# Solutions of kinetic equations related to non-local conservation laws

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**Abstract**

The aim of this paper is to propose kinetic models associated with conservation laws with a non-local flux and to prove the existence of solutions for these kinetic equations. In order to make the article as efficient as possible, we have highlighted the hypotheses that make the proofs work, so that it can be used for other models. We present two types of hypotheses on the kinetic model and two different techniques to obtain an existence result.

**Key-words**: Scalar conservation laws – non-local flux – kinetic equations

**Mathematics Subject Classification**: 35L65, 35A35, 35D30, 82C40

# Contents



# 1 Introduction

## 1.1 Context

Non local fluxes have been introduced recently to model pedestrian or vehicular traffic [10], [11], [13], [9]. These fields of application are emerging [1] and non-local models should appear in the coming years to model more phenomena. On the other hand, kinetic models associated with conservation laws have proven to be very useful both from a theoretical and a numerical point of view ([16], [19]). It is therefore natural to ask the question of kinetic model for these new non-local models. This is the object of this paper: to propose kinetic models associated with conservation laws with a non-local flux and to prove the existence of solutions for these kinetic equations. In order to make the article as efficient as possible, we have highlighted the hypotheses that make the proofs work, so that it can be used for other models.

## 1.2 Models

First of all, let us specify the kind of models that we will study both from the point of view of the law of conservation with non-local flux then from the point of view of the kinetic equation.

For the non-local scalar conservation law, we consider the following models:

$$\partial_t \rho + \partial_x (F(\rho) G(\eta * \rho)) = 0, \tag{1.1}$$

which mean

$$\partial_t \rho(t,x) + \partial_x(F(\rho(t,x))G((\eta * \rho)(t,x))) = 0, \quad (t,x) \in [0,+\infty[\times\mathbb{R}, \tag{1.2}$$

where $F, G \in C^1(\mathbb{R},\mathbb{R})$, $\eta \in C^1(\mathbb{R},\mathbb{R}) \cap L^\infty(\mathbb{R})$ and

$$(\eta * \rho)(t,x) = \int_{\mathbb{R}} \eta(x-y)\rho(t,y)\,dy. \tag{1.3}$$

This term is well defined if $y \mapsto \rho(t,y) \in L^1(\mathbb{R})$ for any $t$.

For the kinetic equation, we consider:

$$\partial_t f_\varepsilon + \partial_x(a(v,\xi)f_\varepsilon) = \frac{\mathcal{M}_{\rho_\varepsilon} - f_\varepsilon}{\varepsilon}, \tag{1.4}$$

which mean

$$\partial_t f_\varepsilon(t,x,v,\xi) + \partial_x(a(v,\xi)f_\varepsilon(t,x,v,\xi)) = \frac{\mathcal{M}_{\rho_\varepsilon}(t,x,v,\xi) - f_\varepsilon(t,x,v,\xi)}{\varepsilon}, \tag{1.5}$$

for $(t,x,v,\xi) \in [0,+\infty[\times\mathbb{R}^3$, where

$$\rho_\varepsilon(t,x) = \iint_{\mathbb{R}^2} f_\varepsilon(t,y,v,\xi)\,d\xi dv \tag{1.6}$$

and $\mathcal{M}$ is the Maxwellian which will be defined later. This kinetic model is of the BGK type. See the references [17], [16], [5], [2] for the case of a scalar conservation law and studies for some systems.

## 1.3 Main results and organization of the paper

The organization of the paper is as follows. In the section 2, we consider the assumptions the kinetic model have to satisfy in order that our study works. First, we need two assumptions (2.1)-(2.2) to assure the consistance between the kinetic equation and the non-local scalar equation. Then, we expose the hypotheses necessary to obtain the existence of solutions for the kinetic equation according to the method used. We will present two proof methods, each one requiring specific hypotheses. For the first existence result, we need (2.3). For the second existence result, we need (2.4)-(2.10). This section ends with a formal proof that justifies the need for consistence assumptions.

In section 3, we study the well-posedness of the kinetic equation with assumption (2.3). In this framework, we can use a fixed point and the proof is relatively usual. We get the following result.

**Theorem 1.1** *Let $F, G \in C^1(\mathbb{R},\mathbb{R})$, $\eta \in C^1(\mathbb{R},\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $f^0 \in L^1(\mathbb{R}^3)$. We consider a maxwellian $\mathcal{M}$ satisfying (2.1) and (2.3). Then there exists $\rho \in L^\infty([0,T],L^1(\mathbb{R}^3))$ for any $T > 0$ solution of (1.4) with initial data $f^0$. Furthermore this solution $\rho \in L^\infty([0,T],L^1(\mathbb{R}^3))$ is unique with the initial data.*

This case is the easiest among the two that we study but most of the models won't verify (2.3) thus it require the study of the second model.

This is why in section 4, we study the existence of a solution for the kinetic equation with assumption (2.4)-(2.10). We get the following result.

**Theorem 1.2** *Let $F, G \in C^1(\mathbb{R}, \mathbb{R})$, $\eta \in C^1(\mathbb{R}, \mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $f^0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ such that $xf^0, \xi f^0, vf^0, a(v,\xi)f^0 \in L^1(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}} \left( \iint_{\mathbb{R}^2} f^0(x,v,\xi)\, dvd\xi \right)^2 dx < +\infty.$$

*We consider a maxwellian $\mathcal{M}$ satisfying (2.1) and (2.4)-(2.10). Assume that there exists a constant $K > 0$ such that*

$$|F(z)| \leq K(|z| + |z|^2) \quad \text{for any } z \in \mathbb{R}. \tag{1.7}$$

*Then there exists $\rho \in L^\infty([0,T], L^1(\mathbb{R}^3))$ for any $T > 0$ solution of (1.4) with initial data $f^0$.*

This proof uses Schauder's theorem and is much more complex and requires solving numerous technical difficulties.

Then, in section 6, we present a model which satisfies the assumptions for the first theorem and in section 5, a model which satisfies the ones for the second theorem.

# 2 General framework for the kinetic model

This section sets the general framework and the assumptions that the kinetic model must satisfy for our study. Then, we present the formal limit of the model to check that the limit equation is indeed the expected one.

## 2.1 Assumptions on the kinetic model

The first assumptions for the kinetic model are consistency type ones. They will ensure that the formal limit is indeed the non-local equation.

**Consistance assumptions.**

We assume that, for any $\rho : \mathbb{R}^2 \to \mathbb{R}$, there exists $\mathcal{M}_\rho : \mathbb{R}^4 \to \mathbb{R}$ such that

$$\iint_{\mathbb{R}^2} \mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv = \rho(t,x) \tag{2.1}$$

and

$$\iint_{\mathbb{R}^2} a(v,\xi)\mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv = F(\rho(t,x))G((\eta * \rho)(t,x)). \tag{2.2}$$

These two previous properties ensure the consistency between the kinetic and the non-local scalar equation.

For the existence of a solution to the kinetic equation, according to the model, we can consider two differents lists of assumptions.

**Assumption for existence 1.**
We assume that for any $\rho_1, \rho_2 \in L^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}} \iint_{\mathbb{R}^2} |\mathcal{M}_{\rho_1}(t,x,v,\xi) - \mathcal{M}_{\rho_2}(t,x,v,\xi)| \, dv d\xi \, dx \leq K \int_{\mathbb{R}} |\rho_1 - \rho_2|(t,x) \, dx \tag{2.3}$$

**Assumptions for existence 2.**
We assume that there exists constants $K_2, K_3, K_4, K_5 > 0$ and $p = 1$ or $2$ such that, for any $\rho : \mathbb{R} \to \mathbb{R}$,

$$\iint_{\mathbb{R}^2} |\mathcal{M}_\rho(t,x,v,\xi)| \, d\xi dv \leq |\rho(t,x)|, \tag{2.4}$$

$$\iint_{\mathbb{R}^2} |a(v,\xi)\mathcal{M}_\rho(t,x,v,\xi)| \, d\xi dv \leq K_2 |F(\rho(t,x))G((\eta * \rho)(t,x))|, \tag{2.5}$$

$$\iint_{\mathbb{R}^2} |\xi||\mathcal{M}_\rho(t,x,v,\xi)| \, d\xi dv \leq K_3 \, |\rho(t,x)| \left(1 + \int_{\mathbb{R}} |\rho(t,y)|^2 \, dy\right), \tag{2.6}$$

$$\iint_{\mathbb{R}^2} |v||\mathcal{M}_\rho(t,x,v,\xi)| \, d\xi dv \leq K_4 \, \rho^2(t,x), \tag{2.7}$$

$$\iint_{\mathbb{R}^2} |\mathcal{M}_\rho(t,x,v,\xi)|^2 \, d\xi dv \leq K_5 \, |\rho(t,x)|^p, \tag{2.8}$$

$$\text{if } \rho_n \to \rho \text{ a.e. } (t,x) \text{ and } |\rho_n| \leq |h| \in L^1(\mathbb{R}), \text{ then } \mathcal{M}_{\rho_n} \to \mathcal{M}_\rho \text{ a.e. } (t,x,v,\xi) \tag{2.9}$$

and

$$\text{the term } a(v,\xi) \text{ allows to apply an averaging lemma .} \tag{2.10}$$

*Remark 2.1 We have to specify what we mean by averaging lemma. It is such a result : if*

$$\partial_t g_n + \partial_x (a(v,\xi) g_n = h_n$$

*with $(g_n)_n$ and $(g_n)_n$ bounded in $L^1(]0,T[\times\mathbb{R} \times \mathbb{R}^2)$ then $\rho_n$ is compact in $L^1_{loc}(]0,T[\times\mathbb{R})$ where*

$$\rho_n(t,x) = \iint_{\mathbb{R}^2} \psi(\xi,v) g_n(t,y,v,\xi) \, d\xi dv$$

*with $\psi \in L^\infty(]0,T[\times\mathbb{R}\times\mathbb{R}^2)$. The function $a(v,\xi)$ must satisfy a non degeneracy condition to pretend getting this kind of result.*

The first assumption allows us to apply contraction technic and the second list of assumptions to apply Schauder's result. If the model satisfies the **Assumption for existence 1**, we should use it but in practice, it is not often the case. Then, we have to consider the **Assumptions for existence 2**.

*Remark 2.2 Notice that (2.1) and (2.4) imply*

$$\iint_{\mathbb{R}^2} |\mathcal{M}_\rho(t,x,v,\xi)|\, d\xi dv = |\rho(t,x)|$$

*and if $K_2 = 1$, then (2.2) and (2.5) imply*

$$\iint_{\mathbb{R}^2} |a(v,\xi)\mathcal{M}_\rho(t,x,v,\xi)|\, d\xi dv = |F(\rho(t,x))G((\eta * \rho)(t,x))|.$$

*Remark 2.3 Notice also that if we have a majoration like*

$$\iint_{\mathbb{R}^2} |\xi||\mathcal{M}_\rho(t,x,v,\xi)|\, d\xi dv \leq K_3\, |\rho(t,x)| \int_{\mathbb{R}} |\rho(t,y)|\, dy,$$

*then it implies (2.6).*

*Remark 2.4 If we want to gives an additional property as for example the positivity of the solution, we have to add this fact in the assumptions. Thus for the important case where we want $\rho \geq 0$, for example for the first existence result, we just need the assumptions:*

$$\iint_{\mathbb{R}^2} \mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv = \rho(t,x), \tag{2.11}$$

$$\iint_{\mathbb{R}^2} a(v,\xi)\mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv = F(\rho(t,x))G((\eta * \rho)(t,x)) \tag{2.12}$$

*and*

$$\int_{\mathbb{R}} \iint_{\mathbb{R}^2} |\mathcal{M}_{\rho_1}(t,x,v,\xi) - \mathcal{M}_{\rho_2}(t,x,v,\xi)|\; dv d\xi\, dx \leq K \int_{\mathbb{R}} |\rho_1 - \rho_2|\,(t,x)\, dx \tag{2.13}$$

**for any** $\rho, \rho_1, \rho_2 \geq \mathbf{0}$.

## 2.2 Formal limit

**Consistency assumptions**, that is to say (2.1) and (2.2) are related to the consistency between kinetic and non-local equation by the following formal limit. Formally, passing to the limit as $\varepsilon \to 0$ in (1.4), gives

$$\mathcal{M}_\rho = f,$$

by noting $f$ the limit of $(f_\varepsilon)$. On the other hand, an integration with respect to $(v,\xi)$ of (1.4) yields

$$\partial_t \iint_{\mathbf{R}^2} f_\varepsilon \, d\xi dv + \partial_x \iint_{\mathbf{R}^2} a(v,\xi) f_\varepsilon \, d\xi dv = 0,$$

since

$$\iint_{\mathbf{R}^2} \mathcal{M}_{\rho_\varepsilon}(t,x,v,\xi) \, d\xi dv = \rho_\varepsilon(t,x) = \iint_{\mathbf{R}^2} f_\varepsilon(t,x,v,\xi) \, d\xi dv.$$

At the limit, we thus have

$$\partial_t \iint_{\mathbf{R}^2} \mathcal{M}_\rho \, d\xi dv + \partial_x \iint_{\mathbf{R}^2} a(v,\xi) \mathcal{M}_\rho \, d\xi dv = 0.$$

Now

$$\iint_{\mathbf{R}^2} \mathcal{M}_\rho(t,x,v,\xi) \, d\xi dv = \rho(t,x)$$

and

$$\iint_{\mathbf{R}^2} a(v,\xi) \mathcal{M}_\rho(t,x,v,\xi) \, d\xi dv = F(\rho) G(\eta * \rho).$$

Finally we get

$$\partial_t \rho + \partial_x (F(\rho) G(\eta * \rho)) = 0.$$

# 3 Well-posedness of the kinetic equation with assumption (2.3)

We consider the case of **Assumption for existence 1.** We also need the first assumption of consistency. Then, we assume that (2.1) and (2.3) are satisfied are we prove that it allows to get existence and unicity of a solution to the kinetic equation.

**Proof of Theorem 1.1.** Equation (1.4) is equivalent to the following integral representation

$$f_\varepsilon(t,x,v,\xi) = e^{-t/\varepsilon} f_\varepsilon(0, x - a(v,\xi)t, v, \xi) + \frac{1}{\varepsilon} \int_0^t e^{(s-t)/\varepsilon} \mathcal{M}_{\rho_\varepsilon}(s, x - a(v,\xi)(t-s), v, \xi) \, ds$$

with

$$\rho_\varepsilon(t,x) = \iint_{\mathbf{R}^2} f_\varepsilon(t,y,v,\xi) \, d\xi dv.$$

Let $\varepsilon > 0$ and $T > 0$. Denote by $\Phi$ the application from $L^\infty([0,T], L^1(\mathbf{R}^3))$ to $L^\infty([0,T], L^1(\mathbf{R}^3))$ which at $f$ associate

$$\Phi(f)(t,x,v,\xi) = e^{-t/\varepsilon} f^0(x - a(v,\xi)t, v, \xi) + \frac{1}{\varepsilon} \int_0^t e^{(s-t)/\varepsilon} \mathcal{M}_\rho(s, x - a(v,\xi)(t-s), v, \xi) \, ds,$$

where

$$\rho(t,x) = \iint_{\mathbf{R}^2} f(t,y,v,\xi)\, d\xi dv.$$

For $f_1, f_2 \in L^\infty([0,T], L^1(\mathbf{R}^3))$, we note

$$\rho_1(t,x) = \iint_{\mathbf{R}^2} f_1(t,y,v,\xi)\, d\xi dv \quad \text{and} \quad \rho_2(t,x) = \iint_{\mathbf{R}^2} f_2(t,y,v,\xi)\, d\xi dv$$

and we have

$$\begin{aligned}
&\iiint_{\mathbf{R}^3} |\Phi(f_1)(t,x,v,\xi) - \Phi(f_2)(t,x,v,\xi)|\, dx d\xi dv \\
\leq\ & \frac{1}{\varepsilon} \iint_{\mathbf{R}^3} \int_0^t e^{(s-t)/\varepsilon} |\mathcal{M}_{\rho_1} - \mathcal{M}_{\rho_2}|(s, x - a(v,\xi)(t-s), v, \xi)\, ds\, dx d\xi dv \\
\leq\ & \frac{1}{\varepsilon} \int_0^t \iint_{\mathbf{R}^2} e^{(s-t)/\varepsilon} \int_{\mathbf{R}} |\mathcal{M}_{\rho_1}(s,x,v,\xi) - \mathcal{M}_{\rho_1}(s,x,v,\xi)|\, dx\, d\xi dv ds \\
\leq\ & \frac{1}{\varepsilon} \int_0^t e^{(s-t)/\varepsilon} \int_{\mathbf{R}} K |\rho_1 - \rho_2|(s,x)\, dx ds \\
\leq\ & K \frac{1}{\varepsilon} \int_0^t e^{(s-t)/\varepsilon} \iiint_{\mathbf{R}^3} |f_1(s,x,v,\xi) - f_2(s,x,v,\xi)|\, dx d\xi dv ds \\
\leq\ & K \frac{1}{\varepsilon} \int_0^t e^{(s-t)/\varepsilon} ds \sup_{s\in[0,t]} \iiint_{\mathbf{R}^3} |f_1(s,x,v,\xi) - f_2(s,x,v,\xi)|\, dx d\xi dv \\
\leq\ & K \left(1 - e^{-t/\varepsilon}\right) \sup_{s\in[0,t]} \iiint_{\mathbf{R}^3} |f_1(s,x,v,\xi) - f_2(s,x,v,\xi)|\, dx d\xi dv.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sup_{t\in[0,T]} \iiint_{\mathbf{R}^3} |\Phi(f_1)(t,x,v,\xi) - \Phi(f_2)(t,x,v,\xi)|\, dx d\xi dv \\
\leq\ & K \left(1 - e^{-T/\varepsilon}\right) \sup_{t\in[0,T]} \iiint_{\mathbf{R}^3} |f_1(t,x,v,\xi) - f_2(t,x,v,\xi)|\, dx d\xi dv.
\end{aligned}$$

Taking

$$T_\varepsilon = -\varepsilon \ln\left(\frac{2K-1}{2K}\right) > 0,$$

we have

$$K\left(1 - e^{-T_\varepsilon/\varepsilon}\right) = \frac{1}{2}$$

and $\Phi$ is a contraction on $L^\infty([0,T_\varepsilon], L^1(\mathbf{R}^3))$. Then we get the existence and uniqueness of a solution in $L^\infty([0,T_\varepsilon], L^1(\mathbf{R}^3))$ to (1.4) with initial data $f^0 \geq 0$.

Since the time $T_\varepsilon$ does not depend on $f^0$, we can restart from the obtained solution at value $T_\varepsilon$ and get a solution on $[T_\varepsilon, 2T_\varepsilon]$ and so on. Finally we get existence and uniqueness of a solution in $X$ on any $[0,T]$ with $T>0$. $\square$

We also have a variant for the important case where $\rho \geq 0$.

**Proposition 3.1** *Let $f^0 \in L^1(\mathbf{R}^3)$ such that $f0 \geq 0$. We consider a maxwellian $\mathcal{M}$ satisfying (2.11) and (2.13). Then there exists $\rho \in L^\infty([0,T], L^1(\mathbf{R}^3))$ for any $T>0$ solution of (1.4) with initial data $f^0$ and such that $\rho \geq 0$.*

**Proof.** We adapt the previous proof by considering the space $X$ of functions $f$ in $L^\infty([0,T_\varepsilon], L^1(\mathbf{R}^3))$ such that $f \geq 0$. For $f \in X$, we have $\Phi(f) \in X$ since then $\rho \geq 0$ and $\mathcal{M}_\rho \geq 0$. $\square$

# 4 Existence of a solution for the kinetic equation with assumptions (2.4)-(2.10)

We consider the case of **Assumption for existence 2.** We also need the first assumption of consistency. Then, we assume that (2.1) and (2.4)-(2.10) are satisfied and we prove that it allows to get existence of a solution to the kinetic equation.

**Proof of Theorem 1.2.**

Let $\varepsilon > 0$ and $T>0$. Denote by $\Phi$ the application from $L^\infty([0,T], L^1(\mathbf{R}^3))$ to $L^\infty([0,T], L^1(\mathbf{R}^3))$ which at $f$ associate

$$\Phi(f)(t,x,v,\xi) = e^{-t/\varepsilon} f^0(x - a(v,\xi)t, v, \xi) + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \mathcal{M}_\rho(s, x - a(v,\xi)(t-s), v, \xi)\, ds,$$

where

$$\rho(t,x) = \iint_{\mathbf{R}^2} f(t,y,v,\xi)\, d\xi dv.$$

There exists constants $C_0^1, \ldots, C_0^6, C_0^a > 0$ such that

$$\iiint_{\mathbf{R}^3} |f^0(x,v,\xi)|\, dv d\xi dx = C_0^1 < +\infty. \tag{4.1}$$

$$\iiint_{\mathbf{R}^3} |x||f^0(x,v,\xi)|\, dv d\xi dx = C_0^2 < +\infty, \tag{4.2}$$

$$\iiint_{\mathbf{R}^3} |\xi||f^0(x,v,\xi)|\, dv d\xi dx = C_0^3 < +\infty, \tag{4.3}$$

$$\iiint_{\mathbf{R}^3} |v||f^0(x,v,\xi)|\, dv d\xi dx = C_0^4 < +\infty, \tag{4.4}$$

$$\iiint_{\mathbf{R}^3} |f^0(x,v,\xi)|^2 \, dvd\xi dx = C_0^5 < +\infty. \tag{4.5}$$

$$\int_{\mathbf{R}} \left( \iint_{\mathbf{R}^2} f^0(x,v,\xi) \, dvd\xi \right)^2 dx = C_0^6 < +\infty \tag{4.6}$$

and

$$\iiint_{\mathbf{R}^3} |a(v,\xi)||f^0(x,v,\xi)| \, dvd\xi dx = C_0^a < +\infty. \tag{4.7}$$

We set

$$\mathcal{G} = 1 + \sup_{z \in \overline{B}(0, \|\eta\|_\infty R_1)} |G(z)| < +\infty \tag{4.8}$$

since $G$ is continuous and $\overline{B}(0, \|\eta\|_\infty R_1)$ is compact. We take

$$R_1 = \max\left(C_0^1, \frac{C_0^a}{K_2 K \mathcal{G}}\right), \quad R_2 = C_0^2, \quad R_3 = \max(C_0^3, K_3 R_1 (1 + R_6)), \tag{4.9}$$

$$R_4 = \max(C_0^4, K_4 K_6), \quad R_5 = \max(C_0^5, K_5 R_1, K_5 R_6) \tag{4.10}$$

and

$$R_6 = \max\left(C_0^6, \frac{C_0^a}{K_2 K \mathcal{G}}\right). \tag{4.11}$$

We denote $R = (R_1, R_2, R_3, R_4, R_5, R_6)$ and $C_R$ the set of all $f \in L^\infty([0,T], L^1(\mathbf{R}^3))$ such that for a.e. $t \in ]0,T[$,

$$\iiint_{\mathbf{R}^3} |f(t,x,v,\xi)| \, dvd\xi dx \leq R_1, \tag{4.12}$$

$$\iiint_{\mathbf{R}^3} |x||f(t,x,v,\xi)| \, dvd\xi dx \leq R_2, \tag{4.13}$$

$$\iiint_{\mathbf{R}^3} |\xi||f(t,x,v,\xi)| \, dvd\xi dx \leq R_3, \tag{4.14}$$

$$\iiint_{\mathbf{R}^3} |v||f(t,x,v,\xi)| \, dvd\xi dx \leq R_4, \tag{4.15}$$

$$\iiint_{\mathbf{R}^3} |f(t,x,v,\xi)|^2 \, dvd\xi dx \leq R_5 \tag{4.16}$$

and

$$\int_{\mathbf{R}} \left( \iint_{\mathbf{R}^2} f(t,x,v,\xi) \, dvd\xi \right)^2 dx \leq R_6. \tag{4.17}$$

We denote also $\tilde{C}_R$ the set of all $f \in C([0,T], L^1(\mathbf{R}^3))$ satisfying (4.12)-(4.17) with

$$\partial_t f + \partial_x(a(v,\xi)f) + \frac{1}{\varepsilon} f \in \frac{C_R}{\varepsilon}. \tag{4.18}$$

The presentation of the proof is divided into seven parts.
**Step 1.** We prove that if $f \in C_R$, then $\mathcal{M}_\rho \in C_R$.
First, using (2.4), we have

$$\begin{aligned} \iiint_{\mathbf{R}^3} |\mathcal{M}_\rho(t,x,v,\xi)|\, dvd\xi dx &\leq \int_{\mathbf{R}} |\rho(t,x)| dx \\ &\leq \int_{\mathbf{R}} \iint_{\mathbf{R}^2} |f(t,x,v,\xi)|\, d\xi dv dx \\ &\leq R_1 \end{aligned}$$

and, using (2.1),

$$\begin{aligned} \int_{\mathbf{R}} \left( \iint_{\mathbf{R}^2} \mathcal{M}_\rho(t,x,v,\xi)\, dvd\xi \right)^2 dx &\leq \int_{\mathbf{R}} \rho(t,x)^2 dx \\ &\leq \int_{\mathbf{R}} \left( \iint_{\mathbf{R}^2} |f(t,x,v,\xi)|\, d\xi dv \right)^2 dx \\ &\leq R_6. \end{aligned}$$

Now, using (2.8), we get

$$\iiint_{\mathbf{R}^3} |\mathcal{M}_\rho(t,x,v,\xi)|^2\, dvd\xi dx \leq \int_{\mathbf{R}} K_5 |\rho(t,x)|^p dx.$$

If $p = 1$, it gives

$$\begin{aligned} \iiint_{\mathbf{R}^3} |\mathcal{M}_\rho(t,x,v,\xi)|^2\, dvd\xi dx &\leq \int_{\mathbf{R}} K_5 \iint_{\mathbf{R}^2} |f(t,x,v,\xi)|\, d\xi dv dx \\ &\leq K_5 R_1 \leq R_5. \end{aligned}$$

Otherwise $p = 2$ and it gives

$$\begin{aligned} \iiint_{\mathbf{R}^3} |\mathcal{M}_\rho(t,x,v,\xi)|^2\, dvd\xi dx &\leq \int_{\mathbf{R}} K_5 \left( \iint_{\mathbf{R}^2} f(t,x,v,\xi)\, d\xi dv \right)^2 dx \\ &\leq K_5 R_6 \leq R_5. \end{aligned}$$

Now, using (2.4), we have

$$\begin{aligned}\iiint_{\mathbf{R}^3} |x||\mathcal{M}_\rho(t,x,v,\xi)|\,dvd\xi dx &\leq \int_{\mathbf{R}} |x||\rho(t,x)|dx \\ &\leq \iiint_{\mathbf{R}^3} |x||f(t,x,v,\xi)|\,dvd\xi dx \\ &\leq R_2.\end{aligned}$$

Furthermore, using (2.6), we have

$$\begin{aligned}&\iiint_{\mathbf{R}^3} |\xi||\mathcal{M}_\rho(t,x,v,\xi)|\,dvd\xi dx \\ \leq\ & \int_{\mathbf{R}} K_3|\rho(t,x)|\left(1+\int_{\mathbf{R}}|\rho(t,y)|^2dy\right)dx \\ \leq\ & K_3\iiint_{\mathbf{R}^3}|f(t,x,v,\xi)|\,d\xi dvdx\left(1+\int_{\mathbf{R}}\left(\iint_{\mathbf{R}^2}|f(t,y,v,\xi)|\,d\xi dv\right)^2 dy\right) \\ \leq\ & K_3R_1(1+R_6) \\ \leq\ & R_3.\end{aligned}$$

Finally, using (2.7), we deduce

$$\begin{aligned}\iiint_{\mathbf{R}^3} |v||\mathcal{M}_\rho(t,x,v,\xi)|\,dvd\xi dx &\leq \int_{\mathbf{R}} K_4|\rho(t,x)|^2\,dx \\ &\leq K_4\int_{\mathbf{R}}\left(\iint_{\mathbf{R}^2} f(t,x,v,\xi)\,d\xi dv\right)^2 dx \\ &\leq K_4K_6 \\ &\leq R_4.\end{aligned}$$

Then we get that $\mathcal{M}_\rho \in C_R$.

**Step 2.** We prove that if $f \in C_R$, then $\Phi(f) \in C_R$.
First, we have

$$\begin{aligned}&\iiint_{\mathbf{R}^3} |\Phi(f)(t,x,v,\xi)|\,dvd\xi dx \\ \leq\ & e^{-t/\varepsilon}\iiint_{\mathbf{R}^3}|f^0(x-a(v,\xi)t,v,\xi)|\,dvd\xi dx \\ & +\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iiint_{\mathbf{R}^3}|\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)|\,dvd\xi dx\,ds\end{aligned}$$

$$
\begin{aligned}
&\leq e^{-t/\varepsilon} \iiint_{\mathbf{R}^3} |f^0(x,v,\xi)|\, dvd\xi dx \\
&\quad + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \iiint_{\mathbf{R}^3} |\mathcal{M}_\rho(s,x,v,\xi)|\, dvd\xi dx\, ds \\
&\leq e^{-t/\varepsilon} R_1 + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} R_1\, ds \\
&\leq e^{-t/\varepsilon} R_1 + R_1(1-e^{-t/\varepsilon}) = R_1.
\end{aligned}
$$

Now notice that

$$
\begin{aligned}
\Phi(f)(t,x,v,\xi) &= e^{-t/\varepsilon} f^0(x - a(v,\xi)t, v, \xi) \\
&\quad + (1-e^{-t/\varepsilon}) \int_0^t \mathcal{M}_\rho(s, x - a(v,\xi)(t-s), v, \xi) \frac{e^{(s-t)/\varepsilon}\, ds}{\int_0^t e^{-\sigma/\varepsilon} d\sigma},
\end{aligned}
$$

then for a convex function $H$, we have

$$
\begin{aligned}
H(\Phi(f)(t,x,v,\xi)) &\leq e^{-t/\varepsilon} H(f^0(x - a(v,\xi)t, v, \xi)) \\
&\quad + (1-e^{-t/\varepsilon}) H\left( \int_0^t \mathcal{M}_\rho(s, x - a(v,\xi)(t-s), v, \xi) \frac{e^{(s-t)/\varepsilon}\, ds}{\int_0^t e^{-\sigma/\varepsilon} d\sigma} \right)
\end{aligned}
$$

and by Jensen's inequality, we get

$$
\begin{aligned}
H(\Phi(f)(t,x,v,\xi)) &\leq e^{-t/\varepsilon} H(f^0(x - a(v,\xi)t, v, \xi)) \\
&\quad + (1-e^{-t/\varepsilon}) \int_0^t H(\mathcal{M}_\rho(s, x - a(v,\xi)(t-s), v, \xi)) \frac{e^{(s-t)/\varepsilon}\, ds}{\int_0^t e^{-\sigma/\varepsilon} d\sigma} \\
&\leq e^{-t/\varepsilon} H(f^0(x - a(v,\xi)t, v, \xi)) \\
&\quad + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} H(\mathcal{M}_\rho(s, x - a(v,\xi)(t-s), v, \xi))\, ds.
\end{aligned}
$$

With $H(z) = z^2$, it gives

$$
\begin{aligned}
(\Phi(f)(t,x,v,\xi))^2 &\leq e^{-t/\varepsilon} (f^0(x - a(v,\xi)t, v, \xi))^2 \\
&\quad + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} (\mathcal{M}_\rho(s, x - a(v,\xi)(t-s), v, \xi))^2\, ds.
\end{aligned}
$$

and, using (2.8),

$$
\begin{aligned}
&\iiint_{\mathbf{R}^3} \Phi(f)(t,x,v,\xi)^2\, dvd\xi dx \\
&\leq e^{-t/\varepsilon} \iiint_{\mathbf{R}^3} |f^0(x - a(v,\xi)t, v, \xi)|^2\, dvd\xi dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iiint_{\mathbf{R}^3}|\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)|^2\,dvd\xi dx\,ds\\
\leq\;& e^{-t/\varepsilon}\iiint_{\mathbf{R}^3}|f^0(x,v,\xi)|^2\,dvd\xi dx\\
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iiint_{\mathbf{R}^3}|\mathcal{M}_\rho(s,x,v,\xi)|^2\,dvd\xi dx\,ds\\
\leq\;& e^{-t/\varepsilon}C_0^5+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}R_5\,ds\\
\leq\;& e^{-t/\varepsilon}R_5+R_5(1-e^{-t/\varepsilon})=R_5.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
&\iint_{\mathbf{R}^2}\Phi(f)(t,x,v,\xi)\,dvd\xi\\
=\;& e^{-t/\varepsilon}\iint_{\mathbf{R}^2}f^0(x-a(v,\xi)t,v,\xi)\,dvd\xi\\
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iint_{\mathbf{R}^2}\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)\,dvd\xi\,ds
\end{aligned}
$$

then, by convexity,

$$
\begin{aligned}
&\left(\iint_{\mathbf{R}^2}\Phi(f)(t,x,v,\xi)\,dvd\xi\right)^2\\
\leq\;& e^{-t/\varepsilon}\left(\iint_{\mathbf{R}^2}f^0(x-a(v,\xi)t,v,\xi)\,dvd\xi\right)^2\\
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\left(\iint_{\mathbf{R}^2}\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)\,dvd\xi\right)^2 ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_{\mathbf{R}}\left(\iint_{\mathbf{R}^2}\Phi(f)(t,x,v,\xi)\,dvd\xi\right)^2 dx\\
\leq\;& e^{-t/\varepsilon}\int_{\mathbf{R}}\left(\iint_{\mathbf{R}^2}f^0(x-a(v,\xi)t,v,\xi)\,dvd\xi\right)^2 dx\\
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\int_{\mathbf{R}}\left(\iint_{\mathbf{R}^2}\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)\,dvd\xi\right)^2 dx\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq e^{-t/\varepsilon}\int_{\mathbf{R}}\left(\iint_{\mathbf{R}^2} f^0(x,v,\xi)\,dvd\xi\right)^2 dx\\
&\quad +\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\int_{\mathbf{R}}\left(\iint_{\mathbf{R}^2}\mathcal{M}_\rho(s,x,v,\xi)\,dvd\xi\right)^2 dx\,ds\\
&\leq e^{-t/\varepsilon}C_0^6 + +\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}R_6\,ds\\
&\leq e^{-t/\varepsilon}R_6 + R_6(1-e^{-t/\varepsilon}) = R_6.
\end{aligned}
$$

Now, using (2.4) and (2.5), we have

$$
\begin{aligned}
&\iiint_{\mathbf{R}^3}|x||\Phi(f)(t,x,v,\xi)|\,dvd\xi dx\\
\leq\ & e^{-t/\varepsilon}\iiint_{\mathbf{R}^3}|x||f^0(x-a(v,\xi)t,v,\xi)|\,dvd\xi dx\\
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iiint_{\mathbf{R}^3}|x||\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)|\,dvd\xi dx\,ds\\
\leq\ & e^{-t/\varepsilon}\iiint_{\mathbf{R}^3}|x+a(v,\xi)t||f^0(x,v,\xi)|\,dvd\xi dx\\
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iiint_{\mathbf{R}^3}|x+a(v,\xi)(t-s)||\mathcal{M}_\rho(s,x,v,\xi)|\,dvd\xi dx\,ds\\
\leq\ & e^{-t/\varepsilon}(C_0^2+tC_0^a)\\
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\int_{\mathbf{R}}(|x||\rho(s,x)|+(t-s)K_2|F(\rho(s,x))G((\eta*\rho)(s,x))|)\,dx\,ds\\
\leq\ & e^{-t/\varepsilon}(C_0^2+tC_0^a)\\
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\left(\iiint_{\mathbf{R}^3}|x||f(s,x,v,\xi)|\,dvd\xi dx+(t-s)K_2\int_{\mathbf{R}}|F(\rho(s,x))G((\eta*\rho)(s,x))|dx\right)ds\\
\leq\ & e^{-t/\varepsilon}(C_0^2+tC_0^a)+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\left(R_2+(t-s)K_2\int_{\mathbf{R}}|F(\rho(s,x))G((\eta*\rho)(s,x))|dx\right)ds.
\end{aligned}
$$

Since

$$
|(\eta*\rho)(s,x)|\leq\|\eta\|_\infty\int_{\mathbf{R}}|\rho(s,y)|\,dy\leq\|\eta\|_\infty R_1,
$$

we note that

$$
|G((\eta*\rho)(s,x))|\leq\mathcal{G},
$$

then, with relation (1.7),

$$
\iiint_{\mathbf{R}^3}|x||\Phi(f)(t,x,v,\xi)|\,dvd\xi dx
$$

$$
\begin{aligned}
&\leq e^{-t/\varepsilon}(C_0^2 + tC_0^a) + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\left(R_2 + (t-s)K_2\mathcal{G}\int_{\mathbf{R}} K(|\rho(s,x)| + |\rho(s,x)|^2)dx\right) ds \\
&\leq e^{-t/\varepsilon}(C_0^2 + tC_0^a) + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\left(R_2 + (t-s)K_2K(R_1+R_6)\mathcal{G}\right) ds.
\end{aligned}
$$

Now

$$
\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}(\alpha + s\beta)\, ds = \alpha - \beta\varepsilon + t\beta + (\beta\varepsilon - \alpha)e^{-t/\varepsilon},
$$

then, with $\alpha = R_2 + tK_2K(R_1+R_6)\mathcal{G}$ and $\beta = -K_2K(R_1+R_6)\mathcal{G}$, we get

$$
\begin{aligned}
&\iiint_{\mathbf{R}^3} |x||\Phi(f)(t,x,v,\xi)|\, dvd\xi dx \\
&\leq e^{-t/\varepsilon}(C_0^2 + tC_0^a) + R_2 + tK_2K(R_1+R_6)\mathcal{G} - \varepsilon R_a + \varepsilon K_2K(R_1+R_6)\mathcal{G} - tK_2K(R_1+R_6)\mathcal{G} \\
&\quad +(R_a\varepsilon - K_2K(R_1+R_6)\mathcal{G}\varepsilon - R_2 - tK_2K(R_1+R_6)\mathcal{G})e^{-t/\varepsilon} \\
&\leq C_0^2 + R_2 + \varepsilon K_2K(R_1+R_6)\mathcal{G} - K_2K(R_1+R_6)\mathcal{G}\varepsilon - R_2 \\
&\quad +t(C_0^a + K_2K(R_1+R_6)\mathcal{G} - K_2K(R_1+R_6)\mathcal{G} - K_2K(R_1+R_6)\mathcal{G}) \\
&\leq C_0^2 + t(C_0^a - K_2K(R_1+R_6)\mathcal{G}) \\
&\leq R_2
\end{aligned}
$$

since $C_0^2 \leq R_2$ and $K_2K(R_1+R_6)\mathcal{G} \geq C_0^a$.

Furthermore we have

$$
\begin{aligned}
&\iiint_{\mathbf{R}^3} |\xi||\Phi(f)(t,x,v,\xi)|\, dvd\xi dx \\
&\leq e^{-t/\varepsilon}\iiint_{\mathbf{R}^3} |\xi||f^0(x - a(v,\xi)t, v, \xi)|\, dvd\xi dx \\
&\quad +\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iiint_{\mathbf{R}^3} |\xi||\mathcal{M}_\rho(s, x - a(v,\xi)(t-s), v, \xi)|\, dvd\xi dx\, ds \\
&\leq e^{-t/\varepsilon}\iiint_{\mathbf{R}^3} |\xi||f^0(x,v,\xi)|\, dvd\xi dx \\
&\quad +\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iiint_{\mathbf{R}^3} |\xi||\mathcal{M}_\rho(s,x,v,\xi)|\, dvd\xi dx\, ds \\
&\leq e^{-t/\varepsilon}C_0^3 + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}R_3\, ds \\
&\leq e^{-t/\varepsilon}R_3 + R_3(1 - e^{-t/\varepsilon}) = R_3.
\end{aligned}
$$

Finally we have

$$
\iiint_{\mathbf{R}^3} |v||\Phi(f)(t,x,v,\xi)|\, dvd\xi dx
$$

$$
\begin{aligned}
&\leq e^{-t/\varepsilon} \iiint_{\mathbf{R}^3} |v||f^0(x-a(v,\xi)t,v,\xi)|\,dvd\xi dx \\
&\quad + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \iiint_{\mathbf{R}^3} |v||\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)|\,dvd\xi dx\,ds \\
&\leq e^{-t/\varepsilon} \iiint_{\mathbf{R}^3} |v||f^0(x,v,\xi)|\,dvd\xi dx \\
&\quad + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \iiint_{\mathbf{R}^3} |v||\mathcal{M}_\rho(s,x,v,\xi)|\,dvd\xi dx\,ds \\
&\leq e^{-t/\varepsilon} C_0 4 + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} R_4\,ds \\
&\leq e^{-t/\varepsilon} R_4 + R_4(1-e^{-t/\varepsilon}) = R_4.
\end{aligned}
$$

Then we get that $\Phi(f) \in C_R$.

**Step 3.** We prove that if $f \in C_R$, then $\Phi(f) \in \tilde{C}_R$.
By step 1 and step 2 and since $\Phi(f)$ satisfies

$$
\partial_t \Phi(f) + \partial_x(a(v,\xi)\Phi(f)) + \frac{1}{\varepsilon}\Phi(f) = \frac{\mathcal{M}_\rho}{\varepsilon}, \tag{4.19}
$$

we get (4.18) for $\Phi(f)$.

**Step 4.** We prove that $\Phi$ is continuous on $C_R$.
Let $g, g_n \in C_R$ such that $g_n \to g$ in $L^\infty([0,T], L^1(\mathbf{R}^3))$. Set

$$
\rho_n(t,x) = \iint_{\mathbf{R}^2} g_n(t,y,v,\xi)\,d\xi dv \quad \text{and} \quad \rho(t,x) = \iint_{\mathbf{R}^2} g(t,y,v,\xi)\,d\xi dv.
$$

Since

$$
\int_{\mathbf{R}} |\rho_n - \rho|\,(t,x,v,\xi)\,dxd\xi dv \leq \iiint_{\mathbf{R}^3} |g_n - g|(t,x,v,\xi)\,ds\,dxd\xi dv,
$$

then $\rho_n \to \rho$ in $L^1(]0,T[\times\mathbf{R})$ and there exists a subsequence $\rho_{\varphi(n)}$ and a function $h \in L^1(\mathbf{R})$ such that $\rho_{\varphi(n)} \to \rho$ and $|\rho_{\varphi(n)}| \leq |h|$ a.e. $t,x$. Thus $\mathcal{M}_{\rho_{\varphi(n)}} \to \mathcal{M}_\rho$ a.e. $t,x,v,\xi$ by (2.9). Furthermore, the sequence $(\mathcal{M}_{\rho_{\varphi(n)}})_n$ is uniformly integrable thanks to (4.16) and tight thanks to (4.13)-(4.15). Then by Vitali's convergence theorem, we get $\mathcal{M}_{\rho_{\varphi(n)}} \to \mathcal{M}_\rho$ in $L^1(]0,T[\times\mathbf{R}^3)$. Now

$$
|\Phi(g_{\varphi(n)}) - \Phi(g)|(t,x,v,\xi) \leq \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \left|\mathcal{M}_{\rho_{\varphi(n)}} - \mathcal{M}_\rho\right|(s,x-a(v,\xi)(t-s),v,\xi)\,ds,
$$

thus

$$\begin{aligned}
&\iiint_{\mathbf{R}^3} \left|\Phi(g_{\varphi(n)})(t,x,v,\xi) - \Phi(g)(t,x,v,\xi)\right| \, dxd\xi dv \\
\leq\ & \frac{1}{\varepsilon} \iint_{\mathbf{R}^3} \int_0^t e^{(s-t)/\varepsilon} |\mathcal{M}_{\rho_{\varphi(n)}} - \mathcal{M}_\rho|(s, x - a(v,\xi)(t-s), v, \xi)\, ds\, dxd\xi dv \\
\leq\ & \frac{1}{\varepsilon} \int_0^t \iint_{\mathbf{R}^2} e^{(s-t)/\varepsilon} \int_{\mathbf{R}} |\mathcal{M}_{\rho_{\varphi(n)}} - \mathcal{M}_\rho|(s,x,v,\xi)\, dx\, d\xi dv ds.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\sup_{t\in[0,T]} \iiint_{\mathbf{R}^3} \left|\Phi(g_{\varphi(n)})(t,x,v,\xi) - \Phi(g)(t,x,v,\xi)\right| \, dxd\xi dv \\
\leq\ & \frac{1}{\varepsilon} \int_0^T \iiint_{\mathbf{R}^3} |\mathcal{M}_{\rho_{\varphi(n)}} - \mathcal{M}_\rho|(s,x,v,\xi)\, dx\, d\xi dv ds
\end{aligned}$$

and we get that $\Phi(g_{\varphi(n)}) \to \Phi(g)$ in $L^\infty([0,T], L^1(\mathbf{R}^3))$, or also in $C([0,T], L^1(\mathbf{R}^3))$. It is enough to get the continuity of $\Phi$ on $C_R$.

**Step 5.** We prove the following properties on the sets $C_R$ and $\tilde{C}_R$: they are convex and not empty, the set $C_R$ is compact for the weak topology of $L^1(]0,T[\times\mathbf{R}^3)$ and the set $\tilde{C}_R$ is closed in $C([0,T], L^1(\times\mathbf{R}^3)$.

The sets $C_R$ and $\tilde{C}_R$ are clearly convex. Since $f^0 \in C_R$, the set $C_R$ is not empty. Since $f^0 \in C_R$, then $\Phi(f^0) \in \tilde{C}_R$ by step 2. Thus the set $\tilde{C}_R$ is not empty.

The uniformly integrability comes from (4.16) and the tightness comes from (4.13)-(4.15), then the set $C_R$ is relatively compact for the weak topology of $L^1(]0,T[\times\mathbf{R}^3)$ by Dunford-Pettis' theorem.

Let us prove now that $C_R$ is closed for the weak topology of $L^1(]0,T[\times\mathbf{R}^3)$. Since $C_R$ is convex, it is enough to prove that $C_R$ is closed for the strong topology of $L^1(]0,T[\times\mathbf{R}^3)$. Let $g_n \in C_R$ such that $g_n \to g$ in $L^1([0,T]\times\mathbf{R}^3)$. After extraction of a subsequence, we have $g_{\varphi(n)} \to g$ a.e. $(t,x,v,\xi)$ and $g_{\varphi(n)}(t,\dot{}) \to g(t,\dot{})$ in $L^1([0,T]\times\mathbf{R}^3)$ a.e. $t$. Since the sequence $(g_{\varphi(n)})_n$ satisfies (4.12)-(4.17) uniformly with respect to $n$, applying Fatou's lemma to each inequality, we get that $g \in C_R$.

We prove similarly that $\tilde{C}_R$ is closed in $C([0,T], L^1(\times\mathbf{R}^3)$.

**Step 6.** We prove that $\Phi(\tilde{C}_R)$ is relatively compact in $C([0,T], L^1(\mathbf{R}^3)$.

Let $f_n \in \Phi(\tilde{C}_R)$ defines a sequence in $\Phi(\tilde{C}_R)$. Then there exists $g_n \in \tilde{C}_R$ such that $f_n = \Phi(g_n)$. Set

$$\rho_n(t,x) = \iint_{\mathbf{R}^2} g_n(t,x,v,\xi)\, d\xi dv.$$

Since $\tilde{C}_R \subset C_R$ and since $C_R$ is compact for the weak topology of $L^1(]0,T[\times\mathbf{R}^3)$, there exists a subsequence $g_{\varphi(n)}$ such that $g_{\varphi(n)} \rightharpoonup g$ in weak $L^1([0,T]\times\mathbf{R}^3)$. Thus $\rho_{\varphi(n)} \rightharpoonup \rho$ in weak $L^1([0,T]\times\mathbf{R})$ since the functions are in $C_R$ where

$$\rho(t,x) = \iint_{\mathbf{R}^2} g(t,x,v,\xi)\, d\xi dv.$$

Since $g_{\varphi(n)} \in \tilde{C}_R$, then, by (4.18),

$$h_{\varphi(n)} = \varepsilon\partial_t g_{\varphi(n)} + \varepsilon\partial_x(a(v,\xi)g_{\varphi(n)}) + g_{\varphi(n)} \in C_R.$$

By (2.10), we get that $\rho_{\varphi(n)}$ is compact in $L^1_{loc}(]0,T[\times\mathbf{R})$, then for a subsequence $\rho_{\varphi\circ\psi(n)} \to \tilde{\rho}$ in $L^1(]0,T[\times K)$ for any compact $K$ of $\mathbf{R}$. We deduce, since the functions are in $C_R$, that $\rho_{\varphi\circ\psi(n)} \to \rho$ in $L^1(]0,T[\times\mathbf{R})$.

Finally we apply the same argument as in step 4 to get that for a subsequence $f_{\varphi\circ\psi\circ\Gamma(n)}\Phi(g_{\varphi\circ\psi\circ\Gamma(n)}) \to \Phi(g)$ in $C([0,T],L^1(\mathbf{R}^3))$. This is how we finalise step 6.

**Step 7.** We conclude by applying Schauder's theorem in $C([0,T],L^1(\mathbf{R}^3))$ to $\Phi : \tilde{C}_R \to \tilde{C}_R$. There exists $f \in C([0,T],L^1(\mathbf{R}^3))$ such that $\Phi(f) = f$. This gives a solution in $[0,T]$ for any $T > 0$, and by extraction of a diagonal subsequence, we obtain a solution in $[0,+\infty[$. □

*Remark 4.1 Notice that (1.7) is satisfied for example if $F' \in L^\infty$ and $F(0) = 0$ since then*

$$|F(z)| = |F(z) - F(0)| \leq \|F'\|_\infty |z|.$$

*But we can also consider more general cases.*

# 5 A model for the contraction result

Let's explicit a model for which the assumptions (2.11)-(2.13) are satisfied.

For the scalar non-local model, we assume that

$$F(0) = 0 \quad \text{and} \quad \eta, \frac{1}{\eta} \in L^\infty(\mathbf{R}), \tag{5.1}$$

that is to say that there exists $\alpha, \beta > 0$ such that

$$\alpha \leq \eta(z) \leq \beta, \quad \text{for any } z \in \mathbf{R}. \tag{5.2}$$

Notice that the term $\eta * \rho$ is well defined as soon as $x \mapsto \rho(t,x) \in L^1(\mathbf{R})$ for any $t$.

For the kinetic model, we take

$$a(v,\xi) = b(v)c(\xi)$$

with

$$b(v) = F'(v), \quad c(\xi) = G(\xi) + \xi G'(\xi) \tag{5.3}$$

and

$$\mathcal{M}_\rho(t,x,v,\xi) = M_{\rho(t,x),(\eta*\rho)(t,x)}(v,\xi), \tag{5.4}$$

where

$$M_{\rho,q}(v,\xi) = M_1(v,\rho)M_2(\xi,q), \tag{5.5}$$

$$M_1(v,\rho) = \begin{cases} \operatorname{sgn}(\rho) & \text{if } (\rho - v)v \geq 0, \\ 0 & \text{if } (\rho - v)v < 0, \end{cases} \tag{5.6}$$

$$M_2(\xi,q) = \begin{cases} \frac{\operatorname{sgn}(q)}{q} & \text{if } (q-\xi)\xi > 0, \\ 0 & \text{if } (q-\xi)\xi \leq 0, \end{cases} \tag{5.7}$$

Remember that

$$\rho(t,x) = \iint_{\mathbf{R}^2} f(t,y,v,\xi)\, d\xi dv.$$

Notice that we write $\mathcal{M}_\rho(t,x,v,\xi)$ and not $\mathcal{M}_{\rho(t,x)}(v,\xi)$ because here the term $\mathcal{M}_\rho(t,x,v,\xi)$ depends on the function $\rho$ for any value at $(t,y)$ because of the term $\eta * \rho$. At the kinetic level, we also have a non-local taking into account the values of $\rho_\varepsilon$.

## 5.1 First properties

First, notice the following properties :

**Proposition 5.1** *The functions $M_1$ and $M_2$ satisfy*

$$\int_{\mathbf{R}} M_1(v,\rho)\, dv = \rho,$$

$$\int_{\mathbf{R}} |M_1(v,\rho) - M_1(v,\tilde{\rho})|\, dv = |\rho - \tilde{\rho}|\,,$$

$$\int_{\mathbf{R}} C'(v) M_1(v,\rho)\, dv = C(\rho) - C(0), \quad \forall C \in C^1(\mathbf{R},\mathbf{R}),$$

$$\int_{\mathbf{R}} M_2(\xi,q)\, d\xi = \mathbb{1}_{q\neq 0},$$

$$\int_{\mathbf{R}} (C(\xi) + \xi C'(\xi)) M_2(\xi,q)\, d\xi = C(q)\mathbb{1}_{q\neq 0}, \quad \forall C \in C^1(\mathbf{R},\mathbf{R}).$$

**Proof.** The four first properties come from classical computations. The last one comes from the following. For $q > 0$, we have

$$\begin{aligned} \int_{\mathbf{R}} (C(\xi) + vC'(\xi)) M_2(\xi,q)\, d\xi &= \int_0^q \frac{1}{q}(C(\xi) + vC'(\xi))\, d\xi = \frac{1}{q}\int_0^q (\xi C(\xi))'\, d\xi \\ &= \frac{1}{q}\left[\xi C(\xi)\right]_0^q = C(q) \end{aligned}$$

and for $q < 0$,

$$\int_{\mathbf{R}} (C(\xi) + vC'(\xi)) M_2(\xi, q)\, d\xi = \int_q^0 \frac{-1}{q} (C(\xi) + vC'(\xi))\, d\xi = C(q). \ \square$$

The most difficult property to deal with is :

**Proposition 5.2** *The function $M$ satisfied*

$$\iint_{\mathbf{R}^2} |M_{\rho,q}(v,\xi) - M_{\tilde{\rho},\tilde{q}}(v,\xi)|\, d\xi dv = |\rho - \tilde{\rho}| + 2\, \frac{\min(\rho, \tilde{\rho})}{\max(q, \tilde{q})}\, |q - \tilde{q}|$$

*for any $\rho, \tilde{\rho} \geq 0$ and $q, \tilde{q} > 0$.*

**Proof.** We have

$$\iint_{\mathbf{R}^2} |M_{\rho,q}(v,\xi) - M_{\tilde{\rho},\tilde{q}}(v,\xi)|\, d\xi dv = \iint_{\mathbf{R}^2} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde{\rho})M_2(\xi,\tilde{q})|\, d\xi dv.$$

For $\tilde{\rho} > \rho > 0$ and $\tilde{q} > q > 0$, we get

$$\begin{aligned}
&\iint_{\mathbf{R}^2} |M_{\rho,q}(v,\xi) - M_{\tilde{\rho},\tilde{q}}(v,\xi)|\, d\xi dv \\
= \ & \int_0^\rho \int_0^q |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde{\rho})M_2(\xi,\tilde{q})|\, d\xi dv \\
& + \int_0^\rho \int_q^{\tilde{q}} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde{\rho})M_2(\xi,\tilde{q})|\, d\xi dv \\
& + \int_\rho^{\tilde{\rho}} \int_0^{\tilde{q}} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde{\rho})M_2(\xi,\tilde{q})|\, d\xi dv \\
= \ & \int_0^\rho \int_0^q \left|\frac{1}{q} - \frac{1}{\tilde{q}}\right| d\xi dv + \int_0^\rho \int_q^{\tilde{q}} \left|0 - \frac{1}{\tilde{q}}\right| d\xi dv + \int_\rho^{\tilde{\rho}} \int_0^{\tilde{q}} \left|0 - \frac{1}{\tilde{q}}\right| d\xi dv \\
= \ & 2\rho \frac{\tilde{q} - q}{\tilde{q}} + (\tilde{\rho} - \rho).
\end{aligned}$$

For $\tilde{\rho} > \rho > 0$ and $q > \tilde{q} > 0$, we get

$$\begin{aligned}
&\iint_{\mathbf{R}^2} |M_{\rho,q}(v,\xi) - M_{\tilde{\rho},\tilde{q}}(v,\xi)|\, d\xi dv \\
= \ & \int_0^\rho \int_0^{\tilde{q}} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde{\rho})M_2(\xi,\tilde{q})|\, d\xi dv \\
& + \int_0^\rho \int_{\tilde{q}}^q |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde{\rho})M_2(\xi,\tilde{q})|\, d\xi dv \\
& + \int_\rho^{\tilde{\rho}} \int_0^{\tilde{q}} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde{\rho})M_2(\xi,\tilde{q})|\, d\xi dv
\end{aligned}$$

$$
\begin{aligned}
&+\int_\rho^{\tilde\rho}\int_{\tilde q}^{q} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde\rho)M_2(\xi,\tilde q)|\, d\xi dv \\
&= \int_0^{\rho}\int_0^{\tilde q}\left|\frac{1}{q}-\frac{1}{\tilde q}\right| d\xi dv + \int_0^{\rho}\int_q^{\tilde q}\left|\frac{1}{q}-0\right| d\xi dv + \int_\rho^{\tilde\rho}\int_0^{\tilde q}\left|0-\frac{1}{\tilde q}\right| d\xi dv + 0 \\
&= 2\rho\frac{q-\tilde q}{q} + (\tilde\rho-\rho).\ \square
\end{aligned}
$$

## 5.2 Assumptions satisfied and existence result

**Proposition 5.3** *Let $F, G, \eta \in C^1(\mathbf{R},\mathbf{R})$ functions such that (5.1)-(5.2). Let $a(v,\xi) = b(v)c(\xi)$ be such that (5.3)-(5.7). Then the model satisfy (2.11)-(2.13).*

**Proof.** First we have

$$
\begin{aligned}
\iint_{\mathbf{R}^2} \mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv &= \int_{\mathbf{R}} M_1(v,\rho(t,x))\, dv \int_{\mathbf{R}} M_2(\xi,(\eta*\rho)(t,x))\, d\xi \\
&= \rho(t,x)\mathbb{1}_{(\eta*\rho)(t,x)\neq 0} = \rho(t,x)
\end{aligned}
$$

since $(\eta*\rho)(t,x) > 0$ as soon as $\rho(t,x) > 0$ (remember that $\eta > 0$) and thus $\rho(t,x) = 0$ a.e. if $(\eta*\rho)(t,x) = 0$ a.e. Thus we get (2.11). Now we have

$$
\begin{aligned}
\iint_{\mathbf{R}^2} b(v)c(\xi)\mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv &= \int_{\mathbf{R}} b(v)M_1(v,\rho_\varepsilon(t,x))\, dv \int_{\mathbf{R}} c(\xi)M_2(\xi,(\eta*\rho_\varepsilon)(t,x))\, d\xi \\
&= F(\rho(t,x))G((\eta*\rho)(t,x))\mathbb{1}_{\eta*\rho(t,x)\neq 0} \\
&= F(\rho(t,x))G((\eta*\rho)(t,x))
\end{aligned}
$$

since $\eta*\rho(t,x) = 0$ a.e. implies $\rho(t,x) = 0$ a.e. and $F(0) = 0$. Thus we get (2.12). Finally we have

$$
\begin{aligned}
&\iint_{\mathbf{R}^2} |\mathcal{M}_{\rho_1}(t,x,v,\xi) - \mathcal{M}_{\rho_2}(t,x,v,\xi)|\, d\xi dv \\
&= \iint_{\mathbf{R}^2} \left|M_{\rho_1(t,x),(\eta*\rho_1)(t,x)}(v,\xi) - M_{\rho_2(t,x),(\eta*\rho_2)(t,x)}(v,\xi)\right| d\xi dv \\
&= |\rho_1-\rho_2|(t,x) + 2\left(\frac{\min(\rho_1,\rho_2)}{\max(\eta*\rho_1,\eta*\rho_2)}\right)(t,x)\,|\eta*\rho_1-\eta*\rho_2|(t,x)
\end{aligned}
$$

from proposition 5.2. As a consequence,

$$
\begin{aligned}
&\int_{\mathbf{R}}\iint_{\mathbf{R}^2} |\mathcal{M}_{\rho_1}(t,x,v,\xi) - \mathcal{M}_{\rho_2}(t,x,v,\xi)|\, d\xi dv \\
&\leq \int_{\mathbf{R}} |\rho_1-\rho_2|(t,x)\, dx + 2\int_{\mathbf{R}}\left(\frac{\min(\rho_1,\rho_2)}{\max(\eta*\rho_1,\eta*\rho_2)}\right)(t,x)\,|\eta*\rho_1-\eta*\rho_2|(t,x)\, dx
\end{aligned}
$$

Since

$$\begin{aligned} |(\eta * \rho_1 - \eta * \rho_2)(t,x)| &\leq \int_{\mathbf{R}} \eta(x-y)\,|(\rho_1-\rho_2)(t,y)|\,dy \\ &\leq \|\eta\|_\infty \int_{\mathbf{R}} |(\rho_1-\rho_2)(t,y)|\,dy, \end{aligned}$$

then we have

$$\begin{aligned} &\int_{\mathbf{R}} \iint_{\mathbf{R}^2} |\mathcal{M}_{\rho_1}(t,x,v,\xi) - \mathcal{M}_{\rho_2}(t,x,v,\xi)|\,d\xi dv \\ \leq\ &\int_{\mathbf{R}} |\rho_1-\rho_2|\,(t,x)\,dx + 2\|\eta\|_\infty \int_{\mathbf{R}} |(\rho_1-\rho_2)(t,y)|\,dy \int_{\mathbf{R}} \left(\frac{\min(\rho_1,\rho_2)}{\max(\eta*\rho_1,\eta*\rho_2)}\right)(t,x)\,dx \end{aligned}$$

From

$$\frac{\min(\rho_1,\rho_2)}{\max(\eta*\rho_1,\eta*\rho_2)} = \frac{\rho_1+\rho_2-|\rho_1-\rho_2|}{\eta*\rho_1+\eta*\rho_2+|\eta*\rho_1-\eta*\rho_2|} \leq \frac{\rho_1+\rho_2}{\eta*\rho_1+\eta*\rho_2}$$

and

$$\eta*\rho_1(t,x)+\eta*\rho_2(t,x) \geq \int_{\mathbf{R}} \eta(x-y)(\rho_1+\rho_2)(t,y)\,dy \geq \alpha \int_{\mathbf{R}} (\rho_1+\rho_2)(t,y)\,dy.$$

we get

$$\int_{\mathbf{R}} \frac{\min(\rho_1,\rho_2)}{\max(\eta*\rho_1,\eta*\rho_2)}\,dx \leq \int_{\mathbf{R}} \frac{\rho_1+\rho_2}{\alpha \int_{\mathbf{R}} (\rho_1+\rho_2)(t,y)\,dy}\,dx = \frac{1}{\alpha}.$$

Therefore we obtain

$$\int_{\mathbf{R}} \iint_{\mathbf{R}^2} |\mathcal{M}_{\rho_1}(t,x,v,\xi) - \mathcal{M}_{\rho_2}(t,x,v,\xi)|\,d\xi dv \leq \left(1+\frac{2\|\eta\|_\infty}{\alpha}\right) \int_{\mathbf{R}} |\rho_1-\rho_2|\,(t,x)\,dx$$

and (2.13). □

Finally, applying Proposition 3.1 we obtain the following result.

**Theorem 5.4** *Let $f^0 \in L^1(\mathbf{R}^3)$ such that $f^0 \geq 0$. Let $F, G, \eta \in C^1(\mathbf{R},\mathbf{R})$ functions such that $F(0)=0$, $\eta, \frac{1}{\eta} \in L^\infty(\mathbf{R})$. Let $a(v,\xi)=b(v)c(\xi)$ such that (5.3)-(5.7). Then there exists $\rho \in L^\infty([0,T],L^1(\mathbf{R}^3))$ for any $T>0$ solution of (1.4) with initial data $f^0$ and such that $\rho \geq 0$.*

# 6 A model for the Schauder result

We make the following assumptions on $F$, $G$ and $\eta$:

$$F \in C^2(\mathbf{R},\mathbf{R}), F(0)=0 \text{ and } F, F' \text{ are strictly monotone functions,} \tag{6.1}$$

$$G \in C^1(\mathbf{R},\mathbf{R}), G, G' \text{ are a strictly increasing functions, } G'>0, \tag{6.2}$$

and such that there exists $X_0<0$, $K_0>0$ and $\gamma>1$ for which

$$|G(x)| \le \frac{K_0}{|x|^\gamma} \quad \text{and} \quad |G'(x)| \le \frac{K_0}{|x|^{\gamma+1}} \quad \text{if } x \le X_0, \tag{6.3}$$

and

$$\eta \in C^1(\mathbf{R},\mathbf{R}) \cap L^\infty(\mathbf{R}) \cap L^2(\mathbf{R}). \tag{6.4}$$

The term $\eta * \rho$ is well defined as soon as $x \mapsto \rho(t,x) \in L^1(\mathbf{R})$ for any $t$.

For the kinetic model, we consider

$$a(v,\xi) = b(v)d'(\xi) \tag{6.5}$$

where

$$b(v) = F'(v), \quad d(\xi) = 2\sum_{n=0}^{+\infty} G(\xi - 2n - 1) \tag{6.6}$$

and

$$\tilde{\mathcal{M}}_\rho(t,x,v,\xi) = \tilde{M}_{\rho(t,x),(\eta*\rho)(t,x)}(v,\xi), \tag{6.7}$$

where

$$\tilde{M}_{\rho,q}(v,\xi) = M_1(v,\rho)M_3(\xi,q), \tag{6.8}$$

$$M_1(v,\rho) = \begin{cases} \operatorname{sgn}(\rho) & \text{if } (\rho - v)v \ge 0, \\ 0 & \text{if } (\rho - v)v < 0, \end{cases} \tag{6.9}$$

$$M_3(\xi,q) = \frac{1}{2}\mathbb{1}_{|\xi-q|<1}(\xi). \tag{6.10}$$

Remember that

$$\rho(t,x) = \iint_{\mathbf{R}^2} f(t,y,v,\xi)\, d\xi dv.$$

*Remark 6.1 Notice that $d$ is well defined thanks to assumption (6.3) because*

$$|G(\xi - 2n - 1)| \le \frac{K_0}{(2n+1-|\xi|)^\gamma} \quad \textit{for } 2n+1 > |\xi|$$

*and is $C^1$ on $\mathbf{R}$ since, for any set $]-\infty,\alpha]$ with $\alpha > 0$, we have, for any $n \ge n_0$ where $2n_0+1>\alpha$,*

$$|G'(\xi - 2n - 1)| \le \frac{K_0}{(2n+1-\alpha)^{\gamma+1}} \quad \textit{if } x \in ]-\infty,\alpha].$$

*Then*

$$d'(\xi) = 2\sum_{n=0}^{+\infty} G'(\xi - 2n - 1) \quad \textit{for any } x \in \mathbf{R},$$

*and $d, d'$ are strictly increasing functions and $d' > 0$.*

*Remark 6.2 We can also consider the case where $G$ is a strictly decreasing function with assumptions on $+\infty$ this time.*

We need to apply averaging lemma, thus we have to assume the following non degeneracy condition : for all $R > 0$, there is a constant $C = C(R)$ such that for $z \in \mathbf{R}$, $\tau \in \mathbf{R}$ with $\sigma^2 + \tau^2 = 1$, then

$$\text{meas}\{(v, \xi) \in \mathbf{R}^2 \text{ s.t. } |v|, |\xi| \leq R \text{ and } |a(v, \xi)\sigma - \tau| \leq \varepsilon\} \leq C\varepsilon. \tag{6.11}$$

We refer to [12], [7], [14], [6], [18], [15] and references within for averaging lemmas.

## 6.1 First properties

The property for $M_1$ is in proposition 5.1. For $M_3$, we have the following result.

**Proposition 6.1** *Let $F, G \in C^1(\mathbf{R}, \mathbf{R})$ such that (6.3) is satisfied. Then we have, for any $q \in \mathbf{R}$,*

$$\int_{\mathbf{R}} M_3(\xi, q)\, d\xi = 1,$$

$$\int_{\mathbf{R}} \tilde{C}'(\xi) M_3(\xi, q)\, d\xi = \frac{1}{2}(C(q+1) - C(q-1)), \quad \forall C \in C^1(\mathbf{R}, \mathbf{R})$$

*and*

$$\int_{\mathbf{R}} d'(\xi) M_3(\xi, q)\, d\xi = G(q).$$

**Proof.** For the first property, we write

$$\int_{\mathbf{R}} \frac{1}{2} \mathbb{1}_{|\xi - q| < 1}(\xi)\, d\xi = \frac{1}{2} \int_{q-1}^{q+1} d\xi = \frac{2}{2} = 1.$$

The second equality comes from the following:

$$\int_{\mathbf{R}} C'(\xi) M_3(\xi, q)\, d\xi = \frac{1}{2} \int_{q-1}^{q+1} C'(\xi)\, d\xi = \frac{1}{2}(C(q+1) - C(q-1)).$$

Then we get the third one since

$$\begin{aligned} d(q+1) - d(q-1) &= 2\sum_{n=0}^{+\infty} G(q + 1 - 2n - 1) - 2\sum_{n=0}^{+\infty} G(q - 1 - 2n - 1) \\ &= 2G(q). \ \square \end{aligned}$$

*Remark 6.3 Notice that we cannot apply contraction technics in this case since we have the following equalities. First*

$$\iint_{\mathbf{R}^2} \left|\tilde{M}_{\rho,q}(v,\xi) - \tilde{M}_{\tilde{\rho},\tilde{q}}(v,\xi)\right| d\xi dv = \iint_{\mathbf{R}^2} |M_1(v,\rho)M_3(\xi,q) - M_1(v,\tilde{\rho})M_3(\xi,\tilde{q})| \, d\xi dv.$$

*For $\tilde{\rho} > \rho > 0$ and $\tilde{q} > q > 0$, we get*

$$\begin{aligned} & \iint_{\mathbf{R}^2} \left|\tilde{M}_{\rho,q}(v,\xi) - \tilde{M}_{\tilde{\rho},\tilde{q}}(v,\xi)\right| d\xi dv \\ = \; & \frac{1}{2}\int_0^{\rho}\int_{\mathbf{R}} |\mathbb{1}_{q-1<\xi<q+1}(\xi) - \mathbb{1}_{\tilde{q}-1<\xi<\tilde{q}+1}(\xi)| \, d\xi dv + \frac{1}{2}\int_{\rho}^{\tilde{\rho}}\int_{\mathbf{R}} \mathbb{1}_{\tilde{q}-1<\xi<\tilde{q}+1}(\xi)\, d\xi dv. \end{aligned}$$

*If $q+2 \leq \tilde{q}$, we have*

$$\iint_{\mathbf{R}^2} \left|\tilde{M}_{\rho,q}(v,\xi) - \tilde{M}_{\tilde{\rho},\tilde{q}}(v,\xi)\right| d\xi dv = 2\rho + (\tilde{\rho} - \rho),$$

*if $\tilde{q} < q+2$, we have*

$$\iint_{\mathbf{R}^2} \left|\tilde{M}_{\rho,q}(v,\xi) - \tilde{M}_{\tilde{\rho},\tilde{q}}(v,\xi)\right| d\xi dv = \rho(\tilde{q} - q) + (\tilde{\rho} - \rho).$$

*Then, by studying the similar cases, we get*

$$\begin{aligned} \iint_{\mathbf{R}^2} \left|\tilde{M}_{\rho,q}(v,\xi) - \tilde{M}_{\tilde{\rho},\tilde{q}}(v,\xi)\right| d\xi dv \; &= \; 2\min(\rho,\tilde{\rho})\mathbb{1}_{\min(q,\tilde{q})+2\leq\max(q,\tilde{q})} \\ &\quad + \min(\rho,\tilde{\rho})|\tilde{q}-q|\,\mathbb{1}_{0<\max(q,\tilde{q})<\min(q,\tilde{q})+2} + |\tilde{\rho}-\rho|. \end{aligned}$$

*The term $2\min(\rho,\tilde{\rho})\mathbb{1}_{\min(q,\tilde{q})+2\leq\max(q,\tilde{q})}$ does not allow a contraction study.*

## 6.2 Assumptions satisfied and existence result

**Proposition 6.2** *Let $F, G, \eta : \mathbf{R} \to \mathbf{R}$, $a : \mathbf{R}^2 \to \mathbf{R}$ satisfying (6.1)-(6.11). Then the model of this section satisfies (2.1)-(2.2) and (2.4)-(2.10).*

**Proof.** First we have

$$\iint_{\mathbf{R}^2} \tilde{\mathcal{M}}_{\rho}(t,x,v,\xi)\, d\xi dv = \int_{\mathbf{R}} M_1(v,\rho(t,x))\, dv \int_{\mathbf{R}} M_3(\xi,(\eta*\rho)(t,x))\, d\xi = \rho(t,x).$$

Thus we get (2.1). Now we have

$$\begin{aligned} \iint_{\mathbf{R}^2} a(v,\xi)\tilde{\mathcal{M}}_{\rho}(t,x,v,\xi)\, d\xi dv \; &= \; \iint_{\mathbf{R}^2} b(v)d'(\xi)\tilde{\mathcal{M}}_{\rho}(t,x,v,\xi)\, d\xi dv \\ &= \; \int_{\mathbf{R}} b(v)M_1(v,\rho(t,x))\, dv \int_{\mathbf{R}} d'(\xi)M_3(\xi,(\eta*\rho)(t,x))\, d\xi \\ &= \; F(\rho(t,))G((\eta*\rho)(t,x)) \end{aligned}$$

and we obtain (2.2). Furthermore, we have

$$\begin{aligned}\iint\limits_{\mathbf{R}^2} |\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|\,d\xi dv &= \int_{\mathbf{R}} |M_1(v,\rho(t,x))|\,dv \int_{\mathbf{R}} |M_3(\xi,(\eta*\rho)(t,x))|\,d\xi \\ &= |\rho(t,x)| \le |\rho(t,x)|,\end{aligned}$$

that is to say (2.4) and

$$\begin{aligned}\iint\limits_{\mathbf{R}^2} |a(v,\xi)\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|\,d\xi dv &= \iint\limits_{\mathbf{R}^2} |b(v)||d'(\xi)||M_1(v,\rho(t,x))||M_3(\xi,(\eta*\rho)(t,x))|\,d\xi dv \\ &= \int_{\mathbf{R}} |b(v)||M_1(v,\rho(t,x))|\,dv \int_{\mathbf{R}} |d'(\xi)||M_3(\xi,(\eta*\rho)(t,x))|\,d\xi \\ &= \int_{\mathbf{R}} |F'(v)||M_1(v,\rho(t,x))|\,dv \left|\int_{\mathbf{R}} d'(\xi) M_3(\xi,(\eta*\rho)(t,x))\,d\xi\right| \\ &\le |F(\rho(t,x))G((\eta*\rho)(t,x))|\end{aligned}$$

thanks to the monotonicity properties of $d$ and $F$ and we have (2.5) with $K_2 = 1$. Now

$$\iint\limits_{\mathbf{R}^2} |\xi||\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|\,d\xi dv = \int_{\mathbf{R}} |M_1(v,\rho(t,x))|\,dv \int_{\mathbf{R}} |\xi| M_3(\xi,(\eta*\rho)(t,x))\,d\xi$$

and since

$$\int_{\mathbf{R}} |\xi| M_3(\xi,q)\,d\xi = \begin{cases} |q| & \text{if } q+1<0 \text{ or } q-1>0, \\ (q^2+1)/2 & \text{if } q-1\le 0\le q+1, \end{cases}$$

we get $\displaystyle\int_{\mathbf{R}} |\xi| M_3(\xi,q)\,d\xi \le q^2+1$ and

$$\begin{aligned}\iint\limits_{\mathbf{R}^2} |\xi||\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|\,d\xi dv &\le |\rho(t,x)|\left((\eta*\rho)(t,x))^2+1\right) \\ &\le |\rho(t,x)|\left(1+\left(\int_{\mathbf{R}} \eta(x-y)\rho(t,y)\,dy\right)^2\right) \\ &\le |\rho(t,x)|\left(1+\int_{\mathbf{R}} \eta(x-y)^2\,dy \int_{\mathbf{R}} |\rho(t,y)|^2\,dy\right) \\ &\le |\rho(t,x)|\left(1+\int_{\mathbf{R}} \eta(y)^2\,dy \int_{\mathbf{R}} |\rho(t,y)|^2\,dy\right) \\ &\le \max\left(1,\int_{\mathbf{R}} \eta(y)^2\,dy\right)|\rho(t,x)|\left(1+\int_{\mathbf{R}} |\rho(t,y)|^2\,dy\right)\end{aligned}$$

that is to say (2.6) with $K_3 = \max\left(1,\displaystyle\int_{\mathbf{R}} \eta(y)^2\,dy\right)$. For the following estimate, we have

$$\iint\limits_{\mathbf{R}^2} |v||\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|\,d\xi dv = \int_{\mathbf{R}} |v||M_1(v,\rho(t,x))|\,dv \int_{\mathbf{R}} M_3(\xi,(\eta*\rho)(t,x))\,d\xi$$

and

$$\int_{\mathbf{R}} |v||M_1(v,\rho(t,x))|\,dv = \begin{cases} \displaystyle\int_0^{\rho} v\,dv & \text{if } \rho > 0, \\ \displaystyle\int_{\rho}^{0} (-v)\,dv & \text{if } \rho \le 0, \end{cases}$$

thus $\int_{\mathbf{R}} |v||M_1(v,\rho(t,x))|\,dv = \rho(t,x)^2/2$ and

$$\iint_{\mathbf{R}^2} |v||\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|\,d\xi dv = \frac{\rho(t,x)^2}{2}$$

that is (2.7) with $K_4 = 1/2$. After this, we write

$$\begin{aligned} \iint_{\mathbf{R}^2} |\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|^2\,d\xi dv &= \int_{\mathbf{R}} M_1(v,\rho(t,x))^2\,dv \int_{\mathbf{R}} M_3(\xi,(\eta*\rho)(t,x))^2\,d\xi \\ &= \int_{\mathbf{R}} |M_1(v,\rho(t,x))|\,dv \int_{\mathbf{R}} \frac{1}{2} M_3(\xi,(\eta*\rho)(t,x))\,d\xi \\ &= \frac{1}{2}|\rho(t,x)| \end{aligned}$$

and we get (2.8) with $K_5 = 1/2$ and $p = 1$. Assuming now that we have functions satisfying $\rho_n \to \rho$ a.e. $(t,x)$ and $|\rho_n| \le |h| \in L^1(\mathbf{R})$, then applying the theorem of dominated convergence, we get that

$$(\eta * \rho_n)(t,x) = \int_{\mathbf{R}} \eta(x-y)\rho_n(t,y)\,dy \to (\eta*\rho)(t,x) = \int_{\mathbf{R}} \eta(x-y)\rho(t,y)\,dy$$

since $\eta \in L^\infty$. Then we get (2.9).

Let $R > 0$. We set

$$K_R = \max\left(8 \sup_{z\in[-R,R]} |F'(z)| \sup_{z\in[-R,R]} |d'(z)|, \frac{4}{\sqrt{7}}\right) \frac{1}{R}.$$

Let $(\sigma,\tau) \in \mathbf{R}^2$ such that $\sigma^2+\tau^2 = 1$. Let $\varepsilon \in ]0,1/2[$. We want to consider set where $|F'(v)d'(\xi)\sigma + \tau| < \varepsilon$. Notice that changing $(\sigma,\tau)$ by $(-\sigma,-\tau)$, we can assume that $\sigma \ge 0$. There exists $\theta \in ]-\pi/2,\pi/2]$ such that $\sigma = \cos\theta$ and $\tau = \sin\theta$. Since $d'$ is strictly increasing and strictly positive, we have

$$0 < d'(-R) < d'(\xi) < d'(R) \quad \text{for any } \xi \in [-R,R].$$

We consider $(v,\xi) \in \mathbf{R}^2$ such that $|v| \le R$ and $|\xi| \le R$ satisfying

$$\sin\theta - \varepsilon < F'(v)d'(\xi)\cos\theta < \sin\theta + \varepsilon.$$

If $\cos\theta = 0$, then the set of $(v,\xi)$ satisfying $\pm 1 - \varepsilon < 0 < \pm 1 + \varepsilon$ is empty since $0 < \varepsilon < 1$. We consider now the case $\cos\theta > 0$. Then we have

$$\frac{\sin\theta - \varepsilon}{d'(\xi)\cos\theta} < F'(v) < \frac{\sin\theta + \varepsilon}{d'(\xi)\cos\theta}$$

and since $F'$ is strictly monotone, we get

$$(F')^{-1}\left(\frac{\sin\theta-\varepsilon}{d'(\xi)\cos\theta}\right) < v < (F')^{-1}\left(\frac{\sin\theta+\varepsilon}{d'(\xi)\cos\theta}\right)$$

or

$$(F')^{-1}\left(\frac{\sin\theta+\varepsilon}{d'(\xi)\cos\theta}\right) < v < (F')^{-1}\left(\frac{\sin\theta-\varepsilon}{d'(\xi)\cos\theta}\right).$$

Consider for example the strictly increasing case. First case: if $0 < 1/\cos(\theta) \leq K_R R$, then we get

$$\begin{aligned}
& \text{meas}\{(v,\xi)\in\mathbf{R}^2 \text{ s.t. } |v|,|\xi|\leq R \text{ and } |a(v,\xi)\sigma-\tau|<\varepsilon\} \\
\leq\ & \int_{d'(-R)}^{d'(R)} \int_{(F')^{-1}((\sin\theta-\varepsilon)/(d'(\xi)\cos\theta))}^{(F')^{-1}((\sin\theta+\varepsilon)/(d'(\xi)\cos\theta))} dv d\xi \\
\leq\ & \int_{g(-R)}^{g(R)} \left((F')^{-1}\left(\frac{\sin\theta+\varepsilon}{d'(\xi)\cos\theta}\right) - (F')^{-1}\left(\frac{\sin\theta-\varepsilon}{d'(\xi)\cos\theta}\right)\right) d\xi \\
\leq\ & \sup_{z\in I_R} |((F')^{-1})'(z)| \int_{g(-R)}^{g(R)} \left(\frac{\sin\theta+\varepsilon}{d'(\xi)\cos\theta} - \frac{\sin\theta-\varepsilon}{d'(\xi)\cos\theta}\right) d\xi
\end{aligned}$$

where $I_R = [(-1-\varepsilon)K_R R/d'(R), (1+\varepsilon)K_R R/d'(-R)]$. It leads to

$$\begin{aligned}
& \text{meas}\{(v,\xi)\in\mathbf{R}^2 \text{ s.t. } |v|,|\xi|\leq R \text{ and } |F'(v)d'(\xi)\sigma-\tau|<\varepsilon\} \\
\leq\ & 2\varepsilon K_R R \sup_{z\in I_R} |((F')^{-1})'(z)| \int_{g(-R)}^{g(R)} \frac{1}{d'(\xi)}\, d\xi.
\end{aligned}$$

Second case: if $1/\cos(\theta) > K_R R$, then we get

$$|F'(v)d'(\xi)\sigma| \leq \sup_{z\in[-R,R]} |F'(z)| \sup_{z\in[-R,R]} |d'(z)| \frac{1}{K_R R} \leq \frac{1}{8}$$

and

$$|\tau| = \sqrt{1-\cos^2\theta} > \sqrt{1-\frac{1}{K_R^2 R^2}} \leq \frac{3}{4}.$$

Thus $|a(v,\xi)\sigma-\tau| > 3/4 - 1/8 > 1/2 > \varepsilon$ and

$$\text{meas}\{(v,\xi)\in\mathbf{R}^2 \text{ s.t. } |v|,|\xi|\leq R \text{ and } |a(v,\xi)\sigma-\tau|<\varepsilon\} = 0.$$

Finally, we get

$$\sup_{\sigma^2+\tau^2=1} \text{meas}\{(v,\xi)\in\mathbf{R}^2 \text{ s.t. } |v|,|\xi|\leq R \text{ and } |a(v,\xi)\sigma-\tau|<\varepsilon\} \leq C_R\varepsilon$$

where

$$\begin{aligned}
C_R &= 2K_R R \sup_{z\in I_R} |((F')^{-1})'(z)| \int_{g(-R)}^{g(R)} \frac{1}{d'(\xi)}\, d\xi \\
&= 2\max\left(8 \sup_{z\in[-R,R]} |F'(z)| \sup_{z\in[-R,R]} |d'(z)|, \frac{4}{\sqrt{7}}\right) \sup_{z\in I_R} |((F')^{-1})'(z)| \int_{g(-R)}^{g(R)} \frac{1}{d'(\xi)}\, d\xi.
\end{aligned}$$

It gives (6.11) and we get (2.10). □

Then applying Theorem 1.2 we settle the following result.

**Theorem 6.3** *Let* $f^0 \in L^1(\mathbf{R}^3) \cap L^2(\mathbf{R}^3)$ *such that* $xf^0, \xi f^0, vf^0, a(v,\xi)f^0 \in L^1(\mathbf{R}^3)$ *and*

$$\int_{\mathbf{R}} \left( \iint_{\mathbf{R}^2} f^0(x,v,\xi)\, dvd\xi \right)^2 dx < +\infty.$$

*Consider* $F, G, \eta : \mathbf{R} \to \mathbf{R}$ *satisfying (6.1)-(6.4), (6.11) and assume that* $\eta \in L^\infty(\mathbf{R})$ *and that there exists a constant* $K > 0$ *such that*

$$|F(z)| \leq K(|z| + |z|^2) \quad \text{for any } z \in \mathbf{R}.$$

*Let* $a(v,\xi) = b(v)d'(\xi)$ *such that (6.6)-(6.10). Then there exists* $\rho \in L^\infty([0,T], L^1(\mathbf{R}^3))$ *for any* $T > 0$ *solution of (1.4) with initial data* $f^0$.

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