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Functional kernel estimation of the conditional extreme value index under random right censoring

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ABSTRACT

Estimation of the extreme-value index of a heavy-tailed distribution is investigated when some functional random covariate (i.e. valued in some infinite-dimensional space) information is available and the scalar response variable is right-censored. A weighted kernel version of Hill's estimator of the extreme-value index is proposed and its asymptotic normality is established under mild assumptions. A simulation study is conducted to assess the finite-sample behavior of the proposed estimator.

KEYWORDS

Kernel estimator; Functional data; Censored data; Conditional extreme value index; Heavy-tailed distributions

1. Introduction

In this paper, we are interested in nonparametric conditional tail index estimation of the conditional distribution of the random variable of interest given a covariate. When some covariate is recorded simultaneously with the quantity of interest Y , the tail index can depend on the covariate and is referred in the sequel to as conditional tail index or conditional extreme value-index. This parameter drives the tail heaviness of the distribution of Y and thus plays a central role in the analysis of extremes, making its estimation a crucial issue. Then, we consider the situation where some covariate information X is available to the investigator, and the distribution of Y depends on X .

Specifically, we focus on the problem of estimating a conditional extreme-value index of a heavy-tailed distribution when some functional covariate information $X \in \mathbf{E}$ is available, where \mathbf{E} is an infinite dimensional space associated with a semi-metric $d(\cdot, \cdot)$. The analysis of functional data have been extensively studied for example in [23] the authors have developed the fundamental theory around the functional data. Parametric model estimators are developed [3,24] and nonparametric estimators have been proposed in [10] by establishing the strong consistency of estimators related to regression function. Despite the greater interest in practice, the study of the

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functional random covariate case have been initiated recently, we refer to the works in [14] based on the estimation of extreme rainfalls given the geographical location and also in [12] they studied the extreme temperatures given topological parameters and the estimation of extreme earthquakes given the location in [22].

Normally, in real-life practical application, the variable of interest may be incomplete. In classical applications such as the analysis of lifetime data (survival analysis, reliability theory, insurance), a typical feature which appears is censorship. For example, in medical follow up, the response variable Y represents the time elapsed from the entry of a patient in, say, a follow-up study until death. If, at the time that the data collection is performed, the patient is still alive or has withdrawn from the study for some reason, the variable of interest Y will not be available. A convenient way to model this situation is the introduction of a random variable C (called a censoring random variable), independent of Y , such that only the pairs (Z_i, δ_i) , $1 \leq i \leq n$ are observed where $Z_i = \min(Y_i, C_i)$ and $\delta_i = \mathbf{1}_{\{Y_i \leq C_i\}}$ and $\mathbf{1}_{\{A\}}$ is the indicator function of the event A .

Estimation of the extreme value-index or high quantiles with censored data when covariate information is not available was the purpose of many investigations. When there is no covariate information, estimation of the extreme-value index from censored data is considered by [2,4,7,9,18,19] and [27]. [1] and [9] additionally address estimation of extreme quantiles. [20] and [21] address estimation of the conditional extreme-value index and conditional extreme quantiles with fixed and random covariates.

Some important literature is devoted to the estimation of the conditional quantile of a scalar response given a functional covariate. [15] dealt with the estimation of conditional quantiles when the covariate is functional and when the order of the quantiles converges to one as the sample size increases. [5] investigated the conditional quantile estimation of a randomly censored scalar response variable given a functional random covariate whenever a stationary ergodic data are considered.

However, based on the literature and our knowledge, estimation of the functional conditional extreme value-index of a heavy-tailed distribution under random right censoring has not yet been addressed, when covariate is a functional random variable. Our methodology combines a kernel version of Hill's estimator of the extreme-value index (such as developed in [17] in the uncensored case) with a weighting term whose role is to correct for censoring (such as in [9] and [4], who estimate the unconditional extreme-value index with censoring). This idea is already used in [21]. Following the same idea, we propose a functional Hill-type estimator depending on a semi-metric $d(.,.)$.

The remainder of this paper is organized as follows. Some notation and definition are introduced in Section 2. The construction of our estimator of the functional conditional extreme value index is summarized in Section 3. Assumptions are introduced and the asymptotic normality of the proposed estimator is established in Section 4. Section 5 illustrates the finite sample behavior of the proposed estimator via simulations. The conclusion and some perspectives are presented in Section 6. All proofs are presented in the Appendix in Section 7.

2. Notation and Definition

Let (X_i, Y_i) $i = 1, \dots, n$ be the independent copies of the random pairs (X, Y) , where Y is positive real random variable and X be a functional random variable, $X \in \mathbf{E}$ is an infinite dimensional space associated to semi-metric. We assume that the variable Y is randomly right censored by a positive random variable C defined on the same probability space $(\Omega, \mathcal{C}, \mathbb{P})$ as Y . Therefore, we really observe independent triplets (X_i, δ_i, Z_i) , where $Z_i = \min(Y_i, C_i)$ and $\delta_i = \mathbf{1}_{\{Y_i \leq C_i\}}$ for $i = 1 \dots, n$ where $\mathbf{1}_A$ is the indicator function of the event A .

We assume that Y and C are independent given $X = x$, where C_1, \dots, C_n are independent each other. Let $F(\cdot|x)$ and $G(\cdot|x)$ be the conditional cumulative distribution functions of random variables Y and C given $X = x$ respectively.

Let $\bar{F}(\cdot|x)$ and $\bar{G}(\cdot|x)$ be the conditional survival function of random variable Y and C given $X = x$ respectively.

In this paper, we focus on heavy tails. More specifically, we assume that the conditional survival function satisfies the following condition

(A1).

$$\bar{F}(t|x) = r_1(x) \exp \left\{ - \int_1^t \left(\frac{1}{\gamma_1(x)} - \varepsilon_1(\mu|x) \right) \frac{d\mu}{\mu} \right\} \quad (1)$$

and

$$\bar{G}(t|x) = r_2(x) \exp \left\{ - \int_1^t \left(\frac{1}{\gamma_2(x)} - \varepsilon_2(\mu|x) \right) \frac{d\mu}{\mu} \right\} \quad (2)$$

where $\gamma_1(x), \gamma_2(x)$ are positive unknown functions of the covariate x , r_1, r_2 are positive functions and $|\varepsilon_1(\mu|x)|, |\varepsilon_2(\mu|x)|$ are continuous and ultimately decreasing to zero. From (1) and (2), we can state that the conditional distribution functions of Y and C given $X = x$ are in Fréchet maximal domain of attraction. Thus, $\gamma_1(x)$ and $\gamma_2(x)$ are taken as the conditional extreme tail index functions. Therefore, for all $t > 0$, $\bar{F}(\cdot|x)$ and $\bar{G}(\cdot|x)$ are regularly varying functions at infinity with index $-\frac{1}{\gamma_1(x)}$ and $-\frac{1}{\gamma_2(x)}$ respectively. Thus,

$$\bar{F}(u|x) = u^{-\frac{1}{\gamma_1(x)}} L_1(u|x) \quad \text{and} \quad \bar{G}(u|x) = u^{-\frac{1}{\gamma_2(x)}} L_2(u|x)$$

where for x fixed, $L_1(\cdot|x)$ and $L_2(\cdot|x)$ are slowly varying functions at infinity, that is, for all $\lambda > 0$,

$$\lim_{y \rightarrow \infty} \frac{L_i(\lambda y|x)}{L_i(y|x)} = 1, \quad i = 1, 2.$$

By conditional independence between Y and C , the conditional survival function $\bar{H}(\cdot|x)$ of Z given $X = x$ is also a regularly varying function at infinity with index

$-\frac{1}{\gamma(x)}$ as expressed as follow:

$$\begin{aligned}\overline{H}(z|x) &= 1 - H(z|x) = \overline{F}(t|x)\overline{G}(t|x) \\ &= r(x) \exp \left\{ - \int_1^z \left(\frac{1}{\gamma(x)} - (\varepsilon(\mu|x)) \right) \frac{d\mu}{\mu} \right\}\end{aligned}$$

with $\gamma(x) = \gamma_1(x)p(x)$ where $p(x) = \frac{\gamma_2(x)}{\gamma_1(x)+\gamma_2(x)}$ is the ultimate proportion of uncensored observations among $Z_i, i = 1, \dots, n$; the proof of this statement is out of scope of presented paper (see [9,20] for more details) and $r(x) = r_1(x)r_2(x), \varepsilon(\mu|x) = \varepsilon_1(\mu|x) + \varepsilon_2(\mu|x)$. In the sequel, we further assume that $L_i(u|x), i = 1, 2$ belong to the Hall class of slowly-varying functions.

3. Construction of the estimator

Let $(X_i, \delta_i, Z_i), i = 1, \dots, n$, be independent realizations of the random vector (X, δ, Z) where $Z_i = \min(Y_i, C_i)$ and $\delta_i = \mathbf{1}_{\{Y_i \leq C_i\}}$ for $i = 1, \dots, n$ and $(X, Z) \in \mathbf{E} \times \mathbb{R}_+^*$. If Z_i were uncensored it means that $Z_i = Y_i$ for all i , in [17], the authors proposed a Hill's version of the conditional extreme value index when the covariate response is in \mathbb{R}^p . Following the same idea, we propose a functional Hill-type estimator depending on a semi-metric $d(.,.)$:

$$\hat{\gamma}_n^H(x) = \sum_{i=1}^n K(h^{-1}d(x, X_i)) (\log Z_i - \log y_n) \mathbf{1}_{\{Z_i > y_n\}} / \sum_{i=1}^n K(h^{-1}d(x, X_i)) \mathbf{1}_{\{Z_i > y_n\}}, \quad (3)$$

where $K(.)$ is a real-valued kernel function on \mathbf{E} , $h = h_n$ is a positive non-random bandwidth sequence such that $h \rightarrow 0$ as $n \rightarrow \infty$ and y_n is non-random threshold sequence with $y_n \rightarrow \infty$ as $n \rightarrow \infty$.

The estimator (3) is not consistent for $\gamma_1(x)$ if it is directly applied to the censored sample $(X_i, \delta_i, Z_i), i = 1, \dots, n$. Indeed, under appropriate regularity assumptions, estimator (3) will converge to the extreme-value index $\gamma(x)$ of the conditional distribution of Z given $X = x$. To accommodate censoring, we suggest, like in [21], to divide (3) by the proportion $\hat{p}_n(x)$ of uncensored observations among the $Z_i, i = 1, \dots, n$ that are larger than y_n , in a neighborhood of x :

$$\hat{p}_n(x) = \frac{\overline{H}_n^1(y_n|x)}{\overline{H}_n(y_n|x)}$$

where $\overline{H}_n(y_n|x) = \sum_{i=1}^n B_i(x) \mathbf{1}_{\{Z_i > y_n\}}$, $\overline{H}_n^1(y_n|x) = \sum_{i=1}^n B_i(x) \mathbf{1}_{\{Z_i > y_n, \delta_i=1\}}$ and $B_i(x)$ are the well-known Nadaraya-Watson weights defined by

$$B_i(x) = \frac{K(h^{-1}d(x, X_i))}{\sum_{j=1}^n K(h^{-1}d(x, X_j))}.$$

The survival functions $\overline{H}_n(y_n|x)$ and $\overline{H}_n^1(y_n|x)$ can be rewritten as follow:

$$\overline{H}_n^1(y_n|x) = \hat{\psi}_n(y_n, x) / \hat{g}_n(x) \quad \text{and} \quad \overline{H}_n(y_n|x) = \hat{\zeta}_n(y_n, x) / \hat{g}_n(x) \quad \text{respectively,}$$

where $\hat{\psi}_n(y_n, x) = \frac{1}{n(\mu_x^{(1)}(h))} \sum_{i=1}^n K\left(\frac{d(x, X_i)}{h}\right) \mathbf{1}_{\{Z_i > y_n, \delta_i = 1\}}$ and $\hat{\zeta}_n(y_n, x) = \frac{1}{n(\mu_x^{(1)}(h))} \sum_{i=1}^n K\left(\frac{d(x, X_i)}{h}\right) \mathbf{1}_{\{Z_i > y_n\}}$ and $\hat{g}_n(x) = \frac{1}{n(\mu_x^{(1)}(h))} \sum_{i=1}^n K\left(\frac{d(x, X_i)}{h}\right)$. Therefore we propose to estimate $\gamma_1(\cdot)$ by

$$\hat{\gamma}_n^{c,H}(x) = \frac{\hat{\gamma}_n^H(x)}{\hat{p}_n(x)}. \quad (4)$$

This estimator depends on the bandwidth h , the threshold y_n and the semi-metric $d(\cdot, \cdot)$. The choice of the semi-metric is a crucial point in nonparametric functional data analysis (see [11]). To the best of our knowledge, theoretical advances for choosing the semi-metric in practical situations have not yet been developed. Once the semi-metric has been chosen, packages are available in the literature (see <https://cran.r-project.org/web/packages/fda.usc/index.html>) to evaluate proximities between functional data. In the simulation study in Section 5, we will discuss the impact of the degree of derivatives on the performance of our estimator when semi-metric based on derivatives are considered for smooth curves as covariates.

Let us introduce the following conditional expectation whose asymptotic expansion will enable us to study the asymptotic expansion of our proposed estimator (4)

$$\mathbb{S}_n(y_n, r; x) = E[(\log Z - \log y_n)^r \mathbf{1}_{\{Z > y_n\}} | X = x]. \quad (5)$$

4. Assumptions and asymptotic results

In this section, we first give some regularity conditions for proving our results. Some of those assumptions have been adapted from different authors including [5, 13, 15]; [21].

(A2). The kernel K is classical kernel density function with support $[0, 1]$, and there exist real number $0 < C < C' < \infty$ such that $C \leq K(t) \leq C'$ for all $t \in [0, 1]$.

(A3). There exist $K_r > 0$, $K_\epsilon > 0$, $K_\gamma > 0$, $\mu_0 > 1$ such that for all $(x, x') \in \mathbf{E} \times \mathbf{E}$ and $\mu > \mu_0$,

$$\begin{aligned} |\log r(x) - \log r(x')| &\leq K_r d(x, x'), \\ |\epsilon(\mu|x) - \epsilon(\mu|x')| &\leq K_\epsilon d(x, x'), \\ \left| \frac{1}{\gamma(x)} - \frac{1}{\gamma(x')} \right| &\leq K_\gamma d(x, x'). \end{aligned}$$

To make everything clearly, we may assume that the integral of K is equal to one. Under condition **(A2)**, we introduce the new notation that $\mu_x^{(j)}(h) = E[K^j(h^{-1}d(x, X))]$ for $j = 1, 2$.

Let $B(t, r)$ be the ball of center t and radius r then we define the function $\varphi_x(h) = P(X \in B(x, h))$ which is the small ball probability. By condition **(A2)** and Lemma 7.2 in the Appendix, it is shown that for $j > 0$, $\mu_x^{(j)}(h)$ has the same

asymptotic order as $\varphi_x(h)$.

(A4). Suppose that there exist a function $\rho(x) < 0$ and a regularly varying function $A(\cdot|x)$ which not changing the sign eventually at infinity with index $\rho(x)$ and with $\lim_{y_n \rightarrow \infty} A(y_n|x) = 0$ such that for all $x > 0$,

$$\lim_{t \rightarrow \infty} \frac{H^{\leftarrow} \left(1 - \frac{1}{ut}|x\right) / H^{\leftarrow} \left(1 - \frac{1}{t}|x\right) - u^{\gamma(x)}}{A(t|x)} = u^{\gamma(x)} \frac{u^{\rho(x)} - 1}{\rho(x)}.$$

The aforementioned condition is called second-order condition in classical extreme value theory.

Considering the condition **(A4)** which control the behavior of $H^{\leftarrow}(\cdot|x)$, we need an additional condition to control the oscillation of $H^{\leftarrow}(u|x)$ when considered as a function of the covariate x . That condition is expressed in terms of conditional expectation $\mathbb{S}_n(y_n, r; x)$ defined in (5) as defined in [16].

(A5). The conditional expectation $\mathbb{S}_n(y_n, r; x)$ satisfies that, for $y_n \rightarrow \infty, h \rightarrow 0$ as $n \rightarrow \infty$ and $(x', x) \in \mathbf{E} \times \mathbf{E}$

$$\Phi_n(y_n, h; x) := \sup_{r \in \{1, 2\}} \sup_{x' \in B(x, h)} \left[\left| \frac{\mathbb{S}_n(y_n, r; x')}{\mathbb{S}_n(y_n, r; x)} - 1 \right| \mathbf{1}_{\{d(x', x) < h\}} \right] \rightarrow 0 \text{ if } n \rightarrow \infty.$$

Remark 1. For sake of simplicity, we choose to deal with the naive kernel estimator. It is easy to construct a smooth version of this naive estimator. To do so, it suffices to change the basic indicator function into a smooth cumulative distribution function. For more details, see [11]. However, the theoretical properties stated in Theorem 4.1 can not be considered as valid for the smooth version of our estimator.

We now derive the limiting distribution of the proposed estimator (4) of $\gamma_1(x)$. The proof is given in the Appendix.

Theorem 4.1. Suppose **(A1)-(A5)** hold. Let y_n be non-random threshold sequence and h be a sequence of bandwidth, such that $y_n \rightarrow \infty$ and $h \rightarrow 0$, $n\bar{H}(y_n|x) \rightarrow \infty$,

$\sigma_n^{-1}(x) \left(A \left(\frac{1}{\bar{H}(y_n|x)} \right) \vee (h \log y_n) \vee \varphi_x(h) \right) \rightarrow 0$ as $n \rightarrow \infty$. For all $x \in \mathbf{E}$, such that $\varphi_x(h) > 0$,

$$\sigma_n^{-1}(x) \left(\hat{\gamma}_n^{c, H}(x) - \gamma_1(x) \right) \xrightarrow{D} \mathcal{N} \left(0, \frac{\gamma_1^3(x)}{\gamma(x)} \right)$$

where

$$\sigma_n(x) = \left(n\bar{H}(y_n|x) \frac{\left(\mu_x^{(1)}(h) \right)^2}{\mu_x^{(2)}(h)} \right)^{-1/2}.$$

The condition $\sigma_n^{-1}(x) \left(A \left(\frac{1}{\bar{H}(y_n|x)} \right) \vee (h \log y_n) \vee \varphi_x(h) \right) \rightarrow 0$ in Theorem 6 is

a combination of three conditions. The first one $\sigma_n^{-1}(x)A\left(\frac{1}{\overline{H}(y_n|x)}\right) \rightarrow 0$ is similar to the one in [21] and [17] and is classical in extreme value theory. Since $\frac{(\mu_x^{(1)}(h))^2}{\mu_x^{(2)}(h)}$ has the same asymptotic order as $\varphi_x(h)$ according to Lemma 7.2 below, $\sigma_n^{-1}(x)((h \log y_n) \vee \varphi_x(h)) \rightarrow 0$ is equivalent to $n\overline{H}(y_n|x)\varphi_x(h)((h \log y_n) \vee \varphi_x(h))^2 \rightarrow 0$. It imposes to $h \log y_n$ and $\varphi_x(h)$ to be negligible compared to the standard deviation $\sigma_n(x)$ of the estimator. For practical implementation, we have to choose the bandwidth h and the non-random threshold sequence such that $y_n = Z_{(n-k)}$ is in the ball $B(x, h)$, where $Z_{(n-k)}$ is the $(n - k)$ th order statistic.

5. Simulation studies

In this section, we assess, via simulations, finite-sample performance of our estimator (4). We also provide comparisons with two simple estimation strategies of the tail index of a heavy-tailed distribution under random censoring.

5.1. Simulation design

The main goal of this section is to demonstrate the efficiency of the proposed estimator in terms of consistency. For this purpose, we consider the simulation of $N = 500$ replications of a sample of size n ($n = 200, n = 500$) of random triple (X_i, δ_i, Z_i) where $Z_i = \min(C_i, Y_i)$ and X_i is a functional covariate $X \in \mathbf{E}$ which is defined by

$$X(t) = \Omega(2 - \cos(\pi W t)) + (1 - \Omega)\cos(\pi W t), \quad \text{for all } t \in [0, 1],$$

where W is normally distributed with mean zero and unit variance and Ω is a random variable Bernoulli distributed with parameter $p = 1/2$ as in [5].

Figure 1 below illustrates some realization of random curves of the given functional random variable $X(\cdot)$.

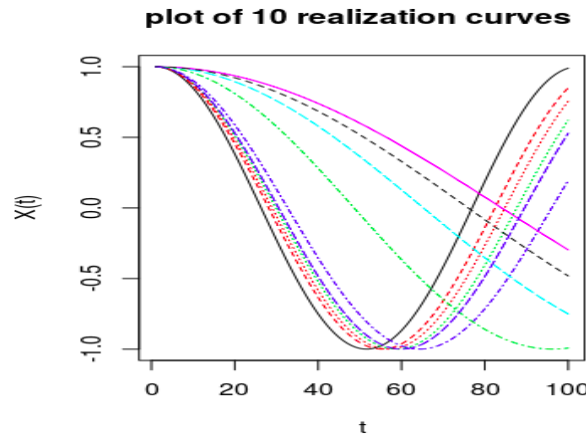


Figure 1. Ten realization of the random curves of function $X(\cdot)$.

The conditional distribution of Y given $X = x$ is a Burr distribution with parameter

$\tau(x) = 2, \lambda_1(x) = 2/(8 \|X\|_2^2 - 3)$, which implies that $\gamma_1(x) = \frac{1}{\tau(x)\lambda_1(x)}$, with

$$\|X\|_2^2 = \int_0^1 X^2(t)dt = 4\Omega^2 - 4\Omega(2\Omega - 1)\frac{\sin(\pi W)}{\pi W} + (2\Omega - 1)^2 \left[\frac{1}{2} + \frac{\sin(2\pi W)}{4\pi W} \right].$$

The conditional distribution of C given $X = x$ is also Burr distribution with parameter $\gamma_2(x) = \frac{1}{\tau(x)\lambda_2(x)}$, where the parameter $\gamma_2(x)$ is chosen to yield various values for the overall censoring percentage c ($c = 10\%, 20\%, 30\%, 40\%$). Since $\gamma(x) = \gamma_1(x)p(x)$ with $p(x) = \frac{\gamma_2(x)}{\gamma_1(x) + \gamma_2(x)} = \frac{\lambda_1(x)}{\lambda_1(x) + \lambda_2(x)}$ is the ultimate proportion of uncensored observations among Z_i for $i = 1, \dots, n$ then $\gamma_1(x)$ is selected, we choose $\gamma_2(x)$ such that $1 - p(x)$ is approximately to (10%, 20%, 30%, 40%) as censoring percentage.

In this simulation, we are interested to calculate the conditional tail index estimator presented in Equation (4). In practice there is some parameters to be fixed. Let K be an asymmetric linear kernel defined as $K(u) = (1.9 - 1.8u)1_{[0,1]}(u)$. Our simulation show that the choice of kernel has no impact. The estimator $\hat{\gamma}_n^{c,H}(x)$ depends on the bandwidth parameter h which is chosen using the cross-validation implemented in [15].

$$h^{opt} = \arg \min_{h \in \mathcal{H}} \sum_{i=1}^n \sum_{j=1}^n \left(\mathbf{1}_{\{Z_i > Z_j\}} - \hat{F}_{n,-i}(Z_j|x_i) \right)^2,$$

with $\hat{F}_{n,-i}$ is the kernel conditional Kaplan-Meier estimator used in [21] which depends on parameter h . The aforementioned estimator is calculated on the sample (X_j, δ_j, Z_j) , for $j = 1, \dots, n$ and $i \neq j$.

In case the bandwidth is already been selected then, next step is to determine the non random threshold y_n . We take y_n as the $(n - k)$ th order statistic $Z_{(n-k)}$, as is classical in extreme value statistics. Several method have been mentioned in literature and in this paper we adopted the method used by [21] as described as follow:

- (1) we compute the estimate $\hat{\gamma}_{n-k}^{c,H}(x)$ with $k = 1, \dots, n - 1$,
- (2) we form several successive "blocks" of estimates $\hat{\gamma}_{n-k}^{c,H}(x)$ (one block for $k \in \{1, \dots, 15\}$, a second block for $k \in \{16, \dots, 30\}$ and so on),
- (3) we calculate the standard deviation of the estimates within each block,
- (4) we determine the k -value to be used (thereafter denoted by k^*) from the block with minimal standard deviation. Precisely, we take the middle value of the k -values in the block (see [16,21])

Finally, we estimate $\gamma_1(x)$ by $\hat{\gamma}_{n-k}^{c,H}(x)$ with $(h, k) = (h^*, k^*)$.

Other thing to discuss is the selection of semi-metric whose choice will become an important stage for the behavior of nonparametric statistics for functional data. According to the literature the semi-metric based on the derivative are used in practice for the smoothing curve while semi-metric based on functional principal analysis and partial least squares are adapted for rough curves for more details, refer to [11]. Since the curves of $X(t)$ are smoothing curves according to [15], the semi-metric distance based on the derivative will be used to determine the distance between two curves X_1

and X_2 . We consider the semi-metric:

$$d_q^{derive}(X_1, X_2) = \sqrt{\int \left(X_1^{(q)}(t) - X_2^{(q)}(t) \right)^2 dt}, \quad (6)$$

where q is the degree of derivative and where $X^{(q)}$ denotes the q th derivative of X .

5.2. Results

The performance of our estimator $\hat{\gamma}_{n-k^*}^{c,H}(x)$ defined in (4) is evaluated using Mean Squared Error (MSE) and Mean Absolute Error (MAE). We also provide the averaged value (over the N samples) of the number of threshold excesses k^* . The accuracy of our estimator depends on the censoring percentage and on the degree of derivative of the semi-metric $d_q^{derive}(.,.)$ defined in (6).

To illustrate the performance of our estimator, we make a comparison with two simple estimation strategies. The first one is a complete-case procedure ("CC" for short): we remove all censored observations from the simulated samples. Then, we compute the estimator proposed in [17] presented in Equation (3). While, the second strategy is the ignored case, where, we consider that δ_i for $i = 1, \dots, n$ equally to one for all observations. We consider the observations Z_i , $i = 1, \dots, n$ as if they were uncensored. That kind of strategy is called Ignored case ("CI" for Censoring-Ignored).

In Table 1, we give the different value of empirical MSE and empirical MAE of our estimators at different sample size with respect to the different censoring percentage and degree of derivative respectively. As expected, the quality of the estimator deteriorates as the censoring percentage increases and the sample size decreases. Our estimator performs well at high level of derivative of semi-metric distance. This is not surprising since semi-metric distance parameter is known to play a key role to guaranty the behavior of nonparametric statistic in functional data analysis specially when curves appear to be smooth. The interested reader is referred to [11].

Considering the results in Table 1, CI and CC estimators of $\gamma_1(x)$ are quite biased, even though when censoring is moderate. As result, our estimator in equation (4) proved a significant result regarding the issues of estimating the functional conditional extreme value index under censorship.

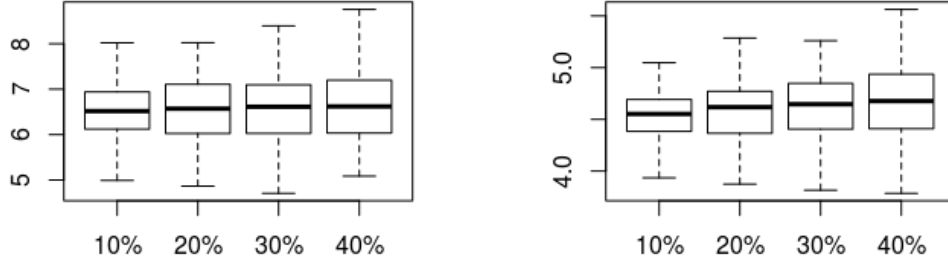
As illustrated from Table 1, the Hill's kernel version estimator under censorship (4) of $\gamma_1(x)$ shows to be well performed in almost simulation cases. As result, it performs quite better on low censoring percentage for large enough sample size and its quality becomes worst as the censored rate increases and sample size decreases. As by [11] advice that in cases of the smoothing curve, the semi-metric based on the derivative is a better one. After going through we have observed that the high order derivative gives a better result.

Figure 2 shows the boxplots of the N realizations of our estimator (4) for different values of the censoring percentage c (10%, 20%, 30%, 40%). This figure illustrates the distribution of the obtained estimates for $N = 500$ replications of estimator 4 for sample size $n = 200$ and $n = 500$. One can observed that the proposed estimator

performs quite well especially when the sample size is large enough and the censoring percentage decreases. This conclusion is expected, since it is confirmed by the results in Table 1.

Table 1. Table of MSE and MAE of the estimators value with sample size $n = 200$ and $n = 500$ for $N = 500$ replications and $\Omega = 1$.

		For sample size of $n = 200$			For sample size of $n = 500$		
		MSE($\hat{\gamma}_n(x)$)	MAE($\hat{\gamma}_n(x)$)	k^*	MSE($\hat{\gamma}_n(x)$)	MAE($\hat{\gamma}_n(x)$)	k^*
For censorship case							
order=4	10%	0.1082	0.2599	65.152	0.0423	0.1653	161.188
	20%	0.2277	0.4072	72.096	0.2054	0.3945	185.388
	30%	0.6504	0.7666	77.024	0.6503	0.7585	199.204
	40%	1.3434	1.1366	81.700	1.3152	1.1357	207.872
order=3	10%	0.1097	0.2698	63.304	0.0449	0.1716	160.132
	20%	0.2392	0.4153	72.684	0.1903	0.4021	179.536
	30%	0.6790	0.7847	76.296	0.6542	0.7901	195.024
	40%	1.3555	1.1429	82.232	1.3478	1.1399	209.060
order=2	10%	0.1098	0.2714	64.032	0.0484	0.1770	162.200
	20%	0.2474	0.4243	75.036	0.1968	0.4029	186.136
	30%	0.7083	0.8029	78.760	0.6677	0.8009	197.180
	40%	1.3692	1.1591	81.028	1.3401	1.1484	207.652
For Ignored case							
order=4	10%	0.1293	0.3010	67.280	0.0933	0.2657	160.836
	20%	0.3584	0.5448	67.756	0.3314	0.4586	166.248
	30%	0.7708	0.8481	71.116	0.7642	0.8320	171.396
	40%	1.3964	1.1642	74.812	1.3918	1.1521	189.216
order=3	10%	0.1351	0.3012	64.256	0.0937	0.2606	162.640
	20%	0.3863	0.5737	68.904	0.3352	0.5544	166.996
	30%	0.7904	0.8604	69.884	0.7790	0.8499	179.536
	40%	1.4184	1.1747	74.756	1.4050	1.1684	191.240
order=2	10%	0.1384	0.3020	66.076	0.0970	0.2638	165.192
	20%	0.3872	0.5763	70.024	0.3503	0.5601	172.144
	30%	0.8068	0.8704	69.996	0.7839	0.8702	179.052
	40%	1.4223	1.1763	75.176	1.3912	1.1699	186.312
For Complete case							
order=4	10%	0.1452	0.3096	58.805	0.0973	0.2572	148.662
	20%	0.3941	0.5624	54.577	0.3498	0.4650	138.925
	30%	0.8326	0.8692	51.255	0.7988	0.8464	128.993
	40%	1.4637	1.1811	47.553	1.4191	1.1798	120.135
order=3	10%	0.1526	0.3190	60.603	0.0975	0.2573	144.235
	20%	0.3999	0.5773	56.256	0.3472	0.5618	134.688
	30%	0.8524	0.8878	52.626	0.7915	0.8746	125.360
	40%	1.5323	1.2115	48.851	1.4318	1.1868	116.549
order=2	10%	0.1545	0.3200	58.339	0.0987	0.2652	144.933
	20%	0.4043	0.5805	54.242	0.3585	0.5631	135.297
	30%	0.8562	0.8899	50.807	0.8078	0.8818	126.054
	40%	1.5907	1.2961	47.148	1.4424	1.1895	117.176



(a) For sample size $n = 200$

(b) For sample size $n = 500$

Figure 2. Boxplot of the $N=500$ estimates of $\gamma_1(x)$ for censored rate 10%, 20%, 30%, 40% respectively for sample size $n = 200$ left and right $n = 500$.

6. Conclusion and further work

We considered the estimation of the conditional extreme value-index when some functional random covariate (i.e. valued in some infinite-dimensional space) information is available and the scalar response variable is right-censored. Its asymptotic properties were established and its finite sample performance was illustrated on a simulation study. Also a comparison with two simple estimation strategies has been provided. In future research, we will focus on a functional Weissman kernel type estimator for high quantiles and establish its asymptotic behavior. The case where the distribution of the dependent variable Y is not heavy-tailed will also be addressed.

7. Appendix

7.1. Preliminary

In order to prove our results, we now start by introducing the following notation: $\overline{H}(y|x) = P(Z > y|X = x)$ as the conditional survival function of Z given $X = x$ and $\overline{H}^1(y|x) = P(Z > y, \delta = 1|X = x)$ is a sub-distribution function of Z given $X = x$. We further define $\tilde{p}_x(y_n) = \frac{\overline{H}^1(y_n|x)}{\overline{H}(y_n|x)}$,

$$\begin{aligned}\mathbb{W}_{n\psi}(x) &= \left(n \frac{\left(\mu_x^{(1)}(h)\right)^2 \bar{H}^1(y_n|x)}{\mu_x^{(2)}(h)} \right)^{1/2} \left(\frac{\hat{\psi}_n(y_n, x) - E\hat{\psi}_n(y_n, x)}{\bar{H}^1(y_n|x)} \right), \\ \mathbb{W}_{n\zeta}(x) &= \left(n \frac{\left(\mu_x^{(1)}(h)\right)^2 \bar{H}(y_n|x)}{\mu_x^{(2)}(h)} \right)^{1/2} \left(\frac{\hat{\zeta}_n(y_n, x) - E\hat{\zeta}_n(y_n, x)}{\bar{H}(y_n|x)} \right).\end{aligned}$$

Lemma 7.1. *Suppose (A1) and (A3) hold. If $y_n \rightarrow \infty$ and $h \log y_n \rightarrow 0$ as $n \rightarrow \infty$, then*

$$\sup_{d(x, x') \leq h} \left| \frac{\bar{H}(y_n|x)}{\bar{H}(y_n|x')} - 1 \right| = \mathcal{O}(h \log y_n).$$

Proof of Lemma 7.1

From (A1), we have

$$\log \frac{\bar{H}(y_n|x)}{\bar{H}(y_n|x')} = \log(r(x)) - \log(r(x')) + \int_1^{y_n} \left(\frac{1}{\gamma(x')} - \frac{1}{\gamma(x)} + \varepsilon(\mu|x') - \varepsilon(\mu|x) \right) \frac{du}{u}.$$

Under (A3), we easily get

$$\begin{aligned} \left| \log \frac{\bar{H}(y_n|x)}{\bar{H}(y_n|x')} \right| &\leq K_r d(x, x') + \int_1^{y_n} \{ (K_\gamma + K_\varepsilon) d(x, x') \} \frac{du}{u} \\ &\leq K_r d(x, x') + \{ (K_\gamma + K_\varepsilon) d(x, x') \} \int_1^{y_n} \frac{du}{u} \\ &\leq \{ (K_r + (K_\gamma + K_\varepsilon) d(x, x')) \} \log(y_n) \\ &\leq (K_r + (K_\gamma + K_\varepsilon) h \log(y_n)). \end{aligned}$$

Thus,

$$\sup_{d(x, x') \leq h} \left| \log \frac{\bar{H}(y_n|x)}{\bar{H}(y_n|x')} \right| = \mathcal{O}(h \log(y_n)) \rightarrow 0, \text{ as } n \rightarrow \infty$$

and considering that $\log(z+1) \sim z$ as $z \rightarrow 0$, we get the desired result.

Lemma 7.2. *Suppose (A2) holds and let $x \in \mathbf{E}$ such that $\varphi_x(h) > 0$, there exist constants $0 < \tilde{C}_1 < \tilde{C}_2 < \infty$ such that*

$$\tilde{C}_1 \varphi_x(h) \leq \frac{\left(\mu_x^{(1)}(h)\right)^2}{\mu_x^{(2)}(h)} \leq \tilde{C}_2 \varphi_x(h).$$

Proof of Lemma 7.2

Under Lemma 3 in [15], we get,

$$\begin{aligned} C_1 \mathbf{1}_{[0,1]}(\frac{d(x, X)}{h}) &\leq K(\frac{d(x, X)}{h}) \leq C_2 \mathbf{1}_{[0,1]}(\frac{d(x, X)}{h}) \\ C_1 \varphi_x(h) &\leq E \left[K(\frac{d(x, X)}{h}) \right] \leq C_2 \varphi_x(h) \\ (C_1 \varphi_x(h))^2 &\leq \left(\mu_x^{(1)}(h) \right)^2 \leq (C_2 \varphi_x(h))^2. \end{aligned} \quad (7)$$

Similarly, we can write

$$C_1^2 \varphi_x(h) \leq \mu_x^{(2)}(h) \leq C_2^2 \varphi_x(h). \quad (8)$$

Using (7) and (8), we prove the desired result.

Lemma 7.3. Suppose (A1)-(A3) hold, let $y_n \rightarrow \infty$ and $h \log y_n \rightarrow 0$ as $n \rightarrow \infty$. Then, $\forall x \in \mathbf{E}$ such that $\varphi_x(h) > 0$.

- (1) $E[\hat{\zeta}_n(y_n, x)] = \overline{H}(y_n|x) [1 + \mathcal{O}(h \log y_n)]$
- (2) $E[\hat{\psi}_n(y_n, x)] = \overline{H}^1(y_n|x) [1 + \mathcal{O}(h \log y_n)]$.

Proof of Lemma 7.3

- (1) Let us first recall that

$$\hat{\zeta}_n(y_n, x) = \frac{1}{n(\mu_x^{(1)}(h))} \sum_{i=1}^n K(\frac{d(x, X_i)}{h}) \mathbf{1}_{\{Z_i > y_n\}}.$$

The couple (X_i, Z_i) , $i = 1, \dots, n$ being identically distributed, using conditional expectation, we can write

$$E[\hat{\zeta}_n(y_n, x)] = \frac{1}{\mu_x^{(1)}(h)} E \left[K(h^{-1}d(x, X)) \overline{H}(y_n|X) \right].$$

Simple calculations yield

$$\begin{aligned} E[\hat{\zeta}_n(y_n, x)] - \overline{H}(y_n|x) &= \frac{1}{\mu_x^{(1)}(h)} E \left[K(h^{-1}d(x, X)) \overline{H}(y_n|X) \right] - \overline{H}(y_n|x) \\ &= \frac{1}{\mu_x^{(1)}(h)} E \left[K(h^{-1}d(x, X)) (\overline{H}(y_n|X) - \overline{H}(y_n|x)) \right] \\ |E[\hat{\zeta}_n(y_n, x)] - \overline{H}(y_n|x)| &\leq \frac{\overline{H}(y_n|x)}{\mu_x^{(1)}(h)} E \left[K(h^{-1}d(x, X)) \left| \frac{\overline{H}(y_n|X)}{\overline{H}(y_n|x)} - 1 \right| \mathbf{1}_{\{d(x, X) < h\}} \right], \end{aligned}$$

under Lemma 7.1

$$\left| \frac{\bar{H}(y_n|X)}{\bar{H}(y_n|x)} - 1 \right| \mathbf{1}_{\{d(x,X) < h\}} \leq h \log y_n$$

thus,

$$E[\hat{\zeta}_n(y_n, x)] - \bar{H}(y_n|x) = \bar{H}(y_n|x) \mathcal{O}(h \log y_n). \quad (9)$$

(2) Calculations for $\hat{\psi}_n(y_n, x)$ are similar as above and are omitted for conciseness.

Lemma 7.4. *Suppose (A1)-(A3) hold, let $y_n \rightarrow \infty$ and $h \log y_n \rightarrow 0$ as $n \rightarrow \infty$. Then, $\forall x \in \mathbf{E}$ such that $\varphi_x(h) > 0$,*

$$\mathbb{W}_{n\psi}(x) \xrightarrow{D} \mathcal{N}(0, 1), \quad \mathbb{W}_{n\zeta}(x) \xrightarrow{D} \mathcal{N}(0, 1).$$

Therefore the random vector $\mathbb{W}_n(x) = (\mathbb{W}_{n\psi}(x), \mathbb{W}_{n\zeta}(x))^T$ converges in distribution to a bivariate Gaussian vector $\mathcal{N}(0, \mathcal{M})$ where,

$$\mathcal{M} := \begin{pmatrix} 1 & \sqrt{p(x)} \\ \sqrt{p(x)} & 1 \end{pmatrix}.$$

Proof of Lemma 7.4.

Let us first rewrite $\mathbb{W}_{n\psi}(x)$ and $\mathbb{W}_{n\zeta}(x)$ as follow:

$$\begin{aligned} \mathbb{W}_{n\zeta}(x) &= \left(\frac{1}{n\mu_x^{(2)}(h)\bar{H}(y_n|x)} \right)^{1/2} \sum_{i=1}^n K\left(\frac{d(x, X_i)}{h}\right) \mathbf{1}_{\{Z_i > y_n\}} - EK\left(\frac{d(x, X_i)}{h}\right) \mathbf{1}_{\{Z_i > y_n\}} \\ &:= \left(\frac{1}{n\mu_x^{(2)}(h)\bar{H}(y_n|x)} \right)^{1/2} \sum_{i=1}^n Q_{i,n} \end{aligned}$$

and

$$\begin{aligned} \mathbb{W}_{n\psi}(x) &= \left(\frac{1}{n\mu_x^{(2)}(h)\bar{H}^1(y_n|x)} \right)^{1/2} \sum_{i=1}^n K\left(\frac{d(x, X_i)}{h}\right) \mathbf{1}_{\{Z_i > y_n, \delta_i=1\}} - EK\left(\frac{d(x, X_i)}{h}\right) \mathbf{1}_{\{Z_i > y_n, \delta_i=1\}} \\ &:= \left(\frac{1}{n\mu_x^{(2)}(h)\bar{H}^1(y_n|x)} \right)^{1/2} \sum_{i=1}^n Q_{i,n}^* \end{aligned}$$

where for all $i = 1, \dots, n$ the random variables $Q_{i,n}^*$ and $Q_{i,n}$ are defined as follow

$$\begin{aligned} Q_{i,n}^* &= K\left(\frac{d(x, X_i)}{h}\right) \mathbf{1}_{\{Z_i > y_n, \delta_i=1\}} - EK\left(\frac{d(x, X_i)}{h}\right) \mathbf{1}_{\{Z_i > y_n, \delta_i=1\}} \\ Q_{i,n} &= K\left(\frac{d(x, X_i)}{h}\right) \mathbf{1}_{\{Z_i > y_n\}} - EK\left(\frac{d(x, X_i)}{h}\right) \mathbf{1}_{\{Z_i > y_n\}}. \end{aligned}$$

Consider that $\{Q_{i,n}, Q_{i,n}^*, \forall i = 1, \dots, n\}$ are the set of centered, independent identically distributed random variables. Now let us focus on the calculation of their variances:

$$\begin{aligned}
\text{Var}(Q_{i,n}) &= \text{Var}\left(K\left(\frac{d(x, X)}{h}\right)\mathbf{1}_{\{Z > y_n\}}\right) \\
&= E\left[K\left(\frac{d(x, X)}{h}\right)\mathbf{1}_{\{Z > y_n\}}\right]^2 - \left(E\left[K\left(\frac{d(x, X)}{h}\right)\mathbf{1}_{\{Z > y_n\}}\right]\right)^2 \\
&= \mathbf{R}_1 - \mathbf{R}_2^2.
\end{aligned} \tag{10}$$

By determining \mathbf{R}_1 and \mathbf{R}_2 separately, we get

$$\begin{aligned}
\mathbf{R}_1 &= E\left[K\left(\frac{d(x, X)}{h}\right)\mathbf{1}_{\{Z > y_n\}}\right]^2 \\
&= E\left[K^2(h^{-1}d(x, X))\overline{H}(y_n|X)\right] \\
&= \overline{H}(y_n|x)E\left[K^2(h^{-1}d(x, X))\right] + \overline{H}(y_n|x)E\left[K^2(h^{-1}d(x, X))\left[\frac{\overline{H}(y_n|X)}{\overline{H}(y_n|x)} - 1\right]\right].
\end{aligned}$$

Under Lipschitz conditions **(A3)** and Lemma 7.1,

$$\sup_{d(x, X) \leq h} \left| \frac{\overline{H}(y_n|X)}{\overline{H}(y_n|x)} - 1 \right| = \mathcal{O}(h \log y_n),$$

then

$$\begin{aligned}
\mathbf{R}_1 &= \overline{H}(y_n|x)E\left[K^2(h^{-1}d(x, X))\right][1 + \mathcal{O}(h \log y_n)] \\
&= \overline{H}(y_n|x)\mu_x^{(2)}(h)[1 + \mathcal{O}(h \log y_n)].
\end{aligned} \tag{11}$$

Simple calculations yield

$$\begin{aligned}
\mathbf{R}_2 &= E\left[K\left(\frac{d(x, X)}{h}\right)\mathbf{1}_{\{Z > y_n\}}\right] \\
&= E\left[K(h^{-1}d(x, X))\overline{H}(y_n|X)\right] \\
&= \overline{H}(y_n|x)E\left[K(h^{-1}d(x, X))\right] + \overline{H}(y_n|x)E\left[K(h^{-1}d(x, X))\left(\frac{\overline{H}(y_n|X)}{\overline{H}(y_n|x)} - 1\right)\right].
\end{aligned}$$

Using the same arguments as above, we get

$$\mathbf{R}_2 = \overline{H}(y_n|x)\mu_x^{(1)}(h)[1 + \mathcal{O}(h \log y_n)]. \tag{12}$$

By substituting the result of Equations (12) and (11) in Equation (10), we obtain

$$\text{Var}(Q_{i,n}) = \overline{H}(y_n|x)\mu_x^{(2)}(h)(1 + \mathcal{O}(h \log y_n)) - \left(\overline{H}(y_n|x)\mu_x^{(1)}(h)(1 + \mathcal{O}(h \log y_n))\right)^2.$$

Then, under Lemma 7.2, the variance of $\mathbb{W}_{n\zeta}(x)$ is given by

$$\begin{aligned}
Var(\mathbb{W}_{n\zeta}(x)) &= \left(\frac{1}{n\mu_x^{(2)}(h)\bar{H}(y_n|x)} \right) nVar(Q_{i,n}) \\
&= \left(\frac{1}{\mu_x^{(2)}(h)\bar{H}(y_n|x)} \right) \bar{H}(y_n|x)\mu_x^{(2)}(h) (1 + \mathcal{O}(h \log y_n)) \\
&\quad - \left(\frac{1}{\mu_x^{(2)}(h)\bar{H}(y_n|x)} \right) \left(\bar{H}(y_n|x)\mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right)^2 \\
&= 1 + \mathcal{O}(h \log y_n) - \left(\frac{1}{\mu_x^{(2)}(h)\bar{H}(y_n|x)} \right) \left(\bar{H}(y_n|x)\mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right)^2 \\
&\rightarrow 1 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

The proof for $Var(\mathbb{W}_{n\psi}(x))$ is similar and is thus omitted. Thus, we have

$$Var(\mathbb{W}_{n\psi}(x)) \rightarrow 1 \quad \text{as } n \rightarrow \infty. \quad (13)$$

Let now determine the covariance of $Cov(\mathbb{W}_{n\psi}(x), \mathbb{W}_{n\zeta}(x))$:

$$\begin{aligned}
Cov(\mathbb{W}_{n\psi}(x), \mathbb{W}_{n\zeta}(x)) &= Cov \left(\left(\frac{1}{n\mu_x^{(2)}(h)\bar{H}^1(y_n|x)} \right)^{1/2} \sum_{i=1}^n Q_{i,n}, \left(\frac{1}{n\mu_x^{(2)}(h)\bar{H}(y_n|x)} \right)^{1/2} \sum_{l=1}^n Q_{l,n}^* \right) \\
&= \frac{1}{n\mu_x^{(2)}(h)\sqrt{\bar{H}^1(y_n|x)\bar{H}(y_n|x)}} Cov \left(\sum_{i=1}^n Q_{i,n}, \sum_{l=1}^n Q_{l,n}^* \right) \\
&= \frac{\sqrt{\tilde{p}_x(y_n)}}{n\mu_x^{(2)}(h)\bar{H}^1(y_n|x)} Cov \left(\sum_{i=1}^n Q_{i,n}, \sum_{l=1}^n Q_{l,n}^* \right) \\
&= \frac{\sqrt{\tilde{p}_x(y_n)}}{n\mu_x^{(2)}(h)\bar{H}^1(y_n|x)} \sum_{l=1}^n \sum_{i=1}^n Cov(Q_{i,n}, Q_{l,n}^*) \\
&= \frac{\sqrt{\tilde{p}_x(y_n)}}{n\mu_x^{(2)}(h)\bar{H}^1(y_n|x)} \sum_{i=1}^n Cov(Q_{i,n}, Q_{i,n}^*)
\end{aligned}$$

since $Cov(Q_{i,n}, Q_{l,n}^*) = 0$ for $i \neq l$. Now let us consider the covariance $Cov(Q_{1,n}, Q_{1,n}^*)$ given by

$$\begin{aligned}
Cov(Q_{1,n}, Q_{1,n}^*) &= Cov \left(K\left(\frac{d(x, X)}{h}\right) \mathbf{1}_{\{Z > y_n, \delta=1\}}, K\left(\frac{d(x, X)}{h}\right) \mathbf{1}_{\{Z > y_n\}} \right) \quad (14) \\
&= E \left(K^2\left(\frac{d(x, X)}{h}\right) \mathbf{1}_{\{Z > y_n, \delta=1\}} \right) \\
&\quad - E \left(K\left(\frac{d(x, X)}{h}\right) \mathbf{1}_{\{Z > y_n, \delta=1\}} \right) E \left(K\left(\frac{d(x, X)}{h}\right) \mathbf{1}_{\{Z > y_n\}} \right) \\
&:= \mathbf{R}_3 - \mathbf{R}_4.
\end{aligned}$$

From the previous result, we obtain

$$\begin{aligned}\mathbf{R}_3 &= \bar{H}^1(y_n|x)\mu_x^{(2)}(h)(1 + \mathcal{O}(h \log y_n)) \\ \mathbf{R}_4 &= \bar{H}^1(y_n|x)\bar{H}(y_n|x) \left(\mu_x^{(1)}(h)(1 + \mathcal{O}(h \log y_n)) \right)^2.\end{aligned}$$

Thus, the covariance $Cov(Q_{1,n}, Q_{1,n}^*)$ is given by

$$Cov(Q_{1,n}, Q_{1,n}^*) = \bar{H}^1(y_n|x)\mu_x^{(2)}(h)(1 + \mathcal{O}(h \log y_n)) - \bar{H}^1(y_n|x)\bar{H}(y_n|x) \left(\mu_x^{(1)}(h)(1 + \mathcal{O}(h \log y_n)) \right)^2 \quad (15)$$

It follows from Equation (15) that

$$\begin{aligned}Cov(\mathbb{W}_{n\psi}(x), \mathbb{W}_{n\zeta}(x)) &= \frac{\sqrt{\tilde{p}_x(y_n)}}{n\mu_x^{(2)}(h)\bar{H}^1(y_n|x)} \sum_{i=1}^n Cov(Q_{i,n}, Q_{i,n}^*) \\ &= \left(\frac{\sqrt{\tilde{p}_x(y_n)}}{\mu_x^{(2)}(h)\bar{H}^1(y_n|x)} \right) \bar{H}^1(y_n|x)\mu_x^{(2)}(h)(1 + \mathcal{O}(h \log y_n)) \\ &\quad - \left(\frac{\sqrt{\tilde{p}_x(y_n)}}{\mu_x^{(2)}(h)\bar{H}^1(y_n|x)} \right) \bar{H}^1(y_n|x)\bar{H}(y_n|x) \left(\mu_x^{(1)}(h)(1 + \mathcal{O}(h \log y_n)) \right)^2.\end{aligned}$$

From Lemma 7.2, the covariance $Cov(\mathbb{W}_{n\psi}(x), \mathbb{W}_{n\zeta}(x)) \rightarrow \sqrt{p(x)}$, since $\lim_{y_n \rightarrow \infty} \tilde{p}_x(y_n) = p(x)$, as $n \rightarrow \infty$ (see [9,20]).

Now, we can prove that Lyapunov's condition for asymptotic normality of the sum of triangular arrays is verified, since we have deduced that $Var(\mathbb{W}_{n\psi}(x)) \rightarrow 1$ as $n \rightarrow \infty$. Let us consider that $\sum_{i=1}^n E|Q_{i,n}^{**}|^3 = nE|Q_{1,n}^{**}|^3 \rightarrow 0$ as $n \rightarrow \infty$ by assuming that $Q_{1,n}^{**}$

is bounded random variable with $Q_{i,n}^{**} = \left(\frac{1}{n\mu_x^{(2)}(h)\bar{H}^1(y_n|x)} \right)^{1/2} Q_{i,n}^*$:

$$\begin{aligned}|Q_{1,n}^{**}| &\leq \frac{2}{\sqrt{n\mu_x^{(2)}(h)\bar{H}^1(y_n|x)}} \\ |Q_{1,n}^{**}|^3 &\leq \frac{2}{\sqrt{n\mu_x^{(2)}(h)\bar{H}^1(y_n|x)}} |Q_{1,n}^{**}|^2 \\ &\leq \frac{2}{\sqrt{n\mu_x^{(2)}(h)\bar{H}^1(y_n|x)}} \left| \left(\frac{1}{n\mu_x^{(2)}(h)\bar{H}^1(y_n|x)} \right)^{1/2} Q_{1,n}^* \right|^2 \\ &\leq \frac{2}{\left(n\mu_x^{(2)}(h)\bar{H}^1(y_n|x) \right)^{3/2}} |Q_{1,n}^*|^2\end{aligned}$$

$$\begin{aligned}
nE|Q_{1,n}^{\star\star}|^3 &\leq \frac{2}{\left(n\mu_x^{(2)}(h)\overline{H}^1(y_n|x)\right)^{3/2}} nVar(Q_{1,n}^{\star}) \\
&\leq \frac{2}{\left(n\mu_x^{(2)}(h)\overline{H}^1(y_n|x)\right)^{1/2}} (1+o(1)) \\
&\quad - \frac{2}{\left(n\mu_x^{(2)}(h)\overline{H}^1(y_n|x)\right)^{1/2}} \overline{H}^1(y_n|x) \frac{(\mu_x^{(1)}(h))^2}{\mu_x^{(2)}(h)} (1+o(1)).
\end{aligned}$$

Since $n\overline{H}^1(y_n|x)\mu_x^{(2)}(h) \rightarrow \infty$, the Lyapunov condition is verified, therefore $\mathbb{W}_{n\psi}(x)/\sqrt{Var(\mathbb{W}_{n\psi}(x))} \xrightarrow{D} \mathcal{N}(0,1)$.

Now, we want to show that $\mathbb{W}_n(x) = (\mathbb{W}_{n\psi}(x), \mathbb{W}_{n\zeta}(x))^T$ converges in distribution to a Gaussian vector $\mathcal{N}(0, \mathcal{M})$ where,

$$\mathcal{M} := \begin{pmatrix} 1 & \sqrt{p(x)} \\ \sqrt{p(x)} & 1 \end{pmatrix}.$$

To prove that $\mathbb{W}_n(x) = (\mathbb{W}_{n\psi}(x), \mathbb{W}_{n\zeta}(x))^T$ converges in distribution to $\mathcal{N}(0, \mathcal{M})$, according to Cramér-Wold's device, it is sufficient to show that $e^T \mathbb{W}_n(x) \xrightarrow{D} \mathcal{N}(0, e^T \mathcal{M} e)$ for all $e = (e_1, e_2)^T$.

We can write that

$$\begin{aligned}
e^T \mathbb{W}_n(x) &= \left(\frac{1}{n\mu_x^{(2)}(h)\overline{H}^1(y_n|x)} \right)^{1/2} \sum_{i=1}^n \left(K\left(\frac{d(x, X_i)}{h}\right) \left[e_1 \mathbf{1}_{\{Z_i > y_n, \delta_i=1\}} + \sqrt{\tilde{p}_x(y_n)} e_2 \mathbf{1}_{\{Z_i > y_n\}} \right] \right) \\
&\quad - \left(\frac{1}{n\mu_x^{(2)}(h)\overline{H}^1(y_n|x)} \right)^{1/2} \sum_{i=1}^n E \left(K\left(\frac{d(x, X_i)}{h}\right) \left[e_1 \mathbf{1}_{\{Z_i > y_n, \delta_i=1\}} + \sqrt{\tilde{p}_x(y_n)} e_2 \mathbf{1}_{\{Z_i > y_n\}} \right] \right) \\
&= \left(\frac{1}{n\mu_x^{(2)}(h)\overline{H}^1(y_n|x)} \right)^{1/2} \sum_{i=1}^n Q_{i,n}^{\star\star\star}
\end{aligned}$$

where

$$\begin{aligned}
Q_{i,n}^{\star\star\star} &= \left(K\left(\frac{d(x, X_i)}{h}\right) \left[e_1 \mathbf{1}_{\{Z_i > y_n, \delta_i=1\}} + \sqrt{\tilde{p}_x(y_n)} e_2 \mathbf{1}_{\{Z_i > y_n\}} \right] \right) \\
&\quad - E \left(K\left(\frac{d(x, X_i)}{h}\right) \left[e_1 \mathbf{1}_{\{Z_i > y_n, \delta_i=1\}} + \sqrt{\tilde{p}_x(y_n)} e_2 \mathbf{1}_{\{Z_i > y_n\}} \right] \right).
\end{aligned}$$

Hence the variance is given by

$$Var(e^T \mathbb{W}_n(x)) = \left(\frac{1}{n\mu_x^{(2)}(h)\overline{H}^1(y_n|x)} \right) nVar(Q_{1,n}^{\star\star\star})$$

where

$$\begin{aligned}
Var(Q_{1,n}^{***}) &= Var \left(K \left(\frac{d(x, X_i)}{h} \right) \left[e_1 \mathbf{1}_{\{Z_i > y_n, \delta_i = 1\}} + \sqrt{\tilde{p}_x(y_n)} e_2 \mathbf{1}_{\{Z_i > y_n\}} \right] \right) \quad (16) \\
&= E \left(K \left(\frac{d(x, X_i)}{h} \right) \left[e_1 \mathbf{1}_{\{Z_i > y_n, \delta_i = 1\}} + \sqrt{\tilde{p}_x(y_n)} e_2 \mathbf{1}_{\{Z_i > y_n\}} \right] \right)^2 \\
&\quad - \left[E \left(K \left(\frac{d(x, X_i)}{h} \right) \left[e_1 \mathbf{1}_{\{Z_i > y_n, \delta_i = 1\}} + \sqrt{\tilde{p}_x(y_n)} e_2 \mathbf{1}_{\{Z_i > y_n\}} \right] \right) \right]^2, \\
&=: \mathbf{R}_5 - \mathbf{R}_6.
\end{aligned}$$

Now, some simple algebra yields

$$\begin{aligned}
\mathbf{R}_5 &= E \left(K^2 \left(\frac{d(x, X_i)}{h} \right) \left[e_1^2 \mathbf{1}_{\{Z_i > y_n, \delta_i = 1\}} + \sqrt{\tilde{p}_x(y_n)} 2e_1 e_2 \mathbf{1}_{\{Z_i > y_n, \delta_i = 1\}} + \tilde{p}_x(y_n) e_2^2 \mathbf{1}_{\{Z_i > y_n\}} \right] \right) \quad (17) \\
&= E \left(K^2 \left(\frac{d(x, X_i)}{h} \right) e_1^2 \mathbf{1}_{\{Z_i > y_n, \delta_i = 1\}} \right) + 2e_1 e_2 \sqrt{\tilde{p}_x(y_n)} E \left(K^2 \left(\frac{d(x, X_i)}{h} \right) \mathbf{1}_{\{Z_i > y_n, \delta_i = 1\}} \right) \\
&\quad + \tilde{p}_x(y_n) e_2^2 E \left(K^2 \left(\frac{d(x, X_i)}{h} \right) \mathbf{1}_{\{Z_i > y_n\}} \right) \\
&= e_1^2 \bar{H}^1(y_n|x) \mu_x^{(2)}(h) (1 + \mathcal{O}(h \log y_n)) + 2e_1 e_2 \sqrt{\tilde{p}_x(y_n)} \bar{H}^1(y_n|x) \mu_x^{(2)}(h) (1 + \mathcal{O}(h \log y_n)) \\
&\quad + e_2^2 \bar{H}^1(y_n|x) \mu_x^{(2)}(h) (1 + \mathcal{O}(h \log y_n)) \\
&= \left(e_1^2 + 2e_1 e_2 \sqrt{\tilde{p}_x(y_n)} + e_2^2 \right) \bar{H}^1(y_n|x) \mu_x^{(2)}(h) (1 + \mathcal{O}(h \log y_n)).
\end{aligned}$$

Similar calculations yield:

$$\begin{aligned}
\mathbf{R}_6 &= e_1^2 \left(E \left(K \left(\frac{d(x, X_i)}{h} \right) \mathbf{1}_{\{Z_i > y_n, \delta_i = 1\}} \right) \right)^2 + \tilde{p}_x(y_n) e_2^2 \left(E \left(K \left(\frac{d(x, X_i)}{h} \right) \mathbf{1}_{\{Z_i > y_n\}} \right) \right)^2 \quad (18) \\
&\quad + 2\sqrt{\tilde{p}_x(y_n)} e_1 e_2 E \left(K \left(\frac{d(x, X_i)}{h} \right) \mathbf{1}_{\{Z_i > y_n, \delta_i = 1\}} \right) E \left(K \left(\frac{d(x, X_i)}{h} \right) \mathbf{1}_{\{Z_i > y_n\}} \right) \\
&= e_1^2 \left(\bar{H}^1(y_n|x) \mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right)^2 + \tilde{p}_x(y_n) e_2^2 \left(\bar{H}^1(y_n|x) \mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right)^2 \\
&\quad + 2\sqrt{\tilde{p}_x(y_n)} e_1 e_2 \left(\bar{H}^1(y_n|x) \mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right) \left(\bar{H}^1(y_n|x) \mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right) \\
&= e_1^2 \left(\bar{H}^1(y_n|x) \mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right)^2 + \frac{1}{\tilde{p}_x(y_n)} e_2^2 \left(\bar{H}^1(y_n|x) \mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right)^2 \\
&\quad + 2 \frac{1}{\sqrt{\tilde{p}_x(y_n)}} e_1 e_2 \left(\bar{H}^1(y_n|x) \mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right) \left(\bar{H}^1(y_n|x) \mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right) \\
&= \left(e_1^2 + 2e_1 e_2 \frac{1}{\sqrt{\tilde{p}_x(y_n)}} + \frac{1}{p(x)} e_2^2 \right) \left(\bar{H}^1(y_n|x) \mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right)^2.
\end{aligned}$$

By substituting the final result of Equation (17) and (18) in Equation (16) then,

the variance of $Var(Q_{1,n}^{***})$ is given by

$$\begin{aligned} Var(Q_{1,n}^{***}) &= (e_1^2 + 2e_1e_2\tilde{p}_x(y_n) + e_2^2) \bar{H}^1(y_n|x) \mu_x^{(2)}(h) (1 + \mathcal{O}(h \log y_n)) \\ &+ \left(e_1^2 + 2e_1e_2 \frac{1}{\sqrt{\tilde{p}_x(y_n)}} + \frac{1}{\tilde{p}_x(y_n)} e_2^2 \right) \left(\bar{H}^1(y_n|x) \mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right)^2. \end{aligned}$$

Then,

$$\begin{aligned} Var(e^T \mathbb{W}_n(x)) &= \frac{1}{n \mu_x^{(2)}(h) \bar{H}^1(y_n|x)} n Var(Q_{1,n}^{***}) \\ &= \frac{e_1^2 + 2e_1e_2\sqrt{\tilde{p}_x(y_n)} + e_2^2}{\mu_x^{(2)}(h) \bar{H}^1(y_n|x)} \bar{H}^1(y_n|x) \mu_x^{(2)}(h) (1 + \mathcal{O}(h \log y_n)) \\ &+ \frac{e_1^2 + 2e_1e_2 \frac{1}{\sqrt{\tilde{p}_x(y_n)}} + \frac{1}{\tilde{p}_x(y_n)} e_2^2}{\mu_x^{(2)}(h) \bar{H}^1(y_n|x)} \left(\bar{H}^1(y_n|x) \mu_x^{(1)}(h) (1 + \mathcal{O}(h \log y_n)) \right)^2. \end{aligned}$$

Using again Lemma 7.2, we deduce that

$Var(e^T \mathbb{W}_n(x)) \rightarrow (e_1^2 + 2e_1e_2\sqrt{p(x)} + e_2^2) = e^T \mathcal{M} e$ as $n \rightarrow \infty$. Therefore, for all $e \in \mathbb{R}^2, e \neq 0$, $e^T \mathbb{W}_n(x)$ converges in distribution to the univariate normal distribution $\mathcal{N}(0, e^T \mathcal{M} e)$. The convergence $\mathbb{W}_n(x) \xrightarrow{D} \mathcal{N}(0, \mathcal{M})$ follows from Cramer-Wold device.

Lemma 7.5. *Suppose (A1)-(A4) hold.*

If $y_n \rightarrow \infty$, $\sigma_n^{-1}(x) (h \log y_n \vee \varphi_x(h) \vee A \left(\frac{1}{\bar{H}(y_n|x)} \right)) \rightarrow 0$. Then for all $x \in \mathbf{E}$, such that $\varphi_x(h) > 0$

$$\sigma_n^{-1}(x) (\hat{p}_n(x) - p(x)) \xrightarrow{D} \mathcal{N}(0, p(x)(1 - p(x))).$$

Proof of Lemma 7.5

Now, let us focus on the term $T_2(x)$ defined as follow $T_2(x) := \sigma_n^{-1}(x) (\hat{p}_n(x) - p(x))$. This term can be rewritten as

$$T_2(x) = \sigma_n^{-1}(x) \left(\hat{p}_n(x) - \frac{\bar{H}^1(y_n|x)}{\bar{H}(y_n|x)} \right) + \sigma_n^{-1}(x) \left(\frac{\bar{H}^1(y_n|x)}{\bar{H}(y_n|x)} - p(x) \right).$$

Let us deal with the first term of the above summation. Applying uniform Delta

method and some algebra, we get

$$\begin{aligned}
\sigma_n^{-1}(x) \begin{pmatrix} \frac{\hat{\psi}_n(y_n, x)}{\bar{H}(y_n|x)} - \frac{\bar{H}^1(y_n|x)}{\bar{H}(y_n|x)} \\ \frac{\hat{\zeta}_n(y_n, x)}{\bar{H}(y_n|x)} - 1 \end{pmatrix} &= \begin{pmatrix} n \frac{\bar{H}(y_n|x)(\mu_x^{(1)}(h))^2}{\mu_x^{(2)}(h)} \end{pmatrix}^{1/2} \begin{pmatrix} \frac{\hat{\psi}_n(y_n, x) - \bar{H}^1(y_n|x)}{\bar{H}(y_n|x)} \\ \frac{\hat{\zeta}_n(y_n, x)}{\bar{H}(y_n|x)} - 1 \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\tilde{p}_x(y_n)} \sqrt{n \frac{\bar{H}^1(y_n|x)(\mu_x^{(1)}(h))^2}{\mu_x^{(2)}(h)}} \left(\frac{\hat{\psi}_n(y_n, x)}{\bar{H}^1(y_n|x)} - 1 \right) \\ \sqrt{n \frac{\bar{H}(y_n|x)(\mu_x^{(1)}(h))^2}{\mu_x^{(2)}(h)}} \left(\frac{\hat{\zeta}_n(y_n, x)}{\bar{H}(y_n|x)} - 1 \right) \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\tilde{p}_x(y_n)} \sqrt{n \frac{\bar{H}^1(y_n|x)(\mu_x^{(1)}(h))^2}{\mu_x^{(2)}(h)}} \left(\frac{\hat{\psi}_n(y_n, x)}{\bar{H}^1(y_n|x)} - \frac{E\hat{\psi}_n(y_n, x)}{\bar{H}^1(y_n|x)} \right) \\ \sqrt{n \frac{\bar{H}(y_n|x)(\mu_x^{(1)}(h))^2}{\mu_x^{(2)}(h)}} \left(\frac{\hat{\zeta}_n(y_n, x)}{\bar{H}(y_n|x)} - \frac{E\hat{\zeta}_n(y_n, x)}{\bar{H}(y_n|x)} \right) \end{pmatrix} \\
&+ \begin{pmatrix} \sqrt{\tilde{p}_x(y_n)} \sqrt{n \frac{\bar{H}^1(y_n|x)(\mu_x^{(1)}(h))^2}{\mu_x^{(2)}(h)}} \left(\frac{\hat{\psi}_n(y_n, x)}{\bar{H}^1(y_n|x)} - 1 \right) \\ \sqrt{n \frac{\bar{H}(y_n|x)(\mu_x^{(1)}(h))^2}{\mu_x^{(2)}(h)}} \left(\frac{E\hat{\zeta}_n(y_n, x)}{\bar{H}(y_n|x)} - 1 \right) \end{pmatrix}.
\end{aligned}$$

Since $\frac{E\hat{\zeta}_n(y, x)}{\bar{H}(y_n|x)} - 1 = \mathcal{O}(h \log y_n)$ and $\frac{E\hat{\psi}_n(y, x)}{\bar{H}^1(y_n|x)} - 1 = \mathcal{O}(h \log y_n)$ according to Lemma 7.3 and by assumption $\sigma_n^{-1}(x) (h \log y_n) \rightarrow 0$, as $n \rightarrow \infty$, it follows that

$$\begin{aligned}
\sigma_n^{-1}(x) \begin{pmatrix} \frac{\hat{\psi}_n(y_n, x)}{\bar{H}(y_n|x)} - \frac{\bar{H}^1(y_n|x)}{\bar{H}(y_n|x)} \\ \frac{\hat{\zeta}_n(y_n, x)}{\bar{H}(y_n|x)} - 1 \end{pmatrix} &= \begin{pmatrix} \sqrt{\tilde{p}_x(y_n)} \sqrt{n \frac{\bar{H}^1(y_n|x)(\mu_x^{(1)}(h))^2}{\mu_x^{(2)}(h)}} \left(\frac{\hat{\psi}_n(y_n, x)}{\bar{H}^1(y_n|x)} - \frac{E\hat{\psi}_n(y_n, x)}{\bar{H}^1(y_n|x)} \right) \\ \sqrt{n \frac{\bar{H}(y_n|x)(\mu_x^{(1)}(h))^2}{\mu_x^{(2)}(h)}} \left(\frac{\hat{\zeta}_n(y_n, x)}{\bar{H}(y_n|x)} - \frac{E\hat{\zeta}_n(y_n, x)}{\bar{H}(y_n|x)} \right) \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\tilde{p}_x(y_n)} \mathbb{W}_{n\psi}(x) \\ \mathbb{W}_{n\zeta}(x) \end{pmatrix} + o(1). \tag{19}
\end{aligned}$$

Using the convergence of $\mathbb{W}_n(x) \xrightarrow{D} \mathcal{N}(0, \mathcal{M})$, Delta method and the fact that $\tilde{p}_x(y_n) \rightarrow p(x)$, the quantity in Equation (19) converges to bivariate Gaussian vector $\mathcal{N}(0, \Sigma)$ where

$$\Sigma := \begin{pmatrix} p(x) & p(x) \\ p(x) & 1 \end{pmatrix}.$$

Let consider a given function $\Phi(\omega_1, \omega_2) = \frac{\omega_1}{\omega_2}$ and apply again uniform Delta method (see [25, 26]), we have

$$\sigma_n^{-1}(x) \begin{pmatrix} \hat{p}_n(x) - \frac{\bar{H}^1(y_n|x)}{\bar{H}(y_n|x)} \\ \hat{\zeta}_n(y_n, x) - \frac{\bar{H}^1(y_n|x)}{\bar{H}(y_n|x)} \end{pmatrix} = \sigma_n^{-1}(x) \left(\Phi \left(\frac{\hat{\psi}_n(y_n, x)}{\bar{H}(y_n|x)}, \frac{\hat{\zeta}_n(y_n, x)}{\bar{H}(y_n|x)} \right) - \Phi \left(\frac{\bar{H}^1(y_n|x)}{\bar{H}(y_n|x)}, 1 \right) \right)$$

converges in distribution to $\mathcal{N}(0, \Phi'(p(x), 1)\Sigma(\Phi'(p(x), 1))^T)$.

By adapting the similar idea in [1], we want to show that

$$\sigma_n^{-1}(x) \left(\frac{\overline{H}^1(y_n|x)}{\overline{H}(y_n|x)} - p(x) \right) = \sigma_n^{-1}(x) \left(\frac{\overline{H}^1(y_n|x)}{\overline{H}(y_n|x)} - p(x) \right) = o_p(1).$$

Let setting that $y_n \in [1, \infty[$

$$\begin{aligned} \frac{\int_{y_n}^{\infty} z^{-1/\gamma_1(x)-1/\gamma_2(x)-1} L(z) dz}{y_n^{-1/\gamma_1(x)-1/\gamma_2(x)} L(y_n|x)} - \gamma(x) &= \frac{\int_{y_n}^{\infty} \frac{\overline{H}(z|x)}{z} dz}{\overline{H}(y_n|x)} - \gamma(x) \\ &= \varepsilon(y_n|x). \end{aligned}$$

By changing variable, we have

$$\lim_{y_n \rightarrow \infty} \frac{\int_{y_n}^{\infty} \frac{\overline{H}(z|x)}{z} dz}{\overline{H}(y_n|x)} = \lim_{y_n \rightarrow \infty} \int_1^{\infty} \frac{\overline{H}(y_n s|x)}{\overline{H}(y_n|x)} \frac{ds}{s}.$$

Since the function $\overline{H}(\cdot|x)$ is regular varying,

$$\begin{aligned} \frac{\overline{H}(y_n s|x)}{\overline{H}(y_n|x)} &= \frac{(y_n s)^{-1/\gamma(x)} L(y_n s|x)}{y_n^{-1/\gamma(x)} L(y_n|x)} \\ &= s^{-1/\gamma(x)} \frac{L(y_n s|x)}{L(y_n|x)} \rightarrow s^{-1/\gamma(x)} \quad \text{as } n \rightarrow \infty \end{aligned}$$

Therefore

$$\lim_{y_n \rightarrow \infty} \frac{\int_{y_n}^{\infty} \frac{\overline{H}(z|x)}{z} dz}{\overline{H}(y_n|x)} = \gamma(x).$$

Hence,

$$\int_{y_n}^{\infty} \frac{\overline{H}(z|x)}{z} dz = \gamma(x) \overline{H}(y_n|x) (1 + \mathcal{O}(\varepsilon(y_n|x)))$$

Remark that

$$\begin{aligned} \overline{H}^1(y_n|x) &= \mathbb{P}(\min(Y, C) > y_n, \delta = 1 | X = x) \\ &= \mathbb{P}(Y > y_n, Y - C < 0 | X = x). \end{aligned}$$

After some calculations, we can write

$$\begin{aligned} \overline{H}^1(y_n|x) &= \int_{y_n}^{\infty} \left(\int_{-\infty}^0 f_Y(u|x) f_C(u - v|x) dv \right) du \\ &= \int_{y_n}^{\infty} f_Y(u|x) \left(\int_u^{\infty} f_C(w|x) dw \right) du \\ &= \int_{y_n}^{\infty} f_Y(u|x) \overline{G}(u|x) du. \end{aligned}$$

Thus,

$$\begin{aligned}\bar{H}^1(y_n|x) &= \int_{y_n}^{\infty} \bar{G}(u|x) dF(u|x) \\ &= \frac{1}{\gamma_1(x)} \int_{y_n}^{\infty} u^{-1/\gamma(x)-1} L_1(u|x) du - \int_{y_n}^{\infty} u^{-1/\gamma(x)} L_1'(u|x) du\end{aligned}$$

Set $\varepsilon_1(y_n|x) = \frac{y_n L_1'(y_n|x)}{L_1(y_n|x)}$. Under the class of Hall, $\varepsilon_1(y_n|x) = \mathcal{O}(\varepsilon(y_n|x))$ and $\varepsilon(y_n|x) = \mathcal{O}\left(A\left(\frac{1}{\bar{H}(y_n|x)}\right)|x\right)$. Hence, we can rewrite as

$$\begin{aligned}\bar{H}^1(y_n|x) &= \frac{1}{\gamma_1(x)} \int_{y_n}^{\infty} \frac{\bar{H}(u|x)}{u} du - \int_{y_n}^{\infty} \frac{\bar{H}(u|x)}{u} \varepsilon_1(u|x) du \\ &= \frac{1}{\gamma_1(x)} \gamma(x) \bar{H}(y_n|x) (1 + \mathcal{O}(\varepsilon(y_n|x))) \\ &= p(x) \bar{H}(y_n|x) (1 + \mathcal{O}(\varepsilon(y_n|x))).\end{aligned}$$

Thus as result,

$$\frac{\bar{H}^1(y_n|x)}{\bar{H}(y_n|x)} - p(x) = \mathcal{O}(\varepsilon(y_n|x)).$$

Consequently, we obtain

$$\sigma_n^{-1}(x) \left(\frac{\bar{H}^1(y_n|x)}{\bar{H}(y_n|x)} - p(x) \right) = o_p(1).$$

7.2. Proof of main results

Proof of Theorem 4.1

We decompose for any fixed $x \in \mathbf{E}$,

$$\begin{aligned}\sigma_n^{-1}(x) (\hat{\gamma}_n^{c,H}(x) - \gamma_1(x)) &= \sigma_n^{-1}(x) \left(\frac{\hat{\gamma}_n^H(x)}{\hat{p}_n(x)} - \gamma_1(x) \right) \\ &= \frac{1}{\hat{p}_n(x)} \sigma_n^{-1}(x) (\hat{\gamma}_n^H(x) - \hat{p}_n(x) \gamma_1(x)) \\ &= \frac{1}{\hat{p}_n(x)} \sigma_n^{-1}(x) (\hat{\gamma}_n^H(x) - \gamma(x) + \gamma(x) - \hat{p}_n(x) \gamma_1(x)) \\ &= \frac{1}{p(x)} \sigma_n^{-1}(x) [(\hat{\gamma}_n^H(x) - \gamma(x))] - \frac{\gamma_1(x)}{p(x)} \sigma_n^{-1}(x) [(\hat{p}_n(x) - p(x))] + o_p(1) \\ &:= \frac{1}{p(x)} T_1(x) - \frac{\gamma_1(x)}{p(x)} T_2(x) + o_p(1).\end{aligned}$$

From here, we do some decomposition

$$\begin{aligned}
T_1(x) &= \sigma_n^{-1}(x) [(\hat{\gamma}_n^H(x) - \gamma(x))] \\
&= \sigma_n^{-1}(x) \left[\sum_{i=1}^n K(h^{-1}d(x, X_i)) \log(Z_i/y_n) \mathbf{1}_{\{Z_i > y_n\}} / \sum_{i=1}^n K(h^{-1}d(x, X_i)) \mathbf{1}_{\{Z_i > y_n\}} - \gamma(x) \right] \\
&= \sigma_n^{-1}(x) \left[\frac{1}{n\mu_x^{(1)}(h)} \sum_{i=1}^n K(h^{-1}d(x, X_i)) \log(Z_i/y_n) \mathbf{1}_{\{Z_i > y_n\}} / \hat{\zeta}_n(y_n, x) - \gamma(x) \right] \\
&= \sigma_n^{-1}(x) \left[\frac{1}{n\mu_x^{(1)}(h)} \sum_{i=1}^n K(h^{-1}d(x, X_i)) \log(Z_i/y_n) \mathbf{1}_{\{Z_i > y_n\}} / \hat{\zeta}_n(y_n, x) - \gamma(x) \frac{\bar{H}(y_n|x)}{\hat{\zeta}_n(y_n, x)} \right] \\
&\quad - \sigma_n^{-1}(x) \left[\gamma(x) - \gamma(x) \frac{\bar{H}(y_n|x)}{\hat{\zeta}_n(y_n, x)} \right].
\end{aligned}$$

Since $\bar{H}(y_n|x)/\hat{\zeta}_n(y_n, x) = 1 + o_p(1)$ (from proof of Lemma 7.5) and using the fact that $\frac{\bar{H}^1(y_n|x)}{\bar{H}(y_n|x)} \rightarrow p(x)$ under Lemma A1 in [21]

$$\begin{aligned}
T_1(x) &= \left(\frac{n \left(\mu_x^{(1)}(h) \right)^2}{\bar{H}(y_n|x) \mu_x^{(2)}(h)} \right)^{1/2} \left[\frac{1}{n\mu_x^{(1)}(h)} \sum_{i=1}^n K(h^{-1}d(x, X_i)) \log(Z_i/y_n) \mathbf{1}_{\{Z_i > y_n\}} - \gamma(x) \bar{H}(y_n|x) \right] \\
&\quad - \gamma(x) \left(\frac{n \left(\mu_x^{(1)}(h) \right)^2}{\bar{H}(y_n|x) \mu_x^{(2)}(h)} \right)^{1/2} \left[\hat{\zeta}_n(y_n, x) - \bar{H}(y_n|x) \right].
\end{aligned}$$

Similar calculations give

$$\begin{aligned}
T_2(x) &= \left(\frac{n \left(\mu_x^{(1)}(h) \right)^2}{\bar{H}(y_n|x) \mu_x^{(2)}(h)} \right)^{1/2} \frac{\bar{H}(y_n|x)}{\hat{\zeta}_n(y_n, x)} \left[\left(\hat{\psi}_n(y_n, x) - \bar{H}^1(y_n|x) \right) - p(x) \left(\hat{\zeta}_n(y_n, x) - \bar{H}(y_n|x) \right) \right] \\
&\quad + \left(\frac{n \bar{H}(y_n|x) \left(\mu_x^{(1)}(h) \right)^2}{\mu_x^{(2)}(h)} \right)^{1/2} \frac{\bar{H}(y_n|x)}{\hat{\zeta}_n(y_n, x)} \left[\frac{\bar{H}^1(y_n|x)}{\bar{H}(y_n|x)} - p(x) \right] \\
&= \left(\frac{n \left(\mu_x^{(1)}(h) \right)^2}{\bar{H}(y_n|x) \mu_x^{(2)}(h)} \right)^{1/2} \left[\left(\hat{\psi}_n(y_n, x) - \bar{H}^1(y_n|x) \right) - p(x) \left(\hat{\zeta}_n(y_n, x) - \bar{H}(y_n|x) \right) \right] + o_p(1).
\end{aligned}$$

Thus, to prove the asymptotic normality of our estimator, it is sufficient to establish asymptotic normality of the random vector

$$\Xi_n(x) := \left(\frac{n \left(\mu_x^{(1)}(h) \right)^2}{\bar{H}(y_n|x) \mu_x^{(2)}(h)} \right)^{1/2} \begin{pmatrix} \hat{\psi}_n(y_n, x) - \bar{H}^1(y_n|x) \\ \frac{1}{n\mu_x^{(1)}(h)} \sum_{i=1}^n K(h^{-1}d(x, X_i)) \log(Z_i/y_n) \mathbf{1}_{\{Z_i > y_n\}} - \gamma(x) \bar{H}(y_n|x) \end{pmatrix}.$$

To prove the asymptotic normality of $\Xi_n(x)$, we use Cramér-Wold's device. Let $e = (e_1, e_2)$ be a vector of real numbers, $e \neq 0$. Then,

$$e^T \Xi_n(x) = \left(\frac{1}{n \bar{H}(y_n|x) \mu_x^{(2)}(h)} \right)^{1/2} \sum_{i=1}^n \chi_{i,n}(x),$$

where,

$$\begin{aligned} \chi_{i,n}(x) &= e_1 \left(K(h^{-1}d(x, X_i)) \mathbf{1}_{\{Z_i > y_n, \delta_i=1\}} \right) + e_2 \left(K(h^{-1}d(x, X_i)) \log(Z_i/y_n) \mathbf{1}_{\{Z_i > y_n\}} \right) \\ &- \sqrt{n \left(\mu_x^{(1)}(h) \right)^2} \left[e_1 \bar{H}^1(y_n|x) + e_2 \gamma(x) \bar{H}(y_n|x) \right]. \end{aligned}$$

From Remark 1.2.3 in [6]

$$\lim_{y_n \rightarrow \infty} E(\log Z - \log y_n | Z > y_n) = \gamma(x).$$

Let us introduce the new notation

$$\mathbb{S}_n(y_n, r; x) = E \left[(\log Z - \log y_n)^r \mathbf{1}_{\{Z > y_n\}} | X = x \right]$$

and

$$\tilde{\mathbb{S}}_n(y_n, r; x) := E \left(K^2(h^{-1}d(x, X)) \log(Z/y_n)^r \mathbf{1}_{\{Z > y_n\}} \right).$$

Consider a rate function $B(\cdot|x)$ with $B(t|x) \rightarrow 0$ for $t \rightarrow \infty$ of constant sign for large values of t , such that for all positive u .

$$\lim_{t \rightarrow \infty} \frac{\bar{H}(ut|x)/\bar{H}(t|x) - u^{-1/\gamma(x)}}{B(t|x)} = u^{-1/\gamma(x)} \frac{u^{\rho(x)/\gamma(x)} - 1}{\gamma(x)\rho(x)}.$$

Let's define

$$\mathbb{T}_n(u; x) := \frac{\bar{H}(ut|x)/\bar{H}(t|x) - u^{-1/\gamma(x)}}{B(t|x)} - u^{-1/\gamma(x)} \frac{u^{\rho(x)/\gamma(x)} - 1}{\gamma(x)\rho(x)}. \quad (20)$$

Some algebraic calculation yield here

$$\begin{aligned} \mathbb{S}_n(y_n, r; x) &= \int_{y_n}^{\infty} (\log z - \log y_n)^r d(H(z|x)) \quad \text{integral by part} \\ &= - \lim_{z \rightarrow \infty} (\log z - \log y_n)^r \bar{H}(z|x) + r \int_{y_n}^{\infty} (\log z - \log y_n)^{r-1} \frac{\bar{H}(z|x)}{z} dz \end{aligned}$$

considering that the first term equal to zero, then

$$\begin{aligned}
\mathbb{S}_n(y_n, r; x) &= r \int_{y_n}^{\infty} (\log z - \log y_n)^{r-1} \frac{\bar{H}(z|x)}{z} dz \\
&= r \bar{H}(y_n|x) \int_{y_n}^{\infty} (\log z - \log y_n)^{r-1} \frac{\bar{H}(z|x)}{z \bar{H}(y_n|x)} dz \\
&= r \bar{H}(y_n|x) \int_1^{\infty} (\log w)^{r-1} \frac{\bar{H}(wy_n|x)}{w \bar{H}(y_n|x)} dw \quad \text{with change of variable } z/y_n = w.
\end{aligned}$$

Considering the ratio $\frac{\bar{H}(wy_n|x)}{\bar{H}(y_n|x)}$ from second order condition of extreme value theory in Equation (20), $\mathbb{S}_n(y_n, r; x)$ can be rewritten as

$$\begin{aligned}
\mathbb{S}_n(y_n, r; x) &= r \bar{H}(y_n|x) \int_1^{\infty} (\log w)^{r-1} w^{-1/\gamma(x)-1} dw \\
&\quad + r \bar{H}(y_n|x) B(y_n|x) \int_1^{\infty} (\log w)^{r-1} w^{-1/\gamma(x)-1} \frac{w^{\rho(x)/\gamma(x)} - 1}{\gamma(x)\rho(x)} dw \\
&\quad + r \bar{H}(y_n|x) B(y_n|x) \int_1^{\infty} (\log w)^{r-1} \mathbb{T}_n(w; x) \frac{dw}{w} \\
&= r \bar{H}(y_n|x) (\mathbb{L}_1 + B(y_n|x)(\mathbb{L}_2 + \mathbb{L}_3))
\end{aligned}$$

with,

$$\begin{aligned}
\mathbb{L}_1 &= \int_1^{\infty} (\log w)^{r-1} w^{-1/\gamma(x)-1} dw \\
\mathbb{L}_2 &= \int_1^{\infty} (\log w)^{r-1} w^{-1/\gamma(x)-1} \frac{w^{\rho(x)/\gamma(x)} - 1}{\gamma(x)\rho(x)} dw \\
\mathbb{L}_3 &= \int_1^{\infty} (\log w)^{r-1} \mathbb{T}_n(w; x) \frac{dw}{w}
\end{aligned}$$

For \mathbb{L}_1 , let $w^{-1/\gamma(x)} = \exp(-v)$ implies that $\log w = \gamma(x)v$ and $w^{-1/\gamma(x)-1} dw = \gamma(x) \exp(-v) dv$, then

$$\begin{aligned}
\mathbb{L}_1 &= \int_0^{\infty} (\gamma(x)v)^{r-1} \gamma(x) \exp(-v) dv \\
&= \gamma^r(x) \int_0^{\infty} v^{r-1} \exp(-v) dv \\
&= \gamma^r(x) \Gamma(r)
\end{aligned}$$

For \mathbb{L}_2 , let $w^{-\frac{1-\rho(x)}{\gamma(x)}} = \exp(-v)$ implies that $\log w = \frac{\gamma(x)}{1-\rho(x)} v$ and $w^{-\frac{1-\rho(x)}{\gamma(x)}-1} dw =$

$\frac{\gamma(x)}{1-\rho(x)} \exp(-v)dv$, then

$$\begin{aligned}\mathbb{L}_2 &= \frac{\gamma(x)}{\rho(x)} \left[\left(\frac{\gamma(x)}{1-\rho(x)} \right)^r \int_0^\infty v^{r-1} \exp(-v) dv - \gamma^r(x) \int_0^\infty v^{r-1} \exp(-v) dv \right] \\ &= \frac{\gamma(x)}{\rho(x)} \left[\left(\frac{\gamma(x)}{1-\rho(x)} \right)^r \Gamma(r) - \gamma^r(x) \Gamma(r) \right] \\ &= \frac{\gamma^{r+1}(x) \Gamma(r)}{\rho(x)} \left[\left(\frac{1}{1-\rho(x)} \right)^r - 1 \right]\end{aligned}$$

Now, from Theorem B 2.18 [8] in [6], we have from a function $B_0(\cdot|x)$, possibly different from the function $B(\cdot|x)$ though with $B_0(y_n|x) \approx B(y_n|x)$, $y_n \rightarrow \infty$ and for each $\epsilon, \delta > 0$, such $|\mathbb{T}_n(w)| \leq \epsilon w^{\rho(x)/\gamma(x)+\delta}$. Then,

$$\begin{aligned}|\mathbb{L}_3| &\leq \epsilon \int_1^\infty (\log w)^{r-1} w^{\frac{1-\rho(x)}{\gamma(x)}+\delta-1} dw \\ &= \epsilon \int_0^\infty v^{r-1} \exp\left(-\left(\frac{1-\rho(x)}{\gamma(x)} - \delta\right)v\right) dv \\ &= \epsilon \Gamma(r) \left(\frac{\gamma(x)}{1-\rho(x)-\gamma(x)\delta}\right)^r\end{aligned}$$

provided $0 < \delta < (1-\rho(x))/\gamma(x)$, thus $\mathbb{L}_3 = o(1)$ for $y_n \rightarrow \infty$. Therefore,

$$\mathbb{S}_n(y_n, r; x) = \gamma^r(x) \Gamma(r+1) \bar{H}(y_n|x) \left(1 + \frac{B(y_n|x)}{\rho(x)} \left[\frac{1}{(1-\rho(x))^r} - 1 \right] (1 + o(1)) \right).$$

The calculation of $E[(\log Z - \log y_n)^r \mathbf{1}_{\{Z > y_n, \delta=1\}} | X = x]$ is similar to $E[(\log Z - \log y_n)^r \mathbf{1}_{\{Z > y_n\}} | X = x]$, then is omitted.

Then,

$$\begin{aligned}\tilde{\mathbb{S}}_n(y_n, r; x) &= E(K^2(h^{-1}d(x, X)) \mathbb{S}_n(y_n, r; X)) \\ \tilde{\mathbb{S}}_n(y_n, r; x) - \mu_x^{(2)}(h) \mathbb{S}_n(y_n, r; x) &= E(K^2(h^{-1}d(x, X)) [\mathbb{S}_n(y_n, r; X) - \mathbb{S}_n(y_n, r; x)]) \\ \tilde{\mathbb{S}}_n(y_n, r; x) - \mu_x^{(2)}(h) \mathbb{S}_n(y_n, r; x) &= \mathbb{S}_n(y_n, r; x) E\left(K^2(h^{-1}d(x, X)) \left[\frac{\mathbb{S}_n(y_n, r; X)}{\mathbb{S}_n(y_n, r; x)} - 1\right]\right) \\ \left|\tilde{\mathbb{S}}_n(y_n, r; x) - \mu_x^{(2)}(h) \mathbb{S}_n(y_n, r; x)\right| &\leq \mathbb{S}_n(y_n, r; x) E\left(K^2(h^{-1}d(x, X)) \left[\left|\frac{\mathbb{S}_n(y_n, r; X)}{\mathbb{S}_n(y_n, r; x)} - 1\right| \mathbf{1}_{\{d(X, x) < h\}}\right]\right).\end{aligned}$$

Thus, Assumption **(A5)** yields,

$$\tilde{\mathbb{S}}_n(y_n, r; x) = \mu_x^{(2)}(h) \mathbb{S}_n(y_n, r; x) (1 + \mathcal{O}(\Phi_n(y_n, h; x))).$$

By considering that $B(y_n|x) \rightarrow 0$ and $y_n \rightarrow \infty$ as $n \rightarrow \infty$, we have

$$\begin{aligned} E(K^2(h^{-1}d(x, X)) \log(Z/y_n) \mathbf{1}_{\{Z > y_n\}}) &= \gamma(x) \bar{H}(y_n|x) \mu_x^{(2)}(h) (1 + \mathcal{O}(\Phi_n(y_n, h; x))) \\ E(K^2(h^{-1}d(x, X)) \log(Z/y_n) \mathbf{1}_{\{Z > y_n, \delta=1\}}) &= \gamma(x) \bar{H}^1(y_n|x) \mu_x^{(2)}(h) (1 + \mathcal{O}(\Phi_n(y_n, h; x))) \\ E(K^2(h^{-1}d(x, X)) (\log(Z/y_n))^2 \mathbf{1}_{\{Z > y_n\}}) &= 2\gamma^2(x) \bar{H}(y_n|x) \mu_x^{(2)}(h) (1 + \mathcal{O}(\Phi_n(y_n, h; x))). \end{aligned}$$

The variance of $e^T \Xi_n(x)$ is given by,

$$Var(e^T \Xi_n(x)) = \frac{1}{n \bar{H}(y_n|x) \mu_x^{(2)}(h)} n Var(\chi_{i,n}(x))$$

where

$$\begin{aligned} Var(\chi_{i,n}(x)) &= e_1^2 \mu_x^{(2)}(h) \bar{H}^1(y_n|x) + 2e_2^2 \gamma^2(x) \mu_x^{(2)}(h) \bar{H}(y_n|x) + 2e_1 e_2 \gamma(x) \mu_x^{(2)}(h) \bar{H}^1(y_n|x) + o(1) \\ &= \mu_x^{(2)}(h) \bar{H}(y_n|x) [e_1^2 p(x) + 2e_2^2 \gamma^2(x) + 2e_1 e_2 p(x) \gamma(x)] + o(1). \end{aligned}$$

Therefore the variance of $e^T \Xi_n(x)$ is given by

$$\begin{aligned} Var(e^T \Xi_n(x)) &= [e_1^2 p(x) + 2e_2^2 \gamma^2(x) + 2e_1 e_2 p(x) \gamma(x)] + o(1) \\ &= e^T \Pi e + o(1) \end{aligned}$$

with

$$\Pi := \begin{pmatrix} p(x) & p(x) \gamma(x) \\ p(x) \gamma(x) & 2\gamma^2(x) \end{pmatrix}.$$

The next step is to establish the asymptotic normality of $e^T \Xi_n(x)$, we check that the Lyapunov's criterion for triangular arrays of random variables holds:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n E \left| \left(\frac{1}{n \bar{H}(y_n|x) \mu_x^{(2)}(h)} \right)^{1/2} \chi_{i,n}(x) \right|^3 = 0.$$

The calculation is the same as in proof of Lemma 7.5, therefore it is omitted.

According to Cramér-Wold's device, for $e = (e_1, e_2) \in \mathbb{R}^2$, with $e \neq 0$ and $e^T \Xi_n(x)$ converges in distribution to $\mathcal{N}(0, e^T \Pi e)$ which implies that $\Xi_n(x)$ converges in distribution $\mathcal{N}(0, \Pi)$. An application of Delta method shows that

$$\sigma_n^{-1}(x) (\hat{\gamma}_n^{c,H}(x) - \gamma_1(x)) = \begin{pmatrix} -\frac{\gamma_1(x)}{p(x)} \\ \frac{1}{p(x)} \end{pmatrix}^T \Xi_n(x)$$

converges in distribution to $\mathcal{N}\left(0, \frac{\gamma_1^3(x)}{\gamma(x)}\right)$.

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