepsilon}\|_{\mathcal{B}(\mathcal{H}_P)} \right. \\ & \quad \left. + \|(L_P + \lambda)^\varepsilon F_\lambda(1 - F_\lambda)^{-1}(L_P + \lambda)^{-1}\|_{\mathcal{B}(\mathcal{H}_P)} \right) + \mu\|R_0(\lambda)\|_{\mathcal{B}(\mathcal{H}_P)} \\ & \leq C\lambda^{\varepsilon-1}. \end{aligned} \quad (41)$$

By Lemma A.4 we have for the off-diagonal part

$$\|S_{\text{od}, \lambda} R_0(\lambda)\|_{\mathcal{B}(\mathcal{H}_P)} \leq C\|d\Gamma(\omega)^{1/2} R_0(\lambda)\|_{\mathcal{B}(\mathcal{H}_P)}. \quad (42)$$

Using Lemma 2.4d) we obtain

$$\begin{aligned} & \|d\Gamma(\omega)^{1/2} R_0(\lambda)\|_{\mathcal{B}(\mathcal{H}_P)} \\ & \leq \|(1 - F_\lambda)^{-1}\|_{\mathcal{B}(d\Gamma(\omega)^{1/2})} \|(L_P + \lambda)^{-1}\|_{\mathcal{B}(\mathcal{H}_P, d\Gamma(\omega)^{1/2})} \|(1 - F_\lambda^*)^{-1}\|_{\mathcal{B}(\mathcal{H}_P)} \\ & \leq C\lambda^{-1/2}. \end{aligned} \quad (43)$$

Altogether, find that

$$\|(S_\lambda - \mu)R_0(\lambda)\| < C\lambda^{-1/2}, \quad (44)$$

and it follows that for large enough $\lambda \geq \max\{\mu, \lambda_0\}$

$$\begin{aligned} (H_P + \lambda - \mu)^{-1} & = R_0(\lambda)(1 + (S_\lambda - \mu)R_0(\lambda))^{-1} \\ & = R_0(\lambda) \sum_{j=0}^{\infty} \left(-(S_\lambda - \mu)R_0(\lambda) \right)^j. \end{aligned} \quad (45)$$

By Lemma 2.8, the sum defines a positivity-preserving operator, which is one-to-one since it is invertible. As $R_0(\lambda)$ improves positivity by Lemma 2.7, we have for every $0 \neq \psi \in \mathcal{C}_+$

$$R_0(\lambda) \underbrace{\left(1 + (S_\lambda - \mu)R_0(\lambda)\right)^{-1}}_{\in \mathcal{C}_+ \setminus \{0\}} \psi > 0. \quad (46)$$

This proves the claim for all $\lambda > \lambda_1$, for some $\lambda_1 > 0$. The property extends to all $\lambda > -\inf \sigma(H_P)$ since, for $\gamma > \lambda_1 \geq \lambda > -\inf \sigma(H_P)$, $(\gamma - \lambda)(H_P + \gamma)^{-1}$ has norm less than one, and thus

$$\begin{aligned} (H_P + \lambda)^{-1} &= (H_P + \gamma)^{-1} \left(1 - (\gamma - \lambda)(H_P + \gamma)^{-1}\right)^{-1} \\ &= (H_P + \gamma)^{-1} \sum_{j=0}^{\infty} \left((\gamma - \lambda)(H_P + \gamma)^{-1}\right)^j \end{aligned} \quad (47)$$

improves positivity. \square

A Technical Lemmas

Here we reprove the key Lemmas of [LS19, Sch20] for the special case of the (massive or massless) Nelson model at fixed momentum.

Lemma A.1. *The family of operators G_λ has the following properties:*

- a) *For every $\lambda > 0$, the operator G_λ is bounded;*
- b) *$\text{ran } G_\lambda \subset D(L_P^s)$ for any $0 \leq s < 1/4$, and for all $\lambda_0 > 0$*

$$\sup_{\lambda \geq \lambda_0} \|(L_P + \lambda)^s G_\lambda\|_{\mathcal{B}(\mathcal{H}_P)} < \infty;$$

- c) *G_λ maps $D(d\Gamma(\omega)^{1/2})$ to itself and there exists $C > 0$ so that for all $\lambda > 0$ and $\psi \in D(d\Gamma(\omega)^{1/2})$*

$$\|d\Gamma(\omega)^{1/2} G_\lambda \psi\|_{\mathcal{H}_P} \leq C \lambda^{-1/4} \|d\Gamma(\omega)^{1/2} \psi\|_{\mathcal{H}_P};$$

- d) *There exists λ_0 so that for all $\lambda > \lambda_0$ the operator $1 - G_\lambda$ is boundedly invertible on \mathcal{H}_P and $D(d\Gamma(\omega)^{1/2})$ and*

$$\sup_{\lambda > \lambda_0} \left(\|(1 - G_\lambda)^{-1}\|_{\mathcal{B}(\mathcal{H}_P)} + \|(1 - G_\lambda)^{-1}\|_{\mathcal{B}(D(d\Gamma(\omega)^{1/2}))} \right) < \infty.$$

Proof. For a) and b) it is sufficient to prove that

$$-a(v)(L_P + \lambda)^{s-1} \quad (48)$$

defines a bounded operator on \mathcal{H}_P , uniformly in λ . To prove this, we insert a factor of $\sqrt{\omega(\xi)/\omega(\eta)}$ and its inverse to obtain for $n \geq 1$

$$\begin{aligned} & \|a(v)(L_P + \lambda)^{s-1}\psi^{(n)}\|^2 \\ &= n \int_{\mathbb{R}^{3(n-1)}} dQ \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} d\eta \frac{v(\eta)\psi^{(n)}(Q, \xi)\omega(\xi)^{1/2}}{(L_P(Q, \eta) + \lambda)^{1-s}\omega(\eta)^{1/2}} \frac{v(\xi)\bar{\psi}^{(n)}(Q, \eta)\omega(\eta)^{1/2}}{(L_P(Q, \xi) + \lambda)^{1-s}\omega(\xi)^{1/2}} \\ &\leq 2n \int_{\mathbb{R}^{3(n-1)}} dQ \int_{\mathbb{R}^3} d\xi \frac{\omega(\xi)|\psi^{(n)}(Q, \xi)|^2}{L_P(Q, \xi) + \lambda} \int_{\mathbb{R}^3} d\eta \frac{|v(\eta)|^2}{(L_P(Q, \eta) + \lambda)^{1-2s}\omega(\eta)}. \end{aligned} \quad (49)$$

By the Hardy-Littlewood rearrangement inequality we have for $s < 1/4$

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|v(\eta)|^2 d\eta}{(L_P(Q, \eta) + \lambda)^{1-2s}\omega(\eta)} &\leq \int_{\mathbb{R}^3} \frac{d\eta}{(\eta^2 + m^2)((P - \eta - \sum_{j=1}^{n-1} k_j)^2 + \lambda)^{2-2s}} \\ &\leq \int_{\mathbb{R}^3} \frac{d\tau}{|\tau|^2(\tau^2 + \lambda)^{1-2s}}, \end{aligned} \quad (50)$$

which is uniformly bounded for $\lambda \geq \lambda_0$. Together with (49) and the symmetry of $\psi^{(n)}$ this gives

$$\begin{aligned} \|a(v)(L_P + \lambda)^{s-1}\psi^{(n)}\|^2 &\leq nC \int_{\mathbb{R}^{3(n-1)}} dQ \int_{\mathbb{R}^3} d\xi \frac{\omega(\xi)|\psi^{(n)}(Q, \xi)|^2}{L_P(Q, \xi) + \lambda} \\ &= C \int_{\mathbb{R}^{3n}} dK \frac{\sum_{j=1}^n \omega(k_j)|\psi^{(n)}(K)|^2}{L_P(K) + \lambda} \\ &\leq C\|\psi^{(n)}\|_{\mathcal{H}_P}^2. \end{aligned} \quad (51)$$

To prove c), we proceed as in (49) to obtain (denoting $\Omega(K) = \sum_{j=1}^n \omega(k_j)$)

$$\begin{aligned} & \|a(v)(L_P + \lambda)^{-1}d\Gamma(\omega)^{1/2}\psi^{(n)}\|^2 \\ &\leq 2n \int_{\mathbb{R}^{3(n-1)}} dQ \int_{\mathbb{R}^3} d\xi \omega(\xi)|\psi^{(n)}(Q, \xi)|^2 \int_{\mathbb{R}^3} d\eta \frac{\Omega(Q, \eta)|v(\eta)|^2}{(L_P(Q, \eta) + \lambda)^2\omega(\eta)}. \end{aligned} \quad (52)$$

By scaling and rearrangement, we have

$$\int_{\mathbb{R}^3} \frac{|v(\eta)|^2 d\eta}{(L_P(Q, \eta) + \lambda)^2} \leq \lambda^{-1} \int_{\mathbb{R}^3} \frac{d\tau}{|\tau|(\tau^2 + 1)^2}, \quad (53)$$

and

$$\int_{\mathbb{R}^3} \frac{|v(\eta)|^2 d\eta}{(L_P(Q, \eta) + \lambda)^2 \omega(\eta)} \leq (\Omega(Q) + \lambda)^{-3/2} \int_{\mathbb{R}^3} \frac{d\tau}{|\tau|^2 (\tau^2 + 1)^2}. \quad (54)$$

Using symmetry as in (51) this implies

$$\|a(v)(L_P + \lambda)^{-1} d\Gamma(\omega)^{1/2} \psi^{(n)}\|^2 \leq C \lambda^{-1/2} \|d\Gamma(\omega)^{1/2} \psi^{(n)}\|^2, \quad (55)$$

and proves c) by taking adjoints.

To prove d) observe that b) and c) imply that

$$\|G_\lambda\|_{\mathcal{B}(\mathcal{H}_P)} + \|G_\lambda\|_{\mathcal{B}(D(d\Gamma(\omega)^{1/2}))} \leq C \lambda^{-s} \quad (56)$$

for $s < 1/4$. Thus for large enough λ the inverse of $1 - G_\lambda$ in both spaces exists and is given by the Neumann series, whose norm is bounded by $(1 - C\lambda_0^{-s})^{-1}$. \square

Lemma A.2. *For any $\varepsilon > 0$ there exists $C > 0$ such that for all $\lambda > 0$, $n \in \mathbb{N}_0$ and $K \in \mathbb{R}^{3n}$*

$$|T_{d,\lambda}(K)| \leq C(L_P(K) + \lambda)^\varepsilon.$$

Proof. We treat only the case $n > 0$, the case $n = 0$ being simpler. We have (with $\Omega(K) = \sum_{j=1}^n \omega(k_j)$)

$$\begin{aligned} |T_{d,\lambda}(K)| &= \left| \int_{\mathbb{R}^3} |v(\xi)|^2 \frac{(P - \sum_{j=1}^n k_j - \xi)^2 - \xi^2 + \Omega(K) + \lambda}{(\xi^2 + \omega(\xi))(L_P(K, \xi) + \lambda)} d\xi \right| \\ &\leq \int_{\mathbb{R}^3} |v(\xi)|^2 \frac{(P - \sum_{j=1}^n k_j)^2 + 2|\xi| |P - \sum_{j=1}^n k_j|}{(\xi^2 + \omega(\xi))(L_P(K, \xi) + \lambda)} d\xi \end{aligned} \quad (57)$$

$$+ \int_{\mathbb{R}^3} |v(\xi)|^2 \frac{\Omega(K) + \lambda}{(\xi^2 + \omega(\xi))(L_P(K, \xi) + \lambda)} d\xi. \quad (58)$$

To simplify the notation, we set $p := P - \sum_{j=1}^n k_j$. The first term in (57) and the term (58) are bounded by almost identical arguments, so we only give the details for one of them.

Scaling out $\sqrt{\Omega(K) + \lambda}$ in (58) and using that $\xi^2 + \omega(x) \geq |\xi|$ we obtain by rearrangement

$$\begin{aligned} (58) &\leq (\Omega(K) + \lambda)^\varepsilon \int_{\mathbb{R}^3} \frac{d\xi}{\xi^2 \left(\left(\frac{p}{\sqrt{\Omega(K) + \lambda}} - \xi \right)^2 + 1 \right)} \\ &\leq (\Omega(K) + \lambda)^\varepsilon \int_{\mathbb{R}^3} \frac{d\xi}{\xi^2 (\xi^2 + 1)}. \end{aligned} \quad (59)$$

For the second term in (57) we have

$$\begin{aligned} & \int_{\mathbb{R}^3} |v(\xi)|^2 \frac{2|\xi||p|}{(\xi^2 + \omega(\xi))((p - \xi)^2 + \Omega(k) + \omega(\xi) + \lambda)} d\xi \\ & \leq \int_{\mathbb{R}^3} \frac{2|p|}{\xi^2((p - \xi)^2 + \lambda)} d\xi. \end{aligned} \quad (60)$$

Scaling by $|p| \neq 0$ then yields

$$\int_{\mathbb{R}^3} \frac{2|p|}{\xi^2((p - \xi)^2 + \lambda)} d\xi \leq \int_{\mathbb{R}^3} \frac{2}{\xi^2(\frac{p}{|p|} - \xi)^2} d\xi = C, \quad (61)$$

where C is independent of p since the last integral is invariant by rotations. Combining these bounds proves the claim. \square

Lemma A.3 (cf. [Sch19, Lem.3.8]). *There is $C > 0$ so that the inequality*

$$\|T_{\text{od},\lambda}\psi^{(n)}\|_{\mathcal{H}_P^{(n)}} \leq C \|d\Gamma(\omega)^{1/2}\psi^{(n)}\|_{\mathcal{H}_P^{(n)}}$$

holds for all $\lambda > 0$ and $n \in \mathbb{N}$.

Proof. We may write

$$T_{\text{od},\lambda}\psi^{(n)}(K) = - \sum_{j=1}^n \int_{\mathbb{R}^3} \frac{v(k_j)\omega(\xi)^{1/2}\psi^{(n)}(\hat{K}_j, \xi)}{(L_P(K, \xi) + \lambda)^{1/2}\omega(k_j)^{1/2}} \frac{v(\xi)\omega(k_j)^{1/2}d\xi}{(L_P(K, \xi) + \lambda)^{1/2}\omega(\xi)^{1/2}}. \quad (62)$$

By the Cauchy-Schwarz inequality we obtain

$$|T_{\text{od},\lambda}\psi^{(n)}(K)|^2 \leq \left(\sum_{j=1}^n \int_{\mathbb{R}^3} \frac{|v(k_j)|^2\omega(\xi)|\psi^{(n)}(\hat{K}_j, \xi)|^2}{(L_P(K, \xi) + \lambda)\omega(k_j)} \right) \quad (63)$$

$$\times \left(\sum_{\ell=1}^n \int_{\mathbb{R}^3} \frac{|v(\eta)|^2\omega(k_\ell)}{(L_P(K, \eta) + \lambda)\omega(\eta)} d\eta \right). \quad (64)$$

The factor (64) is bounded by (writing $\Omega(K) = \sum_{j=1}^n \omega(k_j)$ and $p = P - \sum_{j=1}^n k_j$)

$$\begin{aligned} & \sum_{\ell=1}^n \int_{\mathbb{R}^3} \frac{|v(\eta)|^2\omega(k_\ell)}{(L_P(K, \eta) + \lambda)\omega(\eta)} d\eta \\ & \leq \sum_{\ell=1}^n \int_{\mathbb{R}^3} \omega(k_\ell)(\Omega(K) + \lambda)^{-1/2} \int_{\mathbb{R}^3} \frac{1}{\left(\left(\frac{p}{\sqrt{\Omega(K)+\lambda}} - \eta\right)^2 + 1\right)|\eta|^2} d\eta \\ & \leq \sum_{\ell=1}^n \int_{\mathbb{R}^3} \omega(k_\ell)\Omega(K)^{-1/2} \int_{\mathbb{R}^3} \frac{1}{(\eta^2 + 1)|\eta|^2} d\eta, \end{aligned} \quad (65)$$

where in the final step we have used the Hardy-Littlewood inequality on the integral.

We use this to estimate the integral of $|T_{\text{od},\lambda}\psi^{(n)}(K)|^2$ over K and obtain

$$\begin{aligned} & \|T_{\text{od},\lambda}\psi^{(n)}\|_{\mathcal{H}_P^{(n)}}^2 \\ & \leq C \sum_{j,\ell=1}^n \int_{\mathbb{R}^{3n}} \int_{\mathbb{R}^3} \frac{\omega(k_\ell)\Omega(K)^{-1/2}|v(k_j)|^2\omega(\xi)|\psi^{(n)}(\hat{K}_j,\xi)|^2}{(L_P(K,\xi)+\lambda)\omega(k_j)} d\xi dK \end{aligned} \quad (66)$$

By renaming the variables $k_j = \eta$, $\xi = k_j$ in the j -th integral, and using the symmetry of $\psi^{(n)}$, this becomes

$$\begin{aligned} & \|T_{\text{od},\lambda}\psi^{(n)}\|_{\mathcal{H}_P^{(n)}}^2 \\ & \leq \sum_{j \neq \ell=1}^n \int_{\mathbb{R}^{3n}} \int_{\mathbb{R}^3} \frac{\omega(k_\ell)\Omega(\hat{K}_j,\eta)^{-1/2}|v(\eta)|^2\omega(k_j)|\psi^{(n)}(K)|^2}{(L_P(K,\eta)+\lambda)\omega(\eta)} d\eta dK \end{aligned} \quad (67)$$

$$+ \sum_{j=1}^n \int_{\mathbb{R}^{3n}} \int_{\mathbb{R}^3} \frac{\Omega(\hat{K}_j,\eta)^{-1/2}|v(\eta)|^2\omega(k_j)|\psi^{(n)}(K)|^2}{(L_P(K,\eta)+\lambda)} d\eta dK \quad (68)$$

The first term is bounded by

$$\begin{aligned} (67) & \leq C \sum_{j=1}^n \int_{\mathbb{R}^{3n}} \int_{\mathbb{R}^3} \frac{\Omega(\hat{K}_j)^{1/2}|v(\eta)|^2\omega(k_j)|\psi^{(n)}(K)|^2}{(L_P(K,\eta)+\lambda)\omega(\eta)} d\eta dK \\ & \leq C \int_{\mathbb{R}^{3n}} \sum_{j=1}^n \omega(k_j)\Omega(K)^{-1/2}\Omega(\hat{K}_j)^{1/2}|\psi^{(n)}(K)|^2 dK \\ & \leq C \int_{\mathbb{R}^{3n}} \Omega(K)|\psi^{(n)}(K)|^2 dK, \end{aligned} \quad (69)$$

by the same scaling argument as before.

By the same reasoning, the second term satisfies the bound

$$\begin{aligned} (68) & \leq \sum_{j=1}^n \int_{\mathbb{R}^{3n}} \int_{\mathbb{R}^3} \frac{|v(\eta)|^2\omega(k_j)|\psi^{(n)}(K)|^2}{(L_P(K,\eta)+\lambda)\omega(\eta)^{1/2}} d\eta dK \\ & \leq C \int_{\mathbb{R}^{3n}} \Omega(K)|\psi^{(n)}(K)|^2 dK, \end{aligned} \quad (70)$$

and this proves the claim. \square

Lemma A.4. *There is $C > 0$ so that the inequality*

$$\|S_{\text{od},\lambda}\psi^{(n)}\|_{\mathcal{H}_P^{(n)}} \leq C \|\text{d}\Gamma(\omega)^{1/2}\psi^{(n)}\|_{\mathcal{H}_P^{(n)}}$$

holds for all $\lambda > 0$ and $n \in \mathbb{N}$.

Proof. As $\tau_{+,\lambda}(K) \geq 0$, the proof is identical to that of Lemma A.3. \square

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