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# NEGATIVE RESULTS IN COCONVEX APPROXIMATION OF PERIODIC FUNCTIONS

GERMAN DZYUBENKO, VICTORIA VOLOSHINA,  
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ABSTRACT. We prove, that for each  $r \in \mathbb{N}$ ,  $n \in \mathbb{N}$  and  $s \in \mathbb{N}$  there are a collection  $\{y_i\}_{i=1}^{2s}$  of points  $y_{2s} < y_{2s-1} < \dots < y_1 < y_{2s} + 2\pi =: y_0$  and a  $2\pi$ -periodic function  $f \in C^{(\infty)}(\mathbb{R})$ , such that

$$(1) \quad f''(t) \prod_{i=1}^{2s} (t - y_i) \geq 0, \quad t \in [y_{2s}, y_0],$$

and for each trigonometric polynomial  $T_n$  of degree  $\leq n$  (of order  $\leq 2n + 1$ ), satisfying

$$(2) \quad T_n''(t) \prod_{i=1}^{2s} (t - y_i) \geq 0, \quad t \in [y_{2s}, y_0],$$

the inequality

$$n^{r-1} \|f - T_n\|_{C(\mathbb{R})} \geq c_r \|f^{(r)}\|_{C(\mathbb{R})}$$

holds, where  $c_r > 0$  is a constant, depending only on  $r$ . Moreover, we prove, that for each  $r = 0, 1, 2$  and any such collection  $\{y_i\}_{i=1}^{2s}$  there is a  $2\pi$ -periodic function  $f \in C^{(r)}(\mathbb{R})$ , such that  $(-1)^{i-1}f$  is convex on  $[y_i, y_{i-1}]$ ,  $1 \leq i \leq 2s$ , and, for each sequence  $\{T_n\}_{n=0}^{\infty}$  of trigonometric polynomials  $T_n$ , satisfying (2), we have

$$\limsup_{n \rightarrow \infty} \frac{n^r \|f - T_n\|_{C(\mathbb{R})}}{\omega_4(f^{(r)}, 1/n)} = +\infty,$$

where  $\omega_4$  is the fourth modulus of continuity.

## 1. INTRODUCTION AND THE MAIN RESULTS

Let  $s \in \mathbb{N}$  and  $\mathbb{Y}_s := \{Y_s\}$ , where the collections  $Y_s = \{y_i\}_{i=1}^{2s}$  of points  $y_i \in \mathbb{R}$  are such that  $y_{2s} < \dots < y_1 < y_{2s} + 2\pi =: y_0$ . We say that a  $2\pi$ -periodic function  $f \in C(\mathbb{R})$  is piecewise convex with respect to  $Y_s$ , if it is a convex function on  $[y_1, y_0]$  and changes its convexity at the points  $Y_s$ , that is, if  $(-1)^{i-1}f$  is convex on  $[y_i, y_{i-1}]$ ,  $1 \leq i \leq 2s$ . We denote by  $\Delta^{(2)}(Y_s)$  the collection of all such piecewise convex functions. Note that if, in addition,  $f \in C^{(2)}(\mathbb{R})$ , then  $f \in \Delta^{(2)}(Y_s)$ , if and only if,

$$f^{(2)}(t) \prod_{i=1}^{2s} (t - y_i) \geq 0, \quad t \in [y_{2s}, y_0].$$

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Denote by  $C^{(r)}$ ,  $r \in \mathbb{N}$ , the space of  $2\pi$ -periodic functions  $f \in C^{(r)}(\mathbb{R})$ . We also need the notation  $W^r$ ,  $r \in \mathbb{N}$ , for the Sobolev space of  $2\pi$ -periodic functions  $f \in AC^{(r-1)}(\mathbb{R})$ , such that

$$\|f^{(r)}\| < +\infty,$$

where

$$\|g\| := \operatorname{esssup}_{x \in \mathbb{R}} |g(x)|.$$

If, in addition,  $g$  is continuous, then, of course,

$$\|g\| = \sup_{x \in \mathbb{R}} |g(x)|.$$

Let  $\mathbb{T}_n$  be the space of trigonometric polynomials of degree  $\leq n$  (of order  $2n+1$ ) and, for the function  $g \in \Delta^{(2)}(Y_s)$ , denote by

$$E_n^{(2)}(g, Y_s) := \inf_{T_n \in \mathbb{T}_n \cap \Delta^{(2)}(Y_s)} \|g - T_n\|,$$

the error of the best coconvex approximation of the function  $g$ . It is known [10] that if  $f \in \Delta^{(2)}$ , then

$$(1.1) \quad E_n^{(2)}(f, Y_s) \leq c(s)\omega_3(f, 1/n), \quad n \geq N,$$

where  $c(s)$  is a constant, depending only on  $s$ ,  $N$  is a number, depending only on  $Y_s$ , and

$$\omega_k(f, t) = \sup_{h \in [0, t]} \left\| \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f(\cdot + ih) \right\|, \quad t \geq 0,$$

is the modulus of continuity of a function  $f$  of order  $k \in \mathbb{N}$ . For each  $f \in \Delta^{(2)}(Y_s)$  (1.1) implies

$$(1.2) \quad E_n^{(2)}(f, Y_s) \leq \frac{c(s)}{n} \omega_2(f', 1/n), \quad n \geq N, \quad \text{if } f \in C^{(1)},$$

$$(1.3) \quad E_n^{(2)}(f, Y_s) \leq \frac{c(s)}{n^2} \omega_1(f'', 1/n), \quad n \geq N, \quad \text{if } f \in C^{(2)},$$

and

$$(1.4) \quad E_n^{(2)}(f, Y_s) \leq \frac{c(s, r)}{n^r} \|f^{(r)}\|, \quad n \geq N, \quad \text{if } f \in W^r,$$

$1 \leq r \leq 3$ , where  $N$  is a number, depending only on  $Y_s$ . Leviatan, Motorna and Shevchuk [5] conjectured, that (1.4) holds for all  $r \in \mathbb{N}$ .

However it turns out, that all these estimates in general are invalid with  $N$  independent of  $Y_s$ . In other words, unlike the classical Jackson inequality, (1.4) cannot be had with  $N = 1$ . So, our first result is

**Theorem 1.1.** *Let  $r \in \mathbb{N}$  and  $s \in \mathbb{N}$  be given. For each  $n \geq 1$  there are a collection  $Y_s \in \mathbb{Y}_s$  and a function  $f \in \Delta^{(2)}(Y_s) \cap W^r$ , such that*

$$(1.5) \quad E_n^{(2)}(f, Y_s) > \frac{c(r)}{n^{r-1}} \|f^{(r)}\|,$$

where  $c(r) = \text{const} > 0$  depends only on  $r$ .

Moreover, (1.3) cannot be improved by replacing  $\omega_1$  with  $\omega_k$ ,  $k \geq 4$ . Our second main result is

**Theorem 1.2.** *For each  $Y_s \in \mathbb{Y}$  there is a function  $f \in \Delta^{(2)}(Y_s) \cap C^{(2)}$ , such that*

$$\limsup_{n \rightarrow \infty} \frac{n^2 E_n^{(2)}(f, Y_s)}{\omega_4(f'', 1/n)} = +\infty$$

Clearly,  $f \neq \text{const}$  in all Theorems in this paper. For the completeness we formulate an easy corollary of Example 1 in [2].

**Theorem 1.3.** *For each  $Y_s \in \mathbb{Y}$  there is a function  $f \in \Delta^{(2)}(Y_s) \cap C^{(1)}$ , such that*

$$(1.6) \quad \limsup_{n \rightarrow \infty} \frac{n E_n^{(2)}(f, Y_s)}{\omega_3(f', 1/n)} = +\infty,$$

whence

$$(1.7) \quad \limsup_{n \rightarrow \infty} \frac{E_n^{(2)}(f, Y_s)}{\omega_4(f, 1/n)} = +\infty.$$

We believe that Theorems 1.1 – 1.3 cover all negative results in the question of the validity of Jackson type estimates in the coconvex approximation of periodic functions. In particular, we conjecture, that for each  $k \in \mathbb{N}, r \in \mathbb{N}, r \geq 3, s \in \mathbb{N}$  and  $f \in \Delta^{(2)}(Y_s) \cap C^{(r)}$  we have

$$(1.8) \quad E_n^{(2)}(f, Y_s) \leq \frac{c(k, r, s)}{n^r} \omega_k(f^{(r)}, 1/n), \quad n \geq N(k, r, Y_s).$$

In other words, we conjecture, that the truth table of the validity of Jackson type estimates in the coconvex approximation of periodic functions has the same form as the truth table of Jackson type estimates in the coconvex approximation of non-periodic functions by algebraic polynomials, see [6], Page 114, Fig. 3, or [4], Page 62, Table 24.

**Remark 1.4.** We do not discuss the comonotone approximation in the Introduction, we only note, that in the comonotone approximation of periodic functions more positive, but less negative results are known, see, [7], [8], [1] for more details. However in the last Section we formulate the analogs of Theorems 1.2 and 1.3 for the comonotone (co-1-monotone) approximation. We do not discuss also co- $q$ -monotone approximation of periodic functions for  $q > 2$ , since, in opposite to coconvex (co-2-monotone) approximation, the Jackson type estimates are invalid for all parameters, even if we allow both constants  $c$  and  $N$  in (1.8) to depend on  $f$ . This is recently proved by Leviatan, Motorna and Shevchuk [5].

We prove Theorem 1.1 in the next Section, and Theorems 1.2 and 1.3 in the last Section. In the proofs we apply the ideas from [3] and we have to overcome the constrains and challenges of periodicity.

## 2. PROOF OF THEOREM 1.1

We begin with the following Lemma 2.1

**Lemma 2.1.** *Let an integer  $n \in \mathbb{N}$  and a positive number  $\delta \leq \frac{1}{n}$  be given. If a polynomial  $T \in \mathbb{T}_n$  satisfies*

$$T'(\pm\delta) = 0 \quad \text{and} \quad T'(t) \geq 0, \quad \text{for} \quad \delta \leq |t| \leq \pi,$$

then

$$(2.1) \quad T'(t) \equiv 0.$$

*Proof.* Assume to the contrary, that (2.1) is invalid. Then without loss of generality we assume that  $\|T'\| = 1$ . Put  $\tau := T'$ . By Bernstein inequality,  $\|\tau'\| \leq n$  and  $\|\tau''\| \leq n^2$ . Let  $t_0 \in [-\delta, 2\pi - \delta]$  be a point, such that  $|\tau(t_0)| = \|\tau\| = 1$ . Since, for  $t \in [-\delta, \delta]$ ,

$$(2.2) \quad |\tau(t)| = \left| \tau(t) + \frac{t-\delta}{2\delta}\tau(-\delta) - \frac{t+\delta}{2\delta}\tau(\delta) \right| = \frac{1}{2}|\tau''(\theta)|(\delta-t)(\delta+t) \\ \leq \frac{1}{2}n^2(\delta^2 - t^2) < 1,$$

where  $\theta \in (-\delta, \delta)$ , we conclude, that  $t_0 \notin [-\delta, \delta]$ , whence  $\tau(t_0) = 1$ . Then,

$1 - \tau(t) = \tau(t_0) - \tau(t) \leq n|t - t_0|$ , that is  $\tau(t) \geq 1 - n|t - t_0|$ ,  $t \in \mathbb{R}$ , that implies

$$(-\delta, \delta) \cap (t_0 - 1/n, t_0 + 1/n) = \emptyset$$

and

$$\int_{t_0-1/n}^{t_0+1/n} \tau(t) dt \geq \frac{1}{n}.$$

Therefore

$$0 = T(2\pi - \delta) - T(-\delta) = \int_{-\delta}^{2\pi - \delta} \tau(t) dt \geq \int_{t_0-1/n}^{t_0+1/n} \tau(t) dt + \int_{-\delta}^{\delta} \tau(t) dt \\ \geq \frac{1}{n} - \frac{1}{2}n^2 \int_{-\delta}^{\delta} (\delta^2 - t^2) dt = \frac{1}{n} - \frac{2}{3}n^2\delta^3 \geq \frac{1}{3n} \neq 0$$

– a contradiction. □

*Proof of Theorem 1.1.* Let  $\tilde{S} \in C^{(\infty)}(\mathbb{R})$ , be a monotone odd function, such that  $\tilde{S}(x) = \operatorname{sgn} x$ ,  $|x| \geq 1$ , and

$$S(x) := \frac{1}{2} + \frac{1}{2}\tilde{S}(x).$$

Put

$$s_j := \|S^{(j)}\|, \quad j \in \mathbb{N}_0.$$

Take

$$h := \frac{1}{3n},$$

$$y_{2s} = -3h, \quad y_1 = 3h,$$

and, if  $s \geq 2$ , then let  $-h \leq y_{2s-1} < \dots < y_2 \leq h$ , say

$$y_i = h - 2h \frac{i-2}{2s-3}, \quad i = 2, \dots, 2s-1.$$

We will prove, that the desired  $2\pi$  periodic function  $f$  can be taken in the form

$$f(x) := \int_0^x f'(t) dt,$$

where  $f'$  is a  $2\pi$ -periodic odd function, defined on  $[0, \pi]$  by

$$(2.3) \quad f'(t) := \begin{cases} -S\left(\frac{t-2h}{h}\right), & \text{if } t \in [0, 1], \\ S\left(\frac{t-\pi+h}{h}\right) - 1, & \text{if } t \in [1, \pi]. \end{cases}$$

Note that  $f'(t) = -1$  for  $3h \leq t \leq \pi - 2h$ . Clearly,  $f \in \Delta^{(2)}(Y_s) \cap W^r$ . Since  $3h = \frac{1}{n}$ , Lemma 2.1 yields  $T_n'' \equiv 0$ , if  $T_n \in \Delta^{(2)}(Y_s) \cap \mathbb{T}_n$ , hence (the periodic function)  $T_n \equiv \text{const}$ . Therefore

$$E_n^{(2)}(f, Y_s) \geq \frac{1}{2}(f(0) - f(\pi)) = -\frac{1}{2}f(\pi) \geq \frac{1}{2}(\pi - 5h) > \frac{2}{3}.$$

Thus,

$$\frac{E_n^{(2)}(f, Y_s)}{\|f^{(r)}\|} = E_n^{(2)}(f, Y_s) \frac{h^{r-1}}{s_{r-1}} \geq \frac{2h^{r-1}}{3s_{r-1}} = \frac{2}{3^r s_{r-1} n^{r-1}}.$$

Theorem 1.1 is proved with  $c(r) \geq \frac{2}{3^r s_{r-1}}$ .  $\square$

### 3. AUXILIARY RESULTS

Denote by  $\hat{S}$  an even function  $\hat{S} \in C^{(\infty)}(\mathbb{R})$ , such that  $x\hat{S}'(x) \geq 0, x \in \mathbb{R}$ , and

$$(3.1) \quad \hat{S}(x) = \begin{cases} 0, & \text{if } |x| \leq 1, \\ 1, & \text{if } |x| \geq 2. \end{cases}$$

Put  $\hat{s}_j := \|\hat{S}^{(j)}\|, j \in \mathbb{N}$ . Fix a positive number  $d \leq \pi/4$  and for each positive  $b \leq d/2$  denote by  $q_b$  and  $g_b$  the  $2\pi$ -periodic functions, such that

$$q_b(x) := \left(1 - \hat{S}(x/d)\right) (\cos b - \cos x) \sin x, \quad x \in [-\pi, \pi],$$

and

$$g_b(x) := \hat{S}(x/b) q_b(x), \quad x \in [-\pi, \pi].$$

Clearly,  $q_b$  and  $g_b$  are odd functions,

$$(3.2) \quad \|g_b\| < 1, \quad \|q_b\| < 1,$$

and for each collection  $Y = \{y_i\}_{i=1}^{2s}$ , such that

$$y_2 + 2d \leq y_1 = 0 \leq y_0 - 2d,$$

we have

$$(3.3) \quad g_b(t) \prod_{i=1}^{2s} (t - y_i) \geq 0, \quad t \in [y_{2s}, y_0].$$

Now, for a function  $f \in C[a, b]$  we denote by

$$\|f\|_{[a,b]} := \|f\|_{C[a,b]} = \max_{x \in [a,b]} |f(x)|$$

and formulate Lemma 3.1, which is a particular case of Privalov Theorem (see, e.g., [9], pg. 96-98.)

**Lemma 3.1.** *For each polynomial  $T_n \in \mathbb{T}_n$  and any positive number  $h \leq \pi$ , we have*

$$(3.4) \quad h|T'_n(0)| \leq cn \|T_n\|_{[-h,h]}.$$

Here and in the sequel  $c$  stand for different absolute positive constants.

We conclude the Section with the following

**Lemma 3.2.** *We have*

$$(3.5) \quad \omega_4(g_b, t) \leq c_1(b^3 + (t/d)^4),$$

and for each polynomial  $T_n \in \mathbb{T}_n$ , satisfying  $T'_n(0) \geq 0$ , and a positive number  $h \leq d$ ,

$$(3.6) \quad \|g_b - T_n\|_{[-h,h]} \geq c_2 \frac{hb^2}{n} - 3b^3.$$

where  $c_1$  and  $c_2 \leq 1$  are positive absolute constants.

*Proof.* First we show, that

$$(3.7) \quad \|g_b - q_b\| \leq 3b^3.$$

Indeed, if  $|x| \leq 2b$ , then

$$\begin{aligned} |g_b(x) - q_b(x)| &= \left(1 - \hat{S}(x/b)\right) |(\cos b - \cos x) \sin x| \\ &\leq 2 \left| \sin \frac{x-b}{2} \sin \frac{x+b}{2} \sin x \right| \leq \frac{1}{2} |x(x^2 - b^2)| \leq 3b^3. \end{aligned}$$

If otherwise  $2b \leq |x| \leq \pi$ , then  $g_b(x) = q_b(x)$ , so (3.7) holds.

To prove (3.5) we note, that the equality  $\|\hat{S}^{(j)}(\cdot/d)\| = \hat{s}_j d^{-j}$  yields  $\|q_b^{(4)}\| \leq cd^{-4}$ . Therefore

$$\omega_4(g_b, t) \leq \omega_4(g_b - q_b, t) + \omega_4(q_b, t) \leq 2^4 \|g_b - q_b\| + t^4 \|q_b^{(4)}\| \leq 48b^3 + cd^{-4}t^4,$$

that implies (3.5).

Finally, to prove (3.6), we apply Lemma 3.1. Since  $q_b(x) = (\cos b - \cos x) \sin x$  for  $x \in [-d, d]$ , we get

$$\begin{aligned} 2 \sin^2 \frac{b}{2} &= -q'_b(0) \leq T'_n(0) - q'_b(0) \leq \frac{cn}{h} \|T_n - q_b\|_{[-h,h]} \\ &\leq \frac{cn}{h} (\|T_n - g_b\|_{[-h,h]} + \|q_b - g_b\|) \leq \frac{cn}{h} (\|T - g_b\|_{[-h,h]} + 3b^3), \end{aligned}$$

that yields (3.6). □

#### 4. NEGATIVE RESULT IN COPOSITIVE APPROXIMATION

For each collection  $Y_s = \{y_i\}_{i=1}^{2s} \in \mathbb{Y}_s$  denote by  $\Delta^{(0)}(Y_s)$  the set of all  $2\pi$ -periodic functions  $f \in C(\mathbb{R})$ , such that

$$f(t) \prod_{i=1}^{2s} (t - y_i) \geq 0, \quad t \in [y_{2s}, y_0].$$

For the function  $g \in \Delta^{(0)}(Y_s)$ , denote by

$$E_n^{(0)}(g, Y_s) := \inf_{T_n \in \mathbb{T}_n \cap \Delta^{(0)}(Y_s)} \|g - T_n\|,$$

the error of the best copositive approximation of the function  $g$ . We prove

**Theorem 4.1.** *For each  $Y_s \in \mathbb{Y}_s$  there is a function  $f \in \Delta^{(0)}(Y_s)$ , such that*

$$(4.1) \quad \limsup_{n \rightarrow \infty} \frac{E_n^{(0)}(f, Y_s)}{\omega_4(f, 1/n)} = +\infty$$

and

$$(4.2) \quad \int_{-\pi}^{\pi} f(x) dx = 0.$$

*Proof.* Without loss of generality assume, that  $y_1 = 0$ . Put  $d := \frac{1}{2} \min\{y_0, -y_2, \pi/2\}$ , so that (3.3) implies, for all positive  $b \leq d/2$ ,

$$(4.3) \quad g_b \in \Delta^{(0)}(Y_s),$$

where  $g_b$  is defined in the previous Section. Following [3], pg.343-345, we put

$$(4.4) \quad b_n := \left(\frac{1}{n}\right)^{\frac{4}{3}}, \quad f_n(x) = g_{b_n}(x),$$

and note that (3.5) implies, for all  $n \geq (2/d)^{3/4}$ , so that  $2b_n \leq d$ ,

$$(4.5) \quad \omega_4(g_b; t) \leq c_1 (1 + d^{-4}) t^4, \quad t \geq \frac{1}{n}.$$

We are now in a position to define the desired in Theorem 4.1 function  $f$ . First we put  $\varepsilon = 0.1$  and choose  $n_0 \geq (2/d)^{3/4}$ , so big that

$$(4.6) \quad c_1 (1 + d^{-4}) < n_0^\varepsilon.$$

Set  $d_0 := d$  and

$$(4.7) \quad d_j := \frac{c_2 b_{n_{j-1}} b_{n_j}^2}{4 n_j} d_{j-1}, \quad j \geq 1,$$



where the increasing sequence  $\{n_\nu\}$  is defined by induction as follows. Suppose that  $\{n_0, \dots, n_{\sigma-1}\}$  have been defined, then put

$$F_{\sigma-1}(x) := \sum_{j=1}^{\sigma-1} d_{j-1} f_{n_j}(x), \quad (F_0(x) := 0),$$

and take  $n_\sigma > n_{\sigma-1}$  so big that

$$(4.8) \quad \left\| F_{\sigma-1}^{(4)} \right\| \leq d_{\sigma-1} n_\sigma^\epsilon,$$

and

$$(4.9) \quad c_2 b_{n_{\sigma-1}} > 2n_\sigma^{-\epsilon}.$$

Now put

$$\Phi_\sigma(x) := \sum_{j=\sigma}^{\infty} d_{j-1} f_{n_j}(x),$$

where the uniform convergence of the series is justified by (3.2) and the inequality

$$(4.10) \quad \|\Phi_\sigma\| \leq \sum_{j=\sigma}^{\infty} d_{j-1} \leq d_{\sigma-1} \left( 1 + \frac{1}{4} + \frac{1}{4^2} + \dots \right) < 2d_{\sigma-1}.$$

So we define

$$(4.11) \quad f(x) := \sum_{j=1}^{\infty} d_{j-1} f_{n_j}(x) = F_{\sigma-1}(x) + \Phi_\sigma(x),$$

and note, that (4.3) and (4.4) yield

$$(4.12) \quad f \in \Delta^{(0)}(Y_s).$$

Recall also that  $g_b$  and, hence,  $f_n$ , are odd functions. Therefore,  $f$  is odd as well, which implies (4.2).

It remains to verify (4.1). Inequalities (4.5), (4.10), and (4.6) lead to

$$\omega_4(\Phi_\sigma, 1/n_\sigma) \leq c_1 \left( 1 + \frac{1}{d^4} \right) \frac{1}{n_\sigma^4} \sum_{j=\sigma}^{\infty} d_{j-1} < 2c_1 \left( 1 + \frac{1}{d^4} \right) \frac{d_{\sigma-1}}{n_\sigma^4} < 2 \frac{d_{\sigma-1}}{n_\sigma^{4-\epsilon}},$$

as well as (4.8) provides

$$\omega_4 \left( F_{\sigma-1}, \frac{1}{n_\sigma} \right) \leq \frac{1}{n_\sigma^4} \left\| F_{\sigma-1}^{(4)} \right\| \leq \frac{d_{\sigma-1}}{n_\sigma^{4-\epsilon}}.$$

Hence, for all  $\sigma$ ,

$$(4.13) \quad \omega_4 \left( f, \frac{1}{n_\sigma} \right) \leq 3 \frac{d_{\sigma-1}}{n_\sigma^{4-\epsilon}}.$$

Finally, let us prove that if  $T_{n_\sigma} \in \mathbb{T}_{n_\sigma} \cap \Delta^{(0)}(Y_s)$ , then

$$(4.14) \quad \|f - T_{n_\sigma}\| \geq d_{\sigma-1} \left( \frac{b_{n_\sigma}^2}{n_\sigma^{1+\epsilon}} - 3b_{n_\sigma}^3 \right),$$

$$(4.15) \quad E_{n_\sigma}^{(0)}(f, Y_s) \geq d_{\sigma-1} \left( n_\sigma^{-\varepsilon-11/3} - 3n_\sigma^{-4} \right).$$

Indeed, since by (3.1)  $F_{\sigma-1}$  is zero on  $[-b_{n_{\sigma-1}}, b_{n_{\sigma-1}}] =: J_\sigma$ , we may write

$$f(x) = d_{\sigma-1} f_{n_\sigma}(x) + \Phi_{\sigma+1}(x), \quad x \in J_\sigma.$$

Let  $\tau_{n_\sigma} := T_{n_\sigma}/d_{\sigma-1}$ . Since  $\tau_{n_\sigma} \in \Delta^{(0)}(Y_s)$ , we have  $\tau'_{n_\sigma}(0) \geq 0$ , therefore by virtue of (3.6)

$$(4.16) \quad \|f_{n_\sigma} - \tau_{n_\sigma}\|_{J_\sigma} \geq c_2 \frac{b_{n_\sigma}^2}{n_\sigma} b_{n_{\sigma-1}} - 3b_{n_\sigma}^3.$$

On the other hand, (3.2), (4.10), and (4.7) yield

$$\|\Phi_{\sigma+1}\| < 2d_\sigma = \frac{c_2 b_{n_{\sigma-1}} b_{n_\sigma}^2}{2 n_\sigma} d_{\sigma-1}.$$

Hence

$$\begin{aligned} \|f - T_{n_\sigma}\| &\geq \|T_{n_\sigma} - f\|_{I_\sigma} \geq \|T_{n_\sigma} - d_{\sigma-1} f_{n_\sigma}\|_{I_\sigma} - \|\Phi_{\sigma+1}\| \\ &= d_{\sigma-1} \|\tau_{n_\sigma} - f_{n_\sigma}\|_{I_\sigma} - \|\Phi_{\sigma+1}\| \geq d_{\sigma-1} \left( \frac{c_2 b_{n_{\sigma-1}} b_{n_\sigma}^2}{2n_\sigma} - 3b_{n_\sigma}^3 \right). \end{aligned}$$

Now (4.14) follows from (4.9). Thus, (4.13) and (4.15) lead to

$$\frac{E_{n_\sigma}^{(0)}(f)}{\omega_4(f, 1/n_\sigma)} \geq \frac{1}{3} n_\sigma^{1/3-2\varepsilon} - \frac{1}{n_\sigma^\varepsilon}$$

for all  $\sigma$ , that implies (4.1), since the chosen  $\varepsilon$  is sufficiently small.  $\square$

## 5. NEGATIVE RESULTS IN COMONOTONE APPROXIMATION, PROOFS OF THEOREMS 1.2 AND 1.3

We say that a  $2\pi$ -periodic function  $f \in C(\mathbb{R})$  is piecewise monotone with respect to  $Y_s \in \mathbb{Y}_s$ , if it is a nondecreasing function on  $[y_1, y_0]$  and it changes its monotonicity at the points  $Y_s$ , that is, if  $(-1)^{i-1}f$  is nondecreasing on  $[y_i, y_{i-1}]$ ,  $1 \leq i \leq 2s$ . We denote by  $\Delta^{(1)}(Y_s)$  the collection of all such piecewise monotone functions. For a function  $g \in \Delta^{(1)}(Y_s)$  denote by

$$E_n^{(1)}(g, Y_s) := \inf_{T_n \in \mathbb{T}_n \cap \Delta^{(1)}(Y_s)} \|g - T_n\|,$$

the error of the best comonotone approximation of the function  $g$ .

First we formulate the well known Lemma 5.1.

**Lemma 5.1.** *For each function  $g \in C^{(1)}$  and every polynomial  $T_n \in \mathbb{T}_n$  the inequality*

$$(5.1) \quad \|g' - T_n'\| \leq c(k)(\omega_k(g', 1/n) + n\|g - T_n\|)$$

*holds, where the constant  $c(k)$  depends only on  $k$ .*

Now we prove

**Theorem 5.2.** *For each  $Y_s \in \mathbb{Y}_s$  there is a function  $F \in \Delta^{(1)}(Y_s) \cap C^{(1)}$ , such that*

$$\limsup_{n \rightarrow \infty} \frac{nE_n^{(1)}(F, Y_s)}{\omega_4(F', 1/n)} = +\infty$$

*Proof.* Let  $f$  be a function, guaranteed by Theorem 4.1. Identity (4.2) implies, that the function

$$F(x) := \int_0^x f(t)dt$$

is also  $2\pi$  periodic. Then,  $F' \equiv f$  and  $f \in \Delta^{(0)}(Y_s)$  yield  $F \in \Delta^{(1)}(Y_s)$ . For each polynomial  $T_n \in \Delta^{(1)}(Y_s) \cap \mathbb{T}_n$  (5.1) implies

$$E_n^{(0)}(f, Y_s) \leq \|F' - T_n'\| \leq c(\omega_4(F', 1/n) + n\|F - T_n\|),$$

whence

$$nE_n^{(1)}(F, Y_s) \geq cE_n^{(0)}(f, Y_s) - \omega_4(f, 1/n).$$

Therefore

$$\limsup_{n \rightarrow \infty} \frac{nE_n^{(1)}(F, Y_s)}{\omega_4(F', 1/n)} \geq c \limsup_{n \rightarrow \infty} \frac{E_n^{(0)}(f, Y_s)}{\omega_4(f, 1/n)} - 1 = +\infty.$$

□

We are ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let  $F$  be a function, guaranteed by Theorem 5.2. Since  $F$  is a  $2\pi$ -periodic function and  $F \in \Delta^{(1)}(Y_s) \cap C^{(1)}$ , the function

$$f(x) := \int_0^x F(t)dt - \frac{x}{2\pi} \int_0^{2\pi} F(t)dt$$

is also  $2\pi$ -periodic and  $f \in \Delta^{(2)}(Y_s) \cap C^{(2)}$ . For each polynomial  $T_n \in \Delta^{(2)}(Y_s) \cap \mathbb{T}_n$  (5.1) implies

$$\begin{aligned} E_n^{(1)}(F, Y_s) &\leq \|f' - T_n'\| \leq c(\omega_5(f', 1/n) + n\|f - T_n\|) \\ &\leq c \left( \frac{1}{n} \omega_4(f'', 1/n) + n\|f - T_n\| \right), \end{aligned}$$

whence

$$n^2 E_n^{(2)}(f, Y_s) \geq cnE_n^{(1)}(F, Y_s) - \omega_4(f'', 1/n).$$

Therefore

$$\limsup_{n \rightarrow \infty} \frac{n^2 E_n^{(2)}(f, Y_s)}{\omega_4(f'', 1/n)} \geq c \limsup_{n \rightarrow \infty} \frac{nE_n^{(1)}(F, Y_s)}{\omega_4(F', 1/n)} - 1 = +\infty.$$

□

Our last formulation is

**Theorem 5.3** ([2]). *For each  $Y_s \in \mathbb{Y}$  there is a function  $f \in \Delta^{(1)}(Y_s)$ , such that*

$$\limsup_{n \rightarrow \infty} \frac{E_n^{(1)}(f, Y_s)}{\omega_3(f, 1/n)} = +\infty.$$

*Proof of Theorem 1.3.* In fact we repeat the previous proof with minor changes. So, Let  $F$  be a function, guaranteed by Theorem 5.3. Since  $F$  is a  $2\pi$ -periodic function and  $F \in \Delta^{(1)}(Y_s)$ , the function

$$f(x) := \int_0^x F(t)dt - \frac{x}{2\pi} \int_0^{2\pi} F(t)dt$$

is also  $2\pi$ -periodic and  $f \in \Delta^{(2)}(Y_s) \cap C^{(1)}$ . For each polynomial  $T_n \in \Delta^{(2)}(Y_s) \cap \mathbb{T}_n$  (5.1) implies

$$E_n^{(1)}(F, Y_s) \leq \|f' - T_n'\| \leq c(\omega_3(f', 1/n) + n\|f - T_n\|),$$

whence

$$nE_n^{(2)}(f, Y_s) \geq cE_n^{(1)}(F, Y_s) - \omega_3(F, 1/n).$$

Therefore

$$\limsup_{n \rightarrow \infty} \frac{nE_n^{(2)}(f, Y_s)}{\omega_3(f', 1/n)} \geq c \limsup_{n \rightarrow \infty} \frac{E_n^{(1)}(F, Y_s)}{\omega_3(F, 1/n)} - 1 = +\infty,$$

which is (1.6). Finally, (1.7) follows from (1.6) and the inequality  $\omega_4(f, t) \leq t\omega_3(f', t)$ .  $\square$

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