

Stopping Times of Syracuse Integer Sequences on $\mathbb{Z}/2^n\mathbb{Z}$

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Abstract :

The Collatz Conjecture asserts that for all positive integer m , all Syracuse integer sequence defined by $T(m) = \frac{m}{2}$ if m is even and $T(m) = \frac{3m+1}{2}$ else, have a finite total stopping time. The total stopping time is the number of iterates of T to reach 1. An equivalent formulation of the conjecture is to say that all trajectories of a Syracuse integer sequence has a finite stopping time. The stopping time is the number of iterates to be lower than the starting number. This assertion is obvious for all even number s and can be easily proved for all integer $s \equiv 1 [4]$. But the question becomes extremely complex if we want to analyze the behavior of integers $s \equiv 3 [4]$.

Riho Terras[1] has shown that almost all Syracuse sequence have a finite stopping time, result improve successively by Jean Paul Allouche [2] and Korec [3]. Krasicov and Lagarias [4] has stated that for any enough large x , the number of Syracuse sequences of starting term below x , which satisfy the Collatz conjectures is higher than $x^{0.84}$. Terence Tao [5] gives the most recent major result, proving that almost all Collatz orbits attain almost bounded values.

In the present work, we are going at first to study with a computational approach the behavior of $S(n) = \#\{s \in \mathbb{Z}/2^n\mathbb{Z}, \sigma(s) > n\}$ being the number of residue classes mod 2^n of starting number s such as the stopping time $\sigma(s) > n$ and then to give, based on a work of Mike Winkler [6], an exact formulation of $h_p(p) = \#\{s \in \mathbb{Z}/2^p\mathbb{Z}, \sigma(s) = p\}$, the number of residue classes mod 2^p of starting number s such as $\sigma(s) = p(r)$ with $p(r) = \lfloor r \cdot \log_2(3) \rfloor + 1$

$$z_{p(r)} = h_{p(r)}(p(r)) = \binom{p(r)-2}{r-2} - \sum_{i=2}^{r-1} \binom{p(r)-p(i)}{r-i} \cdot z_{p(i)}$$

and we'll show that $S(n)$ can be fully described thanks to the values of $z_{p(r)}$ for $p \leq n$

$$S(n) = 2^n \cdot \left(1 - \sum_{r=1}^{r=r(n)} \frac{z_{p(r)}}{2^{p(r)}} \right)$$

We'll state that $\frac{S(n)}{2^n}$ tends toward 0 when $n \rightarrow \infty$ and that there is $\theta < 1$ such as for n large, $S(n) < 2^{n\theta}$.

We'll end this work by giving some arguments to conjecture that the Maximal Stopping Time (MST) of all residue classes mod 2^n is lower than $20n$.

Preamble

The $3x+1$ problem is the study of all integer sequences defined by the arithmetic function C :

$$C(s) = \begin{cases} \frac{s}{2} & \text{if } s \text{ is even} \\ 3s+1 & \text{else} \end{cases}$$

$\mathcal{C}(s)$ is the set of all iterate of s with the function C

The two other following formulations can be used

$$T(s) = \begin{cases} \frac{s}{2} & \text{if } n \text{ is even} \\ \frac{(3s+1)}{2} & \text{else} \end{cases}$$

And

$$N(s) = \begin{cases} \frac{s}{2^{\alpha(s)}} & \text{if } n \text{ is even} \\ \frac{(3s+1)}{2^{\alpha(s)}} & \text{else} \end{cases}$$

Where $\alpha(n)$ is the highest integer such as $\frac{(3n+1)}{2^{\alpha(n)}}$ or $\frac{n}{2^{\alpha(n)}}$ is odd.

$\mathcal{T}(s)$ is the set of all iterate of s with the function T and $\mathcal{N}(s)$ is the set of all iterate of s with the function N

For example:

$$\mathcal{C}(7) = \{7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, \dots\}$$

$$\mathcal{T}(7) = \{7, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1, 2, 1, \dots\}$$

$$\mathcal{N}(7) = \{7, 11, 17, 13, 5, 1, 1, \dots\}$$

In the following, $\mathcal{T}_n(s)$ (resp $\mathcal{C}_n(s)$, $\mathcal{N}_n(s)$), will represent the subsequences of the n first iterates of the functions T (resp. C, N) and $C^{n+r}(s) = T^n(s) = N^r(s)$ with $r = \left\lfloor \frac{n}{\log_2 3} \right\rfloor$

The r^{st} iterate of the function N corresponding to the p^{st} iterate of the function T for a starting odd number s is given by:

$$T^p(s) = \frac{3^r}{2^p} \cdot s + \sum_{i=1}^r \frac{3^{r-i}}{2^{\alpha_1 + \dots + \alpha_r}} = N^r(s) \quad \text{with} \quad p = \sum_{i=1}^r \alpha_i \quad (0)$$

In the following, we'll define the stopping time $\sigma(s)$ as the number of iterates of T to be lower than the starting number s .

I - First study cases :

If we start a Syracuse sequence with an integer $2^n \cdot m + s$ with $m \in \mathbb{N}$ and $s \in \mathbb{Z}/2^n\mathbb{Z}$, with a stopping time $\sigma(s) \leq n$, it is quite easy to show that $T^{\sigma(s)}(2^n \cdot m + s) < 2^n \cdot m + s$ for all m . Our idea is to study for each n , the percentage of residue classes $\text{mod}(2^n)$ of starting number s of Syracuse sequences such as $\sigma(s) > n$ and to show that this percentage tends towards 0 when n grows. We are going to start by the first values of n .

Step n=1 :

We begin by the trivial case of integers belonging to $\mathbb{Z}/2\mathbb{Z}$. We compute the stopping time of even and odd numbers.

As $T(2m) = m$, the stopping time of even number $\sigma(2m) = 1$

As $T(2m+1) = 3m+2$ we cannot conclude at this stage regarding the stopping time.

We can state that 50% of the Syracuse integer sequence have a stopping time higher than 1

Step n=2 :

It concerns all residue classes $s \in \mathbb{Z}/2^2\mathbb{Z}$. As we have already treated the even number, we have to analyse the behavior of Syracuse integer sequences beginning by $4m+1$ and $4m+3$

$T^{(2)}(4m+1) = 3m+1$ which is below $4m+1$ for all positive integer m , which means that $\sigma(4m+1) = 2$

But for $T^{(2)}(4m+3) = 9m+8$, we cannot conclude.

At this stage, we can conclude that 25% of the integer sequence have a stopping time higher than 2

Step n=3 :

We have to analyse two families of integers coming from the previous case, $8m+3$ and $8m+7$.

$T^{(3)}(8m+3) = 9m+8$ and $T^{(3)}(8m+7) = 27m+26$

At this stage, we cannot conclude and 25% of the integer sequences still have a stopping time higher than 3

We can also say that no Syracuse integer sequence has a stopping time equal to 3, 3 is not a stopping time value.

Step n=4 :

Here we have to study four residue classes $s \text{ mod } (2^4)$ with $s \in \{3, 7, 11, 15\}$

We see that $T^{(4)}(16m+3) = 9m+8$ which is below $16m+3$ for all positive integer m , thus $\sigma(16m+3) = 4$

But we cannot conclude for the 3 other families :

$T^{(4)}(16m+7) = 27m+13$

$T^{(4)}(16m+11) = 27m+20$

$T^{(4)}(16m+15) = 81m+80$

So at this stage, we have $\frac{3}{16} = 18,75\%$ of residue classes s which has a stopping time higher than 4, $\sigma(s) > 4$

At the fifth step, corresponding to the residue classes $\text{mod}(32)$, we have to compute the stopping time of the following residue classes : $\{3, 7, 11, 15, 19, 23, 27, 31\}$

More generally, if at step n , $S(n) = \#\{s \in \mathbb{Z}/2^n\mathbb{Z}, \sigma(s) > n\}$, at step $n+1$, we have to analyse all $s \in S(n) \cup \{2^n + S(n)\}$.

We have computed $S(n)$ for all $n < 41$ and the limits of our computer.

n	2^n	$S(n)$	$S(n)/2^n$	$\text{Log}_2(S(n))$	$\text{Log}_2(S(n))/n$
1	2	1	50,00%	0,00	0,000000
2	4	1	25,00%	0,00	0,000000
3	8	2	25,00%	1,00	0,333333
4	16	3	18,75%	1,58	0,396241
5	32	4	12,50%	2,00	0,400000
6	64	8	12,50%	3,00	0,500000
7	128	13	10,16%	3,70	0,528634
8	256	19	7,42%	4,25	0,530991
9	512	38	7,42%	5,25	0,583103
10	1024	64	6,25%	6,00	0,600000
11	2048	128	6,25%	7,00	0,636364
12	4096	226	5,52%	7,82	0,651682
13	8192	367	4,48%	8,52	0,655357
14	16384	734	4,48%	9,52	0,679974
15	32768	1295	3,95%	10,34	0,689249
16	65536	2114	3,23%	11,05	0,690360
17	131072	4228	3,23%	12,05	0,708574
18	262144	7495	2,86%	12,87	0,715095
19	524288	14990	2,86%	13,87	0,730090
20	1048576	27328	2,61%	14,74	0,736905
21	2097152	46611	2,22%	15,51	0,738494
22	4194304	93222	2,22%	16,51	0,750381
23	8388608	168807	2,01%	17,37	0,755001
24	16777216	286581	1,71%	18,13	0,755358
25	33554432	573162	1,71%	19,13	0,765143
26	67108864	1037374	1,55%	19,98	0,768635
27	134217728	1762293	1,31%	20,75	0,768482
28	268435456	3524586	1,31%	21,75	0,776751
29	536870912	6385637	1,19%	22,61	0,779531
30	1073741824	12771274	1,19%	23,61	0,786880
31	2147483648	23642078	1,10%	24,49	0,790157
32	4294967296	41347483	0,96%	25,30	0,790666
33	8589934592	82694966	0,96%	26,30	0,797009
34	17179869184	151917636	0,88%	27,18	0,799374
35	34359738368	263841377	0,77%	27,98	0,799288
36	68719476736	527682754	0,77%	28,98	0,804864
37	137438953472	967378591	0,70%	29,85	0,806743
38	274877906944	1934757182	0,70%	30,85	0,811829
39	549755813888	3611585536	0,66%	31,75	0,814102
40	1099511627776	6402835000	0,58%	32,58	0,814402
41	2199023255552	12805670000	0,58%	33,58	0,818928

Table 1 : computational results

The blue lines correspond to the values of n where $h_n(n)=0$. In this case $S(n)=2.S(n-1)$.

On the white lines the relation between $S(n)$ and $S(n-1)$ is given by : $S(n)=2.S(n-1)-h_n(n)$

For instance, $S(5)=4$, the four odd integers which satisfy $\sigma(s)>4$ are $\{7,15,27,31\}$ and as no odd integer has a stopping time $\sigma(s)=6$, the set of integers which satisfy to $\sigma(s)>6$, is exactly $\{7,15,27,31,39,47,59,63\}$

This can be explained by the fact that the stopping time values p satisfy to the following inequalities :

$$2^{p-1} < 3^r < 2^p$$

In this case, if s is such as $\sigma(s)=p$, it means that for all $k < p$, $T^k(s) > s > T^p(s)$

If we have this set of inequalities $3^r < 2^{p-1} < 2^p < 3^{r+1}$, p cannot correspond to a stopping time value because

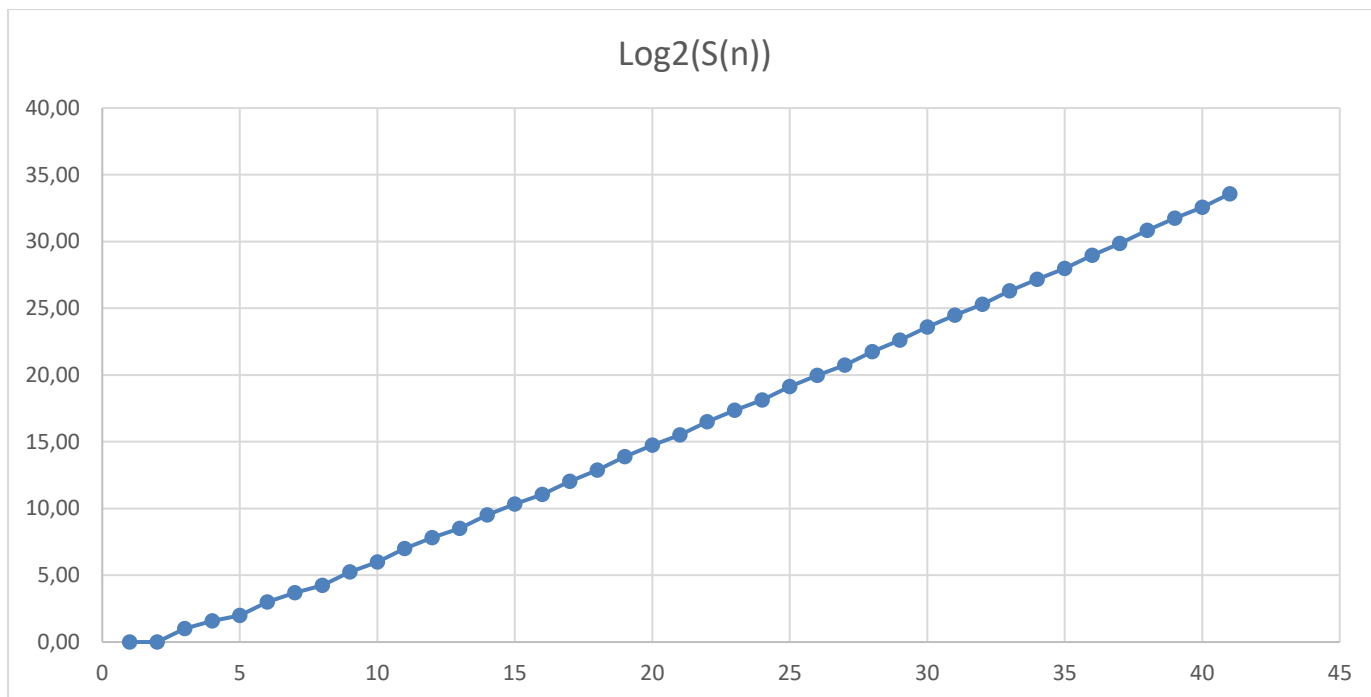
$$s > T^{p-1}(s) > T^p(s)$$

We'll see later a justification of this point.

Thanks to these first results in table 1, we can state that $S(40) < 0.58\%$ which imply that 99,42% of the Syracuse integer sequences has a stopping time lower than 41.

We are convinced that the function linking the number of odd starting terms $s < 2^n$ such as $\sigma(s) > n$ is mainly deterministic. It's what we are going to try to prove in the following pages.

We can illustrate this deterministic behavior with the first graph below which gives $\text{Log}_2(S(n))$ in function of n .



If the slope of this curve tends toward a constant value (< 1), it means that this curve tends asymptotically toward a line with a slope $\theta < 1$, it will imply that for n large, $\frac{S(n)}{2^n} < \frac{1}{2^{n(1-\theta)}}$. We'll develop this point later.

II - Computation of the histograms $H_n(p)$

To go further in the analysis of $S(n)$ behaviour, it seems necessary to compute the histograms

$$H_n = \{h_n(p) \text{ for } p \in \mathbb{N}\} \text{ and } h_n(p) = \#\{s < 2^n \text{ such as } \sigma(s) = p\}$$

where $h_n(p)$ is the number of residue classes mod(2^n) of Syracuse integer sequences of starting number a such as $\sigma(s) = p$

we have computed the $h_n(p)$ for all p and all n up to $n=40$ and the data regarding H_n are presented in table 2a and 2b below for $p \leq 40$ and in annex for $p > 40$.

When we analyse table 2a and 2b, we can remark two points :

- for all stopping time value p and $k > 0$: $h_{p+k}(p) = 2^k \cdot h_p(p)$
- not all integers correspond to stopping time values

n\σ(s)	1	2	4	5	7	8	10	12	13	15	16	18	20
1	1	1	0	0	0	0	0	0	0	0	0	0	0
2	2	1	1	0	0	0	0	0	0	0	0	0	0
3	4	2	1	0	1	0	0	0	0	0	0	0	0
4	8	4	1	1	2	0	0	0	0	0	0	0	0
5	16	8	2	2	2	0	0	0	0	0	0	0	0
6	32	16	4	4	3	1	0	0	0	0	0	0	0
7	64	32	8	8	3	4	0	0	0	1	0	0	0
8	128	64	16	16	6	7	0	1	3	1	0	0	1
9	256	128	32	32	12	14	5	3	4	3	2	1	2
10	512	256	64	64	24	28	12	7	10	5	6	2	3
11	1024	512	128	128	48	56	24	16	21	10	12	6	7
12	2048	1024	256	256	96	112	48	30	41	26	26	17	15
13	4096	2048	512	512	192	224	96	60	85	45	60	32	29
14	8192	4096	1024	1024	384	448	192	120	170	91	115	64	57
15	16384	8192	2048	2048	768	896	384	240	340	173	237	125	100
16	32768	16384	4096	4096	1536	1792	768	480	680	346	476	250	188
17	65536	32768	8192	8192	3072	3584	1536	960	1360	692	952	501	350
18	131072	65536	16384	16384	6144	7168	3072	1920	2720	1384	1904	961	699
19	262144	131072	32768	32768	12288	14336	6144	3840	5440	2768	3808	1922	1342
20	524288	262144	65536	65536	24576	28672	12288	7680	10880	5536	7616	3844	2652
21	1048576	524288	131072	131072	49152	57344	24576	15360	21760	11072	15232	7688	5304
22	2097152	1048576	262144	262144	98304	114688	49152	30720	43520	22144	30464	15376	10608
23	4194304	2097152	524288	524288	196608	229376	98304	61440	87040	44288	60928	30752	21216
24	8388608	4194304	1048576	1048576	393216	458752	196608	122880	174080	88576	121856	61504	42432
25	16777216	8388608	2097152	2097152	786432	917504	393216	245760	348160	177152	243712	123008	84864
26	33554432	16777216	4194304	4194304	1572864	1835008	786432	491520	696320	354304	487424	246016	169728
27	67108864	33554432	8388608	8388608	3145728	3670016	1572864	983040	1392640	708608	974848	492032	339456
28	134217728	67108864	16777216	16777216	6291456	7340032	3145728	1966080	2785280	1417216	1949696	984064	678912
29	268435456	134217728	33554432	33554432	12582912	14680064	6291456	3932160	5570560	2834432	3899392	1968128	1357824
30	536870912	268435456	67108864	67108864	25165824	29360128	12582912	7864320	11141120	5668864	7798784	3936256	2715648
31	1073741824	536870912	134217728	134217728	50331648	58720256	25165824	15728640	22282240	11337728	15597568	7872512	5431296
32	2147483648	1073741824	268435456	268435456	100663296	117440512	50331648	31457280	44564480	22675456	31195136	15745024	10862592
33	4294967296	2147483648	536870912	536870912	201326592	234881024	100663296	62914560	89128960	45350912	62390272	31490048	21725184
34	8589934592	4294967296	1073741824	1073741824	402653184	469762048	201326592	125829120	178257920	90701824	124780544	62980096	43450368
35	17179869184	8589934592	2147483648	2147483648	805306368	939524096	402653184	251658240	356515840	181403648	249561088	125960192	86900736
36	34359738368	17179869184	4294967296	4294967296	1610612736	1879048192	805306368	503316480	713031680	362807296	499122176	251920384	173801472
37	68719476736	34359738368	8589934592	8589934592	3221225472	3758096384	1610612736	1006632960	1426063296	725614592	998244352	503840768	347602944
38	137438953472	68719476736	17179869184	17179869184	6442450944	7516192768	3221225472	2013265920	2852126720	1451229184	1996488704	1007681536	695205888
39	274877906944	137438953472	34359738368	34359738368	12884901888	15032385536	6442450944	4026531840	5704253440	2902458368	3992977408	2015363072	1390411776
40	549755813888	274877906944	68719476736	68719476736	25769803776	30064771072	12884901888	8053063680	11408506880	5804916736	7985954816	4030726144	2780823552

n\σ(s)	21	23	24	26	27	29	31	32	34	35	37	39	40
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	0	0	0	0	0	0
8	1	0	0	0	1	1	2	0	0	0	0	0	1
9	3	0	1	0	2	1	2	0	0	0	0	0	2
10	6	1	4	1	3	1	2	0	0	1	1	1	2
11	9	5	8	4	3	1	3	1	1	2	1	2	2
12	12	8	13	8	7	4	4	3	2	3	3	4	6
13	28	13	27	11	17	6	6	8	3	9	8	5	9
14	54	27	47	31	37	19	12	19	12	20	11	9	12
15	113	63	100	53	78	38	30	38	26	35	20	16	25
16	250	132	196	102	145	75	54	89	55	76	39	27	51
17	507	272	392	212	296	150	116	173	110	158	79	65	104
18	995	551	784	429	594	311	232	363	228	344	159	130	210
19	2022	1102	1578	837	1185	660	488	729	435	646	341	258	393
20	4013	2230	3166	1699	2454	1338	944	1467	835	1290	663	509	802
21	8045	4413	6407	3442	4935	2603	1858	2935	1683	2522	1384	1046	1619
22	16090	8818	12750	6806	9807	5170	3737	5909	3348	5034	2746	1984	3189
23	32180	17637	25427	13453	19613	10347	7449	11676	6607	9959	5418	4023	6299
24	64360	35274	51033	27026	39156	20756	14887	23287	13106	19796	10838	7915	12588
25	128720	70548	102066	54637	78017	41512	29714	46403	26399	39340	21786	15735	25300
26	257440	141096	204132	108950	156415	82960	59439	92651	52796	78435	43312	31524	50320
27	514880	282192	408264	217900	312455	165991	118756	185327	105501	156445	86412	63300	100275
28	1029760	564384	816528	435800	624910	331899	237455	370906	210619	313133	172461	126194	200455
29	2059520	1128768	1633056	871600	1249820	663535	475297	741658	421383	625166	344298	251963	400052
30	4119040	2257536	3266112	1743200	2499640	1327070	950050	1483948	842495	1249751	687092	504060	800442
31	8238080	4515072	6532224	3486400	4999280	2654140	1900470	2968528	1684279	2498989	1374351	1008119	1601779
32	16476160	9030144	13064448	6972800	9998560	5308280	3800940	5936673	3368297	4999512	2748535	2015670	3203945
33	32952320	18060288	26128896	13945600	19997120	10616560	7601880	11873346	6735992	9999087	5498168	4030565	6407961
34	65904640	36120576	52257792	27891200	39994240	21233120	15203760	23746692	13472296	19998291	10996150	8060635	12817360
35	131809280	72241152	104515584	55782400	79988480	42466240	30407520	47493384	26944592	39993895	21995901	16122083	25634043
36	263618560	144482304	209031168	111564800	159976960	84932480	60815040	94986768	53889184	79987790	43993240	32244300	51269020
37	527237120	288964608	418062336	223129600	319953920	169864960	121630080	189973536	107778368	159975580	87986917	64489232	102533100
38	1054474240	577929216	836124672	446259200	639907840	339729920	243260160	379947072	215556736	319951160	175973834	128978066	205071764
39	2108948480	115858432	1672249344	892518400	1279815680	679459840	486520320	759894144	431113472	639902320	351947668	257978502	410116175
40	4217896960	2311716864	3344498688	1785036800	2559631360	1358919680	973040640	1519788288	862226944	1279804640	703895336	515957004	820236724

Table 2a and 2b : $h_n(p)$ values for $n \leq 40$ and $p \leq 40$ (see in annexe for $p > 40$)

Theorem 1 : for all odd integer $a \in \mathbb{Z}/2^n\mathbb{Z}$ verify $\sigma(a)=n$, the set of integers $b=a+2^n$ are the only integers which verify $\sigma(b)=n$, then $h_{n+1}(n) = 2 \cdot h_n(n)$ and $h_{n+k}(n) = 2^k \cdot h_n(n)$ for all positive integers n and k .

Proof : If $a \in \mathbb{Z}/2^n\mathbb{Z}$ such as $\sigma(a)=n$ then by definition of the stopping time:

$$T^n(a) < a < T^{n-1}(a)$$

$$\text{then for all integer } m : \quad T^n(2^n \cdot m + a) < 2^n \cdot m + a < T^{n-1}(2^n \cdot m + a) \quad (1)$$

If we consider r such as : $2^{n-1} < 3^r < 2^n$

$$T^n(2^n \cdot m + a) = 3^r \cdot m + T^n(a) < 2^n \cdot m + a < 3^r \cdot 2 \cdot m + T^{n-1}(a) = T^{n-1}(2^n \cdot m + a)$$

Then if $b = 2^n + a$, we have thanks to (1) :

$$T^n(b) = T^n(2^n + a) = 3^r + T^n(a) < 2^n + a < T^{n-1}(2^n + a)$$

Which means that $\sigma(b)=n$.

Reciprocally if for b such as $2^n < b < 2^{n+1}$ and $\sigma(b)=n$, we have by definition

$$T^n(b) < b < T^{n-1}(b)$$

$$\text{And thanks to (1) for all } m : \quad T^n(2^n \cdot m + b) < 2^n \cdot m + b < T^{n-1}(2^n \cdot m + b)$$

$$\text{We can write :} \quad T^n(2^n \cdot (m+1) + b - 2^n) < 2^n \cdot (m+1) + b - 2^n < T^{n-1}(2^n \cdot (m+1) + b - 2^n)$$

If we replace $a = b - 2^n$ with $0 < a < 2^n$, we have :

$$T^n(2^n \cdot (m+1) + a) < 2^n \cdot (m+1) + a < T^{n-1}(2^n \cdot (m+1) + a) \text{ for all } m$$

which prove that the set of integers b such as $b < 2^{n+1}$ and $\sigma(b)=n$ is exactly the set of integers $a + \varepsilon \cdot 2^n$ with $a < 2^n$ and $\varepsilon \in \{0,1\}$ which means that $h_{n+1}(n) = 2 \cdot h_n(n)$ and consequently :

$$\text{for all } k > 0, \quad h_{n+k}(n) = 2^k \cdot h_n(n) \text{ and for } p < n \quad h_{n+k}(p) = 2^k \cdot h_n(p) \quad (2)$$

From theorem 1, we can affirm that if for $a < 2^n$, $\sigma(a)=n$ then for all positive integer m , all integers $2^n \cdot m + a$ has the same stopping time than a : $\sigma(2^n \cdot m + a) = \sigma(a) = n$.

Theoreme 2 : All stopping time values verify the relation : $p = \left\lfloor \frac{p}{\log_2 3} \right\rfloor \cdot \log_2 3 + 1$ and $p(r) = \lfloor r \cdot \log_2 3 + 1 \rfloor$

As we can see in the above tables, a stopping time is reached for p and r values such as $2^{p-1} < 3^r < 2^p$

If we take the logarithm of these inequalities, we obtain : $(p-1) < r \cdot \log_2 3 < p$ which imply

$$p = \lfloor r \cdot \log_2 3 + 1 \rfloor \quad \text{and} \quad r = \left\lfloor \frac{p}{\log_2 3} \right\rfloor \quad \text{and consequently :} \quad p = \left\lfloor \left\lfloor \frac{p}{\log_2 3} \right\rfloor \cdot \log_2 3 + 1 \right\rfloor \quad (3)$$

In the following, we'll use the arithmetique function $p(r) = \lfloor r \cdot \log_2 3 + 1 \rfloor$ for $r \in \mathbb{N}$

We define the subset of \mathbb{N} of all integer values taken by a stopping time :

$$S_\sigma = \{ p \in \mathbb{N} / \exists s \in \mathbb{N} \text{ and } \sigma(s) = p \}$$

Theoreme 3 : $\Pi(2^n) = \#\{s \in \mathbb{Z}/2^n\mathbb{Z}, \sigma(s) \leq n\}$ and $S(n) = \#\{s \in \mathbb{Z}/2^n\mathbb{Z}, \sigma(s) > n\}$ are totally defined by the values of $h_p(p)$ for $p \leq n$

By definition : $\sum_{r=0}^{\infty} h_n(p(r)) = 2^n$, $\Pi(2^n) = \sum_{r=0}^{r=r(n)} h_n(p(r))$, $S(n) = \sum_{r=r(n+1)}^{\infty} h_n(p(r))$

We can write $\sum_{r=0}^{\infty} h_n(p(r))$ because this power serie has at most 2^n strictly positive terms.

and $S(n) + \Pi(2^n) = 2^n$

Thanks to theorem 1, we have stated that, for $n > p$: $h_n(p) = 2^{n-p} \cdot h_p(p)$ and we replace in (4)

$$\Pi(2^n) = \sum_{r=0}^{r=r(n)} 2^{n-p(r)} \cdot h_{p(r)}(p(r)) = 2^n \cdot \sum_{r=0}^{r=r(n)} \frac{h_{p(r)}(p(r))}{2^{p(r)}} < 2^n \text{ and } S(n) = 2^n \cdot \left(1 - \sum_{r=0}^{r=r(n)} \frac{h_{p(r)}(p(r))}{2^{p(r)}}\right) \quad (4)$$

$$\text{Then } \frac{\Pi(2^n)}{2^n} \text{ is a growing function} \quad \frac{\Pi(2^n)}{2^n} = \sum_{r=0}^{r=r(n)} \frac{h_{p(r)}(p(r))}{2^{p(r)}} < 1 \quad (5)$$

$$\text{And } \frac{S(n)}{2^n} \text{ is a decreasing function} \quad 0 < \frac{S(n)}{2^n} = \left(1 - \sum_{r=0}^{r=r(n)} \frac{h_{p(r)}(p(r))}{2^{p(r)}}\right) < 1 \quad (6)$$

$\Pi(2^n)$ and $S(n)$ is totally defined by the $h_p(p)$ for $p \leq n$. We are going to show in the next part that it's possible to give an exact formulation of $h_p(p)$ for any p .

Collatz Conjecture would be validated if for all n , there is $N(n)$ such as $\sum_{r=0}^{N(n)} h_n(p(r)) = 2^n$ which is equivalent to say that all residue classes mod 2^n , of Syracuse sequences of starting number s has a stopping time $\sigma(s) < N(n)$. We 'll give later some argument to state that $N(n) = 20n$.

$\frac{\Pi(2^n)}{2^n}$ is the percentage of residu classes mod 2^n of Suracuse sequences of starting number s who have a stopping time $\sigma(s) \leq n$. Some residue classes have a stopping time higher than n and if the Collatz conjecture is false, it means than we have at least one Syracuse sequence without stopping time. It concerns a sequence of starting number s which iterate to infinity or to a non trivial cycle.

Olivera e Silva [7] 2005 has computationally verified that the Collatz conjecture holds for all starting number up to 4.10^{18} .

Theoreme 4 : If there is some positive integers $s > 1$ and $p > 1$ such as $T^p(s) = s$ and $T^k(s) \geq s$ for all $k < p$, which means that the Syracuse sequence of starting number s finish on a non-trivial cycle, then s is the only one starting number of the residue classe mod 2^k which satisfy to this condition.

The p^{st} iterate of a Syracuse sequence of starting number s can be literally written as expressed in (0) :

$$T^p(s) = \frac{3^r}{2^p} \cdot s + \sum_{i=1}^r \frac{3^{r-i}}{2^{\alpha_i + \dots + \alpha_r}} \quad \text{with } r = \left\lfloor \frac{p}{\log_2 3} \right\rfloor \text{ and } \sum_{i=1}^r \alpha_i = p$$

If s and p satisfy to $T^p(s) = s$, this implies that :

$$\frac{3^r}{2^p} \cdot s + \sum_{i=1}^r \frac{3^{r-i}}{2^{\alpha_i + \dots + \alpha_r}} = s \quad \Leftrightarrow \quad \left(\frac{3^r}{2^p} - 1\right) \cdot s = - \sum_{i=1}^r \frac{3^{r-i}}{2^{\alpha_i + \dots + \alpha_r}} < 0 \quad \Leftrightarrow \quad 3^r < 2^p$$

$$\text{Then for all } m : \quad T^p(2^p \cdot m + s) = \frac{3^r}{2^p} \cdot (2^p \cdot m + s) + \sum_{i=1}^r \frac{3^{r-i}}{2^{\alpha_i + \dots + \alpha_r}} = 3^r \cdot m + T^p(s) < 2^p \cdot m + s$$

Because $3^r \cdot m < 2^p \cdot m$ and $T^p(s) = s$

We can conclude that if s is the starting number of a Syracuse sequence which finish to a non-trivial cycle, then for all positive integer $m > 0$, $\sigma(2^p \cdot m + s) = p$, which means that s is the only one starting number of the residue classes mod 2^p which finish to a non-trivial cycle, all other integer $2^p \cdot m + s$ of this residue classes having a finite stopping time.

III – An exact formulation of $h_n(n)$:

Mike Winker [6] 2017 has stated in an excellent paper that the number z_r of residue classes (mod 2^{σ_r}) of starting numbers s with finite stopping time $\sigma(s)=\sigma_r$ with $\sigma_r = \lfloor 1 + r \cdot \log_2 3 \rfloor$ will satisfy to :

$$z_r = \frac{(m+r-2)!}{m! \cdot (r-2)!} - \sum_{i=2}^{r-1} \binom{\lfloor \frac{3(r-i)+\delta \rfloor}{2} \rfloor}{r-i} \cdot z_i \quad \text{with } m = \lfloor (r-1) \cdot \log_2 3 \rfloor - (r-1) \quad (7)$$

Where δ can take different values mod[3], but Mike Winker specifies that this value is not obvious to estimate and in his paper, it seems to be not simple to estimate z_r for some $r > 50$

In our work, $z_r = h_{p(r)}(p(r))$ is the number of residue classes (mod $2^{p(r)}$) of starting number s such as $\sigma(s)=p$

We'll give in this paragraph, a new formulation of Winkler formula independent of the parameter δ and which will allow to compute the exact values of z_r for any r . In Winker work, r represents the number of odd integers in the Syracuse integer sequence up to σ_r iterates.

We give a new definition of z_n , as the number of residue classes (mod 2^n) of Syracuse integer sequences of starting number s such as $\sigma(s)=n$. Thanks to this new definition, we are going to show that z_n can be more simply formulated without using the parameter δ . As defined above, z_n corresponds to $h_n(n)$, which is exactly the number of residue classes mod 2^n of starting number s such as $\sigma(s)=n$ with $n(r)=\lfloor r \cdot \log_2 3 + 1 \rfloor$ and $r(n) = \lfloor \frac{n}{\log_2 3} \rfloor$

Theoreme 5 : The number of residue classes mod 2^n of starting number s such as $\sigma(s)=n$ is given by :

$$z_{n(r)} = h_{n(r)}(n(r)) = \binom{n(r)-2}{r-2} - \sum_{i=2}^{r-1} \binom{n(r)-n(i)}{r-i} \cdot z_{n(i)} \quad (9)$$

with $z_1 = 1, z_2 = 1, z_4 = 1$

We consider the subsequence $\mathcal{F}_n(s) = \{T^k(s), k \leq n\} = \{s, T(s), \dots, T^n(s)\}$ and we are going to look for all integers s such as $\sigma(s)=n$. Which is equivalent to find s such as for $k < n, T^k(s) > s > T^n(s)$.

For example, for $s=5, \mathcal{F}_n(s) = \{5, 8, 4\}$ and $\sigma(s)=2$. For $s=7, \mathcal{F}_n(s) = \{7, 11, 17, 26, 13, 20, 10, 5\}$ and $\sigma(s)=7$

We can quickly check that :

$$\sigma(s)=1 \text{ if } s \equiv 0 \pmod{2}$$

$$\sigma(s)=2 \text{ if } s \equiv 1 \pmod{4}$$

$$\sigma(s)=4 \text{ if } s \equiv 3 \pmod{16}$$

$$\sigma(s)=5 \text{ if } s \equiv 11, 23 \pmod{32}$$

$$\sigma(s)=7 \text{ if } s \equiv 7, 15, 59 \pmod{128}$$

$$\sigma(s)=8 \text{ if } s \equiv 39, 79, 95, 123, 175, 199, 219 \pmod{256}$$

We are going to associate to each $\mathfrak{T}_n(s)$ a sequence $T_n(s)=\{t_1, \dots, t_n\}$ of the transitions between the elements of $\mathfrak{T}_n(s)$ where $t_i = 3/2$ if $T^{i-1}(s)$ is odd and $t_i = 1/2$ if $T^{i-1}(s)$ is even.

If $s \in \{7, 15, 59\}$, $\sigma(s)=7$:

To find all integers s such as $\sigma(s)=7$, it corresponds to solve the Diophantine equation :

$$y = \frac{3^4}{2^7} \cdot s + \sum_{i=1}^4 \frac{3^{4-i}}{2^{\alpha_1 + \dots + \alpha_4}} \text{ with } y < s$$

Which can be written :
$$2^7 \cdot y' - 3^4 \cdot s' = \sum_{i=1}^4 3^{4-i} \cdot 2^{7-(\alpha_1 + \dots + \alpha_4)}$$

where $c = \sum_{i=1}^4 3^{4-i} \cdot 2^{7-(\alpha_1 + \dots + \alpha_4)}$ is an odd number.

According to Bachet-Bezout theorem, the equation $2^7 \cdot y' - 3^4 \cdot s' = 1$ where 2^7 and 3^4 are coprime, has an infinity of solution and a unique solution such as $y' < 3^4$ and $s' < 2^7$ and $y' < s'$.

Then $y''=c \cdot y'$ and $s''=c \cdot s'$ are solutions of (10) and if $k = \left\lfloor \frac{c \cdot y'}{3^4} \right\rfloor$, then $y = c \cdot y' - k \cdot 3^4$ and $s = c \cdot s' - k \cdot 2^7$ is the only solution of (10) satisfying to $y < 3^4$ and $s < 2^7$.

Effectively, by construction, $y < 3^4$ and

$$s = 2^7 \cdot \left(\frac{c \cdot s'}{2^7} - k \right) = 2^7 \cdot \left(\frac{c \cdot s'}{2^7} - \left\lfloor \frac{c \cdot y'}{3^4} \right\rfloor \right) = 2^7 \cdot \left(\frac{c}{2^7} \cdot \left(\frac{2^7 \cdot y'}{3^4} - \frac{1}{3^4} \right) - \left\lfloor \frac{c \cdot y'}{3^4} \right\rfloor \right) = 2^7 \cdot \left(\frac{c \cdot y'}{3^4} - \left\lfloor \frac{c \cdot y'}{3^4} \right\rfloor - \frac{c}{2^7 \cdot 3^4} \right) < 2^7 \cdot \left(\frac{c \cdot y'}{3^4} - \left\lfloor \frac{c \cdot y'}{3^4} \right\rfloor \right) < 2^7$$

$$\mathfrak{T}_7(7) = \{7, 11, 17, 26, 13, 20, 10, 5\} \text{ and } T_7(s) = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \right\} \text{ and } c=73$$

$$\mathfrak{T}_7(15) = \{15, 23, 35, 53, 80, 40, 20, 10\} \text{ and } T_7(s) = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \text{ and } c=65=3^4 - 2^4$$

$$\mathfrak{T}_7(59) = \{59, 89, 134, 67, 101, 152, 76, 38\} \text{ and } T_7(s) = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \right\} \text{ and } c=85=5 \cdot (3^4 - 2^6)$$

For $\sigma(s)=8$, the set of s satisfying (10) is $\{39, 79, 95, 123, 175, 199, 219\}$ corresponding to the following $T_8(s)$

$$\mathfrak{T}_8(39) = \{39, 59, 89, 134, 67, 101, 152, 76, 38\} \text{ and } T_8(s) = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \right\} \text{ and } c=251$$

$$\mathfrak{T}_8(79) = \{79, 119, 179, 269, 404, 202, 101, 152, 76\} \text{ and } T_8(s) = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\} \text{ and } c=259$$

$$\mathfrak{T}_8(95) = \{95, 143, 215, 323, 485, 728, 364, 182, 91\} \text{ and } T_8(s) = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \text{ and } c=211=3^5 - 2^5$$

$$\mathfrak{T}_8(123) = \{123, 185, 278, 139, 209, 314, 157, 236, 118\} \text{ and } T_8(s) = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\} \text{ and } c=319=5 \cdot 3^5 - 7 \cdot 2^7$$

$$\mathfrak{T}_8(175) = \{175, 263, 395, 593, 890, 445, 668, 334, 167\} \text{ and } T_7(s) = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \right\} \text{ and } c=227$$

$$\mathfrak{T}_8(199) = \{199, 299, 449, 674, 337, 506, 253, 380, 190\} \text{ and } T_7(s) = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\} \text{ and } c=283$$

$$\mathfrak{T}_8(219) = \{219, 329, 494, 247, 371, 557, 836, 418, 209\} \text{ and } T_7(s) = \left\{ \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \right\} \text{ and } c=287$$

The set of integers s who satisfy $\sigma(s)=8$ is isomorphic to the set of permutations $T_8(s)$ who satisfy to $\prod_{j=1}^8 t_j < 1$ and for all $p < 8$, $\prod_{j=1}^p t_j > 1$

More generally we can characterize the Syracuse sequences of starting number s such as $\sigma(s)=n$ as isomorphic to the set of combinations $T_n(s)=\{t_1, \dots, t_n\}$ satisfying :

$$\prod_{j=1}^{j=n} t_j < 1 \text{ and for all } p < n, \prod_{j=1}^{j=p} t_j > 1$$

Effectively, if $\sigma(s)=n$ then thanks to (0) : $T^n(s) = \frac{3^r}{2^n} \cdot s + \sum_{i=1}^r \frac{3^{r-i}}{2^{\alpha_i + \dots + \alpha_r}} < s < T^p(s)$ for $p < n$ (10)

This implies that $\frac{3^r}{2^n} \cdot s < s \Leftrightarrow \prod_{j=1}^{j=n} t_j < \frac{3^r}{2^n} < 1$

And for all $p < n, \prod_{j=1}^{j=p} t_j > 1$ implies $\frac{3^{r(p)}}{2^p} > 1$ and $T^p(s) = \frac{3^{r(p)}}{2^p} \cdot s + \sum_{i=1}^{r(p)} \frac{3^{r(p)-i}}{2^{\alpha_i + \dots + \alpha_{r(p)}}} > s$

For each $T_n(s)$, there is a unique set $\{\alpha_i\}_{i \leq r}$. And for each $\{\alpha_i\}_{i \leq r}$, the equation above has a unique solution (s, y) with $s < 2^n$ and $y < 3^r$

If s is a starting number such as $\sigma(s)=n$, the sequence $T_n(s)=\{\frac{3}{2}, \dots, \frac{1}{2}\}$ associated to s contains exactly $r = \lfloor \frac{n}{\log_2 3} \rfloor$ elements $\{\frac{3}{2}\}$ and $n-r$ elements $\{\frac{1}{2}\}$ such as $\prod_{j=1}^{j=n} t_j < 1$ and for all $p < n, \prod_{j=1}^{j=p} t_j > 1$.

Thus, $z_{n(r)}$ is exactly the number of solutions who satisfy (10).

We remarks that all sequences $T_n(s)$ for $n > 4$ start with $t_1 = \{\frac{3}{2}\}, t_2 = \{\frac{3}{2}\}$, which implies that the number of combination of $r(n) = \lfloor \frac{n}{\log_2 3} \rfloor$ elements $\{\frac{3}{2}\}$ and $n-r$ elements $\{\frac{1}{2}\}$ is exactly $\binom{n(r)-2}{r-2}$, where $n(r) = \lfloor r \cdot \log_2 3 + 1 \rfloor$.

But we have to deduct the number of integers which have a stopping time lower than $n(r)$. It corresponds, for all $1 < i < r$, to the combinations $T_{n(i)}(s) = \{t_1, \dots, t_{n(i)}\}$ such as $\prod_{j=1}^{j=n(i)} t_j < 1$ and $\prod_{j=1}^{j=p} t_j > 1$ for $p < n(i)$, which is exactly $\binom{n(r)-n(i)}{r-i} \cdot z_{n(i)}$

Which finally gives :
$$z_{n(r)} = \binom{n(r)-2}{r-2} - \sum_{i=2}^{i < r} \binom{n(r)-n(i)}{r-i} \cdot z_{n(i)}$$

We give below the detail of the application of (9) for the first values of n .

$$z_5 = \binom{3}{1} - \binom{1}{1} \cdot z_4 = 2$$

$$z_7 = \binom{5}{2} - \binom{3}{2} \cdot z_4 - \binom{2}{1} \cdot z_5 = 3$$

$$z_8 = \binom{6}{3} - \binom{4}{3} \cdot z_4 - \binom{3}{2} \cdot z_5 - \binom{1}{1} \cdot z_7 = 7$$

$$z_{10} = \binom{8}{4} - \binom{6}{4} \cdot z_4 - \binom{5}{3} \cdot z_5 - \binom{3}{2} \cdot z_7 - \binom{2}{1} \cdot z_8 = 12$$

$$z_{12} = \binom{10}{5} - \binom{8}{5} \cdot z_4 - \binom{7}{4} \cdot z_5 - \binom{5}{3} \cdot z_7 - \binom{4}{2} \cdot z_8 - \binom{2}{1} \cdot z_{10} = 30$$

$$z_{13} = \binom{11}{6} - \binom{9}{6} \cdot z_4 - \binom{8}{5} \cdot z_5 - \binom{6}{4} \cdot z_7 - \binom{5}{3} \cdot z_8 - \binom{3}{2} \cdot z_{10} - \binom{1}{1} \cdot z_{12} = 85$$

$$z_{15} = \binom{13}{7} - \binom{11}{7} \cdot z_4 - \binom{10}{6} \cdot z_5 - \binom{8}{5} \cdot z_7 - \binom{7}{4} \cdot z_8 - \binom{5}{3} \cdot z_{10} - \binom{3}{2} \cdot z_{12} - \binom{2}{1} \cdot z_{13} = 173$$

$$z_{16} = \binom{14}{8} - \binom{12}{8} \cdot z_4 - \binom{11}{7} \cdot z_5 - \binom{9}{6} \cdot z_7 - \binom{8}{5} \cdot z_8 - \binom{6}{4} \cdot z_{10} - \binom{4}{3} \cdot z_{12} - \binom{3}{2} \cdot z_{13} - \binom{1}{1} \cdot z_{15} = 476$$

$$z_{18} = \binom{16}{9} - \binom{14}{9} \cdot z_4 - \binom{13}{8} \cdot z_5 - \binom{11}{7} \cdot z_7 - \binom{10}{6} \cdot z_8 - \binom{8}{5} \cdot z_{10} - \binom{6}{4} \cdot z_{12} - \binom{5}{3} \cdot z_{13} - \binom{3}{2} \cdot z_{15} - \binom{2}{1} \cdot z_{16} = 961$$

Thanks to (9) and (6), we can compute the exact values of $z_n = h_n(n), \frac{\Pi(2^n)}{2^n}, \frac{S(n)}{2^n} = 1 - \frac{\Pi(2^n)}{2^n}$ and $\frac{\log_2(S(n))}{n}$ as far as we want and we'll state later that for larger values of $n, \frac{S(n)}{2^n}$ continue to decrease constantly to 0.

r	n(r)	$Z_n=h_n(n)$	$1-S(n)/2^n$	$S(n)/2^n$
0	1	1	0,5000000000	0,5000000000
1	2	1	0,7500000000	0,2500000000
2	4	1	0,8125000000	0,1875000000
3	5	2	0,8750000000	0,1250000000
4	7	3	0,8984375000	0,1015625000
5	8	7	0,9257812500	0,0742187500
6	10	12	0,9375000000	0,0625000000
7	12	30	0,9448242188	0,0551757813
8	13	85	0,9552001953	0,0447998047
9	15	173	0,9604797363	0,0395202637
10	16	476	0,9677429199	0,0322570801
11	18	961	0,9714088440	0,0285911560
12	20	2652	0,9739379883	0,0260620117
13	21	8045	0,9777741432	0,0222258568
14	23	17637	0,9798766375	0,0201233625
15	24	51033	0,9829184413	0,0170815587
16	26	108950	0,9845419228	0,0154580772
17	27	312455	0,9868698940	0,0131301060
18	29	663535	0,9881058242	0,0118941758
19	31	1900470	0,9889907995	0,0110092005
20	32	5936673	0,9903730389	0,0096269611
21	34	13472296	0,9911572298	0,0088427702
22	35	39993895	0,9923212053	0,0076787947
23	37	87986917	0,9929613944	0,0070386056
24	39	257978502	0,9934306545	0,0065693455
25	40	820236724	0,9941766555	0,0058233445
26	42	1899474678	0,9946085460	0,0053914540
27	43	5723030586	0,9952591793	0,0047408207
28	45	12809477536	0,9956232465	0,0043767535
29	46	38036848410	0,9961637827	0,0038362173
30	48	84141805077	0,9964627144	0,0035372856
31	50	248369601964	0,9966833109	0,0033166891
32	51	794919136728	0,9970363259	0,0029636741
33	53	1857112329035	0,9972425068	0,0027574932
34	54	5636545892795	0,9975553980	0,0024446020
35	56	12732900345928	0,9977321025	0,0022678975
36	58	38088111350198	0,9978642471	0,0021357529
37	59	123110229387834	0,9980778093	0,0019221907
38	61	290838337577435	0,9982039404	0,0017960596
39	62	889949312454085	0,9983969174	0,0016030826
40	64	2029460152095008	0,9985069346	0,0014930654
41	65	6113392816333320	0,9986726385	0,0013273615
42	67	13759389839553008	0,9987658757	0,0012341243
43	69	41156292958100112	0,9988355972	0,0011644028
44	70	13318066714577072	0,9989484056	0,0010515944
45	72	315356241137505268	0,9990151849	0,0009848151
46	73	967303800643232882	0,9991176022	0,0008823978
47	75	2213388970068123188	0,9991761901	0,0008238099
48	77	6687324379116300569	0,9992204431	0,0007795569
49	78	21797112395398269352	0,9992925637	0,0007074363
50	80	52028134169251235063	0,9993356003	0,0006643997
51	81	160509643506854706934	0,9994019856	0,0005980144
52	83	369707873749224505928	0,9994402125	0,0005597875
53	85	1122428422670255691408	0,9994692266	0,0005307734
54	86	3672921591387837707209	0,9995166980	0,0004833020
55	88	8808298119720364971552	0,9995451591	0,0004548409
56	89	27272844922266198818078	0,9995892208	0,0004107792
57	91	63092460692093312467525	0,9996147036	0,0003852964
58	92	192189781828748623023765	0,9996535160	0,0003464840
59	94	438474769118020519475109	0,9996756533	0,0003243467
60	96	1325438036712274130536314	0,9996923827	0,0003076173
61	97	4327322846731848749589802	0,9997196920	0,0002803080
62	99	10359365401268828714082041	0,9997360362	0,0002639638
63	100	32053249939776775765443011	0,9997613217	0,0002386783

Table 3 above gives $z_n = h_n(n), \frac{\Pi(2^n)}{2^n}, \frac{S(n)}{2^n}$ for all n up to 100. The values of z_n in blue correspond to the values obtained by our numerical approach presented in §II and the red one correspond to the additional values of z_n obtained by Mike Winkler. The others are only obtained thanks to (10) and (6).

We have pushed further the computation of $z_n = h_n(n), \frac{S(n)}{2^n}$ and $\frac{\log_2(S(n))}{n}$ up to $n=19475$ and this confirm that $\frac{S(n)}{2^n}$ decrease to 0.

For $n=100$, $\frac{S(n)}{2^n} \approx 0,000225$ and $\frac{\log_2(S(n))}{n} = 0.8788221262$ and $z_{100} \approx 32053249939776775765443011$

For $n=405$ ($r(n)=256$): $\frac{S(n)}{2^n} \approx 9.68160440706356E - 10$ and $\frac{\log_2(S(n))}{n} \approx 0.9260641116$ and

$z_{405} 3476553789120508476368100052260690271283238505581916333757459587755180695960919229021382116342674546834066825086$

For $n=6491$, $\frac{S(n)}{2^n} \approx 3.3659284589631293e - 103$ and $\theta(n) = \frac{\log_2(S(n))}{n} \approx 0.9475569882514053$ and

$Z_{6491} = 1549953482341234298531051149482800040632325009694686038033078427338305145689709906961470300749434637893940359158$
2512318442999308507517003815279709990543329619343613848183973371083841206580716308821530301280999492021941980676965049
6237607137010501268785264252409021727841679305453896616094241954312045376151584561954125233542871895435094852874194945
1553294021820576701491368112788781493912080218089408728037556963606879833715800237988588144539097112848929150007103412
6615242991979534935759552371133200930739959341878888255296619921583933170205087263850124357873566296762654534024833267
5140397669817918885645616831363124123452204304419323433123396558038371249787651679669548556006138118438115285510552329
4654190077942283130089622264439172776649892653067682922787579271521036633862350870813505409115492502543723278116931273
0870029123370032231857541813241621141831619487292688665496626714643926648645015486528556824949365004015540509550651756
0173674103742654986573288372848870177941448219339198000404101615443980450904811397458334096012183355099414500015214500
0835901958473375601043048139736610743341156113338555874923587252763023443905226117793877118909074105544464914508868430
21926231439543116907747100011416373731215382190496322657403290906963388253424048449433404440992765839553800435737401131
8802346190192629996722166970005328248783464695569285405650656319563010761941544695171674431098567331998906707740575021
98487132114489560984995785675392664218323262373537862377812009088829948031280474340521141404943352379348729447342695
5522487565439767629649748194969201002378687149679655692680207964357072115475325256278678120735323512701447495071546929
09172784547453231633253850845828073978645022241210957974050813933647698452432647742813626874841583835986045329777378
405613077471918673506094379013425635388632545161221252205743894014381104050498725002606491.09475569882514053

For $n=19475$, $\frac{S(n)}{2^n} \approx 1,61925631106306E - 299$ and $\theta(n) = \frac{\log_2(S(n))}{n} = 0,949034086315567$ and

$Z_{19475} = 278015778572952955308866376019938626224161942217421628159384264534869383553344461733623620197181932063128166243395402132$
751452768229231425747408892607754365059138041988568408198240864937046684532939304371592351593791017311456284759259099525184066822
24506948993489417895347813636227210018090498543782722533177763220973632083118052731897352532380999943336123990466284388639958988
36181483939509286431777073281953181525739690054258391628617238913312604858753665768822667252350029210167035671641633415176234615
817002036243312562309256244099093682740325443894861440448878845782887925388647518218554218948028391515810602648545756100595522568
186265486156418446044950447278523012029272432717235997593063647785503942649287042037136390138494766029459705824851433352743443172
008100863473877089534951245171500293161735031198382339020959537476077803261172128095727224786839483277455076213690232240230524311
04978363794342803881118736111613517701767374483464455070485066622779439440080862466723163421916944737300119287412460968036616523
647332177435295015078156901131232124107675454990513075351013242541936113743060757147310167815943245127247395709636260224077980542
783724980146556297816716085422876141124739440456245644285524378406809580416117439398118083529653716198499224231050139993412379410
17948022505153449272863475452085258132748293383434160768386510656420430050564370373361048336604599409002406661154692410210021598
535938245074538601669822532124659307137924407771746589979388893381085382625746877882157102113808458659435333926342181273555446307
297491425613698680529270160758290657302372100447677237488287314127493270080805784638307314878911214733961242226885221254446931796
081204010560191478053051927991007847884035026374253914435718111327931509035522579617359988330041127010341510275738248073455797332
945925604776857256023319901772257034728013257887148962925810406563428541595575645380565508182319302418205126384762581295213159759
265439677031806624880862656846726942495816707474597984042198169426789437815858240994805509262737548132326494403320416561164619701
2219345501452824703055447234935132022024014967600493974618528663543568512978387947298026026998240387922146986612211771137747220
77689912269190818260032574963845504818826836532922548840211677444954182759519664255772056290191507330224479912626559897459810696
819384076386884575629217776905466417934323282140336579811586213389787428785660011686284929819574618257136438707233246214809209829
072297381238403030726805262923878351521073560724055050408197985739747349015909175243369107435997566003719692696456515044589825429
840489558446324518253961363920572786071228780127537722652554187033131778828808939150041186732652844510112647955744945822330635423
509023355222129969890874580716184807325445357238758311883457726137107461560856271239617107150627529870187204335659971902237815976
732841271100470694331497367945135112581978812907547344867359984732869608309219446765560438809589863583721801114185536570799038722
609746039510341742948094388582084176989460230762542756614734173438506076254166169134322320251237411351193891218890607434413301984
624837157673299616603553121460643520344257716391596069508440454194127376015092609980946651958689322439468087618789453802371990715
821873537555332042861201052016737741288246583580950869890752088510760235598366480458392390214019929882919194507766144233849412780
536917540960955673799362829349757484093074667727812468071167669564307102175041343121663710498912208883240593337385179491236268519
052605671981268023054814653067773003838084400041716582007114348839031766164588061331380681665993983945755457314736379217535888306
493627030882759424403006583904174515442831130462636541847067979546956917063452841181602280536261923326385504103386805731754502765
902997233882185526643002795805990300621796298208075295974913545096532651231609166429648965213711260079066173292794935633412462714
687526297721835752211096731382552444693377392282080475480624257551348291098821818634289193853903842040614685804584806047157609706
362429159985168335946350858420746637429702319336955032322705632505470064913283290775025552601748342970384725496539656976409824378
31798868117581435733267228909603483205426456850554922874668364378193110938274716998370516861146177949043862023824542131817803610
564456646836717428181872880590797061072396945618862432060781172202035704660374912922406888193469220073082487874289419056181356486
693146860925306139475688237006837426980798563838850641355590218293486769295491460126099745938629103537564398915247846627450176952
811442647217319746788061628210449532671915316733463451645703434181599641634809073061729022180788383361167205118710532095606922811
314308104062921532399421371210831009802086584042207623371034914775921592281009928654547069889018592191184369718600123576085621481
425344341784313224550837502707087854389790409247706396305353680881078477392696530245042261734009107483738672127188719882915722361
1700568946096849366498075861125

At first, how can we interpret this results.

For $n=100$, $\frac{S(n)}{2^n} \approx 0,000225$ means that the percentage of residue classes mod 2^n of starting number s such as $\sigma(s) \leq 100$ is 99,9775%. But it also means that 99,9775% of starting numbers in \mathbb{N} has a stopping time below or equal to n

For $n=19475$, $\alpha_n = \frac{S(n)}{2^n} \approx 1,61925631106306E - 299$ means that the percentage of residue classes mod 2^n of starting number s such as $\sigma(s) \leq 19475$ is equal to $(1-\alpha_n)$. But it also means that $(1-\alpha_n)$ of starting number in \mathbb{N} has a stopping time below or equal to n .

The main interest to be able to compute the exact values of z_n for all n , is the best way to compute $\frac{\Pi(2^n)}{2^n}$, $\frac{S(n)}{2^n} = 1 - \frac{\Pi(2^n)}{2^n}$ in order to state that:
$$\lim_{n \rightarrow \infty} \frac{S(n)}{2^n} = 0$$

$\frac{\Pi(2^n)}{2^n} = \sum_{r=1}^{r=r(n)} \frac{Z_{p(r)}}{2^{p(r)}}$ is a growing function because for all $p(r)$, $Z_{p(r)} > 0$ and $\frac{S(n)}{2^n} = 1 - \frac{\Pi(2^n)}{2^n}$ is a decreasing function.

As given by the last numerical result above for $n > 19475$, $\alpha_n = \frac{S(n)}{2^n} = 1,61925631106306E - 299$

$$1 - \alpha_{19475} < \frac{\Pi(2^n)}{2^n} = \sum_{r=1}^{r=r(n)} \frac{Z_p}{2^p} < 1$$

Which imply for $n > 19475$:

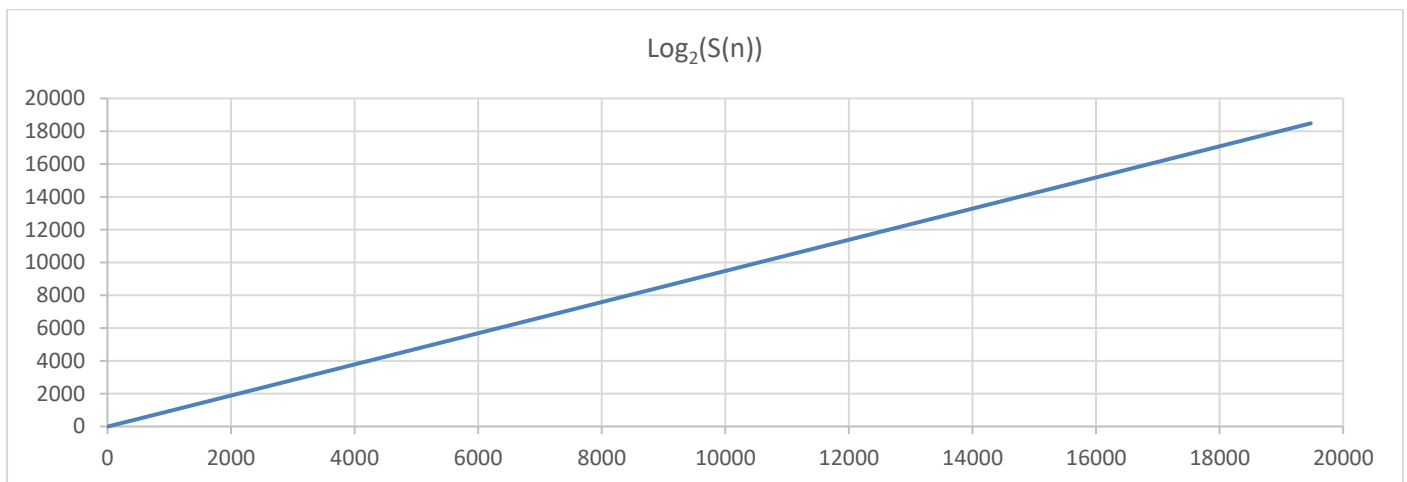
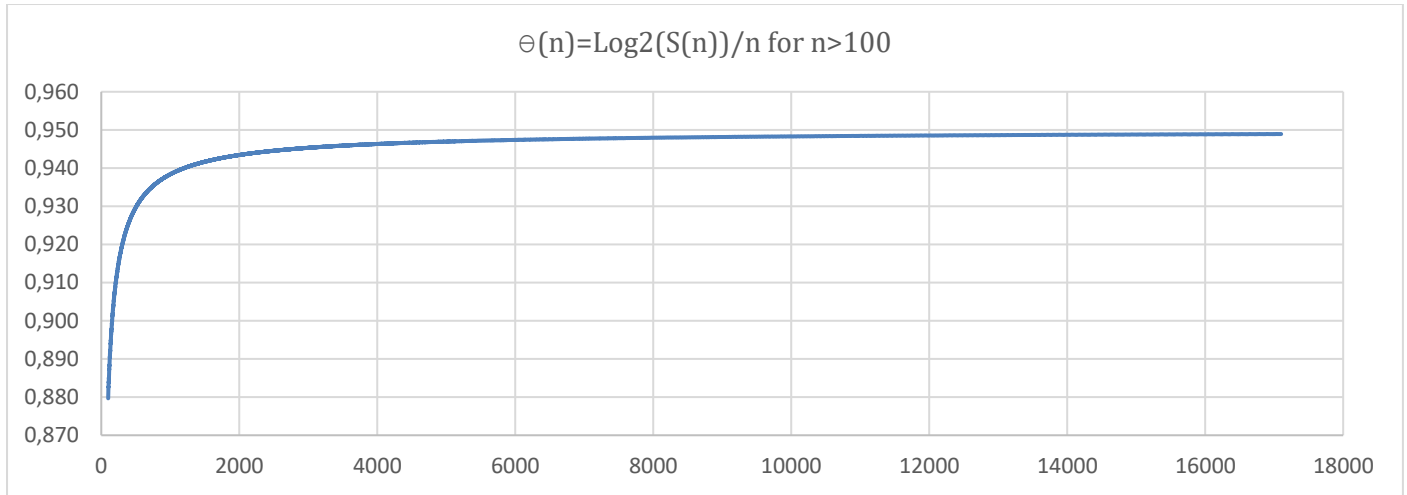
$$0 < \frac{S(n)}{2^n} = 1 - \frac{\Pi(2^n)}{2^n} = 1 - \sum_{r=1}^{r=r(n)} \frac{Z_p}{2^p} < \alpha_{19475}$$

In the following, we'll admit that :

$$\sum_{r=1}^{\infty} \frac{Z_p}{2^p} = 1 \tag{11}$$

IV – Behavior of the function $\text{Log}_2(S(n))/n$

In §I, we have displayed the arithmetic function $\text{Log}_2(S(n))$ for $n < 41$ and have seen that it was very close to a line. Thanks to the developments presented in §III, we are able to compute for any n , the values of z_n and consequently $S(n)$. The graphs below represent respectively the arithmetic functions $\theta(n) = \frac{\log_2(S(n))}{n}$ and $\log_2(S(n))$ up to $n=19475$.



This last graph confirm the strongly determinisc behavior of $S(n)$, we have conjectured in §II.

$$\text{For } n=6491, \theta(n) = \frac{\log_2(S(n))}{n} = 0.9475569882514053 \quad \text{and} \quad \frac{S(n)}{2^n} = 3.3659284589631293e - 103$$

$$\text{for } n=12983, \theta(n) = \frac{\log_2(S(n))}{n} = 0.9486409930166143 \quad \text{and} \quad \frac{S(n)}{2^n} = 1.88368718989396e - 201$$

$$\text{for } n=19475, \theta(n) = \frac{\log_2(S(n))}{n} = 0,949034086315567 \quad \text{and} \quad \frac{S(n)}{2^n} = 1,61925631106306E - 299$$

Theoreme 6

The percentage of residue classes mod 2^n of starting number s such as $\sigma(s) > n$, given by $P(n) = \frac{S(n)}{2^n}$ tends toward 0 when n grows and there is $\theta < 1$ such as $S(n) < 2^{n\theta}$ for n large.

$$\text{Thanks to (6) and (11), } S(n) = 2^n \cdot \left(1 - \sum_{r=1}^{r=r(n)} \frac{h_{p(r)}(p(r))}{2^{p(r)}}\right) = 2^n \cdot \left(1 - \sum_{r=1}^{r=r(n)} \frac{z_{p(r)}}{2^{p(r)}}\right) = 2^n \cdot \left(\sum_{r=r(n+1)}^{\infty} \frac{z_{p(r)}}{2^{p(r)}}\right) \quad (12)$$

We are going to find an upper bound for the last term of (12).

According to (9):
$$z_{p(r)} = \binom{p(r)-2}{r-2} - \sum_{i=2}^{i=r} \binom{p(r)-p(i)}{r-i} \cdot z_{p(i)} < \binom{p(r)-2}{r-2}$$

Thanks to Stirling formula for $0 < x < 1$ and p large :
$$\binom{p}{xp} = \frac{p!}{(xp)!((1-x)p)!} \simeq \frac{1}{(x^x(1-x)^{(1-x)})^p \sqrt{2\pi x(1-x)p}} = a \cdot \frac{q^p}{\sqrt{p}}$$

With $a = \frac{1}{\sqrt{2\pi x(1-x)}}$ and $q = \frac{1}{x^x(1-x)^{(1-x)}}$ and $x = \frac{r-2}{p(r)-2} = \frac{r}{p(r)} \cdot \frac{1-\frac{2}{r}}{1-\frac{2}{p(r)}} \simeq \frac{\ln(2)}{\ln(3)} \cdot \left(1 - \frac{2(1-x)}{x \cdot p(r)} + O\left(\frac{1}{p(r)^2}\right)\right)$

For p enough large, we have : $x \simeq \frac{\ln(2)}{\ln(3)} \simeq 0.6309297536$; $q = 1.9318131064$ and $a = 0.8267325638$

Then we can state that:
$$\frac{z_{p(r)}}{2^{p(r)}} < \frac{1}{2^p} \cdot \binom{p-2}{r-2} \simeq \frac{a \cdot q^{p-2}}{2^p \sqrt{p-2}} = \frac{a}{4 \cdot \sqrt{p-2}} \cdot \left(\frac{q}{2}\right)^{p-2}$$
 where $\frac{q}{2} < 0.9659065532$ (13)

Thanks to (13), we can give the following upper bound of (12) :

This will imply that
$$\frac{S(n)}{2^n} = \sum_{i>r(n)} \frac{z_{p(i)}}{2^{p(i)}} < \sum_{j>n} \frac{z_j}{2^j} < \sum_{j>n} \frac{a}{4 \cdot \sqrt{j-2}} \cdot \left(\frac{q}{2}\right)^{j-2} < \frac{C}{\sqrt{n-1}} \cdot \left(\frac{q}{2}\right)^{n-1}$$
 with $C = \frac{a}{4 \cdot (1-\frac{q}{2})} \simeq 6.07$

We can conclude that :
$$\lim_{n \rightarrow \infty} P(n) = \lim_{n \rightarrow \infty} \frac{S(n)}{2^n} = 0$$
 (14)

Now we can deduce that for n enough large, there is θ such as : $S(n) < 2^{n\theta}$

We have to find θ which satisfy to the relation :
$$\frac{S(n)}{2^n} < \frac{C}{\sqrt{n-1}} \cdot \left(\frac{q}{2}\right)^{n-1} < \frac{2^{n\theta}}{2^n}$$

Which imply for all $n > 40$:
$$\log_2(q) > \log_2(q) - \frac{\log_2(n-1)}{2n} + \frac{\log_2(\frac{2C}{q})}{n} > \frac{\log_2(S(n))}{n}$$

We can define a model of $\theta(n)$:
$$\theta_{\text{mod}}(n) = \log_2(q) - \frac{\log_2(n-1)}{2n} + \frac{\log_2(\frac{2C}{q})}{n}$$

Then we define:
$$\theta = \lim_{n \rightarrow \infty} \theta_{\text{mod}}(n) = \log_2(q)$$

Thus for $n > 40$, we can affirm that there is $\theta = \log_2(q) \simeq 0.9499555271$ such as $S(n) < 2^{n\theta}$ (15)

Corollary 1: For all $\epsilon > 0$ there is n such as $\Pi(2^n) = \#\{s < 2^n / \sigma(s) \leq n\} > 2^{n(1-\epsilon)}$

As defined above, $S(n) = \#\{s < 2^n / \exists j T_i(s) > n\} = 2^n - \Pi(2^n)$

Thanks to (15),
$$\frac{\Pi(2^n)}{2^n} = 1 - \frac{S(n)}{2^n} > 1 - \frac{2^{n\theta}}{2^n}$$

Which is equivalent to
$$\Pi(2^n) > 2^n \cdot \left(1 - \frac{2^{n\theta}}{2^n}\right) = 2^n \cdot 2^{\log_2\left(1 - \frac{2^{n\theta}}{2^n}\right)} = 2^{n \cdot \left(1 + \log_2\left(1 - \frac{2^{n\theta}}{2^n}\right)\right)}$$

Then for any $\epsilon > 0$, there is n such as :
$$\epsilon = -\log_2\left(1 - \frac{2^{n\theta}}{2^n}\right) \quad \text{and} \quad n = \frac{\log_2\left(\frac{2^\epsilon}{2^{\epsilon-1}}\right)}{1-\theta} \simeq -\frac{\log(\epsilon)}{(1-\theta) \cdot \log(2)}$$

Which allows to conclude that for all $\epsilon > 0$, there is n :
$$\Pi(2^n) > 2^{n(1-\epsilon)}$$
 (16)

Corollary 2: More generally, for all real $\epsilon > 0$, there is X enough large such as : $\Pi(X) > X^{(1-\epsilon)}$

This can be linked with the result obtained by Ilia Krasikov & Jeffrey C. Lagaria [3] which state that for x enough large $\pi_1(x) > x^{0.84}$.

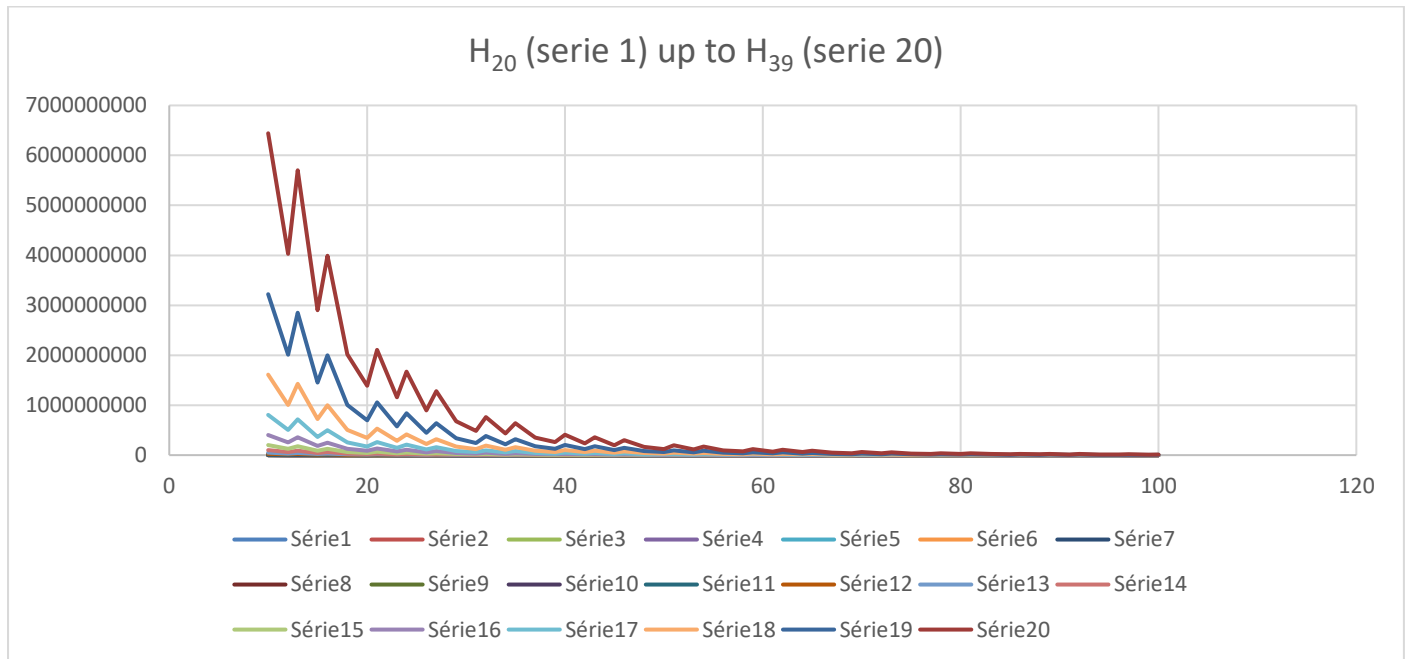
V – A model for H_n estimate :

We have stated in §II that $h_n(p) = 2^n \cdot \frac{h_p(p)}{2^p} = 2^n \cdot \frac{Z_p}{2^p}$ for all $p \leq n$ and $p = \lceil r \cdot \log_2 3 + 1 \rceil$.

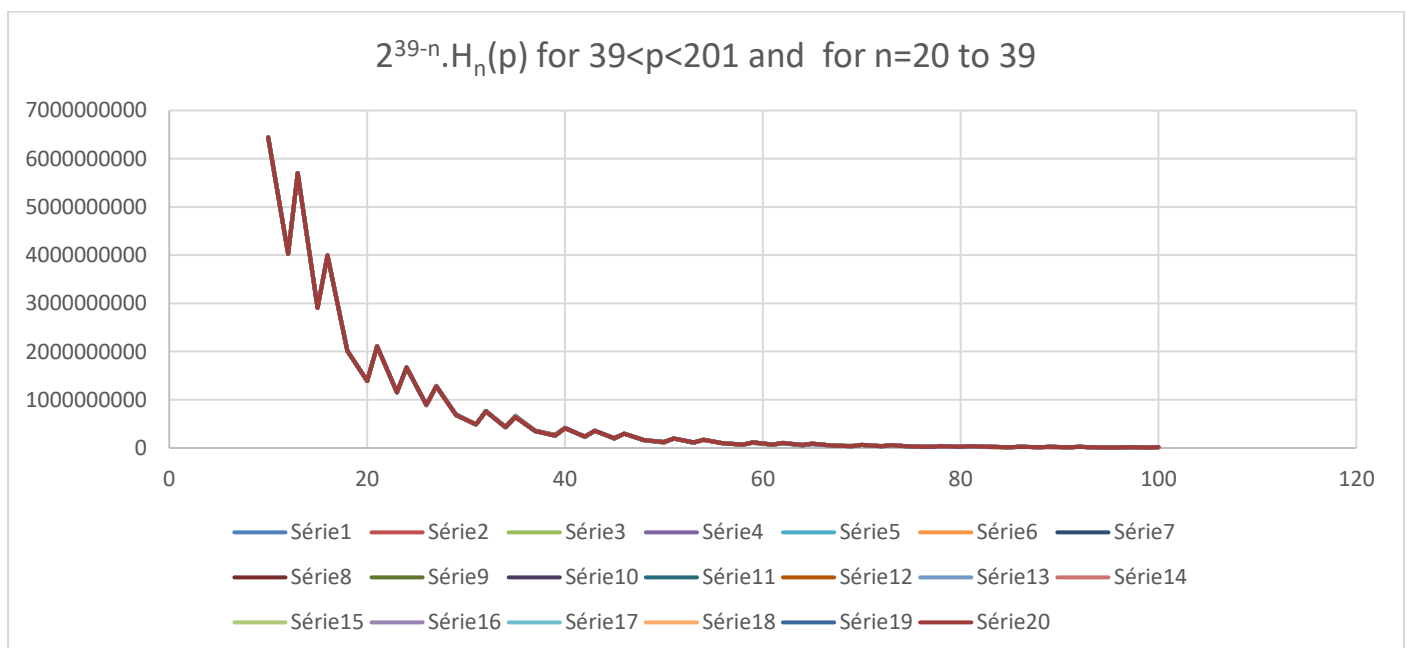
But we can see in table 2a and 2b, that this relation is also a good estimate for $p > n$ and for $h_n(p) \gg 0$:

$$h_n(p) \simeq 2^n \cdot \frac{Z_p}{2^p} \quad \text{and} \quad h_n(p) \simeq 2^{n-p} \cdot h_n(n) \quad (17)$$

This cannot be easily proved, but we are going to show, on the basis of the data of table 2a and 2b that for $p > n$ and $n < 40$, the histograms present an homothetic similarity as shown below

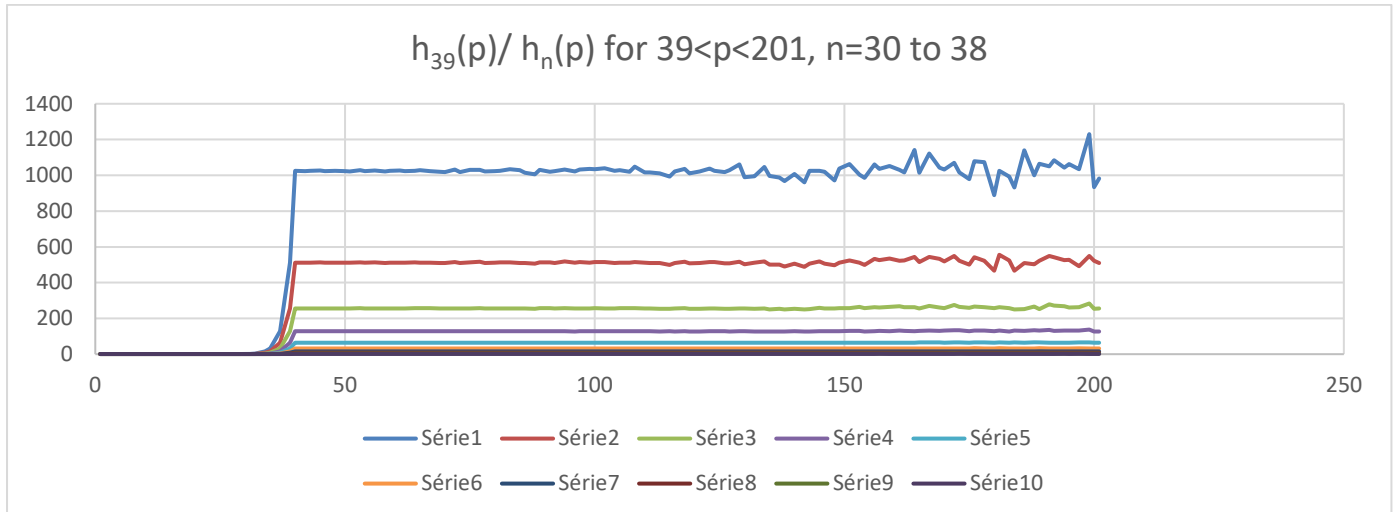


If we normalize these values by multiplying each H_n for $n < 39$ by 2^{39-n} , we obtain very good superimposition of all the resulting functions.



We can see the perfect superimposition of the all histograms after normalization.

The graphic below gives for each p, the ratio $h_{39}(p)/h_n(p)$ for $n=30$ to 39 and confirm that two histograms H_n and H_k are linked with a scale factor 2^{n-k} .



The variations of the ratio are more important for the highest values of p but it concerns some relatively small values of $h_n(p)$. If we compute an average ratio between two histograms weighted by the percentage of $h_n(p)$ in the whole population, given by $h_n(p)/S(n)$ and only limited to $p > n$, we obtain :

$$R_{n,n-k} = \sum_{p>n} \left(\frac{h_n(p)}{h_{n-k}(p)} \right) \cdot \left(\frac{h_n(p)}{S(n)} \right) \quad (18)$$

Remark : We have already seen in §II theorem 1, that for $p < n$, $h_{n+k}(p) = 2^k \cdot h_n(p)$

When we compute this ratio for $n=39$ and $k=1$ to 10 we obtain some values very close to 2^k with a very good accuracy.

(n-k,n)	$R_{n,n-k}$	$(R_{n,n-k} - 2^k)/2^k$
(29,39)	1024,625653	0,061099%
(30,39)	512,179219	0,035004%
(31,39)	256,0753832	0,029447%
(32,39)	128,0629038	0,049144%
(33,39)	64,01436314	0,022442%
(34,39)	32,00314693	0,009834%
(35,39)	16,00131223	0,008201%
(36,39)	8,000456933	0,005712%
(37,39)	4,000224834	0,005621%
(38,39)	2,000081232	0,004062%

We can state that the ratios $R_{n,n-k}$ can be correctly estimated by the relation :

$$R_{n,n-k} = 2^k(1 + o(k)) \approx 2^k = R_k \quad (19)$$

This result confirm the estimate of (17) : $h_n(p) \approx 2^{n-p} \cdot h_n(n)$ for $p > n$ and $h_n(p) \gg 1$

On the next page, we compare in table 5, the sums $\sum_{r=0}^{550} h_n(p(r))$, $2^n \cdot \sum_{r=0}^{550} \frac{z_{p(r)}}{2^{p(r)}}$ and 2^n for $n < 40$

For all n the sum of all $h_n(p)$ is 2^n which is the total number of residue classes mod $(2^n) : 2^n = \sum_{r=0}^{\infty} h_n(p(r))$

We'll see in the next paragraph that we have some arguments to conjecture that the maximal stopping time of the residue classes mod 2^n is lower than $20n$. For this reason, the column $\#H_n$ is the sum of $h_n(p)$ up to $p=20n$

$$\#H_n = \sum_{r=0}^{20n} h_n(p(r))$$

And $\#H_n$ modeled is given by : $\#H_n \text{ modeled up to } 20n = 2^n \cdot \sum_{p(r)=0}^{20n} \frac{z_{p(r)}}{2^{p(r)}}$ and $\#H_n \text{ modeled up to } 800 = 2^n \cdot \sum_{p(r)=1}^{p(r)=800} \frac{z_{p(r)}}{2^{p(r)}}$

In Table 5 below, we see that $\#H_n$ modeled and sum up to 800 gives for each n, the good population 2^n for $n \leq 40$ with an error of 10^{-14} and that $\#H_n$ modeled on $20n$ presents a difference with 2^n lower than 10^{-2} and very stable for all $n \leq 40$.

We can also see that the difference between $S(n)$ deduced from tables 2a and 2b and $S(n)$ estimated thanks to (10) remains under 10^{-2} for $n < 40$

n	2^n	$\#H_n$	$\#H_n$ modeled (sum up to $20n$)	Error	$\#H_n$ modeled (sum up to 800)	Error	$\Pi(n)$	$\Pi(n)$ modeled	Error	$S(n)$	$S(n)$ modeled	Error
2	4	4	3,9767066220	5,82E-03	4,0000000000	-1,42E-14	3	3	1,48E-15	1	1,0000000000	-6,128E-14
3	8	8	7,9846224747	1,92E-03	8,0000000000	-1,42E-14	6	6	1,48E-15	2	2,0000000000	-6,128E-14
4	16	16	15,9893696053	6,64E-04	16,0000000000	-1,42E-14	13	13	0,00E+00	3	3,0000000000	-7,579E-14
5	32	32	31,9923622949	2,39E-04	32,0000000000	-1,42E-14	28	28	8,88E-16	4	4,0000000000	-1,199E-13
6	64	64	63,9940511755	9,30E-05	64,0000000000	-1,42E-14	56	56	8,88E-16	8	8,0000000000	-1,199E-13
7	128	128	127,9951646816	3,78E-05	128,0000000000	-1,42E-14	115	115	7,41E-16	13	13,0000000000	-1,465E-13
8	256	256	256,0000000000	-1,42E-14	256,0000000000	-1,42E-14	237	237	7,20E-16	19	19,0000000000	-2,004E-13
9	512	512	511,9922366075	1,52E-05	512,0000000000	-1,42E-14	474	474	7,20E-16	38	38,0000000000	-2,004E-13
10	1024	1024	1023,9934163219	6,43E-06	1024,0000000000	-1,42E-14	960	960	5,92E-16	64	64,0000000000	-2,363E-13
11	2048	2048	2047,9945548068	2,66E-06	2048,0000000000	-1,42E-14	1920	1920	5,92E-16	128	128,0000000000	-2,363E-13
12	4096	4096	4095,9952935372	1,15E-06	4096,0000000001	-1,42E-14	3870	3870	5,88E-16	226	226,0000000001	-2,676E-13
13	8192	8192	8191,9958499618	5,07E-07	8192,0000000001	-1,42E-14	7825	7825	5,81E-16	367	367,0000000001	-3,296E-13
14	16384	16384	16383,9964070300	2,19E-07	16384,0000000002	-1,42E-14	15650	15650	5,81E-16	734	734,0000000002	-3,296E-13
15	32768	32768	32767,9967948172	9,78E-08	32768,0000000004	-1,42E-14	31473	31473	5,78E-16	1295	1295,0000000005	-3,736E-13
16	65536	65536	65535,9972216543	4,24E-08	65536,0000000009	-1,42E-14	63422	63422	5,74E-16	2114	2114,0000000010	-4,578E-13
17	131072	131072	131071,9975029510	1,91E-08	131072,0000000020	-1,42E-14	126844	126844	5,74E-16	4228	4228,0000000019	-4,578E-13
18	262144	262144	262143,9977061550	8,75E-09	262144,0000000040	-1,42E-14	254649	254649	5,71E-16	7495	7495,0000000039	-5,165E-13
19	524288	524288	524287,9979620870	3,89E-09	524288,0000000070	-1,42E-14	509298	509298	5,71E-16	14990	14990,0000000077	-5,165E-13
20	1048576	1048576	1048575,9981360800	1,78E-09	1048576,0000000100	-1,42E-14	1021248	1021248	5,70E-16	27328	27328,0000000155	-5,666E-13
21	2097152	2097152	2097151,9983455400	7,89E-10	2097152,0000000300	-1,42E-14	2050541	2050541	5,68E-16	46611	46611,0000000309	-6,644E-13
22	4194304	4194304	4194303,9984702900	3,65E-10	4194304,0000000600	-1,42E-14	4101082	4101082	5,68E-16	93222	93222,0000000619	-6,644E-13
23	8388608	8388608	8388607,9985758800	1,70E-10	8388608,0000001200	-1,42E-14	8219801	8219801	5,67E-16	168807	168807,0000001240	-7,338E-13
24	16777216	16777216	16777215,9987142000	7,66E-11	16777216,0000002000	-1,42E-14	16490635	16490635	5,65E-16	286581	286581,0000002480	-8,644E-13
25	33554432	33554432	33554431,9988064000	3,56E-11	33554432,0000005000	-1,42E-14	32981270	32981270	5,65E-16	573162	573162,0000004950	-8,644E-13
26	67108864	67108864	67108863,9989160000	1,62E-11	67108864,0000010000	-1,42E-14	66071490	66071490	5,64E-16	1037374	1037374,0000009900	-9,552E-13
27	134217728	134217728	134217727,9989910000	7,52E-12	134217728,0000020000	-1,42E-14	132455435	132455435	5,62E-16	1762293	1762293,0000019800	-1,125E-12
28	268435456	268435456	268435455,9990520000	3,53E-12	268435456,0000040000	-1,42E-14	264910870	264910870	5,62E-16	3524586	3524586,0000039600	-1,125E-12
29	536870912	536870912	536870911,9991290000	1,62E-12	536870912,0000080000	-1,42E-14	530485275	530485275	5,62E-16	6385637	6385637,0000079300	-1,241E-12
30	1073741824	1073741824	1073741823,9991900000	7,54E-13	1073741824,0000200000	-1,42E-14	1060970550	1060970550	5,62E-16	12771274	12771274,0000159000	-1,241E-12
31	2147483648	2147483648	2147483647,9992800000	3,34E-13	2147483648,0000300000	-1,42E-14	2123841570	2123841570	5,61E-16	23642078	23642078,0000317000	-1,341E-12
32	4294967296	4294967296	4294967295,9993600000	1,48E-13	4294967296,0000600000	-1,42E-14	4253679813	4253679813	5,60E-16	41287483	41287483,0000634000	-1,536E-12
33	8589934592	8589934592	8589934591,9994600000	6,33E-14	8589934592,0001200000	-1,42E-14	8507359626	8507359626	5,60E-16	82574966	82574966,0001268000	-1,536E-12
34	17179869184	17179869184	17179869183,9996000000	2,05E-14	17179869184,0002000000	-1,42E-14	17028191548	17028191548	5,60E-16	151677636	151677636,0002540000	-1,672E-12
35	34359738368	34359738368	34359738367,9999000000	1,55E-15	34359738368,0005000000	-1,42E-14	34096376991	34096376991	5,59E-16	263361377	263361377,0005070000	-1,926E-12
36	68719476736	68719476736	68719476735,0005000000	-7,77E-15	68719476736,0010000000	-1,42E-14	68192753982	68192753982	5,59E-16	526722754	526722754,0010150000	-1,926E-12
37	137438953472	137438953472	137438953471,0020000000	-1,18E-14	137438953472,0020000000	-1,42E-14	136473494881	136473494881	5,59E-16	965458591	965458591,0020290000	-2,102E-12
38	274877906944	274877906944	274877906943,0040000000	-1,38E-14	274877906944,0040000000	-1,42E-14	272946989762	272946989762	5,59E-16	1930917182	1930917182,0040600000	-2,102E-12
39	549755813888	549755813888	549755813887,0080000000	-1,42E-14	549755813888,0080000000	-1,42E-14	546151958026	546151958026	5,59E-16	3603855862	3603855862,0081200000	-2,252E-12

VI – Relation between two consecutive values of H_n :

As we have seen in §II, the histograms H_n has an homothetic link.

For $p \leq n$, $h_{p+k}(p) = 2^k \cdot h_p(p)$ and for $p > n$, $h_{p-k}(p) \approx 2^{-k} \cdot h_p(p)$ if $h_{p-k}(p) \gg 1$

We are going to define an arithmetic function to describe the behavior of the histograms H_n , as independently as possible of n .

We consider the arithmetic function $V_n(p(i))$ for $i \in \mathbb{N}^*$:

$$V_n(p(i)) = \frac{h_n(p(i))}{h_n(p(i-1))} \quad \text{with } p(i) = \lfloor i \cdot \log_2 3 \rfloor + 1 \quad (20)$$

The advantage of this approach is if there is an homothetic factor between H_n and H_{n+k} , we can hope that the different function $V_n(p)$ will be relatively independent of n for n enough large.

According to (2), for $n > p(i)$, $h_{n+k}(p) = 2^k \cdot h_n(p)$ then:

$$V_n(p(i)) = \frac{h_n(p(i))}{h_n(p(i-1))} = \frac{2^{n-p(i)} \cdot h_{p(i)}(p(i))}{2^{n-p(i-1)} \cdot h_{p(i-1)}(p(i-1))} = 2^{p(i)-p(i-1)} \cdot \frac{h_{p(i)}(p(i))}{h_{p(i-1)}(p(i-1))}$$

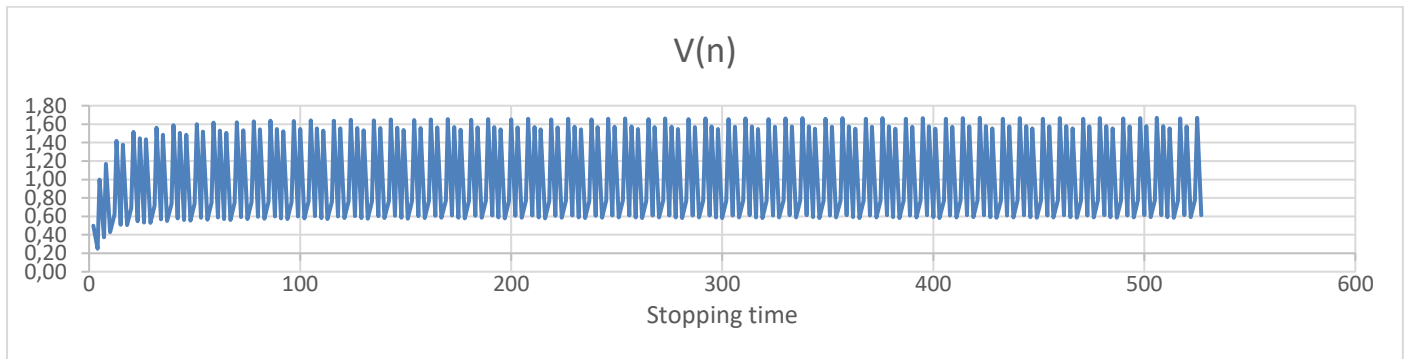
Which is independent of n .

This imply that we can define the arithmetic function

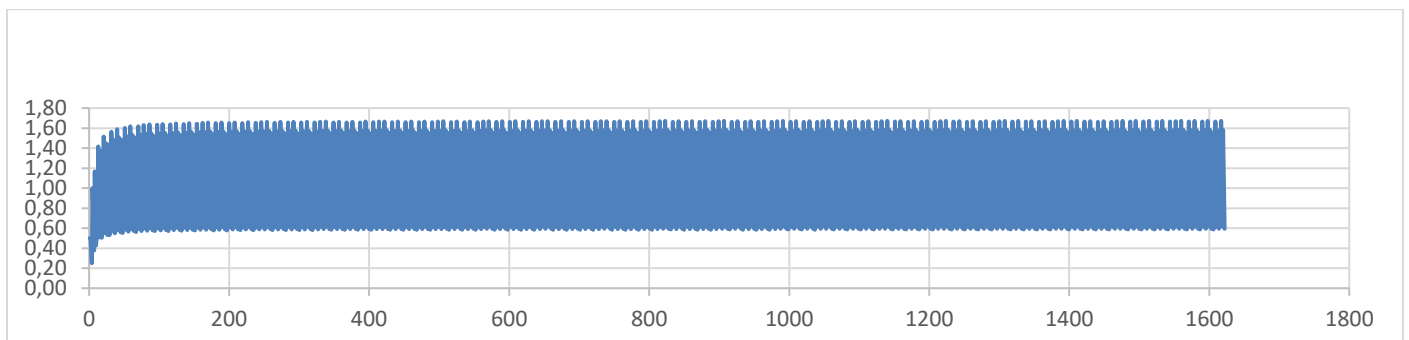
$$V(p(i)) = \lim_{n \rightarrow \infty} (V_n(p(i))) = V_{p(i)}(p(i)) = 2^{p(i)-p(i-1)} \cdot \frac{h_{p(i)}(p(i))}{h_{p(i-1)}(p(i-1))} \quad (21)$$

This function is interesting to describe the local behaviour of the $h_n(p(i))$ when i is growing.

We show below the arithmetic function $V(p(i))$ up to $p=527$ ($i=332$). It illustrates the extrem regularity of the arithmetic function $V(n)$ which seems to be quasi-periodic.



And we have pushed below the computation of the function $V(n)$ up to $n=1622$ ($r(n)=1024$)



These graphs present a quasi periodic behavior and shows that **$V(n)$ remains below 1.8**

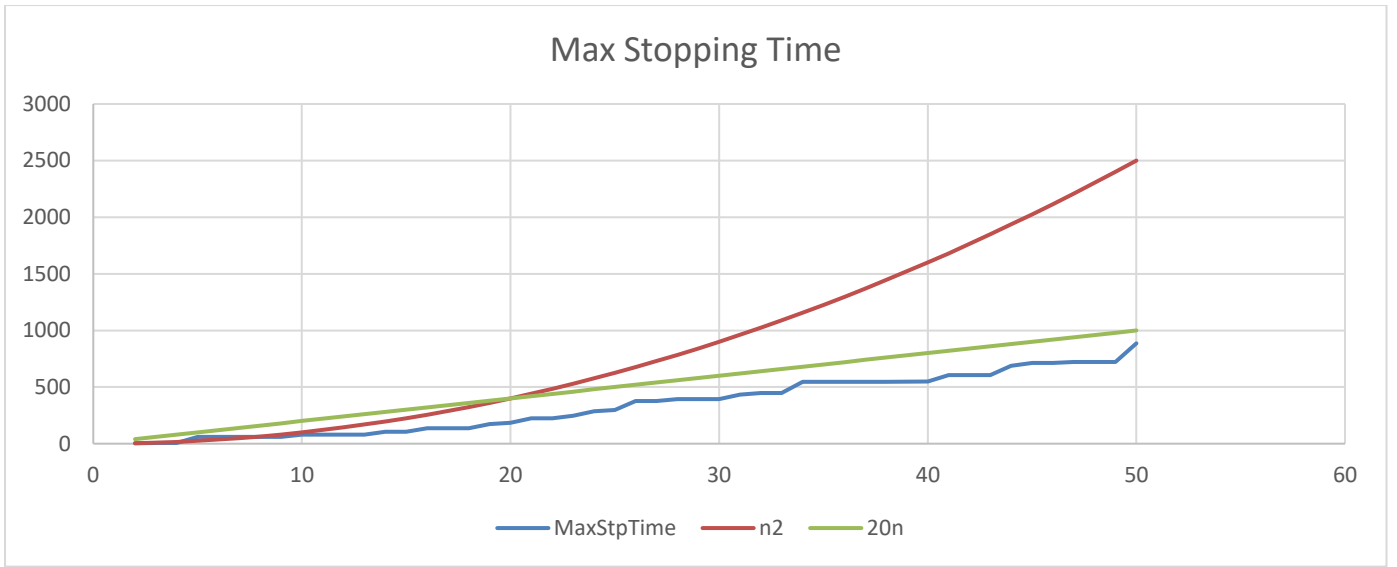
VIII – Maximal Stopping time estimate in $\mathbb{Z}/2^n\mathbb{Z}$.

We have seen previously that we can build the histograms H_n thanks to the exact formulation and computation of all z_n . As the modeled histograms is upper bounded for all n , we could hope that the real histograms H_n too.

We have computed for all $n \leq 38$, the maximum stopping time and use Oliveira E Silva[3]1999 results up to $n=50$.

n	MaxStpTime	n^2	MaxStpTime/ n^2	20n	MaxStpTime/20n
2	4	4	1,0000	40	0,100000
3	7	9	0,7778	60	0,116667
4	7	16	0,4375	80	0,087500
5	59	25	2,3600	100	0,590000
6	59	36	1,6389	120	0,491667
7	59	49	1,2041	140	0,421429
8	59	64	0,9219	160	0,368750
9	59	81	0,7284	180	0,327778
10	81	100	0,8100	200	0,405000
11	81	121	0,6694	220	0,368182
12	81	144	0,5625	240	0,337500
13	81	169	0,4793	260	0,311538
14	105	196	0,5357	280	0,375000
15	105	225	0,4667	300	0,350000
16	135	256	0,5273	320	0,421875
17	135	289	0,4671	340	0,397059
18	135	324	0,4167	360	0,375000
19	173	361	0,4792	380	0,455263
20	183	400	0,4575	400	0,457500
21	224	441	0,5079	420	0,533333
22	224	484	0,4628	440	0,509091
23	246	529	0,4650	460	0,534783
24	287	576	0,4983	480	0,597917
25	298	625	0,4768	500	0,596000
26	376	676	0,5562	520	0,723077
27	376	729	0,5158	540	0,696296
28	395	784	0,5038	560	0,705357
29	395	841	0,4697	580	0,681034
30	395	900	0,4389	600	0,658333
31	433	961	0,4506	620	0,698387
32	447	1024	0,4365	640	0,698438
33	447	1089	0,4105	660	0,677273
34	547	1156	0,4732	680	0,804412
35	547	1225	0,4465	700	0,781429
36	547	1296	0,4221	720	0,759722
37	547	1369	0,3996	740	0,739189
38	547	1444	0,3788	760	0,719737
40	550	1600	0,3438	800	0,687500
41	606	1681	0,3605	820	0,739024
42	606	1764	0,3435	840	0,721429
43	606	1849	0,3277	860	0,704651
44	688	1936	0,3554	880	0,781818
45	712	2025	0,3516	900	0,791111
46	712	2116	0,3365	920	0,773913
47	722	2209	0,3268	940	0,768085
48	722	2304	0,3134	960	0,752083
49	722	2401	0,3007	980	0,736735
50	886	2500	0,3544	1000	0,886000

We can see in these first results that for $n \leq 50$, the arithmetic function $\text{MaxStpTime}/n^2$ is a decreasing function and we have some argument to conjecture that $\text{MaxStpTime} < 20n$ for all n . If this conjecture could be verified, it would imply that all Syracuse sequence with a starting number $a < 2^n$ have a finite stopping time. This will imply that the Collatz conjecture would be true.



How can we estimate that the maximal stopping time are lower than $20n$.

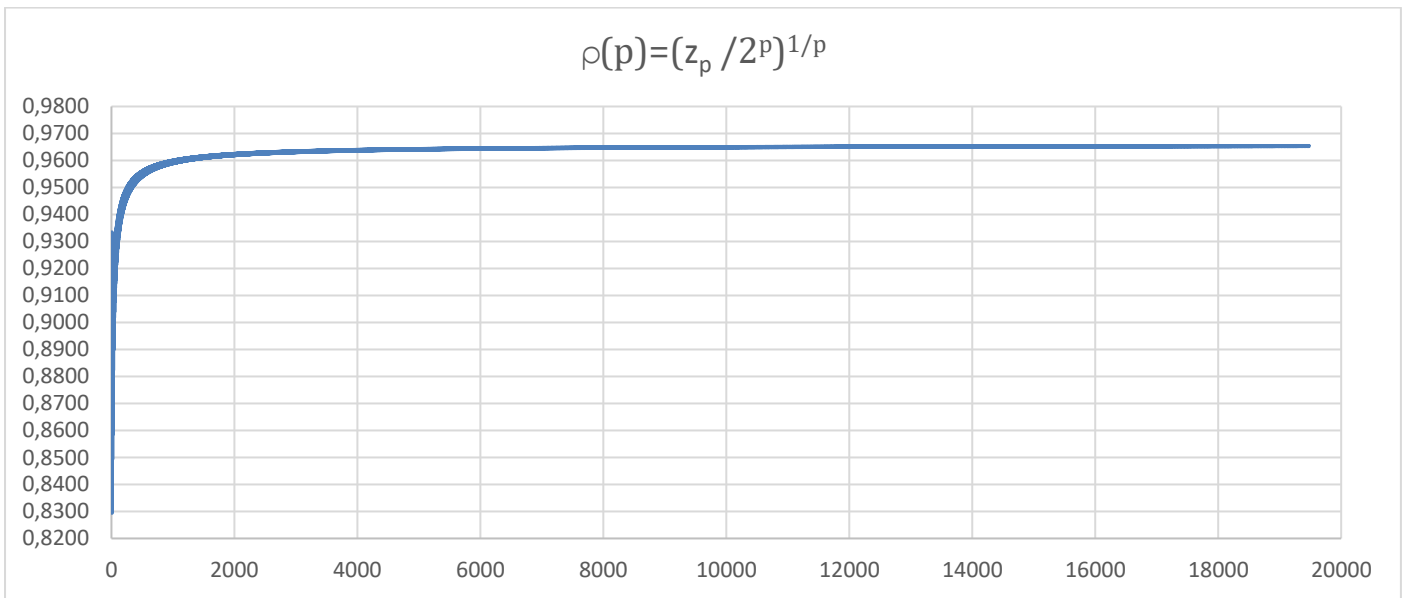
If we consider that the arithmetic function $z_p = h_p(p)$ allows us to generate the histograms H_n for all n as stated in (5), we can suppose that if there is p_0 such as for $p > p_0$, $h_n(p) < 1$, there is no integer $a \in \mathbb{Z}/2^n\mathbb{Z}$ such $\sigma(a) > p_0$.

As for all integers n and p belonging to the set of stopping time value, $h_n(p) = 2^{n-p} \cdot h_p(p) = 2^{n-p} \cdot z_p$, we are

looking for the value of ρ such as
$$\left| \lim_{p \rightarrow \infty} \frac{z_p}{2^p} \right| = 1$$

It corresponds to approximate the behavior of $\frac{z_p}{2^p}$ with a geometric sequence with a common ratio $\rho < 1$.

In the graph below, we represent the arithmetic function $\rho(p) = \sqrt[p]{\frac{z_p}{2^p}}$ and we define ρ as $\lim_{p \rightarrow \infty} \rho(p)$



As stated in (13), $\rho(p) < \left(\frac{a}{4\sqrt{p-2}}\right)^{\frac{1}{p-2}} \cdot \frac{q}{2}$ which implies that $\lim_{p \rightarrow \infty} \rho(p) < \frac{q}{2} = 0,9659065532$

The value of $\rho(p)$ obtained for $p=19475$ is below and coherent with the theoretical results obtained thanks to (13):

$$\rho(19475) = 0,96538167694149 < \left(\frac{a}{4\sqrt{p-2}}\right)^{\frac{1}{p-2}} \cdot \frac{q}{2} = 0,9655834805$$

We are going to try to estimate the maximal stopping time according two approaches :

1- To find $p_0=k.n$ such as for all $p > p_0, h_n(p) < 1$

At first, we are looking for the number $p=k.n$ such as $h_n(p) = 2^n \cdot \frac{z_p}{2^p} < 0,5$, because if we find p verifying this inequality, due to the fact that $V_{p+i} < 1.8$ and $V_{p+i} \cdot V_{p+i+1} < 1$, then all $i > 0$ will verify the inequality $2^n \cdot \frac{z_{p+i}}{2^{p+i}} < 1$.

We want to estimate k such as $p=k.n$.

Thanks to (13), we have stated that: $\frac{z_{p(r)}}{2^{p(r)}} < \frac{a}{4 \cdot \sqrt{p-2}} \cdot \left(\frac{q}{2}\right)^{p-2}$ where $\frac{q}{2} < 0.9659065532$

And we are looking $p=k.n$ such as $h_n(p) = 2^n \cdot \frac{z_p}{2^p} < 0,5$

which is implied by $2^n \cdot \frac{a}{4 \cdot \sqrt{p-2}} \cdot \left(\frac{q}{2}\right)^{p-2} < 0.5 \Leftrightarrow p-2 > \frac{(n-1)\ln(2)+\ln(a)-\frac{\ln(p-2)}{\epsilon}}{\ln\left(\frac{2}{q}\right)}$

Which is true if $p-2 > \frac{(n-1)\ln(2)+\ln(a)}{\ln\left(\frac{2}{q}\right)} \Leftrightarrow k(n) = \frac{p}{n} > \frac{\ln(2)}{\ln\left(\frac{2}{q}\right)} + \frac{\ln(2a/q^2)}{n \cdot \ln\left(\frac{2}{q}\right)}$

We have computed in (12) : $q=1.9318131064$ and $a=0,8267325638$ then $\frac{\ln(2a/q^2)}{\ln\left(\frac{2}{q}\right)} \simeq -23,4636 < 0$

For large values of n , $i(n)$ is a growing function with :

$$k(n) = -\frac{\ln(2)}{\ln\left(\frac{q}{2}\right)} + \frac{\ln(2a/q^2)}{n \cdot \ln\left(\frac{q}{2}\right)} < \lim_{n \rightarrow \infty} (k(n)) = -\frac{\ln(2)}{\ln\left(\frac{q}{2}\right)} = -\frac{1}{\text{Log}_2\left(\frac{q}{2}\right)} \simeq 19,982226664 < 20 \quad (15)$$

2- To find $p_0=k.n$ such as $S_n(p_0) < 1$

We define $S_n(p_0) = 2^n \cdot \left(\sum_{r=r(p_0)}^{\infty} \frac{z_{p(r)}}{2^{p(r)}}\right)$

Thanks to (13) : $S_n(k.n) = 2^n \cdot \sum_{i>r(k.n)} \frac{z_{p(r)}}{2^{p(r)}} < 2^n \cdot \frac{C}{\sqrt{kn-1}} \cdot \left(\frac{q}{2}\right)^{kn-1}$ with $C = \frac{a}{4 \cdot (1-\frac{q}{2})} \simeq 6,07$

We are looking for k such as $S_n(k.n) < 1$.

From (13) we can write : $S_n(k.n) < 2^n \cdot \frac{C}{\sqrt{kn-1}} \cdot \left(\frac{q}{2}\right)^{kn-1} < 2^n \cdot C \cdot \left(\frac{q}{2}\right)^{k.n-1}$

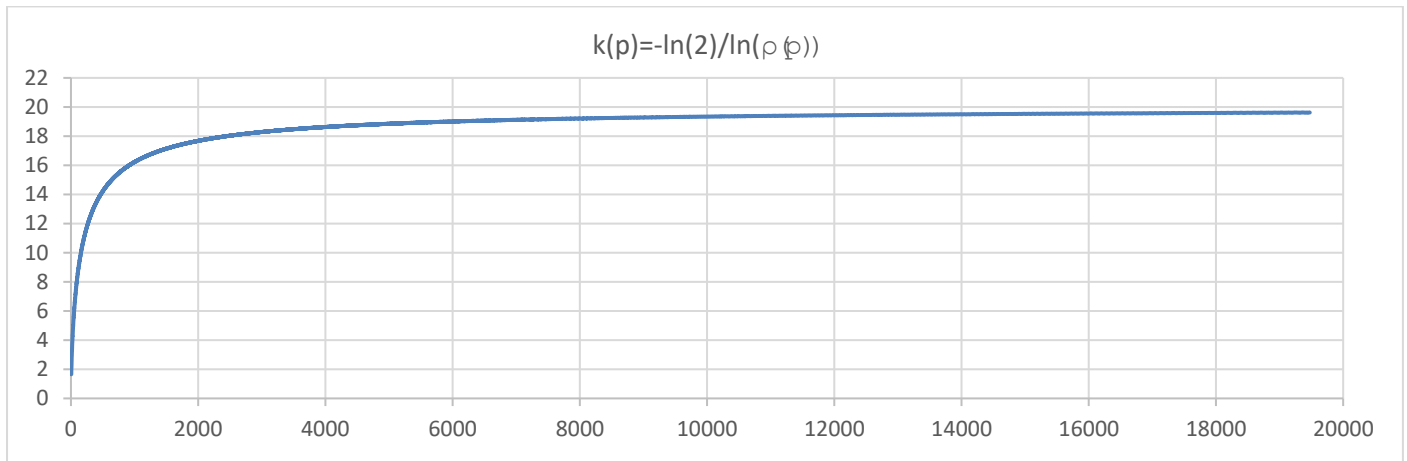
and are looking for k such as $2^n \cdot C \cdot \left(\frac{q}{2}\right)^{k.n-1} < 1$

Then $n \cdot \ln(2) + \ln(C) + (k.n - 1) \cdot \ln\left(\frac{q}{2}\right) < 0 \Leftrightarrow -\frac{\ln(2)}{\ln\left(\frac{q}{2}\right)} + \frac{\ln\left(\frac{q}{2 \cdot C}\right)}{n \cdot \ln\left(\frac{q}{2}\right)} < k$

For n large , $k(n) = -\frac{\ln(2)}{\ln\left(\frac{q}{2}\right)} + o\left(\frac{1}{n}\right) = -\frac{1}{\text{Log}_2\left(\frac{q}{2}\right)} + o\left(\frac{1}{n}\right)$

Then $\lim_{n \rightarrow \infty} (k(n)) = -\frac{1}{\text{Log}_2\left(\frac{q}{2}\right)} \simeq 19,982226664 < 20 \quad (16)$

The two approaches presented above to estimate k arrive to the same result and are coherent with the numerical values obtained with the numerical approach represented in the graph of the arithmetic function $k(p)$ below and deduced from the graph of $\rho(p)$ above.



For this reason, we propose the following conjecture.

Conjecture of the maximal stopping time :

For n enough large, we can state that for $i > 20n$, $2^n \cdot \frac{z_i}{2^i} < 0.5$. This could induce that no Syracuse sequence of starting number $s < 2^n$ could have a stopping time higher than $20n$ and all Syracuse sequences of starting number $s < 2^n$ have a finite stopping time lower than $20n$

Conclusion :

In the present work, thanks to a detailed analysis of computational results regarding the number $h_n(p)$ of residue classes mod 2^n of starting number s of Syracuse sequences such as $\sigma(s) = p$, we have numerically shown that $\frac{S(n)}{2^n}$ with $S(n) = \#\{s \in \mathbb{Z}/2^n\mathbb{Z}, \sigma(s) > n\}$, is a decreasing function who seems to tend towards 0 when n grows and obtain a first result $\frac{S(40)}{2^{40}} < 0,6\%$, which means that 99,4% of the Syracuse sequences have a stopping time lower than 40.

We have stated from this histogram analysis some preliminary results

The stopping time values verify : $p(r) = \lceil r \cdot \log_2 3 + 1 \rceil$ with $r \in \mathbb{N}$

For all stopping time value p and $k > 0$: $h_{p+k}(p) = 2^k \cdot h_p(p)$

And we have stated that $S(n)$ depends only on $h_p(p)$ for $p \leq n$: $S(n) = 2^n \cdot \left(1 - \sum_{r=1}^{r=r(n)} \frac{h_{p(r)}(p(r))}{2^{p(r)}} \right)$

Starting from a previous work of Mike Winkler [2] 2017, regarding the number of residue classes mod (2^n) of starting number s such as $\sigma(s) = n$, we have given an exact formulation of $z_p = h_p(p)$ with $p(r) = \lceil r \cdot \log_2 3 + 1 \rceil$

$$z_{p(r)} = h_{p(r)}(p(r)) = \binom{p(r)-2}{r-2} - \sum_{i=2}^{i < r} \binom{p(r) - p(i)}{r - i} \cdot z_{p(i)}$$

Which has open the possibility to compute $S(n)$ up to $n = 19475$ with $P(n) = \frac{S(n)}{2^n} \approx 1,61925631106306E - 299$ which imply that the percentage of Syracuse sequences which has a stopping time lower than 19475 is $1 - P(19475)$, very close to one. Based on the equality $\lim_{r \rightarrow \infty} \sum_{r=1}^{r=r(n)} \frac{h_{p(r)}(p(r))}{2^{p(r)}} = 1$, we have stated that $\lim_{n \rightarrow \infty} \frac{S(n)}{2^n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\log_2(S(n))}{n} = \theta < 0,9499555271$ and $S(n) < 2^{n\theta}$ for n large.

We conclude this work by presenting some arguments to propose a conjecture which states that the maximal stopping time of the residue classes mod 2^n of starting number s , is upper bounded by $20n$.

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Annexe 1 ; Number of residue classes of stopping time higher than n

```
from math import *
import os
import time

os.remove("Zn_values.txt")
os.remove("Teta_values.txt")

print(time.ctime())

max=12288

nlist=[0]*2*max # list of stopping time values
nlist[0]=1
nlist[1]=2
nlist[2]=4
nlist[3]=5

zlist=[0]*2*max # list of number Zn of residue classes mod 2^n such as stopping time is n
zlist[1]=1
zlist[2]=1
zlist[4]=1

rlist=[0]*2*max # list of number  $P(2^n)=2^n-S(n)$  of residue classes mod 2^n such as stopping time is below or equal to n
rlist[0]=1
rlist[1]=3
rlist[2]=13

Pnlist=[0.0]*2*max # percentage  $S(n)/2^n$  of residue classes mod 2^n such as stopping time is higher than n
Tetalist=[0.0]*2*max

f=open("Zn_values.txt",'wt')
g=open("Teta_values.txt",'wt')

r=3
while r<max:
    n=floor(r*log2(3)+1)
    nlist[r]=n
    sum=0
    i=2
    while i<r:
        l=floor(i*log2(3)+1)
        sum=sum+comb(n-l,r-i)*zlist[l]
        i=i+1
    zlist[n]=comb(n-2,r-2)-sum
    rlist[r]=int(2**(nlist[r]-nlist[r-1]))*rlist[r-1]+zlist[n]
    den=int(2**n)
    Sn = den - rlist[r]
    Pnlist[n]=Sn/den
    Tetalist[n]=log2(Sn)/n
    print(n,r,Pnlist[n],Tetalist[n])
    f.write("%i;" % (zlist[n]))
    g.write("%f;" % (Tetalist[n]))
    r=r+1

print(time.ctime())
```

Annexe 2 : Histograms $H_n = \{h_n(p)\}$ for $40 < p < 566\}$ for $n < 41$

n\p	42	43	45	46	48	50	51	53	54	56	58	59	61	62	64	65	67	69	70	72	73	75	77	78	80
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
7	1	0	1	0	0	0	1	0	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
8	1	0	1	0	0	0	1	0	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
9	1	0	1	0	0	0	1	0	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
10	1	0	1	0	0	0	1	0	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
11	1	1	1	0	0	0	1	0	2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	1	1
12	2	4	1	1	1	1	2	0	3	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	2
13	4	9	1	5	2	5	2	1	5	1	2	1	1	0	1	1	0	1	1	1	1	1	1	1	2
14	8	17	5	7	7	5	6	3	8	2	3	1	1	0	1	1	2	3	1	2	2	1	1	3	3
15	17	27	12	14	14	8	13	6	13	4	6	7	3	3	3	5	9	4	5	4	7	2	1	3	3
16	27	52	27	33	19	15	20	14	20	9	14	14	7	10	8	13	13	10	11	10	8	4	1	6	3
17	54	98	46	63	32	23	44	27	45	23	21	30	13	21	16	19	17	15	23	13	14	7	6	8	5
18	105	176	83	137	69	53	88	57	77	42	42	60	27	47	29	36	29	24	39	25	26	14	10	15	10
19	207	315	174	287	151	114	187	121	163	97	73	104	58	88	53	79	50	47	76	39	58	29	23	33	20
20	429	696	367	568	316	221	371	226	321	185	141	195	115	185	99	151	89	80	131	72	105	56	47	73	32
21	864	1380	746	1139	613	421	716	435	629	359	246	412	230	385	216	311	180	161	247	153	198	124	98	147	87
22	1763	2702	1524	2269	1179	858	1490	844	1298	754	513	852	502	831	443	662	393	319	483	295	415	244	211	304	172
23	3612	5438	3055	4518	2399	1824	3016	1722	2641	1500	1058	1744	1047	1677	919	1363	807	598	930	620	824	496	371	614	337
24	7195	10938	6079	9070	4990	3735	6000	3453	5278	2999	2164	3576	2084	3195	1734	2672	1562	1167	1856	1161	1695	959	742	1202	714
25	14388	21870	12157	18165	10014	7372	12018	6916	10431	5876	4353	7041	4103	6265	3450	5375	3068	2336	3774	2292	3429	2008	1493	2442	1492
26	28949	43533	24381	36147	19927	14729	23837	13748	20809	11775	8702	14111	8307	12785	7249	10895	6253	4652	7650	4590	6975	3964	2999	4914	2965
27	58109	87256	48525	72390	40009	29520	47508	27606	41705	23550	17559	28644	16838	25718	14623	22104	12597	9377	15181	9001	13889	7859	5985	9731	5837
28	116123	174593	97545	144731	80087	59094	95022	55118	83779	47328	35507	57199	33767	51574	29383	44212	25086	18705	30399	17712	27654	15587	11925	19413	11547
29	231926	349086	194980	290500	160368	118531	190152	110133	168031	94615	71150	114583	67537	103661	58984	88627	50066	37617	60944	35578	55292	31226	23591	38775	23116
30	463834	698385	390548	580650	320774	236789	379313	220963	335803	189499	142334	229426	135553	207678	117912	177881	100094	75249	121456	71402	110374	62717	47138	77707	46254
31	928504	1396622	781193	1160969	641708	473266	758013	441800	671797	379932	284647	458628	271190	415674	235738	355272	199920	150115	241963	143519	224048	125873	94473	154842	92717
32	1856318	2793382	1563593	2320432	1282779	946721	1515854	885572	1344375	760504	568701	916243	542068	830132	471887	710575	400534	300241	485356	286713	440361	252391	189713	309990	184979
33	3710431	5587414	3126403	4642380	2566600	1894438	3032305	1771319	2688937	1520179	1135740	1834159	1083578	1658527	944073	1421680	802836	600281	969804	574489	880109	504432	379295	618799	369372
34	7420572	11175809	6253915	9285176	5133732	3788874	6061421	3542976	5374913	3040256	2270521	3667875	2164885	3316071	1887798	2846900	1603998	1199225	1938290	1148778	1760907	1006613	758600	1238581	739216
35	14839951	22352189	12509920	18572007	10267775	7578156	12127047	7085202	10746928	6077154	4544000	7336101	4332797	6632336	3776817	5691783	3208316	2397002	3876645	2296814	3518611	2011586	1519047	2477002	1477972
36	29681811	44707395	25020973	37146809	20540702	15154346	24258231	14167979	21496644	12148639	9081266	14677069	8665385	13266891	7553161	11387600	6416316	4791386	7749568	4589568	7036759	4026023	3038291	4953439	2953663
37	59358657	89421112	50034521	74293444	41082491	30318968	48522088	28339338	43001544	24289101	18166021	29354443	17335601	26521139	15117227	22774141	12822805	9581996	15496000	9178938	14074364	8050750	6077390	9905156	5911211
38	118707849	178833466	100075109	148594796	82165550	60641850	97059093	56679424	86011623	48571843	36328237	58702900	34665799	53041772	30238604	45549311	25640979	19159106	31001505	18357809	28151065	16103256	11261828	19809381	11826154
39	237422468	357686311	200150588	297180981	164342824	121274913	194090923	113355172	172015314	97149928	72642773	117382448	69334930	106082832	60486452	911104291	51271320	38331624	62010235	36715363	56301541	32205306	24327407	39638217	23657820
40	474865550	715365197	400309714	594349827	328656265	245257142	388150146	226708626	344001328	194295650	145273535	234795028	138686978	212193028	120985113	182202624	102542786	76651676	120423711	73429772	112609584	64411082	48656348	79278511	47319498

n\p	81	83	85	86	88	89	91	92	94	96	97	99	100	102	104	105	107	108	110	111	113	115	116	118	119
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
15	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
16	3	2	0	1	1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	1
17	8	2	2	5	3	6	1	3	0	1	2	0	3	0	3	4	0	2	2	1	1	0	2	0	1
18	15	5	5	15	7	15	7	12	6	2	5	3	8	2	3	10	1	5	3	1	2	0	4	0	1
19	30	18	17	33	15	26	13	19	13	10	17	9	19	6	3	13	4	13	6	4	3				

n/p	221	222	224	226	227	229	230	232	233	235	237	238	240	241	243	245	246	248	249	251	253	254	256	257	259	260	262	264	265	267	268	270	272	273	275	276	278	279	281	283	284	286	287	289	291	292	294								
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0					
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0			
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
22	1	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
23	1	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
24	1	3	2	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
25	2	5	4	0	2	2	4	0	1	0	0	0	0	1	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
26	5	15	5	1	8	3	8	1	4	2	1	4	1	5	3	1	2	2	3	2	0	1	0	0	1	2	1	1	1	0	2	1	1	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	14	29	10	2	12	7	15	8	10	4	5	8	7	9	5	2	4	2	6	3	0	2	0	1	3	4	2	2	3	3	2	1	1	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
28	26	42	23	10	25	18	25	13	17	9	5	11	9	14	5	6	8	5	8	5	2	7	2	6	5	9	4	7	11	4	4	2	4	5	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
29	43	74	35	17	47	26	44	26	34	18	15	19	15	23	20	11	24	12	17	12	6	14	5	14	9	19	9	14	18	7	10	4	7	8	5	5	1	2	5	5	7	5	6	2	3	2	0	0	0	0	0	0	0	0	
30	83	122	69	44	95	56	93	51	62	41	31	40	27	58	43	32	42	28	32	29	13	21	10	30	21	31	17	23	28	13	21	8	8	14	7	15	4	6	14	11	12	6	10	2	5	6	5	0	0	0	0	0			
31	153	248	131	102	183	103	181	103	158	87	63	97	51	117	79	59	81	55	73	49	26	52	33	68	43	44	34	27	50	16	42	15	15	26	16	28	12	21	21	17	15	9	18	6	11	19	12	0	0	0	0	0			
32	323	407	272	216	359	218	345	218	305	200	138	208	134	213	144	112	158	103	143	96	62	102	68	129	68	90	50	50	80	38	84	35	34	43	34	50	28	44	35	25	33	17	32	15	18	31	21	0	0	0	0	0			
33	631	900	540	420	713	418	721	403	650	388	287	446	280	404	274	204	329	195	319	192	133	235	130	231	132	170	105	95	141	73	156	64	72	97	58	102	70	85	61	51	66	38	65	35	35	56	33	0	0	0	0	0			
34	1247	1924	1087	831	1422	814	1383	794	1286	774	544	931	533	849	539	412	656	399	665	372	254	467	267	468	273	373	219	184	289	163	291	144	143	203	120	217	128	184	114	86	139	79	129	75	61	105	65	0	0	0	0	0			
35	2437	3806	2259	1713	2858	1650	2687	1623	2474	1555	1087	1829	1053	1674	1038	808	1327	826	1299	754	524	915	553	913	553	814	460	348	580	334	570	305	290	421	249	447	263	391	223	170	272	163	262	157	116	196	132	0	0	0	0	0			
36	4882	7621	4531	3469	5660	3432	5346	3213	4947	3017	2212	3628	2184	3357	2108	1622	26																																						

