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DISTANCE ESTIMATES TO FEASIBLE CONTROLS FOR SYSTEMS WITH FINAL POINT CONSTRAINTS AND SECOND ORDER NECESSARY OPTIMALITY CONDITIONS

H. FRANKOWSKA* AND N. P. OSMOLOVSKII

Abstract. We prove an inverse mapping theorem on a metric space of controls that allows to "control" final points of trajectories of a nonlinear system. More precisely, our result provides local distance estimates of arbitrary controls from feasible ones. As an application we derive second-order necessary optimality conditions for L^1-local minima for the Mayer optimal control problem with a general control constraint $U \subset \mathbb{R}^m$, state constraints described by inequalities and final point constraints, possibly having empty interior. Thanks to this inverse mapping theorem we first get a second-order variational inequality as a necessary optimality condition. Then the separation theorem leads in a straightforward way to second-order necessary conditions.

Keywords. Optimal control, state constraints, second-order optimality conditions, distance estimates on feasible controls, inverse mapping theorem on a metric space.

AMS subject classification. 49J53, 49K15

1. Introduction

Consider the control system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad u(t) \in U \text{ a.e. in } [0,1] \tag{1.1}$$

with the final point constraint

$$x(1) \in K, \tag{1.2}$$

where $K \subset \mathbb{R}^n$ is closed and nonempty, U is an *arbitrary nonempty compact subset* of $\mathbb{R}^m$, $f \in C^2(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$ is fixed. In particular, K can be described by equalities

$$K = \{x \in \mathbb{R}^n \ : \ h_j(x) = 0, \ \forall\, j = 1, ..., k\}, \tag{1.3}$$

where $h_j \in C^2(\mathbb{R}^n, \mathbb{R})$. Then (1.2) becomes $h_j(x(1)) = 0$ for $j = 1, \dots, k$.

Below when we say "a trajectory-control pair" without making precise the control system we always mean a trajectory-control pair of (1.1) with measurable $u(\cdot)$ and absolutely continuous $x(\cdot)$. Let $(\bar{x}, \bar{u})$ be a trajectory-control pair satisfying $\bar{x}(1) \in K$. In this paper we provide a sufficient condition for the existence of $c > 0$, $\varepsilon > 0$ such that for any trajectory-control pair (x, u) with $\|\bar{u} - u\|_{L^1} \le \varepsilon$ we can find a trajectory-control pair $(\widetilde{x}, \widetilde{u})$ such that

$$\widetilde{x}(1) \in K, \quad \|\widetilde{u} - u\|_{L^1} \le c\, \mathrm{d}_K(x(1)),$$

where $\mathrm{d}_K(\cdot)$ denotes the distance function to K. This question is of interest on its own, and is particularly important in problems of optimal control, where to derive optimality conditions one has to perturb optimal controls, while respecting the final point constraints.

As an application we consider the Mayer optimal control problem

$$\text{Minimize} \quad \varphi(x(1)) \tag{1.4}$$

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over trajectories of (1.1), (1.2) satisfying the additional final point and state constraints

$$g_i(x(1)) \leq 0, \quad i = 1, \dots, r, \tag{1.5}$$

$$\Phi_j(x(t)) \leq 0 \quad \text{for all } t \in [0,1], \quad j = 1, \dots, s, \tag{1.6}$$

where $\Phi_j \in C^2(\mathbb{R}^n, \mathbb{R})$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$ are twice differentiable.

This paper can be considered as a companion to [7], where we have investigated second-order optimality conditions for strong local minima in the absence of the constraint (1.2). To derive these conditions we used a second-order linearization of (1.1), a separation theorem in the space of continuous functions and some results from [6] on linearizations of differential inclusions. The advantage of this approach is due to its generality and may be also applied to optimization problems arising in control of PDEs and stochastic control. The case investigated in [7] is simpler than the one of the present work, because for every $i = 1, ..., r$ the interiors of first and second tangent sets to the final point constraint (1.5) are nonempty at every point where the gradient of g_i is nontrivial. In contrast, for constraints involving equalities, interiors of tangent sets are usually empty. In the conference paper [5] *final point equality constraints* were included, however only weak minimizers were studied. Here we fill this gap and address, in particular, strong local minimizers. To handle constraint (1.2) we prove a version of inverse mapping theorem on the space of controls involving the surjectivity assumption (3.3), which, for final point equality constraints becomes (3.14). We use here the very same second-order linearization (3.12) of (1.1) as in [7] and show that not only it allows to approximate some trajectories of (1.1), (1.2), but it also provides estimates on corresponding controls. Thanks to it we get second-order conditions for L^1-local minima that are valid also for strong local minima. In the difference with [7], a finite dimensional separation theorem is applied, instead of the infinite dimensional one. This is done not only for simplification purposes, but permits us to separate an arbitrary convex set from a point outside of it. Recall that in the finite dimensional spaces this is always possible, while in the infinite dimensional spaces, when separating two convex sets, one of them has to have a nonempty interior. We would like to underline that in a recent article by the second author [12] considering final point equality constraints and in the absence of state constraints (1.6), assumption (3.14) is not needed. However in [12] the control set U has a particular structure. Our Corollary 3.12, stated without assumption (3.14) for a general set U is of a different nature.

We do not discuss here the existing literature on this subject, because this was already done in [7], see also [9], [11], [12] for additional references and comments and also [10], where penalization type techniques, close to the Ekeland variational principle, are applied in the presence of equality endpoint constraints. Let us just mention that necessary conditions for a strong local minimum in optimal control problems with equality and inequality endpoint constraints and mixed state-control constraints were stated by the second author (together with the relevant sufficient conditions) in [11, Supplement to Chapter VI, S2] when the gradients with respect to control of active mixed state-control constraints are jointly linearly independent (which excludes the pure state constraints). Much later, similar results were obtained in [3] for optimal control problems in the absence of state constraints, with endpoint constraints of equality and inequality type, as well as control constraints, satisfying the uniform positive independence condition for gradients of active constraints. The transition to problems with positively independent gradients of control constraints was an important step forward, but, unlike the results in [11], some additional, difficult to check and restrictive assumptions, concerning the whole system of constraints were required, see [3, (61) and (80)]. The same can be said about assumptions in [2], devoted to problems with mixed and pure state constraints see [2, Assumptions 4, 5 and Remark 4.7].

The paper is organized as follows. Section 2 is devoted to preliminaries, while in Section 3 we state the main results whose proofs are postponed to Section 5 and the Appendix. Section 4 is concerned with second order tangents to trajectories of (1.1), (1.2).

2. Preliminaries

Let $\mathbb{R}_+$ be the set of all reals $r \geq 0$ and $|\cdot|$, $\langle\cdot,\cdot\rangle$ stand for the Euclidean norm and the scalar product in a finite dimensional space. Below B denotes the closed unit ball in $\mathbb{R}^n$, S^{n-1} the unit sphere in $\mathbb{R}^n$ and $B(x,r)$ the closed ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$. Denote by $W^{1,1}([0,1],\mathbb{R}^n)$ the Sobolev space of absolutely continuous functions $x : [0,1] \to \mathbb{R}^n$, by $L^1([0,1],\mathbb{R}^m)$ the space of Lebesgue integrable functions $u : [0,1] \to \mathbb{R}^m$ with the norm $\|u\|_1 = \int_0^1 |u(t)|\,\mathrm{d}t$ and by $L^\infty([0,1],\mathbb{R}^n)$ the space of measurable, essentially bounded functions with the norm $\|u\|_\infty = \text{ess-sup}_{t\in[0,1]}|u(t)|$. Let $NBV([0,1],\mathbb{R}^n)$ stand for the set of $\psi : [0,1] \to \mathbb{R}^n$ having bounded total variation, that are continuous from the right on $(0,1)$ and satisfying $\psi(0) = 0$. We also denote by $\|\cdot\|_\infty$ the supremum norm on the space $C([0,1],\mathbb{R}^n)$ of continuous functions $x : [0,1] \to \mathbb{R}^n$ and shall use the notation $\Xi := W^{1,1}([0,1],\mathbb{R}^n) \times L^\infty([0,1],\mathbb{R}^m)$.

Any trajectory-control pair $(x,u) \in \Xi$ satisfying (1.1), (1.2), (1.5), (1.6) is called *admissible*. An admissible $(\bar{x},\bar{u})$ is an L^1-*local minimizer* if for some $\varepsilon > 0$ we have $\varphi(x(1)) \geq \varphi(\bar{x}(1))$ for any admissible $(x,u) \in \Xi$ such that $\|u - \bar{u}\|_{L^1} < \varepsilon$. Recall that $(\bar{x},\bar{u})$ is called a *strong local minimizer* if for some $\varepsilon > 0$ we have $\varphi(x(1)) \geq \varphi(\bar{x}(1))$ for any admissible $(x,u) \in \Xi$ such that $\|x - \bar{x}\|_\infty < \varepsilon$. If f is locally Lipschitz and for some $\gamma > 0$ and all $x \in \mathbb{R}^n$, $u \in U$ we have $|f(x,u)| \leq \gamma(|x|+1)$, then the Gronwall lemma implies that every strong local minimizer is also an L^1-*local minimizer*. Define

$$K_i^e = \{x \in \mathbb{R}^n \,:\, g_i(x) \leq 0\},\ i = 1,...,r, \quad K_j = \{x \in \mathbb{R}^n \,:\, \Phi_j(x) \leq 0\},\ j = 1,...,s.$$

Then for every $x \in \partial K_i^e$ we have $g_i(x) = 0$ and similarly for K_j.

Let X be a Banach space and $\emptyset \neq Q \subset X$. Denote by $\mathrm{Int}(Q)$ the interior of Q, by $\mathrm{co}\,Q$ its convex hull and by $\overline{co}\,Q$ its closed convex hull. The distance from $x \in X$ to Q is defined by $\mathrm{d}_Q(x) = \inf_{x'\in Q}|x' - x|$. For a metric space Y and a family $\{A(y)\}_{y\in Y}$ of subsets of X the Peano-Kuratowski upper (resp. lower) set limits when $y \to z$ in the space Y are given by

$$v \in \mathrm{Limsup}_{y\to z}A(y) \Longleftrightarrow \liminf_{y\to z} \mathrm{d}_{A(y)}(v) = 0, \quad v \in \mathrm{Liminf}_{y\to z}A(y) \Longleftrightarrow \lim_{y\to z} \mathrm{d}_{A(y)}(v) = 0.$$

The *adjacent tangent cone* to Q at $\bar{x} \in Q$ is the closed cone

$$T_Q^\flat(\bar{x}) = \{v \in X \,|\, \bar{x} + \delta v \in Q + o(\delta),\ \forall\, \delta > 0\}$$

and the *second-order adjacent set* to Q at $(\bar{x},v) \in Q \times X$ is the closed set

$$T_Q^{\flat(2)}(\bar{x},v) = \{w \in X \,|\, \bar{x} + \delta v + \delta^2 w \in Q + o(\delta^2),\ \forall\, \delta > 0\},$$

where the vectors $o(\delta) \in X$ are so that $\lim_{\delta\to 0+} \frac{1}{\delta}o(\delta) = 0$. See [1] for further information.

Denote by $C_Q(\bar{x})$ and $N_Q(\bar{x})$ respectively the Clarke tangent and normal cones to Q at $\bar{x}$, cf. [13]. By [6, Lemma 2.4] we have

$$T_Q^{\flat(2)}(\bar{x},v) + C_Q(\bar{x}) = T_Q^{\flat(2)}(\bar{x},v). \tag{2.1}$$

We shall need the following two results that follow easily from the separation theorem.

Proposition 2.1. *Let $M_1,...,M_k$ be nonempty convex subsets of $\mathbb{R}^n$ and $\cap_{i=1}^k M_i = \emptyset$. Then there exist $p_i \in (\mathbb{R}^n)^*$ for $i = 1,...,k$ not all equal to zero with $\sum_{i=1}^k p_i = 0$ and $\sum_{i=1}^k \inf p_i(M_i) \geq 0$.*

Proposition 2.2. *Let $b \in \mathbb{R}$, $p,q \in (\mathbb{R}^n)^*$ with $p \neq 0$ and $M = \{x \in \mathbb{R}^n \,:\, p(x) + b < 0\}$. If $\inf q(M) > -\infty$, then there exists $\alpha \geq 0$ such that $q = -\alpha p$ and $\inf q(M) = \alpha b$.*

3. Main Results

Throughout the whole paper we assume

$$\exists\, \gamma > 0, \quad \sup_{u\in U} |f(x,u)| \le \gamma(|x|+1), \ \ \forall\, x \in \mathbb{R}^n. \tag{3.1}$$

Denote by $\mathcal{U}$ the set of all measurable $u : [0,1] \to U$. For any u_1, $u_2 \in \mathcal{U}$ define the distance $\mathrm{d}(u_1,u_2) := \|u_1 - u_2\|_1$. Then $(\mathcal{U}, \mathrm{d})$ is a complete metric space.

Consider the following linearization of (1.1) at $(\bar{x}, \bar{u})$:

$$\begin{cases} \dot{z}(t) = f_x[t]z(t) + v(t), \ \ v(t) \in \mathrm{co} f(\bar{x}(t), U) - f[t], \quad v(\cdot) \text{ is measurable} \\ z(0) = 0, \end{cases} \tag{3.2}$$

where f_x denotes the derivative of f with respect to x and $[t] := (\bar{x}(t), \bar{u}(t))$. Notice that $v(\cdot)$ is integrable by (3.1). The reachable set of (3.2) at time 1 is defined by

$$R^L(1) = \{z(1) : \ z(\cdot) \ \text{ is a trajectory of (3.2)}\}.$$

Recall that it is equal to the reachable set at time 1 of the very same system even after removing the convex hull. In the next result we need less than C^2-regularity of f.

Theorem 3.1. *Let $K \subset \mathbb{R}^n$ be closed. Assume that f is continuous, differentiable with respect to x and $f_x(\cdot,\cdot)$ is continuous. Consider a trajectory-control pair $(\bar{x}, \bar{u})$ with $\bar{x}(1) \in K$. If*

$$0 \in \mathrm{Int}(R^L(1) - C_K(\bar{x}(1))), \tag{3.3}$$

then there exist $\varepsilon > 0$, $c > 0$ such that for every trajectory-control pair (x,u) of (1.1) and $k \in K$ with $\|u - \bar{u}\|_1 + |k - \bar{x}(1)| < \varepsilon$ we can find a trajectory-control pair $(\widetilde{x}, \widetilde{u})$ of (1.1) satisfying

$$\widetilde{x}(1) \in K, \quad \|u - \widetilde{u}\|_1 + |k - \widetilde{x}(1)| \le c\,|x(1) - k|.$$

Remark 3.2. (a) The proof provided in Section 5 is based on a more general inverse mapping theorem [4, Theorem 3.2] in which ε and c are defined explicitly.

(b) Under the assumptions of Theorem 3.1 the set $R^L(1)$ is convex and compact and, because (3.2) is linear with respect to the state, it can be expressed explicitly. Therefore (3.3) is an assumption that can be easily verified, for instance by applying the separation theorem.

Example 3.3. Let $K = \{(a,b) \in \mathbb{R}_+ \times \mathbb{R}, : \ |b| = a^2\}$. Then $C_K(0) = \mathbb{R}_+ \times \{0\}$, $N_K(0) = \mathbb{R}_- \times \mathbb{R}$. Consider $U = \{(0,0), (1,0), (0,1)\}$, $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ with $f(0) = 0$ and having the sublinear growth. For the control system $\dot{x} = f(x) + u(t)$, $u(t) \in U$, $x(0) = 0$ the trajectory-control pair $(\bar{x}, \bar{u}) = (0,0)$ satisfies $\bar{x}(1) \in K$ and (3.2) becomes $\dot{z} = Az + v(t)$, $v(t) \in \mathrm{co}\, U$, $x(0) = 0$, where $A = f_x(0)$. It is not difficult to check (using the separation theorem, the explicit representation of the convex reachable set $R^L(1)$ and the inclusion $0 \in U$) that (3.3) holds true if and only if there exists $u \in \mathrm{co}\, U$ such that $e^{tA}u \in \mathbb{R} \times (-\infty, 0)$ for some $t \in (0,1]$.

From now on we assume that φ, f, g_i, Φ_j, K, U are as in the introduction, satisfying regularity assumptions from the introduction. The *Hamiltonian* is defined by

$$\mathcal{H}(x,u,p) = \langle p, f(x,u)\rangle.$$

Let $\mathcal{H}_x$ stand for the derivative of $\mathcal{H}$ with respect to x. Then $\mathcal{H}_x(x,u,p) = pf_x(x,u)$ for all (x,u,p). Below $\Phi_j'(z)$, resp. $g_i'(z)$, denotes the gradient of Φ_j, resp. g_i, at z.

Let $(\bar{x}, \bar{u})$ be an L^1-local minimizer for problem (1.4), (1.1), (1.2), (1.5), (1.6). Below we shall simply write "let $(\bar{x}, \bar{u})$ be an L^1-local minimizer". Define the set of active indices

$$I_g := \{i = 1, ..., r \ : \ \bar{x}(1) \in \partial K_i^e\}.$$

The first-order necessary optimality condition is as follows: there exist $q \in N_K(\bar{x}(1))$, $\alpha = (\alpha_0, \dots, \alpha_r) \in \mathbb{R}^{r+1}_+$ and positive Borel measures μ_j on $[0,1]$ with

$$\alpha_i = 0 \quad \text{if } i \notin I_g, \quad \operatorname{supp} \mu_j \subset \{t \in [0,1] : \bar{x}(t) \in \partial K_j\}, \quad j = 1, \dots, s, \tag{3.4}$$

not vanishing simultaneously such that for ψ_j, $\psi \in NBV([0,1], \mathbb{R}^n)$ defined by

$$\psi_j(t) := -\int_{[0,t]} \Phi_j'(\bar{x}(\tau))\, \mathrm{d}\mu_j(\tau) \quad \forall\, t \in (0,1], \quad \psi := \sum_{j=1}^{s} \psi_j \tag{3.5}$$

the solution $p \in W^{1,1}([0,1], \mathbb{R}^n)$ of the adjoint system

$$-\dot{p}(t) = \mathcal{H}_x(\bar{x}(t), \bar{u}(t), p(t) + \psi(t)), \quad p(1) + \psi(1) = \alpha_0 \varphi'(\bar{x}(1)) + \sum_{i=1}^{r} \alpha_i g_i'(\bar{x}(1)) + q \tag{3.6}$$

satisfies the minimum principle

$$\inf_{u \in U} \mathcal{H}(\bar{x}(t), u, p(t) + \psi(t)) = \mathcal{H}(\bar{x}(t), \bar{u}(t), p(t) + \psi(t)) \quad \text{a.e. in } [0,1]. \tag{3.7}$$

Below $M([0,1])$ stands for the set of all tuples $(\mu_1, ..., \mu_s)$ such that every μ_j is a positive Borel measure on $[0,1]$. Let $\Lambda(\bar{x}, \bar{u})$ denote the set of all

$$(\alpha, q, p, \psi, \mu) \in \mathbb{R}^{r+1}_+ \times \mathbb{R}^n \times W^{1,1}([0,1], \mathbb{R}^n) \times NBV([0,1], \mathbb{R}^n) \times M([0,1])$$

satisfying (3.4)-(3.7) such that $(\alpha, q, \mu) \neq 0$.

Our proof of second-order necessary conditions will also imply the above first-order one. Let us underline that for the first-order necessary condition alone, less regularity assumptions are required, because only the first order linearizations of the data have to be performed.

Consider the classical linearization of control system (1.1) at $(\bar{x}, \bar{u})$:

$$\begin{cases} \dot{y}(t) = f_x[t]y(t) + f_u[t]u(t),\ u(t) \in T_U^\flat(\bar{u}(t)), \quad u \in L^\infty([0,1], \mathbb{R}^m) \\ y(0) = 0, \end{cases} \tag{3.8}$$

where f_u denotes the derivative of f with respect to u. For every $\eta \geq 0$ and $j = 1, ..., s$ define

$$\mathcal{M}_{j\eta} = \{t \in [0,1] :\ \Phi_j(\bar{x}(t)) \geq -\eta,\ \mathrm{d}_{\partial K_j}(\bar{x}(t)) \leq \eta\}.$$

The *critical cone* $\mathcal{C}_0(\bar{x}, \bar{u})$ is the set of all $(y, u) \in \Xi$ solving the linear system (3.8) such that

$$\langle \varphi'(\bar{x}(1)), y(1) \rangle \leq 0,\ y(1) \in C_K(\bar{x}(1)),\ \langle g_i'(\bar{x}(1)), y(1) \rangle \leq 0, \quad \forall\, i \in I_g,$$

$$\exists\, h_0 > 0,\ c > 0 \text{ such that } \forall\, h \in [0, h_0],\ \mathrm{d}_U(\bar{u}(t) + hu(t)) \leq ch^2 \text{ for a.e. } t \in [0,1], \tag{3.9}$$

$$\exists\, \eta > 0 \quad \text{satisfying} \quad \max_{t \in \mathcal{M}_{j\eta}} \langle \Phi_j'(\bar{x}(t)), y(t) \rangle \leq 0, \quad \forall\, j = 1, \dots, q. \tag{3.10}$$

Remark 3.4. In [12] the set U is described by inequality constraints with linearly independent gradients on ∂U. The critical cone defined there is larger, because (3.9) concerns there only times t with $\bar{u}(t) \in \partial U$.

Lemma 3.5. *Let $(\alpha, q, p, \psi, \mu) \in \Lambda(\bar{x}, \bar{u})$. Then for any $(y, u) \in \mathcal{C}_0(\bar{x}, \bar{u})$ and all i, j*

$$\begin{aligned} \alpha_0 \langle \varphi'(\bar{x}(1)), y(1) \rangle = 0, \quad & \alpha_i \langle g_i'(\bar{x}(1)), y(1) \rangle = 0,\ \langle \mathcal{H}_u(\bar{x}(t), \bar{u}(t), p(t) + \psi(t)), u(t) \rangle = 0 \ \textit{a.e.}, \\ & \langle q, y(1) \rangle = 0,\ \langle \Phi_j'(\bar{x}(t)), y(t) \rangle = 0 \quad \mu_j - \textit{a.e. in } [0,1]. \end{aligned}$$

Proof. Since $\langle \psi_j(1), y(1)\rangle = \int_0^1 y(t)\,\mathrm{d}\psi_j(t) + \int_0^1 \langle \dot y(t), \psi_j(t)\rangle\,\mathrm{d}t$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle p(t), y(t)\rangle &= -\langle (p(t)+\psi(t))f_x[t], y(t)\rangle + \langle p(t), f_x[t]y(t) + f_u[t]u(t)\rangle\\ &= -\langle \psi(t), f_x[t]y(t)\rangle + \langle p(t), f_u[t]u(t)\rangle = -\langle \psi(t), \dot y(t)\rangle + \langle p(t)+\psi(t), f_u[t]u(t)\rangle.\end{aligned}$$

Integrating the above relation yields

$$\int_0^1 \langle \mathcal{H}_u(\bar x(t), \bar u(t), p(t)+\psi(t)), u(t)\rangle\,\mathrm{d}t = \langle p(1)+\psi(1), y(1)\rangle - \sum_{j=1}^s \int_{[0,1]} y(t)d\psi_j(t).$$

Consequently, by (3.5),

$$\begin{aligned}\alpha_0\langle \varphi'(\bar x(1)), y(1)\rangle \quad &+ \textstyle\sum_{i=1}^r \alpha_i \langle g_i'(\bar x(1)), y(1)\rangle + \langle q, y(1)\rangle + \sum_{j=1}^s \int_{[0,1]} \langle \Phi_j'(\bar x(t)), y(t)\rangle d\mu_j(t)\\ &- \textstyle\int_0^1 \langle \mathcal{H}_u(\bar x(t), \bar u(t), p(t)+\psi(t)), u(t)\rangle\,\mathrm{d}t = 0.\end{aligned}$$

All summands in the left-hand-side of this equality being nonpositive, it follows that all of them are equal to zero. Finally, the conditions $\int_0^1 \langle \mathcal{H}_u(\bar x(t), \bar u(t), p(t)+\psi(t)), u(t)\rangle\,\mathrm{d}t = 0$ and $\langle \mathcal{H}_u(\bar x(t), \bar u(t), p(t)+\psi(t)), u(t)\rangle \geq 0$ a.e. complete the proof. $\square$

For any $u(\cdot) : [0,1] \to \mathbb{R}^m$ and any $(\alpha, q, p, \psi, \mu) \in \Lambda(\bar x, \bar u)$ and $t \in [0,1]$, define

$$\Upsilon(u(t), p(t)+\psi(t)) := \inf\Big\{\langle \mathcal{H}_u(\bar x(t), \bar u(t), p(t)+\psi(t)), v\rangle \ :\ v \in T_U^{\flat(2)}(\bar u(t), u(t))\Big\},$$

where, by convention, we set $\inf_\emptyset = +\infty$. For $\alpha \in \mathbb{R}^{r+1}$, $\mu \in M([0,1])$, $\widetilde p \in NBV([0,1], \mathbb{R}^n)$ and $\xi = (y,u) \in \Xi$ define the quadratic form with respect to ξ

$$\begin{aligned}\Omega(\xi, \alpha, \widetilde p, \mu) \quad &:= \alpha_0\langle \varphi''(\bar x(1))y(1), y(1)\rangle + \textstyle\sum_{i=1}^r \alpha_i \langle g_i''(\bar x(1))y(1), y(1)\rangle\\ &+ \textstyle\int_0^1 \langle \mathcal{H}''(\bar x(t), \bar u(t), \widetilde p(t))\xi(t), \xi(t)\rangle\,\mathrm{d}t + \sum_{j=1}^s \int_{[0,1]} \langle \Phi_j''(\bar x(t))\, y(t), y(t)\rangle\,\mathrm{d}\mu_j(t),\end{aligned}$$

where $\mathcal{H}''(\bar x(t), \bar u(t), \widetilde p(t))$ is the Hessian of $\mathcal{H}(\cdot,\cdot,\widetilde p(t))$ at $(\bar x(t), \bar u(t))$ and $\Phi_j''(\bar x(t))$ is the Hessian of Φ_j at $\bar x(t)$. Finally, let

$$V^2(\bar u, u) := \Big\{v : [0,1] \to \mathbb{R}^m \,|\, v \text{ is measurable, } f_u[\cdot]v(\cdot) \text{ is integrable, } v(t) \in T_U^{\flat(2)}(\bar u(t), u(t)) \text{ a.e.}\Big\}.$$

The following theorem contains our main second-order necessary condition.

Theorem 3.6. *Let $(\bar x, \bar u)$ be an L^1-local minimizer satisfying (3.3) and $\xi = (y,u) \in \mathcal{C}_0(\bar x, \bar u)$ be such that $V^2(\bar u, u)$ contains an essentially bounded function. Then for any nonempty convex subset $\Theta \subset T_K^{\flat(2)}(\bar x(1), y(1))$, there exists $(\alpha, q, p, \psi, \mu) \in \Lambda(\bar x, \bar u)$ such that $\Upsilon(u(\cdot), p(\cdot)+\psi(\cdot)) \in L^1([0,1], \mathbb{R})$ and*

$$\frac{1}{2}\Omega(\xi, \alpha, p+\psi, \mu) + \inf_{\theta\in\Theta}\langle -q, \theta\rangle + \int_0^1 \Upsilon(u(t), p(t)+\psi(t))\,\mathrm{d}t \geq 0. \tag{3.11}$$

Remark 3.7. (a) In the absence of state constraint (1.2), in the proof of Theorem 3.6 the constraint qualification (3.3) is not needed. However this case was already investigated in [7].

(b) The interested reader can find in [7, 8] several examples, where, for every $(y,u) \in \mathcal{C}_0(\bar x, \bar u)$, the condition $V^2(\bar u, u) \cap L^\infty([0,1], \mathbb{R}^m) \neq \emptyset$ is fulfilled. For instance, if U is a polytope, then $0 \in T_U^{\flat(2)}(\bar u(t), u(t))$. If U is given by the system of inequalities $\varphi_i(u) \leq 0$, $i = 1, \dots, q$ with C^2-functions $\varphi_i : \mathbb{R}^m \to \mathbb{R}$, having positively independent gradients of active constraints, then $V^2(\bar u, u)$ contains an essentially bounded function.

(c) We would like to underline that in Theorem 3.6 the set Θ is an arbitrary nonempty convex subset $T_K^{\flat(2)}(\bar x(1), y(1))$. Larger it is, more precise necessary optimality conditions are. Using the Zorn lemma it is possible to show that $T_K^{\flat(2)}(\bar x(1), y(1))$ contains maximal convex subsets.

(d) Though we assumed the twice continuous differentiability of Φ_j on $\mathbb{R}^n$, according to the proof provided below, this assumption is needed only on a neighborhood of ∂K_j. A similar remark can be made about functions g_i and the sets ∂K_i^e.

(e) Note that in [12], where U has a particular structure, only quadratic forms with respect to ξ are involved to state necessary conditions.

Example 3.8. This example illustrates the novelty of our result even in the absence of state constraints. Let $U = \{(a,0),(a,b) : a \in [0,1], |b| = a^2\}$, K and f be as in Example 3.3 with $f \in C^2(\mathbb{R}^2,\mathbb{R}^2)$. Consider the control system from Example 3.3 with this new U. Assume that $(1,0)$ is an eigenvector of $A := f_x(0)$ with a nonnegative eigenvalue. Then the solution of the system $z' = Az + (\frac{1}{2},0), z(0) = 0$ satisfies $z(t) \in C_K(0)$ for every $t \in [0,1]$. This and the inclusion $(\frac{1}{2},0) \in \operatorname{Int} \operatorname{co} U$ imply (3.3). Let $\varphi \in C^2(\mathbb{R}^2,\mathbb{R})$ be such that $0 \neq \nabla\varphi(0) \in \{0\} \times \mathbb{R}_-$ and $\langle \varphi''(0)(1,0),(1,0)\rangle = 0$. We claim that $\bar{u} = 0$ is not locally optimal for the optimal control problem (1.1), (1.2), (1.4). Indeed $(\bar{x},\bar{u}) \equiv 0$ and $(y, u = (1,0)) \in \mathcal{C}_0(\bar{x},\bar{u})$. If the conclusions of Theorem 3.6 hold true, then $\psi = 0, \mu = 0$ and for some $0 \neq (\alpha_0, q = (q_1,q_2)) \in \mathbb{R}_+ \times N_K(0)$, we have $p(1) = (p_1,p_2) = \alpha_0(0,b) + (0,q_2)$ for some $b < 0$, $q_2 \in \mathbb{R}$. Hence $p_1 = 0$. By the minimality condition, $p(t) \in \{(a,b) : a \in \mathbb{R}_+, |b| \le a\}$ for every $t \in [0,1]$ and therefore $p_2 = 0$ and $\alpha_0 b + q_2 = 0$. Hence $q = (0,r)$ for some $r > 0$ and $p(\cdot) \equiv 0$. Due to the choice of φ we have $\Omega((y,u),\alpha_0,p+\psi,\mu) = 0$, $\Upsilon(u,p(\cdot)+\psi(\cdot)) = 0$. Since $(0,1) \in T_K^{\flat(2)}(0,(1,0))$, from (2.1) we get $\Theta := \mathbb{R}_+ \times \{1\} = (0,1) + C_K(0) \subset T_K^{\flat(2)}(0,(1,0))$. Then $\inf_{\theta\in\Theta}\langle -q,\theta\rangle = -r < 0$ leading to a contradiction with (3.11). Therefore $\bar{u}$ is not a strong local minimizer.

Recall that if $\alpha_0 > 0$, then the multiplier rule in Theorem 3.6 is called normal and, by normalizing, one can put $\alpha_0 = 1$. To know that necessary conditions are normal is important, because otherwise they do not depend on the cost function. We propose next a sufficient condition for normality. Denote by f'' the Hessian of f. Fix a trajectory-control pair $\xi = (y,u)$ of (3.8). We associate with it a second-order linearization of (1.1) at $((\bar{x},\bar{u}),(y,u))$:

$$\begin{cases} \dot{w}(t) = f_x[t]w(t) + f_u[t]v(t) + \frac{1}{2}\xi(t)^* f''[t]\xi(t) + \pi(t)\kappa(t) \\ v(t) \in T_U^{\flat(2)}(\bar{u}(t),u(t)),\ \pi(t) \ge 0,\ \kappa(t) \in co\, f(\bar{x}(t),U) - f[t] \ \text{ a.e. in } [0,1] \\ w(0) = 0,\ v(\cdot) \in L^\infty([0,1],\mathbb{R}^m),\ \pi \in L^\infty([0,1],\mathbb{R}_+),\ \kappa \text{ is measurable.} \end{cases} \tag{3.12}$$

Below (w,v,κ,π) denotes a trajectory-controls quadruple of (3.12) and $R^{L(2)}(1)$ the reachable set of (3.12) at time 1. By the Aumann theorem it is convex. For every $z \in \mathbb{R}^n$ define

$$I(\bar{x}(1),z) := \{i \in I_g : g_i'(\bar{x}(1)) \neq 0,\ \langle g_i'(\bar{x}(1)),z\rangle = 0\}.$$

Theorem 3.9. *In Theorem 3.6 suppose that* $\min\{\langle g_i'(\bar{x}(1)),y(1)\rangle,\ y(1)^* g_i''(\bar{x}(1))y(1)\} < 0$ *for every* $i = 1,...,r$ *with* $\bar{x}(1)) \in \partial K_i^e$ *and that there exists a solution* w *of (3.12) satisfying*

$$\begin{aligned} &\max_{t\in\mathcal{M}_{j0}} \Big(\langle \Phi_j'(\bar{x}(t)),w(t)\rangle + \tfrac{1}{2}\langle \Phi_j''(\bar{x}(t))y(t),y(t)\rangle\Big) < 0,\ \forall\, j = 1,...,s, \\ &\qquad w(1) \in \Theta,\ \ \langle g_i'(\bar{x}(1)),w(1)\rangle + \tfrac{1}{2}y(1)^* g_i''(\bar{x}(1))y(1) < 0\ \ \forall\, i \in I(\bar{x}(1),y(1)). \end{aligned}$$

Then the conclusion of Theorem 3.6 is valid with $\alpha_0 = 1$.

Example 3.10. Let $h = (h_1,...,h_k) : \mathbb{R}^n \to \mathbb{R}^k$ be continuously differentiable and $\bar{x}(1) \in K$ with K given by (1.3). If the derivative $h'(\bar{x}(1))$ is surjective (that is the matrix $h'(\bar{x}(1))$ has a full rank), then by [1, p.151], $C_K(\bar{x}(1)) = \ker h'(\bar{x}(1))$ and therefore for any $q \in N_K(\bar{x}(1))$ there exists $\beta = (\beta_1,...,\beta_k) \in \mathbb{R}^k$ such that $q = \sum_{i=1}^k \beta_i h_i'(\bar{x}(1))$. Furthermore, by [1, Proposition 4.7.5] if (y,u) is critical and $h \in C^2$, then

$$T_K^{\flat(2)}(\bar{x}(1),y(1)) = \{v\ :\ \langle h_i'(\bar{x}(1)),v\rangle + \frac{1}{2}y(1)^* h_i''(\bar{x}(1))y(1) = 0,\ \forall\, i = 1,...,k\}. \tag{3.13}$$

Note that if $h'(\bar{x}(1))$ is not surjective, then $q := h'(\bar{x}(1))^*\beta = 0$ for some $0 \neq \beta \in \mathbb{R}^k$. Setting $(\alpha, p, \psi, \mu) = 0$ one could claim, as in [12], that the first and second-order necessary optimality conditions are verified with $\beta \neq 0$ (instead of $q \neq 0$). However such necessary conditions do not distinguish at all between optimal and non-optimal controls. To link Theorem 3.6 to the frequently considered in the literature final point equality constraints, i.e. when K is as in (1.3) with $h_j \in C^2$, we provide next two Corollaries that are immediate consequences of Theorem 3.6.

Corollary 3.11. *Assume (1.3), that $h \in C^2$ and let $(\bar{x}, \bar{u})$ be an L^1-local minimizer satisfying*

$$0 \in \operatorname{Int}\left(h'(\bar{x}(1))(R^L(1))\right). \tag{3.14}$$

If $\xi = (y, u) \in \mathcal{C}_0(\bar{x}, \bar{u})$ is such that $V^2(\bar{u}, u)$ contains an essentially bounded function, then there exists $(\alpha, q, p, \psi, \mu) \in \Lambda(\bar{x}, \bar{u})$ with $q := \sum_{i=1}^k \beta_i h_i'(\bar{x}(1))$ and $\beta_i \in \mathbb{R}$ such that the function $\Upsilon(u(\cdot), p(\cdot) + \psi(\cdot))$ is integrable and

$$\frac{1}{2}\Omega(\xi, \alpha, p + \psi, \mu) + \sum_{i=1}^k \frac{\beta_i}{2} y(1)^* h_i''(\bar{x}(1)) y(1) + \int_0^1 \Upsilon(u(t), p(t) + \psi(t))\, \mathrm{d}t \geq 0. \tag{3.15}$$

Condition (3.14) was introduced in [5]. For the sake of completeness we also provide a result not involving (3.14), but with a weaker conclusion involving quadruples (w, v, κ, π) of (3.12).

Corollary 3.12. *Assume (1.3), that $h \in C^2$ and let $(\bar{x}, \bar{u})$ be an L^1-local minimizer. If $\xi = (y, u) \in \mathcal{C}_0(\bar{x}, \bar{u})$ is such that $V^2(\bar{u}, u)$ contains an essentially bounded function, then there exists $(\alpha, q, p, \psi, \mu) \in \Lambda(\bar{x}, \bar{u})$ with $q := \sum_{i=1}^k \beta_i h_i'(\bar{x}(1))$ and $\beta_i \in \mathbb{R}$ such that for any trajectory-controls quadruple (w, v, κ, π) of (3.12) satisfying*

$$\langle h_i'(\bar{x}(1)), w(1)\rangle + \frac{1}{2} y(1)^* h_i''(\bar{x}(1)) y(1) = 0 \quad \forall\, i = 1, ..., k \tag{3.16}$$

and for $[t] := (\bar{x}(t), \bar{u}(t), p(t) + \psi(t))$ we have

$$\frac{1}{2}\Omega(\xi, \alpha, p + \psi, \mu) + \sum_{i=1}^k \frac{\beta_i}{2} y(1)^* h_i''(\bar{x}(1)) y(1) + \int_0^1 \left(\langle \mathcal{H}_u[t], v(t)\rangle + \langle p(t) + \psi(t), \pi(t)\kappa(t)\rangle\right) \mathrm{d}t \geq 0.$$

Proofs of Theorems 3.6, 3.9 and Corollaries 3.11, 3.12 are provided in Section 5.

Remark 3.13. In Example 3.8 consider any $h \in C^2(\mathbb{R}^2, \mathbb{R})$ such that $K = h^{-1}(0) \cap \mathbb{R}_+ \times \mathbb{R}$. That is K is represented by an equality and an inequality. Then using the second order tangents, it is not difficult to check that $\nabla h(0) = 0$ and $\langle h''(0)(1, 0), (1, 0)\rangle = 0$. Hence (3.15) is satisfied with $p = \psi = 0$, $\alpha = 0$, $\beta = 1$. This means that when K is described via an equality and an inequality constraints, (3.15) does not allow to eliminate $\bar{u}$ as a candidate for optimality.

4. Second Order Tangents to Trajectories of (1.1), (1.2)

In this section we show that for any trajectory-control pair $\xi = (y, u)$ of (3.8) and trajectory-control quadruple (w, v, κ, π) of (3.12) such that $w(1) \in T_K^{\flat(2)}(\bar{x}(1), y(1))$, the function $w(\cdot)$ is in the second order tangent to trajectories of (1.1), (1.2) at $((\bar{x}, \bar{u}), (y, u))$.

Theorem 4.1. *Let $(\bar{x}, \bar{u})$ be a trajectory-control pair with $\bar{x}(1) \in K$ satisfying (3.3). Consider a trajectory-control pair $\xi = (y, u)$ of (3.8) and a trajectory-controls quadruple (w, v, κ, π) of (3.12) such that $w(1) \in T_K^{\flat(2)}(\bar{x}(1), y(1))$. Then there exists $C \geq 0$ such that for every small $\delta > 0$ we can find a trajectory-control pair (x_δ, u_δ) satisfying*

$$x_\delta(1) \in K, \quad \|u_\delta - \bar{u}\|_1 \leq C\delta, \quad \lim_{\delta \to 0+} \frac{\|x_\delta - \bar{x} - \delta y - \delta^2 w\|_\infty}{\delta^2} = 0.$$

Proof. By [9, Proposition 4.2] we know that there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0]$ we can find $u_\delta \in \mathcal{U}$ such that the difference quotients $v_\delta := \frac{u_\delta - \bar{u} - \delta u}{\delta^2}$ converge to v a.e. as $\delta \to 0+$ and $\|v_\delta\|_\infty \le 2\|v\|_\infty + c$, with c as in (3.9). Then $\|u_\delta - \bar{u}\|_1 \le C\delta$ for some $C \ge 0$ and all small $\delta > 0$. By [1, Theorem 8.2.15] there exist measurable $\lambda_i : [0,1] \to \mathbb{R}_+,\ u_i : [0,1] \to U$ for $i = 0, ..., n$ such that $\sum_{i=0}^n \lambda_i(t) = 1$ and $\kappa(t) = \sum_{i=0}^n \lambda_i(t)(f(\bar{x}(t), u_i(t)) - f[t])$ a.e. in $[0,1]$.

Consider the augmented control system

$$\left\{ \begin{array}{lcll} \dot{x} & = & f(x, u(t)),\ u(t) \in U, & x(0) = x_0 \\ \dot{z} & = & |u(t) - u_\delta(t)|, & z(0) = 0. \end{array} \right. \tag{4.1}$$

Define $\sigma(t) = \int_0^t \pi(s) \sum_{i=0}^n \lambda_i(s)|u_i(s) - u_\delta(s)|ds$ and denote by $(\hat{x}_\delta, z_\delta)$ the solution of

$$\left\{ \begin{array}{lcll} \dot{x} & = & f(x, u_\delta) + \delta^2\pi(t) \sum_{i=0}^n \lambda_i(t)(f(x, u_i(t)) - f(x, u_\delta(t))), & x(0) = x_0 \\ \dot{z} & = & \delta^2\pi(t) \sum_{i=0}^n \lambda_i(t)|u_i(t) - u_\delta(t)|, & z(0) = 0. \end{array} \right. \tag{4.2}$$

Since that $\delta^2\pi(t) \le 1$ whenever δ is sufficiently small, the pair $(\hat{x}_\delta, z_\delta)$ solves the relaxed (convexified) system (4.2) associated to (4.1) for all small $\delta > 0$. By the relaxation theorem, see for instance [13], we can find controls $u^j(\cdot) \in \mathcal{U}$ such that the corresponding trajectories (x^j, z^j) of (4.1) converge uniformly to $(\hat{x}_\delta, \delta^2\sigma)$. This implies that $\lim_{j\to\infty} \|u^j - u_\delta\|_1 = \delta^2\sigma(1)$ and therefore there exist trajectory-control pairs $(\bar{x}_\delta, \bar{u}_\delta)$ of (1.1) such that for all small $\delta > 0$,

$$\|\bar{x}_\delta - \hat{x}_\delta\|_\infty < \delta^3 \quad \& \quad \|\bar{u}_\delta - u_\delta\|_1 < k\delta^2$$

for a constant $k > 0$ independent from δ. Since

$$\hat{x}_\delta(t) - \bar{x}(t) - \delta y(t) = \int_0^t (f(\hat{x}_\delta(s), u_\delta(s)) - f[s] - \delta f_x[s]y(s) - \delta f_u[s]u(s))ds + o_t(\delta),$$

where $\int_0^1 |o_t(\delta)|\, \mathrm{d}t = o(\delta)$ and $f(\hat{x}_\delta(s), u_\delta(s)) = f[s] + f_x[s](\hat{x}_\delta(s) - \bar{x}(s)) + \delta f_u[s]u(s) + o_\delta(s)$, with $\|o_\delta\|_1 = o(\delta)$, for every $t \in [0,1]$ it holds

$$|\hat{x}_\delta(t) - \bar{x}(t) - \delta y(t)| \le \int_0^t (|f_x[s]||\hat{x}_\delta(s) - \bar{x}(s) - \delta y(s)|)ds + o(\delta),$$

where $o(\delta)$ does not depend on time. This and the Gronwall inequality imply that $\frac{1}{\delta}(\hat{x}_\delta - \bar{x})$ converge to y uniformly on $[0,1]$. Let $\beta(t) := |\hat{x}_\delta(t) - \bar{x}(t) - \delta y(t) - \delta^2 w(t)|$ and observe that

$$\begin{array}{ll} \beta(t) & \le \int_0^t |f(\hat{x}_\delta(s), u_\delta(s)) + \delta^2\pi(s) \sum_{i=0}^n \lambda_i(s)(f(\hat{x}_\delta(s), u_i(s)) - f(\hat{x}_\delta(s), u_\delta(s))) - f[s] \\ & -\delta f_x[s](y(s) + \delta w(s)) - \delta f_u[s](u(s) + \delta v(s)) - \frac{\delta^2}{2}\xi(s)^* f''[s]\xi(s) - \delta^2\pi(s)\kappa(s)|ds \end{array}$$

for every $t \in [0,1]$. On the other hand,

$$\begin{array}{l} f(\hat{x}_\delta(s), u_\delta(s)) = f[s] + f_x[s](\hat{x}_\delta(s) - \bar{x}(s)) + \delta f_u[s](u(s) + \delta v_\delta(s)) \\ \qquad + \frac{1}{2}(\hat{x}_\delta(s) - \bar{x}(s), (\delta u + \delta^2 v_\delta)(s))^* f''[s](\hat{x}_\delta(s) - \bar{x}(s), (\delta u + \delta^2 v_\delta)(s)) + o_\delta(s) \\ \qquad = f[s] + f_x[s](\hat{x}_\delta(s) - \bar{x}(s)) + \delta f_u[s](u(s) + \delta v(s)) + \frac{\delta^2}{2}\xi(s)^* f''[s]\xi(s) + o_\delta(s), \end{array}$$

where $\|o_\delta(\cdot)\|_1 = o(\delta^2)$. Combining two above expressions, we get $\beta(t) \le \int_0^t |f_x[s]|\beta(s)ds + o(\delta^2)$. By the Gronwall inequality $\frac{1}{\delta^2}(\hat{x}_\delta - \bar{x} - \delta y)$ converge to w uniformly on $[0,1]$ and therefore $\frac{1}{\delta^2}(\bar{x}_\delta - \bar{x} - \delta y)$ converge to w uniformly on $[0,1]$. By the choice of y, w there exist $z_\delta \to w(1)$ when $\delta \to 0+$ such that $k_\delta := \bar{x}(1) + \delta y(1) + \delta^2 z_\delta \in K$. This yields $|\bar{x}_\delta(1) - \bar{x}(1) - \delta y(1) - \delta^2 z_\delta| = o(\delta^2)$. Since $\lim_{\delta\to 0+} \|\bar{u}_\delta - \bar{u}\|_1 = 0$, from Theorem 3.1 we deduce the existence of controls $\widetilde{u}_\delta$ such that $\|\widetilde{u}_\delta - \bar{u}_\delta\|_1 = o(\delta^2)$ and the corresponding trajectories $\widetilde{x}_\delta$ of (1.1) satisfy $\widetilde{x}_\delta(1) \in K$.

By the Lipschitz continuity of f for a constant $c_1 > 0$ independent from δ we have $\|\widetilde{x}_\delta - \bar{x}_\delta\|_\infty \le c_1\|\widetilde{u}_\delta - \bar{u}_\delta\|_1 = o(\delta^2)$ and the result follows with (x_δ, u_δ) equal to $(\widetilde{x}_\delta, \widetilde{u}_\delta)$.

5. Proofs of the Main Results

It is not difficult to realise that under assumptions of Theorem 3.1 to every control $u \in \mathcal{U}$ corresponds a unique solution $x_u(\cdot)$ of control system (1.1) defined on $[0,1]$ and that the mapping $\mathcal{U} \ni u \mapsto x_u \in C([0,1],\mathbb{R}^n)$ is continuous.

Theorem 3.1 is an inverse function like theorem. But, because controls belong to the metric space $\mathcal{U}$, it is not possible to differentiate the end-point map $\mathcal{U} \ni u \to x_u(1)$ and to use a classical inverse function theorem. Instead we replace derivatives by variations. Observe that $\mathcal{U} \times K$ is a complete metric space with the metric $D((u_1,k_1),(u_2,k_2)) = \|u_1 - u_2\|_1 + |k_1 - k_2|$. Define the continuous mapping $G : \mathcal{U} \times K \to \mathbb{R}^n$ by $G(u,k) := x_u(1) - k$.

For every $u \in \mathcal{U}, k \in K$, the first-order contingent variation of G at (u,k) is defined by

$$G^{(1)}(u,k) := \text{Limsup}_{\delta \to 0+} \frac{G(B_\delta(u,k)) - G(u,k)}{\delta},$$

where $B_\delta(u,k)$ denotes the closed ball in $\mathcal{U} \times K$ centered at (u,k) of radius $\delta > 0$.

Though the whole set of variations is difficult to compute, for our purposes we only need a subset of variations that can be expressed via the reachable set $R^L(1)$.

Lemma 5.1. *Consider a trajectory-control pair $(\bar{x},\bar{u})$ with $\bar{x}(1) \in K$. Under assumptions of Theorem 3.1 there exist $\epsilon > 0$, $\rho > 0$ such that $\rho B \subset \overline{co}\, G^{(1)}(u,k)$ for every $(u,k) \in \mathcal{U} \times K$ satisfying $\|u - \bar{u}\|_1 + |k - \bar{x}(1)| < \epsilon$.*

The proof is postponed to the appendix.

Proof of Theorem 3.1. The proof follows immediately from Lemma 5.1 and the inverse mapping theorem [4, Theorem 3.2] applied on the metric space $\mathcal{U} \times K$.

Proof of Theorem 3.6. Fix $\xi = (y,u) \in \Xi$ as in Theorem 3.6 and define $g_0 = \varphi$.

Step 1. If for some $i = 1,...,r$, $\bar{x}(1) \notin \partial K_i^e$, then this constraint can be neglected and in the final result we set $\alpha_i = 0$. From now on we assume that every i is active.

For every $i = 0,...,r$ define $Q_i := \mathbb{R}^n$ if $\langle g_i'(\bar{x}(1)), y(1)\rangle < 0$ and

$$Q_i := \{\eta \in \mathbb{R}^n : \ \langle g_i'(\bar{x}(1)), \eta\rangle + \frac{1}{2}\langle g_i''(\bar{x}(1))y(1), y(1)\rangle < 0\} \ \text{ if } \ \langle g_i'(\bar{x}(1)), y(1)\rangle = 0.$$

Then Q_i is open and convex for $i = 0,...,r$. If there exists $i = 0,...,r$ such that $Q_i = \emptyset$, then $g_i'(\bar{x}(1)) = 0$ and $y(1)^* g_i''(\bar{x}(1))y(1) \geq 0$. Set $\alpha_i = 1$ and $\alpha_j = 0$ whenever $j \neq i$. Then the claim of the theorem is verified with $(p,q,\psi,\mu) = 0$.

From now on we assume that $Q_i \neq \emptyset$ for every $i = 0,...,r$.

Step 2. For any $j = 1,\ldots,s$, consider the open convex subset of $C([0,1]),\mathbb{R}^n)$

$$\mathcal{F}_j = \Big\{ w \in C([0,1]),\mathbb{R}^n) : \max_{t \in \mathcal{M}_{j0}} \Big(\langle \Phi_j'(\bar{x}(t)), w(t)\rangle + \frac{1}{2}\langle \Phi_j''(\bar{x}(t))y(t), y(t)\rangle \Big) < 0 \Big\}.$$

If for some $j = 1,...,s$, $\bar{x}(t) \notin \partial K_j$ for every $t \in [0,1]$, then the constraint $\Phi_j(x(t)) \leq 0$ can be neglected and in the final result we set $\mu_j = 0$. If for some j, the set $\mathcal{F}_j = \emptyset$ and $\mathcal{M}_{j0} \neq \emptyset$, then taking $w(\cdot) = -k\Phi_j'(\bar{x}(\cdot))$, for every positive integer k we can find $t_k \in \mathcal{M}_{j0}$ such that

$$-k|\Phi_j'(\bar{x}(t_k))|^2 + \frac{1}{2}\langle \Phi_j''(\bar{x}(t_k))y(t_k), y(t_k)\rangle \geq 0.$$

Let $t \in \mathcal{M}_{j0}$ be the limit of a subsequence of t_k. Since $\frac{1}{2}\langle \Phi_j''(\bar{x}(t_k))y(t_k), y(t_k)\rangle \geq k|\Phi_j'(\bar{x}(t_k))|^2$, we deduce that $\Phi_j'(\bar{x}(t)) = 0$ and $\langle \Phi_j''(\bar{x}(t))y(t), y(t)\rangle \geq 0$. Define $\mu_j = \delta_t$ (the Dirac measure) and set $\mu_i = 0$ for $i \neq j$. Then all the conclusions of our theorem are valid with $(\alpha,q,p,\psi) = 0$.

From now on we assume that $\mathcal{F}_j \neq \emptyset$ for every $j = 1,...,s$.

Step 3. Define $\Gamma : C([0,1],\mathbb{R}^n) \to \mathbb{R}^n$ by $\Gamma(w) = w(1)$. Since Γ is surjective, its adjoint Γ^* is injective. By (2.1), $\Theta_1 := \Theta + C_K(\bar{x}(1))$ is a convex subset of $T_K^{\flat(2)}(\bar{x}(1), y(1))$. We claim that

$$\langle \varphi'(\bar{x}(1)), y(1)\rangle = 0, \quad \langle \varphi'(\bar{x}(1)), w(1)\rangle + \frac{1}{2}\langle \varphi''(\bar{x}(1))y(1), y(1)\rangle \geq 0 \tag{5.1}$$

for any $w(1) \in (\bigcap_{i=1}^r Q_i) \cap \Big(\bigcap_{j=1}^s \Gamma(\mathcal{F}_j)\Big) \cap \Theta_1 \cap R^{L(2)}(1)$. Indeed, fix $w(1)$ in this intersection. By Theorem 4.1, for every $\delta > 0$ there exists a trajectory-control pair (x_δ, u_δ) such that $x_\delta(1) \in K$, $\lim_{\delta\to 0+} \|u_\delta - \bar{u}\|_1 = 0$ and $\lim_{\delta\to 0+} \|r_\delta\|_\infty = 0$, where $r_\delta := (x_\delta - \bar{x} - \delta y - \delta^2 w)/\delta^2$. Since

$$g_i(x_\delta(1)) = g_i(\bar{x}(1)) + \delta\langle g_i'(\bar{x}(1)), y(1)\rangle + \delta^2\Big(\langle g_i'(\bar{x}(1)), w(1)\rangle + \frac{1}{2}\langle g_i''(\bar{x}(1))y(1), y(1)\rangle\Big) + o(\delta^2),$$

by the definition of Q_i for all $\delta > 0$ sufficiently small $g_i(x_\delta(1)) < 0$ for every $i = 1, ..., r$. Similarly to [7, pp. 2368-2369] we verify that $\max_{t\in[0,1]} \Phi_j(x_\delta(t)) \leq 0$ for all small $\delta > 0$ and any $j = 1, \ldots, s$. Therefore (x_δ, u_δ) is admissible whenever $\delta > 0$ is small. Furthermore,

$$\varphi(x_\delta(1)) = \varphi(\bar{x}(1)) + \delta\langle \varphi'(\bar{x}(1)), y(1)\rangle + \delta^2(\langle \varphi'(\bar{x}(1)), w(1)\rangle + \frac{1}{2}\langle \varphi''(\bar{x}(1))y(1), y(1)\rangle) + o(\delta^2).$$

But $\varphi(\bar{x}(1)) \leq \varphi(x_\delta(1))$ for all small $\delta > 0$. Since (y, u) is critical, we deduce (5.1).

Step 4. Observe next that if for some $i = 1, ..., r$ we have $g_i'(\bar{x}(1)) = 0$ or $\langle g_i'(\bar{x}(1)), y(1)\rangle < 0$, then $Q_i = \mathbb{R}^n$. From Step 3 we deduce that

$$Q_0 \cap \left(\bigcap_{i\in I(\bar{x}(1),y(1))} Q_i\right) \cap \left(\bigcap_{j=1}^s \Gamma(\mathcal{F}_j)\right) \cap \Theta_1 \cap R^{L(2)}(1) = \emptyset.$$

By Proposition 2.1 there exist $\zeta_i^* \in (\mathbb{R}^n)^*$ for $i \in \{0\} \cup I(\bar{x}(1), y(1))$, $\eta_j^* \in (\mathbb{R}^n)^*$ for $j = 1, ..., s$ and ζ^*, $p_1^* \in (\mathbb{R}^n)^*$ not vanishing simultaneously such that

$$\zeta_0^* + \sum_{i\in I(\bar{x}(1),y(1))} \zeta_i^* + \sum_{j=1}^s \eta_j^* + \zeta^* + p_1^* = 0, \tag{5.2}$$

$$\sum_{i\in\{0\}\cup I(\bar{x}(1),y(1))} \inf \zeta_i^*(Q_i) + \sum_{j=1}^s \inf \eta_j^*(\Gamma(\mathcal{F}_j)) + \inf \zeta^*(\Theta_1) + \inf p_1^*\left(R^{L(2)}(1)\right) \geq 0. \tag{5.3}$$

If $\varphi'(\bar{x}(1)) = 0$, then $\zeta_0^* = 0$ and set $\alpha_0 = 0$. Then $\inf \zeta_0^*(Q_0) = \frac{\alpha_0}{2} y(1)^*\varphi''(\bar{x}(1))y(1)$. This and Proposition 2.2 imply that for every $i \in \{0\} \cup I(\bar{x}(1), y(1))$ we have

$$\zeta_i^* = -\alpha_i g_i'(\bar{x}(1))^*, \quad \inf \zeta_i^*(Q_i) = \frac{\alpha_i}{2} y(1)^* g_i''(\bar{x}(1))y(1)$$

with some $\alpha_i \geq 0$. For any $i \in \{1, ..., r\}\backslash I(\bar{x}(1), y(1))$ define $\alpha_i = 0$.

From (5.3) we deduce that $\inf \zeta^*(\Theta + C_K(\bar{x}(1))) > -\infty$. Therefore $-\zeta \in N_K(\bar{x}(1))$. Set $q = -\zeta$. Clearly $\inf \zeta^*(\Theta_1) = \inf \zeta^*(\Theta)$ and therefore in (5.3) we may replace Θ_1 by Θ.

Step 5. Fix j. Define $X = C([0,1],\mathbb{R}^n)$, $x^* = \Gamma^*\eta_j^*$, $y_0 = \frac{1}{2}\langle \Phi_j''(\bar{x}(\cdot))y(\cdot), y(\cdot)\rangle \in X$, the linear operator $X \ni w(\cdot) \to A(w(\cdot)) := \Phi_j'(\bar{x}(\cdot))w(\cdot)$ and the convex continuous function $F(w) = \max_{t\in\mathcal{M}_{j0}}((Aw)(t) + y_0(t))$ for $w \in X$. Denote by F^* the Fenchel conjugate of F. Since (5.3) yields $\inf x^*(\mathcal{F}_j) > -\infty$, by [7, Lemma 2.3] there exist $x_1^* \in \operatorname{Dom} F^*$ and $\beta \geq 0$ such that $x^* = -\beta x_1^*$ and $\inf x^*(\mathcal{F}_j) = -\beta F^*(x_1^*)$. Consider the convex positively homogeneous function $\phi(z) = \max_{t\in\mathcal{M}_{j0}} z(t)$ for $z \in C([0,1],\mathbb{R})$. By [7, Lemma 2.5], there exists $x_2^* \in \partial\phi(0)$ such

that $x_1^* = A^* x_2^*$ and $F^*(x_1^*) = -x_2^*(y_0)$. Finally, [7, Lemma 2.6] implies that x_2^* is given by a probability measure $\widetilde{\mu}_j$ with the support contained in $\mathcal{M}_{j0}$. Setting $\mu_j = \beta\widetilde{\mu}_j$ we deduce that

$$\inf x^*(\mathcal{F}_j) = \beta x_2^*(y_0) = \frac{1}{2}\int_{[0,1]} \langle \Phi_j''(\bar{x}(t))\, y(t), y(t)\rangle \,\mathrm{d}\mu_j(t).$$

Define ψ_j, $\psi \in NBV([0,1],\mathbb{R}^n)$ by (3.5). Using that $\Gamma^*\eta_j^*(w) = -\beta x_1^*(w) = -\beta A^* x_2^*(w) = -\beta x_2^*(Aw) = \int_{[0,1]} w(t)\,\mathrm{d}\psi_j(t)$ for every $w \in X$, we finally obtain $\eta_j^*(w(1)) = \int_0^1 w(t)\,\mathrm{d}\psi_j(t)$.

Step 6. From (5.3) it follows that $\inf p_1^*(R^{L(2)}(1))$ is bounded from below. Fix any trajectory w of (3.12) and observe that for any trajectory-control pair (z,v) of (3.2) and any $\lambda \geq 0$ we have $w(1) + \lambda z(1) \in R^{L(2)}(1)$. Therefore $\langle p_1, z(1)\rangle \geq 0$. Consider the solution $p(\cdot)$ of the adjoint system (3.6) with α, q, ψ as defined above. Since by (5.2)

$$p_1^* = \alpha_0 \varphi'(\bar{x}(1))^* + \sum_{i=1}^r \alpha_i g_i'(\bar{x}(1))^* - \sum_{j=1}^s \eta_j^* + q^*,$$

we deduce from (5.3) that for every trajectory-control pair (z,v) of (3.2),

$$\alpha_0 \varphi'(\bar{x}(1))^*(z(1)) + \sum_{i=1}^r \alpha_i g_i'(\bar{x}(1))^*(z(1)) - \sum_{j=1}^s \int_{[0,1]} z(t)\,\mathrm{d}\psi_j(t) + \langle q, z(1)\rangle \geq 0. \tag{5.4}$$

Thus from the equality $\langle \psi_j(1), z(1)\rangle = \int_{[0,1]} z(t)\,\mathrm{d}\psi_j(t) + \int_0^1 \langle \dot{z}(t), \psi_j(t)\rangle\,\mathrm{d}t$ it follows that

$$-\sum_{j=1}^s \int_{[0,1]} z(t)\,\mathrm{d}\psi_j(t) = \int_0^1 \langle \psi(t), \dot{z}(t)\rangle\,\mathrm{d}t - \langle \psi(1), z(1)\rangle.$$

Using (3.6), from (5.4) we obtain $\langle p(1), z(1)\rangle + \int_0^1 \langle \psi(t), \dot{z}(t)\rangle\,\mathrm{d}t \geq 0$ and therefore

$$0 \leq \int_0^1 (\langle \dot{p}(t), z(t)\rangle + \langle p(t), \dot{z}(t)\rangle)\,\mathrm{d}t + \int_0^1 \langle \psi(t), \dot{z}(t)\rangle\,\mathrm{d}t = \int_0^1 \langle p(t) + \psi(t), v(t)\rangle)\,\mathrm{d}t.$$

The measurable selection theorem implies that $(\alpha, q, p, \psi, \mu) \in \Lambda(\bar{x}, \bar{u})$.

Step 7. Consider a trajectory-controls quadruple $(w, v, 0, 0)$ of (3.12). Using results of the above calculations in inequality (5.3) we deduce that

$$\begin{aligned} &\textstyle\sum_{i\in\{0\}\cup I(\bar{x}(1),y(1))} \frac{\alpha_i}{2} y(1)^* g_i''(\bar{x}(1)) y(1) + \inf_{\theta\in\Theta}\langle -q, \theta\rangle + \\ &\qquad \textstyle\frac{1}{2}\sum_{j=1}^s \int_{[0,1]} \langle \Phi_j''(\bar{x}(t))\, y(t), y(t)\rangle\,\mathrm{d}\mu_j(t) + p_1^*(w(1)) \geq 0. \end{aligned} \tag{5.5}$$

On the other hand,

$$\begin{aligned} p_1^*(w(1)) &= \langle p(1), w(1)\rangle + \langle \psi(1), w(1)\rangle - \textstyle\sum_{j=1}^s \int_0^1 w(t)\,\mathrm{d}\psi_j(t) = \langle p(1), w(1)\rangle + \int_0^1 \langle \psi(t), \dot{w}(t)\rangle\,\mathrm{d}t \\ &= -\textstyle\int_0^1 \langle (p(t) + \psi(t)) f_x[t], w(t)\rangle\,\mathrm{d}t + \int_0^1 \langle p(t) + \psi(t), \dot{w}(t)\rangle\,\mathrm{d}t \\ &= \textstyle\int_0^1 \langle p(t) + \psi(t)), f_u[t] v(t) + \frac{1}{2}\xi(t)^* f''[t]\xi(t)\rangle\,\mathrm{d}t. \end{aligned}$$

This and (5.5) yield

$$\frac{1}{2}\Omega(\xi, \alpha, p + \psi, \mu) + \inf_{\theta\in\Theta}\langle -q, \theta\rangle + \int_0^1 \langle \mathcal{H}_u[t], v(t)\rangle\,\mathrm{d}t \geq 0,$$

where $[t] := (\bar{x}(t), \bar{u}(t), p(t) + \psi(t))$. Observe that for every $\widetilde{v} \in V^2(\bar{u}, u)$ there exist essentially bounded $v_i \in V^2(\bar{u}, u)$ such that $\langle \mathcal{H}_u[\cdot], v_i(\cdot)\rangle$ converge to $\langle \mathcal{H}_u[\cdot], \widetilde{v}(\cdot)\rangle$ in $L^1([0,1],\mathbb{R})$. Thus

$$\frac{1}{2}\Omega(\xi, \alpha, p + \psi, \mu) + \inf_{\theta\in\Theta}\langle -q, \theta\rangle + \inf_{v\in V^2(\bar{u},u)} \int_0^1 \langle \mathcal{H}_u[t], v(t)\rangle\,\mathrm{d}t \geq 0.$$

The conclusion (3.11) follows from the measurable selection theorem as in [8, end of Section 5].

Proof of Theorem 3.9. By Step 1 of the proof of Theorem 3.6, if $Q_0 = \emptyset$, then the conclusion of Theorem 3.6 holds with $\alpha_0 = 1$. Assume next that $Q_0 \neq \emptyset$. Let $w(\cdot)$ be as in our assumptions. Hence $Q_i \neq \emptyset$ for all $i = 1, ..., r$ and $\mathcal{F}_j \neq \emptyset$ for $j = 1, ..., s$. By Step 4 of the proof of Theorem 3.6 we have only to show that $\zeta_0^* \neq 0$. Indeed, otherwise, by (5.3),

$$\sum_{i\in I(\bar{x}(1),y(1))} \inf \zeta_i^*(Q_i - w(1)) + \sum_{j=1}^{s} \inf \eta_j^*(\Gamma(\mathcal{F}_j) - w(1)) + \inf \zeta^*(\Theta_1 - w(1)) \geq 0.$$

This and the choice of $w(1)$ yield $\zeta_i^* = 0$ for every $i \in I(\bar{x}(1), y(1))$ and $\eta_j^* = 0$ for every $j = 1, ..., s$. Consequently, by (5.2), $p_1 = -\zeta \in N_K(\bar{x}(1))$. This and (5.3) imply that

$$\inf p_1^* \left(R^{L(2)}(1)\right) = \inf_{z\in R^{L(2)}} \langle -\zeta, z\rangle > -\infty.$$

Since $R^{L(2)}(1) + \lambda R^L(1) \subset R^{L(2)}(1)$ for any $\lambda \geq 0$ we obtain $\sup \zeta(R^L(1)) \leq 0$ contradicting (3.3).

Proof of Corollary 3.11. By (3.14) $h'(\bar{x}(1))$ is surjective. This and the separation theorem imply (3.3). Let $(\alpha, q, p, \psi, \mu)$ be as in the conclusions of Theorem 3.6 for $\Theta = T_K^{\flat(2)}(\bar{x}(1), y(1))$, see (3.13). Then $q = \sum_{i=1}^k \beta_i h_i'(\bar{x}(1))$ for some $\beta \in \mathbb{R}^k$. On the other hand, for any $v \in T_K^{\flat(2)}(\bar{x}(1), y(1))$ we have $\sum_{i=1}^k \beta_i \langle h_i'(\bar{x}(1)), v\rangle + \frac{1}{2}\sum_{i=1}^k \beta_i \langle h_i''(\bar{x}(1))y(1), y(1)\rangle = 0$. Theorem 3.6 implies the result.

Proof of Corollary 3.12. If (3.14) is verified, then, as in the proof of Corollary 3.11, (3.3) holds true. Let $(\alpha, q, p, \psi, \mu)$ be as in the proof of Theorem 3.6. Then $q = \sum_{i=1}^k \beta_i h_i'(\bar{x}(1))$ for some $\beta \in \mathbb{R}^k$ and $\langle -q, v\rangle = \frac{1}{2}\sum_{i=1}^k \beta_i \langle h_i''(\bar{x}(1))y(1), y(1)\rangle$ for any $v \in T_K^{\flat(2)}(\bar{x}(1), y(1))$. Pick any trajectory-controls quadruple (w, v, κ, π) of (3.12) (even not necessarily satisfying (3.16)). The same arguments as in Step 7 of the proof of Theorem 3.6 imply the inequality of Corollary 3.12 in this case. (In Step 7 we have set $\kappa = 0$, $\pi = 0$ because, by the minimum principle (3.7), this extra term would not have any impact on the final results).

Assume next that 0 is the boundary point of the compact set $h'(\bar{x}(1))(R^L(1))$. Then, by the separation theorem, there exists $q_0 \in \mathbb{R}^k$ different from zero such that

$$\min_{z\in R^L(1)} \langle q_0, h'(\bar{x}(1))z\rangle \geq 0. \tag{5.6}$$

Set $(\alpha, \psi, \mu) = 0$, $\beta = q_0$ and consider the solution p of (3.6) with $q := \sum_{i=1}^k \beta_i h_i'(\bar{x}(1))$. Then, as before, (5.6) and the measurable selection theorem imply (3.7). Furthermore, if a trajectory-controls quadruple (w, v, κ, π) of (3.12) satisfies (3.16), then

$$\langle p(1), w(1)\rangle + \frac{1}{2}\sum_{i=1}^k \beta_i y(1)^* h_i''(\bar{x}(1))y(1) = 0.$$

On the other hand, as in Step 7 of the proof of Theorem 3.6,

$$\langle p(1), w(1)\rangle = \int_0^1 \left(\langle \mathcal{H}_u[t], v(t)\rangle + \frac{1}{2}\langle \mathcal{H}''[t]\xi(t), \xi(t)\rangle + \langle p(t), \pi(t)\kappa(t)\rangle\right) \mathrm{d}t.$$

Hence

$$\frac{1}{2}\int_0^1 \xi(t)^* \mathcal{H}''[t]\xi(t)\,\mathrm{d}t + \frac{1}{2}\sum_{i=1}^k \beta_i y(1)^* h_i''(\bar{x}(1))y(1) + \int_0^1 (\langle \mathcal{H}_u[t], v(t)\rangle + \langle p(t), \pi(t)\kappa(t)\rangle)\,\mathrm{d}t = 0$$

completing the proof.

6. Appendix.

Proof of Lemma 5.1. We claim that

$$0 \in \mathrm{Int}(R^L(1) - C_K(\bar{x}(1)) \cap B). \tag{6.1}$$

Indeed, since $0 \in R^L(1) \cap (C_K(\bar{x}(1)) \cap B)$, if (6.1) does not hold, then, by the separation theorem, for some $0 \neq p \in \mathbb{R}^n$ we have $\inf\langle p, R^L(1) - C_K(\bar{x}(1)) \cap B\rangle \geq 0$ implying that $\inf\langle p, R^L(1)\rangle \geq 0$ and $\sup\langle p, C_K(\bar{x}(1)) \cap B\rangle \leq 0$. The set $C_K(\bar{x}(1))$ being a cone we deduce that $\sup\langle p, C_K(\bar{x}(1))\rangle \leq 0$. Therefore $\inf\langle p, R^L(1) - C_K(\bar{x}(1))\rangle \geq 0$ in contradiction with assumption (3.3).

Fix $k \in K$, $u \in \mathcal{U}$ and the corresponding trajectory $x_u(\cdot)$ of (1.1). Let $M > 0$ be such that $U \subset B(0, M)$, $t \in (0, 1]$ be a Lebesgue point of $f(x_u(\cdot), u(\cdot))$ and pick any $u_0 \in U$. For any $\delta \in (0, 2Mt)$ define the control

$$u_\delta(s) = \begin{cases} u_0 & \text{if} \qquad s \in [t - \delta/2M, t] \\ u(s) & \text{otherwise.} \end{cases}$$

Observe that $\mathrm{d}(u_\delta, u) \leq \delta$ and therefore $(u_\delta, k) \in B_\delta(u, k)$. Let x^δ be the corresponding trajectory of the control system (1.1). Then the usual well known calculation yields

$$\lim_{\delta \to 0+} \frac{x^\delta(t) - x_u(t)}{\delta} = \frac{1}{2M} \left(f(x_u(t), u_0) - f(x_u(t), u(t)) \right).$$

Denoting by $X_u(\cdot)$ the fundamental solution of the system $X' = f_x(x_u(t), u(t))X$, $X(0) = Id$, from the variational equation we deduce that

$$\lim_{\delta \to 0+} \frac{x^\delta(1) - x_u(1)}{\delta} = \frac{1}{2M} X_u(1) X_u(t)^{-1} \left(f(x_u(t), u_0) - f(x_u(t), u(t)) \right).$$

Therefore, $X_u(1)X_u(t)^{-1}(f(x_u(t), u_0) - f(x_u(t), u(t))) \in 2MG^{(1)}(u, k)$. Since $u_0 \in U$ is arbitrary,

$$X_u(1)X_u(t)^{-1}(f(x_u(t), U) - f(x_u(t), u(t))) \subset 2MG^{(1)}(u, k)$$

for a.e $t \in [0, 1]$. Hence for every measurable selection $v(t) \in f(x_u(t), U) - f(x_u(t), u(t))$ and for a.e. $t \in [0, 1]$ we have $X_u(1)X_u(t)^{-1}v(t) \in 2MG^{(1)}(u, k)$. Integrating, we obtain

$$\int_0^1 X_u(1)X_u(t)^{-1}v(t)\,\mathrm{d}t \in 2M\overline{co}\, G^{(1)}(u, k).$$

Denote by $R^L_u(1)$ the reachable set at time 1 of the following control system

$$\begin{cases} \dot{z}(t) = f_x(x_u(t), u(t))z(t) + v(t), \quad v(t) \in f(x_u(t), U) - f(x_u(t), u(t)) \text{ a.e.} \\ z(0) = 0. \end{cases}$$

We have shown that for any $k \in K$, $R^L_u(1) \subset 2M\overline{co}\, G^{(1)}(u, k)$. Let $v \in T_K(k)$ satisfy $|v| < 1$. Then there exist $\delta_i \to 0+$, $v_i \to v$ such that $k + \delta_i v_i \in K$ and $|v_i| < 1$ for all i. Therefore $(u, k + \delta_i v_i) \in B_{\delta_i}(u, k)$ and the very definition of $G^{(1)}(u, k)$ yields $-v \in G^{(1)}(u, k)$. We have shown that $-T_K(k) \cap \mathrm{Int}(B) \subset G^{(1)}(u, k)$. Since $T_K(k)$ is a cone and $G^{(1)}(u, k)$ is closed, it follows that $-T_K(k) \cap B \subset G^{(1)}(u, k)$ for any $k \in K$. Consequently, $-co\,(T_K(k) \cap B) \subset \overline{co}\, G^{(1)}(u, k)$. Combining with $R^L_u(1) \subset 2M\overline{co}\, G^{(1)}(u, k)$, we obtain, by the arbitrariness of $u \in \mathcal{U}$ and $k \in K$, that

$$R^L_u(1) - co\,(T_K(k) \cap B) \subset (2M + 1)\overline{co}\, G^{(1)}(u, k) \quad \forall\, u \in \mathcal{U},\ k \in K. \tag{6.2}$$

By the separation theorem and (6.1) for some $\delta > 0$, $\sup(\langle p, R^L(1) - C_K(\bar{x}(1)) \cap B\rangle) \geq 3\delta$ for each $p \in S^{n-1}$. Using the boundedness of sets $R^L(1)$ and $C_K(\bar{x}(1)) \cap B$, we can find $\{b_1, ..., b_\ell\} \subset R^L(1)$ and $\{a_1, ..., a_\ell\} \subset C_K(\bar{x}(1)) \cap B$ such that for every $p \in S^{n-1}$ we have

$$\sup(\langle p, co\{b_1, ..., b_\ell\} - co\{a_1, ..., a_\ell\}\rangle) \geq 2\delta. \tag{6.3}$$

Denote by $\to_K$ the convergence in K. Recalling that $C_K(\bar{x}(1)) = \text{Liminf}_{k \to_K \bar{x}(1)} T_K(k)$, see for instance [1, Chapter 4], by (6.3), for some $\varepsilon > 0$ and every $k \in K \cap B(\bar{x}(1), \varepsilon)$

$$\sup(\langle p, co\{b_1, ..., b_\ell\} - co\,(T_K(k) \cap B)\rangle) \geq \delta \tag{6.4}$$

for any $p \in S^{n-1}$. By the separation theorem, this is equivalent

$$\delta B \subset co\{b_1, ..., b_\ell\} - co\,(T_K(k) \cap B) \;\; \forall\, k \in K \cap B(\bar{x}(1), \varepsilon). \tag{6.5}$$

To prove our lemma it remains to show that for some $\rho > 0$ and all $u \in \mathcal{U}$ close to $\bar{u}$ in $L^1([0,1], \mathbb{R}^m)$ and $k \in K$ close to $\bar{x}(1)$ we have $\rho B \subset R^L_u(1) - co(T_K(k) \cap B)$. Assume by a contradiction that we can find $u_i \to \bar{u}$ in L^1, $k_i \in K$ converging to $\bar{x}(1)$ and $z_i \in \mathbb{R}^n$ converging to zero such that $z_i \notin R^L_{u_i}(1) - \overline{co}\,(T_K(k_i) \cap B)$. Denote by x_i the trajectory of (1.1) corresponding to the control u_i. By our assumptions, $x_i \to \bar{x}$ uniformly on $[0,1]$. Taking a subsequence and keeping the same notation we may assume that $u_i \to \bar{u}$ almost everywhere. Then $\lim_{i\to\infty} f_x(x_i(t), u_i(t)) = f_x(\bar{x}(t), \bar{u}(t))$ and $\lim_{i\to\infty} f(x_i(t), u_i(t)) = f(\bar{x}(t), \bar{u}(t))$ a.e.. This and the Gronwall inequality imply that X_{u_i} converge to $X_{\bar{u}}$ uniformly on $[0,1]$. Consequently also $X^{-1}_{u_i}$ converge to $X^{-1}_{\bar{u}}$ uniformly on $[0,1]$.

Fix $1 \leq j \leq \ell$ and let $v(t) \in f(\bar{x}(t), U) - f[t]$ be a control such that the corresponding trajectory $z(\cdot)$ satisfies $z(1) = b_j$ (such $v(\cdot)$ does exist because after removing the convex hull in (3.2) the reachable set remains the same). By the measurable selection theorem, for some $u \in \mathcal{U}$ we have $v(t) = f(\bar{x}(t), u(t)) - f[t]$ a.e. in $[0,1]$. Notice next that $\lim_{i\to\infty} f(x_i(t), u(t)) = f(\bar{x}(t), u(t))$ for all $t \in [0,1]$ and define

$$y_i(1) := \int_0^1 X_{u_i}(1) X_{u_i}(t)^{-1} \left(f(x_i(t), u(t)) - f(x_i(t), u_i(t))\right) \mathrm{d}t.$$

Then $y_i(1) \in R^L_{u_i}(1)$ and $\lim_{i\to\infty} y_i(1) = b_j$. Since $j \in \{1, ..., \ell\}$ is arbitrary, the convexity of $R^L_{u_i}(1)$ and (6.5) imply that for all i sufficiently large and all $k \in K$ near $\bar{x}(1)$ we have

$$\frac{\delta}{2} B \subset R^L_{u_i}(1) - \overline{co}\,(T_K(k) \cap B).$$

This contradicts the choice of z_i and completes the proof.

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