

Stopping Times of Syracuse Integer Sequences on $\mathbb{Z}/2^n\mathbb{Z}$

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Abstract :

The Collatz Conjecture affirm that for all positive integer m , the integer sequence defined by $T(m) = \frac{m}{2}$ if m is even and $T(m) = \frac{3m+1}{2}$ else, there is an integer k such as $T^k(m)=1$. It has been proved that another equivalent formulation of the conjecture is to say that all trajectories of a Collatz integer sequence has a finite stopping time, which means that all trajectories finish to be lower than the starting term of the integer sequence. This consideration is obvious for all even number m and can be easily proved for all integer $m \equiv 1 [4]$. But the question becomes extremely complex if we want to analyze the behavior of integers $m \equiv 3 [4]$.

I. Krasikov and J. C. Lagarias [1]2002 proved in 2002 that the number of integers below x , $\pi_1(x)$, which reach 1 under iteration satisfy the inequality $\pi_1(x) > x^{0.84}$.

In the present work, we have based our approach on the second formulation of the conjecture which affirm that all trajectories has a finite stopping time.

We are going to analyze the behavior of all Syracuse integer sequences starting from odd integers belonging to $\mathbb{Z}/2^n\mathbb{Z}$ for successive integer values of n and we'll state that the percentage of integers $P(n) = S(n)/2^n$ tends toward 0 when $n \rightarrow \infty$, where $S(n) = \#\{a \in \mathbb{Z}/2^n\mathbb{Z}, \sigma(a) > n\}$ and $\sigma(a)$ is the stopping time of a Syracuse sequence starting from a .

We'll start on a computational approach to study the behavior of $P(n)$ for n up to 41 and in a second time thanks to an exact formulation of $h_n(n) = \#\{a \in \mathbb{Z}/2^n\mathbb{Z}, \sigma(a) = n\}$, we'll state that $P(n)$ tends toward 0 when $n \rightarrow \infty$,

Introduction : \mathbb{Z}

As clearly rephrased by Chamberland [1], the $3x+1$ problem is true if and only if every positive integer has a finite stopping time. The stopping time $\sigma(a)$ of a Syracuse sequence starting from an integer a is the number of steps to iterate below a .

$$\sigma(a) = \inf \{ k \in \mathbb{N} : T^k(a) < a \}$$

We'll compute for each integer n the histogram $H_n = \{h_n(p), p \in \mathbb{N}\}$ where $h_n(p) = \#\{a \in \mathbb{Z}/2^n\mathbb{Z}, \sigma(a) = p\}$.

In a first part, we'll use a computational approach to compute all H_n as far as possible and we'll show that $P(n)$ the percentage of integers of $\mathbb{Z}/2^n\mathbb{Z}$ which has a finite stopping time $\sigma(a) \geq n$ decreases when n grows and becomes lower than 0.6% for $n=40$. Thanks to this result, we'll be able to conclude that 99.4% of the Syracuse integers sequences has a stopping time below $n=41$ and we'll show that for $n=1$ to 40, the arithmetic function $\frac{\log_2(S(n))}{n}$ has a mainly deterministic behavior.

We'll state that the stopping time can take certain integer values which will satisfy the equality below :

$$p = \left\lfloor \left\lfloor \frac{p}{\log_2 3} \right\rfloor \cdot \log_2 3 \right\rfloor + 1$$

We'll state that the histograms H_n and H_{n+k} are strongly correlated and can be linked by the following relation :

$$h_{n+k}(p) = R_{n,n+k} \cdot h_n(p) \simeq R_k \cdot h_n(p) \text{ with } R_k = 2^k$$

We'll show that the number of integers $S(n) = \#\{a \in \mathbb{Z}/2^n\mathbb{Z}, \sigma(a) > n\}$ can be expressed as a function of $S(n-1)$ and $h_{n-1}(n) = \#\{a \in \mathbb{Z}/2^{n-1}\mathbb{Z}, \sigma(a) = n\}$ by the relation :

$$S(n) \approx 2 \cdot (S(n-1) - h_{n-1}(n)) = S_{e1}(n) \text{ and } \frac{S(n) - S_{e1}(n)}{S(n)} = o(n)$$

From the homotetic similarity between the histograms H_n for the different values of n , we'll define an arithmetic function $V_n(p) = \frac{h_n(p)}{h_n(p-1)}$ and will show that $V_n(p)$ is strongly independent of n for n enough large. We'll define the arithmetic function $V(p) = \lim_{n \rightarrow \infty} V_n(p)$ and will use it to build a good estimator of $S(n+k)$ in function of $S(n)$ and $h_n(n)$:

$$S(n+k) \approx 2^k \cdot (S(n) - h_n(n) \cdot \sum_{p>n+1}^{n+k} (\prod_{i=n+1}^p V(i)))$$

In the last part, based on a recent work of Mike Winker [2]2017 who proposed an interesting approach to compute the number of residue classes of Syracuse sequences starting from $a \in \mathbb{Z}/2^n\mathbb{Z}$ such $\sigma(a) = n$, we'll improve the Winker formula to gives an explicit formulation $h_n(n)$, available for all n :

$$h_n(n) = \binom{n-2}{m(n)} - \sum_{\substack{i \leq n \\ p(i)=i}} \binom{n-i}{m(n)-m(i)} \cdot z_i \quad \text{where } m(n) \text{ is the number of even iterates before } T^n(a) < a$$

and we'll show that we can state that $\lim_{n \rightarrow \infty} \left(\frac{S(n)}{2^n}\right) = 0$ where $\frac{S(n)}{2^n}$ is the percentage of residue classes $a \in \mathbb{Z}/2^n\mathbb{Z}$ such $\sigma(a) > n$.

We'll finish this work by showing that we have some clues to estimate that the maximum stopping time for all $a \in \mathbb{Z}/2^n\mathbb{Z}$ is upper bounded by the function $f(n) = 20n$.

I - First study cases :

If we start a Collatz sequence with an integer $2^n \cdot m + a$ with $m \in \mathbb{N}$ and $a \in \mathbb{Z}/2^n\mathbb{Z}$, with a stopping time $\sigma(a) \leq n$, it is quite easy to show that $T^{\sigma(a)}(2^n \cdot m + a) < 2^n \cdot m + a$ for all m . Our idea is to study for each n , the percentage of integers $a \in \mathbb{Z}/2^n\mathbb{Z}$ which satisfy to $\sigma(a) > n$ and to show that this percentage tends towards 0 when n grows. We are going to start by the first values of n .

Case n=1 :

We begin by the trivial case of integers belonging to $\mathbb{Z}/2\mathbb{Z}$. We compute the stopping time of even and odd numbers.

As $T(2m) = m$, the stopping time of even number $\sigma(2m) = 1$

As $T(2m+1) = 3m+2$ we cannot conclude at this stage regarding the stopping time.

We can state that 50% of the integer sequence have a stopping time higher than 1

Case n=2 :

It concerns all integers $a \in \mathbb{Z}/2^2\mathbb{Z}$. As we have already treated the even number, we have to analyse the behavior of Syracuse integer sequences beginning by $4m+1$ and $4m+3$

$T^{(2)}(4m+1) = 3m+1$ which is below $4m+1$ for all positive integer m , which means that $\sigma(4m+1) = 2$

But for $T^{(2)}(4m+3) = 9m+8$, we cannot conclude.

At this stage, we can conclude that 25% of the integer sequence have a stopping time higher than 2

Case n=3 :

We have to analyse two families of integers coming from the previous case, $8m+3$ and $8m+7$.

$T^{(3)}(8m+3) = 9m+8$ and $T^{(3)}(8m+7) = 27m+26$

At this stage, we cannot conclude and 25% of the integer sequences still have a stopping time higher than 3

We can also say that as no integer sequence has a stopping time equal to 3, 3 is not a stopping time value.

Case n=4 :

Here we have four integer families $4m+a$ to study with $a \in \{3, 7, 11, 15\}$

We see that $T^{(4)}(16m+3) = 9m+8$ which is below $16m+3$ for all positive integer m , thus $\sigma(16m+3) = 4$

But we cannot conclude for the 3 other families :

$T^{(4)}(16m+7) = 27m+13$

$T^{(4)}(16m+11) = 27m+20$

$T^{(4)}(16m+15) = 81m+80$

So at this stage, we have $\frac{3}{16} = 18,75\%$ integer families which doesn't satisfy $\sigma(m) \leq 4$

We'll see in below that for all n such as $3^q < 2^{n-1} < 2^n < 3^{q+1}$, n will not be an available stopping time value and we'll give in paragraph II.

We see that the odd integers we have to test at a certain step n , are the set of integers $a \in \mathbb{Z}/2^{n-1}\mathbb{Z}$ such as $\sigma(a) > n-1$ and all $a+2^{n-1}$.

For instance, at the step $n=4$, three integers don't satisfy $\sigma(a) \leq 4$, which means that $S(4)=3$. It implies that we have to check 6 values at next step $n=5$, $\{7, 11, 15, 23, 27, 31\}$.

The table below gives for all n up to n=41, S(n), number of integers $a \in \mathbb{Z}/2^n\mathbb{Z}$ such as $\sigma(a) > n$.

n	2^n	S(n)	$S(n)/2^n$	$\text{Log}_2(S(n))$	$\text{Log}_2(S(n))/n$
1	2	1	50,00%	0,00	0,000000
2	4	1	25,00%	0,00	0,000000
3	8	2	25,00%	1,00	0,333333
4	16	3	18,75%	1,58	0,396241
5	32	4	12,50%	2,00	0,400000
6	64	8	12,50%	3,00	0,500000
7	128	13	10,16%	3,70	0,528634
8	256	19	7,42%	4,25	0,530991
9	512	38	7,42%	5,25	0,583103
10	1024	64	6,25%	6,00	0,600000
11	2048	128	6,25%	7,00	0,636364
12	4096	226	5,52%	7,82	0,651682
13	8192	367	4,48%	8,52	0,655357
14	16384	734	4,48%	9,52	0,679974
15	32768	1295	3,95%	10,34	0,689249
16	65536	2114	3,23%	11,05	0,690360
17	131072	4228	3,23%	12,05	0,708574
18	262144	7495	2,86%	12,87	0,715095
19	524288	14990	2,86%	13,87	0,730090
20	1048576	27328	2,61%	14,74	0,736905
21	2097152	46611	2,22%	15,51	0,738494
22	4194304	93222	2,22%	16,51	0,750381
23	8388608	168807	2,01%	17,37	0,755001
24	16777216	286581	1,71%	18,13	0,755358
25	33554432	573162	1,71%	19,13	0,765143
26	67108864	1037374	1,55%	19,98	0,768635
27	134217728	1762293	1,31%	20,75	0,768482
28	268435456	3524586	1,31%	21,75	0,776751
29	536870912	6385637	1,19%	22,61	0,779531
30	1073741824	12771274	1,19%	23,61	0,786880
31	2147483648	23642078	1,10%	24,49	0,790157
32	4294967296	41347483	0,96%	25,30	0,790666
33	8589934592	82694966	0,96%	26,30	0,797009
34	17179869184	151917636	0,88%	27,18	0,799374
35	34359738368	263841377	0,77%	27,98	0,799288
36	68719476736	527682754	0,77%	28,98	0,804864
37	137438953472	967378591	0,70%	29,85	0,806743
38	274877906944	1934757182	0,70%	30,85	0,811829
39	549755813888	3611585536	0,66%	31,75	0,814102
40	1099511627776	6402835000	0,58%	32,58	0,814402
41	2199023255552	12805670000	0,58%	33,58	0,818928

Table 1 : computational results

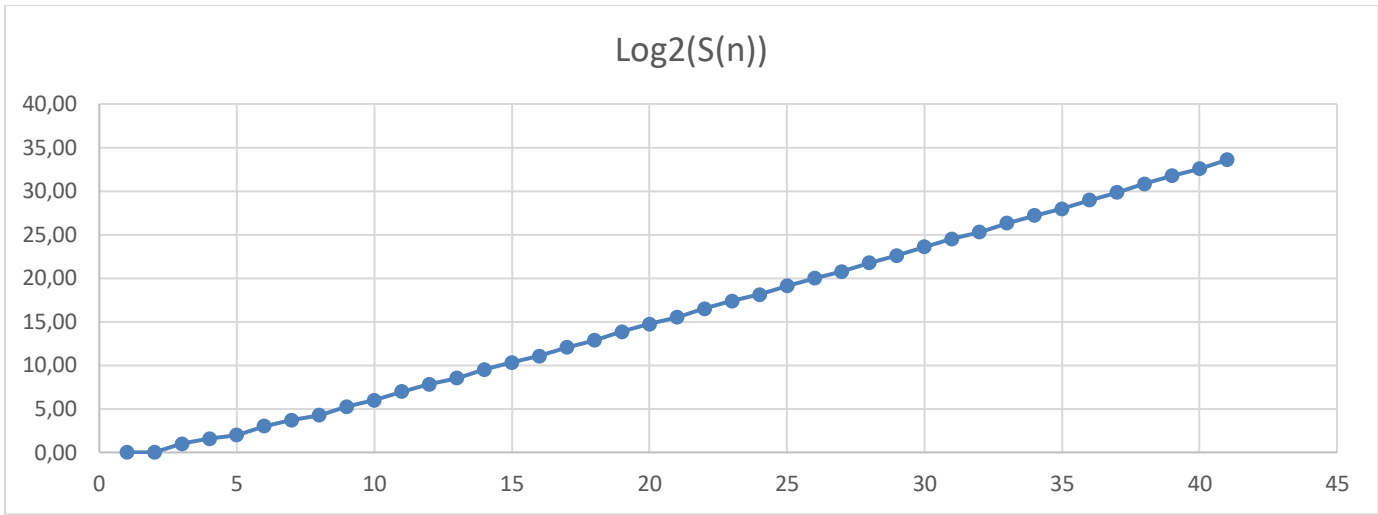
The blue lines correspond to the values of n where $h_n(n)=0$. In this case $S(n)=2.S(n-1)$. For instance, $S(5)=4$, the four odd integers which satisfy $\sigma(a) > 4$ are $\{7,15,27,31\}$ and as no odd integer has a stopping time $\sigma(a)=6$, the set of integers which satisfy to $\sigma(a) > 6$, is exactly $\{7,15,27,31,39,47,59,63\}$

This can be explained by the fact that when two consecutive powers of 2 are surrounded by two consecutive powers of 3, only the first one gives the number of iteration to satisfy to $\sigma(a) \leq n$. For instance, if we have $3^q < 2^n < 2^{n+1} < 3^{q+1}$, all integer sequences beginning by an integer a such as $\sigma(a)=n$ has exactly n iterations such as $T^{(n)}(a) < a$. For this reason if there is an integer b such as $\sigma(b)=n+1$, then $T^{(n+1)}(b) < b < T^{(n)}(b)$ but as $3^q < 2^n < 2^{n+1}$, we have also $T^{(n+1)}(b) < T^{(n)}(b) < b$ which is impossible.

Thanks to these first results in table 1, we can state that if we represent all integers $m' \in \mathbb{N}$ as $m' = 2^{41}.m+a$ with $a < 2^{41}$ and $m \in \mathbb{N}$ then 99,42% of $m' \in \mathbb{N}$ satisfy to $\sigma(a) \leq 41$.

We are convinced that the function linking the number of odd integers $a < 2^n$ such as $\sigma(a) > n$ is a mainly deterministic function of n. It's what we are going to try to prove in the following parts.

We can illustrate this deterministic link with the graph below which gives $\text{Log}_2(S(n))$ in function of n. We don't see any stochastic behavior.



If the slope of this curve tends toward a constant value, it means that this curve tends asymptotically toward a line with a slope $\theta < 1$, it will imply that for n large, $\frac{S(n)}{2^n} < \frac{1}{2^{n(1-\theta)}}$. We'll develop this point later.

II - Computation of the histograms $H_n(p)$

We have computed the $h_n(p)$ for all p and all n up to $n=40$.

$n \setminus \sigma(a)$	1	2	4	5	7	8	10	12	13	15	16	18	20
1	1	1	0	0	0	0	0	0	0	0	0	0	0
2	2	1	1	0	0	0	0	0	0	0	0	0	0
3	4	2	1	0	1	0	0	0	0	0	0	0	0
4	8	4	1	1	2	0	0	0	0	0	0	0	0
5	16	8	2	2	2	0	0	0	0	0	0	0	0
6	32	16	4	4	3	1	0	0	0	0	0	0	0
7	64	32	8	8	3	4	0	0	0	1	0	0	0
8	128	64	16	16	6	7	0	1	3	1	0	0	1
9	256	128	32	32	12	14	5	3	4	3	2	1	2
10	512	256	64	64	24	28	12	7	10	5	6	2	3
11	1024	512	128	128	48	56	24	16	21	10	12	6	7
12	2048	1024	256	256	96	112	48	30	41	26	26	17	15
13	4096	2048	512	512	192	224	96	60	85	45	60	32	29
14	8192	4096	1024	1024	384	448	192	120	170	91	115	64	57
15	16384	8192	2048	2048	768	896	384	240	340	173	237	125	100
16	32768	16384	4096	4096	1536	1792	768	480	680	346	476	250	188
17	65536	32768	8192	8192	3072	3584	1536	960	1360	692	952	501	350
18	131072	65536	16384	16384	6144	7168	3072	1920	2720	1384	1904	961	699
19	262144	131072	32768	32768	12288	14336	6144	3840	5440	2768	3808	1922	1342
20	524288	262144	65536	65536	24576	28672	12288	7680	10880	5536	7616	3844	2652
21	1048576	524288	131072	131072	49152	57344	24576	15360	21760	11072	15232	7688	5304
22	2097152	1048576	262144	262144	98304	114688	49152	30720	43520	22144	30464	15376	10608
23	4194304	2097152	524288	524288	196608	229376	98304	61440	87040	44288	60928	30752	21216
24	8388608	4194304	1048576	1048576	393216	458752	196608	122880	174080	88576	121856	61504	42432
25	16777216	8388608	2097152	2097152	786432	917504	393216	245760	348160	177152	243712	123008	84864
26	33554432	16777216	4194304	4194304	1572864	1835008	786432	491520	696320	354304	487424	246016	169728
27	67108864	33554432	8388608	8388608	3145728	3670016	1572864	983040	1392640	708608	974848	492032	339456
28	134217728	67108864	16777216	16777216	6291456	7340032	3145728	1966080	2785280	1417216	1949696	984064	678912
29	268435456	134217728	33554432	33554432	12582912	14680064	6291456	3932160	5570560	2834432	3899392	1968128	1357824
30	536870912	268435456	67108864	67108864	25165824	29360128	12582912	7864320	11141120	5668864	7798784	3936256	2715648
31	1073741824	536870912	134217728	134217728	50331648	58720256	25165824	15728640	22282240	11337728	15597568	7872512	5431296
32	2147483648	1073741824	268435456	268435456	100663296	117440512	50331648	31457280	44564480	22675456	31195136	15745024	10862592
33	4294967296	2147483648	536870912	536870912	201326592	234881024	100663296	62914560	89128960	45350912	62390272	31490048	21725184
34	8589934592	4294967296	1073741824	1073741824	402653184	469762048	201326592	125829120	178257920	90701824	124780544	62980096	43450368
35	17179869184	8589934592	2147483648	2147483648	805306368	939524096	402653184	251658240	356515840	181403648	249561088	125960192	86900736
36	34359738368	17179869184	4294967296	4294967296	1610612736	1879048192	805306368	503316480	713031680	362807296	499122176	251920384	173801472
37	68719476736	34359738368	8589934592	8589934592	3221225472	3758096384	1610612736	1006632960	1426063360	725614592	998244352	503840768	347602944
38	137438953472	68719476736	17179869184	17179869184	6442450944	7516192768	3221225472	2013265920	2852126720	1451229184	1996488704	1007681536	695205888
39	274877906944	137438953472	34359738368	34359738368	12884901888	15032385536	6442450944	4026531840	5704253440	2902458368	3992977408	2015363072	1390411776
40	549755813888	274877906944	68719476736	68719476736	25769803776	30064771072	12884901888	8053063680	11408506880	5804916736	7985954816	4030726144	2780823552

n\σ(a)	21	23	24	26	27	29	31	32	34	35	37	39	40
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	0	0	0	0	0	0
8	1	0	0	0	1	1	2	0	0	0	0	0	1
9	3	0	1	0	2	1	2	0	0	0	0	0	2
10	6	1	4	1	3	1	2	0	0	1	1	1	2
11	9	5	8	4	3	1	3	1	1	2	1	2	2
12	12	8	13	8	7	4	4	3	2	3	3	4	6
13	28	13	27	11	17	6	6	8	3	9	8	5	9
14	54	27	47	31	37	19	12	19	12	20	11	9	12
15	113	63	100	53	78	38	30	38	26	35	20	16	25
16	250	132	196	102	145	75	54	89	55	76	39	27	51
17	507	272	392	212	296	150	116	173	110	158	79	65	104
18	995	551	784	429	594	311	232	363	228	344	159	130	210
19	2022	1102	1578	837	1185	660	488	729	435	646	341	258	393
20	4013	2230	3166	1699	2454	1338	944	1467	835	1290	663	509	802
21	8045	4413	6407	3442	4935	2603	1858	2935	1683	2522	1384	1046	1619
22	16090	8818	12750	6806	9807	5170	3737	5909	3348	5034	2746	1984	3189
23	32180	17637	25427	13453	19613	10347	7449	11676	6607	9959	5418	4023	6299
24	64360	35274	51033	27026	39156	20756	14887	23287	13106	19796	10838	7915	12588
25	128720	70548	102066	54637	78017	41512	29714	46403	26399	39340	21786	15735	25300
26	257440	141096	204132	108950	156415	82960	59439	92651	52796	78435	43312	31524	50320
27	514880	282192	408264	217900	312455	165991	118756	185327	105501	156445	86412	63300	100275
28	1029760	564384	816528	435800	624910	331899	237455	370906	210619	313133	172461	126194	200455
29	2059520	1128768	1633056	871600	1249820	663535	475297	741658	421383	625166	344298	251963	400052
30	4119040	2257536	3266112	1743200	2499640	1327070	950050	1483948	842495	1249751	687092	504060	800442
31	8238080	4515072	6532224	3486400	4999280	2654140	1900470	2968528	1684279	2498989	1374351	1008119	1601779
32	16476160	9030144	13064448	6972800	9998560	5308280	3800940	5936673	3368297	4999512	2748535	2015670	3203945
33	32952320	18060288	26128896	13945600	19997120	10616560	7601880	11873346	6735992	9999087	5498168	4030565	6407961
34	65904640	36120576	52257792	27891200	39994240	21233120	15203760	23746692	13472296	19998291	10996150	8060635	12817360
35	131809280	72241152	104515584	55782400	79988480	42466240	30407520	47493384	26944592	39993895	21995901	16122083	25634043
36	263618560	144482304	209031168	111564800	159976960	84932480	60815040	94986768	53889184	79987790	43993240	32244300	51269020
37	527237120	288964608	418062336	223129600	319953920	169864960	121630080	189973536	107778368	159975580	87986917	64489232	102533100
38	1054474240	577929216	836124672	446259200	639907840	339729920	243260160	379947072	215556736	319951160	175973834	128978066	205071764
39	2108948480	1155858432	1672249344	892518400	1279815680	679459840	486520320	759894144	431113472	639902320	351947668	257978502	410116175
40	4217896960	2311716864	3344498688	1785036800	2559631360	1358919680	973040640	1519788288	862226944	1279804640	703895336	515957004	820236724

Table 2a and 2b above includes the $h_n(p)$ values for $n=1$ to 40 and in annex for $\sigma(a) = 41$ to 600.

We define the subset of \mathbb{N} of all integer values taken by a stopping time :

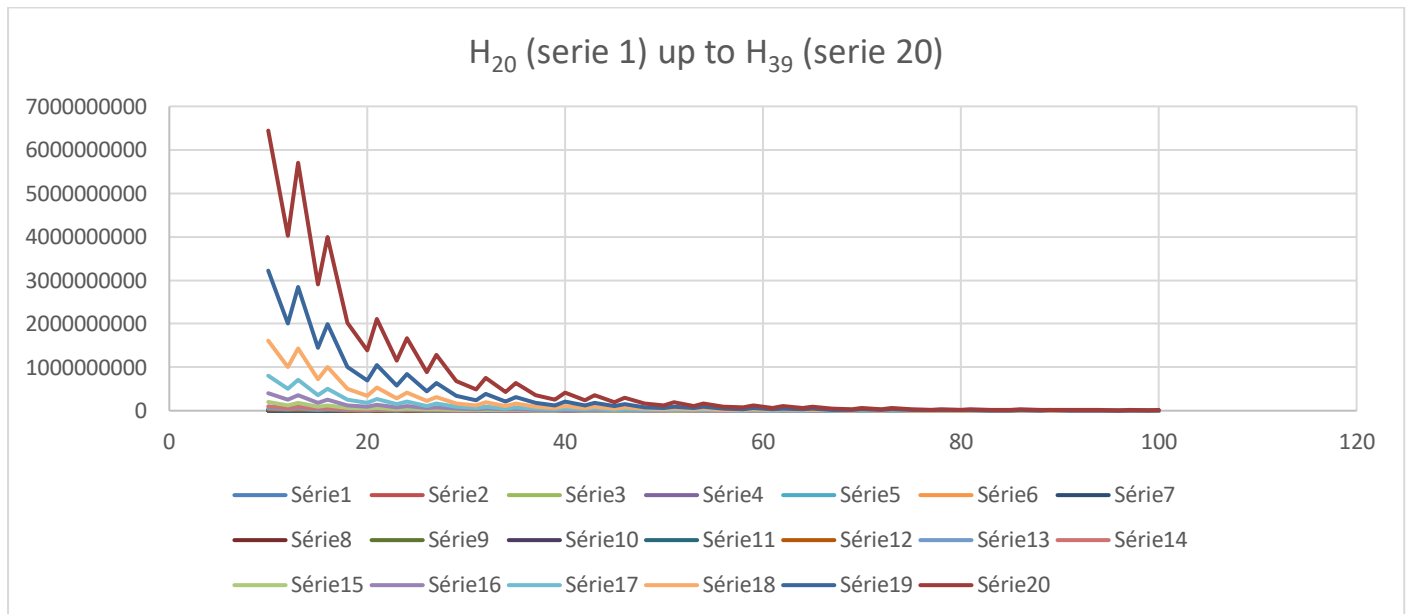
$$S_\sigma = \{ p \in \mathbb{N} / \exists a \in \mathbb{N} / \sigma(a) = p \}$$

As we can see in table 2a and 2b, we have only displayed on the first line, the values of p which belong to S_σ . These values satisfy the equality below :

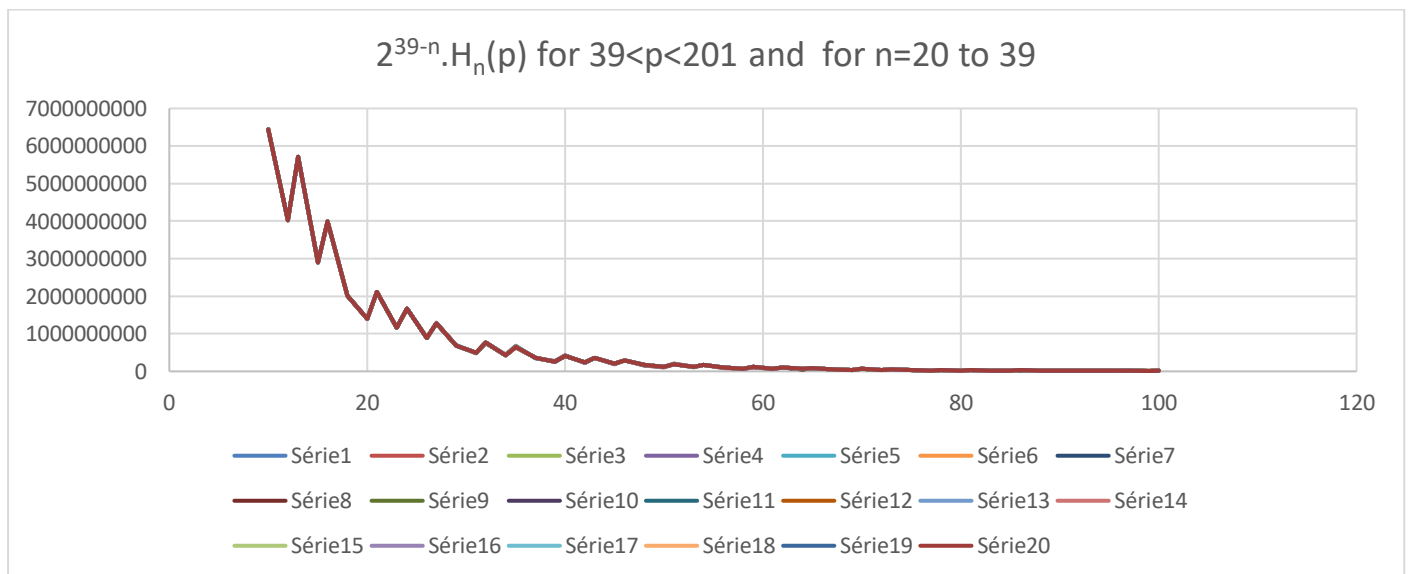
$$p = \left\lceil \left\lfloor \frac{p}{\log_2 3} \right\rfloor \cdot \log_2 3 \right\rceil + 1$$

III - Correlation between the different histograms H_n for different values of n

If we look at more precisely the different histograms H_n , we can see a very important similarity between the shape of the H_n with mainly a difference due to a scale factor. The graphs below gives $H_n(p)$ for $n=20$ to 39 , and for all $h_n(p)$ such as $p > 10$.

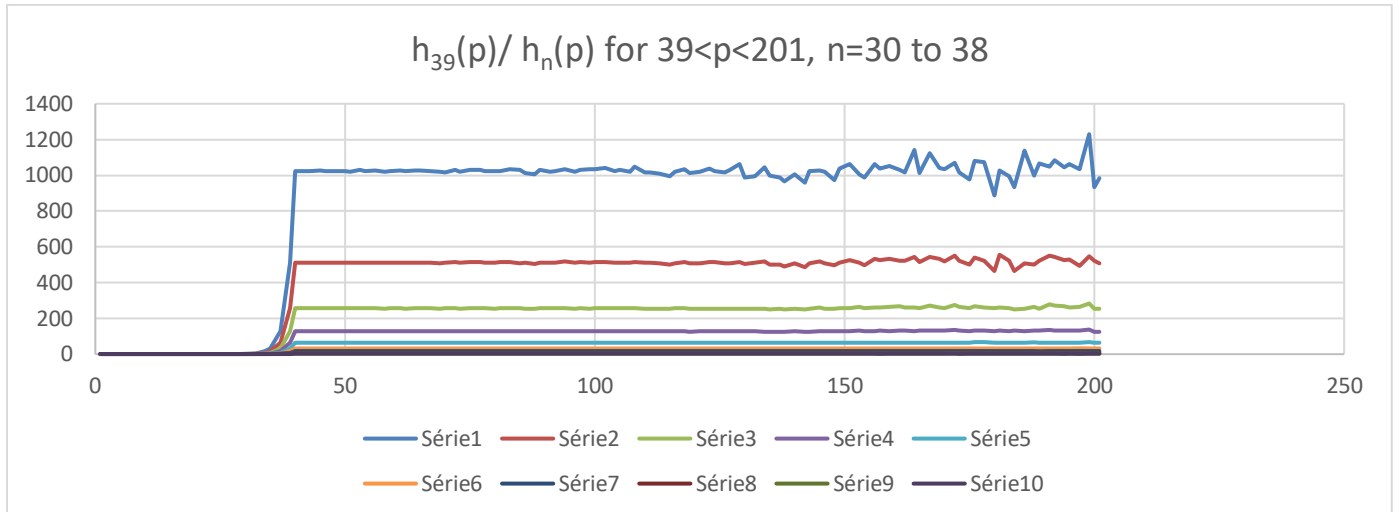


If we normalize these values by multiplying each H_n for $n < 39$ by 2^{39-n} , we obtain perfect superimposition of all the resulting functions.



We can see the perfect superimposition of the all histograms after normalization. It confirms the strong correlation between the different histograms H_n .

The graphic below gives for each p, the ratio $h_{39}(p)/h_n(p)$ for $n=30$ to 39 and confirm that two histograms H_n and H_k are linked with a scale factor 2^{n-k} .



The variations of the ratio are more important for the highest values of p but it concerns some relatively small values of $h_n(p)$. If we compute an average ratio between two histograms weighted by the percentage of $h_n(p)$ in the whole population, given by $h_n(p)/S(n)$, we obtain :

$$R_{n,n-k} = \sum_{p>n} \left(\frac{h_n(p)}{h_{n-k}(p)} \right) \cdot \left(\frac{h_n(p)}{S(n)} \right) \quad (1)$$

When we compute this ratio for $n=39$ and $k=1$ to 10 we obtain some values very close to 2^k with a very good accuracy.

(n,n-k)	$R_{n,n-k}$	$(R_{n,n-k} - 2^k)/2^k$
(29,39)	1024,625653	0,061099%
(30,39)	512,179219	0,035004%
(31,39)	256,0753832	0,029447%
(32,39)	128,0629038	0,049144%
(33,39)	64,01436314	0,022442%
(34,39)	32,00314693	0,009834%
(35,39)	16,00131223	0,008201%
(36,39)	8,000456933	0,005712%
(37,39)	4,000224834	0,005621%
(38,39)	2,000081232	0,004062%

We can state that the ratios $R_{n,n-k}$ can be correctly estimated by the relation :

$$R_{n,n-k} = 2^k(1 + o(k)) \approx 2^k = R_k$$

In these conditions we can write :

$$S(n+k) \approx R_k \cdot \left(S(n) - \sum_{p>n}^{p \leq n+k} h_n(p) \right) \approx 2^k \cdot \left(S(n) - \sum_{p>n}^{p \leq n+k} h_n(p) \right) = S_e(n+k) \quad (2)$$

Thanks to (2), we have estimated $S(n+k)$ for $n=19$ to 23 and $k=1$ to 20 .

n	2 ⁿ	S(n)	S(19+k)	Er ₁₉ (k)	S(20+k)	Er ₂₀ (k)	S(21+k)	Er ₂₁ (k)	S(22+k)	Er ₂₂ (k)	S(23+k)	Er ₂₃ (k)
1	2	1	1		1		1		1		1	
2	4	1	1		1		1		1		1	
3	8	2	2		2		2		2		2	
4	16	3	3		3		3		3		3	
5	32	4	4		4		4		4		4	
6	64	8	8		8		8		8		8	
7	128	13	13		13		13		13		13	
8	256	19	19		19		19		19		19	
9	512	38	38		38		38		38		38	
10	1024	64	64		64		64		64		64	
11	2048	128	128		128		128		128		128	
12	4096	226	226		226		226		226		226	
13	8192	367	367		367		367		367		367	
14	16384	734	734		734		734		734		734	
15	32768	1295	1295		1295		1295		1295		1295	
16	65536	2114	2114		2114		2114		2114		2114	
17	131072	4228	4228		4228		4228		4228		4228	
18	262144	7495	7495		7495		7495		7495		7495	
19	524288	14990	14990		14990		14990		14990		14990	
20	1048576	27328	27296	-0,117096%	27328		27328		27328		27328	
21	2097152	46611	46504	-0,229560%	46630	0,040763%	46611		46611		46611	
22	4194304	93222	93008	-0,229560%	93260	0,040763%	93222	0,000000%	93222		93222	
23	8388608	168807	168384	-0,250582%	168680	-0,075234%	168792	-0,008886%	168808	0,000592%	168807	
24	16777216	286581	286272	-0,107823%	286704	0,042920%	286328	-0,088282%	286616	0,012213%	286760	0,062461%
25	33554432	573162	572544	-0,107823%	573408	0,042920%	572656	-0,088282%	573232	0,012213%	573520	0,062461%
26	67108864	1037374	1037952	0,055718%	1038080	0,068056%	1035168	-0,212652%	1037568	0,018701%	1039416	0,196843%
27	134217728	1762293	1772544	0,581685%	1762048	-0,013902%	1754496	-0,442435%	1761312	-0,055666%	1765024	0,154969%
28	268435456	3524586	3545088	0,581685%	3524096	-0,013902%	3508992	-0,442435%	3522624	-0,055666%	3530048	0,154969%
29	536870912	6385637	6414336	0,449430%	6363136	-0,352369%	6351616	-0,532774%	6383488	-0,033654%	6397888	0,191852%
30	1073741824	12771274	12828672	0,449430%	12726272	-0,352369%	12703232	-0,532774%	12766976	-0,033654%	12795776	0,191852%
31	2147483648	23642078	23658496	0,069444%	23519232	-0,519607%	23503872	-0,584576%	23620608	-0,090813%	23684608	0,179891%
32	4294967296	41347483	41345024	-0,005947%	41029632	-0,768731%	40996864	-0,847981%	41190400	-0,379909%	41391104	0,105499%
33	8589934592	82694966	82690048	-0,005947%	82059264	-0,768731%	81993728	-0,847981%	82380800	-0,379909%	82782208	0,105499%
34	17179869184	151917636	151126016	-0,521085%	150437888	-0,974046%	150200320	-1,130426%	151048192	-0,572313%	152033280	0,076123%
35	34359738368	263841377	259915776	-1,487864%	258605056	-1,984647%	259080192	-1,804563%	260857856	-1,130801%	263274496	-0,214857%
36	68719476736	527682754	519831552	-1,487864%	517210112	-1,984647%	518160384	-1,804563%	521715712	-1,130801%	526548992	-0,214857%
37	137438953472	967378591	950272000	-1,768345%	947519488	-2,052878%	945618944	-2,249341%	953450496	-1,439777%	964329472	-0,315194%
38	274877906944	1934757182	1900544000	-1,768345%	1895038976	-2,052878%	1891237888	-2,249341%	1906900992	-1,439777%	1928658944	-0,315194%
39	549755813888	3611535862	3530555392	-2,242272%	3523215360	-2,445511%	3508273152	-2,859246%	3553755136	-1,599893%	3593666560	-0,494784%
40	1099511627776	6403518464	6236930048	-2,582420%	6205472768	-3,073766%	6167724032	-3,663382%	6271533056	-2,041939%	6361710592	-0,633413%

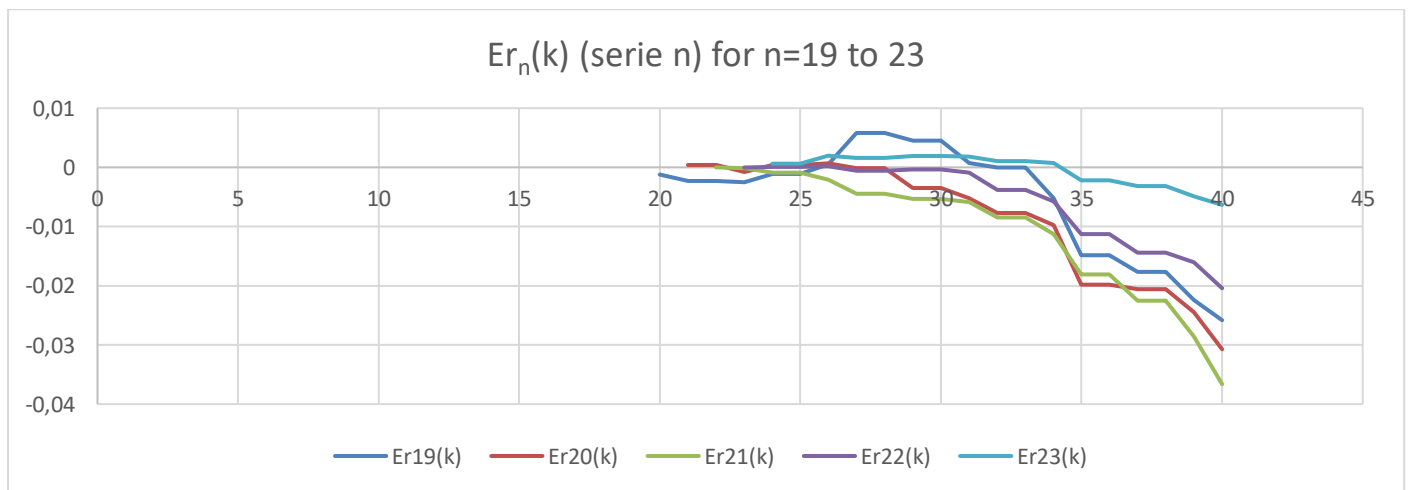
Table 3 : in green S(n+k) for n=19 to 23

We have build the relation between S(n) and S(n+k) as following:

$$S(n+k) = \sum_{p \geq n+k+1} h_{n+k}(p) \approx 2^k \cdot \sum_{p \geq n+k+1} h_n(p) = 2^k \cdot \left(S(n) - \sum_{p \geq n+1}^{p \leq n+k} h_n(p) \right) = S_e(n+k)$$

Where S_e(n+k) is an estimator of S(n+k) and $Er_n(k) = \frac{S(n+k) - S_e(n+k)}{S(n+k)}$

We see in table 3 above that we can estimate S(n+k) with an accuracy better than 0,6% for n=19 to 24 and k≤10 and an accuracy better than 3,7% for k≤19. And we also see that higher is n, flatter is the function Er_n(k) as shown below .



IV - Relation between $S(n)$ and $S(n-1)$ and application :

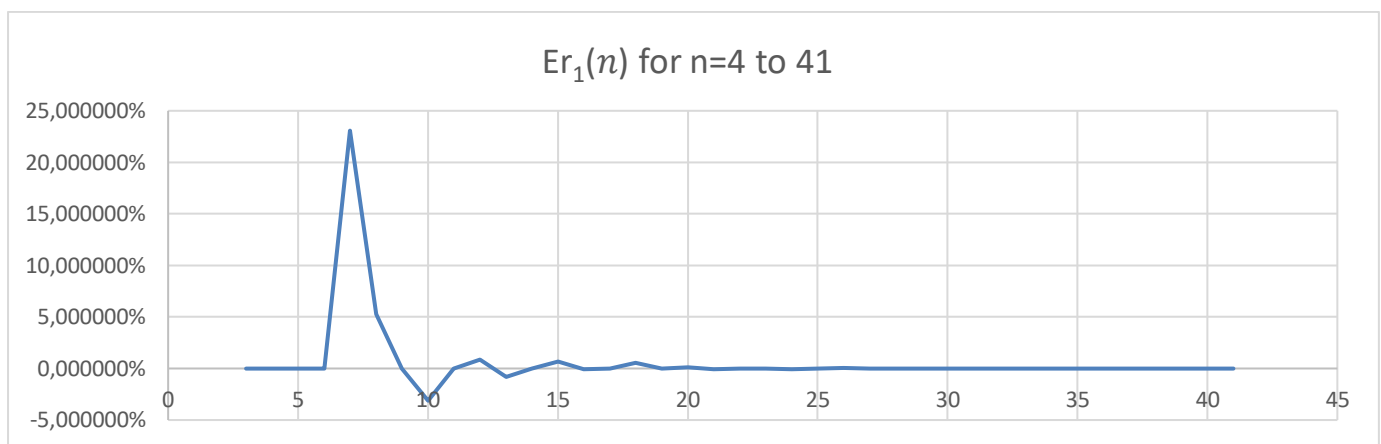
We can estimate the link between two consecutive values of $S(n)$ on the range of n where we have computed the histograms H_n and try to use these results to extrapolate the values of $S(n)$ for $n > 40$, where the computational time becomes high.

In the table below we give for $n=6$ to 39, $S(n)$ and the estimation $S_e(n)$ computed with the data of the step $n-1$. Thanks to (2), we can write:

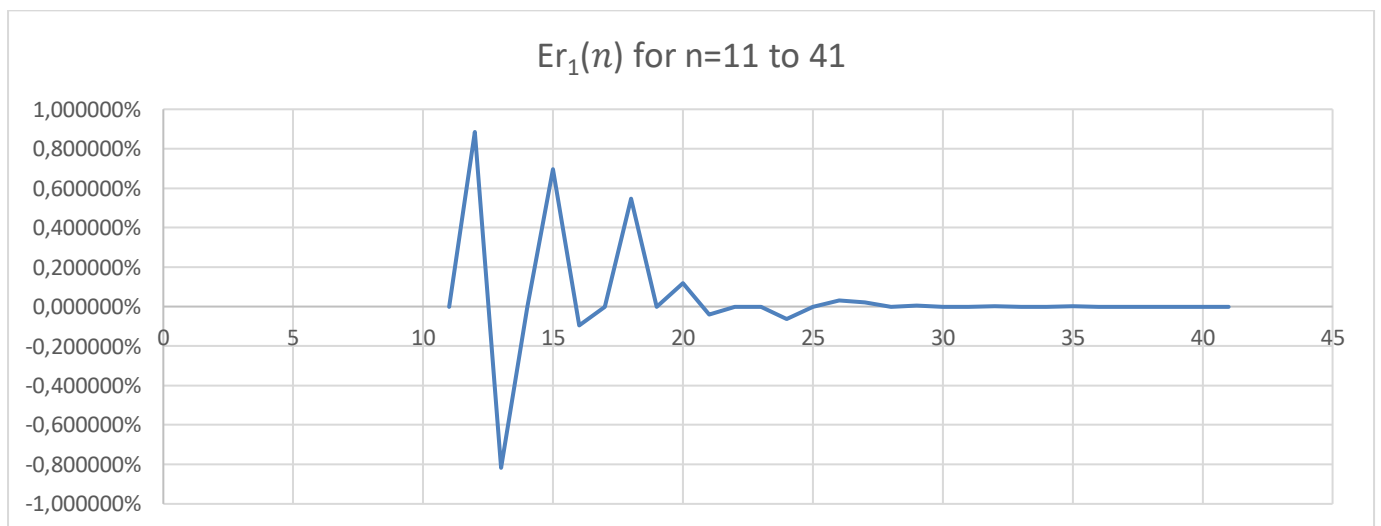
$$\text{If } n \in S_\sigma \quad S_e(n) = R_1 \cdot (S(n-1) - \sum_{p=n}^n h_{n-1}(p)) \approx 2 \cdot (S(n-1) - h_{n-1}(n)) \quad (3)$$

$$\text{Else} \quad S_e(n) = R_1 \cdot S(n-1) \approx 2 \cdot S(n-1)$$

The table below gives for $n \leq 41$, the values of $S(n)$, $S_e(n)$ and $Er_1(n) = \frac{(S(n) - S_e(n))}{S(n)}$



In the second graph below, we display $Er_1(n)$ from $n=11$ to have a more accurate view of its behavior.



This two representations above show clearly that $Er_1(n)$ tends toward 0 when n grows. $S_e(n)$ gives a very accurate estimate of $S(n)$ which reach 0,00068% for $n=39$, as presented in table 7 below and for $20 < n < 42$,

$$Er_1(n) = \frac{(S(n) - S_e(n))}{S(n)} < 0,1\% = Er_1$$

n	S(n)	S _e (n)	(S(n)-S _e (n))/S(n)
3	2	2	0,000000%
4	4	4	0,000000%
5	4	4	0,000000%
6	8	8	0,000000%
7	13	10	23,076923%
8	19	18	5,263158%
9	38	38	0,000000%
10	64	66	-3,125000%
11	128	128	0,000000%
12	226	224	0,884956%
13	367	370	-0,817439%
14	734	734	0,000000%
15	1295	1286	0,694981%
16	2114	2116	-0,094607%
17	4228	4228	0,000000%
18	7495	7454	0,547031%
19	14990	14990	0,000000%
20	27328	27296	0,117096%
21	46611	46630	-0,040763%
22	93222	93222	0,000000%
23	168807	168808	-0,000592%
24	286581	286760	-0,062461%
25	573162	573162	0,000000%
26	1037374	1037050	0,031233%
27	1762293	1761918	0,021279%
28	3524586	3524586	0,000000%
29	6385637	6385374	0,004119%
30	12771274	12771274	0,000000%
31	23642078	23642448	-0,001565%
32	41347483	41347100	0,000926%
33	82694966	82694966	0,000000%
34	151917636	151917948	-0,000205%
35	263841377	263838690	0,001018%
36	527682754	527682754	0,000000%
37	967378591	967379028	-0,000045%
38	1934757182	1934757182	0,000000%
39	3611535862	3611558232	-0,000619%
40	6402835000	6402839374	-0,000068%
41	12805670000	12805670000	0,000000%

By definition of $Er_1(n)$

$$\text{If } n \in S_\sigma \quad S(n) = S_e(n) \cdot (1 + Er_1(n)) = 2 \cdot (S(n-1) - h_{n-1}(n)) \cdot (1 + Er_1(n))$$

$$S(n) < 2 \cdot (S(n-1) - h_{n-1}(n)) \cdot (1 + |Er_1(n)|)$$

$$\text{Else} \quad S(n) = 2 \cdot S(n-1) = 2 \cdot S_e(n-1) \cdot (1 + Er_1(n-1))$$

$$S(n) = 4 \cdot (S(n-2) - h_{n-2}(n-1)) \cdot (1 + Er_1(n-1))$$

$$S(n) < 4 \cdot (S(n-2) - h_{n-2}(n-1)) \cdot (1 + |Er_1(n)|)$$

Which gives some upper bounds for $S(n)$.

V -Relation between two consecutive values $h_n(p(i))$ and $h_n(p(i+1))$:

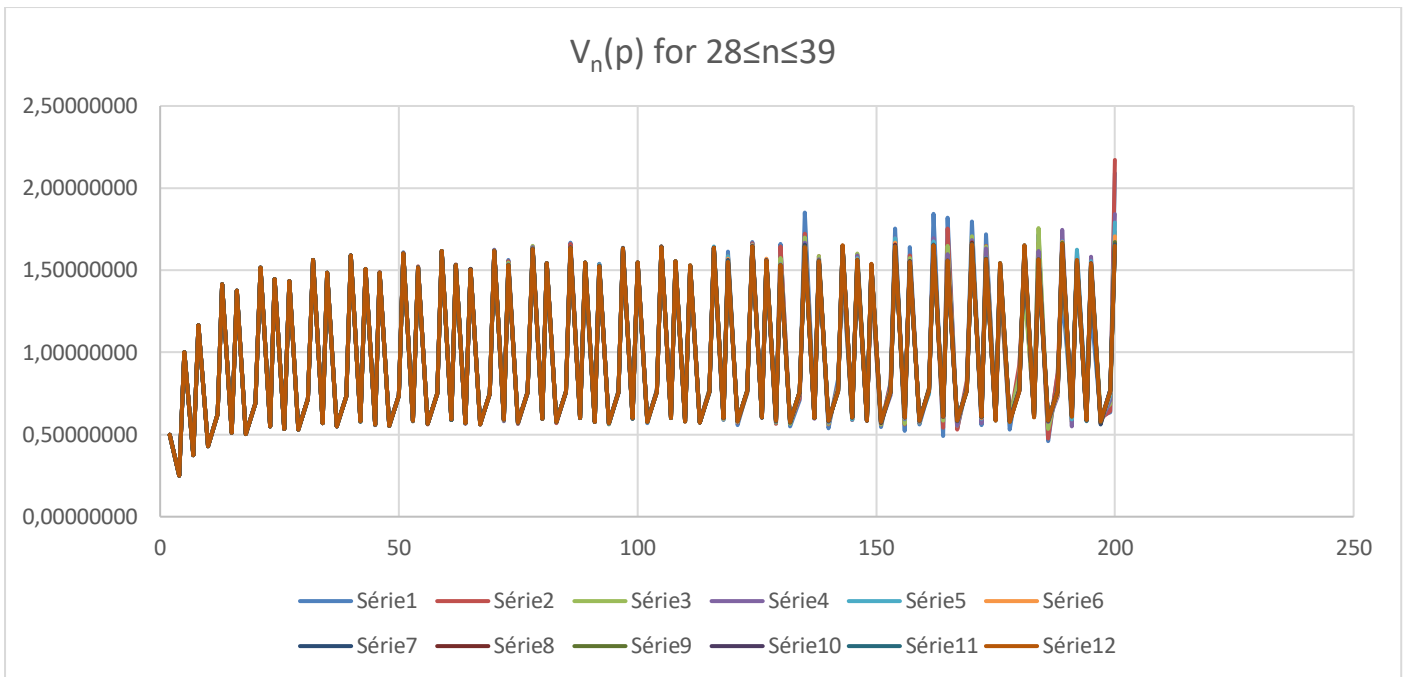
We have studied in §III, the different histograms H_n and shown that we could link different histograms H_n and H_{n+k} thanks to an homothetic factor 2^k . When we look at the superimposition of the different histograms in graph 3, we are obliged to recognize that the $h_n(p)$ haven't a stochastic but a strongly deterministic behavior. We are going to try to study in this part, the discrete variations of $h_n(p)$ for consecutive values of $p \in S_\sigma$.

We'll define a new arithmetic function $V_n(p)$ and will analyze its behavior on \mathbb{N} :

$$\text{For } p(i) = [i \cdot \log_2 3] + 1 \quad V_n(p(i)) = \frac{h_n(p(i))}{h_n(p(i-1))} \quad \text{thus} \quad h_n(p(i)) = h_n(p(i-1)) \cdot V_n(p(i)) \quad (4)$$

The advantage of this approach is if there is an homothetic factor between H_n and H_{n+k} , we can hope that the different function $V_n(p)$ will be relatively independent of n for n enough large.

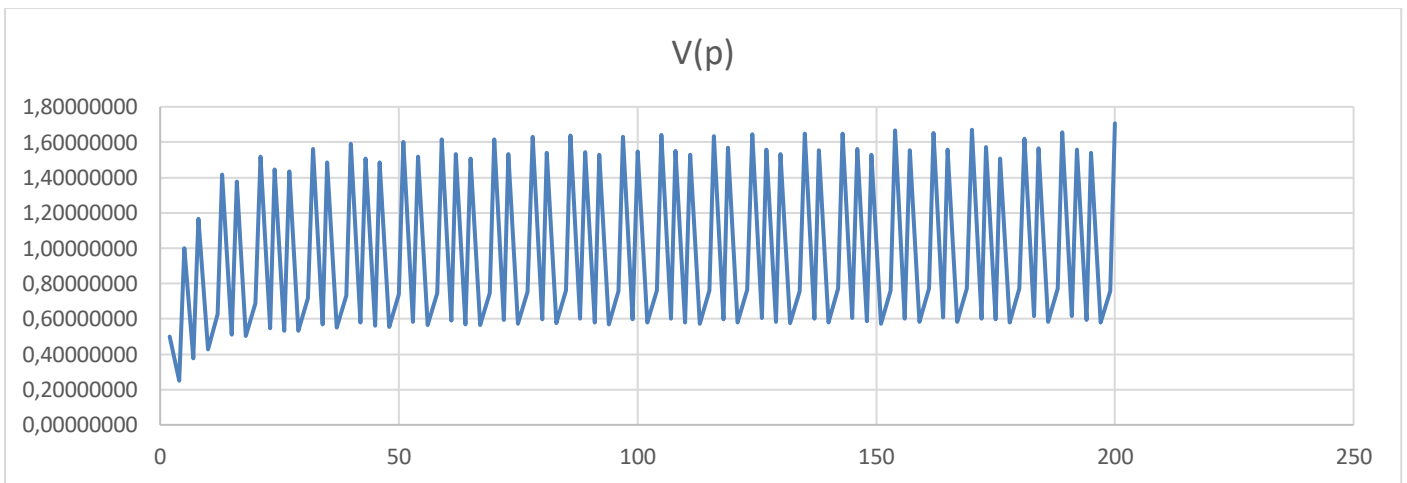
We display below these functions $V_n(p(i))$ for n such as $28 < n < 39$, and we can see the very good superimposition of this set of 12 functions.



Thanks to the very high similarity of the functions $V_n(p)$ for n from 28 to 39, we can define a new function

$$\text{For all integer } p \text{ such as } p(i)=i \quad V(p(i)) = \text{Med}_{n=30}^{39}(V_n(p(i)))$$

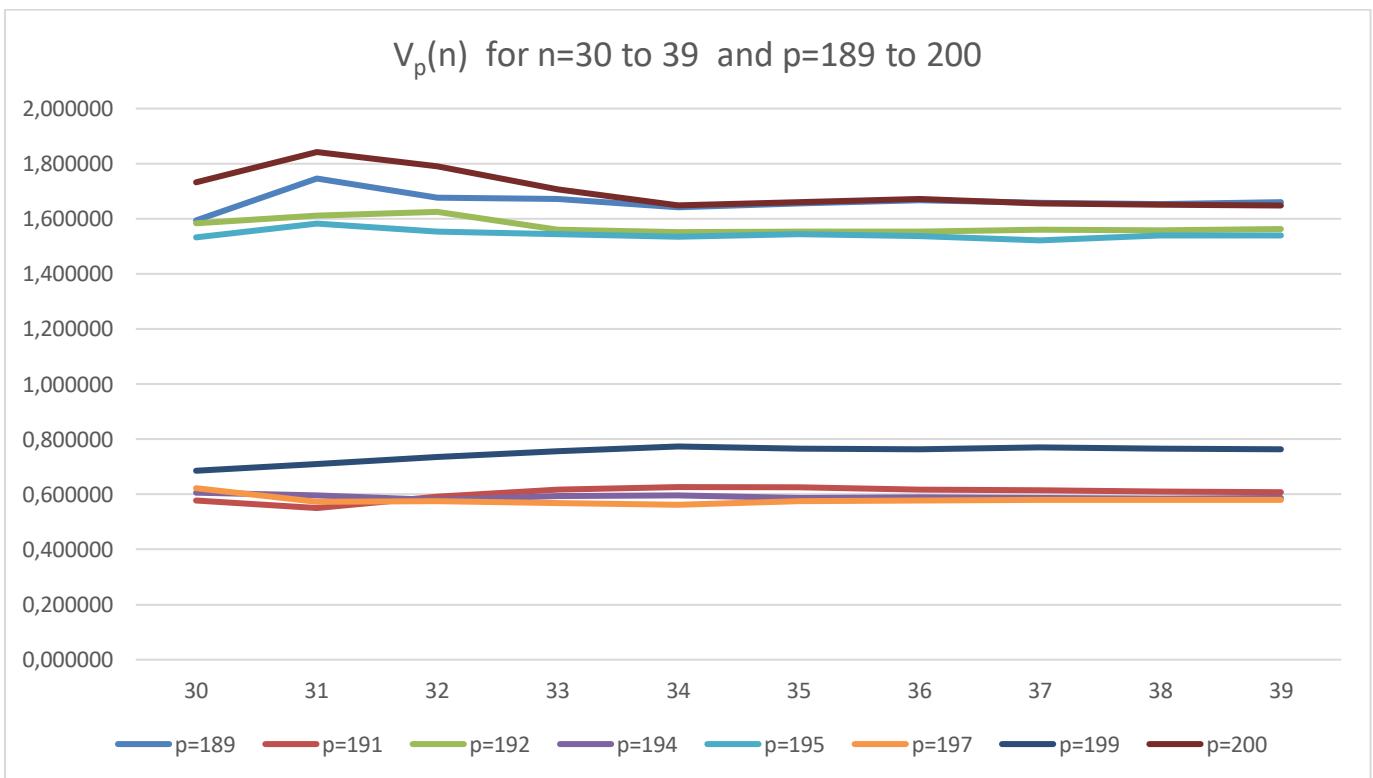
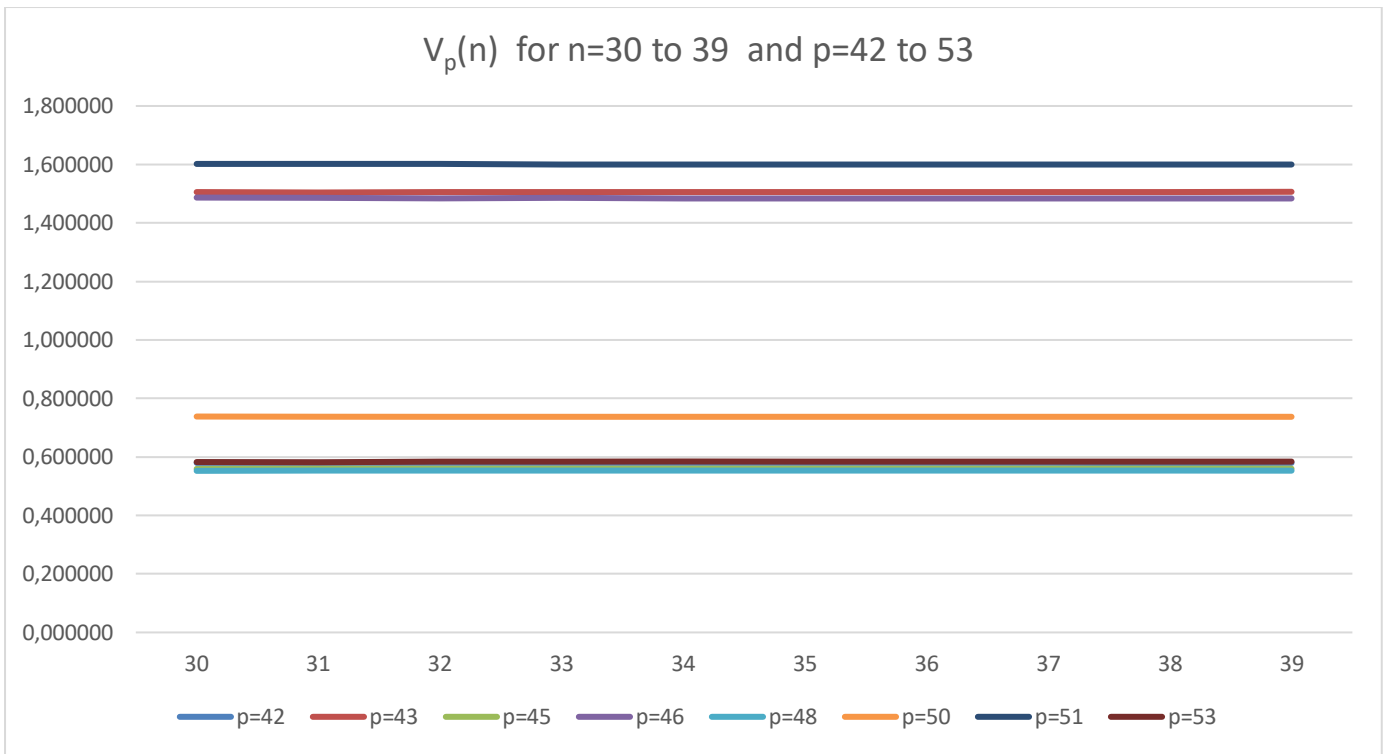
where Med is the median function applied for each $p > n$ on the set of $n=30$ to 39.



On the previous graph, we see that the link between consecutive values $h_n(p(i))$ and $h_n(p(i+1))$ with $p(i+1) > p(i)$ is not stochastic but strongly deterministic and relatively independent on n . $V(p)$ present a quasi-periodic behavior.

We display below the values of $V(p)$ for p from 42 to 173. We'll see later that the median is probably not the best estimator of $V(p)$.

p	42	43	45	46	48	50	51	53	54	56	58	59	61	62	64	65	67	69	70	72	73
$V_{30}(p)$	0,57947	1,50568	0,55922	1,48676	0,55244	0,73818	1,60190	0,58253	1,51973	0,56432	0,75111	1,61188	0,59084	1,53208	0,56776	1,50859	0,56270	0,75178	1,61405	0,58788	1,54581
$V_{31}(p)$	0,57967	1,50416	0,55934	1,48615	0,55273	0,73751	1,60166	0,58284	1,52059	0,56555	0,74921	1,61122	0,59131	1,53278	0,56712	1,50706	0,56272	0,75088	1,61185	0,59314	1,53574
$V_{32}(p)$	0,57939	1,50480	0,55975	1,48404	0,55282	0,73802	1,60116	0,58421	1,51809	0,56569	0,74779	1,61112	0,59162	1,53142	0,56845	1,50582	0,56368	0,74960	1,61655	0,59073	1,53589
$V_{33}(p)$	0,57903	1,50587	0,55954	1,48489	0,55286	0,73811	1,60064	0,58415	1,51804	0,56535	0,74711	1,61495	0,59078	1,53060	0,56922	1,50590	0,56471	0,74770	1,61558	0,59238	1,53199
$V_{34}(p)$	0,57895	1,50606	0,55959	1,48470	0,55290	0,73804	1,59979	0,58451	1,51706	0,56564	0,74682	1,61543	0,59023	1,53175	0,56929	1,50805	0,56342	0,74765	1,61629	0,59268	1,53285
$V_{35}(p)$	0,57892	1,50622	0,55967	1,48458	0,55286	0,73805	1,60026	0,58425	1,51681	0,56548	0,74706	1,61588	0,59061	1,53073	0,56946	1,50703	0,56368	0,74712	1,61729	0,59247	1,53195
$V_{36}(p)$	0,57894	1,50622	0,55966	1,48463	0,55296	0,73777	1,60074	0,58405	1,51727	0,56514	0,74751	1,61619	0,59040	1,53102	0,56932	1,50766	0,56345	0,74675	1,61740	0,59224	1,53321
$V_{37}(p)$	0,57892	1,50645	0,55954	1,48484	0,55298	0,73800	1,60038	0,58405	1,51738	0,56484	0,74791	1,61590	0,59056	1,52987	0,57001	1,50650	0,56304	0,74726	1,61720	0,59234	1,53333
$V_{38}(p)$	0,57886	1,50650	0,55960	1,48483	0,55295	0,73804	1,60040	0,58402	1,51751	0,56471	0,74793	1,61590	0,59053	1,53009	0,57009	1,50633	0,56293	0,74721	1,61811	0,59216	1,53347
$V_{39}(p)$	0,57892	1,50654	0,55957	1,48479	0,55301	0,73794	1,60042	0,58403	1,51749	0,56477	0,74774	1,61589	0,59068	1,53001	0,57018	1,50619	0,56278	0,74762	1,61773	0,59209	1,53346
$V(p)$	0,57894	1,50614	0,55958	1,48481	0,55288	0,73803	1,60053	0,58405	1,51750	0,56524	0,74777	1,61566	0,59064	1,53088	0,56931	1,50677	0,56323	0,74764	1,61688	0,59229	1,53340
	75	77	78	80	81	83	85	86	88	89	91	92	94	96	97	99	100	102	104	105	107
$V_{30}(p)$	0,56822	0,75160	1,64850	0,59524	1,53554	0,57606	0,76576	1,63333	0,60482	1,52593	0,57833	1,53218	0,56079	0,76556	1,62108	0,60293	1,53738	0,57801	0,76898	1,63883	0,60073
$V_{31}(p)$	0,57109	0,75054	1,63901	0,59878	1,53688	0,57587	0,76028	1,63888	0,60281	1,52901	0,57688	1,53549	0,56516	0,76428	1,62742	0,60124	1,53519	0,57921	0,76332	1,63532	0,60231
$V_{32}(p)$	0,57315	0,75166	1,63399	0,59673	1,53587	0,57765	0,75910	1,63823	0,60042	1,54060	0,57631	1,53443	0,56686	0,76295	1,62843	0,59760	1,53597	0,58267	0,75994	1,64065	0,60495
$V_{33}(p)$	0,57315	0,75192	1,63145	0,59692	1,53907	0,57621	0,75999	1,63936	0,60019	1,54265	0,57792	1,52846	0,56991	0,75634	1,63241	0,59690	1,54686	0,58063	0,76101	1,63973	0,60314
$V_{34}(p)$	0,57164	0,75362	1,63272	0,59682	1,54050	0,57645	0,75818	1,64230	0,59960	1,54466	0,57777	1,52776	0,56879	0,75593	1,63615	0,59758	1,54846	0,58005	0,75991	1,64158	0,60237
$V_{35}(p)$	0,57170	0,75555	1,62977	0,59668	1,54155	0,57632	0,75910	1,63726	0,59969	1,54698	0,57847	1,52555	0,56974	0,75558	1,63321	0,59861	1,54686	0,57897	0,76156	1,64541	0,60030
$V_{36}(p)$	0,57214	0,75466	1,63034	0,59629	1,54331	0,57574	0,75923	1,63720	0,59971	1,54704	0,57868	1,52486	0,56987	0,75624	1,63083	0,59892	1,54639	0,57874	0,76185	1,64384	0,60080
$V_{37}(p)$	0,57202	0,75488	1,62984	0,59678	1,54266	0,57581	0,75883	1,63764	0,59958	1,54790	0,57848	1,52388	0,57017	0,75623	1,63164	0,59870	1,54492	0,57926	0,76132	1,64363	0,60154
$V_{38}(p)$	0,57203	0,75524	1,62882	0,59700	1,54256	0,57585	0,75893	1,63734	0,59905	1,54871	0,57820	1,52409	0,57037	0,75634	1,63126	0,59819	1,54721	0,57858	0,76210	1,64223	0,60126
$V_{39}(p)$	0,57201	0,75539	1,62936	0,59684	1,54259	0,57584	0,75895	1,63597	0,59936	1,54859	0,57821	1,52402	0,57060	0,75573	1,63150	0,59846	1,54749	0,57826	0,76192	1,64177	0,60177
$V(p)$	0,57201	0,75414	1,63089	0,59680	1,54102	0,57597	0,75910	1,63749	0,59970	1,54582	0,57820	1,52666	0,56981	0,75629	1,63138	0,59854	1,54662	0,57909	0,76170	1,64168	0,60165
	108	110	111	113	115	116	118	119	121	123	124	126	127	129	130	132	134	135	137	138	140
$V_{30}(p)$	1,54681	0,58438	1,52992	0,57443	0,77326	1,60521	0,59378	1,57785	0,57779	0,754361	1,645785	0,613872	1,557844	0,572325	1,574796	0,566390	0,748823	1,699511	0,602179	1,587573	0,565161
$V_{31}(p)$	1,55119	0,58396	1,52841	0,57567	0,76275	1,61943	0,59696	1,57503	0,57958	0,760945	1,642642	0,609344	1,548350	0,580915	1,535125	0,580553	0,753948	1,667296	0,597405	1,568252	0,574679
$V_{32}(p)$	1,55153	0,58124	1,52035	0,57851	0,75255	1,64458	0,59277	1,58443	0,57781	0,754690	1,657909	0,604498	1,563789	0,578570	1,536379	0,580598	0,761006	1,641646	0,601827	1,560883	0,575691
$V_{33}(p)$	1,55352	0,57814	1,53251	0,57213	0,76140	1,63413	0,59754	1,57031	0,57726	0,758396	1,653636	0,603285	1,572390	0,580518	1,531281	0,578329	0,759680	1,648907	0,599092	1,552138	0,580328
$V_{34}(p)$	1,55107	0,58080	1,52705	0,57168	0,76038	1,63573	0,59884	1,56288	0,57937	0,762589	1,649323	0,604910	1,562065	0,581875	1,527761	0,575488	0,756563	1,656311	0,599444	1,553008	0,582573
$V_{35}(p)$	1,55447	0,58050	1,52446	0,57316	0,75820	1,63773	0,59971	1,55500	0,58181	0,765366	1,642939	0,604883	1,558630	0,582562	1,526293	0,572947	0,761448	1,648350	0,600590	1,558579	0,581981
$V_{36}(p)$	1,55489	0,58045	1,52570	0,57313	0,76003	1,63358	0,59950	1,55615	0,58136	0,766120	1,642904	0,603903	1,554618	0,583038	1,529736	0,574953	0,758444	1,645383	0,600517	1,559924	0,583377
$V_{37}(p)$	1,55261	0,58069	1,52679	0,57342	0,75818	1,63731	0,60022	1,55538	0,58029	0,765856	1,642884	0,603803	1,554957	0,583264	1,530594	0,575079	0,758920	1,644509	0,601545	1,555295	0,583444
$V_{38}(p)$	1,55346	0,58048	1,52740	0,57316	0,75818	1,63826	0,60097	1,55344	0,58038	0,764536	1,642587	0,604383	1,554983	0,584100	1,529936	0,576242	0,758419	1,644652	0,601684	1,554332	0,583122
$V_{39}(p)$	1,55333	0,58070	1,52752	0,57307	0,75865	1,63754	0,60118	1,55133	0,58033	0,763494	1,645203	0,604264	1,557176	0,582913	1,533525	0,575586	0,759643	1,642423	0,602147	1,554744	0,583108
$V(p)$	1,55297	0,58070	1,52723	0,57316	0,75934	1,63652	0,59917	1,55952	0,57994	0,76304	1,64407	0,60444	1,55751	0,58222	1,53094	0,57554	0,75868	1,64687	0,60107	1,55694	0,58228
	142	143	145	146	148	149	151	153	154	156	157	159	161	162	164	165	167	169	170	172	173
$V_{30}(p)$	0,792998	1,591651	0,590292	1,600613	0,595086	1,496515	0,561448	0,780472	1,688471	0,567070	1,576430	0,573673	0,784703	1,649819	0,585704	1,650062	0,552453	0,785519	1,706087	0,573904	1,646536
$V_{31}(p)$	0,779499	1,631011	0,589200	1,588461	0,585182	1,523208	0,574107	0,741834	1,698984	0,594069	1,554111	0,578408	0,755016	1,694067	0,607078	1,598558	0,554135	0,796472	1,682283	0,569114	1,631673
$V_{32}(p)$	0,773321	1,650037	0,592659	1,572843	0,584154	1,525552	0,573744	0,753196	1,696417	0,597752	1,533264	0,591160	0,747436	1,676731	0,617679	1,530035	0,581024	0,775422	1,653994	0,596333	1,572975
$V_{33}(p)$	0,771018	1,639823	0,602442	1,566837	0,585516	1,532590	0,572327	0,758945	1,667287	0,599405	1,547										



The behavior of the $V_n(p) = V_p(n)$ as a function of n , for different values of p , shows for all p that $\Delta V_p(n) = V_p(n) - V_p(n-1)$ tends toward 0 when n grows. It means that we can define $V(p)$ better than previously as :

$$\text{For all integer } p, \quad V(p) = \lim_{n \rightarrow \infty} (V_n(p)) \quad (5)$$

We shall state later in paragraph VI that for all integer p , $V(p) = \lim_{n \rightarrow \infty} (V_n(p)) = V_p(p)$

We can apply the function $V(p)$ to estimate $S(n)$ for different values of n . We can see in the table below that if we replace in the definition of $S(n)$, $V_n(p)$ by $V(p)$, we obtain a good estimator of $S(n)$ with an error relatively small which remains under 1% for a set of 18 consecutive values of n .

In table 6 below, we have estimated $S(n-k)$ for $k=39$ up to 22, only with $h_{39}(39)$ and the function $V(p)$ and the accuracy of this model varies from 0.00084% for $n=37$ up to 1.04% for $n=22$ and the accuracy is better than 0.012% for all $n=31$ up to 39. At this stage we could estimate up to $S(60)$ with an accuracy close to 1%.

n	S_e(n)	S(n)	(S(n)-S_e(n))/S(n)
40	6402349649	6402835000	0,007580%
39	3611411298	3611535862	0,003449%
38	1934804148	1934757182	-0,002427%
37	967377775	967378591	0,000084%
36	527703562	527682754	-0,003943%
35	263846883	263841377	-0,002087%
34	151922506	151917636	-0,003206%
33	82691196	82694966	0,004559%
32	41345393	41347483	0,005054%
31	23639252	23642078	0,011955%
30	12765948	12771274	0,041701%
29	6382553	6385637	0,048297%
28	3527839	3524586	-0,092285%
27	1764736	1762293	-0,138632%
26	1040641	1037374	-0,314941%
25	576330	573162	-0,552789%
24	287615	286581	-0,360710%
23	169440	168807	-0,374921%
22	94191	93222	-1,039214%

Table 6:

Thanks to (4), for n,k such as n and $n+k \in S_\sigma$, which means that n and $n+k$ are two stopping time values, we can write :

$$h_n(n+k) = h_n(n) \cdot \left(\prod_{p(i)>n}^{p(i)=n+k} V_n(p(i)) \right) \quad (6)$$

$$S(n) = \sum_{p(i)>n} h_n(p(i)) = \sum_{p(i)>n} h_n(n) \cdot \left(\prod_{p(j)>n}^{p(j)=p(i)} V_n(p(j)) \right) = h_n(n) \cdot \sum_{p(i)>n} \left(\prod_{p(j)>n}^{p(j)=p(i)} V_n(p(j)) \right)$$

If $V(p(i))$ is defined as in (5), from (6) we can define a new estimator $S_e(n)$ of $S(n)$:

$$S_e(n) = h_n(n) \cdot \sum_{p(i)>n} \left(\prod_{p(j)>n}^{p(j)<p(i)} V(p(j)) \right) \quad (7)$$

Which imply that we could estimate for all n , $S(n)$ with only $h_n(n)$ and the function $V(p(i))$.

Table 6 above, gives the numerical values of $S(n)$ and $S_e(n)$ for $n=22$ to 39.

In these conditions, from (2) and (7), we build a new estimator of $S(n+k)$

$$S_e(n+k) = R_k \cdot \left(S(n) - \sum_{p(i)>n}^{p(i)=n+k} h_n(p(i)) \right) \approx R_k \cdot \left(S(n) - h_n(p(n)) \cdot \sum_{p(j)>n+1}^{p(j)=n+k} \left(\prod_{p(i)=n+1}^{p(i)} V(i) \right) \right) \quad (8)$$

For $k=1$, (3) can be written:

$$\text{If } n \in S_\sigma \quad S_e(n+1) = R_1 \cdot (S(n) - h_n(n+1)) = 2 \cdot (S(n) - h_n(n) \cdot V(n+1))$$

$$\text{Else} \quad S_e(n+1) = R_1 \cdot S(n) = 2 \cdot S(n)$$

VI – An exact formulation of $h_n(n)$ and $V(n)$:

Mike Winker [2] 2017 has conjectured in an excellent paper that the number z_r of residue classes (mod 2^{σ_r}) of starting numbers s with finite stopping time $\sigma(s)=\sigma_r$ with $\sigma_r = \lfloor 1 + r \cdot \log_2 3 \rfloor$ will satisfy to :

$$z_r = \frac{(m+r-2)!}{m! \cdot (r-2)!} - \sum_{i=2}^{r-1} \binom{\lfloor \frac{3(r-i)+\delta}{2} \rfloor}{r-i} \cdot z_i \quad \text{with } m = \lfloor (r-1) \cdot \log_2 3 \rfloor - (r-1) \quad (9)$$

Where δ can take different values mod[3], but Mike Winker specifies that this value is not obvious to estimate and in his paper, it seems to be not simple to estimate z_r for some values of r larger than 50.

We'll give in the following a new formulation of Winkler formula which avoids this issue due to the estimate of δ and which allows to compute the exact values of z_r for any r . In Winker's work, r represents the number of odd integers in the Syracuse integer sequence up to σ_r iterates.

We are going to show that if we define z_n , the number of residue classes (mod 2^n) of starting number s with finite stopping time $\sigma(s)=n$, z_n can be simply formulated without using the parameter δ .

If we look at more precisely the Winker formula, the first term is $\frac{(m+r-2)!}{m! \cdot (r-2)!} = \binom{m+r-2}{m}$ and we must redefine $z_n = h_n(n)$ where n is a stopping time value. In these conditions, the first values of z_n are :

$$z_1 = 1, z_2 = 1, z_4 = 1, z_5 = 2, z_7 = 3, z_8 = 7, z_{10} = 12, z_{12} = 30 \dots \dots$$

We see that only the values of n corresponding to a stopping time are taken into account. We can also say that for all integer n such as n is not a stopping time value (like 3,6,9,11...), $z_n = 0$. We'll not take into account these terms in the following.

Thanks to that we can re-write the Winker formula as below :

$$z_n = h_n(n) = \binom{n-2}{m(n)} - \sum_{\substack{i=4 \\ p(i)=i}}^{i \leq n} \binom{n-i}{m(n)-m(i)} \cdot z_i \quad (10)$$

Where $r(n) = \lfloor \frac{n}{\log_2 3} \rfloor$ and $m(n) = \lfloor \frac{n}{\log_2 3} \rfloor \cdot \log_2 3 - \lfloor \frac{n}{\log_2 3} \rfloor + 1$ and $m(n)+r(n)=n$

$r(n)$ is the number of odd iterates and $m(n)$ the number of even iterates before reaching the stopping time.

We can easily check that for the first values of $n = \lfloor r \cdot \log_2 3 \rfloor + 1$, the numerical values of z_n correspond exactly to the $h_n(n)$ given by the histograms H_n .

We give below the detail of the application of (10) for the first values of n .

$$z_5 = h_5(5) = \binom{3}{2} - \binom{1}{0} \cdot z_4 = 2$$

$$z_7 = h_7(7) = \binom{5}{3} - \binom{3}{1} \cdot z_4 - \binom{2}{1} \cdot z_5 = 3$$

$$z_8 = h_8(8) = \binom{6}{3} - \binom{4}{1} \cdot z_4 - \binom{3}{1} \cdot z_5 - \binom{1}{0} \cdot z_7 = 7$$

$$z_{10} = h_{10}(10) = \binom{8}{4} - \binom{6}{2} \cdot z_4 - \binom{5}{2} \cdot z_5 - \binom{3}{1} \cdot z_7 - \binom{2}{1} \cdot z_8 = 12$$

$$z_{12} = h_{12}(12) = \binom{10}{5} - \binom{8}{3} \cdot z_4 - \binom{7}{3} \cdot z_5 - \binom{5}{2} \cdot z_7 - \binom{4}{2} \cdot z_8 - \binom{2}{1} \cdot z_{10} = 30$$

$$z_{13} = h_{13}(13) = \binom{11}{5} - \binom{9}{3} \cdot z_4 - \binom{8}{3} \cdot z_5 - \binom{6}{2} \cdot z_7 - \binom{5}{2} \cdot z_8 - \binom{3}{1} \cdot z_{10} - \binom{1}{0} \cdot z_{12} = 85$$

The other values of n which don't correspond to a stopping time are equal to 0 : $z_3 = z_6 = z_9 = z_{11} = \dots = 0$

We are going to show that we can deduce the arithmetic function $V(d)$ from the expression of $h_n(n)$

$$\text{If } n \in S_\sigma \quad V_{n(r)}(n(r)) = \frac{h_{n(r)}(n(r))}{h_{n(r)}(n(r-1))} \approx \frac{h_{n(r)}(n(r))}{2^{n(r)-n(r-1)} \cdot h_{n(r-1)}(n(r-1))} = \frac{z_{n(r)}}{2^{n(r)-n(r-1)} \cdot z_{n(r-1)}} \quad (11)$$

Which means that if, thanks to (10), we can compute all $z_{n(r)} = h_{n(r)}(n(r))$, we are able to build an exact formulation of $V_{n(r)}(n(r))$. As we have given an exact formulation of $z_n = h_n(n)$ for all $n \in S_\sigma$, we can also compute the exact values of $V_{n(r)}(n(r))$ for all $n \in S_\sigma$

In the table below, we gives the values of z_n obtained by us in §II up to $n=39$ completed by Winker results given in his work up to $n=50$.

n	r(n)	$h_n(n)$ & Winker results	Z_n thanks to (10)	$V_n(n)$
1	1	1	1	
2	1	1	1	0,500000
4	2	1	1	0,250000
5	3	2	2	1,000000
7	4	3	3	0,375000
8	5	7	7	1,166667
10	6	12	12	0,428571
12	7	30	30	0,625000
13	8	85	85	1,416667
15	9	173	173	0,508824
16	10	476	476	1,375723
18	11	961	961	0,504727
20	12	2652	2652	0,689906
21	13	8045	8045	1,516780
23	14	17637	17637	0,548073
24	15	51033	51033	1,446760
26	16	108950	108950	0,533723
27	17	312455	312455	1,433938
29	18	663535	663535	0,530904
31	19	1900470	1900470	0,716040
32	20	5996673	5996673	1,577682
34	21	13472296	13472296	0,561657
35	22	39993895	39993895	1,484301
37	23	87986917	87986917	0,550002
39	24	257978502	257978502	0,733002
40	25	820236724	820236724	1,589739
42	26	1899474678	1899474678	0,578941
43	27	5723030586	5723030586	1,506477
45	28	12809477536	12809477536	0,559558
46	29	38036848410	38036848410	1,484715
48	30	84141805077	84141805077	0,553028
50	31	248369601964	248369601964	0,737949

These values of $h_n(n)$ correspond exactly to the values of z_n predicted by (10).

In the table below, thanks to formula (10), we have computed z_n , $\frac{S(n)}{2^n}$ and $1 - \frac{S(n)}{2^n}$ up to $n=100$

n	Z_n=h_n(n)	1-S(n)/2ⁿ	S(n)/2ⁿ
1	1	0,5000000000	0,5000000000
2	1	0,7500000000	0,2500000000
4	1	0,8125000000	0,1875000000
5	2	0,8750000000	0,1250000000
7	3	0,8984375000	0,1015625000
8	7	0,9257812500	0,0742187500
10	12	0,9375000000	0,0625000000
12	30	0,9448242188	0,0551757813
13	85	0,9552001953	0,0447998047
15	173	0,9604797363	0,0395202637
16	476	0,9677429199	0,0322570801
18	961	0,9714088440	0,0285911560
20	2652	0,9739379883	0,0260620117
21	8045	0,9777741432	0,0222258568
23	17637	0,9798766375	0,0201233625
24	51033	0,9829184413	0,0170815587
26	108950	0,9845419228	0,0154580772
27	312455	0,9868698940	0,0131301060
29	663535	0,9881058242	0,0118941758
31	1900470	0,9889907995	0,0110092005
32	5936673	0,9903730389	0,0096269611
34	13472296	0,9911572298	0,0088427702
35	39993895	0,9923212053	0,0076787947
37	87986917	0,9929613944	0,0070386056
39	257978502	0,9934306545	0,0065693455
40	820236724	0,9941766555	0,0058233445
42	1899474678	0,9946085460	0,0053914540
43	5723030586	0,9952591793	0,0047408207
45	12809477536	0,9956232465	0,0043767535
46	38036848410	0,9961637827	0,0038362173
48	84141805077	0,9964627144	0,0035372856
50	248369601964	0,9966833109	0,0033166891
51	794919136728	0,9970363259	0,0029636741
53	1857112329035	0,9972425068	0,0027574932
54	5636545892795	0,9975553980	0,0024446020
56	12732900345928	0,9977321025	0,0022678975
58	38088111350198	0,9978642471	0,0021357529
59	123110229387834	0,9980778093	0,0019221907
61	290838337577435	0,9982039404	0,0017960596
62	889949312454085	0,9983969174	0,0016030826
64	2029460152095008	0,9985069346	0,0014930654
65	6113392816333320	0,9986726385	0,0013273615
67	13759389839553008	0,9987658757	0,0012341243
69	41156292958100112	0,9988355972	0,0011644028
70	133180667145777072	0,9989484056	0,0010515944
72	315356241137505268	0,9990151849	0,0009848151
73	967303800643232882	0,9991176022	0,0008823978
75	2213388970068123188	0,9991761901	0,0008238099
77	6687324379116300569	0,9992204431	0,0007795569
78	21797112395398269352	0,9992925637	0,0007074363
80	52028134169251235063	0,9993356003	0,0006643997
81	160509643506854706934	0,9994019856	0,0005980144
83	369707873749224505928	0,9994402125	0,0005597875
85	1122428422670255691408	0,9994692266	0,0005307734
86	3672921591387837707209	0,9995166980	0,0004833020
88	8808298119720364971552	0,9995451591	0,0004548409
89	27272844922266198818078	0,9995892208	0,0004107792
91	63092460692093312467525	0,9996147036	0,0003852964
92	192189781828748623023765	0,9996535160	0,0003464840
94	438474769118020519475109	0,9996756533	0,0003243467
96	1325438036712274130536314	0,9996923827	0,0003076173
97	4327322846731848749589802	0,9997196920	0,0002803080
99	10359365401268828714082041	0,9997360362	0,0002639638
100	32053249939776775765443011	0,9997613217	0,0002386783

We see that $S(n)$ is a strictly decreasing function of n which tends toward 0 when n grows.

For $n=100$, $\frac{S(n)}{2^n} < 0,0225\%$. We'll compute et see later that for larger values of n , $\frac{S(n)}{2^n}$ continue to decrease constantly.

Thanks to (10), we can compute the exact values of z_n , $\frac{S(n)}{2^n}$ and $1 - \frac{S(n)}{2^n}$ as far as we want.

In the next part, we are going to show that all $h_m(n)$ for $m > n$ are totally defined by $h_n(n)$

Theorem : If an integer $a \in \mathbb{Z}/2^n\mathbb{Z}$ verify $\sigma(a)=n$, then $b \in \mathbb{Z}/2^n\mathbb{Z}$ such as $2^n < b < 2^{n+1}$ satisfy to $\sigma(b)=n$ if and only if $b=a+2^n$ and implies that $h_{n+1}(n) = 2 \cdot h_n(n)$ and $h_{n+k}(n) = 2^k \cdot h_n(n)$ for all positive integers n and k .

Proof : If $a \in \mathbb{Z}/2^n\mathbb{Z}$ such as $\sigma(a)=n$ then by definition of the stopping time:

$$T^n(2^n \cdot m + a) < a < T^{n-1}(2^n \cdot m + a) \text{ for all } m \quad (12)$$

Then if $b=a+2^n$, we have thanks to (12) :

$$T^n(2^{n+1} \cdot m + b) = T^n(2^n \cdot (2m + 1) + a) < 2^n \cdot (2m + 1) + a < T^{n-1}(2^n \cdot (2m + 1) + a) < T^{n-1}(2^{n+1} \cdot m + b)$$

Whicj means that $\sigma(b)=n$ and reciprocally if for b such as $2^n < b < 2^{n+1}$ and $\sigma(b)=n$, we have by definition

$$T^n(2^n \cdot m + b) < 2^n \cdot m + b < T^{n-1}(2^n \cdot m + b) \text{ for all } m$$

$$T^n(2^n \cdot (m + 1) + b - 2^n) < 2^n \cdot (m + 1) + b - 2^n < T^{n-1}(2^n \cdot (m + 1) + b - 2^n) \text{ for all } m$$

If we replace $a= b - 2^n$ with $0 < a < 2^n$, we can write :

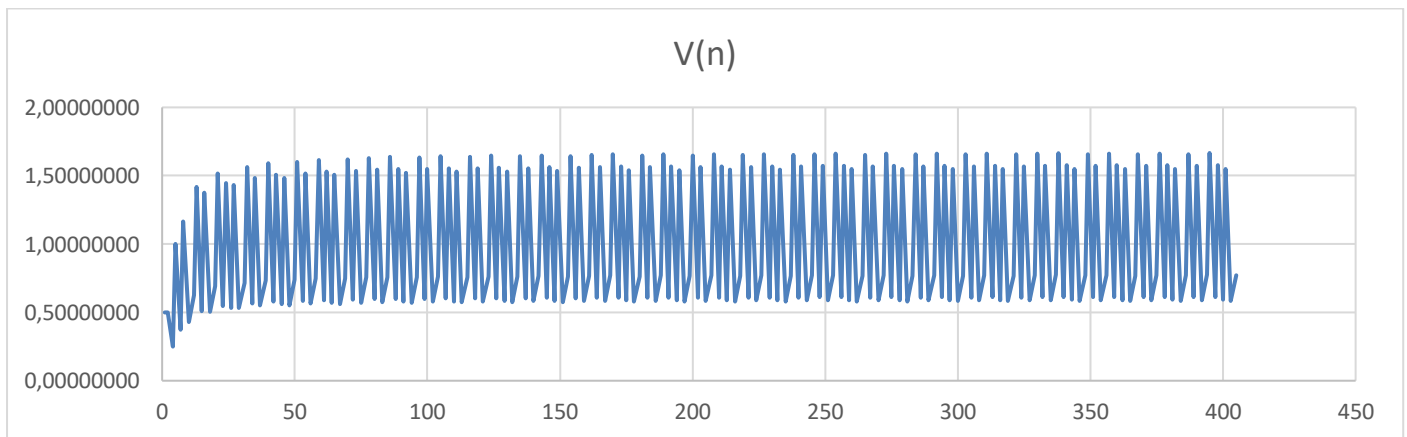
$$T^n(2^n \cdot (m + 1) + a) < 2^n \cdot (m + 1) + a < T^{n-1}(2^n \cdot (m + 1) + a) \text{ for all } m$$

which prove the fact the set of integers b such as $2^n < b < 2^{n+1}$ and $\sigma(b)=n$ is exactly the set of integers $a+\varepsilon \cdot 2^n$ with $\varepsilon \in \{0,1\}$ which means that $h_{n+1}(n) = 2 \cdot h_n(n)$ and consequently for all $k > 0$, $h_{n+k}(n) = 2^k \cdot h_n(n)$

Thanks to this result, we can deduce that $V_m(n) = V_n(n)$ for $m > n$.

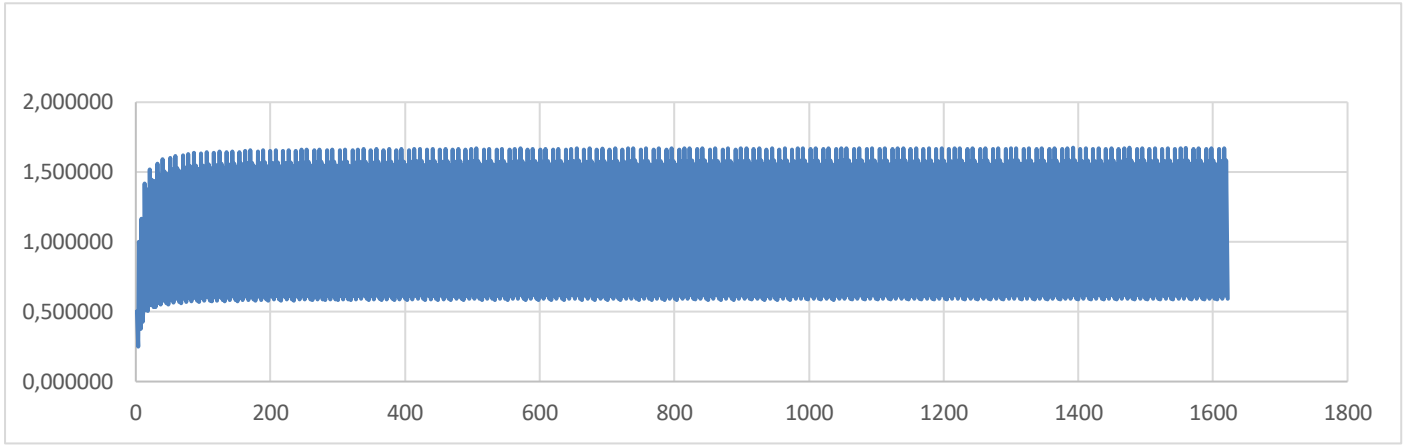
As we have speculated in (5) that $V(p) = \lim_{n \rightarrow \infty} (V_n(p))$, then we can conclude, for all n , that $V(n) = V_n(n)$.

Thanks to these previous results we have compute $V(n)$ up to $n=405$ ($r=256$) and $n=1622$ ($r=1024$) represented by the two graphs below. It illustrates the extrem regularity of the arithmetic function $V(n)$.



Thanks to these result, we state that for $n=405$ ($r(n)=256$): $\frac{S(n)}{2^n} = 9.68160440706356E - 10$ and $z_n = 3476553789120508476368100052260690271283238505581916333757459587755180695960919229021382116342674546834066825086$

And we have pushed below the computation of the function $V(n)$ up to $n=1622$ ($r(n)=1024$)



In the first part of this paper, to obtain the first estimation of $S(n)$, we have been obliged to compute all Syracuse sequences up to 2^{40} with a computation time of more than 48 hours.

We present in the next paragraph an exact formulation of $S(n)$ depending only on the values of $z_p = h_p(p)$ for $p \geq n$

VII – New formulation of $S(n)$:

We are now going to state that $S(n)$ can be expressed as a function of the different z_i .

According to (7) and (11) :

$$S(n) = h_n(n) \cdot \sum_{p(i) > n} \left(\prod_{p(j) > n}^{p(j)=p(i)} V(p(j)) \right) = h_n(n) \cdot \sum_{p(i) > n} \left(\prod_{p(j) > n}^{p(j)=p(i)} \frac{h_{p(j)}(p(j))}{2^{p(i)-p(j-1)} \cdot h_{p(j-1)}(p(j-1))} \right)$$

$$S(n) = h_n(n) \cdot \sum_{p(i) > n} \frac{h_{p(i)}(p(i))}{2^{p(i)-n} \cdot h_n(n)} = \sum_{p(i) > n} \frac{h_{p(i)}(p(i))}{2^{p(i)-n}} = \sum_{p(i) > n} \frac{z_{p(i)}}{2^{p(i)-n}} = 2^n \cdot \sum_{p(i) > n} \frac{z_{p(i)}}{2^{p(i)}}$$

$$\mathbf{S(n) = 2^n \cdot \sum_{p(i) > n} \frac{z_{p(i)}}{2^{p(i)}} \text{ and } h_n(p(i)) = 2^n \cdot \frac{z_{p(i)}}{2^{p(i)}}} \quad (13)$$

We can easily see that the total population of the histogram satisfy the following equality :

$$\sum_{p(i) > 0} \frac{z_{p(i)}}{2^{p(i)}} = 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{2}{32} + \frac{3}{128} + \frac{7}{256} + \frac{12}{1024} + \frac{30}{4096} + \frac{85}{8192} + \dots \quad (14)$$

Thus, we can deduce that : $\lim_{n \rightarrow \infty} \frac{S(n)}{2^n} = \lim_{n \rightarrow \infty} (1 - \sum_{i \leq n} \frac{z_i}{2^i}) = \lim_{n \rightarrow \infty} (\sum_{i > n} \frac{z_i}{2^i}) = 0$ (15)

And as we can compute z_i , for any n as far as possible depending on computers capabilities, thanks to (10) and (11), we have computed $S(n) = \#\{a < 2^n / \sigma(a) \leq n\}$ up to $n=6491$.

For $n=6491$, $\frac{\#\{a < 2^n \text{ and } \sigma(a) > n\}}{2^n} = \frac{S(n)}{2^n} = \alpha_n = 3.3659284589631293e - 103$ $\frac{\log_2(S(n))}{n} = 0.9475569882514053$ and

$Z_{6491} = 1549953482341234298531051149482800040632325009694686038033078427338305145689709906961470300749434637893940359158251231844299930$
 $850751700381527970999054332961934361384818397337108384120658071630882153030128099949202194198067696504962376071370105012687852642524090$
 $217278416793054538966160942419543120453761515845619541252335428718954350948528741949451553294021820576701491368112788781493912080218089$
 $40872803755696360687983371580023798858814453909711284892915000710341266152429919795349357595523711332009307399593418788825529661992158$
 $393317020508726385012435787356629676265453402483326751403976698179188856456168313631241234522043044193234331233965580383712497876516796$
 $69548560061381184381152855105523294654190077942283130089622264439172776649892653067682922787579271521036633862350870813505409115492502$
 $543723278116931273087002912337003223185754181324162114183161948729268866549662671464392664864501548652855682494936500401554050955065175$
 $601736741037426549865732883728488701779414482193391980004041016154439804509048113974583340960121833550994145000152145000835901958473375$
 $601043048139736610743341156113338555874923587252763023443905226117793877118909074105544464914508868430219262314395431169077471000114163$
 $737312153821904963226574032909069633882534240484494334044409927658395538004353740113188023461901926299967221669700053282487834646955692$
 $854056506563195630107619415446951716744310985673319989067077405750219848713211448956098499578567539266421832326237353786237781200908888$
 $299480312804743405211414049433552379348729447342695552248756543976762964974819496920100237868714967965569268020796435707211547532525627$
 $867812073532351270144749507154692909172784547453231633325385084582807397864502224121095797405081393364769845243264774281362687484158383$
 $598604532977737840561307747191867350650943790134256353886325451612212522057438940143811040504987250026064910.9475569882514053$

VIII – A model for H_n estimate :

Thanks to (13), we can estimate for all $p > n$, $h_n(p) = 2^n \cdot \frac{z_p}{2^p}$ with $h_n(n)$. But what is the accuracy of this approach.

We are going to estimate the accuracy of our model by comparing the histograms computed in paragraph II and the histograms predicted by our model. In the table below we give a comparison between $\#H_n$ real and modeled.

n	$\#H_n$	$\#H_n$ modeled	Error	$\#H_n$ modeled (nearest integer of each $h_n(p)$)	Error	$\#\{a < 2^n / \sigma(a) \leq n\}$	$\#\{a < 2^n / \sigma(a) \leq n\}$ modeled
2	4	4,000	9,77E-12	3	2,500000E-01	3	3
3	8	8,000	9,77E-12	8	0,000000E+00	6	6
4	16	16,000	9,77E-12	14	1,250000E-01	12	12
5	32	32,000	9,77E-12	30	6,250000E-02	28	28
6	64	64,000	9,77E-12	62	3,125000E-02	56	56
7	128	128,000	9,77E-12	125	2,343750E-02	115	115
8	256	256,000	9,77E-12	254	7,812500E-03	237	237
9	512	512,000	9,77E-12	509	5,859375E-03	474	474
10	1024	1024,000	9,77E-12	1020	3,906250E-03	960	960
11	2048	2048,000	9,77E-12	2044	1,953125E-03	1920	1920
12	4096	4096,000	9,77E-12	4094	4,882813E-04	3870	3870
13	8192	8192,000	9,77E-12	8187	6,103516E-04	7825	7825
14	16384	16384,000	9,77E-12	16379	3,051758E-04	15650	15650
15	32768	32768,000	9,77E-12	32760	2,441406E-04	31473	31473
16	65536	65536,000	9,77E-12	65533	4,577637E-05	63422	63422
17	131072	131072,000	9,77E-12	131069	2,288818E-05	126844	126844
18	262144	262144,000	9,77E-12	262140	1,525879E-05	254649	254649
19	524288	524288,000	9,77E-12	524283	9,536743E-06	509298	509298
20	1048576	1048576,000	9,77E-12	1048572	3,814697E-06	1021248	1021248
21	2097152	2097152,000	9,77E-12	2097150	9,536743E-07	2050541	2050541
22	4194304	4194304,000	9,77E-12	4194294	2,384186E-06	4101082	4101082
23	8388608	8388608,000	9,77E-12	8388602	7,152557E-07	8219801	8219801
24	16777216	16777216,000	9,77E-12	16777210	3,576279E-07	16490635	16490635
25	33554432	33554432,000	9,77E-12	33554429	8,940697E-08	32981270	32981270
26	67108864	67108864,999	9,77E-12	67108861	4,470348E-08	66071490	66071490
27	134217728	134217728,999	9,77E-12	134217719	6,705523E-08	132455435	132455435
28	268435456	268435456,997	9,77E-12	268435454	7,450581E-09	264910870	264910870
29	536870912	536870912,995	9,77E-12	536870907	9,313226E-09	530485275	530485275
30	1073741824	1073741824,990	9,77E-12	1073741819	4,656613E-09	1060970550	1060970550
31	2147483648	2147483648,979	9,77E-12	2147483644	1,862645E-09	2123841570	2123841570
32	4294967296	4294967296,958	9,77E-12	4294967291	1,164153E-09	4253679813	4253679813
33	8589934592	8589934592,916	9,77E-12	8589934582	1,164153E-09	8507359626	8507359626
34	17179869184	17179869184,832	9,77E-12	17179869181	1,746230E-10	17028191548	17028191548
35	34359738368	34359738368,664	9,77E-12	34359738364	1,164153E-10	34096376991	34096376991
36	68719476736	68719476736,329	9,77E-12	68719476728	1,164153E-10	68192753982	68192753982
37	137438953472	137438953472,657	9,77E-12	137438953462	7,275958E-11	136473494881	136473494881
38	274877906944	274877906944,314	9,77E-12	274877906938	2,182787E-11	272946989762	272946989762
39	549755813888	549755813888,628	9,77E-12	549755813888	0,000000E+00	546151958026	546151958026

In the table above, we can see that our model give an exact value of the $\Pi(2^n) = \#\{a < 2^n / \sigma(a) \leq n\} = 2^n - S(n)$ (the two last columns) and a good estimate of the whole population of the histograms (two first columns). The accuracy grows when n grows.

Thanks to the previous results, we can say that for each histogram H_n , the values of $h_n(p)$ can be computed exactly thanks to z_p for all $p \leq n$ and can be estimated thanks to $h_n(p) = 2^n \cdot \frac{z_p}{2^p}$ for all $p > n$. For this reason, for each n, the value of $S(n)$ computed thanks to (15) is the exact value and not an estimate.

IX – Analogy with I. Krasikov & J.C. Lagarias result :

Ilia Krasikov and Jeffrey C. Lagarias has proved that $\Pi(x) = \#\{a < x / \exists j T^j(a) = 1\} > x^c$ with $c=0,84$.

We are going to state the following result :

For all $\varepsilon > 0$ there is n such as $\Pi(2^n) = \#\{a < 2^n / \exists j \leq n T^j(a) \leq a\} = \#\{a < 2^n / \sigma(a) \leq n\} > 2^{n(1-\varepsilon)}$

As defined above, $S(n) = \#\{a < 2^n / \exists j T^j(a) > n\} = 2^n - \Pi(2^n)$

For all $\varepsilon > 0$, we are looking n and α_n , such as :

$$\frac{\#\{a < 2^n \text{ and } \sigma(a) \leq n\}}{2^n} = \frac{\Pi(2^n)}{2^n} = 1 - \alpha_n > \frac{1}{2^{n\varepsilon}}$$

With $\alpha_n = \frac{S(n)}{2^n} = \sum_{i > n} \frac{z_i}{2^i}$

We'll see in paragraph X that $\frac{z_n}{2^n} < \frac{1}{2^n} \cdot \binom{n-2}{m(n)} \simeq \frac{a \cdot q^{n-2}}{2^{n\sqrt{n-2}}} = \frac{a}{4\sqrt{n-2}} \cdot \left(\frac{q}{2}\right)^{n-2}$ with $\frac{q}{2} < 1$

This will imply that $\alpha_n = \frac{S(n)}{2^n} = \sum_{i > n} \frac{z_i}{2^i} < \sum_{i > n} \frac{a}{4\sqrt{i-2}} \cdot \left(\frac{q}{2}\right)^{i-2} < C \left(\frac{q}{2}\right)^{n-1}$ where $C = \sum_{i > n} \frac{a}{4\sqrt{i-2}} \cdot \left(\frac{q}{2}\right)^{i-n-1}$

Which prove that $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \frac{S(n)}{2^n} < \lim_{n \rightarrow \infty} C \left(\frac{q}{2}\right)^{n-1} = 0$

For n large, as α_n and $\varepsilon \ll 1$, that :

$$\ln(1 - \alpha_n) > -n\varepsilon \cdot \ln(2) \Leftrightarrow -\alpha_n > -n\varepsilon \cdot \ln(2) \Leftrightarrow \alpha_n < n\varepsilon \cdot \ln(2)$$

As $\alpha_n \rightarrow 0$ when $n \rightarrow \infty$ we can insure that for all real $\varepsilon > 0$, there N such as for all $n > N$: $\alpha_n < n\varepsilon \cdot \ln(2)$

In this conditions, we can affirm that for all real $\varepsilon > 0$, there is N such as for all $n > N$:

$$\Pi(2^n) = \#\{a < 2^n / \exists j T^j(a) \leq n\} > 2^{n(1-\varepsilon)}$$

And more generally, for all real $\varepsilon > 0$, there exist X enough large and $N = \ln(X)/\ln(2)$, such as for all $n > N$:

$$\Pi(X) = \#\{a < X / \exists j T^j(a) \leq n\} > X^{(1-\varepsilon)}$$

Appelgate and Lagarias result was regarding the total stopping time . Our result above regards the stopping time.

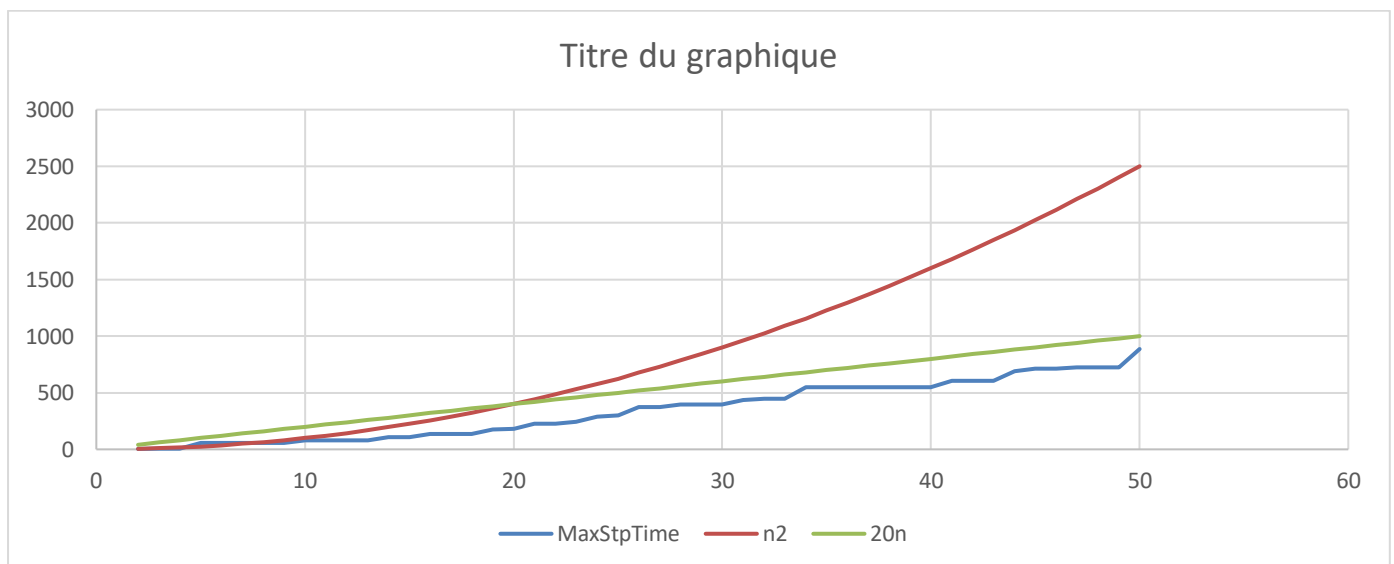
X - Maximal Stopping time estimate in $\mathbb{Z}/2^n\mathbb{Z}$,

We have seen previously that we can build the histograms H_n thanks to the exact computation of all z_n . As the modeled histograms is upper bounded for all n , we could hope that the real histograms H_n

We have computed for all $n \leq 38$, the maximum stopping time and use Oliveira E Silva[3]1999 results up to $n=50$.

n	MaxStpTime	n^2	MaxStpTime/ n^2	20n	MaxStpTime/20n
2	4	4	1,0000	40	0,100000
3	7	9	0,7778	60	0,116667
4	7	16	0,4375	80	0,087500
5	59	25	2,3600	100	0,590000
6	59	36	1,6389	120	0,491667
7	59	49	1,2041	140	0,421429
8	59	64	0,9219	160	0,368750
9	59	81	0,7284	180	0,327778
10	81	100	0,8100	200	0,405000
11	81	121	0,6694	220	0,368182
12	81	144	0,5625	240	0,337500
13	81	169	0,4793	260	0,311538
14	105	196	0,5357	280	0,375000
15	105	225	0,4667	300	0,350000
16	135	256	0,5273	320	0,421875
17	135	289	0,4671	340	0,397059
18	135	324	0,4167	360	0,375000
19	173	361	0,4792	380	0,455263
20	183	400	0,4575	400	0,457500
21	224	441	0,5079	420	0,533333
22	224	484	0,4628	440	0,509091
23	246	529	0,4650	460	0,534783
24	287	576	0,4983	480	0,597917
25	298	625	0,4768	500	0,596000
26	376	676	0,5562	520	0,723077
27	376	729	0,5158	540	0,696296
28	395	784	0,5038	560	0,705357
29	395	841	0,4697	580	0,681034
30	395	900	0,4389	600	0,658333
31	433	961	0,4506	620	0,698387
32	447	1024	0,4365	640	0,698438
33	447	1089	0,4105	660	0,677273
34	547	1156	0,4732	680	0,804412
35	547	1225	0,4465	700	0,781429
36	547	1296	0,4221	720	0,759722
37	547	1369	0,3996	740	0,739189
38	547	1444	0,3788	760	0,719737
40	550	1600	0,3438	800	0,687500
41	606	1681	0,3605	820	0,739024
42	606	1764	0,3435	840	0,721429
43	606	1849	0,3277	860	0,704651
44	688	1936	0,3554	880	0,781818
45	712	2025	0,3516	900	0,791111
46	712	2116	0,3365	920	0,773913
47	722	2209	0,3268	940	0,768085
48	722	2304	0,3134	960	0,752083
49	722	2401	0,3007	980	0,736735
50	886	2500	0,3544	1000	0,886000

We can see in these first results that for $n \leq 50$, the arithmetic function $\text{MaxStpTime}/n^2$ is a decreasing function and we have some argument to conjecture that $\text{MaxStpTime} < 20n$ for all n . If this conjecture could be verified, it would imply that all Syracuse sequence with a starting number $a < 2^n$ have a finite stopping time. This will imply that the Collatz conjecture would be true.



How can we estimate that the maximal stopping time are lower than $20n$.

If we consider that the arithmetic function $V(p)$ allows to generate the histograms H_n for all n as stated in (5), we can suppose that there is p_0 such as for $p > p_0$, $h_n(p) < 1$, there is no integer $a \in \mathbb{Z}/2^n\mathbb{Z}$ such $\sigma(a) > p_0$.

As for all integers i and p belonging to the set of stopping time value, $h_n(p) = 2^n \cdot \prod_{i < p} V(i)$, we can looking for

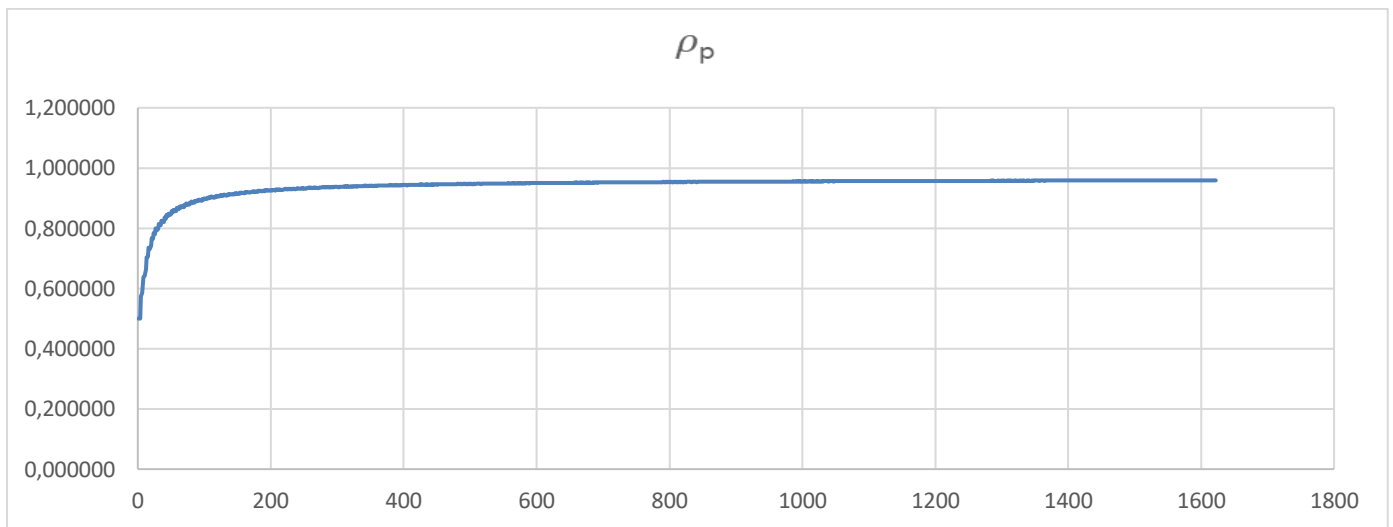
the value of ρ such as

$$\left| \lim_{p \rightarrow \infty} \frac{(\prod_{i < p} V(i))}{\rho^p} \right| = \left| \lim_{p \rightarrow \infty} \frac{\frac{2^p}{\rho^p}}{\rho^p} \right| < 1 + \epsilon$$

It corresponds to approximate the behavior of $\prod_{i < p} V(i)$ with a geometric sequence with a common ratio $\rho < 1$.

In the two graphs below, we have tried to estimate the value of ρ

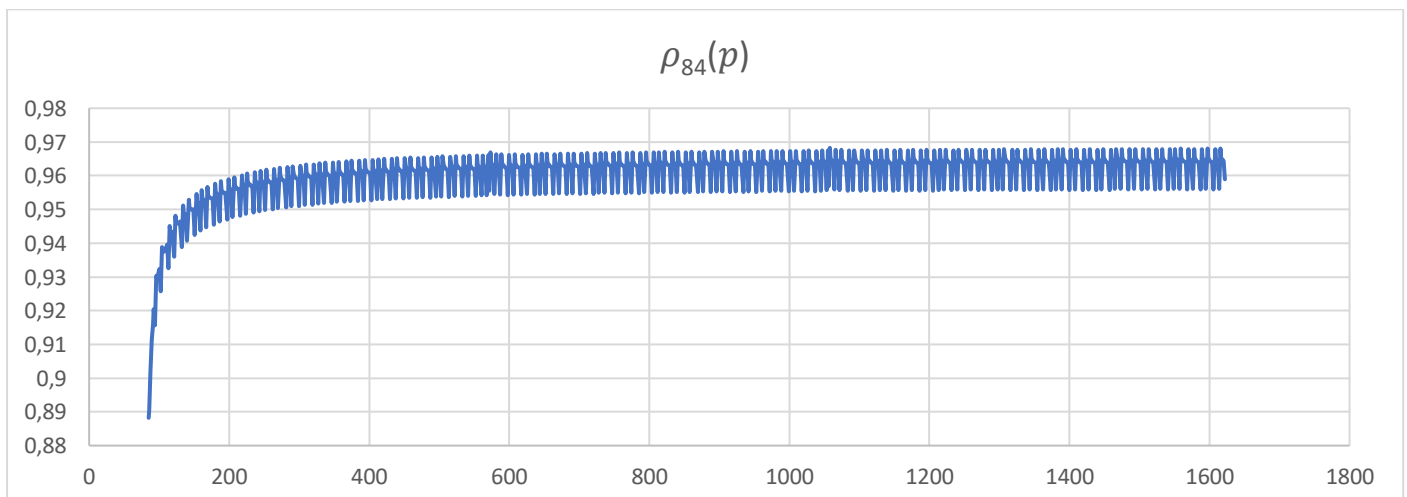
The first curve represent the function $\rho_p(p) = \sqrt[p]{\prod_{i < p} V(i)}$



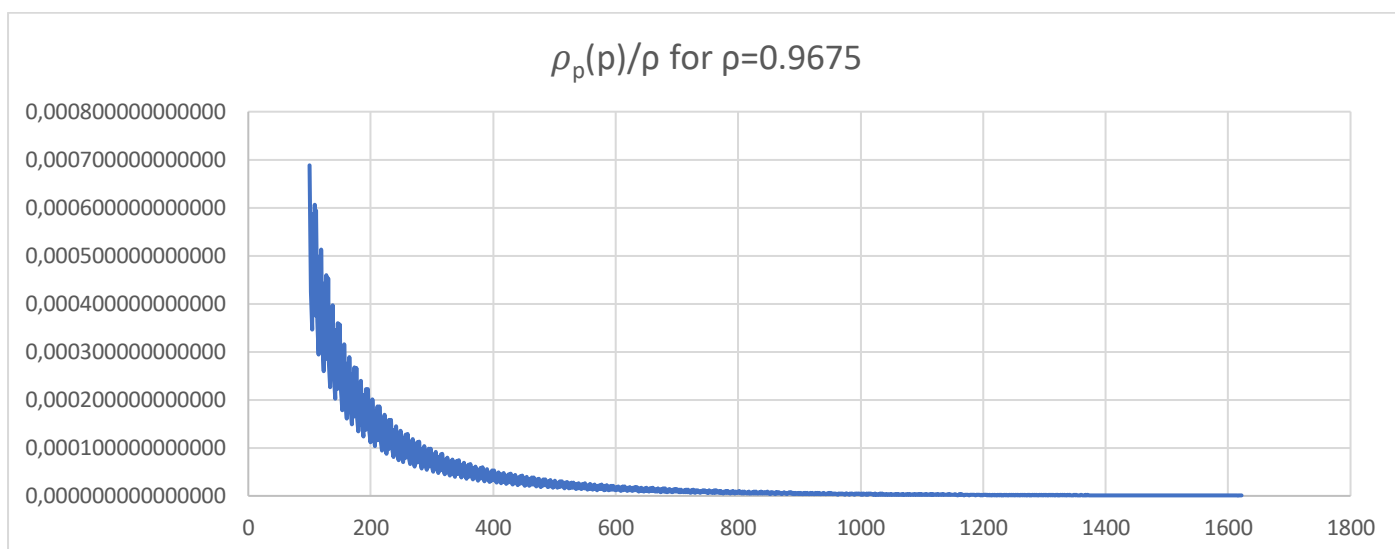
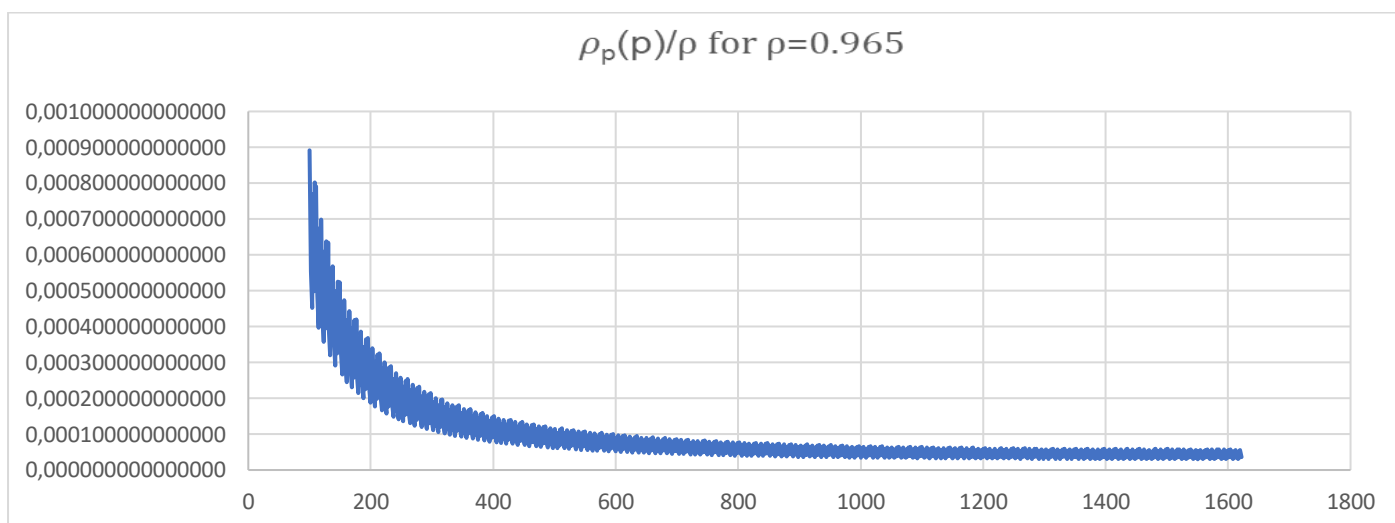
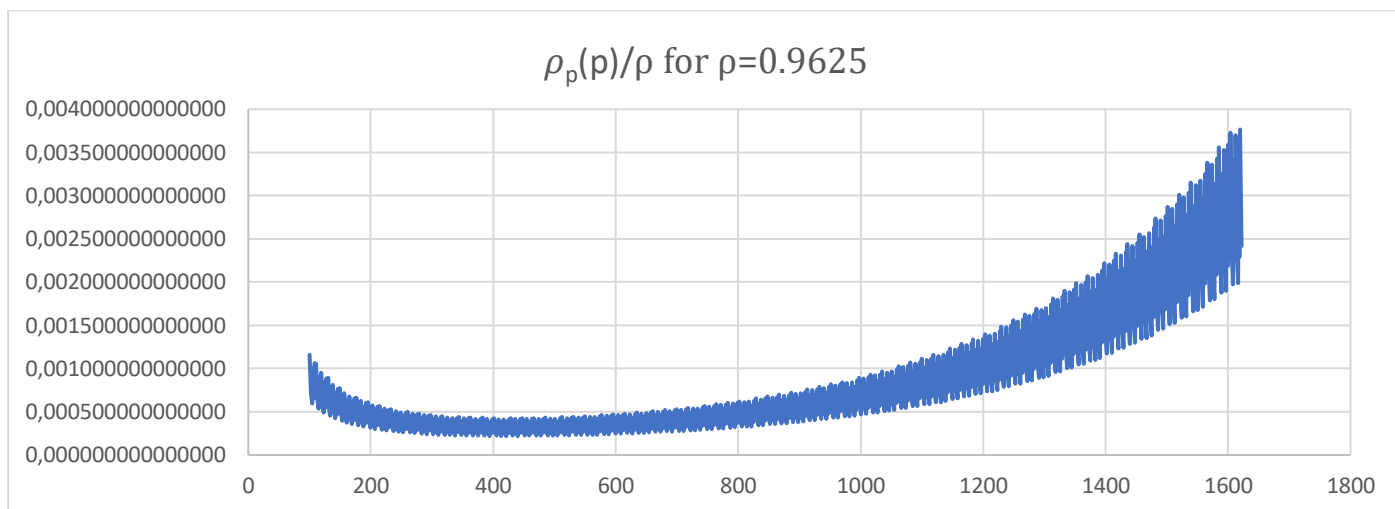
The problem of this estimate of ρ is due to the fact the convergence speed is impacted by the fact that the arithmetic function $V(p)$ reach a stable behavior for $p > 100$ at least. For this reason it seems more reasonable to estimate ρ on a fixed window. To choose this window we can take into account the fact that $V(p)$ present a quasi periodic behavior.

19 and 84 are two quasi periods we can use to estimate ρ . Larger is the window of $V(p)$ we use to estimate ρ , smoother is the curve giving ρ .

We can also estimate $\rho_{84}(p) = \sqrt[84]{\prod_{0 < p-84 < i < p} V(i)}$



This last curve gives an indication that the good candidat for ρ should be between $[0.96;0.97]$. To get a better estimate of ρ , we have displayed the three following representations of $\frac{\rho_p(p)}{\rho}$ for $\rho=0,9625$; $0,965$; $0,9675$. And we see that for $\rho=0,9625$ the curve is growing for p large, for $\rho=0,9675$ the curve converge toward 0 and for $\rho=0,965$ the curve seems to be stable for p large.



We think that the optimal value of ρ is $\rho=0.9649649694$ such as $\rho^{84}=0.05$.

As defined in (13), the percentage of integers $a \in \mathbb{Z}/2^n\mathbb{Z}$ such $\sigma(a)=i$ is given by $\frac{z_i}{2^n}$. Thanks to (10), we are going to estimate an upper value of $\frac{z_n}{2^n}$.

$$\frac{z_n}{2^n} = \frac{1}{2^n} \cdot \binom{n-2}{m(n)} - \frac{1}{2^n} \cdot \sum_{\substack{i \leq n \\ p(i)=i}} \binom{n-i}{m(i)} \cdot z_i < \frac{1}{2^n} \cdot \binom{n-2}{m(n)} = \frac{1}{2^n} \cdot \binom{n-2}{x(n-2)} \quad \text{with} \quad x = \frac{m(n)}{n-2}$$

Thanks to Stirling formula for $0 < x < 1/2$: $\binom{n}{xn} = \frac{n!}{(xn)! \cdot ((1-x)n)!} \simeq \frac{1}{(x^x \cdot (1-x)^{(1-x)})^n \cdot \sqrt{2\pi \cdot x \cdot (1-x)n}} = a \cdot \frac{q^n}{\sqrt{n}}$

With $a = \frac{1}{\sqrt{2\pi \cdot x \cdot (1-x)}}$ and $q = \frac{1}{x^x \cdot (1-x)^{(1-x)}}$ and $x = \frac{m(n)}{n-2} = \frac{n-r(n)}{n-2} \simeq \left(1 - \frac{\ln(2)}{\ln(3)}\right) \cdot \left(1 + \frac{2}{n-2}\right) = 1 - \frac{\ln(2)}{\ln(3)} + O\left(\frac{1}{n}\right)$

For n enough large, we have : $x \simeq 1 - \frac{\ln(2)}{\ln(3)} \simeq 0.369070$; $q=1.9318131$ and $a=0,8267326$

Thanks to this results we can state that: $\frac{z_n}{2^n} < \frac{1}{2^n} \cdot \binom{n-2}{m(n)} \simeq \frac{a \cdot q^{n-2}}{2^n \sqrt{n-2}} = \frac{a}{4 \cdot \sqrt{n-2}} \cdot \left(\frac{q}{2}\right)^{n-2}$ where $\frac{q}{2} < 0,966$ (16)

We are going to look for the number $i=kn$ such as $2^n \cdot \frac{z_i}{2^i} < 0,5$, because if we find i verifying this inequality, due to the fact that $V_{i+1} < 1.8$ and $V_{i+1} \cdot V_{i+2} < 1$, then all $j > i$ will verify the inequality $2^n \cdot \frac{z_j}{2^j} < 1$.

We want to estimate k such as $i=kn$.

Thanks to (16), we can replace $\frac{z_i}{2^i}$ by $\frac{a}{4 \cdot \sqrt{i-2}} \cdot \left(\frac{q}{2}\right)^{i-2}$ and deduce that $i - 2 > \frac{(n-1) \ln(2) + \ln(a)}{\ln\left(\frac{2}{q}\right)}$

For large values of n , $i(n)$ is a growing function with $\lim_{n \rightarrow \infty} \left(\frac{i(n)}{n}\right) \rightarrow \frac{\ln(2)}{\ln\left(\frac{2}{q}\right)} \simeq 19,982248 < 20$ (17)

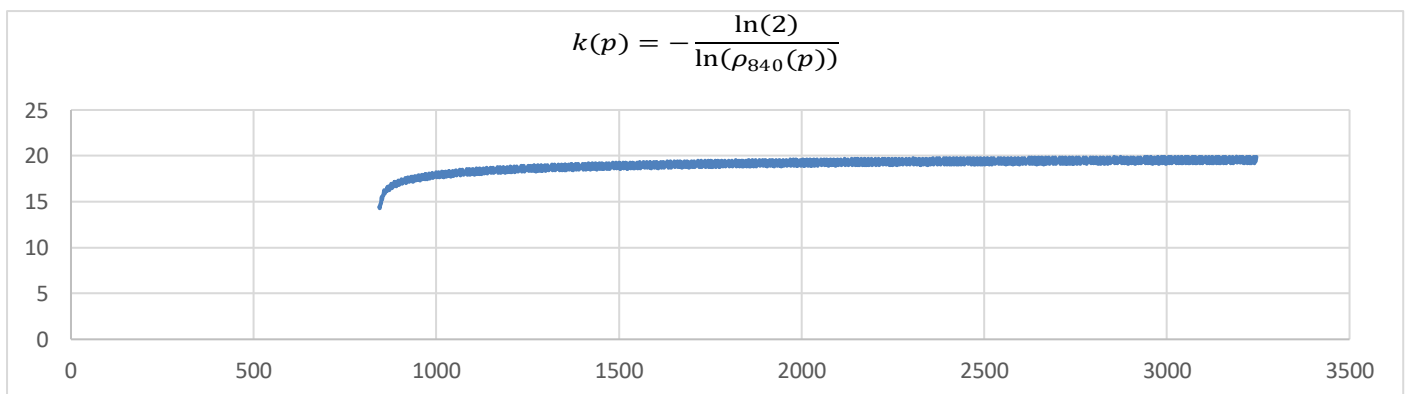
Conjecture of the maximal stopping time :

For n enough large, we can state that for $i > 20n$, $2^n \cdot \frac{z_i}{2^i} < 0.5$ which could induce that no Syracuse sequence could have a stopping time higher that $20n$.

This corroborate what we can see in the graph page 18. To try to confirm the idea that $\text{Max}_{\sigma_n} = \text{Max} \{\sigma(a), a \in \mathbb{Z}/2^n\mathbb{Z}\}$ is lower than $20n$, we have computed for $840 < p < 3245$.

$$\rho_{840}(p) = \sqrt[840]{\prod_{0 < p-840 < i < p} V(i)}$$

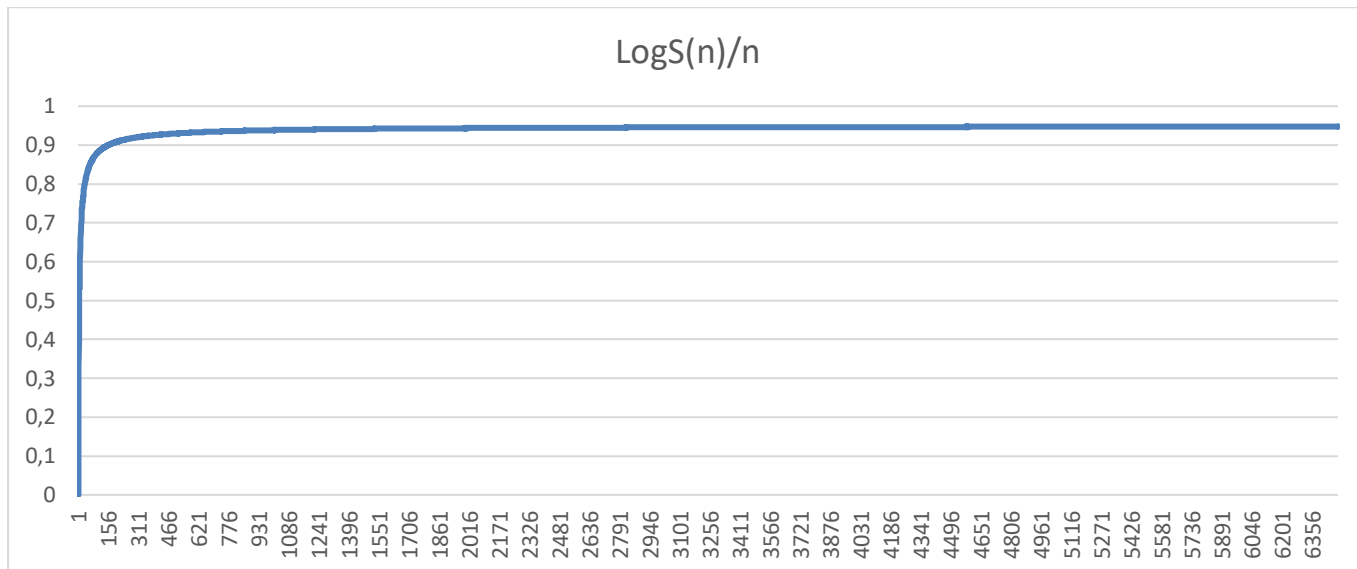
And we are defining the arithmetic function $k(p)$ such as $k(p) = -\frac{\ln(2)}{\ln(\rho_{840}(p))}$



We conjecture that the limit of $k(p)$ when p grows is less than 20.

XI – Behavior of the function $\text{Log}(S(n))/n$

In paragraph II, we have displayed the arithmetic function $\text{Log}_2(S(n))$ in function of n for $n < 41$ and have seen that it was very close to a line. Thanks to the developments presented in the last paragraphs, we are able to compute for any n , the values of z and consequently the values of $S(n)$. The curve above where we have computed $S(n)$ up to $n=6492$, gives an idea of the limit which can be reached by the slope θ of this line.



For $n=6491$, $\theta = \frac{\log_2(S(n))}{n} = 0.9475569882514053$ and $\frac{S(n)}{2^n} = .3659284589631293e - 103$

We can find a link between θ and the value of ρ expressed above.

θ can be define as a real below 1 such as for n large $S(n) \simeq (2^n)^\theta$

We have define ρ as a real below 1 such as z_n could be assimilated as geometric sequence $z_n \simeq (2\rho)^n$

$$\frac{S(n)}{2^n} = \sum_{p(i)>n} \frac{z_{p(i)}}{2^{p(i)}} \simeq \sum_{p(i)>n} \rho^{p(i)} = \frac{\rho^{p(I)}}{1-\rho} \quad \text{where } I = \inf\{i \in \mathbb{N} / p(i) > n\}$$

Thus

$$\theta = \frac{\log_2(S(n))}{n} \simeq \frac{\log_2\left(\frac{2^n \rho^{p(I)}}{1-\rho}\right)}{n} = 1 + \frac{p(I) \cdot \log_2(\rho) - \log_2(1-\rho)}{n}$$

When n is large $p(I) \simeq n+1$ then :

$$\theta \simeq 1 + \frac{(n+1) \cdot \log_2(\rho) - \log_2(1-\rho)}{n} = 1 + \log_2(\rho) + O\left(\frac{1}{n}\right) \quad (18)$$

We have seen above that $\rho \simeq 0,965$ and thanks to (18) we can compute $\theta \simeq 0,9486$ which is upper and coherent with the value of θ computed above for $n=6491$.

Conclusion :

In the present work, we have first computed up to $n=40$, the percentage of integers $a < 2^n$ such as the stopping time $\sigma(a) > n$ and seen that this percentage was decreasing constantly. We have state that the integers p which was corresponding to a stopping time value was satisfying to the equality :

$$p = \left\lfloor \left\lfloor \frac{p}{\log_2 3} \right\rfloor \cdot \log_2 3 \right\rfloor + 1$$

We have computed in detail all the histograms H_n such as $h_n(p) = \#\{a < 2^n / \sigma(a) = p\}$ for $n < 41$ and shown that the behavior of these histograms was strongly deterministic and these histograms for different values of n was linked by an homothetic factor.

$$R_{n,n-k} = \sum_{p > n} \left(\frac{h_n(p)}{h_{n-k}(p)} \right) \cdot \left(\frac{h_n(p)}{S(n)} \right) \simeq 2^k$$

. We have shown that it was possible to define a function $V_n(i)$

$$h_n(p(i)) = h_n(p(i-1)) \cdot V_n(p(i)) \quad \text{where } p(i) = i$$

which link two consecutive values of H_n . We have also shown that $V_n(i)$ was converging to $V(i)$ when n was growing. And thanks to Winker work, proved that $V(i) = V_i(i)$ and that we can give an exact formulation of

$$z_n = h_n(n) = \binom{n-2}{m(n)} - \sum_{\substack{i \leq n \\ p(i)=i}} \binom{n-i}{m(n)-m(i)} \cdot z_i \quad \text{for all } n.$$

Thanks to this formulation , we gave a new formulation of $S(n)$ and have stated that $\frac{S(n)}{2^n}$ tends toward 0 when n tend toward ∞ .

$$\frac{S(n)}{2^n} = \sum_{i > n} \frac{z_i}{2^i} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{S(n)}{2^n} = 0$$

We have stated that for all integers $a < 2^n$ and $n < 51$ that the maximum stopping time is lower than $20n$ and have given an heuristic where we have approximate $h_n(p)/2^n$ by a geometric serie of common ration ρ to deduce that $h_n(p)/2^n$ should be equal to 0 for p enough large. And finally, we have given some argument to conjecture that the max stopping time of Syracuse sequence beginning by $a < 2^n$, should be upper bounded by $20n$.

References

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- 3 – Mike Winker – Facultaät für Mathematik, Ruhr Universität Bochum , Germany – “New results on the stopping time behaviour of the Collatz $3x+1$ function” – arXiv:1504.00212v2 [math.GM]-December 31, 2017
- 4 – David Appelgate & Jeffrey C. Lagarias – “Density bounds for the $3x+1$ problems II. Krasilov Inequalities”- AT&TBell Laboratories, Murray Hill, NJ 07974 – December 15, 1993.
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Annexe 1 : Histograms $H_n(p)$ for $n < 41$ and $p < 201$

n\p	46	48	50	51	53	54	56	58	59	61	62	64	65	67	69	70	72	73	75	77	78	80
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	1	0	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	1	0	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	1	0	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	1	0	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	1	0	2	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1
12	1	1	1	2	0	3	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	2
13	5	2	5	2	1	5	1	2	1	1	0	1	1	0	1	1	1	1	0	1	1	2
14	7	7	5	6	3	8	2	3	1	1	0	1	1	2	3	1	2	2	1	1	3	3
15	14	14	8	13	6	13	4	6	7	3	3	3	5	9	4	5	4	7	2	1	3	3
16	33	19	15	20	14	20	9	14	14	7	10	8	13	13	10	11	10	8	4	1	6	3
17	63	32	23	44	27	45	23	21	30	13	21	16	19	17	15	23	13	14	7	6	8	5
18	137	69	53	88	57	77	42	42	60	27	47	29	36	29	24	39	25	26	14	10	15	10
19	287	151	114	187	121	163	97	73	104	58	88	53	79	50	47	76	39	58	29	23	33	20
20	568	316	221	371	226	321	185	141	195	115	185	99	151	89	80	131	72	105	56	47	73	32
21	1139	613	421	716	435	629	359	246	412	230	385	216	311	180	161	247	153	198	124	98	147	87
22	2269	1179	858	1490	844	1298	754	513	852	502	831	443	662	393	319	483	295	415	244	211	304	172
23	4518	2399	1824	3016	1722	2641	1500	1058	1744	1047	1677	919	1363	807	598	930	620	824	496	371	614	337
24	9070	4990	3735	6000	3453	5278	2999	2164	3576	2084	3195	1734	2672	1562	1167	1856	1161	1695	959	742	1202	714
25	18165	10014	7372	12018	6916	10431	5876	4353	7041	4103	6265	3450	5375	3068	2336	3774	2292	3429	2008	1493	2442	1492
26	36147	19927	14729	23837	13748	20809	11775	8702	14111	8307	12785	7249	10895	6253	4652	7650	4590	6975	3964	2999	4914	2965
27	72390	40009	29520	47508	27606	41705	23550	17559	28644	16838	25718	14623	22104	12597	9377	15181	9001	13889	7859	5985	9731	5837
28	144731	80087	59094	95022	55118	83779	47328	35507	57199	33767	51574	29383	44212	25086	18705	30399	17712	27654	15587	11925	19413	11547
29	290500	160368	118531	190152	110133	168031	94615	71150	114583	67537	103661	58984	88627	50066	37617	60944	35578	55292	31226	23591	38775	23116
30	580650	320774	236789	379313	220963	335803	189499	142334	229426	135553	207678	117912	177881	100094	75249	121456	71402	110374	62717	47138	77707	46254
31	1160969	641708	473266	758013	441800	671797	379932	284647	458628	271190	415674	235738	355272	199920	150115	241963	143519	220408	125873	94473	154842	92717
32	2320432	1282779	946721	1515854	885572	1344375	760504	568701	916243	542068	830132	471887	710575	400534	300241	485356	286713	440361	252391	189713	309990	184979
33	4642380	2566600	1894438	3032305	1771319	2688937	1520179	1135740	1834159	1083578	1658527	944073	1421680	802836	600281	969804	574489	880109	504432	379295	618799	369372
34	9285176	5133732	3788874	6061421	3542976	5374913	3040256	2270521	3667875	2164885	3316071	1887798	2846900	1603998	1199225	1938290	1148778	1760907	1006613	758600	1238581	739216
35	18572007	10267775	7578156	12127047	7085202	10746928	6077154	4540000	7336101	4332797	6632336	3776817	5691783	3208316	2397002	3876645	2296814	3518611	2015886	1519847	2477002	1477972
36	37146809	20540702	15154346	24258231	14167979	21496644	12148639	9081266	14677069	8665385	13266891	7553161	11387600	6416316	4791386	7749568	4589568	7036759	4026023	3038291	4953439	2953663
37	74293444	41082491	30318958	48522008	28339338	43001544	24289101	18166021	29354443	17335601	26521139	15117227	22774141	12822805	9581996	15496000	9178938	14074364	8050750	6077390	9905156	5911211
38	148594796	82165550	60641850	97050993	56679424	86011623	48571843	36328237	58702900	34665799	53041772	30238604	45549311	25640979	19159106	31001505	18357809	28151065	16103256	12161828	19809381	11826154
39	297180981	164342824	121274913	194099223	113355172	172015314	97149928	72642773	117382448	69334930	106082832	60486452	91104291	51271320	38331624	62010235	36715363	56301541	32205306	24327407	39638217	23657820
40	594349827	328656265	242571542	388150146	226708626	344001328	194295650	145273535	234795028	138686978	212193028	120895113	182202624	102542786	76651676	124023711	73429772	112609584	64410182	48656348	79278511	47314998

n\p	81	83	85	86	88	89	91	92	94	96	97	99	100	102	104	105	107	108	110	111	113	115	116	118	119
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	2	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
15	3	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
16	3	2	0	1	1	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1
17	8	2	2	5	3	6	1	3	0	1	2	0	3	0	3	4	0	2	2	1	1	0	2	0	1
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19	30	18	17	33	15	26	13	19	13	10	17	9	19	6	3	13	4	13	6	4	3	1	4	2	2
20	70	39	33	53	27	37	22	43	27	19	29	19	30	14	7	23	8	20	12	9	7	4	10	4	5
21	145	74	51	96	56	73	47	80	54	32	58	32	57	28	20	33	16	35	22	25	12	12	24	9	9
22	289	152	102	188	118	156	91	181	96	79	117	68	112	54	35	76	31	73	30	64	31	18	39	18	32
23	568	313	229	390	236	346	227	343	189	145	231	133	223	109	85	154	92	133	75	119	56	43	66	41	66
24	1125	627	497	782	473	734	443	665	357	287	465	287	419	227	179	310	179	269	151	243	118	87	153	86	158
25	2260	1251	981	1568	959	1441	853	134																	

n\p	121	123	124	126	127	129	130	132	134	135	137	138	140	142	143	145	146	148	149	151	153	154	156	157	159
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	1	1	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	1	1	2	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	1	1	3	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	4	1	3	1	2	1	1	1	0	2	1	1	0	0	0	0	2	0	1	0	1	1	0	0	0
20	7	4	6	5	4	2	2	3	0	6	1	3	2	0	1	2	4	0	2	2	2	2	0	1	1
21	12	7	14	7	6	7	10	7	1	12	2	7	4	1	5	5	6	0	4	4	3	5	0	2	1
22	20	10	21	14	13	17	21	12	9	20	6	18	10	8	11	12	15	6	10	7	7	8	2	5	3
23	39	37	54	29	39	32	49	28	24	39	22	42	27	13	28	24	31	16	17	14	11	19	9	14	6
24	93	78	110	66	98	78	110	57	55	73	52	88	57	29	54	36	61	30	40	29	27	33	20	28	12
25	186	160	232	149	233	147	211	127	90	142	108	172	107	64	98	74	115	66	92	55	43	66	38	67	36
26	367	287	486	305	461	282	442	232	177	299	216	340	206	145	197	147	215	127	180	105	86	139	79	126	75
27	757	591	959	583	929	541	884	471	357	630	424	654	353	290	403	266	406	244	358	198	150	277	153	248	138
28	1531	1172	1961	1177	1840	1050	1744	960	680	1259	787	1217	657	551	807	514	779	474	700	383	313	549	287	471	264
29	3147	2365	3939	2396	3690	2085	3432	1965	1419	2445	1486	2359	1322	1059	1640	991	1558	955	1375	773	623	1044	588	935	538
30	6306	4757	7829	4806	7487	4285	6748	3822	2862	4864	2929	4650	2628	2084	3317	1958	3134	1865	2791	1567	1223	2065	1171	1846	1059
31	12654	9629	15817	9638	14923	8669	13308	7726	5825	9712	5802	9099	5229	4076	6648	3917	6222	3641	5546	3184	2362	4013	2384	3705	2143
32	25266	19068	31613	19110	29884	17290	26564	15423	11737	19268	11596	18100	10420	8058	13296	7880	12394	7240	11045	6337	4773	8097	4840	7421	4387
33	50293	38142	63073	38051	59831	34733	53186	30759	23367	38530	23083	35828	20792	16031	26288	15837	24814	14529	22267	12744	9672	16126	9666	14958	8728
34	100560	76686	126480	76509	119512	69541	106242	61141	46257	76616	45927	71325	41552	31883	52545	31782	49603	28994	44356	25528	19446	32039	19415	29955	17458
35	201224	154010	253029	153053	238553	138972	212112	121529	92538	152535	91611	142783	83097	63842	105139	63514	99005	57874	88954	50798	38693	64113	38747	60241	35184
36	402305	308214	506366	305796	475396	277174	424003	243782	184895	304223	182691	284984	166253	122720	209865	126844	198044	116266	177934	101814	78096	128458	77932	120937	70681
37	804049	615786	1011665	610846	949839	554007	847960	487644	370083	608605	366103	569398	332212	254272	419356	254077	396712	232277	356301	204576	156340	257197	155554	242098	141196
38	1607865	1229271	2019185	1220361	1897640	1108411	1695798	977190	741119	1218883	733383	1139921	664713	508482	839283	508143	794526	464188	713106	410534	312751	514283	311507	483642	282518
39	3213359	2453381	4036311	2438996	3797947	2213874	3395032	1954133	1484443	2438083	1468085	2282496	1330941	1017253	1679948	1016703	1588451	928064	1427094	822044	625524	1029760	624194	969342	566218
40	6422049	4907107	8077605	4880493	7603263	4431069	6792135	3906149	2969716	4875589	2938690	4567046	2661171	2036445	3359081	2034378	3176859	1857741	2853086	1645653	1253621	2061197	1247875	1939437	1132799

n\p	161	162	164	165	167	169	170	172	173	175	176	178	180	181	183	184	186	188	189	191	192	194	195	197	199	200	201
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	1	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	4	1	2	1	0	1	0	2	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
21	1	5	2	2	1	1	2	0	4	1	2	2	1	1	2	2	1	1	0	0	0	0	0	0	0	0	4
22	2	5	3	6	3	3	4	0	4	4	5	5	3	2	3	4	1	3	0	0	0	1	0	0	1	7	
23	6	7	6	12	6	4	6	2	6	7	9	8	5	6	5	7	2	4	1	2	2	1	1	0	0	2	14
24	12	18	16	21	10	8	12	2	15	13	16	14	8														

Annexe 2 : $\#\{a < 2^n / \sigma(a) \leq n\}$; $\#\{a < 2^n / \sigma(a) > n\}$ for n up to 405

$\sigma_r = p(n)$	$V(p(n))$	$IV(p(i)) i < n$	$(IV(p(i)))^{1/n} i < n$	$\%a < 2^n / \sigma(a) \leq n$	$\%a < 2^n / \sigma(a) > n$	$\sigma_r = p(n)$	$V(p(n))$	$IV(p(i)) i < n$	$(IV(p(i)))^{1/n} i < n$	$\%a < 2^n / \sigma(a) \leq n$	$\%a < 2^n / \sigma(a) > n$
1	0,50000000	0,500000000000		0,500000000000	0,500000000000	102	0,57818710	0,000014619782	0,96597368408	0,999775941499	0,000224058501
2	0,50000000	0,250000000000	0,500000000000	0,750000000000	0,250000000000	104	0,76168573	0,000011335680	0,896132599135	0,999787077178	0,000212922822
4	0,25000000	0,062500000000	0,500000000000	0,812500000000	0,187500000000	105	1,64139002	0,000018277993	0,901312739579	0,999805355171	0,000194644829
5	1,00000000	0,062500000000	0,574349177499	0,875000000000	0,125000000000	107	0,60202889	0,000011003880	0,898792225117	0,999816359051	0,0000183640949
7	0,37500000	0,023437500000	0,584965406379	0,898437500000	0,101562500000	108	1,55347936	0,000017094300	0,903357666358	0,999833453352	0,000166546648
8	1,16666667	0,027343750000	0,637686553429	0,925781250000	0,074218750000	110	0,58087634	0,000009929675	0,900570239568	0,999843383026	0,000156616974
10	0,42857143	0,011718750000	0,641044426993	0,937500000000	0,062500000000	111	1,52849391	0,000015177447	0,904872471260	0,999858560474	0,000141439526
12	0,62500000	0,007324218750	0,663840037823	0,944824218750	0,051575812500	113	0,57290848	0,000008695288	0,902017373578	0,999867255762	0,000132744238
13	1,41666667	0,010375976563	0,703698855641	0,955200195313	0,044799804688	115	0,75840954	0,000006594590	0,901466223095	0,999873850351	0,000126149649
15	0,50882353	0,005279541016	0,704973963827	0,960479736328	0,039520263672	116	1,63745305	0,000010798331	0,906116646117	0,999884648682	0,000115351318
16	1,37572254	0,007263183594	0,735056020232	0,967742919922	0,032257080078	118	0,60081108	0,000006487757	0,903721677219	0,999891136439	0,000108863561
18	0,50472689	0,003665924072	0,732279471126	0,971408843994	0,028591156006	119	1,55194996	0,000010068674	0,907837609157	0,999901205113	0,000098794887
20	0,68990635	0,002529144287	0,741564071638	0,973937988281	0,026062011719	121	0,58046425	0,000005844505	0,905211330585	0,999907049618	0,000092950382
21	1,51677979	0,003836154938	0,767268617467	0,977774143219	0,022225856781	123	0,76408835	0,000004465718	0,904697065263	0,999911515336	0,000088484664
23	0,54807334	0,002102494240	0,764886799444	0,979876637459	0,020123362541	124	1,64586268	0,000007349959	0,909073654773	0,999918865295	0,000081134705
24	1,44675965	0,003041803837	0,785471563409	0,982918441296	0,017081558704	126	0,60408928	0,000004440031	0,906815504367	0,999923305327	0,000076694673
26	0,53372328	0,001623481512	0,781105171840	0,984541922808	0,015458077192	127	1,55779849	0,000006916674	0,910687279356	0,999930222001	0,000069777999
27	1,43393759	0,002327971160	0,798878349956	0,986869893968	0,013130106032	129	0,58288921	0,000004031655	0,908201137413	0,999934253656	0,000065746344
29	0,53090445	0,001235930249	0,7948823460313	0,988105824217	0,011894175783	130	1,53273705	0,000006179469	0,911864679150	0,999940433126	0,000059566874
31	0,71603985	0,000884975307	0,797102017144	0,988990799524	0,011009200476	132	0,57488374	0,000003552476	0,909318748509	0,999943985602	0,000056014398
32	1,56189600	0,001382239396	0,814035279292	0,990373038920	0,009626961080	134	0,76049143	0,000002701628	0,908751135840	0,999946687230	0,000053312770
34	0,56733359	0,000784190837	0,810325421207	0,991157229757	0,008842770243	135	1,64132239	0,000004434242	0,912739426421	0,999951121472	0,000048878528
35	1,48430138	0,001163975539	0,824460444978	0,992321205296	0,007678794704	137	0,60259148	0,000002672037	0,910584005521	0,999953793509	0,000046206491
37	0,55000218	0,000640189079	0,819754873112	0,992961394375	0,007038605625	138	1,55567997	0,000004156834	0,914124874114	0,999957950343	0,000042049657
39	0,73300245	0,000469260161	0,821583356793	0,993430654536	0,006569345464	140	0,58220186	0,000002420116	0,911768417493	0,999960370459	0,000039629541
40	1,58973852	0,000746000955	0,835253846999	0,994176655491	0,005823344509	142	0,76591993	0,000001853615	0,911242445812	0,999962224075	0,00003775925
42	0,57894100	0,000431890539	0,831552837382	0,994608546030	0,005391453970	143	1,64927295	0,000003057118	0,915030877344	0,999965281192	0,000034718808
43	1,50647720	0,000650633250	0,843124172513	0,995259179280	0,004740820720	145	0,60566146	0,000001851578	0,912898551945	0,999967132771	0,000032867229
45	0,55955832	0,000364067248	0,838652051323	0,995623246528	0,004376753472	146	1,56109636	0,000002890492	0,916350140458	0,999970023263	0,000029976737
46	1,48471506	0,000540536127	0,849130519399	0,996163782655	0,003836217345	148	0,58442804	0,000001689285	0,914109015524	0,999971712548	0,000028287452
48	0,55302824	0,000298931742	0,844450928103	0,996462714397	0,003537285603	149	1,53598563	0,000002594717	0,912798474340	0,999974307265	0,000025692735
50	0,73794947	0,000220596521	0,845029699862	0,996683310917	0,003316689083	151	0,57639789	0,000001495589	0,915003169899	0,999975802854	0,000024197146
51	1,60027461	0,000353015011	0,855676628510	0,997036325928	0,002963674072	153	0,76182896	0,00000139773	0,91440971057	0,999976942628	0,000023053772
53	0,58405699	0,000206180887	0,852035160682	0,997242506815	0,002757493185	154	1,64429826	0,000001874127	0,917931732389	0,999978816755	0,000021183245
54	1,51755653	0,000312891151	0,861191779435	0,997555397967	0,002444602033	156	0,60396344	0,000001131904	0,915974525482	0,999979948659	0,000020051341
56	0,56474748	0,000176704489	0,857011330965	0,997732102456	0,002267897544	157	1,55855962	0,000001764140	0,919080843470	0,999981712799	0,000018287201
58	0,74782866	0,000132144682	0,857277738722	0,997864247138	0,002135752862	159	0,58354606	0,000001029457	0,916945341084	0,999982742256	0,000017257744
59	1,61612410	0,000213562205	0,866539868885	0,998077809343	0,001922190657	161	0,76733958	0,000000789943	0,916424874565	0,999983532100	0,000016467800
61	0,59060555	0,000126131023	0,863135636561	0,998203940366	0,001796059634	162	1,65192182	0,000001304924	0,919764069099	0,999984837124	0,000015162876
62	1,52997249	0,000192976996	0,871141691784	0,998396917361	0,001603082596	164	0,60688520	0,000000791939	0,917903238003	0,999985629063	0,000014370937
64	0,57010555	0,000110017255	0,867257022345	0,998506934617	0,001493065383	165	1,56366857	0,000001238331	0,920871452440	0,999986867394	0,000013132606
65	1,50616232	0,000165703844	0,874653192998	0,998672638461	0,001327361539	167	0,58563081	0,000000725205	0,918832378151	0,999987592599	0,000012407401
67	0,56267404	0,000093237252	0,870652033896	0,998765875713	0,001234124287	169	0,76926472	0,000000557874	0,918326795300	0,999988150473	0,000011849527
69	0,74778557	0,000069721472	0,870480277007	0,998835597185	0,001164402815	170	1,65517262	0,000000923378	0,921514627298	0,999989073851	0,000010926149
70	1,61798667	0,000112808413	0,878223144909	0,998948405598	0,001051594402	172	0,60827316	0,000000561666	0,919728739753	0,999989635517	0,000010364483
72	0,59197076	0,000066779282	0,875001732837	0,999015184880	0,000984815120	173	1,56635888	0,000000879771	0,922563686082	0,999990515288	0,000009484712
73	1,53366840	0,000102417274	0,881754283728	0,999117602154	0,000882397846	175	0,58681250	0,000000516261	0,920605439709	0,999991031549	0,000008968451
75	0,57205114	0,000058587918	0,878154182663	0,999176190072	0,000823809928	176	1,54087232	0,000000795492	0,923303581865	0,999991827040	0,000008172960
77	0,75532639	0,000044253001	0,877917692992	0,999220443072	0,000779556928	178	0,57862753	0,000000460293	0,921295757382	0,999992287334	0,000007712666
78	1,62973344	0,000072120595	0,884908152458	0,999292563668	0,000707436332	180	0,76439219	0,000000351845	0,920759893439	0,999992639170	0,000007360827
80	0,59673196	0,000043036664	0,881907391346	0,999335600331	0,000664399669	181	1,64851767	0,000000580022	0,923727541122	0,999993219200	0,000006780800
81	1,54252739	0,000066385233	0,888015682533	0,999401985564	0,000598014436	183	0,60588465	0,000000351426	0,92200880710	0,999993570627	0,000006429373
83	0,57583436	0,000038226898	0,884658247768	0,999440212462	0,000559787538	184	1,56253955	0,000000549118	0,924648017560	0,999994119745	0,000005880255
85	0,75899683	0,000029014095	0,884339308601	0,999469226557	0,000530773443	186	0,58538576	0,000000321446	0,922766833645	0,999994441190	0,000005558810
86	1,63614958	0,000047471399	0,890688723176	0,999516697956	0,000483302044	188	0,76926235	0,000000247276	0,922268449302	0,999994688466	0,000005311534
88	0,59954303	0,000028461146	0,887858541527	0,999545159102	0,000454840898	189	1,65548196	0,000000409361	0,925127566622	0,999995097827	0,000004902173
89	1,54813362	0,000044061657	0,893422454078	0,999589220759	0,000410779241	191	0,60852001	0,000000249104	0,923476994491	0,999995346932	0,000004653068
91	0,57834506	0,000025482842	0,890264810297	0,999614703601	0,000385296399	192	1,56708198	0,000000390367	0,926024028719	0,999995737299	0,000004262701
92	1,52308041	0,000038812417	0,895476162112	0,999653516019	0,000346483981	194	0,58721904	0,000000229231	0,924218387284	0,999995966530	0,000004033470
94	0,57036691	0,000022137319	0,892236639253	0,999675653337							

$\sigma_i=p(n)$	$V(p(n))$	$PV(p(i)) i < n$	$(PV(p(i)))^{1/n} i < n$	$\%a < 2^n \sigma(a) \leq n$	$\%a < 2^n \sigma(a) > n$	$\sigma_i=p(n)$	$V(p(n))$	$PV(p(i)) i < n$	$(PV(p(i)))^{1/n} i < n$	$\%a < 2^n \sigma(a) \leq n$	$\%a < 2^n \sigma(a) > n$
203	1,56378061	0,000000244979	0,92756778919	0,999997341214	0,000002658786	305	0,60903488	0,000000002359	0,936944144774	0,99999953904	0,000000046096
205	0,58598542	0,000000143554	0,926018349681	0,999997484769	0,000002515231	306	1,56925844	0,000000003701	0,938524609230	0,99999957606	0,000000042394
207	0,76991737	0,000000110525	0,925536455174	0,999997595294	0,000002404706	308	0,58856624	0,000000002179	0,937296879959	0,99999959784	0,000000040216
208	1,65673691	0,000000183111	0,928130837801	0,999997778405	0,000002221595	310	0,77266798	0,000000001683	0,936908751851	0,99999961468	0,000000038532
210	0,60911259	0,000000111535	0,926600300723	0,999997889940	0,000002110060	311	1,66191508	0,000000002798	0,938636967191	0,99999964265	0,000000035735
211	1,56835530	0,000000174927	0,928914242543	0,999998064866	0,000001935134	313	0,61152522	0,000000001711	0,937542590873	0,99999965976	0,000000034024
213	0,58782534	0,000000102826	0,927241751893	0,999998167693	0,000001832307	314	1,57346930	0,000000002692	0,939089844143	0,99999968668	0,000000031332
214	1,54317607	0,000000158679	0,929451491846	0,999998326372	0,000001673628	316	0,59023636	0,000000001589	0,937897292308	0,99999970257	0,000000029743
216	0,57976029	0,000000091996	0,927736949571	0,999998418368	0,000001581632	317	1,54831671	0,000000002460	0,939381595361	0,99999972716	0,000000027284
218	0,76564875	0,000000070437	0,927239094128	0,999998488804	0,000001511196	319	0,58218109	0,000000001432	0,938157643777	0,99999974149	0,000000025851
219	1,65094593	0,000000116287	0,929684857596	0,999998605091	0,000001394909	321	0,76822923	0,000000001100	0,937760274845	0,99999975249	0,000000024751
221	0,60703781	0,000000070591	0,928199622323	0,999998675682	0,000001324318	322	1,65579834	0,000000001822	0,939417508189	0,99999977071	0,000000022929
222	1,56503157	0,000000110476	0,930386450657	0,999998786158	0,000001213842	324	0,60929624	0,000000001110	0,938343996249	0,99999978181	0,000000021819
224	0,58657655	0,000000064803	0,928771549844	0,999998850961	0,000001149039	325	1,56981642	0,000000001742	0,939589249717	0,99999979293	0,000000020077
226	0,77054946	0,000000049934	0,928307828629	0,999998900895	0,000001099105	327	0,58883088	0,000000001026	0,938666183054	0,99999980949	0,000000019051
227	1,65792956	0,000000082787	0,930682591213	0,999998983682	0,000001016318	329	0,77295201	0,000000000793	0,938292652325	0,99999981742	0,000000018258
229	0,60966922	0,000000050473	0,929246510594	0,999999034154	0,000000965846	330	1,66245313	0,000000001318	0,939920401521	0,99999983646	0,000000016939
230	1,56953747	0,000000079219	0,931376626031	0,999999113373	0,000000886627	332	0,61177735	0,000000000807	0,938880647373	0,99999983867	0,000000016133
232	0,58838341	0,000000046611	0,929819505573	0,999999159984	0,000000840016	333	1,57400696	0,000000001270	0,940338569858	0,99999985137	0,000000014863
233	1,54436778	0,000000071984	0,931846481160	0,999999231968	0,000000768032	335	0,59049123	0,000000000750	0,939205878183	0,99999985886	0,000000014114
235	0,586032201	0,000000041774	0,932049841958	0,999999273740	0,000000726258	337	0,77443162	0,000000000581	0,938843128386	0,99999986467	0,000000013533
237	0,76624823	0,000000032009	0,929772495340	0,999999305752	0,000000694248	338	1,66487961	0,000000000967	0,940435676931	0,99999987433	0,000000012567
238	1,65207393	0,000000052882	0,932020910079	0,999999358633	0,000000641367	340	0,61278760	0,000000000592	0,939421348052	0,99999988026	0,000000011974
240	0,60756300	0,000000032129	0,930633623270	0,999999390762	0,000000609238	341	1,57590784	0,000000000933	0,940847602615	0,99999988959	0,000000011041
241	1,56614486	0,000000050319	0,932645758203	0,999999441081	0,000000558919	343	0,59130626	0,000000000552	0,939741528528	0,99999989511	0,000000010489
243	0,58710117	0,000000029542	0,931138246928	0,999999470623	0,000000529377	344	1,55043434	0,000000000856	0,941110300501	0,99999990367	0,000000009633
245	0,77110878	0,000000022780	0,930692810352	0,999999493404	0,000000506596	346	0,58312259	0,000000000499	0,939974125681	0,99999990866	0,000000009134
246	1,65898268	0,000000037792	0,932882247106	0,999999531196	0,000000468804	348	0,76917464	0,000000000384	0,939599746504	0,99999991260	0,000000008750
248	0,61015993	0,000000023059	0,931547520189	0,999999554255	0,000000445745	349	1,65749580	0,000000000636	0,941129143057	0,99999991886	0,000000008114
249	1,57057786	0,000000036216	0,933503769859	0,999999590471	0,000000409529	351	0,61005699	0,000000000388	0,940129950487	0,99999992274	0,000000007726
251	0,58887395	0,000000021327	0,932047292903	0,999999611798	0,000000388202	352	1,57136488	0,000000000610	0,941502905459	0,99999992884	0,000000007116
253	0,77270695	0,000000016479	0,931615954423	0,999999628277	0,000000371723	354	0,58953760	0,000000000360	0,940418777589	0,99999993243	0,000000006757
254	1,66162935	0,000000027383	0,933740672807	0,999999655660	0,000000344340	356	0,77368021	0,000000000278	0,940065561710	0,99999993522	0,000000006478
256	0,61127145	0,000000016738	0,932446366284	0,999999672398	0,000000327602	357	1,66378823	0,000000000463	0,941570086490	0,99999993984	0,000000006016
257	1,57269129	0,000000026324	0,934344870511	0,999999698722	0,000000301278	359	0,61238595	0,000000000283	0,940600219498	0,99999994268	0,000000005732
259	0,58978809	0,000000015526	0,932931174671	0,999999714247	0,000000285753	360	1,57526682	0,000000000446	0,941948494764	0,99999994714	0,000000005286
260	1,54719452	0,000000024021	0,934748092550	0,999999738268	0,000000261732	362	0,59107411	0,000000000264	0,940892115966	0,99999994978	0,000000005022
262	0,58159517	0,000000013971	0,933297062914	0,999999752239	0,000000247761	363	1,55008091	0,000000000409	0,942187015185	0,99999995387	0,000000004613
264	0,76754495	0,000000010723	0,932849983704	0,999999762962	0,000000237038	365	0,58300407	0,000000000238	0,941102297090	0,99999995626	0,000000004374
265	1,65442772	0,000000017740	0,934869114263	0,999999780703	0,000000219297	367	0,76909756	0,000000000183	0,940740463896	0,99999995809	0,000000004191
267	0,60862730	0,000000010797	0,933602991205	0,999999791500	0,000000208500	368	1,65741865	0,000000000304	0,942189371328	0,99999996113	0,000000003887
268	1,56833261	0,000000016934	0,935411740178	0,999999808434	0,000000191566	370	0,61004574	0,000000000185	0,942346196262	0,99999996298	0,000000003702
270	0,58810728	0,000000009959	0,934036280425	0,999999818393	0,000000181607	371	1,57139347	0,000000000291	0,942535806242	0,99999996590	0,000000003410
272	0,77215423	0,000000007690	0,933617119381	0,999999826082	0,000000173918	373	0,58957004	0,000000000172	0,941500346600	0,99999996762	0,000000003238
273	1,66091164	0,000000012772	0,935589218609	0,999999838854	0,000000161146	375	0,77373518	0,000000000133	0,941159046870	0,99999996895	0,000000003105
275	0,61104374	0,000000007804	0,934367187982	0,999999846659	0,000000153341	376	1,66392140	0,000000000221	0,942586435298	0,99999997116	0,000000002884
276	1,57241800	0,000000012272	0,936130946386	0,999999858930	0,000000141070	378	0,61245929	0,000000000135	0,941659223294	0,99999997251	0,000000002749
278	0,58972899	0,000000007237	0,934798103626	0,999999866167	0,000000133833	379	1,57544709	0,000000000213	0,942938787896	0,99999997465	0,000000002535
279	1,54720895	0,000000011197	0,936487885593	0,999999877364	0,000000122636	381	0,59116846	0,000000000126	0,941929205095	0,99999997591	0,000000002409
281	0,58164948	0,000000006513	0,935120313424	0,999999883877	0,000000116123	382	1,55030275	0,000000000196	0,943158645884	0,99999997786	0,000000002214
283	0,76765153	0,000000005000	0,934689995340	0,999999888877	0,000000111123	384	0,58311635	0,000000000114	0,942121920916	0,99999997901	0,000000002099
284	1,65469504	0,000000008273	0,936571657305	0,999999897149	0,000000102851	386	0,76922592	0,000000000088	0,941772645401	0,99999997988	0,000000002012
286	0,60877599	0,000000005036	0,935376338764	0,999999902186	0,000000097814	387	1,65767318	0,000000000145	0,943149579153	0,99999998134	0,000000001866
287	1,56869907	0,000000007900	0,937063011416	0,999999910086	0,000000089914	389	0,61016940	0,000000000089	0,942236068967	0,99999998222	0,000000001778
289	0,58829847	0,000000004648	0,935765279801	0,999999914734	0,000000085266	390	1,57166709	0,000000000139	0,943472988360	0,99999998362	0,000000001638
291	0,77237787	0,000000003590	0,935361797387	0,999999918323	0,000000081677	392	0,58970351	0,000000000082	0,942482476196	0,99999998444	0,000000001556
292	1,66136148	0,000000005964	0,937203770629	0,999999924287	0,000000075713	394	0,77388256	0,000000000064	0,942152762500	0,99999998508	0,000000001492
294	0,61126426	0,000000003646	0,936048863358	0,999999927933	0,000000072067	395	1,66420692	0,000000000106	0,943510765899	0,99999998614	0,000000001386
295	1,57290936	0,000000005734	0,937697170640	0,999999933667	0,000000066333	397	0,61259561	0,000000000065	0,942622916643	0,99999998679	0,000000001321
297	0,58996963	0,000000003383	0,936438194328	0,999999937050	0,000000062950	398	1,57574345	0,000000000102	0,943840624301	0,999999	