

Statics and vibrations in the post-buckling domain with the extensible Kirchhoff model

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Abstract

We list here detailed analytical calculations for the post-buckled equilibrium solution (1st mode) and the first vibration mode around this equilibrium solution, using the Kirchhoff extensible model.

1 Introduction

The beam has length L , width w , thickness h , Young's modulus E and density ρ . The bending rigidity is then $EI = Eh^3w/12$ and the stretching rigidity $EA = Ehw$, where A is the surface area and I the second moment of area of the cross-section. We consider slender beams, with a small ratio $\eta = I/(AL^2) = (1/12)(h/L)^2$. We use the arc-length S of the beam in its reference configuration as a Lagrangian variable, that is $S \in (0, L)$ always. The beam is clamped horizontally at its left end ($S = 0$), which lies at the origin. The right end ($S = L$) is constrained to lie on the horizontal axis, with a horizontal tangent. An axial displacement D is imposed and we compute the equilibrium shape and the vibrations around this shape. We then study how equilibrium and vibrations vary as D is varied.

Kirchhoff equations for the dynamics of the beam write

$$Y' = (1 + e) \sin \theta \quad , \quad X' = (1 + e) \cos \theta \quad (1)$$

$$N'_Y = \rho A \ddot{Y} \quad , \quad N'_X = \rho A \ddot{X} \quad (2)$$

$$EI\theta' = M \quad , \quad M' = (1 + e) (N_X \sin \theta - N_Y \cos \theta) \quad (3)$$

$$EAe = N_X \cos \theta + N_Y \sin \theta \quad (4)$$

where the internal force is $\mathbf{N} = (N_X, N_Y, 0)$, the bending moment is $\mathbf{M} = (0, 0, M)$. All variables depend on both S and time T with the derivatives noted $' \stackrel{\text{def}}{=} d/dS$ and $\dot{\cdot} \stackrel{\text{def}}{=} d/dT$. Rotation inertia is neglected. We use the following scheme to non-dimensionalize the equations

$$y = Y/L \quad , \quad x = X/L \quad (5)$$

$$\mathbf{n} = \mathbf{N}L^2/(EI) \quad , \quad m = ML/(EI) \quad (6)$$

$$t = T/\tau \quad , \quad \text{with } \tau = L^2 \sqrt{\rho A/(EI)} \quad (7)$$

Using (4), we isolate:

$$e = \eta(n_x \cos \theta + n_y \sin \theta) \quad (8)$$

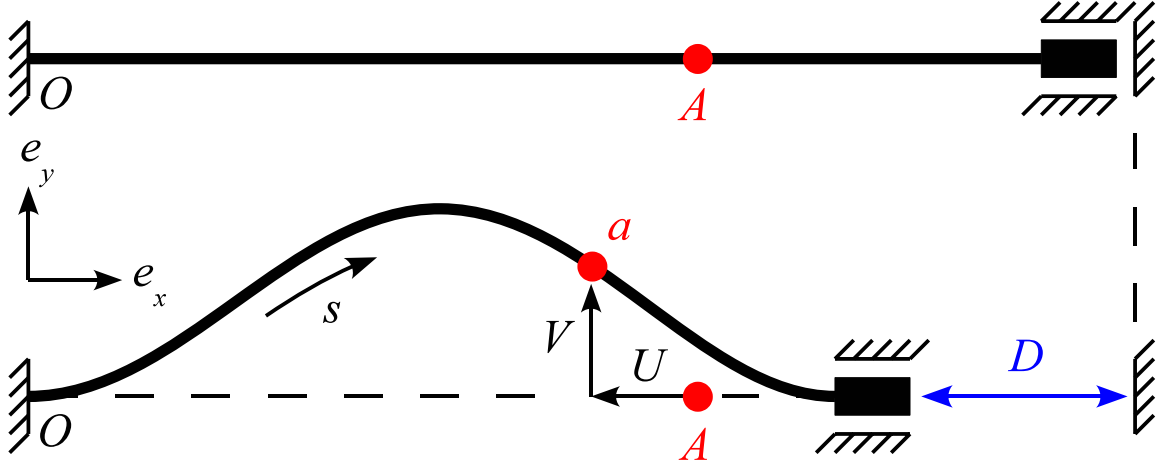


Figure 1: Buckling of a clamped-clamped beam. Displacement are $V = Y$ et $U = X - S$.

and we obtain

$$y' = (1 + \eta(n_x \cos \theta + n_y \sin \theta)) \sin \theta \quad (9a)$$

$$x' = (1 + \eta(n_x \cos \theta + n_y \sin \theta)) \cos \theta \quad (9b)$$

$$n'_y = \ddot{y} \quad (9c)$$

$$n'_x = \ddot{x} \quad (9d)$$

$$\theta' = m \quad (9e)$$

$$m' = (1 + \eta(n_x \cos \theta + n_y \sin \theta)) (n_x \sin \theta - n_y \cos \theta) \quad (9f)$$

where

$$\eta = \frac{I}{AL^2} \quad (10)$$

Boundary conditions read

$$y(s=0, t) = 0 \quad y(s=1, t) = 0 \quad \forall t \quad (11a)$$

$$x(s=0, t) = 0 \quad x(s=1, t) = 1 - d \quad \forall t \quad (11b)$$

$$\theta(s=0, t) = 0 \quad \theta(s=1, t) = 0 \quad \forall t \quad (11c)$$

2 Equilibrium and vibrations

2.1 Equilibrium

Equilibrium equations are :

$$y'_E = (1 + e_E) \sin \theta_E \quad , \quad x'_E = (1 + e_E) \cos \theta_E \quad (12)$$

$$n'_{yE} = 0 \quad , \quad n'_{xE} = 0 \quad (13)$$

$$\theta'_E = m_E \quad , \quad m'_E = (1 + e_E) (n_{xE} \sin \theta_E - n_{yE} \cos \theta_E) \quad (14)$$

$$e_E = \eta (n_{xE} \cos \theta_E + n_{yE} \sin \theta_E) \quad (15)$$

2.2 Small amplitude vibrations around equilibrium

Solutions of (9) having the form:

$$x(s, t) = x_E(s) + \delta x(s)e^{i\omega t} \text{ with } |\delta x(s)| \ll 1 \forall s \quad (16)$$

$$y(s, t) = y_E(s) + \delta y(s)e^{i\omega t} \text{ with } |\delta y(s)| \ll 1 \forall s \quad (17)$$

$$\dots \quad (18)$$

fulfil the vibration equations

$$\delta n'_y = -\omega^2 \delta y \quad (19a)$$

$$\delta n'_x = -\omega^2 \delta x \quad (19b)$$

$$\delta \theta' = \delta m \quad (19c)$$

$$\delta m' = (1 + e_E) [\delta n_x \sin \theta_E - \delta n_y \cos \theta_E + \delta \theta (n_{yE} \sin \theta_E + n_{xE} \cos \theta_E)] + \delta e (n_{xE} \sin \theta_E - n_{yE} \cos \theta_E) \quad (19d)$$

$$\delta y' = (1 + e_E) \cos \theta_E \delta \theta + \delta e \sin \theta_E \quad (19e)$$

$$\delta x' = -(1 + e_E) \sin \theta_E \delta \theta + \delta e \cos \theta_E \quad (19f)$$

where $e_E = \eta (n_{yE} \sin \theta_E + n_{xE} \cos \theta_E)$ and

$$\delta e = \eta (\delta n_y \sin \theta_E + \delta n_x \cos \theta_E + \delta \theta (n_{yE} \cos \theta_E - n_{xE} \sin \theta_E))$$

2.3 Boundary conditions

For equilibrium we have

$$y_E(0) = 0 \quad y_E(1) = 0 \quad (20)$$

$$x_E(0) = 0 \quad x_E(1) = 1 - d \quad (21)$$

$$\theta_E(0) = 0 \quad \theta_E(1) = 0 \quad (22)$$

while for vibrations, we have

$$\delta y(0) = 0 = \delta y(1) \quad (23a)$$

$$\delta x(0) = 0 = \delta x(1) \quad (23b)$$

$$\delta \theta(0) = 0 = \delta \theta(1) \quad (23c)$$

System (19) is 6D, and we have an additional parameter ω . With the six above conditions, we have then 1D paths of solutions, but system (19) being linear, we may norm the initial conditions, or just fix one of the initial conditions to 1:

$$\delta y''(0) = 1 \quad (24)$$

Only isolated solutions remain: the vibration modes.

3 Post-buckled equilibrium

We solve the equilibrium equations in the near post-buckling domain, for p just above p_0 ($p_0 \simeq 4\pi^2$, see (35)). System (12)–(14) with $n_{yE} = 0$ and $n_{xE} = -p$ boils down to

$$\theta_E'' = -p \sin \theta_E + \frac{1}{2}\eta p^2 \sin 2\theta_E \quad \text{with } \theta_E(0) = 0 = \theta_E(1) \quad (25)$$

$$y_E' = \sin \theta_E - \frac{1}{2}\eta p \sin 2\theta_E \quad \text{with } y_E(0) = 0 = y_E(1) \quad (26)$$

As $m_E(s) = -p y_E(s) + cte$, boundary conditions on $y_E(s)$ are equivalent to condition

$$m_E(0) = \theta_E'(0) = m_E(1) = \theta_E'(1) \quad (27)$$

We introduce the series for $\theta_E(s)$, $y_E(s)$, and p

$$\theta_E(s) = \epsilon \theta_1(s) + \epsilon^2 \theta_2(s) + \epsilon^3 \theta_3(s) + O(\epsilon^4) \quad (28)$$

$$y_E(s) = \epsilon y_1(s) + \epsilon^2 y_2(s) + \epsilon^3 y_3(s) + O(\epsilon^4) \quad (29)$$

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + O(\epsilon^4) \quad (30)$$

with ϵ a small parameter measuring the 'distance' from buckling. At each order, we extract equations for $\theta_i(s)$, $y_i(s)$, and p_i . Boundary conditions have to be verified at all orders. In order to fix a length scale for ϵ , we use the norm condition:

$$\epsilon = -4 \int_0^1 y_E(s) \cos 2\pi s \, ds \quad (31)$$

3.1 order ϵ

We have to solve

$$y_1' = (1 - \eta p_0) \theta_1 \quad \text{with } y_1(0) = 0 = y_1(1) \quad (32)$$

$$\theta_1'' = -p_0 (1 - \eta p_0) \theta_1 \quad \text{with } \theta_1(0) = 0 = \theta_1(1) \quad (33)$$

Solutions are

$$p_0(1 - \eta p_0) = 4\pi^2 \quad (34)$$

$$p_0 = \frac{1 - \sqrt{1 - 16\pi^2\eta}}{2\eta} \quad (35)$$

$$y_1(s) = \frac{1}{2}(1 - \cos 2\pi s) \quad (36)$$

$$\theta_1(s) = \frac{p_0}{4\pi} \sin 2\pi s \quad (37)$$

Eq. (31) imposes $A_1 = \frac{p_0}{4\pi}$.

3.2 order ϵ^2

At order 2, we have to solve

$$y_2' = (1 - \eta p_0) \theta_2 - \frac{\eta p_1 p_0}{4\pi} \sin(2\pi s) \text{ with } y_2(0) = 0 = y_2(1) \quad (38)$$

$$\theta_2'' = -4\pi^2 \theta_2 - \frac{(8\pi^2 - p_0)p_1}{4\pi} \sin 2\pi s \text{ with } \theta_2(0) = 0 = \theta_2(1) \quad (39)$$

Solutions are

$$p_1 = 0 \quad (40)$$

$$y_2(s) = \frac{A_2}{2\pi} (1 - \eta p_0) (1 - \cos 2\pi s) \quad (41)$$

$$\theta_2(s) = A_2 \sin 2\pi s \quad (42)$$

and Eq. (31) imposes $A_2 = 0$.

3.3 order ϵ^3

At order 3, equations are

$$y_3' = (1 - \eta p_0) \theta_3 - \frac{1 - 4\eta p_0}{6} \left(\frac{p_0}{4\pi}\right)^3 \sin^3(2\pi s) - \frac{\eta p_0 p_2}{4\pi} \sin(2\pi s) \quad (43)$$

$$y_3(0) = 0 \quad (44)$$

$$y_3(1) = 0 \quad (45)$$

$$\theta_3'' = -4\pi^2 \theta_3 + \frac{p_0(1 - 4\eta p_0)}{6} \left(\frac{p_0}{4\pi}\right)^3 \sin^3(2\pi s) - \frac{p_0 p_2}{4\pi} (1 - 2\eta p_0) \sin 2\pi s \quad (46)$$

$$\theta_3(0) = 0 \quad (47)$$

$$\theta_3(1) = 0 \quad (48)$$

and Eq. (31) gives A_3 . The solution is

$$A_3 = \frac{p_0^3(16\pi^2 - 3p_0)(88\pi^2 + p_0)}{49152\pi^5(8\pi^2 - p_0)} \quad (49)$$

$$p_2 = \frac{p_0^4(1 - 4\eta p_0)}{128\pi^2(8\pi^2 - p_0)} = \frac{\pi^4}{2} + 2\pi^6\eta + O(\eta^2) \quad (50)$$

$$\theta_3(s) = \frac{p_0^3(16\pi^2 - 3p_0)[96\pi^2 \sin(2\pi s) - (8\pi^2 - p_0) \sin(6\pi s)]}{48(4\pi)^5(8\pi^2 - p_0)} \quad (51)$$

$$y_3(s) = p_0^2 \frac{16\pi^2 - 3p_0}{4096\pi^4} \sin^2(3\pi s) \quad (52)$$

3.4 order ϵ^4

At this order, we find

$$p_3 = 0 \quad (53)$$

$$y_4(s) = 0 \quad (54)$$

$$\theta_4(s) = 0 \quad (55)$$

3.5 order ϵ^5

We find

$$p_4 = -p_0^5 \frac{3p_0^4 + 720p_0^3\pi^2 - 14528p_0^2\pi^4 + 90112p_0\pi^6 - 180224\pi^8}{128(4\pi)^6(-p_0 + 8\pi^2)^3} \quad (56)$$

$$= \frac{21\pi^6}{128} + \frac{3\pi^8}{4}\eta + O(\eta^2) \quad (57)$$

3.6 Solution up to ϵ^4

$$\theta_E(s) = \epsilon \frac{p_0}{4\pi} \sin 2\pi s + \epsilon^3 \frac{p_0^3(16\pi^2 - 3p_0)[96\pi^2 \sin(2\pi s) - (8\pi^2 - p_0) \sin(6\pi s)]}{48 \cdot (4\pi)^5 (8\pi^2 - p_0)} + 0\epsilon^4 + O(\epsilon^5) \quad (58)$$

$$\theta_E(s) = \epsilon\pi \sin 2\pi s [1 + 4\pi^2\eta + O(\eta^2)] + \epsilon^3\pi^3 \left(\frac{24 \sin(2\pi s) - \sin(6\pi s)}{192} + \eta\pi^2 \frac{\sin 2\pi s}{2} + O(\eta^2) \right) + 0\epsilon^4 + O(\epsilon^5) \quad (59)$$

$$y_E(s) = \frac{\epsilon}{2}(1 - \cos 2\pi s) + \epsilon^3 p_0^2 \frac{16\pi^2 - 3p_0}{4096\pi^4} \sin^2(3\pi s) + 0\epsilon^4 + O(\epsilon^5) \quad (60)$$

$$y_E(s) = \frac{\epsilon}{2}(1 - \cos 2\pi s) + \epsilon^3 \frac{\pi^2}{64} [1 - 4\pi^2\eta + O(\eta^2)] \sin^2(3\pi s) + 0\epsilon^4 + O(\epsilon^5) \quad (61)$$

$$p = p_0 + \epsilon^2 \frac{p_0^3(16\pi^2 - 3p_0)}{128\pi^2(8\pi^2 - p_0)} + 0\epsilon^3 + p_4\epsilon^4 + O(\epsilon^5) \quad (62)$$

$$p = 4\pi^2 + 16\pi^4\eta + O(\eta^2) + \epsilon^2 \left(\frac{\pi^4}{2} + 2\pi^6\eta + O(\eta^2) \right) + 0\epsilon^3 + \epsilon^4 \left(\frac{21\pi^6}{128} + \frac{3\pi^8}{4}\eta + O(\eta^2) \right) + O(\epsilon^5) \quad (63)$$

$$p_4 = -p_0^5 \frac{3p_0^4 + 720p_0^3\pi^2 - 14528p_0^2\pi^4 + 90112p_0\pi^6 - 180224\pi^8}{128(4\pi)^6(-p_0 + 8\pi^2)^3} = \frac{21\pi^6}{128} + \frac{3\pi^8}{4}\eta + O(\eta^2) \quad (64)$$

$$p_0 = \frac{1 - \sqrt{1 - 16\pi^2\eta}}{2\eta} = 4\pi^2 + 16\pi^4\eta + O(\eta^2) \quad (65)$$

Maximal vertical deflection (at the mid span) take the value

$$y_E(1/2) = \epsilon + p_0^2 \frac{16\pi^2 - 3p_0}{4096\pi^4} \epsilon^3 + O(\epsilon^5) = \epsilon + \frac{\pi^2}{64} [1 - 4\pi^2\eta + O(\eta^2)] \epsilon^3 + O(\epsilon^5) \quad (66)$$

Current position $x_E(s)$ is found by integrating $(1 + e_E(s)) \cos \theta_E(s)$ with

$$\cos \theta_E(s) = 1 - \frac{p_0^2}{32\pi^2} \sin^2(2\pi s) \epsilon^2 + 0\epsilon^3 + O(\epsilon^4) \quad (67)$$

$$e_E(s) = -\eta p_0 - \eta \epsilon^2 \left[\frac{p_0^4(1 - 4\eta p_0)}{128\pi^2(8\pi^2 - p_0)} - \frac{p_0^3}{32\pi^2} \sin^2(2\pi s) \right] + 0\epsilon^3 + O(\epsilon^4) \quad (68)$$

Hence

$$x_E(s) = s(1 - \eta p_0) - \epsilon^2 p_0 \frac{2\pi(p_0^2 + 4p_0\pi^2 - 64\pi^4)s + (8\pi^2 - p_0)^2 \sin 4\pi s}{256\pi^3(8\pi^2 - p_0)} + 0\epsilon^3 + O(\epsilon^4) \quad (69)$$

$$x_E(s) = [s - 4\pi^2\eta s + O(\eta^2)] + \left[\frac{\pi}{16}(\sin 4\pi s - 4\pi s) - \frac{\pi^4}{2}s\eta + O(\eta^2) \right] \epsilon^2 + 0\epsilon^3 + O(\epsilon^4) \quad (70)$$

We note $d \stackrel{\text{def}}{=} 1 - (x_E(1) - x_E(0)) = 1 - x_E(1)$ the end-shortening (longitudinal displacement), and have

$$d = \eta p_0 + \epsilon^2 p_0 \frac{2\pi(p_0^2 + 4p_0\pi^2 - 64\pi^4)}{256\pi^3(8\pi^2 - p_0)} + 0\epsilon^3 + O(\epsilon^4) \quad (71)$$

$$= [4\pi^2\eta + O(\eta^2)] + \epsilon^2 \frac{\pi^2}{4} [1 + 2\pi^2\eta + O(\eta^2)] + 0\epsilon^3 + O(\epsilon^4) \quad (72)$$

Going back to the longitudinal displacement $u(s) \stackrel{\text{def}}{=} x(s) - s$, von Karman kinematics writes $e(s) = u'(s) + (1/2)v'^2(s)$, where vertical displacement is written $v(s) \stackrel{\text{def}}{=} y(s)$. This kinematics is not strictly verified:

$$e_E(s) - u'_E(s) - \frac{1}{2}v'^2_E(s) = \frac{1}{8}\eta p_0^2 \epsilon^2 \sin^2(2\pi s) + 0\epsilon^3 + O(\epsilon^4) \quad (73)$$

or

$$x'_E(s) - 1 + \eta p + \frac{1}{2}y'^2_E(s) = \frac{\eta^2 p_0^4}{32\pi^2} \epsilon^2 \sin^2(2\pi s) + 0\epsilon^3 + O(\epsilon^4) \quad (74)$$

Von Karman differential equation for the vertical displacement is not even verified at order ϵ

$$y''''_E(s) + p_E y''_E(s) = 2\pi^2 \eta p_0^2 \epsilon \cos(2\pi s) + 0\epsilon^2 + O(\epsilon^3) \quad (75)$$

4 Vibrations around an equilibrium solution

We set :

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 + O(\epsilon^4) \quad (76)$$

$$\delta y(s) = \delta y_0 + \epsilon\delta y_1 + \epsilon^2\delta y_2 + \epsilon^3\delta y_3 + O(\epsilon^4) \quad (77)$$

Same expansion for all the other variables $\delta x(s)$, $\delta\theta(s)$, ...

4.1 Restriction to the first vibration mode: $\omega_0 = 0$

order ϵ^0

Equations are

$$\delta x'_0 = \eta\delta n_{x0} \text{ with } \delta x_0(0) = 0 = \delta x_0(1) \quad (78a)$$

$$\delta n'_{x0} = 0 \quad (78b)$$

$$0 = \delta y_0'''' + 4\pi^2\delta y_0'' \text{ with } \delta y_0(0) = \delta y_0(1) = \delta y'_0(0) = \delta y'_0(1) = 0 \quad (78c)$$

Solutions are

$$\delta x_0(s) = 0 \quad (79a)$$

$$\delta n_{x0}(s) = 0 \quad (79b)$$

$$\delta y_0(s) = A_0(1 - \cos 2\pi s) \quad (79c)$$

which yields

$$\delta\theta_0(s) = \frac{2\pi A_0}{1 - \eta p_0} \sin 2\pi s \quad (80a)$$

$$\delta m_0(s) = \frac{4\pi^2 A_0}{1 - \eta p_0} \cos 2\pi s \quad (80b)$$

$$\delta n_{y0}(s) = 0 \quad (80c)$$

The norm condition (24) implies

$$A_0 = \frac{1}{4\pi^2} \quad (81)$$

order ϵ^1

Equations are

$$\delta x_1' = \eta \delta n_{x1} - p_0 \frac{1 - 2\eta p_0}{8\pi^2(1 - \eta p_0)} \sin^2 2\pi s, \text{ with } \delta x_1(0) = 0 = \delta x_1(1) \quad (82a)$$

$$\delta n_{x1}' = 0 \quad (82b)$$

$$0 = \delta y_1'''' + 4\pi^2 \delta y_1'', \text{ with } \delta y_1(0) = \delta y_1(1) = \delta y_1'(0) = \delta y_1'(1) = 0 \quad (82c)$$

Longitudinal mode solutions are

$$\delta x_1(s) = p_0 \frac{1 - 2\eta p_0}{1 - \eta p_0} \frac{\sin 4\pi s}{64\pi^3} \quad (83)$$

$$\delta n_{x1}(s) = p_0 \frac{1 - 2\eta p_0}{1 - \eta p_0} \frac{1}{16\pi^2\eta} \quad (84)$$

$$(85)$$

Transverse variable:

$$\delta y_1(s) = A_1(1 - \cos 2\pi s) \quad (86)$$

The norm condition (24) forces $A_1 = 0$. We then have

$$\delta y_1(s) = 0 = \delta\theta_1(s) = \delta m_1(s) = \delta n_{y1}(s) \quad (87)$$

order ϵ^2

For longitudinal variable, equations are

$$\delta x_2' = \eta \delta n_{x2} \text{ with } \delta x_2(0) = 0 = \delta x_2(1) \quad (88)$$

$$\delta n_{x2}' = 0 \quad (89)$$

The solution vanishes

$$\delta x_2(s) = 0 \quad (90a)$$

$$\delta n_{x2}(s) = 0 \quad (90b)$$

For transverse variables, equations boils down to

$$\delta y_2'''' + 4\pi^2 \delta y_2'' + A \cos 6\pi s + B \cos 2\pi s + C = 0 \quad (91a)$$

$$\text{with } \delta y_2(0) = \delta y_2(1) = \delta y_2'(0) = \delta y_2'(1) = 0 \quad (91b)$$

$$A = -\frac{27p_0^2}{128} \left(16 - \frac{3p_0}{\pi^2} \right) \quad (91c)$$

$$B = \frac{p_0^2}{16\pi^2} \frac{p_0^2 + 4p_0\pi^2 - 64\pi^4}{4(p_0 - 4\pi^2)} + \frac{4\pi^2\omega_1^2}{p_0^2} \quad (91d)$$

$$C = -\frac{4\pi^2\omega_1^2}{p_0^2} \quad (91e)$$

Solution is

$$1152\pi^4 \delta y_2(s) = A - 90B + 144C\pi^2 s - 144C\pi^2 s^2 + 288\pi^2 K + 90B \cos 2\pi s - 288\pi^2 K \cos 2\pi s - A \cos 6\pi s - 72C\pi \sin 2\pi s + 72B\pi s \sin 2\pi s \quad (92)$$

$$\delta y_2(s) = \frac{1 + 2\pi^2(s-1)s - \cos 2\pi s + \pi(1-2s) \sin 2\pi s}{96\pi^2\eta} + O(\eta^0) \quad (93)$$

Boundary conditions $\delta y_1'(1) = 0$ imposes $B = 2C$, which in return yields ω_1

$$\omega_1^2 = \frac{p_0^4(p_0^2 + 4p_0\pi^2 - 64\pi^4)}{768\pi^4(-p_0 + 4\pi^2)} \quad (94)$$

which yields, with (65)

$$\omega_1 = \sqrt{\frac{2}{3}}\pi^2 \left[\frac{1}{\sqrt{\eta}} + \pi^2\sqrt{\eta} \right] + O(\eta^{3/2}) \quad (95)$$

The norm condition (24) fixes

$$K = \frac{12C - A}{32\pi^2} \quad (96)$$

order ϵ^3

Solutions are

$$\delta x_3(s) = \frac{s}{64\eta} + \frac{\sin \pi s}{384\pi\eta} (-7 \cos \pi s + \cos 3\pi s + 8\pi(1-2s) \sin^3 \pi s) + O(\eta^0) \quad (97)$$

$$\delta n_{x3}(s) = n_{x3K} - p_0 \frac{p_0 - 8\pi^2}{1024\pi^6} \omega_1^2 \cos 4\pi s \quad (98)$$

$$\omega_2 = 0 \quad (99)$$

$$\delta y_3(s) = 0 = \delta \theta_3(s) = \delta m_3(s) = \delta n_{y3}(s) \quad (100)$$

order ϵ^4

For the longitudinal variables, equations are

$$\delta x_4' = \eta \delta n_{x4} \text{ with } \delta x_4(0) = 0 = \delta x_4(1) \quad (101)$$

$$\delta n_{x4}' = 0 \quad (102)$$

The solution vanishes

$$\delta x_4(s) = 0 \quad (103a)$$

$$\delta n_{x4}(s) = 0 \quad (103b)$$

For the transverse variables, equations boils down to

$$\delta y_4'''' + 4\pi^2 \delta y_4'' + f(\omega_3, s, s^2, \sin(2, 4, 6\pi s), s \sin(2, 4, 6\pi s), \cos(2, 4, 6, 10\pi s), \dots) = 0 \quad (104a)$$

$$\text{with } \delta y_4(0) = \delta y_4(1) = \delta y_4'(0) = \delta y_4'(1) = 0 \quad (104b)$$

We find ω_3 by enforcing the condition $\delta y_4'(1) = 0$

$$\omega_3 = \left(\frac{p_0}{4\pi}\right)^6 \frac{M}{41472\pi^4(p_0 - 8\pi^2)(p_0 - 4\pi^2)^2\omega_1} \quad (105)$$

$$M = 9p_0^6 - 13680p_0^4\pi^4 - 65536\pi^{12}(-285 + 16\pi^2) + 2p_0^5\pi^2(-87 + 16\pi^2) + 768p_0^2\pi^8(399 + 16\pi^2) + 8192p_0\pi^{10}(-831 + 32\pi^2) - 32p_0^3\pi^6(-4233 + 176\pi^2) \quad (106)$$

which yields, for small η

$$\omega_3 = \pi^2 \frac{-21 + 2\pi^2}{216\sqrt{6}\eta^{3/2}} + \frac{\pi^4(141 - 10\pi^2)}{216\sqrt{6}\eta^{1/2}} + O(\eta^{1/2}) \quad (107)$$

where we note that the first term is exactly the same as in the von Karman model.

4.2 General case, $\omega_0 \neq 0$

todo !