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Groups of piecewise linear homeomorphisms of flows

Nicolás Matte Bon and Michele Triestino

ABSTRACT

To every dynamical system (X, φ) over a totally disconnected compact space, we associate a left-orderable group $T(\varphi)$. It is defined as a group of homeomorphisms of the suspension of (X, φ) which preserve every orbit of the suspension flow and act by dyadic piecewise linear homeomorphisms in the flow direction. We show that if the system is minimal, the group is simple, and if it is a subshift then the group is finitely generated. The proofs of these two statements are short and elementary, providing straightforward examples of finitely generated simple left-orderable groups. We show that if the system is minimal, every action of the corresponding group on the circle has a fixed point. These constitute the first examples of finitely generated left-orderable groups with this fixed point property. We show that for every system (X, φ) , the group $T(\varphi)$ does not have infinite subgroups with Kazhdan's property (T) .

In addition, we show that for every minimal subshift, the corresponding group is never finitely presentable. Finally if (X, φ) has a dense orbit then the isomorphism type of the group $T(\varphi)$ is a complete invariant of flow equivalence of the pair $\{\varphi, \varphi^{-1}\}$.

1. Introduction

A group is *left-orderable* if it admits a total order which is invariant under left multiplication. For instance, the group $\text{Homeo}_+(\mathbb{R})$ of orientation-preserving homeomorphisms of the real line is left-orderable, as well as all of its subgroups. Conversely, if a non-trivial countable group is left-orderable, then it admits a faithful orientation-preserving action on \mathbb{R} without global fixed points. This provides a dynamical point of view on left-orderability. See Deroin, Navas and Rivas [DNR] for a detailed treatment.

One recurrent difficulty in this dynamical approach is the lack of compactness of \mathbb{R} . As a possible solution to this issue, Deroin [Der13] showed that every finitely generated left-orderable group can be realised as a group of homeomorphisms of a compact space Y , without global fixed points, which preserves each orbit of a flow $\Phi: \mathbb{R} \times Y \rightarrow Y$. The goal of this paper is to explore how this point of view can be conversely used to define new examples of left-orderable groups. We introduce and study a family of left-orderable groups associated to suspension flows of homeomorphisms of the Cantor set (more generally, of a Stone space). These groups turn out to satisfy interesting properties in the setting of group actions on one-manifolds: simplicity and finite generation (Theorems A and B), and a fixed point property for actions on the circle (Theorem D).

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Throughout the paper, let X be a non-empty Stone space, that is, a totally disconnected compact Hausdorff space, and let φ be a self-homeomorphism of X . The dynamical system (X, φ) will be called a *Stone system*, or a *Cantor system* in the case X is homeomorphic to a Cantor set. The *suspension* (or *mapping torus*) of (X, φ) is the space $Y^\varphi := (X \times \mathbb{R})/\mathbb{Z}$, where the quotient is taken with respect to the diagonal action of \mathbb{Z} on $X \times \mathbb{R}$ given by $n \cdot (x, t) = (\varphi^n(x), t - n)$. The natural projection of $(x, t) \in X \times \mathbb{R}$ to Y^φ is denoted by $[x, t]$. The suspension comes with a flow $\Phi: \mathbb{R} \times Y^\varphi \rightarrow Y^\varphi$, given by $\Phi^t([x, s]) = [x, s + t]$.

We denote by $\mathbf{H}(\varphi)$ the group of homeomorphisms of Y^φ , and by $\mathbf{H}_0(\varphi)$ its subgroup consisting of elements $g \in \mathbf{H}(\varphi)$ that are isotopic to the identity. This condition forces every element $g \in \mathbf{H}_0(\varphi)$ to preserve every Φ -orbit, since Φ -orbits are precisely the path-components of Y^φ .¹ Observe that $\mathbf{H}_0(\varphi)$ is usually not closed in $\mathbf{H}(\varphi)$ (with respect to the compact open topology). A concrete characterisation of $\mathbf{H}_0(\varphi)$ is the following (see Boyle, Carlsen and Eilers [BCE17, Theorem 3.1]): a homeomorphism $g \in \mathbf{H}(\varphi)$ belongs to $\mathbf{H}_0(\varphi)$ if and only if there exists a continuous function $\tau: Y^\varphi \rightarrow \mathbb{R}$ such that g is given by

$$g(y) = \Phi^{\tau(y)}(y).$$

The groups $\mathbf{H}_0(\varphi)$ can be thought as generalisations of the group $\mathbf{Homeo}_+(\mathbb{S}^1)$ of orientation-preserving homeomorphisms of the circle (indeed, it reduces to it when X is reduced to a point). For instance, it is a classical result by Schreier and Ulam [SU34] that $\mathbf{Homeo}_+(\mathbb{S}^1)$ is simple, and after a result of Aliste-Prieto and Petite [APP16] the group $\mathbf{H}_0(\varphi)$ is also simple as soon as (X, φ) is minimal (i.e. every orbit is dense). On the other hand, unlike $\mathbf{Homeo}_+(\mathbb{S}^1)$, the group $\mathbf{H}_0(\varphi)$ is often left-orderable (see Proposition 3.3). For instance if the system (X, φ) has a dense orbit, the restriction of the natural action of $\mathbf{H}_0(\varphi)$ to any dense Φ -orbit yields a faithful action on the real line.

The group $\mathbf{Homeo}_+(\mathbb{S}^1)$ has a natural finitely generated subgroup which is also simple, namely Thompson's group T (whose definition is recalled in Section 3). Pursuing this analogy, we define a group $\mathbf{T}(\varphi) \leq \mathbf{H}_0(\varphi)$. It is defined as the subgroup of $\mathbf{H}_0(\varphi)$ formed by homeomorphisms that in the direction of the flow are locally defined by dyadic piecewise linear homeomorphisms, and whose displacement is locally constant in the totally disconnected direction (see Definition 3.1). The goal of this paper is to study properties of the group $\mathbf{T}(\varphi)$.

1.1 Finite generation and simplicity

Let us begin with a criterion that ensures that the group $\mathbf{T}(\varphi)$ is finitely generated. Recall that a *subshift* over a finite alphabet A is a Stone system (X, φ) where $X \subset A^{\mathbb{Z}}$ is a non-empty closed shift-invariant subset, and φ is the restriction of the shift to X . (We allow X to admit isolated points, for instance it can be countable).

THEOREM A. *If the Stone system (X, φ) is conjugate to a subshift, the group $\mathbf{T}(\varphi)$ is finitely generated.*

Note that the converse is also true (see Proposition 6.3).

It has been until recently an open question, attributed to Rhemtulla, whether there exists a finitely generated left-orderable group which is simple (this question first appeared in [Notices AMS 28 (June 1982), p. 327]). The first examples of such groups have been constructed by Hyde

¹Conversely, an element preserving every Φ -orbit needs not be isotopic to the identity in general (see [BCE17]), but it is if (X, φ) is minimal [APP16]. This subtle problem will be irrelevant in this paper, with the exception of the appendix.

and Lodha [HL]. They define a family of groups G_ρ parametrised by a sequence $\rho \in \{a^{\pm 1}, b^{\pm 1}\}^{\mathbb{Z}}$ on a four-letter alphabet, required to be quasi-periodic and to verify a palindromicity condition. Each G_ρ is defined as the group generated by six explicit piecewise linear homeomorphisms of the real line with infinitely many pieces, which depend in a natural way on the sequence ρ . In relation to the present setting, if we let (X, φ) be the subshift given by the orbit-closure of the sequence $\rho \in \{a^{\pm 1}, b^{\pm 1}\}^{\mathbb{Z}}$, then the group G_ρ can be shown to be isomorphic to a (strict) subgroup of $\mathbb{T}(\varphi)$ (this is close to the construction of Deroin [Der13]).

Note that several natural classes of left-orderable groups are actually *locally indicable* (that is, every finitely generated non-trivial subgroup surjects to \mathbb{Z}). Historically the first example which is not locally indicable (due to Thurston [Thu74], rediscovered by Bergman [Ber91]) is the central extension $\widetilde{\Gamma}_{2,3,7}$ in $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ of the $(2, 3, 7)$ -triangle group in $\mathrm{PSL}(2, \mathbb{R})$, which is finitely generated and perfect. Another example with the same property is the central extension \widetilde{T} of Thompson’s group T (defined in Section 8), see Navas [Nav11, Remark 2.2.51].

Many examples of simple groups arise as groups of homeomorphisms $G \leq \mathrm{Homeo}_+(Z)$ of some (Hausdorff) space Z whose action is *micro-supported*, meaning that every non-empty open subset $U \subset Z$ has a non-trivial *rigid stabiliser* $\mathrm{RiSt}(U)$, defined as the subgroup of elements with support in U . This property helps in proving simplicity using a method *à la Higman–Epstein* [Eps70, Hig54] (see Section 2). The starting observation is a “Double Commutator Lemma” (see Lemma 2.1), stating that every non-trivial normal subgroup of a micro-supported group G must contain the derived subgroup $\mathrm{RiSt}_G(U)'$ of the rigid stabiliser of some non-empty open set. As a consequence, to prove simplicity it is enough to show that G is generated by commutators of elements of sufficiently “small” support (cf. Proposition 2.2). However it is easy to see that a *finitely generated* simple group cannot admit a micro-supported action on any *non-compact*, locally compact space (see Remark 2.3). This is an important technical difficulty when establishing simplicity of finitely generated simple subgroups of $\mathrm{Homeo}_+(\mathbb{R})$. In particular the proof of simplicity of the groups G_ρ given by Hyde and Lodha [HL] goes through a quite involved combinatorial analysis of the action of the groups G_ρ on the real line \mathbb{R} .

The groups $\mathbb{T}(\varphi)$ provide a more conceptual way to obtain such examples, which makes this difficulty disappear. In fact, the action of the group $\mathbb{T}(\varphi)$ on the *compact* suspension space Y^φ is micro-supported. This yields a very short proof of the following result.

THEOREM B. *If the Stone system (X, φ) is minimal, then $\mathbb{T}(\varphi)$ is simple.*

Note that minimality is also necessary: if (X, φ) is not minimal then $\mathbb{T}(\varphi)$ admits a non-trivial proper quotient (see Corollary 9.2).

Combining with Theorem A, we have:

COROLLARY C. *For any infinite minimal subshift (X, φ) , the group $\mathbb{T}(\varphi)$ is simple, finitely generated, and left-orderable.*

Remark 1.1. Theorems A and B are analogous to the results of Matui [Mat06] on the derived subgroup of the *topological full group* $[[\varphi]]$ of a Cantor minimal system (X, φ) , and were largely inspired by these (with the difference that here the group $\mathbb{T}(\varphi)$ always coincides with its derived subgroup). Recall that the topological full group $[[\varphi]]$ is the group of all homeomorphisms of X that locally coincide with a power of φ . Note that the group $[[\varphi]]$ is very far from being torsion-free (it contains infinite locally finite subgroups), thus it cannot be left-orderable. Another important difference is that $\mathbb{T}(\varphi)$ always contains non-abelian free subgroups and is therefore non-amenable (see Corollary 8.2), whereas Juschenko and Monod [JM13] proved that

the topological full group $[[\varphi]]$ is amenable. Note that by a result of Witte Morris [WM06] (see also Deroin [Der13]), no amenable left-orderable group can be simple, as such a group is locally indicable. Finally, Grigorchuk and Medynets proved that the topological full group $[[\varphi]]$ of a Cantor minimal system (X, φ) is *locally embeddable into finite groups (LEF)* [GM14], but in contrast the group $\mathbb{T}(\varphi)$ is not, as it contains subgroups isomorphic to Thompson’s group F (see Section 4; F is not LEF because it is finitely presented and not residually finite, therefore [CSC10, Proposition 7.3.8] applies).

1.2 Actions of $\mathbb{T}(\varphi)$ on the circle and on the real line

Our next result analyzes actions of the group $\mathbb{T}(\varphi)$ on the circle. Clearly every group of homeomorphisms of the real line can also be seen as acting on the circle $\mathbb{S}^1 \simeq \mathbb{R} \cup \{\infty\}$ by fixing the point at infinity. The next result says that conversely, actions of the group $\mathbb{T}(\varphi)$ on the circle always reduce to actions on the real line.

THEOREM D. *If the Stone system (X, φ) is infinite and minimal, every continuous action of the group $\mathbb{T}(\varphi)$ on the circle has a fixed point.*

In combination with Theorem A, this provides the first examples of non-trivial, finitely generated subgroups of $\text{Homeo}_+(\mathbb{S}^1)$ with this property.

Remark 1.2. We remark here, and this will be silently used throughout the text, that if G is a perfect group, every one-dimensional continuous action of G preserves the orientation.

Theorem D implies a related property for actions of the group $\mathbb{T}(\varphi)$ on the real line. Recall (see Deroin, Navas and Rivas [DNR, §3.5]) that every action of a finitely generated group G on the real line by orientation-preserving homeomorphisms, without global fixed points, is exactly of one of the following three types.

- (I) The action preserves a non-zero Radon measure on \mathbb{R} .
- (II) The action is semi-conjugate to a minimal action on \mathbb{R} which commutes with the action of \mathbb{Z} on \mathbb{R} by translations, and passes to the quotient to a *proximal*² action on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.
- (III) The action is semi-conjugate to a minimal and proximal action on \mathbb{R} .

(The notion of semi-conjugacy is recalled in Section 8.) While actions of type (III) are in some sense the “generic” ones, all finitely generated, left-orderable groups known so far were also known to admit actions of type (I) or (II). This lead Deroin, Navas and Rivas to ask whether there exist finitely generated, left-orderable groups whose actions on the real line, without global fixed points, are all of type (III) [DNR, Question 3.5.11]. A discussion on this question can also be found in Navas’ survey [Nav18, see Question 4].

For a given finitely generated group G , it is clear from the above trichotomy that the following statements are equivalent.

- (i) Every orientation-preserving action of G on the real line, without global fixed point, is of type (III).
- (ii) Every orientation-preserving action of G on the circle, which lifts to the real line, has a fixed point.

²In this context, a minimal G -action on $X = \mathbb{R}/\mathbb{Z}$ or \mathbb{R} is *proximal* if for all compact intervals $I, J \subset X$ there exists $g \in G$ such that $g(I) \subset J$.

Thus Theorem D answers the question of Deroin, Navas and Rivas in the affirmative.

COROLLARY E. *If the Stone system (X, φ) is infinite and minimal, every action of the group $\mathbb{T}(\varphi)$ on the real line, without global fixed points, is of type (III).*

The proof of Theorem D is based on an analogue of a property known to hold for Thompson’s group T , namely that every non-trivial action of the group T on the circle is semi-conjugate to the standard one (Theorem 8.6). Using arguments from a proof of this fact by Le Boudec and the first named author [LBMB18], we show that every action of $\mathbb{T}(\varphi)$ on \mathbb{S}^1 without global fixed points, must give rise to a continuous surjective map $\mathbb{S}^1 \rightarrow Y^\varphi$. But here the space Y^φ is not path-connected, thus such an action cannot exist. The action of the group $\mathbb{T}(\varphi)$ on the space Y^φ (rather than its action on the real line) therefore plays a crucial role also here.

Remark 1.3. After the first version of this work, James Hyde, Yash Lodha, Andrés Navas and Cristóbal Rivas informed us of the work [HLNR] which independently establishes the analogue of Corollary E for the groups G_ρ of Hyde and Lodha [HL].

We mention the following related question.

Question 1. Assume that the Stone system (X, φ) is infinite minimal. Is every action of $\mathbb{T}(\varphi)$ on the real line, without global fixed points, semi-conjugate to its action on some infinite Φ -orbit?

At the moment of writing we are unable to answer this question. Let us point out that a similar statement (previously unnoticed?) holds for the central extension \tilde{T} of Thompson’s group T (see Theorem 8.7), which embeds into $\mathbb{T}(\varphi)$ as soon as φ has infinite order (see Proposition 8.1).

Let us also mention that an example close in spirit to Theorem D appears in Calegari [Cal06]: let $\tilde{\Gamma}_{2,3,7} \subset \tilde{\text{PSL}}(2, \mathbb{R})$ be the central extension of the $(2, 3, 7)$ -triangle group in $\text{PSL}(2, \mathbb{R})$; it is not true that every action of $\tilde{\Gamma}_{2,3,7}$ on the circle has a global fixed point, but this happens for the restriction to $\tilde{\Gamma}_{2,3,7}$ of actions of a finitely generated, circularly left-orderable overgroup.

1.3 Non-existence of subgroups with Kazhdan’s property (T)

Recall that a countable group G has Kazhdan’s property (T) if every G -action on a Hilbert space by affine isometries has a fixed point. It is an open question whether an infinite group with Kazhdan’s property (T) can act faithfully on a one-dimensional manifold by homeomorphisms. Navas [Nav02] proved that this is not the case if one restricts to actions on the circle by smooth diffeomorphisms. By results in [LMBT, Cor], this extends to actions by piecewise smooth diffeomorphisms with finitely many pieces. Moreover, every action of a Kazhdan group on the circle by homeomorphisms which are individually smooth on the complement of a countable closed set must have a finite orbit [LMBT]. Here we prove that also the groups $\mathbb{T}(\varphi)$ do not have Kazhdan’s property (T) .

THEOREM F. *For every Stone system (X, φ) the group $\mathbb{T}(\varphi)$ does not contain infinite subgroups with Kazhdan’s property (T) . The same holds more generally for the larger group $\text{PL}(\varphi)$ (Definition 3.1).*

Although we cannot apply the results of [Nav02, LMBT, Cor] to the group $\mathbb{T}(\varphi)$ directly (unless φ is periodic), we show that it can be approximated by groups to which such results do apply, and which therefore do not have Kazhdan’s property (T) . Then the proof relies on the fact (proved independently by Shalom [Sha00, Theorem 6.7] and Gromov [Gro03, §3.8.B]) that the set of property (T) groups is open in the space of marked groups (in the sense of Grigorchuk [Gri84]).

Remark 1.4. Since the groups G_ρ defined by Hyde and Lodha [HL] arise as subgroups of $\mathbb{T}(\varphi)$ for some (X, φ) , Theorem F also implies that they do not have Kazhdan’s property (T).

1.4 Further results

By the same limit argument used to prove Theorem F, we also obtain the following in Section 9.

THEOREM G. *For every minimal subshift (X, φ) , the group $\mathbb{T}(\varphi)$ is not finitely presented.*

In Section 10, we clarify how the isomorphism class of the group $\mathbb{T}(\varphi)$ depends on (X, φ) . Two Stone systems (X_1, φ_1) and (X_2, φ_2) are said to be *flow equivalent* if there exists a homeomorphism $q: Y^{\varphi_1} \rightarrow Y^{\varphi_2}$ between the suspensions which sends orbits to orbits in an orientation-preserving fashion. By a result of Parry and Sullivan [PS75] (see Theorem 10.2), this is equivalent to the existence of clopen subsets $X'_i \subset X_i$ intersecting every orbit, with the property that the *first return maps* to X'_1 and to X'_2 are topologically conjugate. Flow equivalence is well-studied in symbolic dynamics and C^* -algebras. By combining the Parry–Sullivan characterisation of flow equivalence with a general reconstruction result for groups of homeomorphisms by Ben Ami and Rubin [BAR10] (see Theorem 10.1), we observe the following.

THEOREM H. *For $i = 1, 2$, let (X_i, φ_i) be a Stone system. Assume that (X_i, φ_i) admits a dense orbit for $i = 1, 2$. Then the groups $\mathbb{T}(\varphi_1)$ and $\mathbb{T}(\varphi_2)$ are isomorphic if and only if the system (X_1, φ_1) is flow equivalent to (X_2, φ_2) or to (X_2, φ_2^{-1}) .*

In [APP16], Aliste-Prieto and Petite observed that [BAR10] implies the analogous result for $\mathbb{H}_0(\varphi)$.

Let us conclude this introduction with a question, which aims at clarifying a connection between the outer automorphism group $\text{Out}(\mathbb{T}(\varphi))$ and the *mapping class group* of subshifts, extensively studied in [APP16, Boy02, BCE17, BC18, SY].

For simplicity, let us restrict to the case where (X, φ) is a topologically transitive subshift. Note that by topological transitivity, every element $g \in \mathbb{H}(\varphi)$ must simultaneously preserve or reverse the orientation of every Φ -orbit, as this is determined by what it does on any dense Φ -orbit. The *complete mapping class group* of (X, φ) is defined as the quotient $\text{MCG}_\pm(\varphi) = \mathbb{H}(\varphi)/\mathbb{H}_0(\varphi)$ ³. This group is countable, see [BC18, Corollary 2.3].

The result of Ben Ami and Rubin [BAR10] allows to identify the automorphism group $\text{Aut}(\mathbb{T}(\varphi))$ with the normaliser of $\mathbb{T}(\varphi)$ in $\mathbb{H}(\varphi)$. In particular, we have a homomorphism $\text{Out}(\mathbb{T}(\varphi)) = \text{Aut}(\mathbb{T}(\varphi))/\mathbb{T}(\varphi) \rightarrow \text{MCG}_\pm(\varphi)$.

Question 2. Let (X, φ) be a topologically transitive subshift. Is the map $\text{Out}(\mathbb{T}(\varphi)) \rightarrow \text{MCG}_\pm(\varphi)$ a group isomorphism?

Of course the answer might depend non-trivially on (X, φ) . The result of Brin [Bri96] can be reinterpreted by saying that for Thompson’s group T we have indeed an isomorphism $\text{Out}(T) \rightarrow \text{MCG}_\pm(\mathbb{S}^1) = \mathbb{Z}/2\mathbb{Z}$.

Finally this paper contains an appendix (Appendix A), in which we endow the group $\mathbb{H}_0(\varphi)$ with a natural Polish group topology, and show that the group $\mathbb{T}(\varphi)$ is dense in $\mathbb{H}_0(\varphi)$ with respect to this topology. This will be useful for the discussion in Section 4.

³The mapping class group of (X, φ) is more usually defined as its index 2 subgroup $\text{MCG}_+(\varphi) = \mathbb{H}_+(\varphi)/\mathbb{H}_0(\varphi)$ of mapping classes of elements that send each Φ -orbit to a Φ -orbit in an orientation-preserving way.

2. Preliminaries on simplicity of groups of homeomorphisms

In this short section, we recall a well-known method to study normal subgroups in groups of homeomorphisms and prove their simplicity. Moreover, this serves in part as a motivation for our construction.

For every group G acting by homeomorphisms on a topological space Y , and every open subset $U \subset Y$, the *rigid stabiliser* $\text{RiSt}(U)$ is defined as the subgroup consisting of elements that move only points in U . The action of G on Y is said to be *micro-supported* if $\text{RiSt}(U)$ is non-trivial for every non-empty open subset U .

The short argument giving the following lemma is classical and has been used in many proofs of simplicity. With this formulation, it is exactly [Nek18, Lemma 4.1].

LEMMA 2.1 (Double Commutator Lemma). *Let G be a group of homeomorphisms of a Hausdorff space Y , and $N \trianglelefteq G$ be a non-trivial normal subgroup. Then there exists a non-empty open subset $U \subset Y$ such that the derived subgroup $\text{RiSt}(U)'$ of the rigid stabilizer is contained in N .*

Proof. Take a non-trivial element $g \in N$ and an open subset $U \subset Y$ such that $g(U) \cap U = \emptyset$. Then for every two elements h_1 and $h_2 \in \text{RiSt}(U)$, the element gh_1g^{-1} is supported in $g(U)$ and therefore commutes with h_1 and h_2 . Using this, one readily checks that the commutator $[h_1, h_2] = h_1h_2h_1^{-1}h_2^{-1}$ equals $[[h_1, g], h_2]$, and therefore belongs to N . \square

Note that the lemma is valid without any assumptions on the action of G on Y , but its conclusion is trivial unless the action is micro-supported.

We will use the following immediate consequence of the Double Commutator Lemma, which is a variant of Epstein's simplicity criterion [Eps70]. We say that a group of homeomorphisms of a Hausdorff space Y is *fragmentable* if for every open cover \mathcal{U} of Y , the group G is generated by the derived subgroups $\text{RiSt}(U)'$ for $U \in \mathcal{U}$.

PROPOSITION 2.2. *Let G be a non-trivial group of homeomorphisms of a Hausdorff space Y . If G is fragmentable and acts minimally on Y , then G is simple.*

Proof. Let $N \trianglelefteq G$ be a non-trivial normal subgroup of G , and $V \subset Y$ be a non-empty open set given by the Double Commutator Lemma (Lemma 2.1). Observe that for every $g \in G$, the group N contains $g\text{RiSt}(V)'g^{-1} = \text{RiSt}(g(V))'$. By minimality, the subsets $g(V)$ form an open cover of Y , and thus $N = G$. \square

Remark 2.3. Note that if G is a *finitely generated* simple group, then G cannot admit any micro-supported action by homeomorphisms on any locally compact, *non-compact* Hausdorff space Y . To wit, assume by contradiction that this is the case. Let N be the subgroup of G generated by all rigid stabiliser $\text{RiSt}(U)$ of relatively compact open subsets $U \subset Y$. Then N is non-trivial and normal, so that by simplicity $N = G$. However, N cannot be finitely generated unless Y is compact.

3. The group $\mathbb{T}(\varphi)$ and its first properties

As mentioned earlier, we are assuming that (X, φ) is a dynamical system consisting of a non-empty Stone space X and of a homeomorphism $\varphi: X \rightarrow X$. We denote by $Y^\varphi = (X \times \mathbb{R}) / \mathbb{Z}$ its suspension, and by Φ the suspension flow on Y^φ . We do not make any further assumptions on (X, φ) unless specified.

A subset $C \subset X$ is *clopen* if it is both open and closed. Given a clopen subset C , its *first return time* is the quantity

$$\tau_C := \min\{n \geq 1 : \exists x \in C, \varphi^n(x) \in C\} \in \mathbb{N} \cup \{\infty\}. \quad (1)$$

Let $C \subset X$ be a clopen subset, and $J \subset \mathbb{R}$ be an open interval of length $|J| < \tau_C$ (in particular, this is always verified if $|J| < 1$). For such a pair, the inclusion of $C \times J$ into $X \times \mathbb{R}$ defines an injective map $\pi_{C,J} : C \times J \hookrightarrow Y^\varphi$, that we call a *chart*. The image $U_{C \times J} = \pi_{C,J}(C \times J)$ will be called the *domain* associated with (C, J) .

The domains satisfy the relation

$$U_{C \times J} = U_{\varphi^n(C) \times (J-n)}, \quad n \in \mathbb{Z}. \quad (2)$$

Here $J - n$ stands for the translate $J - n = \{x - n : x \in J\}$.

We call an interval $I \subset \mathbb{R}$ *dyadic* if it is bounded and its endpoints are dyadic rationals, i.e. rationals of the form $p/2^q$ for $p, q \in \mathbb{Z}$. A chart $\pi_{C,J}$ and a domain $U_{C \times J}$ are *dyadic* if J is dyadic. Given two intervals I and $J \subset \mathbb{R}$, a *piecewise linear (PL) homeomorphism* $f : I \rightarrow J$ is a homeomorphism which is piecewise affine, with finitely many discontinuity points for the derivatives (also called *breakpoints*). We will further say that $f : I \rightarrow J$ is *dyadic* if both I and J are dyadic and its breakpoints are dyadic rationals and the restriction of f to every piece is of the form $x \mapsto 2^m x + b$ for some $m \in \mathbb{Z}$ and b dyadic rational. Recall that the group of all dyadic PL homeomorphisms of $[0, 1]$ is *Thompson's group* F , whereas the analogous definition on the circle gives *Thompson's group* T . Both groups are finitely generated. For preliminaries on Thompson's groups we refer to [CFP96, BS16].

DEFINITION 3.1. We let $\text{PL}(\varphi)$ (resp. $\text{T}(\varphi)$) be the group consisting of all elements $g \in \text{H}_0(\varphi)$ such that for every $y \in Y^\varphi$, there exist a dyadic chart $\pi_{C,J}$ with $y \in U_{C \times J}$, and a PL homeomorphism (resp. a dyadic PL homeomorphism) $f : J \rightarrow f(J) \subset \mathbb{R}$ such that $g(U_{C \times J}) = U_{C \times f(J)}$ and such that the map $\pi_{C,f(J)}^{-1} \circ g \circ \pi_{C,J} : C \times J \rightarrow C \times f(J)$ is given in coordinates by $(x, t) \mapsto (x, f(t))$.

Remark 3.2. Clearly we have the inclusions $\text{T}(\varphi) \leq \text{PL}(\varphi) \leq \text{H}_0(\varphi)$. While the groups $\text{H}_0(\varphi)$ and $\text{PL}(\varphi)$ are always uncountable, the group $\text{T}(\varphi)$ is countable as soon as X is metrisable. In fact, in this case the set of clopen subsets of X is countable, and so is the set of dyadic charts. Since every $g \in \text{T}(\varphi)$ is determined by a finite collection of dyadic charts and of dyadic PL-homeomorphisms, the conclusion follows.

Let us clarify when these groups are left-orderable. The system (X, φ) is said to be *topologically free* if the set of infinite orbits is dense in X . By Baire Category Theorem, this is equivalent to the fact that for every $n \geq 1$, the set of points of period n has empty interior.

PROPOSITION 3.3. *If (X, φ) is topologically free, then $\text{H}_0(\varphi)$ (and hence $\text{T}(\varphi)$) is left-orderable. If, moreover, X is metrisable, then $\text{H}_0(\varphi)$ is isomorphic to a subgroup of $\text{Homeo}_+(\mathbb{R})$.*

Proof. Observe that if (X, φ) has a dense collection of non-periodic orbits, then also (Y^φ, Φ) does. Let $(\mathcal{O}_\lambda)_{\lambda \in \Lambda}$ be a collection of non-periodic Φ -orbits, whose union is dense in Y^φ . Every orbit \mathcal{O}_λ can be identified with the real line \mathbb{R} , and the restriction of the action of $\text{H}_0(\varphi)$ defines a morphism $\rho_\lambda : \text{H}_0(\varphi) \rightarrow \text{Homeo}_+(\mathcal{O}_\lambda)$. Observe that density of $\bigcup \mathcal{O}_\lambda$ implies that the morphism $(\rho_\lambda) : \text{H}_0(\varphi) \rightarrow \prod \text{Homeo}_+(\mathcal{O}_\lambda)$ is injective. Thus $\text{H}_0(\varphi)$ embeds in a product of left-orderable groups and is therefore left-orderable. If moreover X is metrisable, we can extract a countable collection $(\mathcal{O}_n)_{n \in \mathbb{N}}$ from (\mathcal{O}_λ) whose union remains dense. Identifying each \mathcal{O}_n with the interval $(n, n+1)$, we obtain a faithful action of $\text{H}_0(\varphi)$ on \mathbb{R} . \square

We will use the following fact.

LEMMA 3.4 (Minimality). *For every $y \in Y^\varphi$, the $\mathbb{T}(\varphi)$ -orbit of y is dense in the Φ -orbit of y . In particular the $\mathbb{T}(\varphi)$ -action on Y^φ is minimal if the system (X, φ) is so.*

Proof. The group $\mathbb{T}(\varphi)$ contains the translation $[x, s] \mapsto [x, s + r]$ for every dyadic rational r (cf. Proposition 8.1). That is, the group $\mathbb{T}(\varphi)$ contains a dense set of times of the flow Φ , so its action on every Φ -orbit must be minimal. \square

4. A family of subgroups of $\mathbb{T}(\varphi)$

In this section we introduce an (infinite) family of subgroups of $\mathbb{T}(\varphi)$ isomorphic to Thompson's group F (see Definition 4.3), which constitute a generating system for $\mathbb{T}(\varphi)$, and derive some useful properties of this family.

For every open dyadic interval $J \subset \mathbb{R}$, we will denote by F_J the group of dyadic PL self-homeomorphisms of \mathbb{R} which are the identity on the complement of J . Defined this way, F_J is naturally a subgroup of $\text{Homeo}_+(\mathbb{R})$. If J is non-empty, F_J is always conjugate to Thompson's group $F = F_{(0,1)}$, a conjugacy being given by any dyadic PL homeomorphism which maps J onto $(0, 1)$. Its derived subgroup F'_J is simple and consists of elements whose *closed* support is contained in J . More precisely, we have the following: for every open dyadic interval $J \subset \mathbb{R}$, one has

$$F'_J = \bigcup_{\bar{I} \subset J} F_I = \bigcup_{\bar{I} \subset J} F'_I, \quad (3)$$

where I varies over open dyadic intervals whose closure is contained in J . We will use repeatedly the following well-known properties of Thompson's group F :

LEMMA 4.1. *For any open dyadic intervals I, J_1 and J_2 such that $I \supset \overline{J_1}, \overline{J_2}$, there exists an element $h \in F'_I$ such that $h(J_1) = J_2$ and thus $hF_{J_1}h^{-1} = F_{J_2}$.*

Proof. See Cannon, Floyd and Parry [CFP96, Lemma 4.2]. \square

LEMMA 4.2 (Fragmentation Lemma for F). *Let J be an open dyadic interval covered by a family $\{J_i\}_{i \in \mathcal{L}}$ of open dyadic intervals $J_i \subset J$. Then the group F'_J is contained in the group generated by $\bigcup_{i \in \mathcal{L}} F'_{J_i}$.*

Proof. Although this is well-known we prefer to include a proof here. Let us denote by K the group generated by $\bigcup_{i \in \mathcal{L}} F'_{J_i}$. By (3), it is enough to show that $F'_I \leq K$ for every open dyadic interval such that $\bar{I} \subset J$. Fix such an I and extract from \mathcal{L} a finite subcover J_1, \dots, J_n of \bar{I} of minimal cardinality. We argue by induction on the number n . If $n = 1$, we have $F_I \leq F_{J_1} \leq K$. Assume that $n \geq 2$, and let a_i be the leftmost point of J_i , for $i = 1, \dots, n$. These are pairwise distinct: if J_i and J_j share an endpoint, we would have $J_i \subset J_j$ or $J_j \subset J_i$, contradicting the minimality of n . Without loss of generality we can assume that $a_1 < a_2 < \dots < a_n$, and that $J_i \cap J_{i+1} \neq \emptyset$ for $i = 1, \dots, n$. Let a be the leftmost point of I and choose $g \in F'_{J_1}$ such that $g(a) \in J_2$ (this exists by Lemma 4.1). Then $g(\bar{I}) \subset J_2 \cup \dots \cup J_n$, so that $gF'_I g^{-1} = F'_{g(I)} \leq K$ by the induction hypothesis. It follows that $F'_I \leq K$. \square

DEFINITION 4.3. Every dyadic chart $\pi_{C,J}$ defines an embedding of the group F_J into $\mathbb{T}(\varphi)$ obtained by letting F_J act on $C \times J$ with trivial action on C , and transporting this action on $U_{C \times J}$. We denote the image of this embedding by $F_{C,J} \leq \mathbb{T}(\varphi)$.

PROPOSITION 4.4. *The group $\mathbb{T}(\varphi)$ is generated by the derived subgroups $F'_{C,J}$, defined by all dyadic charts $\pi_{C,J}$.*

Remark 4.5. We postpone the proof of Proposition 4.4 to the appendix (Appendix A), since it becomes cleaner after introducing a suitable topology on the group $H_0(\varphi)$. In the relevant special case where (X, φ) is minimal, a more direct proof will appear in Section 7 (see Remark 7.4). We point out however that Proposition 4.4 is orthogonal to all other results in this paper: the reader may as well take as definition that the group $\mathbb{T}(\varphi)$ is the subgroup of $H_0(\varphi)$ generated by the subgroups $F_{C,J} \leq H_0(\varphi)$, and continue reading. We decided to include it for completeness, since we believe that Definition 3.1 is more natural.

We thank the referee for pointing out that, at this stage, Proposition 4.4 already yields the following, which can be seen as a weak form of simplicity.

COROLLARY 4.6. *The group $\mathbb{T}(\varphi)$ is perfect and admits no non-trivial finite quotient.*

Proof. After Proposition 4.4, the group $\mathbb{T}(\varphi)$ is generated by commutators, and thus is perfect. Let $N \triangleleft \mathbb{T}(\varphi)$ be a proper normal subgroup. After Proposition 4.4, there exists a dyadic chart $\pi_{C,J}$ such that $(\mathbb{T}(\varphi) \setminus N) \cap F'_{C,J} \neq \emptyset$. As $F'_{C,J} \cong F'$ is simple, we must have $F'_{C,J} \cap N = \{\text{id}\}$. This implies that the quotient $\mathbb{T}(\varphi)/N$ is infinite. \square

We conclude this section with two lemmas on the subgroups $F'_{C,J}$, that will be repeatedly used later.

LEMMA 4.7 (Intersection Lemma). *Let $\pi_{C,I}$ and $\pi_{D,J}$ be dyadic charts such that $C \cap D \neq \emptyset$ and $I \cap J \neq \emptyset$. Then the group $\langle F'_{C,I}, F'_{D,J} \rangle$ contains the subgroups $F'_{C \cap D, I \cap J}$, $F'_{C \cap D, I}$ and $F'_{C \setminus D, I}$.*

Proof. Let us first show that $F'_{C \cap D, I \cap J} \leq \langle F'_{C,I}, F'_{D,J} \rangle$. Observe that for every dyadic intervals I and $J \subset \mathbb{R}$, one has $F'_I \geq F'_{I \cap J}$. Thus the group $F'_{C,I}$ contains $F'_{C, I \cap J}$ as subgroup, and similarly $F'_{D,J}$ contains $F'_{D, I \cap J}$. Therefore, it suffices to prove that $F'_{C \cap D, I \cap J}$ is contained in the group generated by $F'_{C, I \cap J}$ and $F'_{D, I \cap J}$. To simplify notations, we will assume $I = J = I \cap J$. Take $f \in F'_{C,I}$ and $g \in F'_{D,I}$, defined respectively by f_0 and $g_0 \in F'_I$. Then the commutator $[f, g]$ is supported on $U_{(C \cap D) \times I}$, where it is defined on the flow direction by the commutator $[f_0, g_0] \in F'_I$. As commutators generate $[F'_I, F'_I] = F'_I$, we have the statement.

Let us now show that $F'_{C \cap D, I} \leq \langle F'_{C,I}, F'_{D,J} \rangle$. We already know that $\langle F'_{C,I}, F'_{D,J} \rangle$ contains $F'_{C \cap D, I \cap J}$. Fix an open dyadic interval L such that $\bar{L} \subset I$. By Lemma 4.1 we can find $h \in F'_I$ such that $h(L) \subset I \cap J$. Let $g \in F'_{C,I}$ be the element given in coordinates by $\text{id} \times h$. Then $F'_{C \cap D, L} = g^{-1} F'_{C \cap D, h(L)} g$ is also contained in $\langle F'_{C,I}, F'_{D,J} \rangle$. Since L was arbitrary, by (3) we have that $F'_{C \cap D, I} \leq \langle F'_{C,I}, F'_{D,J} \rangle$.

Finally, we show that $F'_{C \setminus D, I} \leq \langle F'_{C,I}, F'_{D,J} \rangle$. Take an element $g \in F'_{C \setminus D, I}$, defined in the flow direction by a dyadic PL homeomorphism $f : I \rightarrow I$. Take the elements $g_1 \in F'_{C,I}$ and $g_2 \in F'_{C \cap D, I}$, defined by the same f . Then we have the equality $g = g_1 g_2^{-1}$. As $g \in F'_{C \setminus D, I}$ was arbitrary, this proves that $F'_{C \setminus D, I}$ is also a subgroup of $\langle F'_{C,I}, F'_{D,J} \rangle$. \square

LEMMA 4.8 (Fragmentation Lemma). *Assume that $C \times J$ is covered by a family $\{C_i \times J_i\}_{i \in \mathcal{L}}$, for clopen subsets $C_i \subset C$ and open dyadic intervals $J_i \subset J$. Then the group $F'_{C,J}$ is contained in the group generated by $\bigcup_{i \in \mathcal{L}} F'_{C_i, J_i}$.*

Proof. Let us denote by $H \leq \mathbb{T}(\varphi)$ the subgroup generated by $\bigcup_{i \in \mathcal{L}} F'_{C_i, J_i}$, and let us show that $H = F'_{C,J}$. Assume first that $J_i = J$ for every i . By compactness we can find a finite

cover C_1, \dots, C_r of C . Taking finitely many intersections and symmetric differences and using the Intersection Lemma (Lemma 4.7) we can replace it by a refinement D_1, \dots, D_s consisting of pairwise disjoint clopen subsets such that $F_{D_i, J} \leq H$ for every i . Every element of $F'_{C, J}$ is a commuting product of elements in $F'_{D_i, J}$, whence the conclusion. Assume next that the intervals J_i are arbitrary.

CLAIM. *For any $t \in J$ there exists a subinterval $I \subset J$ such that $t \in I$ and $F'_{C, I} \leq H$.*

Proof of Claim. Using compactness, we can extract a finite subfamily $\mathcal{F} \subset \mathcal{L}$, such that $C \times \{t\} \subset \bigcup_{j \in \mathcal{F}} C_j \times J_j$ and such that $t \in J_j$ for every $j \in \mathcal{F}$. Let $I = \bigcap_{j \in \mathcal{F}} J_j$. Since $F_{C_j, I} \leq F_{C_j, J_j}$, the previous case (applied to $J = I$) implies that $F'_{C, I} \leq H$. \square

By Lemma 4.2, we deduce that $F'_{C, J} \leq H$. \square

5. Simplicity

Theorem B will be an application of Proposition 2.2 (we refer to Section 2 for the definition of fragmentable).

PROPOSITION 5.1. *Let (X, φ) be a Stone system. Then the group $\mathbb{T}(\varphi)$, as group of homeomorphisms of the suspension Y^φ , is fragmentable.*

Proof. Let \mathcal{U} be an open cover of Y^φ . Let K be the subgroup of $\mathbb{T}(\varphi)$ generated by $\text{RiSt}(U)'$, for $U \in \mathcal{U}$, and let us show that $K = \mathbb{T}(\varphi)$. By Proposition 4.4, it is enough to show that $F'_{C, I} \leq K$ for every dyadic chart $\pi_{C, I}$. Since $U_{C \times I}$ is covered by \mathcal{U} and dyadic domains form a basis of the topology, we can find a cover $C \times I = \bigcup_{i \in \Lambda} C_i \times I_i$, with $C_i \subset C$ and $I_i \subset I$, such that each dyadic domain $U_{C_i \times I_i}$ is contained in some element $U_i \in \mathcal{U}$. Note that $F_{C_i, I_i} \leq \text{RiSt}(U_i)'$. By the Fragmentation Lemma (Lemma 4.8), we have $F'_{C, I} \leq \langle F_{I_i, C_i} \rangle_{i \in \Lambda} \leq \langle \text{RiSt}(U_i)' \rangle_{i \in \Lambda} \leq K$, as desired. \square

Proof of Theorem B. Suppose that (X, φ) is minimal. By Lemma 3.4, the action of $\mathbb{T}(\varphi)$ on Y^φ is minimal, and after Proposition 5.1, the group $\mathbb{T}(\varphi)$ is fragmentable. Hence we can apply Proposition 2.2 and obtain that $\mathbb{T}(\varphi)$ is simple. \square

6. Finite generation

In this section we prove Theorem A. We first recall a classical characterisation of subshifts in terms of existence of generating partitions. Let (X, φ) be a Stone system. A *generating partition* for (X, φ) is a clopen partition $X = C_1 \sqcup \dots \sqcup C_d$ such that the sets $\varphi^n(C_i)$, for $n \in \mathbb{Z}$, and $i = 1, \dots, d$, separate points of X , or equivalently generate the boolean algebra of clopen subsets of X . It is well-known that a generating partition exists if and only if (X, φ) is conjugate to a subshift. Indeed, if $X = C_1 \sqcup \dots \sqcup C_d$ is generating, the map $X \rightarrow \{1, \dots, d\}^{\mathbb{Z}}$ sending $x \in X$ to the unique sequence $(i_n) \in \{1, \dots, d\}^{\mathbb{Z}}$ such that $\varphi^n(x) \in C_{i_n}$ provides such a conjugation. Conversely if $X \subset \{1, \dots, d\}^{\mathbb{Z}}$ is a subshift, the subsets $C_i = \{(x_n)_{n \in \mathbb{Z}} \in X : x_0 = i\}$ provide a generating partition for (X, φ) .

In what follows we let (X, φ) be a Stone system conjugate to a subshift. Fix the dyadic intervals $I_0 = (-1/4, 1/2)$ and $I_1 = (1/4, 1+1/8)$. The only relevant properties of these two dyadic intervals will be that $|I_0|, |I_1| < 1$, they intersect non-trivially and their union is a neighbourhood of $[0, 1]$.

LEMMA 6.1. *Let (X, φ) be a Stone system. Let $K \leq \mathsf{T}(\varphi)$ be the group generated by F_{X, I_0} and F_{X, I_1} . Let $C \subset X$ be a clopen subset and I, J be open dyadic intervals of length $|I|, |J| < 1$. Then there exists an element $k \in K$ such that $k(U_{C \times J}) = U_{C \times I}$.*

Proof. Observe that $F_{X, I_i} = F_{\varphi^n(X), I_i - n} = F_{X, I_i - n}$ for every $i = 0, 1$ and $n \in \mathbb{Z}$, by the domain relations (2). Thus K contains $F_{X, L}$ whenever L is an interval obtained by translating I_0 or I_1 by an integer. Let U be the smallest interval containing both I and J . Without loss of generality we assume that the rightmost point of U coincides with that of J . Choose a cover of \bar{U} by a sequence L_1, \dots, L_m of intervals, each of which is a translate of I_0 or I_1 by an integer, such that $L_i \cap L_{i+1} \neq \emptyset$, for every $i = 1, \dots, m-1$. We argue by induction on m . If $m = 1$, by Lemma 4.1 we can find an element $f \in F_{L_1}$ such that $f(I) = f(J)$. The element $k \in F_{X, L_1}$ which acts as f in the flow direction will then verify the conclusion. If $m \geq 2$, we can choose an element $f \in F_{L_m}$ which maps the leftmost point of J into L_{m-1} , so that I and $f(J)$ are in the interval covered by L_1, \dots, L_{m-1} . Letting $g \in F_{X, L_m}$ be the element corresponding to f , we have $g(U_{C \times J}) = U_{C \times f(J)}$, and we conclude by the induction hypothesis applied to the intervals I and $f(J)$. \square

PROPOSITION 6.2. *Let (X, φ) be a Stone system. Assume that (X, φ) is conjugate to a subshift, and that $X = C_1 \sqcup \dots \sqcup C_d$ is a generating clopen partition of (X, φ) . Then the group $\mathsf{T}(\varphi)$ is generated by the subgroups F_{X, I_0} and F_{C_i, I_1} , for $i = 1, \dots, d$.*

Proof. Let $H \leq \mathsf{T}(\varphi)$ be the group generated by the subgroups in the statement, and let us show that $H = \mathsf{T}(\varphi)$. We begin by observing that every element $g \in F_{X, I_1}$ can be written as a commuting product $g = g_1 \cdots g_d$ of elements $g_i \in F_{C_i, I_1}$, so that H contains the group $K := \langle F_{X, I_0}, F_{X, I_1} \rangle$ as in the statement of Lemma 6.1. Thus conjugating the subgroups F_{C_i, I_1} by elements of K , we deduce that H contains $F_{C_i, J}$ for every $i = 1, \dots, d$ and every dyadic interval J such that $|J| < 1$. By the domain relations (2), this is actually equivalent to say that H contains $F_{\varphi^n(C_i), J}$ for every $n \in \mathbb{Z}$, every $i = 1, \dots, d$, and every J with $|J| < 1$. Using now the Intersection Lemma (Lemma 4.7), we obtain that $F_{D, J} \leq H$ whenever $D \subset X$ is a clopen subset that can be written as a finite intersection of the form $D = \bigcap_{j=1}^k \varphi^{n_j}(C_{I_j})$ with $n_j \in \mathbb{Z}$ and $i_j = 1, \dots, d$ (and $|J| < 1$ is an open dyadic interval). Now the fact that the partition C_1, \dots, C_d is generating means exactly that an arbitrary clopen set $C \subset X$ can be written as a finite union $C = D_1 \cup \dots \cup D_\ell$ of sets of that form. So the Fragmentation Lemma (Lemma 4.8) implies that $F_{C, J} \leq H$ for every clopen subset $C \subset X$ and every open dyadic interval $|J| < 1$. By Proposition 4.4, we deduce that $H = \mathsf{T}(\varphi)$. \square

Proof of Theorem A. Since the subgroups F_{X, I_0} and F_{C_i, I_1} , for every $i = 1, \dots, d$, are all isomorphic to Thompson's group F , and the latter is finitely generated, Proposition 6.2 implies Theorem A. \square

The converse of Theorem A is also true: namely if (X, φ) is not conjugate to a subshift, then the group $\mathsf{T}(\varphi)$ is never finitely generated.

PROPOSITION 6.3. *Let (X, φ) be a Stone system which is not conjugate to a subshift. Then the group $\mathsf{T}(\varphi)$ is not finitely generated.*

Proof. The proof is a standard argument in the setting of topological full groups, see the beginning of the proof of Theorem 5.4 in Matui [Mat06]. We sketch how to adapt the argument to this setting, following an approach kindly suggested by the referee. Given a Stone system (X, φ) , the projection $X \times \mathbb{R} \rightarrow \mathbb{R}$ passes to the quotient to a map $\eta: Y^\varphi \rightarrow \mathbb{R}/\mathbb{Z}$. For $t \in \mathbb{R}/\mathbb{Z}$ let

$X_t = \eta^{-1}(\{t\})$. Now let \mathcal{B} be a φ -invariant Boolean algebra of clopen subsets of X . We can translate \mathcal{B} to a Boolean algebra \mathcal{B}_t of X_t for each t , which only depends on t modulo 1 by φ -invariance of \mathcal{B} . For every $g \in \mathsf{T}(\varphi)$ and every $t \in \mathbb{R}/\mathbb{Z}$, there exists a clopen partition $X_t = C_1 \sqcup \cdots \sqcup C_n$ with the property that for every i , the set C_i is entirely contained in the domain $U_{C \times I}$ of a chart where g is given in coordinates by $\text{id} \times f$ for some PL homeomorphism $f: I \rightarrow f(I)$. We say that g is \mathcal{B} -admissible if for every $t \in \mathbb{R}/\mathbb{Z}$ such a partition can be chosen with $C_i \in \mathcal{B}_t, i = 1, \dots, n$. The set of \mathcal{B} -admissible elements is a subgroup $\mathsf{T}(\varphi, \mathcal{B})$ of $\mathsf{T}(\varphi)$, and it is easy to see that it is a strict subgroup unless \mathcal{B} separates points of X (i.e. coincides with the Boolean algebra of all clopen subsets of X). Moreover every finitely generated subgroup $G \leq \mathsf{T}(\varphi)$ is contained in $\mathsf{T}(\varphi, \mathcal{B})$ for some \mathcal{B} which is *finitely generated* as a φ -invariant Boolean algebra (that is, it is generated as a Boolean algebra by a finite number of clopen subsets together with all their φ -translates). Thus if $\mathsf{T}(\varphi)$ is itself finitely generated, we deduce that the Boolean algebra of all clopen subsets of X is itself finitely generated as a φ -invariant Boolean algebra, which implies that (X, φ) must be conjugate to a subshift. \square

Note that the same argument provides the following statement, that shows that the study of arbitrary finitely generated subgroups of $\text{PL}(\varphi)$ can essentially be reduced to the case where (X, φ) is a subshift. We record it here as it will be useful in Section 9.

LEMMA 6.4. *Let (X, φ) be a Stone system, and let G be a finitely generated subgroup of $\text{PL}(\varphi)$. Then there exists a subshift $(\bar{X}, \bar{\varphi})$ and a homomorphism $G \rightarrow \text{PL}(\bar{\varphi})$ with abelian kernel.*

Proof. In the proof we keep the same setting and notations from the proof of Proposition 6.3. For a φ -invariant Boolean algebra \mathcal{B} , we have a subgroup $\text{PL}(\varphi, \mathcal{B}) \leq \text{PL}(\varphi)$ which is defined analogously to $\mathsf{T}(\varphi, \mathcal{B})$.

Fix a Stone system (X, φ) and write $Y := Y^\varphi$. Let $G \leq \text{PL}(\varphi)$ be a finitely generated subgroup. Then we have $G \leq \text{PL}(\varphi, \mathcal{B})$ for some finitely generated φ -invariant Boolean algebra \mathcal{B} . By Stone duality, there exists a Stone system $(\bar{X}, \bar{\varphi})$ and a factor map $p: X \rightarrow \bar{X}$ from (X, φ) to $(\bar{X}, \bar{\varphi})$ such that \mathcal{B} coincides with the Boolean algebra of preimages of clopen subsets of \bar{X} via p . The system $(\bar{X}, \bar{\varphi})$ is conjugate to a subshift (using that \mathcal{B} is finitely generated). Set $\bar{Y} := Y^{\bar{\varphi}}$, denote by $\bar{\eta}: \bar{Y} \rightarrow \mathbb{R}/\mathbb{Z}$ the associated projection to the circle, and $\bar{X}_t := \bar{\eta}^{-1}(t)$ its fibers for $t \in \mathbb{R}/\mathbb{Z}$. The map $p: X \rightarrow \bar{X}$ then defines a map $\hat{p}: Y \rightarrow \bar{Y}$, which in restriction to each fiber X_t coincides with the quotient map $X_t \rightarrow \bar{X}_t$ associated to the Boolean algebra \mathcal{B}_t . Every element $g \in \text{PL}(\varphi, \mathcal{B})$ (and hence every $g \in G$) descends via the map $\hat{p}: Y \rightarrow \bar{Y}$ to an element $\bar{g} \in \text{PL}(\bar{\varphi})$, providing the desired homomorphism $G \rightarrow \text{PL}(\bar{\varphi})$. To see that its kernel must be abelian, let $g \in G$ be such that $\bar{g} = \text{id}$. Since the map \hat{p} sends X_t to \bar{X}_t for every $t \in \mathbb{R}/\mathbb{Z}$ and \bar{g} acts trivially on \bar{Y} , we have that $g(X_t) = X_t$ for every t . This implies that in restriction to each Φ -orbit g must coincide with a translation by an integer time of the flow Φ . Thus the kernel of $G \rightarrow \text{PL}(\varphi)$ acts on each Φ -orbit as a cyclic group of integer translations. This implies that it embeds in a product of cyclic groups, showing that it is abelian. \square

7. Actions on the circle

In this section we study actions on the circle of $\mathsf{T}(\varphi)$ and prove Theorem D. We begin with a lemma.

LEMMA 7.1. *Let (X, φ) be a Stone system. Let $g \in \mathsf{T}(\varphi)$ be an element whose closed support is contained in a domain $U_{C \times I}$. Then there exists a partition $C = C_1 \sqcup \cdots \sqcup C_r$ of C into clopen*

subsets such that g can be written as a commuting product $g = g_1 \cdots g_r$ with $g_j \in F_{C_j, I}$ for $j = 1, \dots, r$.

Proof. Fix $x \in C$. The element g must preserve the fiber $\pi_{C \times I}(\{x\} \times I)$ and act on it by an element $f_x \in F'_I$ (because its closed support is contained in I). Furthermore, for every $y = \pi_{C, I}(x, t) \in U_{C \times I}$ there exists a clopen neighbourhood $C_y \subset C$ of x and an interval $J_y \subset I$ containing t , such that for every x and $x' \in C_y$ we have $f_x|_{J_y} = f_{x'}|_{J_y}$. By compactness choose $y_1, \dots, y_\ell \in \pi_{C, I}(\{x\} \times I)$ such that $J_{y_1}, \dots, J_{y_\ell}$ cover the support of f_x and set $C_x = \bigcap_{i=1}^{\ell} C_{y_i}$. The clopen subset $C_x \subset C$ verifies $f_x = f_{x'}$ for every $x' \in C_x$. Since x was arbitrary, we can cover X with finitely many clopen subsets with this property, and refine this cover to obtain a partition $C = C_1 \sqcup \cdots \sqcup C_r$ which verifies the desired conclusion. \square

We will use the following notation. Given a group G acting by homeomorphism on a topological space Z , and given a point $z \in Z$ we denote by G_z the stabiliser of z in G , and by G_z^0 the *germ-stabiliser*, namely the subgroup of G_z consisting of elements that fix pointwise a neighbourhood of z . The next two lemmas analyse the germ-stabilisers for the action of $\mathbb{T}(\varphi)$ on Y^φ .

LEMMA 7.2 (First Return Decomposition). *Let (X, φ) be a minimal Stone system, and fix $y \in Y^\varphi$. For every finite subset $P \subset \mathbb{T}(\varphi)_y^0$, there exist disjoint dyadic domains $U_{C_1 \times I_1}, \dots, U_{C_r \times I_r}$ not containing y such that P is contained in the direct product $F'_{C_1, I_1} \times \cdots \times F'_{C_r, I_r} \leq \mathbb{T}(\varphi)_y^0$.*

In particular, the germ-stabilizer $\mathbb{T}(\varphi)_y^0$ is perfect, has no non-abelian free subgroup, and no non-trivial finite quotient.

Proof. Let $P \subset \mathbb{T}(\varphi)_y^0$ be any finite subset. Choose a dyadic domain $U_{C \times J} \ni y$ with $J = (a, b)$ and $|J| < 1/2$ such that every element $g \in P$ fixes pointwise a neighbourhood of the closure $\overline{U_{C \times J}}$. Write $\partial \overline{U_{C \times J}} = C_a \sqcup C_b$, where C_a corresponds to $C \times \{a\}$ and C_b to $C \times \{b\}$. By minimality, the Φ -orbit of every point in C_b must eventually return to $U_{C \times J}$, and it must enter through a point in C_a . For $x \in C$, let $\tau(x) = \inf\{t > 0 : \Phi^{b+t}([x, 0]) \in C_a\}$. Note that $\tau(x)$ is of the form $n_x - (b - a)$ for some positive integer n_x (equal to the first return time of x to C under φ). The function $\tau : C \rightarrow \mathbb{N} - (b - a)$ is locally constant (hence continuous and bounded). Let $C = C_1 \sqcup \cdots \sqcup C_r$ be the partition of C into level sets of τ , with $\tau = \tau_j$ on C_j . By construction, the interval $I_j = (b, b + \tau_j)$ is such that the chart π_{C_j, I_j} is defined (injective), and the domains $U_{C_j \times I_1}, \dots, U_{C_r \times I_r}$ form a partition of $Y^\varphi \setminus \overline{U_{C \times J}}$. Every $g \in P$ must preserve every domain $U_{C_j \times I_j}$ and is the commuting product of elements g_j supported in $U_{C_j \times I_j}$. By Lemma 7.1, upon refining the partition $C = \bigsqcup C_j$ we can assume that $g_j \in F'_{C_j \times I_j}$ for all j and all $g \in P$. After doing so, P is contained in the subgroup of $\mathbb{T}(\varphi)_y^0$ generated by the commuting subgroups $F'_{C_j \times I_j} \simeq F'$.

As a consequence, similarly to the proof of Corollary 4.6, we deduce that the group $\mathbb{T}(\varphi)_y^0$ is perfect and has no non-trivial finite quotient. Moreover, the group F' does not contain non-abelian free subgroups [BS85], so neither does $\mathbb{T}(\varphi)_y^0$. \square

LEMMA 7.3. *Let (X, φ) be a Stone system without finite orbits. For every two distinct points y and $z \in Y^\varphi$, we have*

$$\mathbb{T}(\varphi) = \left(\mathbb{T}(\varphi)_y^0 \mathbb{T}(\varphi)_z^0 \right) \cup \left(\mathbb{T}(\varphi)_z^0 \mathbb{T}(\varphi)_y^0 \right).$$

In particular $\mathbb{T}(\varphi)$ is generated by $\mathbb{T}(\varphi)_y^0$ and $\mathbb{T}(\varphi)_z^0$.

Proof. Let $g \in \mathbb{T}(\varphi)$. The points y and $g(y)$ lie on the same Φ -orbit, and delimit a unique interval on this orbit (possibly reduced to a point).

Assume at first that z does not lie on this interval (in particular, this holds if y and z lie on different Φ -orbits, or if $y = g(y)$). Choose a sufficiently small domain $U_{C \times I} \ni y$ such that g is given in coordinates by $\text{id} \times f$ for a PL homeomorphism $f: I \rightarrow f(I)$, and write $y = [x, t]$, with $x \in C$ and $t \in I$. Let $J \subset \mathbb{R}$ be an open dyadic interval which contains both t and $f(t)$. Since we assume that φ has no periodic orbits, we can find a clopen neighbourhood D such that $x \in D \subset C$, and whose first return time is greater than $|J|$ (this was defined in (1)). With this choice, the domain $U_{D \times J}$ is well-defined and $y \in U_{D \times J}$. Furthermore, since z does not lie on the arc between y and $g(y)$, we can shrink this domain so that it avoids a neighbourhood of z . Now let $k \in F'_J$ be an element that coincides with f^{-1} on a neighbourhood of $f(t)$, and $h = \text{id} \times k \in F'_{D,J}$. Then $h \in \mathbb{T}(\varphi)_z^0$ and $hg \in \mathbb{T}(\varphi)_y^0$, so that $g = h^{-1}hg \in \mathbb{T}(\varphi)_z^0 \mathbb{T}(\varphi)_y^0$.

We now consider the case where z lies on the interval between y and $g(y)$ on the corresponding Φ -orbit. Let \mathcal{O} be this orbit. Assume that $y < g(y)$ with respect to the natural orientation on \mathcal{O} , so that $y < z \leq g(y)$. Since g acts on \mathcal{O} in an orientation-preserving way, this implies that $g(y) < g(z)$, and thus y does not lie on the interval between z and $g(z)$. Thus we can exchange the roles of z and y and repeat the same argument as above to see that $g \in \mathbb{T}(\varphi)_y^0 \mathbb{T}(\varphi)_z^0$. The case $g(y) < y$ is treated similarly. \square

Remark 7.4. The previous two lemmas imply Proposition 4.4 in the special case where (X, φ) is an infinite minimal Stone system (note that the proposition has not been used in their proofs). Indeed, in this case Lemma 7.2 implies that every germ-stabiliser $\mathbb{T}(\varphi)_y^0$ is contained in the subgroup generated by the subgroups $F_{C,I}$, when $\pi_{C,I}$ varies over dyadic charts. By Lemma 7.3, the germs stabilisers generate $\mathbb{T}(\varphi)$, and Proposition 4.4 follows.

LEMMA 7.5. *Let (X, φ) be a minimal Stone system. For every $y \in Y^\varphi$, every action of the group $\mathbb{T}(\varphi)_y^0$ on the circle has a fixed point.*

Proof. By Lemma 7.2, the group $\mathbb{T}(\varphi)_y^0$ does not contain non-abelian free subgroups. Thus, a result of Margulis [Mar00] implies that every action of the group $\mathbb{T}(\varphi)_y^0$ on the circle preserves a Borel probability measure μ . Therefore, the rotation number $g \in \mathbb{T}(\varphi)_y^0 \mapsto \text{rot}(g) = \mu[\vartheta, g(\vartheta)) \pmod{\mathbb{Z}}$ defines a group homomorphism to \mathbb{R}/\mathbb{Z} (which does not depend on the choice of $\vartheta \in \mathbb{S}^1$). By Lemma 7.2, the only possibility is that the image of rot is $\{0\}$. Therefore every $g \in \mathbb{T}(\varphi)_y^0$ fixes a point, and μ must be supported on fixed points of g , for every $g \in \mathbb{T}(\varphi)_y^0$. Thus there exists a global fixed point (any point in the support of μ). \square

To conclude the proof of Theorem D we will work in the *Chabauty space* of subgroups of $\mathbb{T}(\varphi)$, by adapting arguments from [LMB18, §4.1.4]. For every group G , we denote by $\text{Sub}(G)$ the set of subgroups of G . The *Chabauty topology* on $\text{Sub}(G)$ is the topology induced by the inclusion of $\text{Sub}(G)$ into the set $\{0, 1\}^G$ of all subsets of G , endowed with the product topology, and this makes $\text{Sub}(G)$ a compact space. Given a compact space Z , a map $H: Z \rightarrow \text{Sub}(G)$ is said to be upper (respectively lower) semi-continuous if for every net $(z_i)_i$ converging to a limit $z \in Z$, every cluster point K of $(H(z_i))_i$ in $\text{Sub}(G)$ verifies $K \leq H(z)$ (respectively $H(z) \leq K$). If G acts on Z by homeomorphisms, the stabiliser map $z \mapsto G_z$ is always upper semi-continuous, while the germ-stabiliser map $z \mapsto G_z^0$ is always lower semi-continuous.

Proof of Theorem D. Let (X, φ) be a minimal Stone system, with X infinite. Consider an action of the group $\mathbb{T}(\varphi)$ on \mathbb{S}^1 . If it has a finite orbit, by simplicity of the group $\mathbb{T}(\varphi)$ (Theorem B, although Corollary 4.6 is enough) we deduce that it has a fixed point, and the conclusion follows. So we assume that the action does not have finite orbits. By general properties of groups of circle homeomorphisms, every group action on the circle without finite orbit is either minimal

or semi-conjugate to a minimal action (see [Ghy01, Propositions 5.6 and 5.8]). We can therefore assume that the action of $\mathbb{T}(\varphi)$ on \mathbb{S}^1 is minimal and show that this leads to a contradiction.

CLAIM. *Assume that $\mathbb{T}(\varphi)$ acts minimally on \mathbb{S}^1 . For every $\vartheta \in \mathbb{S}^1$ there exists a unique point $q(\vartheta) \in Y^\varphi$ such that $\mathbb{T}(\varphi)_{q(\vartheta)}^0 \leq \mathbb{T}(\varphi)_\vartheta$. Moreover, the map $q: \mathbb{S}^1 \rightarrow Y^\varphi$ defined in this way is equivariant and continuous.*

Before proving the claim, let us observe that it leads to the desired contradiction. Indeed the image $q(\mathbb{S}^1) \subset Y^\varphi$ would be a compact (hence closed) $\mathbb{T}(\varphi)$ -invariant subset of Y^φ , which by minimality (Lemma 3.4) would be equal to the whole space Y^φ . But the space Y^φ is not path-connected, and therefore cannot be a continuous image of \mathbb{S}^1 , reaching a contradiction. We now prove the claim.

Proof of Claim. Let us first observe that if the point $q(\vartheta)$ as in the statement exists, it must be unique. Indeed if $y \neq y' \in Y^\varphi$ are such that $\mathbb{T}(\varphi)_y^0$ and $\mathbb{T}(\varphi)_{y'}^0$ fix the point ϑ , then by Lemma 7.3 we deduce that the whole group $\mathbb{T}(\varphi)$ fixes ϑ , contradicting minimality of the action on \mathbb{S}^1 .

Let us now show the existence of the point $q(\vartheta)$. To this end fix $y_0 \in Y^\varphi$, and let $\vartheta_0 \in \mathbb{S}^1$ be a point fixed by $\mathbb{T}(\varphi)_{y_0}^0$ (which exists by Lemma 7.5), so that $\mathbb{T}(\varphi)_{y_0}^0 \leq \mathbb{T}(\varphi)_{\vartheta_0}$. Let $\vartheta \in \mathbb{S}^1$ be an arbitrary point. By minimality, and upon extracting a subnet, we can choose a net $(g_i)_i \subset \mathbb{T}(\varphi)$ such that:

- $(g_i(\vartheta_0))_i \subset \mathbb{S}^1$ converges to ϑ ,
- $(g_i(y_0))_i \subset Y^\varphi$ converges to a limit $y \in Y^\varphi$,
- the nets $(\mathbb{T}(\varphi)_{g_i(y_0)}^0)_i$ and $(\mathbb{T}(\varphi)_{g_i(\vartheta_0)}^0)_i$ both converge in $\text{Sub}(\mathbb{T}(\varphi))$ to limit subgroups denoted respectively by H and K .

Since

$$\mathbb{T}(\varphi)_{g_i(y_0)}^0 = g_i \mathbb{T}(\varphi)_{y_0}^0 g_i^{-1} \leq g_i \mathbb{T}(\varphi)_{\vartheta_0} g_i^{-1} = \mathbb{T}(\varphi)_{g_i(\vartheta_0)},$$

we have $H \leq K$, and by lower semi-continuity of the germ-stabiliser map and upper semi-continuity of the stabiliser map we obtain $\mathbb{T}(\varphi)_y^0 \leq H \leq K \leq \mathbb{T}(\varphi)_\vartheta$. Thus $q(\vartheta) := y$ verifies the desired conclusion.

Equivariance of the map q is clear. Let us show that it is continuous. Assume that $(\vartheta_i)_i \subset \mathbb{S}^1$ is a net converging to a limit $\vartheta \in \mathbb{S}^1$. Let $y \in Y^\varphi$ be a cluster point of the net $(q(\vartheta_i))_i$. We have $\mathbb{T}(\varphi)_{q(\vartheta_i)}^0 \leq \mathbb{T}(\varphi)_{\vartheta_i}$, and using semi-continuity exactly as in the proof of existence of $q(\vartheta)$ we obtain that $\mathbb{T}(\varphi)_y^0 \leq \mathbb{T}(\varphi)_\vartheta$. By uniqueness of $q(\vartheta)$ we must have $y = q(\vartheta)$. Since y was an arbitrary cluster point of $(q(\vartheta_i))_i$, we deduce that the net $(q(\vartheta_i))_i$ converges to $q(\vartheta)$, showing continuity of the the map q . \square

This concludes the proof of the theorem. \square

8. Central extensions of Thompson's group T

In this section we explain that the group $\mathbb{T}(\varphi)$ always contains subgroups isomorphic to a central extension of Thompson's group T acting on the circle. In particular it always contains free subgroups, and is therefore non-amenable.

Let $\tilde{T} \leq \text{Homeo}_+(\mathbb{R})$ be the natural lift of Thompson's group T to the real line under the covering map $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$. Explicitly, \tilde{T} consists of all homeomorphisms of \mathbb{R} which commute

with integer translations and are PL dyadic with a (possibly infinite) discrete closed set of breakpoints for the derivative. It is a non-trivial central extension of Thompson's group T :

$$0 \rightarrow \mathbb{Z} \rightarrow \tilde{T} \rightarrow T \rightarrow 1.$$

For each $k \geq 1$, let $T^{(k)}$ be the quotient of \tilde{T} by the subgroup consisting of translations by a multiple of k . Note that $T^{(k)}$ identifies with the lift of T under the k -fold self-covering of the circle $\mathbb{S}^1 \rightarrow \mathbb{S}^1, x \mapsto kx$. We have the following.

PROPOSITION 8.1. *Let (X, φ) be a Stone system. If φ has infinite order, then $\mathbb{T}(\varphi)$ contains a subgroup isomorphic to the group \tilde{T} . If φ has order $k \geq 1$, the group $\mathbb{T}(\varphi)$ contains a subgroup isomorphic to $T^{(k)}$.*

Proof. Let \tilde{T} act on $X \times \mathbb{R}$ diagonally with trivial action on X and by its natural action on \mathbb{R} . This action commutes with the diagonal \mathbb{Z} -action on $X \times \mathbb{R}$ given by $n \cdot (x, t) \mapsto (\varphi^n(x), t - n)$, and thus passes to the quotient to an action on the suspension Y^φ , which gives rise to a homomorphism $\tilde{T} \rightarrow \mathbb{T}(\varphi)$. Moreover, given an orbit $\mathcal{O} \subset Y^\varphi$ of the suspension flow Φ , the action of \tilde{T} on \mathcal{O} is faithful unless \mathcal{O} is periodic, and in this case its kernel consists exactly of the subgroup of translations $k\mathbb{Z}$, where k is the period of \mathcal{O} . It follows that if φ is periodic of period k , the homomorphism $\tilde{T} \rightarrow \text{PL}(\varphi)$ factors through an injective homomorphism $T^{(k)} \rightarrow \text{PL}(\varphi)$. Otherwise φ admits infinite orbits or finite orbits of arbitrarily large period, so that the homomorphism $\tilde{T} \rightarrow \text{PL}(\varphi)$ is injective. \square

COROLLARY 8.2. *For every Stone system (X, φ) , the group $\mathbb{T}(\varphi)$ contains free subgroups.*

Proof. The group T contains free subgroups, therefore so do its extensions \tilde{T} and $T^{(k)}$ for $k \geq 1$. We conclude by Proposition 8.1. \square

We next prove Theorem 8.7, which states that every action of the central extension \tilde{T} on \mathbb{R} , without global fixed point, is semi-conjugate to the standard action. Before this, let us clarify what is a semi-conjugacy in this setting (in particular, for us a semi-conjugacy can reverse the orientation, cf. the definitions in [KKM18]):

DEFINITION 8.3. Let $\rho_\nu : G \rightarrow \text{Homeo}_+(\mathbb{R})$, $\nu = 1, 2$, be two representations. They are *semi-conjugate* if the following holds: there exists a weakly monotone proper map $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$h \rho_1(g) = \rho_2(g) h, \quad \text{for any } g \in G.$$

We also need the analogue notion for actions on the circle.

DEFINITION 8.4. Let $\rho_\nu : G \rightarrow \text{Homeo}_+(\mathbb{S}^1)$, $\nu = 1, 2$, be two representations. They are *semi-conjugate* if the following holds: there exist

- a weakly monotone map $h : \mathbb{R} \rightarrow \mathbb{R}$ either commuting or anti-commuting with the integer translations (this depends on the monotonicity of h) and
- two corresponding central lifts $\hat{\rho}_\nu : \hat{G} \rightarrow \text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ to homeomorphisms of the real line commuting with integer translations,

such that

$$h \hat{\rho}_1(\hat{g}) = \hat{\rho}_2(\hat{g}) h, \quad \text{for any } \hat{g} \in \hat{G}.$$

Remark 8.5. Both for actions on the real line and on the circle, if a representation ρ_1 is semi-conjugate to a *minimal* representation ρ_2 , then the map h realizing the semi-conjugacy is automatically continuous and surjective. (This is forced by the fact that the image of h must be dense, together with the monotonicity of h). Here we will only use the notion of semi-conjugacy in this case.

The following result appears in [Ghy01, LBMB18]:

THEOREM 8.6. *Every non-trivial action of Thompson's group T on the circle is semi-conjugate to the standard action.*

Let us observe that it implies the following:

THEOREM 8.7. *Let \tilde{T} denote the central extension of Thompson's T . Every action of \tilde{T} by homeomorphisms of the real line, without global fixed points, is semi-conjugate to the action lifting the action of T on the circle.*

Proof. Let $\rho : \tilde{T} \rightarrow \text{Homeo}_+(\mathbb{R})$ be an action without global fixed points. Let us denote by $\gamma \in \tilde{T}$ the central element corresponding to the translation $t \mapsto t + 1$ in the standard action, so that $\langle \gamma \rangle$ is the center of \tilde{T} .

CLAIM. *The homeomorphism $\rho(\gamma)$ has no fixed point.*

Proof of Claim. Assume for contradiction that $Z := \text{Fix}(\rho(\gamma)) \neq \emptyset$. As γ is central, the subset $Z \subset \mathbb{R}$ is $\rho(\tilde{T})$ -invariant, and the corresponding action of \tilde{T} on Z passes to the quotient to an action of $T = \tilde{T}/\langle \gamma \rangle$ which preserves the order of points of $Z \subset \mathbb{R}$. As T is simple and contains torsion elements, this implies that the induced action on Z is trivial, that is, $\text{Fix}(\rho(\gamma)) = \text{Fix}(\rho(\tilde{T}))$. Thus the action has a global fixed point, giving the desired contradiction. \square

Now after conjugating ρ by an element of $\text{Homeo}_+(\mathbb{R})$ we can assume that $\rho(\gamma) = \gamma^{\pm 1}$, so that $\rho(\langle \gamma \rangle) = \langle \gamma \rangle$. Then the action $\rho : \tilde{T} \rightarrow \text{Homeo}_+(\mathbb{R})$ descends to a non-trivial action $\bar{\rho} : T \rightarrow \text{Homeo}_+(\mathbb{S}^1)$, where $\mathbb{S}^1 = \mathbb{R}/\langle \gamma \rangle$, which by Theorem 8.6 is semi-conjugate to the standard action. Thus, upon replacing again ρ with a semi-conjugate representation, we can assume that ρ is a homomorphism $\rho : \tilde{T} \rightarrow \tilde{T} \leq \text{Homeo}_+(\mathbb{R})$ such that $\bar{\rho}(\langle \gamma \rangle) = \langle \gamma \rangle$, and which passes to the quotient to the identity map $T \rightarrow T$. This implies that for every $g \in \tilde{T}$ we have $\rho(g) = g\gamma^{n_g}$ for some integer $n_g \in \mathbb{Z}$. Using that γ is central one readily checks that the map $g \mapsto n_g$ is a homomorphism $\tilde{T} \rightarrow \mathbb{Z}$, and since \tilde{T} is perfect, we must have $n_g = 0$ for all $g \in \tilde{T}$. Thus ρ is the identity. \square

9. Approximation by subshifts of finite type and lack of Kazhdan subgroups

In this section we prove the non-existence of infinite subgroups with Kazhdan's property (T) in the group $\text{PL}(\varphi)$. As another consequence of the arguments used, we also observe that for every infinite minimal subshift (X, φ) the group $\text{T}(\varphi)$ is not finitely presentable.

Let us set up some extra notation. Given a Stone system (X, φ) and a closed subset $Z \subset X$ which is φ -invariant, we denote by $\varphi|_Z$ the restriction of φ to Z and by $Y_Z^\varphi \subset Y^\varphi$ the suspension of $(Z, \varphi|_Z)$, which is a closed Φ -invariant subset of Y^φ .

We begin with two lemmas that hold for any Stone system (X, φ) .

LEMMA 9.1. *Let (X, φ) be a Stone system and let $Z \subset X$ be a φ -invariant closed subset. We let G be either $\text{PL}(\varphi)$ or $\text{T}(\varphi)$ and $G_Z = \text{PL}(\varphi|_Z)$ or $\text{T}(\varphi|_Z)$ accordingly. The restriction of the action of G to Y_Z^φ defines an epimorphism $G \rightarrow G_Z$, whose kernel is infinite unless $Z = X$.*

Proof. Consider the case $G = \text{T}(\varphi)$. It is clear that the restriction of the action defines a group homomorphism $\text{T}(\varphi) \rightarrow \text{T}(\varphi|_Z)$, and we need to show that it is surjective. Let $C \subset Z$ be a clopen subset and J be a dyadic interval of length $|J| < 1$. Choosing a clopen subset $D \subset X$ such that $C = D \cap Z$, we see that the subgroup $F'_{C,J} \leq \text{T}(\varphi|_Z)$ is the image of the subgroup $F'_{D,J} \leq \text{T}(\varphi)$. Therefore the image of $\text{T}(\varphi)$ in $\text{T}(\varphi|_Z)$ contains a generating set of $\text{T}(\varphi|_Z)$. Finally, taking a clopen subset $C \subset (X \setminus Z)$ and any interval I with $|I| < 1$, we see that $F_{C,I} \leq \text{T}(\varphi)$ acts trivially on Y_Z^φ and therefore the kernel of the map $\text{T}(\varphi) \rightarrow \text{T}(\varphi|_Z)$ is infinite.

The proof is identical for $G = \text{PL}(\varphi)$ (one needs to replace the groups $F'_{C,I}$ with groups defined similarly for $\text{PL}(\varphi)$). \square

We point out that this implies that minimality is a necessary condition in Theorem B.

COROLLARY 9.2. *If the system (X, φ) is not minimal, the group $\text{T}(\varphi)$ admits non-trivial proper quotients.*

LEMMA 9.3. *Let (X, φ) be a Stone system. Consider a nested decreasing sequence of φ -invariant closed subsets $(Z_n)_n$ of X and let $Z = \bigcap Z_n$. We let G_Z be either $\text{PL}(\varphi|_Z)$ or $\text{T}(\varphi|_Z)$ and $G_{Z_n} = \text{PL}(\varphi|_{Z_n})$ or $\text{T}(\varphi|_{Z_n})$ accordingly. Then G_Z is isomorphic to the direct limit $\varinjlim G_{Z_n}$ with respect to the restriction maps.*

Proof. Let G be either $\text{PL}(\varphi)$ or $\text{T}(\varphi)$, depending on the choice of G_Z . Fix $g \in G$ which projects trivially to G_Z . By Lemma 9.1, this implies that every point $y \in Y_Z^\varphi$ is contained in a domain $U_{C \times I}$ on which g acts trivially. Thus g acts trivially on a neighbourhood U of Y_Z^φ in Y^φ . By compactness, for m large enough, we have $Y_{Z_m}^\varphi \subset U$, and therefore the projection of g to G_{Z_m} is eventually trivial. This means exactly that G_Z is the direct limit of the sequence $(G_{Z_n})_n$. \square

We now restrict to the case where (X, φ) is a subshift over a finite alphabet A . Recall that a subshift $X \subset A^{\mathbb{Z}}$ over a finite alphabet is called a *subshift of finite type* if there exists $n \geq 1$ and a finite set $L \subset A^n$ of the set of words of length n such that X consists exactly of all sequences whose subwords of length n all belong to L .

Assume that $X \subset A^{\mathbb{Z}}$ is an arbitrary subshift (not necessarily of finite type). For each $n \geq 1$ let $L_n \subset A^n$ be the collection of finite words of length n that appear in some sequence of X , and let (X_n, φ_n) be the subshift of finite type defined by L_n . This provides a nested sequence of subshifts:

$$X_1 \supset X_2 \supset \cdots \supset X = \bigcap_{n \geq 1} X_n.$$

In this specific setting, we rewrite Lemma 9.3 for further reference as follows:

LEMMA 9.4. *Let (X, φ) be a subshift, and $((X_n, \varphi_n))_n$ be the sequence of subshifts of finite type defined above. Then the group $\text{PL}(\varphi)$ is the direct limit of the sequence of groups $(\text{PL}(\varphi_n))_n$ with respect to the restriction homomorphisms $\text{PL}(\varphi_1) \rightarrow \text{PL}(\varphi_2) \rightarrow \cdots$. Similarly the group $\text{T}(\varphi)$ is the direct limit of the sequence of groups $(\text{T}(\varphi_n))_n$.*

We point out the following consequence:

COROLLARY 9.5. *If (X, φ) is a subshift which is not of finite type, the group $\mathbb{T}(\varphi)$ is not finitely presentable. In particular for every infinite minimal subshift (X, φ) , the group $\mathbb{T}(\varphi)$ is not finitely presentable.*

Proof. If (X, φ) is not of finite type, the inclusion $X_n \supset X$ is strict for every $n \geq 1$, so that the natural epimorphism $\mathbb{T}(\varphi_n) \rightarrow \mathbb{T}(\varphi)$ has a non-trivial kernel. Since $\mathbb{T}(\varphi) = \varinjlim \mathbb{T}(\varphi_n)$ is the direct limit of finitely generated groups (by Theorem A applied to every $\mathbb{T}(\varphi_n)$), the group $\mathbb{T}(\varphi)$ cannot be finitely presented. Moreover a subshift of finite type has periodic orbits, thus it is never minimal (unless it is finite and consists of a single periodic orbit). \square

We now turn to study Kazhdan's property (T) for subgroups of $\text{PL}(\varphi)$. For preliminaries on Kazhdan's property (T) , we refer to [BdlHV08]. We will freely use some basic facts, in particular that a discrete group with Kazhdan's property (T) is finitely generated, that homomorphic images of a property (T) group still have property (T) , and that amenable groups with property (T) are finite.

In addition we will need the following result from [Cor] and [LMBT]. We denote by $\text{PL}(\mathbb{S}^1)$ the group of piecewise linear orientation-preserving homeomorphisms of the circle.

THEOREM 9.6. *Every subgroup of $\text{PL}(\mathbb{S}^1)$ with Kazhdan's property (T) is finite cyclic.*

To proceed to the proof of Theorem F we begin by considering the case where (X, φ) is a subshift of finite type. Recall that a subshift of finite type defined by the set $L \subset A^n$ is *irreducible* if it is not reduced to a single periodic orbit and if for every w_1 and $w_2 \in L$, there exists a finite word $u \in A^*$ which admits w_1 as a prefix and w_2 as a suffix, and such that all subwords of u of length n belong to L . It is well-known and not difficult to see that an irreducible subshift of finite type has a dense set of periodic orbits, and admits dense infinite orbits.

LEMMA 9.7. *Assume that (X, φ) is an irreducible subshift of finite type. Then $\text{PL}(\varphi)$ has no non-trivial subgroup with Kazhdan's property (T) .*

Proof. Let $H \leq \text{PL}(\varphi)$ be a subgroup with Kazhdan's property (T) . The subshift (X, φ) has a dense collection of periodic orbits, hence the suspension (Y^φ, Φ) has a dense collection of closed Φ -orbits. The restriction of the action of H to a closed orbit \mathcal{C} gives rise to a homomorphism $\rho_{\mathcal{C}} : H \rightarrow \text{PL}(\mathbb{S}^1)$ to the group of PL homeomorphisms of the circle. Using Theorem 9.6, the image $\rho_{\mathcal{C}}(H)$ must be finite cyclic. But the density of closed orbits implies that H acts faithfully on the union of closed orbits. Therefore H embeds in an infinite product of finite cyclic groups. We deduce that H is abelian (hence amenable), and since it has property (T) , it must be finite. However an irreducible subshift of finite type has a dense infinite orbit, so that by Proposition 3.3 we deduce that $\text{PL}(\varphi)$ is torsion-free and H is actually trivial. \square

LEMMA 9.8. *Let (X, φ) be an infinite minimal subshift. Then $\text{PL}(\varphi)$ has no non-trivial subgroup with Kazhdan's property (T) .*

Proof. Let $H \leq \text{PL}(\varphi)$ have Kazhdan's property (T) , in particular H is finitely generated, let S be a finite generating set of H . Let $((X_n, \varphi_n))_n$ be the associated sequence of subshifts of finite type as explained above. Then each (X_n, φ_n) is irreducible: indeed any two words w_1 and $w_2 \in L_n$ appear infinitely often in every sequence of X , as a consequence of the minimality of (X, φ) . By Lemma 9.4 we have $\text{PL}(\varphi) = \varinjlim \text{PL}(\varphi_n)$. Since the connecting homomorphisms are surjective, we can choose a finite set $S_1 \subset \text{PL}(\varphi_1)$ which projects bijectively to S . Let H_1 be the group generated by S_1 and $H_n \leq \text{PL}(\varphi_n)$ be its image. Then each H_n is finitely generated and

$H = \varinjlim H_n$. By [Sha00, Theorem 6.7] and [Gro03, §3.8.B], a group with property (T) cannot be a direct limit of finitely generated groups without property (T). We deduce that H_n must also have property (T) for n large enough. By Lemma 9.7, we deduce that H_n is trivial for n large enough, and thus so is H . \square

PROPOSITION 9.9. *For every subshift (X, φ) , every subgroup of $\text{PL}(\varphi)$ with Kazhdan's property (T) is finite abelian.*

Proof. Fix $H \leq \text{PL}(\varphi)$ with Kazhdan's property (T). For $x \in X$ we let $\mathcal{O}_x \subset Y^\varphi$ be the corresponding Φ -orbit in the suspension, and let $\rho_x: H \rightarrow \text{Homeo}_+(\mathcal{O}_x)$ be the homomorphism obtained by restricting the H -action to \mathcal{O}_x . Let us prove that, for every $x \in X$, the image of ρ_x must be finite cyclic. As in Lemma 9.7, this implies that H is abelian, hence finite by Kazhdan's property (T). If x is periodic, so that \mathcal{O}_x is a closed orbit, then the image of ρ_x is finite cyclic by Theorem 9.6. Thus we can assume that x has an infinite φ -orbit (in this case, we will prove that the image of ρ_x is trivial).

Choose a closed minimal φ -invariant subset $Z \subset X$ contained in the orbit-closure of x (note that Z could be reduced to a single finite orbit). Consider the restriction homomorphism $H \rightarrow \text{PL}(\varphi|_Z)$ (Lemma 9.1). If Z is a finite orbit, Theorem 9.6 implies that its image is finite cyclic, and if Z is infinite Lemma 9.8 implies that its image is trivial. In both cases, denote by $K \trianglelefteq H$ its kernel, which has finite index in H . The group K still has property (T), and in particular it is finitely generated.

CLAIM. *There exists an open subset $V \supset Y_Z^\varphi$ such that K fixes V pointwise.*

Proof of Claim. Assume first that Z is infinite, so that Y_Z^φ does not contain closed orbits. Fix $g \in K$. For every $y \in Y_Z^\varphi \subset Y^\varphi$, there exists a domain $U_{C \times I} \subset Y^\varphi$ containing y , in restriction to which g is given by a single PL homeomorphism $f: I \rightarrow f(I)$. As g is the identity in restriction to $Y_Z^\varphi \cap U_{C \times I}$ and Y_Z^φ does not contain closed orbits, f must necessarily be the identity on I , so that g actually fixes $U_{C \times I}$ pointwise. Since y was arbitrary, we have that g is the identity in restriction to an open neighbourhood of Y_Z^φ . Finally, since K is finitely generated, we can find an open neighbourhood of Y_Z^φ which is fixed by all elements in some finite generating set of K and the conclusion follows.

Assume now that Z is a finite orbit of period $N > 0$, so that Y_Z^φ consists of a single closed orbit of length N . Again fix $g \in K$. Let $y \in Y_Z^\varphi \subset Y^\varphi$ and choose a domain $U_{C \times I} \subset Y^\varphi$ containing y , where g is given by a single PL homeomorphism $f: I \rightarrow f(I)$. As g belongs to the kernel of the restriction homomorphism (Lemma 9.1), the PL homeomorphism f must coincide on I with a translation by an integer $n_{g,y}$ which is a multiple of N . In other words, the germ of g at the point y coincides with the germ of the flow $\Phi^{n_{g,y}}$. (Note that in particular the integer $n_{g,y}$ does not depend on the choice of the domain, as every neighbourhood of y must intersect non-trivially the orbit \mathcal{O}_x which is not periodic, so that the flow Φ acts freely on it.) The map $g \mapsto n_{g,y}$ is a homomorphism $K \rightarrow \mathbb{Z}$. Using that K has Kazhdan's property (T), this homomorphism must have trivial image, so that we must have $n_{g,y} = 0$ for every $g \in K$. In particular, every g must fix a neighbourhood of y , and we can repeat the previous reasoning. \square

To conclude the proof of the proposition, let $V \supset Y_Z^\varphi$ be a neighbourhood fixed by K . Since \mathcal{O}_x accumulates on Y_Z^φ , the intersection $\mathcal{O}_x \cap V$ is non-empty. It follows that the action of $\rho_x(K)$ on \mathcal{O}_x admits global fixed points. Assume towards a contradiction that $\rho_x(K)$ is non-trivial, and let $\xi \in \mathcal{O}_x$ a point lying on the boundary of the set of global fixed points. Since the action of elements of $\rho_x(K)$ is piecewise linear, we have a non-trivial homomorphism $\rho_x(K) \rightarrow \mathbb{R}$ given by

the logarithm of a one-sided derivative at ξ . This is in contradiction with the fact that K has Kazhdan's property (T). We deduce that $\rho_x(K)$ is trivial. In particular, $\rho_x(H)$ is a homomorphic image of H/K , and therefore is finite cyclic. Since moreover $\rho_x(H)$ is torsion-free, we deduce that it is actually trivial. This concludes the proof. \square

The assumption that (X, φ) is a subshift is not essential, due to Lemma 6.4:

Proof of Theorem F. Let (X, φ) be an arbitrary Stone system, and assume that $G \leq \text{PL}(\varphi)$ is a subgroup with Kazhdan's property (T). Then G is finitely generated, so that by Lemma 6.4 we can find a subshift (X', φ') and a homomorphism $G \rightarrow \text{PL}(\varphi')$ with abelian kernel. By Proposition 9.9, the image of this homomorphism is finite. Therefore G is virtually abelian, and thus property (T) implies that it is finite. \square

10. Isomorphisms correspond to flow equivalences

We now prove Theorem H, which will be obtained by combining two results. The first is the following:

THEOREM 10.1 (Ben Ami and Rubin [BAR10]). *For $i = 1, 2$, let G_i be a fragmentable group of homeomorphisms of a locally compact Hausdorff space Y_i whose action on Y_i has no global fixed points. Then for every group isomorphism $\rho: G_1 \rightarrow G_2$ there exists a homeomorphism $q: Y_1 \rightarrow Y_2$ such that $\rho(g) = q \circ g \circ q^{-1}$.*

The second is a characterisation of flow equivalence due to Parry and Sullivan [PS75], which we recall now. Given a Stone system (X, φ) , a *clopen transversal* is a clopen subset $C \subset X$ which intersects all φ -orbits. It follows that for every point $x \in X$ the first entry time

$$T_C(x) = \min\{n \geq 1: \varphi^n(x) \in C\}$$

is finite and locally constant as a function of x , and that the corresponding *first return map* $\varphi_C: C \rightarrow C, x \mapsto \varphi^{T_C(x)}(x)$ is a homeomorphism.

THEOREM 10.2 (Parry and Sullivan [PS75]). *Two Stone systems (X_i, φ_i) , for $i = 1, 2$, are flow equivalent if and only if they admit clopen transversals $C_i \subset X_i$ whose first return maps are topologically conjugate.*

See also [BCE17] for a more recent and detailed exposition of this result.

Proof of Theorem H. For $i = 1, 2$, let (X_i, φ_i) be a Stone system with suspension flow (Y_i, Φ_i) , and assume that each (X_i, φ_i) admits a dense orbit. Assume that the groups $\mathbb{T}(\varphi_i)$ are isomorphic. By Proposition 5.1, the groups $\mathbb{T}(\varphi_i)$ are fragmentable. Thus by Theorem 10.1 there exists a homeomorphism $q: Y_1 \rightarrow Y_2$ which conjugates the actions. Note that q must send Φ_1 -orbits to Φ_2 -orbits, as these are the connected components of Y_1 and Y_2 , respectively. Moreover it must simultaneously preserve or reverse the orientations of all orbits, as this is uniquely determined by whether it does so on any dense orbit of Y_1 . Thus (X_1, φ_1) is flow equivalent to (X_2, φ_2) or to (X_2, φ_2^{-1}) .

Conversely assume that (X_1, φ_1) is flow equivalent to (X_2, φ_2) and let us show that $\mathbb{T}(\varphi_1)$ and $\mathbb{T}(\varphi_2)$ are isomorphic (note that it is enough to consider this case, since $\mathbb{T}(\varphi_2) = \mathbb{T}(\varphi_2^{-1})$). To this end we need to check that a homeomorphism $q: Y_1 \rightarrow Y_2$ realising the flow equivalence can be chosen so that it conjugates $\mathbb{T}(\varphi_1)$ to $\mathbb{T}(\varphi_2)$. By Theorem 10.2, the two systems admit

clopen transversals whose first return maps are topologically conjugate, and we can identify these two clopen transversals with a same subset $C \subset X_1$ and $C \subset X_2$ with common first return map equal to $\psi: C \rightarrow C$; let $T^{(i)}: C \rightarrow \mathbb{N}$ be the first return time with respect to φ_i . Choose a partition $C = \bigsqcup_{j=1}^r C_j$ into clopen subsets, such that in restriction to every C_j , the functions $T^{(1)}$ and $T^{(2)}$ are both constant and equal to integers n_j and m_j respectively. We can write Y_1 and Y_2 as the union of the closures of domains $Y_1 = \bigcup_{j=1}^r \overline{U_{C_j \times (0, n_j)}^{(1)}}$ and $Y_2 = \bigcup_{j=1}^r \overline{U_{C_j \times (0, m_j)}^{(2)}}$. For every j , the choice of a homeomorphism $f_j: [0, n_j] \rightarrow [0, m_j]$ induces a homeomorphism $q_j: \overline{U_{C_j \times (0, n_j)}^{(1)}} \rightarrow \overline{U_{C_j \times (0, m_j)}^{(2)}}$, and by construction the homeomorphisms q_j are compatible on the boundary and give rise to a homeomorphism $q: Y_1 \rightarrow Y_2$. If we choose the homeomorphisms f_j to be PL dyadic, the resulting homeomorphism q will conjugate $\mathsf{T}(\varphi_1)$ to $\mathsf{T}(\varphi_2)$. \square

Appendix A. Topology on $\mathsf{H}_0(\varphi)$

In this appendix we endow the group $\mathsf{H}_0(\varphi)$ with a topology defined by a complete metric. When X is metrisable, this topology is separable, and thus makes $\mathsf{H}_0(\varphi)$ a Polish group. We then observe that the subgroup $\mathsf{T}(\varphi)$ is dense in it with respect to this topology. This is analogous to the fact that Thompson's group T is dense in $\mathsf{Homeo}_+(\mathbb{S}^1)$ for the C^0 topology [BS16, Corollary A5.8]. As application, we prove Proposition 4.4.

For any closed interval $I \subset \mathbb{R}$, the group $\mathsf{Homeo}_+(I)$ is a Polish group with respect to the distance $\delta(g, h) = \sup_{x \in I} |f(x) - g(x)| + \sup_{x \in I} |f^{-1}(x) - g^{-1}(x)|$. The topology on $\mathsf{H}_0(\varphi)$ is a natural generalization of this.

By [BCE17, Theorem 3.1], for every $g \in \mathsf{H}_0(\varphi)$ there exists a continuous function $\tau: Y^\varphi \rightarrow \mathbb{R}$ such that for every $y \in Y^\varphi$ we have $g(y) = \Phi^{\tau(y)}(y)$. We will say that τ is a *displacement function* for g . Note that the value of $\tau(y)$ is uniquely determined by g and y , provided y belongs to a non-closed orbit. Thus, by continuity, a displacement function for g is uniquely determined by g provided the union of non-periodic orbits of (X, φ) is dense, that is, if (X, φ) is topologically free. However, if this assumption is not satisfied, an element g can admit more than one displacement function. If τ and τ' are two different displacement functions for g , then $\sigma := \tau - \tau'$ takes integer values, and its value at every point $y \in Y^\varphi$ is a multiple of the period of y . Conversely given any continuous function $\sigma: Y^\varphi \rightarrow \mathbb{Z}$ with this property and a displacement function τ for g , then $\tau' := \tau + \sigma$ is still a displacement function for g .

Finally note that if τ_1 is a displacement function for g_1 and τ_2 for g_2 , then a displacement function for $g_3 = g_2 g_1$ is given by the formula

$$\tau_3(y) = \tau_2(g_1(y)) + \tau_1(y). \quad (4)$$

For any two g and $h \in \mathsf{H}_0(\varphi)$, set

$$d_+(g, h) = \inf_{\tau, \lambda} \sup_{y \in Y^\varphi} |\tau(y) - \lambda(y)|$$

where the infimum is taken over all possible displacement functions τ and λ , for g and h respectively. This defines a distance on $\mathsf{H}_0(\varphi)$, which is however not complete. To ensure completeness, set $d_-(g, h) := d_+(g^{-1}, h^{-1})$, and

$$d(g, h) = d_+(g, h) + d_-(g, h)$$

which is again a distance on $\mathsf{H}_0(\varphi)$.

PROPOSITION A.1. *The distance d is complete and defines a group topology on $\mathsf{H}_0(\varphi)$.*

Proof. Let $(g_n)_n$ be a Cauchy sequence. It follows that we can find a sequence of displacement functions $(\tau_n)_n$ for $(g_n)_n$ and $(\lambda_n)_n$ for $(g_n^{-1})_n$ that are Cauchy with respect to the distance of uniform convergence. Therefore, $(\tau_n)_n$ and $(\lambda_n)_n$ converge uniformly to continuous functions $\tau, \lambda: Y^\varphi \rightarrow \mathbb{R}$ respectively, and we can define continuous maps $h, k: Y^\varphi \rightarrow Y^\varphi$ by $h(y) = \Phi^{\tau(y)}(y)$ and $k(y) = \Phi^{\lambda(y)}(y)$ respectively. Passing to the limit in the equality

$$y = g_n g_n^{-1}(y) = \Phi^{\tau_n(\Phi^{\lambda_n(y)}(y)) + \lambda_n(y)}(y)$$

we see that $hk(y) = y$, and similarly $kh(y) = y$. Thus h and k are homeomorphisms inverse to each other, and g_n tends to $h \in \mathbf{H}_0(\varphi)$ as $n \rightarrow \infty$.

Let us now check that d defines a group topology. It is clear that the map $g \mapsto g^{-1}$ is continuous. Let $(g_n)_n$ and $(h_n)_n$ be sequences converging to g and h respectively, and let us check that $(g_n h_n)_n$ tends to gh as $n \rightarrow \infty$. Let us show that $d_+(g_n h_n, gh) \rightarrow 0$. Fix displacement functions τ for g , and λ for h , as well as two sequences $(\tau_n)_n$ and $(\lambda_n)_n$ of displacement functions for $(g_n)_n$ and $(h_n)_n$ that converge to τ and λ uniformly. Set $\delta(y) = \tau(h(y)) + \lambda(y)$ and $\delta_n(y) = \tau_n(h_n(y)) + \lambda_n(y)$, these define displacement functions for gh and for $g_n h_n$ respectively (see (4)). We have

$$|\delta_n(y) - \delta(y)| \leq |\tau_n(h_n(y)) - \tau(h_n(y))| + |\tau(h_n(y)) - \tau(h(y))| + |\lambda_n(y) - \lambda(y)|.$$

The first and third summand tend to zero uniformly on y because $\tau_n \rightarrow \tau$ and $\lambda_n \rightarrow \lambda$ uniformly. Moreover the second term also tends to zero uniformly by uniform continuity of τ , and because $h_n(y) = \Phi^{\lambda_n(y)}(y)$ tends to $h(y)$ uniformly. Thus $\sup_y |\delta_n(y) - \delta(y)|$ tends to zero, so that $d_+(g_n h_n, gh) \rightarrow 0$. Similarly one proves that $d_-(g_n h_n, gh) \rightarrow 0$, showing that $d(g_n h_n, gh) \rightarrow 0$. \square

PROPOSITION A.2. *The group $\mathbf{T}(\varphi)$ is a dense subgroup of $\mathbf{H}_0(\varphi)$. In particular if X is metrisable, the group $\mathbf{H}_0(\varphi)$ is Polish.*

Proof. The proof proceeds by reducing to the classical remark that Thompson's group F is dense in $\mathbf{Homeo}_+([0, 1])$ [BS16, Corollary A5.8].

Let us show that every $g \in \mathbf{H}_0(\varphi)$ is in the closure of $\mathbf{T}(\varphi)$. Assume first that g is supported in a domain $U_{C \times I}$. Fix $\varepsilon > 0$. For every $x \in C$ the element g fixes the fiber $\pi_{C,I}(\{x\} \times I)$ and acts on it by a homeomorphism $f_x \in \mathbf{Homeo}_+(I)$ which varies continuously with respect to x . Thus we can find a clopen partition $C = \bigsqcup_{i=1}^r C_i$ such that whenever x and $y \in C_i$, we have that $\delta(f_x, f_y) < \varepsilon/2$, where δ is the distance in $\mathbf{Homeo}_+(I)$. For $i = 1, \dots, r$, choose $x_i \in C_i$ and an element $h_i \in F_I$ such that $\delta(h_i, f_{x_i}) < \varepsilon/2$. If we identify h_i with the corresponding element of $F_{C_i, I}$ and set $h = h_1 \cdots h_r \in \mathbf{T}(\varphi)$, we have $d(g, h) < \varepsilon$. Since ε is arbitrary, we deduce that g is in the closure of $\mathbf{T}(\varphi)$.

Next, assume that $d(g, \text{id}) < 1/8$. We consider the domains $U_{X \times (-1/4, 1/4)} \subset U_{X \times (-1/2, 1/2)}$. Then $g(U_{X \times (-1/4, 1/4)}) \subset U_{X \times (-1/2, 1/2)}$. Find $g_1 \in \mathbf{H}_0(\varphi)$ supported in $U_{X \times (-1/2, 1/2)}$ which coincides with g in restriction to $U_{X \times (-1/4, 1/4)}$. Then $g_2 := g g_1^{-1}$ is supported in $U_{X \times (1/4, 3/4)}$ and therefore $g = g_2 g_1$ is in the closure of $\mathbf{T}(\varphi)$ by the previous case.

Finally since $\mathbf{H}_0(\varphi)$ is connected, it is generated by any neighbourhood of the identity. Therefore every element $g \in \mathbf{H}_0(\varphi)$ is in the closure of $\mathbf{T}(\varphi)$, that is, $\mathbf{T}(\varphi)$ is dense.

If X is metrisable, it has only countably many clopen subsets. Moreover there are countably many dyadic intervals in \mathbb{R} , and countably many dyadic PL homeomorphisms between them. It follows from Definition 3.1 (using compactness) that in this case the group $\mathbf{T}(\varphi)$ is countable, and therefore $\mathbf{H}_0(\varphi)$ is separable. \square

We now prove Proposition 4.4 for an arbitrary Stone system (X, φ) (see also Remark 7.4 for the case where (X, φ) is minimal).

Proof of Proposition 4.4. Let $K \leq \mathsf{T}(\varphi)$ be the subgroup generated by the groups $F'_{C,J}$. Fix $g \in \mathsf{T}(\varphi)$ and let us show that $g \in K$. Assume first that $d(g, \text{id}) < 1/4$. Let $X_0 \subset Y^\varphi$ be the projection of $X \times \{0\}$ and cover it by disjoint domains $U_{C_1 \times I_1}, \dots, U_{C_r \times I_r}$ with $0 \in I_i$ so that in restriction to each of them, g is given by a dyadic PL homeomorphism $f_i: I_i \rightarrow f_i(I_i)$. Since $-1/4 < f_i(0) < 1/4$, we can choose the intervals I_i small enough so that each closure $\overline{I_i} \cup f_i(I_i)$ is contained in $(-1/4, 1/4)$. Choose $k_i \in F_{(-1/4, 1/4)}$ that extends $f_i|_{I_i}$. Let $h_i \in F_{C_i, J_i}$ correspond to k_i , and $h = h_1 \cdots h_r$. Replace g by $h^{-1}g$. After doing this, we obtain that g fixes pointwise a domain of the form $U_{X \times I}$ for some $I \ni 0$. Equivalently, g is supported in a domain of the form $U_{X \times J}$ for some $J \subset (0, 1)$. Now the conclusion follows from Lemma 7.1.

Assume now that g is arbitrary. Since $\mathsf{H}_0(\varphi)$ is connected and hence generated by any neighbourhood of the identity, we can write $g = g_1 \cdots g_r$ for some $g_i \in \mathsf{H}_0(\varphi)$ with $d(g_i, \text{id}) < 1/4$. By density of $\mathsf{T}(\varphi)$ we can choose $h_i \in \mathsf{T}(\varphi)$ close enough to g_i so that $d(h_i, \text{id}) < 1/4$ and the element $h = h_1 \cdots h_r$ satisfies $d(gh^{-1}, \text{id}) < 1/4$. It follows from the previous case that h and gh^{-1} both belong to K and thus $g \in K$. \square

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