

# Statics and vibrations in the post-buckling domain with the order-3 model

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## Résumé

We list here detailed analytical calculations for the post-buckled equilibrium solution (1st mode) and the first vibration mode around this equilibrium solution, using the Kirchhoff extensible order 3 model.

## 1 Introduction

The beam has length  $L$ , width  $w$ , thickness  $h$ , Young's modulus  $E$  and density  $\rho$ . The bending rigidity is then  $EI = Eh^3w/12$  and the stretching rigidity  $EA = Ehw$ , where  $A$  is the surface area and  $I$  the second moment of area of the cross-section. We consider slender beams, with a small ratio  $\eta = I/(AL^2) = (1/12)(h/L)^2$ . We use the arc-length  $S$  of the beam in its reference configuration as a Lagrangian variable, that is  $S \in (0, L)$  always. The beam is clamped horizontally at its left end ( $S = 0$ ), which lies at the origin. The right end ( $S = L$ ) is constrained to lie on the horizontal axis, with a horizontal tangent. An axial displacement  $D$  is imposed and we compute the equilibrium shape and the vibrations around this shape. We then study how equilibrium and vibrations vary as  $D$  is varied. Equations for this proposed order 3 model write

$$X'(S, T) = 1 + \frac{N_x(S, T)}{EA} - \frac{1}{2}Y'^2(S, T) \quad (1a)$$

$$N'_x(S, T) = \rho A \ddot{X}(S, T) \quad (1b)$$

$$EI \left[ Y(S, T)''' + F_3 \right] = N_x(S, T)Y'(S, T) - N_y(S, T)X'(S, T) \quad (1c)$$

$$N'_y(S, T) = \rho A \ddot{Y}(S, T) \quad (1d)$$

$$\text{with } F_3 = Y'(S, T)Y''(S, T)^2 + \frac{1}{2}Y'(S, T)^2Y'''(S, T) \quad (1e)$$

where  $T$  is the physical time and  $' \stackrel{\text{def}}{=} d/dS$  et  $\cdot \stackrel{\text{def}}{=} d/dT$ . We study the dynamics of the beam with fixed longitudinal displacement

$$X(0, T) = 0, \quad \forall T \quad (2)$$

$$X(L, T) = L - D, \quad \forall T \quad (3)$$

$$Y(0, T) = 0 = Y'(0, T), \quad \forall T \quad (4)$$

$$Y(L, T) = 0 = Y'(L, T), \quad \forall T \quad (5)$$

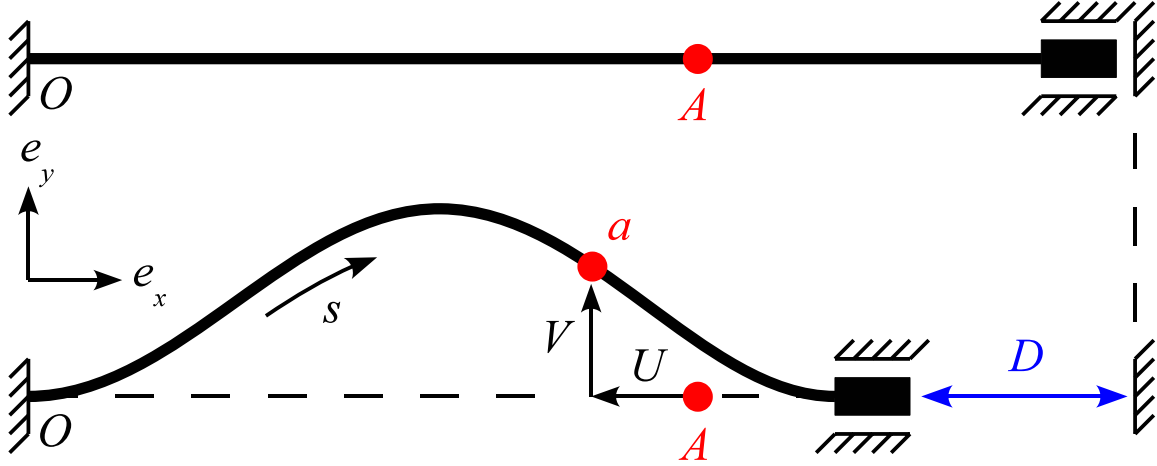


FIGURE 1 – Buckling of a clamped-clamped beam. Displacement are  $V = Y$  et  $U = X - S$ .

which implies that, during vibrations for example, the axial thrust is fluctuating in time  $P = P(T) = -N_x(L, T)$ .

### 1.1 Non-dimensionalisation

We work with adim variables

$$y = Y/L \quad , \quad x = x/L \quad , \quad \text{et} \quad s = S/L \quad (6)$$

$$n_{x,y} = N_{x,y} L^2 / (EI) \quad (7)$$

$$t = T/\tau \quad , \quad \text{with} \quad \tau = L^2 \sqrt{\rho A / (EI)} \quad (8)$$

and obtain the system

$$x'(s, t) = 1 + \eta n_x(s, t) - \frac{1}{2} y'^2(s, t) \quad (9a)$$

$$n'_x(s, t) = \ddot{x}(s, t) \quad (9b)$$

$$y(s, t)''' + f_3 = n_x(s, t) y'(s, t) - n_y(s, t) x'(s, t) \quad (9c)$$

$$n'_y(s, t) = \ddot{y}(s, t) \quad (9d)$$

$$\text{with} \quad f_3 = y'(s, t) y''(s, t)^2 + \frac{1}{2} y'(s, t)^2 y'''(s, t) \quad (9e)$$

where the small ratio  $\eta$  appears

$$\eta = \frac{I}{AL^2} = \frac{1}{12} \frac{h^2}{L^2} \quad (10)$$

Note : we also use  $u(s, t)$  which is defined as

$$u(s, t) = x(s, t) - s \quad (11)$$

## 2 Equations for equilibrium and vibrations

### 2.1 Equilibrium

Equilibrium equations read

$$x'_E = 1 + \eta n_{xE} - \frac{1}{2} y_E'^2 \quad (12a)$$

$$n'_{xE} = 0 \Rightarrow n_{xE} = -p_E \quad (12b)$$

$$y_E''' + f_{3E} = n_{xE} y'_E - n_{yE} x'_E \quad (12c)$$

$$n'_{yE} = 0 \Rightarrow n_{yE} = 0 \quad (12d)$$

$$\text{with } f_{3E} = y'_E y_E''^2 + \frac{1}{2} y_E'^2 y_E''' \quad (12e)$$

where we only consider the first buckling mode, which has  $n_{yE} = 0$ .

### 2.2 Small amplitude vibrations

We look for the solutions of (9) having the form

$$x(s, t) = x_E(s) + \delta x(s) e^{i\omega t} \text{ with } |\delta x(s)| \ll 1 \forall s \quad (13)$$

$$y(s, t) = y_E(s) + \delta y(s) e^{i\omega t} \text{ with } |\delta y(s)| \ll 1 \forall s \quad (14)$$

$$n_x(s, t) = n_{xE}(s) + \delta n_x(s) e^{i\omega t} \text{ with } |\delta n_x(s)| \ll 1 \forall s \quad (15)$$

$$n_y(s, t) = n_{yE}(s) + \delta n_y(s) e^{i\omega t} \text{ with } |\delta n_y(s)| \ll 1 \forall s \quad (16)$$

and we have

$$\delta u(s) = \delta x(s) \quad (17)$$

The equations for the vibrations then read

$$\delta x' = \eta \delta n_x - y'_E \delta y' \quad (18a)$$

$$\delta n'_x = -\omega^2 \delta x \quad (18b)$$

$$\delta y''' + \delta f_3 = \delta n_x y' + n_{xE} \delta y' - \delta n_y x' - n_{yE} \delta x' \quad (18c)$$

$$\delta n'_y = -\omega^2 \delta y \quad (18d)$$

$$\text{with } \delta f_3 = \delta y' y_E''^2 + 2y'_E \delta y'' y_E'' + \delta y' y'_E y_E''' + \frac{1}{2} y_E'^2 \delta y''' \quad (18e)$$

### 2.3 Conditions de bord

The tangent is horizontal at both clamps. Left end is fixed, right end is always on the  $y$  axis, with a imposed axial displacement  $d = D/L$ . For equilibrium we have then

$$y_E(0) = 0 \quad y_E(1) = 0 \quad (19a)$$

$$y'_E(0) = 0 \quad y'_E(1) = 0 \quad (19b)$$

$$x_E(0) = 0 \quad x_E(1) = 1 - d \quad (19c)$$

System (12) is order 6. We have 6 boundary conditions, we will then obtain isolated solutions for each given value of  $d$ . Boundary condition for vibrations read

$$\delta y(0) = 0 = \delta y(1) \quad (20a)$$

$$\delta y'(0) = 0 = \delta y'(1) \quad (20b)$$

$$\delta x(0) = 0 = \delta x(1) \quad (20c)$$

System (18) is order 6, and there is an additional parameter  $\omega$ . With the six above boundary conditions, we will then a priori have a 1D family of solutions. But as the system is linear, we may enforce a norm condition. We choose

$$\delta y''(0) = 1 \quad (21)$$

We finally have 7 conditions for 7 unknowns, and only isolated solution : the vibrations modes.

### 3 Post-buckled equilibrium

We solve equilibrium system (12) for compression forces values slightly above the buckling threshold :  $p \gtrsim 4\pi^2$ . We expand  $x_E(s)$ ,  $y_E(s)$ , et  $p_E = -n_{xE}$  in power of  $\epsilon$ , a small parameter giving the 'distance' from buckling

$$x_E(s) = x_0(s) + \epsilon x_1(s) + \epsilon^2 x_2(s) + \epsilon^3 x_3(s) + O(\epsilon^4) \quad (22a)$$

$$y_E(s) = \epsilon y_1(s) + \epsilon^2 y_2(s) + \epsilon^3 y_3(s) + O(\epsilon^4) \quad (22b)$$

$$p_E = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + O(\epsilon^4) \quad (22c)$$

At each  $\epsilon$  order, we extract and solve the equations for the variables  $x_i(s)$ ,  $y_i(s)$ ,  $\dots$ . Boundary conditions have to be verified at all orders. In order to fix a length scale for  $\epsilon$ , we use the norm condition :

$$\epsilon = -4 \int_0^1 y_E(s) \cos 2\pi s ds \quad (23)$$

#### 3.1 order $\epsilon^0$

Equations are

$$u_0'(s) = -\eta p_0 \quad \text{with} \quad u_0(0) = 0 \quad (24)$$

Solutions is

$$u_0(s) = -\eta p_0 s \quad (25)$$

#### 3.2 order $\epsilon^1$

Equations are

$$y_1'''' + p_0 y_1'' = 0 \quad \text{with} \quad y_1(0) = 0 = y_1(1) \quad \text{et} \quad y_1'(0) = 0 = y_1'(1) \quad (26)$$

$$u_1'(s) = -\eta p_1 \quad \text{with} \quad u_1(0) = 0 \quad (27)$$

With the norm condition (23) taken into account, solutions are

$$p_0 = 4\pi^2 \quad (28)$$

$$y_1(s) = \frac{1}{2}(1 - \cos 2\pi s) \quad (29)$$

$$u_1(s) = -\eta p_1 s \quad (30)$$

### 3.3 order $\epsilon^2$

Equations are

$$y_2'''' + 4\pi^2 y_2'' + 2\pi^2 p_1 \cos 2\pi s = 0 \text{ with } y_2(0) = y_2(1) = y_2'(0) = y_2'(1) = 0 \quad (31)$$

$$u_2'(s) = -\eta p_2 - \frac{\pi^2}{2} \sin^2 2\pi s \text{ with } u_2(0) = 0 \quad (32)$$

With the norm condition (23) taken into account, solutions are

$$p_1 = 0 \quad (33)$$

$$y_2(s) = 0 \quad (34)$$

$$u_2(s) = -\eta p_2 s - \frac{\pi^2}{4} s + \frac{\pi}{16} \sin 4\pi s \quad (35)$$

Up to now, the solution is the same as with the von Karman model. Things change at order 3.

### 3.4 order $\epsilon^3$

Equations are

$$y_3'''' + \left( y_3''^3 + 3y_3' y_3'' y_3''' + \frac{1}{2} y_3'^2 y_3'''' \right) + p_0 y_3'' + p_2 y_1'' = 0 \quad (36)$$

$$\text{with } y_3(0) = y_3(1) = y_3'(0) = y_3'(1) = 0 \quad (37)$$

$$u_3'(s) = -\eta p_3 \text{ with } u_3(0) = 0 \quad (38)$$

With the norm condition (23) taken into account, solutions are

$$p_2 = \frac{\pi^4}{4} \quad (39)$$

$$y_3(s) = \frac{\pi^2}{64} \sin^2 3\pi s \quad (40)$$

$$u_3(s) = -\eta p_3 s \quad (41)$$

### 3.5 order $\epsilon^4$

At this order, we find

$$p_3 = 0 \quad (42)$$

$$y_4(s) = 0 \quad (43)$$

$$u_4(s) = -\eta p_4 s - \frac{3\pi^3}{512} \sin 4\pi s + \frac{3\pi^3}{1024} \sin 8\pi s \quad (44)$$

### 3.6 Solution up to order $\epsilon^4$

We obtain

$$y_E(s) = \frac{\epsilon}{2} (1 - \cos 2\pi s) + \epsilon^3 \frac{\pi^2}{64} \sin^2 3\pi s + 0\epsilon^4 + O(\epsilon^5) \quad (45)$$

$$u_E(s) = -\eta 4\pi^2 s - \epsilon^2 \frac{\pi}{16} (4\pi s + 8\pi^3 \eta s - \sin 4\pi s) + u_4(s)\epsilon^4 + O(\epsilon^5) \quad (46)$$

$$p_E = 4\pi^2 + \epsilon^2 \frac{\pi^4}{2} + p_4\epsilon^4 + O(\epsilon^5) \quad (47)$$

where order 5 is needed to obtain  $p_4$ .

For the end-shortening we have

$$d = -u_E(1) = \eta 4\pi^2 + \epsilon^2 \frac{\pi^2}{4} (1 + 2\pi^2 \eta) + \eta p_4 \epsilon^4 + O(\epsilon^5) \quad (48)$$

And the mid span vertical deflection is

$$y_E(1/2) = \epsilon + \epsilon^3 \frac{\pi^2}{64} + 0\epsilon^4 + O(\epsilon^5) \quad (49)$$

Considering only the lowest  $\eta$  order, the solutions for  $y_1, y_3, u_0, u_2, p_0, p_2$  are the same as in the full (extensible) Kirchhoff model.

For  $u_4$  and  $p_4$ , we would need order  $\epsilon^5$ .

## 4 Vibrations around post-buckled equilibrium

We set :

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 + O(\epsilon^4) \quad (50)$$

$$\delta x(s) = \delta u(s) = \delta u_0 + \epsilon\delta u_1 + \epsilon^2\delta u_2 + \epsilon^3\delta u_3 + O(\epsilon^4) \quad (51)$$

$$\delta y(s) = \delta y_0 + \epsilon\delta y_1 + \epsilon^2\delta y_2 + \epsilon^3\delta y_3 + O(\epsilon^4) \quad (52)$$

$$\delta n_x(s) = \delta n_{x0} + \epsilon\delta n_{x1} + \epsilon^2\delta n_{x2} + \epsilon^3\delta n_{x3} + O(\epsilon^4) \quad (53)$$

$$\delta n_y(s) = \delta n_{y0} + \epsilon\delta n_{y1} + \epsilon^2\delta n_{y2} + \epsilon^3\delta n_{y3} + O(\epsilon^4) \quad (54)$$

order  $\epsilon^0$

Equations are

$$\delta u'_0 = \eta \delta n_{x0} \text{ with } \delta u_0(0) = \delta u_0(1) = 0 \quad (55a)$$

$$\delta n'_{x0} = 0 \quad (55b)$$

$$\delta n'_{y0} = 0 \quad (55c)$$

$$-(1 - 4\pi^2\eta)\delta n_{y0} = \delta y_0''' + 4\pi^2\delta y_0' \text{ with } \delta y_0(0) = \delta y_0(1) = \delta y_0'(0) = \delta y_0'(1) = 0 \quad (55d)$$

Using the norm condition (21), we obtain the solutions

$$\delta u_0(s) = 0 \quad (56a)$$

$$\delta n_{x0}(s) = 0 \quad (56b)$$

$$\delta n_{y0}(s) = 0 \quad (56c)$$

$$\delta y_0(s) = \frac{1 - \cos 2\pi s}{4\pi^2} \quad (56d)$$

**order  $\epsilon^1$**

Equations are

$$\delta u'_1 = \eta \delta n_{x1} - \frac{\sin^2 2\pi s}{2} \text{ with } \delta u_1(0) = \delta u_1(1) = 0 \quad (57a)$$

$$\delta n'_{x1} = 0 \quad (57b)$$

$$\delta n'_{y1} = 0 \quad (57c)$$

$$-(1 - 4\pi^2\eta)\delta n_{y1} = \delta y_1''' + 4\pi^2\delta y_1' \text{ with } \delta y_1(0) = \delta y_1(1) = \delta y_1'(0) = \delta y_1'(1) = 0 \quad (57d)$$

Using the norm condition (21), we obtain the solutions

$$\delta u_1(s) = \frac{\sin 4\pi s}{16\pi} \quad (58a)$$

$$\delta n_{x1}(s) = \frac{1}{4\eta} \quad (58b)$$

$$\delta n_{y1}(s) = 0 \quad (58c)$$

$$\delta y_1(s) = 0 \quad (58d)$$

**order  $\epsilon^2$**

Equations are

$$\delta u'_2 = \eta \delta n_{x2} \text{ with } \delta u_2(0) = \delta u_2(1) = 0 \quad (59a)$$

$$\delta n'_{x2} = 0 \quad (59b)$$

$$\delta n'_{y2} = -\frac{\omega_1^2 \sin^2 \pi s}{2\pi^2} \quad (59c)$$

$$\delta y_2''' + 4\pi^2\delta y_2' = -(1 - 4\pi^2\eta)\delta n_{y2} + c_1 \sin 2\pi s + c_2 \sin 6\pi s \quad (59d)$$

$$\text{with } \delta y_2(0) = \delta y_2(1) = \delta y_2'(0) = \delta y_2'(1) = 0 \quad (59e)$$

Using the norm condition (21), we obtain the solutions

$$\omega_1^2 = \frac{2\pi^4}{3\eta} \frac{1 + 2\pi^2\eta}{1 - 4\pi^2\eta} \quad (60a)$$

$$\delta u_2(s) = 0 \quad (60b)$$

$$\delta n_{x2}(s) = 0 \quad (60c)$$

$$\delta y_2(s) = 1 + s + s^2 \cos 2\pi s + \cos 6\pi s + \sin 2\pi s + s \sin 2\pi s \quad (60d)$$

$$\delta y_2(s) = \frac{1 + 2\pi^2(s-1)s - \cos 2\pi s + \pi(1-2s)\sin 2\pi s}{96\pi^2\eta} + O(\eta^0) \quad (60e)$$

order  $\epsilon^3$

Equations are

$$\delta u_3' = \eta \delta n_{x3} + \dots \text{ with } \delta u_1(0) = \delta u_1(1) = 0 \quad (61a)$$

$$\delta n_{x3}' = -\frac{\omega_1^2}{16\pi} \sin 4\pi s \quad (61b)$$

$$\delta n_{y3}' = -\frac{\omega_1 \omega_2 \sin^2 \pi s}{\pi^2} \quad (61c)$$

$$-(1 - 4\pi^2 \eta) \delta n_{y3} = \delta y_3''' + 4\pi^2 \delta y_3' \quad (61d)$$

$$\text{with } \delta y_2(0) = \delta y_2(1) = \delta y_2'(0) = \delta y_2'(1) = 0 \quad (61e)$$

Using the norm condition (21), we obtain the solutions

$$\omega_2 = 0 \quad (62a)$$

$$\delta u_3(s) = \frac{\sin \pi s}{384\pi\eta} (-7 \cos \pi s + \cos 3\pi s + 8\pi(1 - 2s) \sin^3 \pi s) + O(\eta^0) \quad (62b)$$

$$\delta n_{x3}(s) = cte + \frac{1}{64\pi^2} \omega_1^2 \cos 4\pi s \quad (62c)$$

$$\delta y_3(s) = 0 \quad (62d)$$

$$\delta n_{y3}(s) = 0 \quad (62e)$$

order  $\epsilon^4$

At this order  $p_4$  pops up. It would need to be calculated in the static solution.

### Conclusion for vibrations

At the lowest order in  $\eta$ , vibrations solutions at order 1, 2, and 3 are the same as in the solutions of the full Kirchhoff model, except for  $\delta u_3$ . A term involving  $y'^4$  would have been needed in (9a) to have the same solution for  $\delta u_3$ .