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POINTWISE CONVERGENCE OF NONCOMMUTATIVE FOURIER SERIES

GUIXIANG HONG, SIMENG WANG AND XUMIN WANG

Abstract. This paper is devoted to the study of pointwise convergence of Fourier series for non-abelian compact groups, group von Neumann algebras and quantum groups. It is well-known that a number of approximation properties of groups can be interpreted as summation methods and mean convergence of the associated noncommutative Fourier series. Based on this framework, this work studies the refined counterpart of pointwise convergence of these Fourier series. As a key ingredient, we develop a noncommutative bootstrap method and establish a general criterion of maximal inequalities for approximative identities of noncommutative Fourier multipliers. Based on this criterion, we prove that for any countable discrete amenable group, there exists a sequence of finitely supported positive definite functions tending to 1 pointwise, so that the associated Fourier multipliers on noncommutative L_p-spaces satisfy the pointwise convergence for all $p > 1$. In a similar fashion, we also obtain results for a large subclass of groups (as well as quantum groups) with the Haagerup property and the weak amenability. We also consider the analogues of Fejér means and Bochner-Riesz means in the noncommutative setting. Our approach heavily relies on the noncommutative ergodic theory in conjunction with abstract constructions of Markov semigroups, inspired by quantum probability and geometric group theory. Even back to the Fourier analysis on Euclidean spaces and non-abelian compact groups, our results are novel and yield new insights and problems. On the other hand, we obtain as a byproduct the dimension free bounds of the noncommutative Hardy-Littlewood maximal inequalities associated with convex bodies.

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1. Introduction and main results

The study of convergence of Fourier series goes back to the very beginning of Fourier analysis. Recall that for an integrable function f on the unit circle $\mathbb{T}$, the Dirichlet summation method is defined as

$$(D_N f)(z) = \sum_{k=-N}^{N} \hat{f}(k) z^k, \quad z \in \mathbb{T}, \quad N \in \mathbb{N},$$

where $\hat{f}$ denotes the Fourier transform of f. This summation method is quite intuitive, but very intricate to deal with. Indeed, the mean convergence of these sums is equivalent to the boundedness of the Hilbert transform, which is a typical example of Calderón-Zgymund singular integral operators; the corresponding pointwise convergence problem is much more complicated and was solved by Carleson and Hunt, which is now well-known as the Carleson-Hunt theorem. In order to study these Dirichlet means and their higher-dimensional version, there have appeared numerous related problems together with other summation methods, which have always been motivating the development of harmonic analysis. For instance, as averages of Dirichlet means, the Fejér means stand out

$$(F_N f)(z) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right) \hat{f}(k) z^k, \quad z \in \mathbb{T}, \quad N \in \mathbb{N}.$$

It is well-known that F_N defines a positive and contractive operator on $L_p(\mathbb{T})$, and $F_N f$ converges almost everywhere to f for all $1 \leq p \leq \infty$ (see e.g. [Gra08]). In the case of higher dimensions, the ball multiplier problem was solved negatively by Fefferman [Fef71]; and people considered instead the Bochner-Riesz means which can be viewed as fractional averages of ball multipliers. However, that whether the Bochner-Riesz means with critical index still have the desired mapping properties remains one of the famous open problems in three and higher dimensions, which is closely related to many other open problems in harmonic analysis, PDEs, additive combinatorics, number theory etc (see e.g. [KT02, Tao99b, Tao99a, Tao04] and the references therein). These problems have been stimulating the further development of analysis and beyond.

In recent decades, similar topics have been fruitfully developed in the setting of operator algebras and geometric group theory. The study was initiated in the groundbreaking work of Haagerup [Haa79], motivated by the approximation properties of group von Neumann algebras. Indeed, let Γ be a countable discrete group with left regular representation $\lambda : \Gamma \to B(\ell_2(\Gamma))$ given by $\lambda(g)\delta_h = \delta_{gh}$, where the δ_g's form the unit vector basis of $\ell_2(\Gamma)$. The corresponding group von Neumann algebra $VN(\Gamma)$ is defined to be the weak operator closure of the linear span of $\lambda(\Gamma)$. For $f \in VN(\Gamma)$ we set $\tau(f) = \langle \delta_e, f\delta_e \rangle$ where e denotes the identity of Γ. Then τ is a faithful normal tracial state on $VN(\Gamma)$. Any such f admits a formal Fourier series

$$\sum_{g \in \Gamma} \hat{f}(g)\lambda(g) \quad \text{with} \quad \hat{f}(g) = \tau(f\lambda(g^{-1})).$$

The convergence and summation methods of these Fourier series at the operator algebraic level (i.e. at the $L_\infty(VN(\Gamma))$ level) are deeply linked with the geometric and analytic properties of Γ, and in the noncommutative setting they are usually interpreted as various *approximation properties* for groups (see e.g. [BO08, CCJ+01]). More precisely, for a function $m : \Gamma \to \mathbb{C}$ we may formally define the corresponding Fourier multiplier by

$$T_m : \sum_{g\in\Gamma} \hat{f}(g)\lambda(g) \mapsto \sum_{g\in\Gamma} m(g)\hat{f}(g)\lambda(g). \tag{1.1}$$

We may consider among others the following approximate properties:

(1) Γ is *amenable* if there exists a family of finitely supported functions $(m_N)_{N\in\mathbb{N}}$ on Γ so that T_{m_N} defines a unital completely positive map on $VN(\Gamma)$ and $T_{m_N}f$ converges to f in the w*-topology for all $f \in VN(\Gamma)$ (equivalently, m_N converges pointwise to 1).
(2) Γ has the *Haagerup property* if there exists a family of c_0-functions $(m_N)_{N\in\mathbb{N}}$ on Γ so that T_{m_N} defines a unital completely positive map on $VN(\Gamma)$ and m_N converges pointwise to 1.
(3) Γ is *weakly amenable* if there exists a family of finitely supported functions $(m_N)_{N\in\mathbb{N}}$ on Γ so that T_{m_N} defines a completely bounded map on $VN(\Gamma)$ with $\sup_N \|T_{m_N}\|_{cb} < \infty$ and m_N converges pointwise to 1.

If we take $\Gamma = \mathbb{Z}$ (in this case $VN(\mathbb{Z}) = L_\infty(\mathbb{T})$) and $m_N(k) = (1 - |k|/N)_+$, then T_{m_N} recovers the Fejér means F_N and obviously satisfies the above conditions. These approximation properties play an essential role in the modern theory of von Neumann algebras, as well as in geometric group theory. For example, the work of Cowling-Haagerup [CH89] on the weak amenability solves the isomorphism problems of various group von Neumann algebras; the Haagerup property and its opposite Kazhdan property (T) are amongst the central tools in Popa's deformation/rigidity theory [Pop07]; also, the weak amenability is a key ingredient in the modern approach to the strong solidity and uniqueness of Cartan subalgebras [OP10, CS13, PV14].

Despite the remarkable progress in this field, it is worthy mentioning that only the convergence of $T_{m_N}f$ in the w*-topology was studied in the aforementioned work. A standard argument also yields the convergence in norm in the corresponding noncommutative L_p-spaces $L_p(VN(\Gamma))$ for $1 \le p < \infty$. On the other hand, the analogue of almost everywhere convergence in the noncommutative setting was introduced by Lance in his study of noncommutative ergodic theory [Lan76]; this type of convergence is usually called the *almost uniform convergence* (abbreviated as *a.u. convergence*; see Section 2.2). Keeping in mind the aforementioned impressive results already obtained from the mean convergence, it is natural to develop a refined theory of pointwise convergence of noncommutative Fourier series, and to seek applications in geometric group theory, operator algebras and harmonic analysis. More precisely, it is known that for the previous maps T_{m_N} and for $f \in L_p(VN(\Gamma))$, there exists a subsequence $(N_k)_k$ (possibly depending on f and p) such that $T_{m_{N_k}}f$ converges a.u. to f. From the viewpoint of analysis, the following problem naturally arises: *can we choose N_k to be independent of f, or even can we choose N_k to be k?* If G is abelian, this is exactly the classical pointwise convergence problem. As mentioned previously, the study of the pointwise convergence problem is much more difficult than the mean convergence problem as in the case of Dirichlet means; it still remains one of the major subjects of harmonic analysis nowadays, for instance the study of Bochner-Riesz means and maximal Schrödinger operators, see e.g. [Tao02, LS15, DZ19, LW19]

and the references therein. So the above problem should be regarded as one of the initial steps to develop Fourier analysis on noncommutative L_p-spaces.

However, compared to the classical setting, the pointwise convergence problem on noncommutative L_p-spaces remains essentially unexplored, up to sporadic contributions [JX07, CXY13]. The reason for this lack of development might be explained by numerous difficulties one may encounter when dealing with maximal inequalities for noncommutative Fourier multipliers. Indeed, in the commutative setting, the pointwise convergence problem almost amounts to the validity of maximal inequalities [Ste61], and the arguments for maximal inequalities depend in their turn on the explicit expressions or the pointwise estimates of the kernels. However, the kernels of noncommutative Fourier multipliers are only formal elements in a noncommutative L_1-space, which are in general no longer related to classical functions and cannot be pointwisely comparable, so the usual methods for classical maximal inequalities do not apply to the noncommutative setting any more. Although the notion of noncommutative maximal inequality has been formulated successfully thanks to the theory of vector-valued noncommutative L_p spaces [Pis98, Jun02], the approaches to these inequalities are very limited, except the noncommutative Doob inequality in martingale theory [Jun02] and its analogue in ergodic theory [JX07, HLW20], where some additional nice properties of the underlying operators are available.

In this paper we would like to provide a new approach to the maximal inequalities and pointwise convergence theorems for noncommutative Fourier series. To our best knowledge, the current trend of investigation on noncommutative Fourier multipliers mainly relies on various transference methods and quantum probability theory (see e.g. [NR11, CXY13, JMP14, JMP18]). The method presented in this paper is completely independent of all these preceding works, so is entirely new. The strategy turns out to be efficient in a very general setting; roughly speaking, it allows us to deal with all Fourier-like structures including quantum groups, twisted crossed products and free Gaussian systems. In many cases, we may give an explicit answer to the pointwise convergence problem raised previously. Back to the classical setting, this approach also yields new results, insights and problems.

In the following part of this section we will present some of our main results.

Criteria for maximal inequalities of Fourier multipliers. Our key technical theorem gives a criterion for maximal inequalities of noncommutative Fourier multipliers. This criterion only focuses on the regularity and decay information of symbols of multipliers in terms of length functions. Hence it is relatively easy to verify. As mentioned previously, the theorem can be extended to *all Fourier-like expansions in general von Neumann algebras*. For simplicity we only present the results for group von Neumann algebras $VN(\Gamma)$ as illustration, and we refer to Theorem 4.2, Theorem 4.3 and Theorem 4.18 for a complete statement.

Let Γ be a discrete group and let $\ell : \Gamma \to [0, \infty)$ be a conditionally negative definite function on it. We consider a family of real valued unital positive definite functions $(m_t)_{t\in\mathbb{R}_+}$. It is known that the associated operators $(T_{m_t})_{t\in\mathbb{R}_+}$ defined as in (1.1) extend to contractive maps on $L_p(VN(\Gamma))$ for all $1 \le p \le \infty$ (see Section 4 for more details). In this framework we present the following result. We refer to Section 2 for the notions of noncommutative L_p-spaces $L_p(VN(\Gamma))$ and noncommutative maximal norms $\| \sup_n^+ x_n\|_p$ for $(x_n)_n \subset L_p(VN(\Gamma))$.

Theorem 1.1. *Let Γ, ℓ and $(T_{m_t})_{t\in\mathbb{R}_+}$ be as above. Assume that there exist $\alpha, \beta > 0$ and $\eta \in \mathbb{N}_+$ such that for all $g \in \Gamma$ and $1 \le k \le \eta$ we have*

$$|1 - m_t(g)| \le \beta \frac{\ell(g)^\alpha}{t}, \quad |m_t(g)| \le \beta \frac{t}{\ell(g)^\alpha}, \quad \left|\frac{d^k m_t(g)}{dt^k}\right| \le \beta \frac{1}{t^k},$$

then for all $1 + \frac{1}{2\eta} < p \le \infty$ there exists a constant c such that for all $f \in L_p(VN(\Gamma))$,

$$\|\sup_{t\in\mathbb{R}_+}{}^+ T_{m_t} f\|_p \le c\|f\|_p \qquad \text{and} \qquad T_{m_t} f \to f \text{ a.u. as } t \to \infty,$$

and for all $1 < p \le \infty$ there exists a constant c such that for all $f \in L_p(VN(\Gamma))$,

$$\|\sup_{N\in\mathbb{N}}{}^+ T_{m_{2^N}} f\|_p \le c\|f\|_p \qquad \text{and} \qquad T_{m_{2^N}} f \to f \text{ a.u. as } N \to \infty,$$

Similar results hold for uniformly bounded (but not necessarily positive) Fourier multipliers (T_{m_t}) if we restrict ourselves to the case $p \ge 2$. The study of Theorem 1.1 relies on the analysis of lacunary subsequences $(T_{m_{2^N}})_{N\in\mathbb{N}}$. This type of lacunarity seems to be insufficient in the further study of abstract analysis on groups. The theorem below is more suitable for the abstract setting, which applies to other sequences without being of the form $(T_{m_{2^N}})_{N\in\mathbb{N}}$ in Theorem 1.1 and will play a prominent role in the remaining part of this work.

Theorem 1.2. *Let Γ and ℓ be as above. Let $(m_N)_{N\in\mathbb{N}}$ be a sequence of real valued unital positive definite functions. If there exist $\alpha, \beta > 0$ such that for all $g \in \Gamma$,*

$$|1 - m_N(g)| \le \beta \frac{\ell(g)^\alpha}{2^N}, \quad |m_N(g)| \le \beta \frac{2^N}{\ell(g)^\alpha},$$

then for all $1 < p \le \infty$ there exists a constant $c > 0$ such that for all $f \in L_p(VN(\Gamma))$,

$$\|\sup_{N\in\mathbb{N}}{}^+ T_{m_N} f\|_p \le c\|f\|_p \qquad \text{and} \qquad T_{m_N} f \to f \text{ a.u. as } N \to \infty.$$

The main idea of the proof will be to compare the Fourier multipliers with certain quantum Markov semigroups, and then apply the ergodic theory of the latter developed by [JX07]. This is first loosely inspired by the study of variational inequalities (in particular the comparison between averaging operators and martingales in [CDHX17, DHL17, HM17]), and then by Bourgain's approach to the dimension-free bounds of Hardy-Littlewood maximal inequalities [Bou86a, Bou86b, Bou87, DGM18]. Bourgain's work is based on a careful study of the $L_p(\ell_\infty)$-norm estimate of differences between ball averaging operators and Poisson semigroups on Euclidean spaces. In this paper we will develop similar techniques for noncommutative Fourier multipliers and abstract quantum Markov semigroups. This method based on ergodic theory seems to be new even for the study of pointwise convergence of commutative Fourier multipliers; see in particular Corollaries 1.7 and 1.8.

As a key point of the proof, we will develop a *bootstrap* argument in the noncommutative setting for the first time. The so-called bootstrap methods have had a deep impact on classical harmonic analysis since the original work [NSW78]; see e.g. [DRdF86, Car86, BMSW18]. Though not explicitly mentioned in the original papers, the aforementioned work by Bourgain [Bou86a, Bou86b, Bou87] can be essentially compared with the previous ones and recognized with hindsight as a certain bootstrap argument with independent techniques. Motivated by Bourgain's method, we will develop a bootstrap argument based on the almost orthogonality principle: deducing the desired L_p-estimates for $p < 2$ from the L_2-estimates by a delicate study of suitable decompositions of $(T_{m_N} f - T_{e^{-t_N}\sqrt{\ell}} f)_N$ and certain differences of $(T_{m_N} f)_N$.

It is relevant to remark that there is no straightforward way to extend the classical bootstrap arguments directly to the noncommutative setting. In particular, as a noncommutative variant of the vector-valued L_p-space $L_p(\Omega;\ell_2)$ for the study of noncommutative square functions, the space $L_p(\mathcal{M};\ell_2^{cr})$ (to be defined in Subsection 3) does not coincide with the interpolation space of the form $(L_p(\mathcal{M};\ell_{q_1}), L_p(\mathcal{M};\ell_{q_2}))_\theta$ unless the underlying von Neumann algebra $\mathcal{M}$ is commutative, but the corresponding interpolation method for the commutative case plays a key role in realizing classical bootstrap arguments; also, as a fundamental tool, the Littlewood-Paley-Stein square function estimate of sharp growth order for $p<2$ is itself a quite involved topic in noncommutative analysis. Our proof is consequently more intricate than the classical ones, and involves more modern techniques or ideas from operator space theory, maximal inequalities and noncommutative square function estimates.

Approximation properties, pointwise convergence of Fourier series and explicit examples. Based on the preceding theorems, we may provide answers to the noncommutative pointwise convergence problems for a wide class of Fourier multipliers on quantum groups. Again we only present here the particular case of group von Neumann algebras for simplicity; the general version for *quantum groups* can be found in Thereom 5.9 and Corollary 5.11. We refer to Section 5 for the notion of groups with the ACPAP, which form a large subclass of groups with the Haagerup property and the weak amenability.

Theorem 1.3. (1) *Any countable discrete amenable group Γ admits a sequence of finitely supported unital positive definite functions $(m_N)_{N\in\mathbb{N}}$ so that $T_{m_N}f$ converges to f a.u. for all $f\in L_p(VN(\Gamma))$ with $1<p\leq\infty$.*

More generally, for any sequence of finitely supported unital positive definite functions $(m_N)_{N\in\mathbb{N}}$ on Γ pointwise converging to 1, there exists a subsequence $(m_{N_k})_{k\in\mathbb{N}}$ such that $T_{m_{N_k}}f$ converges to f a.u. for all $f\in L_p(VN(\Gamma))$ with $1<p\leq\infty$.

(2) *Any countable discrete group Γ with the ACPAP admits a sequence of completely contractive Fourier multipliers $(T_{m_N})_{N\in\mathbb{N}}$ on $VN(\Gamma)$ so that m_N are finitely supported and $T_{m_N}f$ converges to f a.u. for all $f\in L_p(VN(\Gamma))$ with $2\leq p\leq\infty$.*

Our approach to the above theorem differs greatly from usual strategies in the study of pointwise convergence problems. The key idea is to construct an *abstract* Markov semigroup whose symbols are sufficiently close to $(m_N)_N$ so that Theorem 1.2 becomes applicable. In hindsight, the construction is essentially inspired by geometric group theory and operator algebras; in particular we would like to mention several related works in this setting [JM04, CS15, DFSW16, CS17], where an interplay between Fourier multipliers, approximation properties and abstract Markov semigroups has been highlighted. Our method applies to quite general classes of Fourier multipliers as soon as the symbols satisfy a suitable convergence rate. Together with the comments after Theorem 1.2, our approach might be viewed as an application of ergodic theory of genuinely abstract semigroups to pointwise convergence problems, which is novel even in the classical setting (see Theorem 1.6 and Corollary 1.8).

Our method is also useful for the study of pointwise convergence of Dirichlet means in the noncommutative setting. Taking an increasing sequence $(K_N)_{N\in\mathbb{N}}$ of finite subsets of Γ, one may consider the partial sums $D_Nf=\sum_{g\in K_N}\hat{f}(g)\lambda(g)$ for $f\in VN(\Gamma)$. As in the classical case, in general f cannot be approximated by D_Nf in the uniform norm $\|\ \|_\infty$ even for elements f in the reduced C*-algebra $C_r^*(\Gamma)$ generated by $\lambda(\Gamma)$. On the other hand, the problem of convergence of D_Nf in L_p-norms in the noncommutative setting is also very subtle (see e.g. [JNRX04, BF06]). In [BC09, BC12, CWW15], the uniform convergence of $(D_N)_{N\in\mathbb{N}}$

on some smooth dense subalgebras of $C_r^*(\Gamma)$ was studied. However, if we replace the uniform convergence by the almost uniform one and choose appropriately the family $(K_N)_{N\in\mathbb{N}}$, it seems that the result can be largely improved; in particular we may obtain the almost uniform convergence for more general measurable operators contained in $L_2(VN(\Gamma))$. The proof will be given in the general setting of quantum groups in Proposition 5.12.

Theorem 1.4. *Any countable discrete group Γ with the ACPAP admits an increasing sequence $(K_N)_{N\in\mathbb{N}}$ of finite subsets of Γ such that the series $\sum_{g\in K_N}\hat{f}(g)\lambda(g)$ converges a.u. to f as N tends to infinity for all $f\in L_2(VN(\Gamma))$.*

As in the classical setting, it would be interesting to deal with more explicit examples of Fourier multipliers. Apart from the above abstract result, our method is also useful for the study of concrete multipliers in the noncommutative setting.

Example 1.5. i) Generalized Fejér means: We may introduce the following analogue of Fejér means on non-abelian discrete amenable groups. Let $(K_N)_{N\in\mathbb{N}}$ be a Følner sequence, that is, $K_N\subset\Gamma$ are subsets so that

$$m_N(g) := \frac{\mathbb{1}_{K_N} * \mathbb{1}_{K_N}(g)}{|K_N|} = \frac{|K_N\cap gK_N|}{|K_N|} \to 1, \quad \text{as } N\to\infty,$$

where $|K_N|$ denotes the cardinality of K_N. Then m_N is finitely supported and the associated multiplier T_{m_N} is unital completely positive with m_N finitely supported. And there is a subsequence $(N_k)_{k\in\mathbb{N}}$ such that

$$T_{m_{N_k}}f\to f \text{ a.u.} \qquad f\in L_p(VN(\Gamma))$$

for all $1<p\le\infty$. For instance if Γ is a group of polynomial growth with finite generating set S, we may take $K_N=S^N$. If moreover Γ is a 2-step nilpotent group and $p>3/2$, we may take $N_j=j$. We refer to Section 6.1 for more details.

ii) Noncommutative Bochner-Riesz means: Let Γ be a hyperbolic group so that the word length function $|\ |$ is conditionally negative definite. For example we may take Γ to be a non-abelian free group or a hyperbolic Coxeter group. The following Bochner-Riesz means are introduced in [MdlS17]: for a fixed $\delta>1$ we take

$$B_N^\delta f = \sum_{g\in\Gamma:|g|\le N}\left(1-\frac{|g|^2}{N^2}\right)^\delta \hat{f}(g)\lambda(g), \quad f\in L_p(VN(\Gamma)).$$

We have $B_N^\delta f\to f$ almost uniformly for all $2\le p\le\infty$. We refer to Section 6.3.1 for more details.

iii) Smooth positive definite radial kernels on free groups: Let $\mathbb{F}$ be a non-abelian free group. As before, $|\ |$ denotes its natural word length function. Let ν be an arbitrary positive Borel measure supported on $[-1,1]$ with $\nu([-1,1])=1$ and write $d\nu_t(x)=d\nu(tx)$ for all $t>0$. For any $t>0$, set

$$m_t(g)=\int_{\mathbb{R}} x^{|g|}d\nu_t(x-e^{-\frac{2}{t}}) = \int_{-1}^1\left(\frac{y}{t}+e^{-\frac{2}{t}}\right)^{|g|}d\nu(y), \qquad g\in\mathbb{F}.$$

Then for all $1<p\le\infty$ and all $f\in L_p(VN(\mathbb{F}))$,

$$T_{m_t}f\to f \text{ a.u. as } t\to\infty.$$

The proof can be founded in Section 6.3.2. Note that if ν is the Dirac measure on 0, then this statement amounts to the almost uniform convergence of Poisson semigroups on $VN(\mathbb{F})$ proved in [JX07].

Applications to classical analysis on compact groups. As mentioned previously, we will indeed establish Theorem 1.3 in the general setting of Woronowicz's compact quantum groups. As a particular case our result applies to Fourier series of non-abelian compact groups. Recall that for a compact group G, any function $f \in L_p(G)$ admits a Fourier series

$$f(x) \sim \sum_{\pi \in \mathrm{Irr}(G)} \dim(\pi)\mathrm{Tr}(\hat{f}(\pi)\pi(x)), \quad x \in G \quad \text{with } \hat{f}(\pi) = \int_G f(x)\pi(x^{-1})dx,$$

where $\mathrm{Irr}(G)$ denotes the collection of equivalence classes of irreducible unitary representations of G. The study of pointwise summability of above Fourier series is much more intricate than the abelian case. To our best knowledge, the pointwise convergence theorems in this setting were studied for differentiable or continuous functions in [HC66, Sug71, Hua18], and for some p-integrable functions on compact Lie groups for example in [Cle74, ST78]. However, from the viewpoint of amenable quantum groups, our approach easily establishes the following pointwise convergence theorem for general p-integral functions. The summation method does not rely on the Lie algebraic structure, which is a novel aspect compared to previous works; moreover it can be extended to the general setting of compact quantum groups.

Theorem 1.6. *Let G be a compact second countable group. There exists a sequence of finitely supported functions $m_N : \mathrm{Irr}(G) \to \mathbb{R}$ so that*

$$(F_N f)(x) := \sum_{\pi \in \mathrm{Irr}(G)} m_N(\pi) \dim(\pi)\mathrm{Tr}(\hat{f}(\pi)\pi(x)), \quad x \in G, f \in L_p(G)$$

defines unital positive operators on $L_p(G)$ and

$$\lim_{N\to\infty} F_N f = f \ \ a.e., \quad f \in L_p(G)$$

for all $1 < p \leq \infty$.

Moreover, there exists an increasing sequence of finite subsets $K_N \subset \mathrm{Irr}(G)$ such that for all $f \in L_2(G)$ we have

$$f(x) = \lim_{N\to\infty} \sum_{\pi \in K_N} \dim(\pi)\mathrm{Tr}(\hat{f}(\pi)\pi(x)), \quad a.e.\ x \in G.$$

The functions m_N can be explicitly determined by the representation theory of G. More details and examples will be given in Section 6.2. Note that though the above result is stated in a totally classical setting, the role of noncommutative analysis on quantum groups is still non-avoidable in the proof.

The classical Euclidean setting. In the classical Euclidean setting, our approach provides the following new type of approximate identities. We refer to Section 6.4 for more details.

Corollary 1.7. *Let B be a symmetric convex body in $\mathbb{R}^d$ with volume 1. Let Φ be the inverse Fourier transform of the convolution $\mathbb{1}_B * \mathbb{1}_B$, where $\mathbb{1}_B$ denotes the characteristic function of B, and let $\Phi_t = t^{-d}\Phi(t^{-1}\cdot)$ for $t > 0$. Then we have*

$$\lim_{t\to 0} \Phi_t * f = f \ \ a.e., \quad f \in L_p(\mathbb{R}^d)$$

with $3/2 < p < \infty$*, and*

$$\lim_{j\to\infty} \Phi_{2^{-j}} * f = f \ \ a.e., \quad f \in L_p(\mathbb{R}^d)$$

with $1 < p < \infty$*.*

These approximate identities might be regarded as generalized Fejér means; in particular, if B is the unit cube, the convergence of $(\Phi_t * f)_{t>0}$ also holds for $1 < p < \infty$ and we recover the classical Fejér means. We can alternatively deal with the problem by using classical estimate of Φ if the boundary of B satisfies some smooth conditions (see Remark **??**), but our approach based on Theorem 1.1 seems more efficient for general convex bodies without smooth boundaries. To our knowledge, these approximate identities have not yet been widely studied by harmonic analysts. However, from the viewpoint of geometric theory of amenable groups, they arise very naturally when we study Følner sets in $\mathbb{R}^d$ other than cubes, such as balls and rectangles. It seems that one needs more efforts and new tools to obtain further results, which suggests new challenges to classical harmonic analysts and (convex) geometers. We will give the proof and mention a few problems in Section 6.4.

On the other hand, Theorem 1.3 (1) and Theorem 1.4 indeed provide the following general abstract answers to the classical pointwise convergence problems. Indeed, if we take $\Gamma = \mathbb{Z}^d$, then $L_p(VN(\Gamma))$ coincides with $L_p(\mathbb{T}^d)$ and the theorems amount to the following facts:

Corollary 1.8. (1) *Let* $(\Phi_N)_{N\in\mathbb{N}} \subset L_1(\mathbb{T}^d)$ *be an arbitrary sequence of positive trigonometric polynomials with* $\lim_N \|\Phi_N * f - f\|_1 = 0$ *for all* $f \in L_1(\mathbb{T}^d)$*. Then there exists a subsequence* $(N_k)_{k\in\mathbb{N}}$ *such that for all* $1 < p \leq \infty$ *and all* $f \in L_p(\mathbb{T}^d)$*, we have*

$$\lim_{k\to\infty} \Phi_{N_k} * f = f \ \ a.e.$$

(2) *There exists a subsequence* $(N_k)_{k\in\mathbb{N}}$ *such that*

$$\sum_{j\in\mathbb{Z}^d:\sqrt{|j_1|^2+\cdots+|j_d|^2}\leq N_k} \hat{f}(j)e^{2\pi i\langle j,\cdot\rangle}$$

converges a.e. to f *as* k *tends to infinity for all* $f \in L_2(\mathbb{T}^d)$*.*

From the proof of Theorem 1.4, the number N_k in the above assertion (2) can be actually chosen to be 2^k, which partially recovers [CRdFV88, Theorem B]; while the problem whether N_k can be equal to k, was solved by Carleson when $d = 1$ - known as the Carleson theorem - and is still a well-known open problem in classical harmonic analysis for $d \geq 2$.

Note that when reduced to this classical setting, our method is still novel, which unavoidably involves the ergodic theory and the bootstrap method in Theorem 1.2 as well as the aforementioned genuinely abstract Markov semigroups.

Dimension free bounds of noncommutative Hardy-Littlewood maximal inequalities. We remark that Theorem 1.1 also implies as a byproduct the dimension free bounds of noncommutative Hardy-Littlewood maximal operators. The noncommutative version of Hardy-Littlewood maximal inequalities was studied in [Mei07] for balls respect to Euclidean metrics and in [HLW20] for general doubling metric spaces. The dimension free bounds in this noncommutative setting were studied by the first author in [Hon20] for Euclidean balls; because of various difficulties in noncommutative analysis as mentioned before, the general case for convex bodies remained unexplored before our work. Our following result establishes the desired maximal inequalities for general convex bodies in $\mathbb{R}^d$ with dimension free estimates.

Theorem 1.9. *Let B be a symmtric convex body in $\mathbb{R}^d$ and $\mathcal{N}$ a semifinite von Neumman algebra. Define $\Phi_r : L_p(\mathbb{R}^d; L_p(\mathcal{N})) \to L_p(\mathbb{R}^d; L_p(\mathcal{N}))$ by*

$$\Phi_r(f)(x) = \frac{1}{\mu(B)} \int_B f(x - ry)dy.$$

Then there exist constants $C_p > 0$ independent of d and B such that the following holds:

(1) *For any $1 < p < \infty$,*

$$\|\sup_{j\in\mathbb{Z}}{}^{+} \Phi_{2^j}(f)\|_p \leq C_p \|f\|_p, \qquad f \in L_p(\mathbb{R}^d; L_p(\mathcal{N})).$$

(2) *For any $\frac{3}{2} < p < \infty$,*

$$\|\sup_{r\in\mathbb{R}_+}{}^{+} \Phi_r(f)\|_p \leq C_p \|f\|_p, \qquad f \in L_p(\mathbb{R}^d; L_p(\mathcal{N})).$$

(3) *If B is the ℓ_q-ball $\{(x_i)_{i=1}^d : \sum_{i=1}^d |x_i|^q \leq 1\}$ with $q \in 2\mathbb{N}$, then for any $1 < p < \infty$,*

$$\|\sup_{r\in\mathbb{R}_+}{}^{+} \Phi_r(f)\|_p \leq C_p \|f\|_p, \qquad f \in L_p(\mathbb{R}^d; L_p(\mathcal{N})).$$

Before ending the introduction, we would like to mention the following order estimate for square function inequalities, which is of independent interest. This order of constants $(p-1)^{-6}$ is of crucial use in the proof of Theorem 1.1. The following remarkable result is established in [JLMX06] without specifying the order of the constants. Our proof is based on a slight adaption of the arguments in [JLMX06], which will be given in Section 3.

Theorem 1.10. *Let $\mathcal{M}$ be a finite von Neumann algebra. Let $(P_t)_{t\in\mathbb{R}_+}$ be the subordinate Poisson semigroup of a semigroup $(S_t)_{t\in\mathbb{R}_+}$ of unital completely positive trace preserving and symmetric operators on $\mathcal{M}$. Then there exists an absolute positive constant c such that for all $1 < p < 2$ and $x \in L_p(\mathcal{M})$ we have*

$$\inf\left\{ \left\| \left(\int_0^\infty |t\partial P_t(x^c)|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \right\|_p + \left\| \left(\int_0^\infty |(t\partial P_t(x^r))^*|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \right\|_p \right\} \leq c(p-1)^{-6} \|x\|_p$$

where the infimum runs over all $x_c, x_r \in L_p(\mathcal{M})$ such that $x = x_c + x_r$.

The rest of the paper is divided as follows. In Section 2 we will recall the background and prove some preliminary results on noncommutative vector-valued L_p-spaces and pointwise convergences. Section 3 is devoted to the proof of square function estimates in Theorem 1.10. In Section 4 we will establish the key criterion for maximal inequalities of Fourier multipliers, i.e., Theorem 1.1. Lastly, in Section 5 we will prove Theorem 1.3 and in Section 6 we will establish various examples of maximal inequalities and pointwise convergence theorems of noncommutative Fourier multipliers, including Theorem 1.6 and Theorem 1.9.

Notation: In all what follows, we write $X \lesssim Y$ if $X \leq CY$ for an *absolute* constant $C > 0$, and $X \lesssim_{\alpha,\beta,\dots} Y$ if $X \leq CY$ for a constant $C > 0$ depending only on the parameters indicated. Also, we write $X \asymp Y$ if $C^{-1}Y \leq X \leq CY$ for an *absolute* constant $C > 0$.

2. Preliminaries

Let $\mathcal{M}$ denote a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $\mathcal{S}_{\mathcal{M}+}$ denote the set of all $x \in \mathcal{M}_+$ such that $\tau(\operatorname{supp} x) < \infty$, where $\operatorname{supp} x$ denotes the support projection of x. Let $\mathcal{S}_{\mathcal{M}}$ be the linear span of $\mathcal{S}_{\mathcal{M}+}$. Then $\mathcal{S}_{\mathcal{M}}$ is a w*-dense $*$-subalgebra of $\mathcal{M}$. Given $1 \le p < \infty$, we define

$$\|x\|_p = [\tau(|x|)^p]^{1/p}, \qquad x \in \mathcal{S}_{\mathcal{M}},$$

where $|x| = (x^*x)^{1/2}$ is the modulus of x. Then $(\mathcal{S}_{\mathcal{M}}, \|\cdot\|_p)$ is a normed space, whose completion is the noncommutative L_p-space associated with $(\mathcal{M}, \tau)$, denoted by $L_p(\mathcal{M}, \tau)$ or simply by $L_p(\mathcal{M})$. As usual, we set $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ equipped with the operator norm. Let $L_0(\mathcal{M})$ denote the space of all closed densely defined operators on H measurable with respect to $(\mathcal{M}, \tau)$, where H is the Hilbert space on which $\mathcal{M}$ acts. Then the elements of $L_p(\mathcal{M})$ can be viewed as closed densely defined operators on H. A more general notion of Haagerup L_p-spaces on arbitrary von Neumann algebras can be found in Section 4.5.2. We refer to [PX03] for more information on noncommutative L_p-spaces. We say that a map $T : L_p(\mathcal{M}, \tau) \to L_p(\mathcal{M}, \tau)$ is *n-positive* (resp. *n-bounded*) for some $n \in \mathbb{N}$ if $T \otimes \mathrm{id}_{M_n}$ extends to a positive (resp. bounded) map on $L_p(\mathcal{M} \otimes M_n, \tau \otimes \mathrm{Tr})$, where M_n denotes the algebra of all $n \times n$ complex matrices and Tr denotes the usual trace on it, and we say that T is *completely positive* (resp. *completely bounded*) if it is n-positive (resp. n-bounded) for all $n \in \mathbb{N}$. We will denote by $\|T\|_{cb}$ the supremum of the norms of $T \otimes \mathrm{id}_{M_n}$ on $L_p(\mathcal{M} \otimes M_n, \tau \otimes \mathrm{Tr})$ over all $n \in \mathbb{N}$.

2.1. Noncommutative ℓ_∞-valued L_p-spaces.

In classical analysis, the pointwise properties of measurable functions are often studied by estimating the norms of maximal functions of the form $\|\sup_n |f_n|\|_p$. However, these maximal norms in the noncommutative setting require a specific definition, since $\sup_n |x_n|$ does not make sense for a sequence $(x_n)_n$ of operators. This difficulty is overcome by considering the spaces $L_p(\mathcal{M}; \ell_\infty)$, which are the noncommutative analogs of the usual Bochner spaces $L_p(X; \ell_\infty)$. These spaces were first introduced by Pisier [Pis98] for injective von Neumann algebras and then extended to general von Neumann algebras by Junge [Jun02]. See also [JX07, Section 2] for more details.

Given $1 \le p \le \infty$, we define $L_p(\mathcal{M}; \ell_\infty)$ to be the space of all sequences $x = (x_n)_{n\in\mathbb{N}}$ in $L_p(\mathcal{M})$ which admit a factorization of the following form: there exist $a, b \in L_{2p}(\mathcal{M})$ and a bounded sequence $y = (y_n) \subset \mathcal{M}$ such that

$$x_n = a y_n b, \qquad n \in \mathbb{N}.$$

The norm of x in $L_p(\mathcal{M}; \ell_\infty)$ is given by

$$\|x\|_{L_p(\mathcal{M};\ell_\infty)} = \inf \left\{ \|a\|_{2p} \sup_{n\in\mathbb{N}} \|y\|_\infty \|b\|_{2p} \right\}$$

where the infimum runs over all factorizations of x as above. We will adpot the convention that the norm $\|x\|_{L_p(\mathcal{M};\ell_\infty)}$ is denoted by $\|\sup_n^+ x_n\|_p$. As an intuitive description, it is worth remarking that a selfadjoint sequence $(x_n)_{n\in\mathbb{N}}$ of $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M}; \ell_\infty)$ if and only if there exists a positive element $a \in L_p(\mathcal{M})$ such that $-a \le x_n \le a$ for any $n \in \mathbb{N}$. In this case, we have

$$\|\sup_{n\in\mathbb{N}}{}^+ x_n\|_p = \inf\{\|a\|_p : a \in L_p(\mathcal{M})_+, -a \le x_n \le a, \forall n \in \mathbb{N}\}. \tag{2.1}$$

The subspace $L_p(\mathcal{M}, c_0)$ of $L_p(\mathcal{M};\ell_\infty)$ is defined as the space of all family $(x_n)_{n\in\mathbb{N}} \subset L_p(\mathcal{M})$ such that there are $a, b \in L_{2p}(\mathcal{M})$ and $(y_n) \subset \mathcal{M}$ verifying

$$x_n = ay_nb \quad \text{and} \quad \lim_{n\to\infty} \|y_n\|_\infty = 0.$$

It is easy to check that $L_p(\mathcal{M}, c_0)$ is a closed subspace of $L_p(\mathcal{M};\ell_\infty)$. It is indeed the closure of the subspace of all finitely supported sequences.

On the other hand, we may also consider the space $L_p(\mathcal{M};\ell_\infty^c)$ for $2 \le p \le \infty$. This space $L_p(\mathcal{M};\ell_\infty^c)$ is defined to be the family of all sequences $(x_n)_{n\in\mathbb{N}} \subset L_p(\mathcal{M})$ which admits $a \in L_p(\mathcal{M})$ and $(y_n) \subset L_\infty(\mathcal{M})$ such that

$$x_n = y_na \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|y_n\|_\infty < \infty.$$

$\|(x_n)\|_{L_p(\mathcal{M};\ell_\infty^c)}$ is then defined to be the infimum of $\{\sup_{n\in\mathbb{N}} \|y_n\|_\infty \|a\|_p\}$ over all factorization of (x_n) as above. It is easy to check that $\| \ \|_{L_p(\mathcal{M};\ell_\infty^c)}$ is a norm, which makes $L_p(\mathcal{M};\ell_\infty^c)$ a Banach space. Moreover, $(x_n) \in L_p(\mathcal{M};\ell_\infty^c)$ iff $(x_n^*x_n) \in L_{p/2}(\mathcal{M};\ell_\infty)$. Indeed, we have

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_\infty^c)} = \|(x_n^*x_n)\|^{1/2}_{L_{p/2}(\mathcal{M};\ell_\infty)}. \tag{2.2}$$

We define similarly the subspace $L_p(\mathcal{M};c_0^c)$ of $L_p(\mathcal{M};\ell_\infty^c)$.

We define the space $L_p(\mathcal{M};\ell_\infty^r) := \{(x_n) : (x_n^*) \in L_p(\mathcal{M};\ell_\infty^c)\}$ for $2 \le p \le \infty$ with the norm $\|(x_n)\|_{L_p(\mathcal{M};\ell_\infty^r)} = \|(x_n^*)\|_{L_p(\mathcal{M};\ell_\infty^c)}$. The following interpolation theorem was firsted studied by Pisier in [Pis96] and then generalized by Junge and Parcet in [JP10].

Lemma 2.1 ([JP10, Theorem A]). *For any $2 \le p \le \infty$, we have isometrically*

$$L_p(\mathcal{M};\ell_\infty) = (L_p(\mathcal{M};\ell_\infty^c), L_p(\mathcal{M};\ell_\infty^r))_{1/2}.$$

Another Banach space $L_p(\mathcal{M};\ell_1)$ is also defined in [Jun02]. Given $1 \le p \le \infty$, a sequence $x = (x_n)_{n\in\mathbb{N}}$ belongs to $L_p(\mathcal{M};\ell_1)$ if there are $u_{kn}, v_{kn} \in L_{2p}(\mathcal{M})$ such that

$$x_n = \sum_{k\ge 0} u_{kn}^* v_{kn}, \qquad n \ge 0$$

and

$$\|(x_n)_n\|_{L_p(\mathcal{M};\ell_1)} := \inf\left\{ \left\| \sum_{k,n\ge 0} u_{kn}^* u_{kn} \right\|_p^{1/2} \left\| \sum_{k,n\ge 0} v_{kn}^* v_{kn} \right\|_p^{1/2} \right\} < \infty.$$

Specially, for a positive sequence $x = (x_n)$, we have

$$\|x\|_{L_p(\mathcal{M};\ell_1)} = \|\sum_{n\ge 0} x_n\|_p.$$

The following proposition will be useful in this paper.

Proposition 2.2 ([JX07, JP10]). *Let $1 \le p, p' \le \infty$ and $1/p + 1/p' = 1$.*

(1) *$L_p(\mathcal{M};\ell_\infty)$ is the dual space of $L_{p'}(\mathcal{M};\ell_1)$ when $p' \ne \infty$. The duality bracket is given by*

$$\langle x, y\rangle = \sum_{n\ge 0} \tau(x_ny_n), \quad x \in L_p(\mathcal{M};\ell_\infty),\ y \in L_{p'}(\mathcal{M};\ell_1).$$

In particular for any positive sequence $(x_n)_n$ in $L_p(\mathcal{M};\ell_\infty)$, we have

$$\|\sup_n{}^+ x_n\|_p = \sup\{\sum_n \tau(x_ny_n) : y_n \in L_{p'}^+(\mathcal{M}) \text{ and } \|\sum_n y_n\|_{p'} \le 1\}.$$

(2) *Each element in the unit ball of* $L_p(\mathcal{M};\ell_\infty)$ *(resp.* $L_p(\mathcal{M};\ell_1)$*) is a sum of sixteen (resp. eight) positive elements in the same ball.*

(3) *Let* $1 \le p_0 < p < p_1 \le \infty$ *and* $0 < \theta < 1$ *be such that* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$*. Then the following complex interpolation holds: we have isometrically*

$$L_p(\mathcal{M};\ell_\infty) = (L_{p_0}(\mathcal{M};\ell_\infty), L_{p_1}(\mathcal{M};\ell_\infty))_\theta .$$

Similar complex interpolations also hold for $L_p(\mathcal{M};\ell_\infty^c)$ *with* $2 \le p \le \infty$*.*

We remark that we may define the spaces $L_p(\mathcal{M};\ell_\infty(I)), L_p(\mathcal{M};c_0(I))$ and $L_p(\mathcal{M};\ell_\infty^c(I))$, $L_p(\mathcal{M};c_0^c(I))$ for any uncountable index set I in the same way. The above properties still hold for these spaces. We will simply denote the spaces by the same notation $L_p(\mathcal{M};\ell_\infty), L_p(\mathcal{M};c_0)$ and so on if no confusion can occur.

Remark 2.3. It is known that a family $(x_i)_{i\in I} \subset L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M};\ell_\infty)$ if and only if

$$\sup_{J\subset I \text{ finite}} \Big\| \sup_{i\in J}{}^+ x_i \Big\|_p \quad < \quad \infty,$$

and in this case

$$\Big\| \sup_{i\in I}{}^+ x_i \Big\|_p = \sup_{J \text{ finite}} \Big\| \sup_{i\in J}{}^+ x_i \Big\|_p . \tag{2.3}$$

Similar observations hold for $L_p(\mathcal{M};\ell_\infty^c)$. As a conseqeunce, for any $1 \le p < \infty$ and any $(x_t)_{t\in\mathbb{R}_+} \in L_p(\mathcal{M};\ell_\infty)$ such that the map $t \mapsto x_t$ from $\mathbb{R}_+$ to $L_p(\mathcal{M})$ is continuous, we have

$$\|(x_t)_{t\in\mathbb{R}_+}\|_{L_p(\mathcal{M};\ell_\infty)} = \lim_{a\to 1^+} \|(x_{a^j})_{j\in\mathbb{Z}}\|_{L_p(\mathcal{M};\ell_\infty)}.$$

To see this, we note that $\|(x_t)_{t\in\mathbb{R}_+}\|_{L_p(\mathcal{M};\ell_\infty)} \ge \limsup_{a\to 1^+} \|(x_{a^j})_{j\in\mathbb{Z}}\|_{L_p(\mathcal{M};\ell_\infty)}$; thus by (2.3) it suffices to show that $\|(x_{t_k})_{1\le k\le n}\|_{L_p(\mathcal{M};\ell_\infty)}$ is dominated by $\liminf_{a\to 1^+} \|(x_{a^j})_{j\in\mathbb{Z}}\|_{L_p(\mathcal{M};\ell_\infty)}$ for any (finitely many) elements $t_1,\dots,t_n$. This is obvious since for any $\varepsilon > 0$, by continuity we may find a scalar $a_0 \in \mathbb{R}_+$ sufficiently close to 1 such that for all $1 \le k \le n$, $\|x_{t_k} - x_{a_0^{j_k}}\|_p < \varepsilon/n$ with some $j_k \in \mathbb{Z}$ and $\|(x_{a_0^j})_{j\in\mathbb{Z}}\|_{L_p(\mathcal{M};\ell_\infty)} \le \liminf_{a\to 1^+} \|(x_{a^j})_{j\in\mathbb{Z}}\|_{L_p(\mathcal{M};\ell_\infty)} + \varepsilon$, which implis

$$\begin{aligned}\|(x_{t_k})_{1\le k\le n}\|_{L_p(\mathcal{M};\ell_\infty)} &\le \|(x_{a_0^{j_k}})_{1\le k\le n}\|_{L_p(\mathcal{M};\ell_\infty)} + \sum_{1\le k\le n} \|x_{t_k} - x_{a_0^{j_k}}\|_p \\ &\le \liminf_{a\to 1^+} \|(x_{a^j})_{j\in\mathbb{Z}}\|_{L_p(\mathcal{M};\ell_\infty)} + 2\varepsilon.\end{aligned}$$

Similarly, for $2 \le p < \infty$, we have

$$\|(x_t)_{t\in\mathbb{R}_+}\|_{L_p(\mathcal{M};\ell_\infty^c)} = \lim_{a\to 1^+} \|(x_{a^j})_{j\in\mathbb{Z}}\|_{L_p(\mathcal{M};\ell_\infty^c)}.$$

2.2. **Maximal inequalities and pointwise convergence.** The standard tool in the study of pointwise convergence is the following type of *maximal inequalities*.

Definition 2.4. Let $1 \le p \le \infty$. Consider a family of maps $\Phi_n : L_p(\mathcal{M}) \to L_0(\mathcal{M})$ for $n \in \mathbb{N}$.

(1) We say that $(\Phi_n)_{n\in\mathbb{N}}$ is of *strong type* (p,p) with constant C if

$$\|\sup_{n\in\mathbb{N}}{}^+ \Phi_n(x)\|_p \le C\|x\|_p, \qquad x \in L_p(\mathcal{M}).$$

(2) We say that $(\Phi_n)_{n\in\mathbb{N}}$ is of *weak type* (p,p) $(p < \infty)$ with constant C if for any $x \in L_p(\mathcal{M})$ and any $\alpha > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\|e\Phi_n(x)e\|_\infty \le \alpha, \quad n \in \mathbb{N} \quad \text{and} \quad \tau(e^\perp) \le \left[C\frac{\|x\|_p}{\alpha}\right]^p .$$

(3) We say that $(\Phi_n)_{n\in\mathbb{N}}$ is of *restricted weak type* (p,p) $(p<\infty)$ with constant C if for any projection $f\in\mathcal{M}$ and any $\alpha>0$, there is a projection $e\in\mathcal{M}$ such that

$$\|e\Phi_n(f)e\|_\infty\le\alpha \quad n\in\mathbb{N} \quad \text{and} \quad \tau(e^\perp)\le\left(\frac{C}{\alpha}\right)^p\tau(f).$$

It is easy to see that for any $1<p<\infty$,

$$\text{strong type } (p,p)\Rightarrow \text{ weak type } (p,p)\Rightarrow \text{ restricted weak type } (p,p).$$

Here is a simple but useful proposition.

Proposition 2.5. *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of positive linear maps on $L_p(\mathcal{M})$. Then*

$$\|(\Phi_n)_n : L_p(\mathcal{M};\ell_\infty)\to L_p(\mathcal{M};\ell_\infty)\|\asymp\|(\Phi_n)_n : L_p(\mathcal{M})\to L_p(\mathcal{M};\ell_\infty)\|.$$

Proof. By setting $x_n=x$, it is obvious to see that

$$\|(\Phi_n)_n : L_p(\mathcal{M})\to L_p(\mathcal{M};\ell_\infty)\|\le\|(\Phi_n)_n : L_p(\mathcal{M};\ell_\infty)\to L_p(\mathcal{M};\ell_\infty)\|.$$

For the inverse direction, we consider positive elements first. Let $(x_n)_n\in L_p(\mathcal{M};\ell_\infty)_+$. For any $\varepsilon>0$, by (2.1), we can find an element $a\in L_p(\mathcal{M})_+$ such that,

$$0\le x_n\le a,\ \forall n \quad \text{and} \quad \|a\|_p\le\|\sup_{n\in\mathbb{N}}{}^+x_n\|_p+\varepsilon.$$

By linearity and positivity of $(\Phi_n)_n$, we have $0\le\Phi_nx_n\le\Phi_na$. Therefore

$$\|\sup_n{}^+\Phi_nx_n\|\le\|\sup_n{}^+\Phi_na\|\le\|(\Phi_n)_n : L_p(\mathcal{M})\to L_p(\mathcal{M};\ell_\infty)\|(\|\sup_n{}^+x_n\|_p+\varepsilon).$$

Thus, by arbitrariness of ε and Proposition 2.2 (2), we get

$$\|(\Phi_n)_n : L_p(\mathcal{M};\ell_\infty)\to L_p(\mathcal{M};\ell_\infty)\|\le 16\|(\Phi_n)_n : L_p(\mathcal{M})\to L_p(\mathcal{M};\ell_\infty)\|.$$

□

The Marcinkiewicz interpolation theorem plays an important role in the study of maximal inequalities. Its analogue for the noncommutative setting was first established by Junge and Xu in [JX07], and then was generalized in [BCO17] and [Dir15]. We present Dirksen's version here.

Theorem 2.6 ([Dir15, Corollary 5.3])**.** *Let $1\le p<r<q\le\infty$. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a family of positive linear maps from $L_p(\mathcal{M})+L_q(\mathcal{M})$ into $L_0(\mathcal{M})$. If $(\Phi_n)_{n\in\mathbb{N}}$ is of restricted weak type (p,p) and of strong type (q,q) with constants C_p and C_q, then it is of strong type (r,r) with constant*

$$C_r\lesssim\max\{C_p,C_q\}\left(\frac{rp}{r-p}+\frac{rq}{q-r}\right)^2.$$

We need an appropriate analogue for the noncommutative setting of the usual almost everywhere convergence. This is the notion of almost uniform convergence introduced by Lance [Lan76].

Definition 2.7. Let $x_n,x\in L_0(\mathcal{M})$. $(x_n)_{n\in\mathbb{N}}$ is said to converge *almost uniformly* (*a.u.* in short) to x if for any $\varepsilon>0$ there is a projection $e\in\mathcal{M}$ such that

$$\tau(e^\perp)<\varepsilon \quad \text{and} \quad \lim_{n\to\infty}\|(x_n-x)e\|_\infty=0.$$

$(x_n)_{n\in\mathbb{N}}$ is said to converge *bilaterally almost uniformly* (*b.a.u.* in short) to x if for any $\varepsilon>0$ there is a projection $e\in\mathcal{M}$ such that

$$\tau(e^\perp)<\varepsilon \quad \text{and} \quad \lim_{n\to\infty}\|e(x_n-x)e\|_\infty=0.$$

It is obvious that the a.u. convergence implies the b.a.u. convergence, so we will be mainly interested in the former. Note that in the commutative case, both notions are equivalent to the usual almost everywhere convergence in terms of Egorov's Theorem in the case of probability space.

It is nowadays a standard method of deducing pointwise convergence from maximal inequalities. We will use the following facts.

Lemma 2.8 ([DJ04]). (1) *If a family $(x_i)_{i\in I}$ belongs to $L_p(\mathcal{M}, c_0)$ with some $1 \le p < \infty$, then x_i conveges b.a.u. to* 0.

(2) *If a family $(x_i)_{i\in I}$ belongs to $L_p(\mathcal{M}, c_0^c)$ with some $2 \le p < \infty$, then x_i conveges a.u. to* 0.

Proposition 2.9. (1) *Let $1 \le p < \infty$ and $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of positive linear maps on $L_p(\mathcal{M})$. Assume that $(\Phi_n)_{n\in\mathbb{N}}$ is of weak type (p,p). If $(\Phi_n x)_{n\in\mathbb{N}}$ converges a.u. to 0 for all elements x in a dense subspace of $L_p(\mathcal{M})$, then $(\Phi_n x)_{n\in\mathbb{N}}$ converges a.u. for all $x \in L_p(\mathcal{M})$.*

(2) *Let $1 \le p < \infty$ and $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of linear maps on $L_p(\mathcal{M})$. Assume that $(\Phi_n)_{n\in\mathbb{N}}$ satisfies the following* one sided weak type (p,p) *maximal inequalities, i.e. there exists $C > 0$ such that for any $x \in L_p(\mathcal{M})$ and $\alpha > 0$ there exists a projection $e \in \mathcal{M}$ such that*

$$\|\Phi_n(x)e\|_\infty \le \alpha, \quad n \in \mathbb{N} \quad \text{and} \quad \tau(e^\perp) \le \left[C\frac{\|x\|_p}{\alpha}\right]^p. \tag{2.4}$$

If $(\Phi_n x)_{n\in\mathbb{N}}$ converges a.u. to 0 for all elements x in a dense subspace of $L_p(\mathcal{M})$, then $(\Phi_n x)_{n\in\mathbb{N}}$ converges a.u. for all $x \in L_p(\mathcal{M})$.

Proof. The assertion (1) is given by [CL16, Theorem 3.1].

The second part is standard and is implicitly established in the proof of [JX07, Remark 6.5] and [CXY13, Theorem 5.1] for which we provide a brief argument for the convenience of the reader. Let $x \in L_p(\mathcal{M})$ and $\varepsilon > 0$. For any $m \ge 1$, take $y_m \in L_p(\mathcal{M})$ such that $\|x - y_m\|_p < 2^{-2m/p}C^{-1}\varepsilon^{1/p}$ and $(\Phi_n y_m)_n$ converges a.u. to 0 as $n \to \infty$. Denote $z_m = x - y_m$. By the estimation of one side weak type (p,p), we may find a projection $e_m \in \mathcal{M}$ such that

$$\sup_n \|\Phi_n(z_m)e_m\|_\infty \le 2^{-m/p} \quad \text{and} \quad \tau(e_m^\perp) \le \left[C\frac{\|z_m\|_p}{2^{-m/p}}\right]^p < 2^{-m}\varepsilon.$$

We may also find a projection $f_m \in \mathcal{M}$ such that

$$\tau(f_m^\perp) < 2^{-m}\varepsilon \quad \text{and} \quad \lim_{n\to\infty} \|\Phi_n(y_m)f_m\|_\infty = 0.$$

Let $e = \bigwedge_m (e_m \wedge f_m)$. Then

$$\tau(e^\perp) \le \sum_{m\ge 1} (\tau(e_m^\perp) + \tau(f_m^\perp)) < \varepsilon$$

and for any $m \ge 1$,

$$\limsup_{n\to\infty} \|\Phi_n(x)e\|_\infty \le \lim_{n\to\infty} (\|\Phi_n(y_m)f_m\| + \|\Phi_n(z_m)\|) \le 2^{-m/p},$$

which means that $\lim_{n\to\infty} \|\Phi_n(x)e\|_\infty = 0$. Therefore, $\Phi_n(x)$ converge a.u. to 0. □

We recall the following well-known fact, which is of essential use for our arguments. The following maximal inequalities and the a.u. convergence on dense subspaces are given in [JX07], and the a.u. convergence on L_p-spaces then follows from Proposition 2.9 (1). We recall that a map T is said to be *symmetric* if $\tau(T(x)^*y) = \tau(x^*T(y))$ for any $x, y \in \mathcal{S}_\mathcal{M}$.

Proposition 2.10. *Let $(S_t)_{t\in\mathbb{R}_+}$ be a semigroup of unital completely positive trace preserving and symmetric maps on $\mathcal{M}$. We have*

$$\|(S_t(x))_t\|_{L_p(\mathcal{M};\ell_\infty)} \le c_p\|x\|_p, \qquad x\in L_p(\mathcal{M}), \quad 1<p<\infty,$$

and

$$\|(S_t(x))_t\|_{L_p(\mathcal{M};\ell_\infty^c)} \le \sqrt{c_p}\|x\|_p, \qquad x\in L_p(\mathcal{M}), \quad 2<p<\infty,$$

where $c_p \le Cp^2(p-1)^{-2}$ with C an absolute constant. Moreover, $(S_t(x))_t$ converges a.u. to x as $t\to 0$ for all $x\in L_p(\mathcal{M})$ with $1<p<\infty$.

3. Noncommutative Hilbert space valued L_p-spaces and square function estimates

In this section we will collect some preliminary results on noncommutative square functions, which are among the essential tools in this paper. Some of the results proved in this section might be folkloric for experts, but we include them here for the convenience of the reader.

The noncommutative Hilbert space valued L_p-spaces provide a suitable framework for studying square functions in the noncommutative setting. In this paper we will only use the following concrete representations of these spaces; for a more general description we refer to the papers [LPP91, LP86, JLMX06].

First, for a finite sequence $(x_n)_n \subset L_p(\mathcal{M})$, we define

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_2^c)} = \left\|\left(\sum_n x_n^*x_n\right)^{1/2}\right\|_p, \qquad \|(x_n)\|_{L_p(\mathcal{M};\ell_2^r)} = \left\|\left(\sum_n x_nx_n^*\right)^{1/2}\right\|_p.$$

We alert the reader that the two norms above are not comparable at all if $p\neq 2$. Let $L_p(\mathcal{M};\ell_2^c)$ (resp. $L_p(\mathcal{M};\ell_2^r)$) be the completion of the space of all finite sequences in $L_p(\mathcal{M})$ with respect to $\|\ \|_{L_p(\mathcal{M};\ell_2^c)}$ (resp. $\|\ \|_{L_p(\mathcal{M};\ell_2^r)}$). The space $L_p(\mathcal{M};\ell_2^{cr})$ is defined in the following way. If $2\le p\le\infty$, we set

$$L_p(\mathcal{M};\ell_2^{cr}) = L_p(\mathcal{M};\ell_2^c)\cap L_p(\mathcal{M};\ell_2^r)$$

equipped with the norm

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_2^{cr})} = \max\{\|(x_n)\|_{L_p(\mathcal{M};\ell_2^c)}, \|(x_n)\|_{L_p(\mathcal{M};\ell_2^r)}\}.$$

If $1\le p\le 2$, we set

$$L_p(\mathcal{M};\ell_2^{cr}) = L_p(\mathcal{M};\ell_2^c) + L_p(\mathcal{M};\ell_2^r)$$

equipped with the norm

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_2^{cr})} = \inf\{\|(y_n)\|_{L_p(\mathcal{M};\ell_2^c)} + \|(z_n)\|_{L_p(\mathcal{M};\ell_2^r)}\}$$

where the infimum runs over all decompositions $x_n = y_n + z_n$ in $L_p(\mathcal{M})$.

Second, for the Borel measure space $(\mathbb{R}_+\setminus\{0\}, \frac{dt}{t})$, we may consider the norms

$$\|(x_t)_t\|_{L_p(\mathcal{M};L_2^c(\frac{dt}{t}))} = \left\|\left(\int_0^\infty x_t^*x_t\frac{dt}{t}\right)^{1/2}\right\|_p, \quad \|(x_t)_t\|_{L_p(\mathcal{M};L_2^r(\frac{dt}{t}))} = \left\|\left(\int_0^\infty x_tx_t^*\frac{dt}{t}\right)^{1/2}\right\|_p.$$

We refer to [JLMX06, Section 6.A] for the rigorous meaning of the integral appeared in the above norm. Then we may define the spaces $L_p(\mathcal{M};L_2^c(\frac{dt}{t}))$, $L_p(\mathcal{M};L_2^r(\frac{dt}{t}))$ and $L_p(\mathcal{M};L_2^{cr}(\frac{dt}{t}))$ in a similar way.

We recall the following basic properties.

Proposition 3.1. (1) (Duality) *Let $1 \le p < \infty$ and p' such that $1/p' + 1/p = 1$. Then*

$$(L_p(\mathcal{M};\ell_2^c))^* = L_{p'}(\mathcal{M};\ell_2^r), \quad (L_p(\mathcal{M};\ell_2^r))^* = L_{p'}(\mathcal{M};\ell_2^c), \quad (L_p(\mathcal{M};\ell_2^{cr}))^* = L_{p'}(\mathcal{M};\ell_2^{cr}).$$

The duality bracket is given by

$$\langle (x_n)_n, (y_n)_n \rangle = \sum_n \tau(x_n y_n), \qquad (x_n)_n \subset L_p(\mathcal{M}),\ (y_n)_n \subset L_{p'}(\mathcal{M}).$$

(2) (Complex interpolation [Pis82]) *Let $1 \le p, q \le \infty$ and $0 < \theta < 1$. Let $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$. Then we have the isomorphism with absolute constants*

$$(L_p(\mathcal{M};\ell_2^{cr}), L_q(\mathcal{M};\ell_2^{cr}))_\theta = L_r(\mathcal{M};\ell_2^{cr}).$$

Similar complex interpolations also hold for $L_p(\mathcal{M};\ell_2^c)$ and $L_p(\mathcal{M};\ell_2^r)$.

A sequence of independent random variables (ε_n) on a probalility space (Ω, P) is called a *Rademarcher* sequence if $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = \frac{1}{2}$ for any $n \ge 1$. The following noncommutative Khintchine inequalities are well-known.

Proposition 3.2 ([LP86, LPP91, Pis98])**.** *Let (ε_n) be a Rademarcher sequence on a probability space (Ω, P). Let $1 \le p < \infty$ and (x_n) be a sequence in $L_p(\mathcal{M};\ell_2^{cr})$.*

(1) *If $1 \le p \le 2$, then there exists an absolute constant $c > 0$ such that*

$$c\|(x_n)_n\|_{L_p(\mathcal{M};\ell_2^{cr})} \le \|\sum_n \varepsilon_n x_n\|_{L_p(\Omega;L_p(\mathcal{M}))} \le \|(x_n)_n\|_{L_p(\mathcal{M};\ell_2^{cr})}.$$

(2) *If $2 \le p < \infty$, then there exists an absolute constant $c > 0$ such that*

$$\|(x_n)_n\|_{L_p(\mathcal{M};\ell_2^{cr})} \le \|\sum_n \varepsilon_n x_n\|_{L_p(\Omega;L_p(\mathcal{M}))} \le c\sqrt{p}\|(x_n)_n\|_{L_p(\mathcal{M};\ell_2^{cr})}.$$

The following proposition will be useful for our further studies.

Proposition 3.3. *Let $(x_n)_{n\in\mathbb{N}} \in L_p(\mathcal{M};\ell_\infty)$. Then there exists an absolute constant $c > 0$ such that for any $1 \le p \le \infty$,*

$$\|(x_n)_n\|_{L_p(\mathcal{M};\ell_\infty)} \le \|(x_n)_n\|_{L_p(\mathcal{M};\ell_2^{cr})};$$

and for any $2 \le p \le \infty$,

$$\|(x_n)_n\|_{L_p(\mathcal{M};\ell_\infty^c)} \le \|(x_n)_n\|_{L_p(\mathcal{M};\ell_2^c)}.$$

Proof. We start with the proof of the first inequality. It is trivial for the case $p = \infty$:

$$\|(x_n)_n\|_{L_\infty(\mathcal{M};\ell_\infty)} = \sup_n \|x_n\|_\infty \le \left\| \left(\sum_n x_n^* x_n \right)^{1/2} \right\|_\infty \le \|(x_n)_n\|_{L_\infty(\mathcal{M};\ell_2^{cr})}.$$

Recall that by the Hölder inequality (see also [Jun02, Lemma 3.5]), for any $1 \le p \le \infty$ and for any sequence $(x_n)_n$ in $L_p(\mathcal{M};\ell_1)$,

$$\|\sum_n x_n\|_p \le \|(x_n)_n\|_{L_p(\mathcal{M};\ell_1)}.$$

On the other hand, for any $1 \le p \le \infty$, by the definition of $\|\ \|_{L_p(\mathcal{M};\ell_1)}$, one can easily get $\|(\varepsilon_n x_n)\|_{L_p(\mathcal{M};\ell_1)} = \|(x_n)\|_{L_p(\mathcal{M};\ell_1)}$ with $\varepsilon_n \in \{\pm 1\}$. Now we set $(\varepsilon_n)_n$ to be a Rademarcher

sequence on a probability space (Ω, P). It is folkloric that

$$\begin{aligned}\|(x_n)\|_{L_\infty(\mathcal{M};\ell_2^{cr})} &= \|(\varepsilon_n x_n)\|_{L_\infty(\mathcal{M};\ell_2^{cr})} \leq \|\sum_n \varepsilon_n x_n\|_{L_2(\Omega;L_\infty(\mathcal{M}))} \\ &\leq \|\sum_n \varepsilon_n x_n\|_{L_\infty(\Omega;L_\infty(\mathcal{M}))} = \sup_{\omega\in\Omega}\left\|\sum_n \varepsilon_n(\omega)x_n\right\|_\infty \\ &\leq \sup_{\omega\in\Omega}\|(\varepsilon_n(\omega)x_n)_n\|_{L_\infty(\mathcal{M};\ell_1)} = \|(x_n)_n\|_{L_\infty(\mathcal{M};\ell_1)}.\end{aligned}$$

Let $(y_n)_n \in L_1(\mathcal{M};\ell_\infty)$. By duality and the above inequality we have

$$\begin{aligned}\|(y_n)_n\|_{L_1(\mathcal{M};\ell_\infty)} &= \sup\left\{\frac{\sum_n \tau(x_n y_n)}{\|(x_n)\|_{L_\infty(\mathcal{M},\ell_1)}} : (x_n) \in L_\infty(\mathcal{M},\ell_1)\right\} \\ &\leq \sup\left\{\frac{\sum_n \tau(x_n y_n)}{\|(x_n)\|_{L_\infty(\mathcal{M};\ell_2^{cr})}} : (x_n) \in L_1(\mathcal{M};\ell_2^{cr})\right\} \\ &\leq \|(y_n)\|_{L_1(\mathcal{M};\ell_2^{cr})}.\end{aligned}$$

By interpolation we immediately get the first inequality in the lemma.

The above arguments tell that $\|\ \|_{L_p(\mathcal{M};\ell_\infty)} \leq \|\ \|_{L_p(\mathcal{M};\ell_2^{cr})}$ for $1 \leq p \leq \infty$. As before, by a duality argument we indeed get $\|\ \|_{L_p(\mathcal{M};\ell_\infty)} \leq \|\ \|_{L_p(\mathcal{M};\ell_2^{cr})} \leq \|\ \|_{L_p(\mathcal{M};\ell_1)}$. Therefore, we obtain the second inequality:

$$\begin{aligned}\|(x_n)\|_{L_p(\mathcal{M};\ell_\infty^c)} &= \|(x_n^* x_n)\|^{1/2}_{L_{p/2}(\mathcal{M};\ell_\infty)} \leq \|(x_n^* x_n)\|^{1/2}_{L_{p/2}(\mathcal{M};\ell_1)} \\ &= \left\|\left(\sum_n x_n^* \otimes e_{1,n}\right)\left(\sum_n x_n \otimes e_{n,1}\right)\right\|^{1/2}_{L_{p/2}(\mathcal{M}\otimes B(\ell_2))} \\ &\leq \left\|\left(\sum_n x_n \otimes e_{n,1}\right)^*\right\|^{1/2}_{L_p(\mathcal{M}\otimes B(\ell_2))}\left\|\left(\sum_n x_n \otimes e_{n,1}\right)\right\|^{1/2}_{L_p(\mathcal{M}\otimes B(\ell_2))} \\ &= \|x_n\|_{L_p(\mathcal{M};\ell_2^c)}.\end{aligned}$$

□

In the noncommutative setting usually we do not have the analogue of the complex interpolation $(L_p(\ell_{q_1}), L_p(\ell_{q_2}))_\theta = L_p(\ell_2^{cr})$ with $1/2 = (1-\theta)/q_1 + \theta/q_2$, which is an essential obstruction to generalize many classical methods on maximal inequalities in [Bou86a, Bou87, Car86]. Nevertheless, we still have the following weaker property, which will be enough for our purpose in this paper. More precisely, we will compare the norms of positive symmetric maps on $L_p(\mathcal{M};\ell_2^{cr})$ with those on $L_{\frac{p}{2-p}}(\mathcal{M};\ell_\infty)$. Note that if T is a symmetric and selfadjoint map on $\mathcal{M}$ (by selfadjointness we mean that $T(x^*) = T(x)^*$ for all $x \in \mathcal{M}$), then

$$\tau(T(x)y) = \tau([T(x^*)]^* y) = \tau((x^*)^* T(y)) = \tau(xT(y)) \qquad x, y \in S_{\mathcal{M}}. \tag{3.1}$$

Therefore, T equals its predual operator on $L_1(\mathcal{M})$. In particular, T extends to $L_1(\mathcal{M})$ with the same norm, and by interpolation it also extends to a bounded map on $L_p(\mathcal{M})$ with $1 < p < \infty$. In this context we state the following property (note that a positive map is automatically selfajoint).

Lemma 3.4. *Let $(\Phi_j)_j$ be a sequence of unital positive and symmetric maps on $\mathcal{M}$. Denote $\Phi : (x_j)_j \mapsto (\Phi_j x_j)_j$. Let $1 < p < 2$. Then*

$$\|\Phi : L_p(\mathcal{M};\ell_2^{cr}) \to L_p(\mathcal{M};\ell_2^{cr})\| \le 2\|\Phi : L_{\frac{p}{2-p}}(\mathcal{M};\ell_\infty) \to L_{\frac{p}{2-p}}(\mathcal{M};\ell_\infty)\|^{1/2}.$$

Similar inequalities hold for the spaces $L_p(\mathcal{M};\ell_2^c)$ and $L_p(\mathcal{M};\ell_2^r)$ if $(\Phi_j)_j$ is a sequence of unital 2-positive maps.

Proof. Since Φ_j are unital positive maps, by Kadison's Cauchy-Schwarz inequality [Kad52], for any selfadjoint element $x_j \in L_p(\mathcal{M})$, we have

$$\Phi_j(x_j)^2 \le \Phi_j(x_j^2).$$

Assume that $(x_j)_j \in L_{p'}(\mathcal{M};\ell_2^{cr})$ is a sequence of selfadjoint elements. Then the conjugate index p' is greater than 2 and

$$\begin{aligned}\|(\Phi_j x_j)_j\|_{L_{p'}(\mathcal{M};\ell_2^{cr})} &= \left\|\left(\sum_j (\Phi_j x_j)^2\right)^{1/2}\right\|_{p'} \le \left\|\left(\sum_j \Phi_j(x_j^2)\right)^{1/2}\right\|_{p'} = \left\|\big(\Phi_j(x_j^2)\big)_j\right\|_{L_{p'/2}(\mathcal{M};\ell_1)}^{1/2} \\ &\le \|\Phi : L_{p'/2}(\mathcal{M};\ell_1) \to L_{p'/2}(\mathcal{M};\ell_1)\|^{1/2}\|(x_j)_j\|_{L_{p'}(\mathcal{M};\ell_2^{cr})}.\end{aligned}$$

For general $(x_j)_j \in L_{p'}(\mathcal{M};\ell_2^{cr})$, we may decompose it into two sequence of selfadjoint elements. Note that $\|(x_j)_j\|_{L_{p'}(\mathcal{M};\ell_2^{cr})} = \|(x_j^*)_j\|_{L_{p'}(\mathcal{M};\ell_2^{cr})}$ for $p' > 2$. Therefore,

$$\|\Phi : L_{p'}(\mathcal{M};\ell_2^{cr}) \to L_{p'}(\mathcal{M};\ell_2^{cr})\| \le 2\|\Phi : L_{p'/2}(\mathcal{M};\ell_1) \to L_{p'/2}(\mathcal{M};\ell_1)\|^{1/2}.$$

As explained in (3.1), the dual operator of Φ equals itself and we obtain

$$\|\Phi : L_p(\mathcal{M};\ell_2^{cr}) \to L_p(\mathcal{M};\ell_2^{cr})\| \le 2\|\Phi : L_{\frac{p}{2-p}}(\mathcal{M};\ell_\infty) \to L_{\frac{p}{2-p}}(\mathcal{M};\ell_\infty)\|^{1/2},$$

as desired.

For the spaces $L_p(\mathcal{M};\ell_2^c)$ and $L_p(\mathcal{M};\ell_2^r)$, similar arguments still work for non selfadjoint elements (x_j) if the maps Φ_j are 2-positive, since in this case we can use the following Cauchy-Schwarz inequality $|\Phi_j(x_j)|^2 \le \Phi_j(|x_j|^2)$ (see e.g. [Pau02, Proposition 3.3]). □

The square function estimates for noncommutative diffusion semigroups has been established in [JLMX06]. In this section we will slightly adapt the arguments of [JLMX06] so as to obtain a refined version of this result for our further purpose. Throughout this subsection, $(S_t)_{t\in\mathbb{R}_+}$ always denotes a semigroup of unital completely positive trace preserving and symmetric maps on $\mathcal{M}$ with the negative infinitesimal generator A. Let (P_t) denote the subordinate Poisson semigroup of (S_t), i.e. the negative generator of P_t is $-(-A)^{1/2}$.

For notational simplicity, the vector-valued spaces $L_p(\mathcal{M};L_2^c(\mathbb{R};\frac{dt}{t}))$, $L_p(\mathcal{M};L_2^r(\mathbb{R};\frac{dt}{t}))$ and $L_p(\mathcal{M};L_2^{cr}(\mathbb{R};\frac{dt}{t}))$ are denoted respectively by $L_p(L_2^c(\frac{dt}{t}))$, $L_p(L_2^r(\frac{dt}{t}))$ and $L_p(L_2^{cr}(\frac{dt}{t}))$ in this section.

To state our theorem, we recall the dilation property. Let $(\mathcal{M},\tau),(\mathcal{N},\tau')$ be two von Neumann algebra where τ and τ' are normal faithful semifinite traces. If $\pi\colon(\mathcal{M},\tau)\to(\mathcal{N},\tau')$ is a normal unital faithful trace preserving $*$-representation, then it (uniquely) extends to an isometry from $L_p(\mathcal{M})$ into $L_p(\mathcal{N})$ for any $1 \le p < \infty$. We call the adjoint $\mathbb{E}\colon\mathcal{N}\to\mathcal{M}$ of the embedding $L_1(\mathcal{M}) \hookrightarrow L_1(\mathcal{N})$ induced by π *the conditional expectation associated with* π. Moreover $\mathbb{E}\colon\mathcal{N}\to\mathcal{M}$ is unital and completely positive.

Definition 3.5. Let $T\colon \mathcal{M} \to \mathcal{M}$ be a bounded operator. We say that T satisfies *Rota's dilation property* if there exist a von Neumann algebra $\mathcal{N}$ equipped with a normal semifinite faithful trace, a normal unital faithful $*$-representation $\pi\colon \mathcal{M} \to \mathcal{N}$ which preserves traces, and a decreasing sequence $(\mathcal{N}_m)_{m\geq 1}$ of von Neumann subalgebras of $\mathcal{N}$ such that

$$T^m = \hat{\mathbb{E}} \circ \mathbb{E}_m \circ \pi, \qquad m \geq 1, \tag{3.2}$$

where $\mathbb{E}_m\colon \mathcal{N} \to \mathcal{N}_m \subset \mathcal{N}$ is the canonical conditional expectation onto $\mathcal{N}_m$, and where $\hat{\mathbb{E}}\colon \mathcal{N} \to \mathcal{M}$ is the conditional expectation associated with π.

We recall the following typical examples of operators with Rota's dilation property.

Lemma 3.6. (1) ([JRS14, Dab10]) *If $\mathcal{M}$ is a finite von Neumann algebra and τ is a normal faithful state on $\mathcal{M}$, then for all $t \in \mathbb{R}_+$, the operator S_t satisfies Rota's dilation property.*

(2) ([Ste70]) *If $\mathcal{M}$ is a commutative von Neumann algebra and $\mathcal{L}$ is another semifinite von Neumann algebra, then for all $t \in \mathbb{R}_+$, the operator $S_t \otimes \mathrm{Id}_{\mathcal{L}}$ on $\mathcal{M}\overline{\otimes}\mathcal{L}$ satisfies Rota's dialtion property.*

We aim to prove the following square function estimates, which are essentially established in [JLMX06, JW17], without specifying the order $(p-1)^{-6}$. However, we will see that the methods in [JLMX06], together with the sharp constants of various martingale inequalities, are enough to obtain this order. The outline of our proof is slightly different from that of [JLMX06], but all the ingredients are already available in the latter.

Proposition 3.7. *Assume that for all $t \in \mathbb{R}_+$, the operator S_t satisfies Rota's dilation property. Then for all $1 < p < 2$ and $x \in L_p(\mathcal{M})$ we have*

$$\inf\left\{ \left\| \Big(\int_0^\infty |t\partial_t P_t(x^c)|^2 \frac{dt}{t}\Big)^{\frac{1}{2}} \right\|_p + \left\| \Big(\int_0^\infty |(t\partial_t P_t(x^r))^*|^2 \frac{dt}{t}\Big)^{\frac{1}{2}} \right\|_p \right\} \leq c(p-1)^{-6}\|x\|_p \tag{3.3}$$

where the infimum runs over all $x_c, x_r \in L_p(\mathcal{M})$ such that $x = x_c + x_r$, and $c > 0$ is an absolute constant.

Remark 3.8. When the underlying von Neumann algebra is commutative, it is known from Stein [Ste70] that the optimal order here is $(p-1)^{-1}$. In the noncommutative case, we believe that the order $(p-1)^{-6}$ is not optimal. However, this order is sufficient for our purpose in the sequel.

Our study of the semigroup $(P_t)_t$ is based on the analysis of the ergodic averages as follows:

$$M_t = \frac{1}{t}\int_0^t S_u du.$$

We will need the following claim.

Lemma 3.9. *For any $y \in L_p(\mathcal{M})$,*

$$\|(t\partial P_t y)_t\|_{L_p(L_2^c(\frac{dt}{t}))} \leq c\|(t\partial M_t y)_t\|_{L_p(L_2^c(\frac{dt}{t}))},$$

where $c > 0$ is an absolute constant. The inequality remains true if we replace the norm of $L_p(L_2^c(\frac{dt}{t}))$ by $L_p(L_2^r(\frac{dt}{t}))$ or $L_p(L_2^{cr}(\frac{dt}{t}))$.

Proof. This claim is well known to experts. We only give a sketch of its arguments. Set $\varphi(s) = \frac{1}{2\sqrt{\pi}}\frac{e^{-1/4s}}{s^{3/2}}$. Using integration by parts we have

$$P_t = \frac{1}{t^2}\int_0^\infty \varphi(\frac{s}{t^2})(\partial(sM_s))ds = -\int_0^\infty s\varphi'(s)M_{t^2s}ds.$$

Therefore

$$t\partial P_t = -2\int_0^\infty t^2s^2\varphi'(s)\partial M_{t^2s}ds = -2\int_0^\infty s\varphi'(s)(t^2s\partial M_{t^2s})ds \tag{3.4}$$

which yields the claim by noting that $\|\cdot\|_{L_p(L_2^c(\frac{dt}{t}))}$ is a norm and $s\varphi'(s)$ is absolutely integrable. □

We need the following auxiliary result.

Proposition 3.10. *Assume that for all $t \in \mathbb{R}_+$, the operator S_t satisfies Rota's dialtion property. Then for $1 < p < 2$ and $x \in L_p(\mathcal{M})$, we have*

$$\|(t\partial P_t(x))_{t>0}\|_{L_p(L_2^{cr}(\frac{dt}{t}))} \leq c(p-1)^{-2}\|x\|_p, \tag{3.5}$$

where $c > 0$ is an absolute constant.

Proof. This result has been essentially obtained in [JLMX06], together with the optimal estimates for martingale inequalities in [Ran02, JX05]. Indeed, let $(\mathbb{E}_n)_{n\in\mathbb{N}}$ be a monotone sequence of conditional expectations on $\mathcal{M}$ and $x_n = \mathbb{E}_{n+1}(x) - \mathbb{E}_n(x)$ be a sequence of martingale differences with $x \in L_p(\mathcal{M})$. By the estimate for noncommutative martingale transform in [Ran02, Theorem 4.3] and the Khintchine inequality in Lemma 3.2, we have

$$\|(x_n)_{n\in\mathbb{N}}\|_{L_p(\ell_2^{cr})} \leq \frac{c}{p-1}\|x\|_p,$$

and by the noncommutative Stein inequality [JX05, Theorem 8] we have for any sequence $(y_n)_{n\in\mathbb{N}} \subset L_p(\mathcal{M})$,

$$\|(\mathbb{E}_n y_n)_{n\in\mathbb{N}}\|_{L_p(\ell_2^{cr})} \leq \frac{c}{p-1}\|(y_n)_{n\in\mathbb{N}}\|_{L_p(\ell_2^{cr})} \quad 1 < p < 2,$$

where $c > 0$ is an absolute constant. Then tracing the order in the proof of [JLMX06, Corollary 10.9], we obtain that for all $\varepsilon > 0$,

$$\|(\sqrt{m}D_m^\varepsilon(x))_{m\geq 1}\|_{L_p(\ell_2^{cr})} \leq \frac{c'}{(p-1)^2}\|x\|_p,$$

where $c' > 0$ is an absolute constant, and where

$$D_m^\varepsilon = \rho_m^\varepsilon - \rho_{m-1}^\varepsilon \quad \text{and} \quad \rho_m^\varepsilon = \frac{1}{m+1}\sum_{k=0}^m S_{k\varepsilon}.$$

By a standard discretization argument (see e.g. [JLMX06, Lemma 10.11]), we get the following inequality

$$\|(t\partial M_t x)_t\|_{L_p(L_2^{cr}(\frac{dt}{t}))} \leq c'(p-1)^{-2}\|x\|_p.$$

Then the desired result follows from Lemma 3.9. □

Estimate (3.5) is weaker than (3.3). The remaining task for proving Proposition 3.7 is to show

$$\inf\left\{\left\|\left(\int_0^\infty |t\partial P_t(x^c)|^2\,\frac{dt}{t}\right)^{\frac12}\right\|_p+\left\|\left(\int_0^\infty |(t\partial P_t(x^r))^*|^2\,\frac{dt}{t}\right)^{\frac12}\right\|_p\right\} \tag{3.6}$$
$$\lesssim (p-1)^{-4}\|(t\partial P_t(x))_{t>0}\|_{L_p(L_2^{cr}(\frac{dt}{t}))}$$

where the infimum runs over all decompositions $x = x^c + x^r$ in $L_p(\mathcal{M})$. This inequality is essentially proved in [JLMX06, Theorem 7.8]; the order $(p-1)^{-4}$ is not stated there, but it follows from a careful computation on all the related constants appearing in the argument therein. For the convenience of the reader, we will recall some parts of the proof and clarify all the constants in the proof which are concerned with the precise order.

For notational simplicity, we say that a family $\mathcal{F} \subset B(L_p(\mathcal{M}))$ *is Col-bounded (resp. Row-bounded)* if there is a constant C such that for any sequence $(T_k)_k \subset \mathcal{F}$, we have

$$\|(T_k)_k : L_p(\mathcal{M};\ell_2^c) \to L_p(\mathcal{M};\ell_2^c)\| \le C \ \ (\text{resp. } \|(T_k)_k : L_p(\mathcal{M};\ell_2^r) \to L_p(\mathcal{M};\ell_2^r)\| \le C)\,, \tag{3.7}$$

and the least constant C will be denoted by $\mathrm{Col}(\mathcal{F})$ (resp. $\mathrm{Row}(\mathcal{F})$).

We quote a useful result from [CdPSW00, Lemma 3.2] (see also [JLMX06, Lemma 4.2]).

Lemma 3.11. *Let $\mathcal{F} \subset B(L_p(\mathcal{M}))$ be a Col-bounded (resp. Row-bounded) collection with $\mathrm{Col}(\mathcal{F}) = M$ (resp. $\mathrm{Row}(\mathcal{F}) = M$). Then the closure of the complex absolute convex hull of $\mathcal{F}$ in the strong operator topology is also Col-bounded (resp. Row-bounded) with the constant $\mathrm{Col}(\mathcal{F}) \le 2M$ (resp. $\mathrm{Row}(\mathcal{F}) \le 2M$).*

For any $\theta \in (0,\pi)$, we let

$$\Sigma_\theta = \{z \in \mathbb{C}^* : |\mathrm{Arg}(z)| < \theta\}.$$

Without the concrete order of growth of the constant on p, the following lemma is contained in [JLMX06, Theorem 5.6]. Note that the present area $\mathbb{C}\backslash\Sigma_{\nu_p}$ in the following lemma is contained in the optimal area $\mathbb{C}\backslash\overline{\Sigma_{\omega_p}}$ described in [JLMX06, Theorem 5.6].

Lemma 3.12. *Let $1 < p < 2$. Let $(S_t)_t$ be a semigroup of unital completely positive trace preserving and symmetric maps. Let A be the negative infinitesimal generator of $(S_t)_t$. Then the set $\mathcal{F}_p = \{z(z-A)^{-1} : z \in \mathbb{C}\backslash\Sigma_{\nu_p}\} \subset B(L_p(\mathcal{M}))$ with $\nu_p = \frac{(p+1)\pi}{4p}$ is Col-bouneded and Row-bounded with constants*

$$\mathrm{Col}(\mathcal{F}_p) \le c(p-1)^{-2} \quad \textit{and} \quad \mathrm{Row}(\mathcal{F}_p) \le c(p-1)^{-2},$$

where c is an absolute constant.

Proof. Let $s_1, \cdots, s_n$ be some nonnegative real numbers. For any $z \in \mathbb{C}^*$ with $0 \le \mathrm{Arg}(z) \le \frac{\pi}{2}$, we define a map $U(z)$ with

$$U(z) : L_2(\mathcal{M};\ell_2^c) \cap L_q(\mathcal{M};\ell_2^c) \to L_2(\mathcal{M};\ell_2^c) + L_q(\mathcal{M};\ell_2^c)$$
$$(x_k)_k \mapsto (S_{zs_k}(x_k))_k.$$

Note that for any $x \in L_2(\mathcal{M})$, the function $z \mapsto U(z)x$ is continuous and bounded in the area $\{z \in \mathbb{C}^* : 0 \le \mathrm{Arg}(z) \le \pi/2\}$ by [JLMX06, Proposition 5.4 and Lemma 3.1]. This $U(z)$ is well defined. On the one hand, for any $t > 0$, we have

$$\|U(te^{i\frac{\pi}{2}}) : L_2(\mathcal{M};\ell_2^c) \to L_2(\mathcal{M};\ell_2^c)\| \le 1.$$

On the other hand, by duality and Lemma 3.4 and Proposition 2.10, we may find an absolute constant c such that for any $t>0,\ 2<q<\infty$,

$$\begin{aligned}\|U(t):L_q(\mathcal{M};\ell_2^c)\to L_q(\mathcal{M};\ell_2^c)\| &= \|U(t):L_{\frac{q}{q-1}}(\mathcal{M};\ell_2^c)\to L_{\frac{q}{q-1}}(\mathcal{M};\ell_2^c)\| \\ &\le \|U(t):L_{\frac{q}{q-2}}(\mathcal{M};\ell_\infty)\to L_{\frac{q}{q-2}}(\mathcal{M};\ell_\infty)\|^{1/2} \\ &\le cq.\end{aligned}$$

Let $p'=\frac{p}{p-1}$ be the conjugate number of p. We fix $\beta_p=\frac{\pi}{2p'}$. Let $q=p'(\frac{\pi-2\beta_p}{\pi-p'\beta_p})=2(p'-1)$ and $\alpha=\frac{2\beta_p}{\pi}=\frac{1}{p'}$. These numbers satisfy $\frac{1-\alpha}{q}+\frac{\alpha}{2}=\frac{1}{p'}$. By complex interpolation, we know that

$$L_{p'}(\mathcal{M};\ell_2^c)=[L_q(\mathcal{M};\ell_2^c),L_2(\mathcal{M};\ell_2^c)]_\alpha.$$

By the 'sectorial' form of Stein's interpolation principle (see for instance [JLMX06, Lemma 5.3]), we have

$$\|U(e^{i\beta_p}):L_{p'}(\mathcal{M};\ell_2^c)\to L_{p'}(\mathcal{M};\ell_2^c)\|\le (cq)^{1-\alpha}\le cq.$$

Thus,

$$\|(S_{s_ke^{i\beta_p}}(x_k))_k\|_{L_{p'}(\mathcal{M};\ell_2^c)}\le cq\|(x_k)\|_{L_{p'}(\mathcal{M};\ell_2^c)}.$$

Similarly, we have

$$\|(S_{s_ke^{-i\beta_p}}(x_k))_k\|_{L_{p'}(\mathcal{M};\ell_2^c)}\le cq\|(x_k)\|_{L_{p'}(\mathcal{M};\ell_2^c)}.$$

Namely, $(S_z)_{z\in\partial\Sigma_{\beta_p}}$ is Col-bounded with constant cq. Then we get that the set

$$\{S_z:L_{p'}(\mathcal{M})\to L_{p'}(\mathcal{M}):z\in\Sigma_{\beta_p}\}$$

is also Col-bounded. Indeed, by a standard argument (see e.g. [Wei01, Proposition 2.8]), we see that any S_z with $z\in\Sigma_{\beta_p}$ can be approximated by convex combinations of $\{S_z:z\in\partial\Sigma_{\beta_p}\}$. Therefore, by Lemma 3.11 we get that

$$\mathrm{Col}(\{S_z:L_{p'}(\mathcal{M})\to L_{p'}(\mathcal{M}):z\in\Sigma_{\beta_p}\})\le 2cq=2cp'(\frac{\pi-2\beta_p}{\pi-p'\beta_p})\le 2c(p-1)^{-1}.$$

By duality, we have

$$\mathrm{Row}\left(\{S_z:L_p(\mathcal{M})\to L_p(\mathcal{M}):z\in\Sigma_{\beta_p}\}\right)\le 2c(p-1)^{-1}. \tag{3.8}$$

Set $\omega_p=\frac{\pi}{p}-\frac{\pi}{2}$ and $\nu_p=\frac{(p+1)\pi}{4p}$. Then $0<\frac{\pi}{2}-\nu_p<\beta_p<\frac{\pi}{p'}=\frac{\pi}{2}-\omega_p$. By the Laplace formula, we have that for any $z\in\mathbb{C}\backslash\overline{\Sigma_{\pi/2}}$,

$$(z-A)^{-1}=-\int_0^\infty e^{tz}S_tdt.$$

Denote $\Gamma_{\beta_p}^+=\{u=te^{i\beta_p}:t\in\mathbb{R}_+\}$. By [JLMX06, Proposition 5.4 and Lemma 3.1], $u\mapsto S_u$ is analytic on the area $\Sigma_{\frac{\pi}{2}-\omega_p}$. Note that $\Gamma_{\beta_p}^+\subset\Sigma_{\frac{\pi}{2}-\omega_p}$. Then by the Cauchy theorem, we have that for any $z\in\mathbb{C}\backslash\overline{\Sigma_{\pi/2}}$,

$$z(z-A)^{-1}=-\int_0^\infty ze^{tz}S_tdt=-\int_{\Gamma_{\beta_p}^+}ze^{uz}S_udu. \tag{3.9}$$

Note that for any $z\in\mathbb{C}$ with $\nu_p\le\mathrm{Arg}(z)\le\pi/2$, we have

$$\mathrm{Re}(uz)=|z|t\cos(\mathrm{Arg}(z)+\beta_p)=-|z|t\sin(\mathrm{Arg}(z)-\pi/2p),$$

and hence,

$$\int_{\Gamma^+_{\beta_p}} \|ze^{uz}S_u du\| \leq \sup_{u\in\Gamma^+_{\beta_p}} \|S_u\| \int_0^\infty re^{-rt\sin(\mathrm{Arg}(z)-\pi/2p)}dt \leq \frac{\sup_{u\in\Gamma^+_{\beta_p}} \|S_u\|}{\sin(\nu_p-\pi/2p)} < \infty.$$

Hence, (3.9) holds for any $z \in \mathbb{C}\backslash\Sigma_{\nu_p}$. Note that $\frac{\nu_p-\pi/2p}{\sin(\nu_p-\pi/2p)} \lesssim 1$ since $0 < \nu_p - \pi/2p < \frac{\pi}{8}$. By Lemma 3.11 and (3.8), we get

$$\begin{aligned}\mathrm{Row}(\mathcal{F}_p) &\leq 2\left|\int_{\Gamma^+_{\beta_p}} ze^{uz}du\right| \cdot \mathrm{Row}(\{S_z : L_p(\mathcal{M}) \to L_p(\mathcal{M}) : z \in \Sigma_{\beta_p}\}) \\ &\lesssim \frac{(p-1)^{-1}}{\sin(\nu_p-\pi/2p)} \lesssim \frac{(p-1)^{-1}}{\nu_p-\pi/2p} \lesssim (p-1)^{-2}.\end{aligned}$$

A similar proof shows that

$$\mathrm{Col}(\mathcal{F}_p) \lesssim (p-1)^{-2}.$$

□

Now let us prove the desired proposition.

Proof of (3.6) *and Proposition 3.7.* Set $F(z) = ze^{-z}$, $G(z) = 4F(z)$ and $\widetilde{G}(z) = \overline{G(\bar{z})}$. Let $B = -(-A)^{1/2}$ be the negative infinitesimal generator of $(P_t)_t$. We have

$$F(tB)x = tBe^{-tB}x = -t\frac{\partial}{\partial t}(P_t(x)).$$

Let $\omega_p = \frac{\pi}{p} - \frac{\pi}{2}$, $\nu_p = \frac{(p+1)\pi}{4p}$ and $\xi_p = \frac{(3p+1)\pi}{8p}$. These numbers satisfy $\omega_p < \nu_p < \xi_p < \frac{\pi}{2}$. Note that $(P_t)_{t\in\mathbb{R}_+}$ is again a semigroup of unital completely positive trace preserving and symmetric maps. By [JLMX06, Corollary 11.2], $B : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ admits a bounded $H^\infty(\Sigma_{\xi_p})$ functional calculus. By [JLMX06, Theorem 7.6 (1)], B satisfies the dual square function estimate $(\mathcal{S}^*_{\widetilde{G}})$, since $G \in H^\infty_0(\Sigma_{\xi_p})$. Note that $F \in H^\infty_0(\Sigma_{\xi_p})$ and $\int_0^\infty G(t)F(t)\frac{dt}{t} = 1$. By the proof of [JLMX06, Theorem 7.8], we get that

(3.10)
$$\inf\left\{\left\|\left(\int_0^\infty |t\partial P_t(x^c)|^2\frac{dt}{t}\right)^{\frac{1}{2}}\right\|_p + \left\|\left(\int_0^\infty |(t\partial P_t(x^r))^*|^2\frac{dt}{t}\right)^{\frac{1}{2}}\right\|_p\right\} \leq 2C\|(t\partial P_t(x))_{t>0}\|_{L_p(L_2^{cr}(\frac{dt}{t}))}$$

where the infimun runs over all decompositions of $x = x^c + x^r$ in $L_p(\mathcal{M})$ and $C = \max\{\|T_c\|, \|T_r\|\}$, with T_c and T_r being defined as

$$\begin{aligned}T_c : L_p(L_2^c(\frac{dt}{t})) &\to L_p(L_2^c(\frac{dt}{t})) \\ (x_t)_t &\mapsto \left(\int_0^\infty F(sB)G(tB)x_t\frac{dt}{t}\right)_s\end{aligned}$$

and

$$\begin{aligned}T_r : L_p(L_2^r(\frac{dt}{t})) &\to L_p(L_2^r(\frac{dt}{t})) \\ (x_t)_t &\mapsto \left(\int_0^\infty F(sB)G(tB)x_t\frac{dt}{t}\right)_s.\end{aligned}$$

Let $\nu_p < \gamma < \xi_p$. Denote

$$f_\gamma(t) = \begin{cases} -te^{i\gamma}, & t \in \mathbb{R}_-, \\ te^{-i\gamma}, & t \in \mathbb{R}_+, \end{cases}$$

and let $\Gamma_\gamma = \{f_\gamma(t) : t \in \mathbb{R}\} \subset \mathbb{C}$. Set

$$T_\Phi : L_p(L_2^c(\mathbb{R}, \left|\frac{dt}{t}\right|)) \to L_p(L_2^c(\mathbb{R}, \left|\frac{dt}{t}\right|))$$
$$(x_t)_{t\in\mathbb{R}} \mapsto \left(\frac{f_\gamma(t)(f_\gamma(t)-B)^{-1}x_t}{2\pi i}\right)_{t\in\mathbb{R}}.$$

Denote $K_1 = \int_{\Gamma_\gamma} |F(z)| \left|\frac{dz}{z}\right|$ and $K_2 = \int_{\Gamma_\gamma} |G(z)| \left|\frac{dz}{z}\right|$. By the proof of [JLMX06, Theorem 4.14], we have

$$\|T_c\| \le K_1 K_2 \|T_\Phi\|.$$

And the proof of [JLMX06, Proposition 4.4] shows that

$$\|T_\Phi\| \le \mathrm{Col}(\mathcal{O})$$

where $\mathcal{O} = \left\{\frac{1}{\mu(I)} \int_I f_\gamma(t)(f_\gamma(t)-B)^{-1} d\mu(t) : I \subset \mathbb{R}, 0 < \mu(I) < \infty\right\}$. Moreover, by Lemma 3.11,

$$\mathrm{Col}(\mathcal{O}) \le 2\,\mathrm{Col}(\{z(z-B)^{-1} : z \in \Gamma_\gamma\}).$$

Since $\gamma > \nu_p$, by Lemma 3.12,

$$\mathrm{Col}(\{z(z-B)^{-1} : z \in \Gamma_\gamma\}) \lesssim (p-1)^{-2}.$$

On the other hand,

$$K_1 = \int_{\Gamma_\gamma} |F(z)| \left|\frac{dz}{z}\right| = 2\int_0^\infty |te^{-i\gamma} e^{-t(\cos(\gamma) - i\sin(\gamma))}| \frac{dt}{t} = 2\int_0^\infty e^{-t\cos(\gamma)} dt \lesssim \frac{1}{\cos(\gamma)},$$

and $K_2 = 4K_1$. Note that $\gamma < \xi_p = \frac{(3p+1)\pi}{8p}$ and $0 < \frac{\pi}{2} - \xi_p < \frac{\pi}{16}$. We have

$$\frac{1}{\cos(\gamma)} \le \frac{1}{\sin(\frac{\pi}{2} - \xi_p)} \lesssim (p-1)^{-1}.$$

Therefore,

$$\|T_c\| \lesssim (p-1)^{-4}.$$

Similarly,

$$\|T_r\| \lesssim (p-1)^{-4}.$$

Thus $C \lesssim (p-1)^{-4}$ and we obtain (3.6). As mentioned previously, this implies Proposition 3.7. The proof is complete. □

4. Proof of Theorem 1.1 and Theorem 1.2

This section is devoted to the study of general criteria for maximal inequalities and pointwise convergences given by Theorem 1.1. Our argument does not essentially rely on the group theoretic structure. Note that there are a number of typical structures with Fourier-like expansions in noncommutative analysis, which are not given by group algebras. Hence instead of the framework in Theorem 1.1, we would like to state and prove results in a quite general setting.

To proceed with our study, we will only require the following simple framework. In the sequel, we fix a von Neumann algebra $\mathcal{M}$ equipped with a normal semifinite faithful trace τ,

and an isometric isomorphism of Hilbert spaces $U : L_2(\mathcal{M}) \to L_2(\Omega, \mu; H)$ for some distinguished regular Borel measure space (Ω, μ) and Hilbert space H. Assume additionally that $U^{-1}(C_c(\Omega; H))$ is a dense subspace in $L_p(\mathcal{M})$ for all $1 \leq p \leq \infty$ (for $p = \infty$ we refer to the w*-density), where $C_c(\Omega; H)$ denotes the space of H-valued continuous functions with compact supports. Given a measurable function $m \in L_\infty(\Omega; \mathbb{C})$, we denote by T_m the linear operator on $L_2(\mathcal{M})$ determined by

$$T_m : L_2(\mathcal{M}) \to L_2(\mathcal{M}), \quad U(T_m x) = mU(x), \quad x \in L_2(\mathcal{M}). \tag{4.1}$$

We call m the *symbol* of T_m. The operator T_m is obviously a generalization of a classical Fourier multiplier. Moreover, for a discrete group Γ, taking

$$\mathcal{M} = VN(\Gamma), \quad (\Omega, \mu) = (\Gamma, \text{counting measure}), \quad H = \mathbb{C}, \quad U : \lambda(g) \mapsto \delta_g,$$

the above framework coincides with that considered in Theorem 1.1.

Example 4.1. Apart from group von Neumann algebras, this framework applies to various typical models in the study of noncommutative analysis. As an illustration we recall briefly some of them.

(1) Twisted crossed product ([BC09, BC12]): Let Γ be a discrete group with a twisted dynamical system Σ on a von Neumann algebra $\mathcal{N} \subset B(L_2(\mathcal{N}))$. Then we may consider the von Neumann algebra $\mathcal{M}$ generated by the associated regular covariant representation of Γ and the natural representation of $\mathcal{N}$ on $\ell_2(\Gamma; L_2(\mathcal{N}))$. Each $x \in \mathcal{M}$ admits a Fourier series $\sum_{g\in\Gamma} \hat{x}(g)\lambda_\Sigma(g)$ with $\hat{x}(g) \in \mathcal{N}$. Take $\Omega = \Gamma$, $H = L_2(\mathcal{N})$ and $U : x \mapsto \hat{x}$. It is easy to see that for any $m \in \ell_\infty(\Gamma)$, the multiplier in (4.1) is given by

$$T_m : \sum_{g\in\Gamma} \hat{x}(g)\lambda_\Sigma(g) \mapsto \sum_{g\in\Gamma} m(g)\hat{x}(g)\lambda_\Sigma(g),$$

which is the usual Fourier multiplier considered in [BC09]. As a particular case, this also coincides with the Fourier multipliers on quantum tori studied by [CXY13].

(2) Rigid C*-tensor category ([PV15, AdLW18]): Let $\mathcal{C}$ be a rigid C*-tensor category, $A(\mathcal{C})$ its Fourier algebra and $\mathcal{M}$ the von Neumann algebra generated by the image of the left regular representation of $\mathbb{C}[\mathcal{C}]$. Set $(\Omega, \mu) = (\mathrm{Irr}(\mathcal{C}), d)$ where d denotes the intrinsic dimension, and set $U : L_2(\mathcal{M}) \to \ell_2(\Omega)$ by $U(\alpha) = \delta_\alpha$ for $\alpha \in \mathrm{Irr}(\mathcal{C})$. Then for any $m \in \ell_\infty(\Omega)$, it is easy to check that T_m is the dual map of the multiplication operator $\theta \mapsto m\theta$ for $\theta \in A(\mathcal{C})$, which gives the Fourier multiplier studied in [PV15, AdLW18].

(3) Clifford algebras, free semicircular systems and q-deformations: Let $\mathcal{M} = \Gamma_q(H)$ be a q-deformed von Neumann algebra in the sense of Bożejko and Speicher [BS91, BS94]. The case $q = 0$ corresponds to Voiculescu's free Gaussian von Neumann algebra and the case $q = -1$ to the usual Clifford algebras. We choose the canonical orthonormal basis of $L_2(\mathcal{M})$ with index set Ω according to the Fock representation $\bigoplus_n H^{\otimes n}$, and denote by U the corresponding isomorphism. We view $m : \mathbb{N} \to \mathbb{C}$ naturally as a function on Ω by setting the value $m(n)$ on indexes of basis in $H^{\otimes n}$. Then for any such m, the operator T_m coincides with the radial Fourier multiplier studied in [JLMX06, Section 9].

(4) Quantum Euclidean spaces [GPJP20]: Let $\mathcal{M} = \mathcal{R}_\Theta$ be the quantum Euclidean space associated with an antisymmetric $n\times n$-matrix Θ. Take $\Omega = \mathbb{R}^n$ and let $U : L_2(\mathcal{R}_\Theta) \to L_2(\mathbb{R}^n)$ be the canonical isomorphism. Then the operator T_m coincides with a usual quantum Fourier multiplier on $\mathcal{R}_\Theta$.

(5) The framework also applies to non-abelian compact groups, compact quantum groups and von Neumann algebras of locally compact groups, where we may take U to be the usual Fourier transform. We will discuss some of them in more details in the next sections.

Our criterion is based on comparisons with symbols of a symmetric Markov semigroup. To state our results, we fix a semigroup $(S_t)_{t\in\mathbb{R}_+}$ of unital completely positive trace preserving and symmetric maps on $\mathcal{M}$ of the form

$$S_t = T_{e^{-t\ell}} : L_2(\mathcal{M}) \to L_2(\mathcal{M}), \quad U(S_t x) = e^{-t\ell(\cdot)}(Ux), \quad x \in L_2(\mathcal{M}),$$

for a distinguished continuous function $\ell : \Omega \to [0,\infty)$. We will also consider the subordinate Poisson semigroup $(P_t)_t$ of $(S_t)_t$, that is,

$$P_t = T_{e^{-t\sqrt{\ell}}} : L_2(\mathcal{M}) \to L_2(\mathcal{M}), \quad U(P_t x) = e^{-t\sqrt{\ell(\cdot)}}(Ux), \quad x \in L_2(\mathcal{M}).$$

We will consider the family of operators $(T_{m_N})_{N\in\mathbb{N}}$ (resp. $(T_{m_t})_{t\in\mathbb{R}_+}$) induced by a sequence of measurable functions $(m_N)_{N\in\mathbb{N}}$ (resp. $(m_t)_{t\in\mathbb{R}_+}$) on Ω. Recall that we are interested in the following types of conditions for the symbols $(m_N)_{N\in\mathbb{N}}$ (resp. $(m_t)_{t\in\mathbb{R}_+}$) in Theorem 1.1:

(A1) There exist $\alpha > 0$ and $\beta > 0$ such that for all $N \in \mathbb{N}$ and almost all $\omega \in \Omega$, we have

$$|1 - m_N(\omega)| \le \beta \frac{\ell(\omega)^\alpha}{2^N}, \quad |m_N(\omega)| \le \beta \frac{2^N}{\ell(\omega)^\alpha}. \tag{4.2}$$

(A2) There exist $\alpha > 0$, $\beta > 0$ and $\eta \in \mathbb{N}_+$ such that $t \mapsto m_t(\omega)$ is piecewise η-differentiable for almost all $\omega \in \Omega$, and for all $1 \le k \le \eta$, all $t \in \mathbb{R}_+$ and almost all $\omega \in \Omega$ we have

$$|1 - m_t(\omega)| \le \beta \frac{\ell(\omega)^\alpha}{t}, \quad |m_t(\omega)| \le \beta \frac{t}{\ell(\omega)^\alpha}, \quad \left|\frac{d^k m_t(\omega)}{dt^k}\right| \le \beta \frac{1}{t^k}. \tag{4.3}$$

Intuitively, **(A1)** is motivated by considering the subsequence $(m_{2^N})_{N\in\mathbb{N}}$ of $(m_t)_{t\in\mathbb{R}_+}$ in **(A2)**, but the present form in **(A1)** is slightly more general. Indeed we will see in Section 5 other abstract and important constructions of symbols satisfying **(A1)** but without being of the aforementioned form $(m_{2^N})_{N\in\mathbb{N}}$.

We split our study into two parts. The first part mainly deals with the L_2-theory of the above multipliers. Note that **(A1)** (resp. **(A2)**) implies that $(m_N)_{N\in\mathbb{N}}$ (resp. $(m_t)_{\mathbb{R}_+}$) is uniformly bounded with respect to the $\| \ \|_\infty$-norm: for any $N \in \mathbb{N}$ and almost all $\omega \in \Omega$,

$$|m_N(\omega)| \le \min\{|m_N(\omega)| + 1, |1 - m_N(\omega)| + 1\} \le \beta + 1.$$

Similar arguments hold for $(m_t)_{t\in\mathbb{R}_+}$. This implies that the operators $(T_{m_N})_{N\in\mathbb{N}}$ and $(T_{m_t})_{t\in\mathbb{R}_+}$ are uniformly bounded on $L_2(\mathcal{M})$. In this section we will always assume that *the operators* $(T_{m_N})_{N\in\mathbb{N}}$ *and* $(T_{m_t})_{t\in\mathbb{R}_+}$ *extend to uniformly bounded maps on* $\mathcal{M}$ and for notational convenience we set

$$\gamma = \|T_{m_N} : \mathcal{M} \to \mathcal{M}\|, \ (\text{resp. } \gamma = \|T_{m_t} : \mathcal{M} \to \mathcal{M}\|).$$

Then by complex interpolation they also extend to uniformly bounded maps on $L_p(\mathcal{M})$ for all $2 \le p \le \infty$. In this setting we have the following result. A more precise estimate on the endpoint case $p = 2$ can be found in Subsection 4.1.

Theorem 4.2. *Let* $(T_{m_N})_{N\in\mathbb{N}}$ *and* $(T_{m_t})_{t\in\mathbb{R}_+}$ *be the uniformly bounded maps on* $\mathcal{M}$ *given above.*

(1) *If* $(m_N)_N$ *satisfies* **(A1)**, *then for any* $2 \le p < \infty$ *there exists a constant* $c > 0$ *depending only on* p, α, β *and* γ, *such that for all* $x \in L_p(\mathcal{M})$,

$$\|(T_{m_N} x)_N\|_{L_p(\mathcal{M};\ell_\infty)} \le c\|x\|_p, \qquad \text{and} \qquad T_{m_N} x \to x \text{ a.u. as } N \to \infty.$$

(2) *If* $(m_t)_t$ *satisfies* **(A2)**, *then for any* $2 \le p < \infty$ *there exists a constant* $c > 0$ *depending only on* p, α, β *and* γ, *such that for all* $x \in L_p(\mathcal{M})$,

$$\|(T_{m_t}x)_t\|_{L_p(\mathcal{M};\ell_\infty)} \le c\|x\|_p, \qquad \text{and} \qquad T_{m_t}x \to x \text{ a.u. as } t \to \infty.$$

In order to obtain similar results for general $p > 1$, we need to assume the positivity of the maps $(T_{m_N})_{N\in\mathbb{N}}$ and $(T_{m_t})_{t\in\mathbb{R}_+}$. Note that if the maps extend to positive and symmetric contractions on $\mathcal{M}$, then by the argument before Lemma 3.4, they also extend to contractions on $L_p(\mathcal{M})$ for all $1 \le p \le \infty$. In this framework we have the following results.

Theorem 4.3. *Assume that the operators* $(T_{m_N})_{N\in\mathbb{N}}$ *and* $(T_{m_t})_{t\in\mathbb{R}_+}$ *extend to positive and symmetric contractions on* $\mathcal{M}$. *Assume additionally that for all* $t \in \mathbb{R}_+$, *the operator* S_t *satisfies Rota's dialtion property.*

(1) *If* $(m_N)_N$ *satisfies* **(A1)**, *then for any* $1 < p < \infty$ *there is a constant* $c > 0$ *depending only on* p, α, β *such that for all* $x \in L_p(\mathcal{M})$,

$$\|(T_{m_N}x)_N\|_{L_p(\mathcal{M};\ell_\infty)} \le c\|x\|_p \qquad \text{and} \qquad T_{m_N}x \to x \text{ a.u. as } N \to \infty.$$

(2) *If* $(m_t)_t$ *satisfies* **(A2)**, *then for any* $1+\frac{1}{2\eta} < p < \infty$ *there is a constant* $c > 0$ *depending only on* p, α, β, η *such that for all* $x \in L_p(\mathcal{M})$

$$\|(T_{m_t}x)_t\|_{L_p(\mathcal{M};\ell_\infty)} \le c\|x\|_p \qquad \text{and} \qquad T_{m_t}x \to x \text{ a.u. as } t \to \infty.$$

The above theorems recover Theorem 1.1 and Theorem 1.2. Indeed, if $\mathcal{M}$ is a finite von Neumann algebra, the additional assumption on Rota's dilation property is fulfilled by Lemma 3.6 (1). Note that for any positive definite function m on Γ, the associated map T_m on $VN(\Gamma)$ is completely positive (see e.g. [BO08, Theorem 2.5.11]). Also, by the Schoenberg theorem, for any conditionally negative definite function $\ell : \Gamma \to [0, \infty)$, the associated map $\lambda(g) \mapsto e^{-t\ell(g)}\lambda(g)$ forms a semigroup of unital completely positive trace preserving and symmetric maps on $VN(\Gamma)$. On the other hand, for any function $m : \Gamma \to \mathbb{C}$ with $m(e) = 1$, the map T_m is τ-preserving; if T_m is unital positive on $VN(\Gamma)$, then it extends to positive contractions to $L_p(VN(\Gamma))$ for all $1 \le p \le \infty$ (see e.g. [JX07, Lemma 1.1]). Moreover, if m is real-valued, then one may easily check that T_m is a symmetric map. So the assumptions of Theorem 1.1 coincide with those of the above theorems.

Before starting the proof, we give several remarks on the statement of the above theorems.

Remark 4.4. Instead of continuous families $(m_t)_{t\in\mathbb{R}_+}$ in **(A2)**, we may also consider maximal inequalities of families $(m_N)_{N\in\mathbb{N}}$ with suitable conditions on their differences, which we will frequently use in further discussions. Let $(m_N)_{N\in\mathbb{N}}$ be a family of measurable functions on Ω satisfying the following assumption: there exist $\alpha > 0$ and $\beta > 0$ such that for almost all $\omega \in \Omega$ we have

$$|1 - m_N(\omega)| \le \beta\frac{\ell(\omega)^\alpha}{N}, \quad |m_N(\omega)| \le \beta\frac{N}{\ell(\omega)^\alpha}, \quad |m_{N+1}(\omega) - m_N(\omega)| \le \beta\frac{1}{N}. \tag{4.4}$$

Then for any $2 \le p < \infty$, there exists a constant $c > 0$ depending only on p, α, β and γ, such that for all $x \in L_p(\mathcal{M})$, we have

$$\|(T_{m_N}x)_N\|_{L_p(\mathcal{M};\ell_\infty)} \le c\|x\|_p, \qquad \text{and} \qquad T_{m_N}x \to x \text{ a.u. as } N \to \infty.$$

If moreover the operators $(T_{m_N})_{N\in\mathbb{N}}$ extend to positive and symmetric contractions on $\mathcal{M}$, then the assertion holds for all $3/2 < p < \infty$ as well.

This follows immediately from the previous theorems since (4.4) leads to a special case of **(A2)**. Indeed, for $0 \le t < 1$, set $m_t = m_0 = 0$. For $t \ge 1$, we write $t = N_t + r_t$ with $N_t \in \mathbb{N}$ and $0 \le r_t < 1$, and we define

$$m_t = (1 - r_t)m_{N_t} + r_t m_{N_t+1}.$$

It is obvious that $(m_t)_{t\in\mathbb{R}_+}$ satisfies **(A2)** with $\eta = 1$.

One may also study more general conditions associated with higher order differences, which might be parallel to the case $\eta > 1$ in **(A2)**. However the computation seems to be much more intricate and we would like to leave it to the reader.

Remark 4.5. The statement in **(A1)** and **(A2)** can be flexibly adjusted, which we will frequently use in further discussions:

(1) For $\alpha \ge 1$ and for the maps $(T_{m_N})_{N\in\mathbb{N}}$ and $(T_{m_t})_{t\in\mathbb{R}_+}$ given in Theorem 4.2 or Theorem 4.3, we will indeed establish the corresponding maximal inequalities under the following weaker conditions (4.5) or (4.6). Indeed, for $\alpha \ge 1$, **(A1)** implies that for almost all $\omega \in \Omega$ we have

$$|1 - m_N(\omega)| \lesssim_\beta \frac{\ell(\omega)}{2^{N/\alpha}}, \quad |m_N(\omega)| \lesssim_\beta \frac{2^{N/\alpha}}{\ell(\omega)}. \tag{4.5}$$

To see this, recall that we have $|m_N(\omega)| \le \beta + 1$, so we see that

$$(\beta+1)^{-1}|m_N(\omega)| \le ((\beta+1)^{-1}|m_N(\omega)|)^{\frac{1}{\alpha}} \le (\beta+1)^{-\frac{1}{\alpha}}\beta^{\frac{1}{\alpha}}\frac{2^{N/\alpha}}{\ell(\omega)}.$$

Similarly, using $|1 - m_N(\omega)| \le \beta + 1$ and repeating the above argument, we see that

$$|1 - m_N(\omega)| \lesssim_\beta \frac{\ell(\omega)}{2^{N/\alpha}}.$$

In the same way, **(A2)** implies

$$|1 - m_t(\omega)| \lesssim_{\beta,\gamma} \frac{\ell(\omega)}{t^{1/\alpha}}, \quad |m_t(\omega)| \lesssim_{\beta,\gamma} \frac{t^{1/\alpha}}{\ell(\omega)}, \quad \left|\frac{d^k m_t(\omega)}{dt^k}\right| \le \beta\frac{1}{t^k}. \tag{4.6}$$

On the other hand, the proof of the above theorems for the case of $0 < \alpha < 1$ can be always reduced to that of $\alpha \ge 1$. To see this it suffices to take $\tilde{\ell} = \ell^\alpha$ for $0 < \alpha < 1$ and consider the new semigroup of unital completely positive trace preserving and symmetric maps given by $\tilde{S}_t := T_{e^{-t\tilde{\ell}}}$ (see [Yos95]); if the multipliers satisfy **(A1)** or **(A2)** with respect to ℓ for $0 < \alpha < 1$, then they also satisfy the same condition with respect to $\tilde{\ell}$ for $\alpha = 1$.

(2) Theorem 4.2 (1) and Theorem 4.3 (1) still hold with the index set $\mathbb{N}$ replaced by $\mathbb{Z}$ in **(A1)**. This can be seen from their proofs; alternatively, we may deduce this easily from a standard re-indexation argument. Indeed, let $(m_N)_{N\in\mathbb{Z}}$ be a sequence of measurable functions on Ω satisfying (4.2) for all $N \in \mathbb{Z}$. Take $N_0 \in \mathbb{Z}$ and write $\widetilde{m}_j = m_{N_0+j}$ for $j \in \mathbb{N}$. Then (4.2) implies that for all $j \in \mathbb{N}$ and almost all $\omega \in \Omega$,

$$|1 - \widetilde{m}_j(\omega)| \le \beta\frac{2^{N_0}\ell(\omega)^\alpha}{2^j}, \quad |\widetilde{m}_j(\omega)| \le \beta\frac{2^j}{2^{N_0}\ell(\omega)^\alpha}.$$

Note that $\tilde{\ell} = 2^{N_0/\alpha}\ell$ yields again a semigroup of unital completely positive trace preserving and symmetric maps $\tilde{S}_t^{(N_0)} := T_{e^{-t\tilde{\ell}}}$. Then applying Theorem 4.2 (1) or Theorem 4.3 (1) to $\widetilde{m}$ and $\tilde{\ell}$, we see that $(\widetilde{m}_j)_{j\in\mathbb{N}} = (m_N)_{N\ge N_0}$ satisfies the corresponding maximal inequality with

constant independent of $\widetilde{\ell}$ and N_0. Thus the similar maximal inequalities and a.u. convergence still hold for $(m_N)_{N\in\mathbb{Z}}$.

Remark 4.6. The completely bounded version of the above two theorems holds true as well. In other words, if $\mathcal{N}$ is another semifinite von Neumann algebra and if we replace T_{m_N} by $T_{m_N}\otimes \mathrm{Id}_{\mathcal{N}}$, T_{m_t} by $T_{m_t}\otimes \mathrm{Id}_{\mathcal{N}}$ and $\mathcal{M}$ by $\tilde{\mathcal{M}}=\mathcal{M}\overline{\otimes}\mathcal{N}$, then the above two theorems still hold true. Indeed, it suffices to consider a larger Hilbert space $\tilde{H}=H\otimes L_2(\mathcal{N})$ and apply the above theorems to $\tilde{M}$ and $\tilde{H}$.

The following result on mean convergences is an easy consequence of our assumptions.

Proposition 4.7. *Let $(T_{m_N})_N$, $(T_{m_t})_t$ and $1<p<\infty$ be given as in Theorem 4.2 or Theorem 4.3 which satisfy* **(A1)** *or* **(A2)** *correspondingly.*

(1) *The family $(T_{m_t})_t$ is strongly continuous on $L_p(\mathcal{M})$, i.e., for any $x\in L_p(\mathcal{M})$ the function $t\mapsto T_{m_t}x$ is continuous from $\mathbb{R}_+$ to $L_p(\mathcal{M})$.*

(2) *We have*

$$\lim_{N\to\infty}\|T_{m_N}x-x\|_p=0,\quad \lim_{t\to\infty}\|T_{m_t}x-x\|_p=0,\quad x\in L_p(\mathcal{M}).$$

Proof. Let $x\in U^{-1}(C_c(\Omega;H))$ and $E=\mathrm{supp}(U(x))\subset\Omega$. By the Hölder inequality, for any $t_0\geq 0$ and $2\leq p<\infty$,

$$\begin{aligned}\|T_{m_t}x-T_{m_{t_0}}x\|_p &\leq \|T_{m_t}x-T_{m_{t_0}}x\|_2^{2/p}\|T_{m_t}x-T_{m_{t_0}}x\|_\infty^{1-2/p}\\ &\lesssim \|(m_t-m_{t_0})\mathbb{1}_E\|_\infty^{2/p}\|x\|_2^{2/p}\|x\|_\infty^{1-2/p}.\end{aligned}$$

By the continuity of m_t and the compactness of E, the above quantity tends to 0 as $t\to t_0$. Similar arguments work for $p<2$ by using the Hölder inequality with endpoints $p=1,2$. Similarly, by the continuity of ℓ and the compactness of E, we have

$$\lim_{N\to\infty}\|T_{m_N}x-x\|_p=0,\quad \lim_{t\to\infty}\|T_{m_t}x-x\|_p=0.$$

For general elements $x\in L_p(\mathcal{M})$, it suffices to note that the operators $(T_{m_N})_N$ and $(T_{m_t})_t$ extend to uniformly bounded operators on $L_p(\mathcal{M})$. Thus the desired results follow from a standard density argument. □

Now we are ready to proceed with the proof of the previous theorems.

4.1. L_2-estimates under lacunary conditions.

Proposition 4.8. *Let $(m_N)_{N\in\mathbb{Z}}\subset L_\infty(\Omega)$. Assume that there exist a function $f:\Omega\to[0,\infty)$ and a positive number $a>1$ such that for almost all $\omega\in\Omega$,*

$$|m_N(\omega)|\leq \beta\frac{a^N f(\omega)}{(a^N+f(\omega))^2}. \tag{4.7}$$

Then,

$$\|(T_{m_N}x)_{N\in\mathbb{Z}}\|_{L_2(\mathcal{M};\ell_2^{c,r})}\lesssim \beta\sqrt{\frac{a^2}{a^2-1}}\|x\|_2,\qquad x\in L_2(\mathcal{M}).$$

Proof. We have

$$\begin{aligned}\|(T_{m_N}x)_N\|^2_{L_2(\mathcal{M};\ell_2^{cr})} =&\tau\left(\sum_{N\in\mathbb{Z}}|T_{m_N}x|^2\right)\\ =&\sum_{N\in\mathbb{Z}}\|T_{m_N}x\|^2_{L_2(\mathcal{M})}=\sum_{N\in\mathbb{Z}}\|m_NU(x)\|^2_{L_2(\Omega;H)}\\ =&\int_\Omega\sum_{N\in\mathbb{Z}}\|m_N(\omega)(Ux)(\omega)\|^2_H d\mu(\omega)\\ \le&\|\sum_{N\in\mathbb{Z}}|m_N|^2\|_{L_\infty(\Omega)}\|Ux\|^2_{L_2(\Omega;H)}=\|\sum_{N\in\mathbb{Z}}|m_N|^2\|_{L_\infty(\Omega)}\|x\|^2_{L_2(\mathcal{M})}.\end{aligned}$$

However, by (4.7) we see that for almost all $\omega\in\Omega$ with $f(\omega)>0$,

$$\sum_{N\in\mathbb{Z}}|m_N(\omega)|^2\le\sum_{N<\log_a f(\omega)}\beta^2\frac{a^{2N}}{f(\omega)^2}+\sum_{N\ge\log_a f(\omega)}\beta^2\frac{f(\omega)^2}{a^{2N}}\lesssim\beta^2\frac{a^2}{a^2-1},$$

while $m_N(\omega)=0$ if $f(\omega)=0$ by (4.7). Thus we obtain the desired inequality. □

Below we show a more precise L_2-estimate.

Lemma 4.9. *Assume that $t\mapsto m_t(\omega)$ is differentiable for almost all $\omega\in\Omega$. Choose an arbitrary measurable function $f:\Omega\to[0,\infty)$. For $j\in\mathbb{Z}$, define*

$$a_j=\sup_t\left\{\sup_{2^{j-2}<\frac{f(\omega)}{t}\le 2^j}|m_t(\omega)|\right\},\qquad b_j=\sup_t\left\{\sup_{2^{j-2}<\frac{f(\omega)}{t}\le 2^j}t\cdot\left|\frac{\partial m_t(\omega)}{\partial t}\right|\right\}.$$

Assume

$$K=\sum_{j\in\mathbb{Z}}a_j^{1/2}(a_j^{1/2}+b_j^{1/2})<\infty.$$

Then for $x\in L_2(\mathcal{M})$, we have the following maximal inequalities

$$\|(T_{m_t}x)_t\|_{L_2(\mathcal{M};\ell_\infty)}\lesssim K\|x\|_2\quad\text{and}\quad\|(T_{m_t}x)_t\|_{L_2(\mathcal{M};\ell_\infty^c)}\lesssim K\|x\|_2.$$

Proof. We prove the second assertion first. Let $\{\eta_j\}_{j\in\mathbb{Z}}$ be a partition of unity of $\mathbb{R}_+$ satisfying

$$\sum_j\eta_j=1,\qquad\operatorname{supp}\eta_j\subset[2^{j-2},2^j],\qquad 0\le\eta_j\le 1\quad\text{and}\quad|\eta_j'|<C2^{-j}.$$

Define $m_{t,j}(\omega)=m_t(\omega)\eta_j(\frac{f(\omega)}{t})\in L_\infty(\Omega)$. For notational simplicity, denote by $T_{t,j}$ the operators with symbols $m_{t,j}$; that is,

$$U(T_{t,j}x)=m_{t,j}U(x),\quad x\in L_2(\mathcal{M}).$$

Then we have

$$\|(T_{m_t}x)_t\|_{L_2(\mathcal{M};\ell_\infty^c)}=\left\|\left(\sum_{j\in\mathbb{Z}}T_{t,j}x\right)_t\right\|_{L_2(\mathcal{M};\ell_\infty^c)}\le\sum_{j\in\mathbb{Z}}\|(T_{t,j}x)_t\|_{L_2(\mathcal{M};\ell_\infty^c)}.\tag{4.8}$$

From now on we fix an arbitrary $j\in\mathbb{Z}$. In the sequel of this proof we denote

$$U_k(x)(\omega)=U(x)(\omega)\cdot\mathbb{1}_{[2^{k-2},2^{k+1}]}(f(\omega)),\quad\omega\in\Omega\quad\text{and}\quad x_k=U^*(U_k(x)),\quad x\in L_2(\mathcal{M}).$$

Since $\operatorname{supp}\eta_j \subset [2^{j-2}, 2^j]$, for $v \in \mathbb{Z}$ and $t \in [2^v, 2^{v+1})$ we have

$$m_{t,j}(\omega)U(x)(\omega) = m_t(\omega)\eta_j(\frac{f(\omega)}{t})U(x)(\omega) = m_t(\omega)\eta_j(\frac{f(\omega)}{t})U_{v+j}(x)(\omega).$$

We may rewrite the above equality as

$$T_{t,j}x = T_{t,j}x_{v+j}, \qquad t \in [2^v, 2^{v+1}).$$

Choose an integer A_j such that $\frac{a_j+b_j}{a_j} \leq A_j \leq \frac{2(a_j+b_j)}{a_j}$ and we divide the interval $[2^v, 2^{v+1}]$ into A_j parts:

$$2^v = \gamma_0 < \gamma_1 < \gamma_2 \cdots < \gamma_{A_j} = 2^{v+1} \quad \text{with} \quad \gamma_{k+1} - \gamma_k = 2^v \cdot A_j^{-1}.$$

For any $t \in [2^v, 2^{v+1})$, there exists $0 \leq k(t) \leq A_j - 1$ such that $t \in [\gamma_{k(t)}, \gamma_{k(t)+1})$. By the convexity of the operator square function, we have

$$\begin{aligned}
|T_{t,j}x_{v+j}|^2 &= |U^*(m_{t,j}U_{v+j}(x))|^2 \\
&= \left| U^* \left(\int_{\gamma_{k(t)}}^t \left(\frac{\partial m_{s,j}}{\partial s} \right) U_{v+j}(x) ds + m_{\gamma_{k(t)},j} \cdot U_{v+j}(x) \right) \right|^2 \\
&\leq 2(t - \gamma_{k(t)}) \int_{\gamma_{k(t)}}^t \left| \frac{\partial T_{s,j}(x_{v+j})}{\partial s} \right|^2 ds + 2|T_{\gamma_{k(t)},j}(x_{v+j})|^2 \\
&\leq 2 \left(\frac{2^v}{A_j} \int_{\gamma_{k(t)}}^{\gamma_{k(t)+1}} \left| \frac{\partial T_{s,j}(x_{v+j})}{\partial s} \right|^2 ds + |T_{\gamma_{k(t)},j}(x_{v+j})|^2 \right) \\
&\leq 2 \sum_{k=0}^{A_j-1} \left(\frac{2^v}{A_j} \int_{\gamma_k}^{\gamma_{k+1}} \left| \frac{\partial T_{s,j}(x_{v+j})}{\partial s} \right|^2 ds + |T_{\gamma_k,j}(x_{v+j})|^2 \right) \\
&= \frac{2^{v+1}}{A_j} \int_{2^v}^{2^{v+1}} \left| \frac{\partial T_{s,j}(x_{v+j})}{\partial s} \right|^2 ds + 2 \sum_{k=0}^{A_j-1} |T_{\gamma_k,j}(x_{v+j})|^2.
\end{aligned}$$

We denote

$$y_v = \frac{2^{v+1}}{A_j} \int_{2^v}^{2^{v+1}} \left| \frac{\partial T_{s,j}(x_{v+j})}{\partial s} \right|^2 ds + 2 \sum_{k=0}^{A_j-1} |T_{\gamma_k,j}(x_{v+j})|^2.$$

Then

$$|T_{t,j}x_{v+j}|^2 \leq y_v. \tag{4.9}$$

Similarly, we have

$$|(T_{t,j}x_{v+j})^*|^2 \leq y_v' = \frac{2^{v+1}}{A_j} \int_{2^v}^{2^{v+1}} \left| \left(\frac{\partial T_{s,j}(x_{v+j})}{\partial s} \right)^* \right|^2 ds + 2 \sum_{k=0}^{A_j-1} |T_{\gamma_k,j}(x_{v+j})^*|^2.$$

Let us estimate the quantities $\|y_v\|_1$ and $\|y_v'\|_1$. We have

$$\|y_v\|_1 = \tau(y_v) = \frac{2^{v+1}}{A_j} \int_{2^v}^{2^{v+1}} \left\| \frac{\partial T_{s,j}(x_{v+j})}{\partial s} \right\|_2^2 ds + 2 \sum_{k=0}^{A_j-1} \|T_{\gamma_k,j}(x_{v+j})\|_2^2,$$

$$\|y_v'\|_1 = \tau(y_v') = \frac{2^{v+1}}{A_j}\int_{2^v}^{2^{v+1}} \left\|\left(\frac{\partial T_{s,j}(x_{v+j})}{\partial s}\right)^*\right\|_2^2 ds + 2\sum_{k=0}^{A_j-1}\|T_{\gamma_k,j}(x_{v+j})^*\|_2^2.$$

Hence, $\|y_v\|_1 = \|y_v'\|_1$. Note that

$$\begin{aligned}\int_{2^v}^{2^{v+1}} \left|\left(\frac{\partial m_{s,j}(\omega)}{\partial s}\right)\right|^2 ds &= \int_{2^v}^{2^{v+1}} \left|\frac{\partial}{\partial s}\left(m_s(\omega)\eta_j(\frac{f(\omega)}{s})\right)\right|^2 ds \\ &= \int_{2^v}^{2^{v+1}} \left|\frac{\partial m_s(\omega)}{\partial s}\cdot\eta_j\left(\frac{f(\omega)}{s}\right) - \eta_j'\left(\frac{f(\omega)}{s}\right)\frac{f(\omega)}{s^2}\cdot m_s(\omega)\right|^2 ds \\ &\lesssim \int_{2^v}^{2^{v+1}} \left(\left|\frac{b_j}{s}\right| + \left|\frac{a_j}{s}\right|\right)^2 ds \quad (\text{since } \operatorname{supp}\eta_j \subset [2^{j-2}, 2^j]) \\ &\lesssim \frac{1}{2^{v+1}}(b_j+a_j)^2.\end{aligned}$$

By the Fubini theorem, we have

$$\begin{aligned}\frac{2^{v+1}}{A_j}\int_{2^v}^{2^{v+1}} \left\|\frac{\partial T_{s,j}(x_{v+j})}{\partial s}\right\|_2^2 ds &= \frac{2^{v+1}}{A_j}\int_{2^v}^{2^{v+1}} \left\|\frac{\partial m_{s,j}}{\partial s}U_{v+j}(x)\right\|_{L_2(\Omega;H)}^2 ds \\ &= \frac{2^{v+1}}{A_j}\int_\Omega\left(\int_{2^v}^{2^{v+1}}\left|\left(\frac{\partial m_{s,j}(\omega)}{\partial s}\right)\right|^2 ds\right)|U_{v+j}(x)(\omega)|^2 d\mu(\omega) \\ &\lesssim \frac{(b_j+a_j)^2}{A_j}\|x_{v+j}\|_2^2,\end{aligned}$$

and

$$\begin{aligned}2\sum_{k=0}^{A_j-1}\|T_{\gamma_k,j}(x_{v+j})\|_2^2 &= \sum_{k=0}^{A_j-1}\int_\Omega |m_{\gamma_k}(\omega)|^2\left|\eta_j\left(\frac{f(\omega)}{\gamma_k}\right)\right|^2 |U_{v+j}(x)(\omega)|^2 d\mu(\omega) \\ &\le 2A_j a_j^2\|x_{v+j}\|_2^2.\end{aligned}$$

Therefore,

$$\|y_v'\|_1 = \|y_v\|_1 \lesssim \left(\frac{(b_j+a_j)^2}{A_j} + A_j a_j^2\right)\|x_{v+j}\|_2^2.$$

Recall that (4.9) asserts that $|T_{t,j}x_{v+j}|^2 \le y_v \le \sum_{u\in\mathbb{Z}} y_u$. So

$$\begin{aligned}\|(T_{t,j}x)_t\|_{L_2(\mathcal{M};\ell_\infty^c)} = \|(|T_{t,j}x_{v+j}|^2)_t\|_{L_1(\mathcal{M};\ell_\infty)}^{1/2} &\le \left\|\sum_{u\in\mathbb{Z}} y_u\right\|_1^{1/2} \le \left(\sum_{u\in\mathbb{Z}}\|y_u\|_1\right)^{1/2} \\ &\lesssim \left(\sum_{u\in\mathbb{Z}}\left(\frac{(b_j+a_j)^2}{A_j} + A_j a_j^2\right)\|x_{u+j}\|_2^2\right)^{\frac{1}{2}}.\end{aligned} \tag{4.10}$$

Note that

$$[2^{j+u-2}, 2^{j+u+1}] = \cup_{l=0}^2 [2^{j+u+l-2}, 2^{j+u+l-1}].$$

We have

$$\begin{aligned}\sum_{u\in\mathbb{Z}}\|x_{u+j}\|_2^2 &= \sum_{u\in\mathbb{Z}}\int_\Omega \left|U(x)(\omega)\cdot \mathbb{1}_{[2^{j+u-2},2^{j+u-2}]}(f(\omega))\right|^2 d\mu(\omega)\\ &\le \sum_{l=0}^{2}\int_\Omega \left|U(x)(\omega)\cdot \sum_{u\in\mathbb{Z}}\mathbb{1}_{[2^{j+u+l-2},2^{j+u+l-1}]}(f(\omega))\right|^2 d\mu(\omega)\\ &= \sum_{l=0}^{2}\|U(x)\|_2^2\end{aligned}$$

where the last equality holds since

$$\sum_{u\in\mathbb{Z}}\mathbb{1}_{[2^{j+u+l-2},2^{j+u+l-1}]}(f(\omega)) = 1.$$

Thus,

$$\sum_{u\in\mathbb{Z}}\|x_{u+j}\|_2^2 \le 3\|x\|_2^2.$$

Recall that $\frac{a_j+b_j}{a_j} \le A_j \le \frac{2(a_j+b_j)}{a_j}$. Together with (4.10), we have

$$\|(T_{t,j}x)_t\|_{L_2(\mathcal{M};\ell_\infty^c)} \lesssim a_j^{1/2}(a_j+b_j)^{1/2}\left(\sum_{u\in\mathbb{Z}}\|x_{u+j}\|_2^2\right)^{1/2} \lesssim a_j^{1/2}(a_j+b_j)^{1/2}\|x\|_2.$$

By (4.8), the proof is complete for the second maximal inequality.

Similarly, we have

$$\|(T_{m_t}x)_t\|_{L_2(\mathcal{M};\ell_\infty^r)} \lesssim K\|x\|_2.$$

Let us recall Lemma 2.1 which shows that the space $L_2(\mathcal{M};\ell_\infty)$ is the complex interpolation space of $L_2(\mathcal{M};\ell_\infty^c)$ and $L_2(\mathcal{M};\ell_\infty^r)$. Therefore we have

$$\begin{aligned}&\|(T_{m_t}) : L_2(\mathcal{M}) \to L_2(\mathcal{M};\ell_\infty)\|\\ &\le \|(T_{m_t}) : L_2(\mathcal{M}) \to L_2(\mathcal{M};\ell_\infty^c)\|^{1/2}\|(T_{m_t}) : L_2(\mathcal{M}) \to L_2(\mathcal{M};\ell_\infty^r)\|^{1/2}\\ &\lesssim K.\end{aligned}$$

□

4.2. **Proof of Theorem 4.2.** Now we are ready to conclude Theorem 4.2.

First assume that $(m_N)_{N\in\mathbb{N}}$ satisfies **(A1)**. Set $T_{\phi_N} = T_{m_N} - P_{2^{-N/\alpha}}$ with the symbol $\phi_N = m_N - e^{-\frac{\sqrt{\ell}}{2^{N/2\alpha}}}$.By Remark 4.5 (1) and **(A1)**, we can easily see that

$$|\phi_N(\omega)| \le |1-m_N(\omega)| + |1-e^{-\frac{\sqrt{\ell(\omega)}}{2^{N/2\alpha}}}| \lesssim_\beta \frac{\sqrt{\ell(\omega)}}{2^{N/2\alpha}}, \tag{4.11}$$

$$|\phi_N(\omega)| \le |m_N(\omega)| + |e^{-\frac{\sqrt{\ell(\omega)}}{2^{N/2\alpha}}}| \lesssim_\beta \frac{2^{N/2\alpha}}{\sqrt{\ell(\omega)}}, \tag{4.12}$$

Therefore, $|\phi_N(\omega)| \le \frac{2^{N/2\alpha}\sqrt{\ell(\omega)}}{\left(2^{N/2\alpha}+\sqrt{\ell(\omega)}\right)^2}$. By Proposition 3.3, Proposition 4.8 and Proposition 2.2 (3), we get for any $2 \le p < \infty$ and $x \in L_p(\mathcal{M})$,

$$\|(T_{\phi_N}(x))_N\|_{L_p(\mathcal{M};\ell_\infty)} \lesssim_{\alpha,\beta,\gamma,p} \|x\|_p, \quad \text{and} \quad \|(T_{\phi_N}(x))_N\|_{L_p(\mathcal{M};\ell_\infty^c)} \lesssim_{\alpha,\beta,\gamma,p} \|x\|_p.$$

Applying Proposition 2.10, we also get the strong type (p,p) estimate for T_{m_N} with a constant c depending only on α,β,γ,p.

Assume that $(m_t)_{t\in\mathbb{R}_+}$ satisfies **(A2)**. Again set $T_{\phi_t}=T_{m_t}-P_{t^{-1/\alpha}}$ with $\phi_t=m_t-e^{-\frac{\sqrt{\ell}}{t^{1/2\alpha}}}$. We have the following estimates similar to (4.11) and (4.12): for almost all $\omega\in\Omega$

$$|\phi_t(\omega)|\lesssim_\beta \min\left\{\left(\frac{\ell(\omega)^\alpha}{t}\right)^{1/2\alpha},\left(\frac{t}{\ell(\omega)^\alpha}\right)^{1/2\alpha}\right\},$$

$$|\frac{\partial\phi_t(\omega)}{\partial t}|\le\beta\frac{1}{t}+\frac{1}{t}\cdot(\frac{\sqrt{\ell(\omega)}}{\alpha t^{1/2\alpha}}e^{-\frac{\sqrt{\ell(\omega)}}{t^{1/2\alpha}}})\lesssim_{\alpha,\beta}\frac{1}{t}.$$

Applying Lemma 4.9, we get

$$\|\sup_t{}^+T_{\phi_t}x\|_2\le K\|x\|_2\qquad x\in L_2(\mathcal{M}),$$

where

$$K\lesssim_{\alpha,\beta}\sum_{j\in\mathbb{Z}}(2^{-|j|/2\alpha}(2^{-|j|/2\alpha}+1))^{1/2}\lesssim_{\alpha,\beta}1.$$

By Proposition 2.2 (3), for any $2\le p<\infty$, $(T_{\phi_t})_{t\in\mathbb{R}_+}$ is of strong type (p,p) with constant c depending only on α,β,γ,p. Similarly, for any $2\le p<\infty$, we have

$$\|(T_{\phi_t}(x))_{t\in\mathbb{R}_+}\|_{L_p(\mathcal{M};\ell_\infty^c)}\lesssim_{\alpha,\beta,\gamma,p}\|x\|_p\qquad x\in L_p(\mathcal{M}).\tag{4.13}$$

Therefore, we conclude the strong type (p,p) estimate for $(T_{m_t})_t$ thanks to Proposition 2.10.

Now the desired a.u. convergence follows immediately from the above maximal inequalities by an argument in [HLW20]. For instance, we consider the symbols $(m_t)_{t\in\mathbb{R}_+}$ satisfying **(A2)**. Let $x\in U^{-1}(C_c(\Omega;H))$ and set $E=\operatorname{supp}U(x)$. Note that E is a compact set. We consider the maps $T_{\psi_t}=T_{m_t}-\mathrm{id}$ with $\psi_t=m_t-1$. As the proof of Proposition 4.7, by **(A2)**, we have

$$\|T_{\psi_t}x\|_p\le 2\gamma^{1-(2/p)}\|(m_t-1)\mathbb{1}_E\|_{L_\infty(E)}^{2/p}\|x\|_2^{2/p}\|x\|_\infty^{1-2/p}\lesssim_{\alpha,\gamma}\frac{\|\ell^\alpha\mathbb{1}_E\|_{L_\infty(E)}^{2/p}}{t^{2/p}}\|x\|_2^{2/p}\|x\|_\infty^{1-2/p}.$$

By the continuity of ℓ and the compactnesss of E, we have

$$\begin{aligned}\lim_{M\to\infty}\int_M^\infty\|T_{\psi_t}x\|_p^p dt&\lesssim_{\alpha,\gamma}\lim_{M\to\infty}\int_M^\infty\frac{\|\ell^\alpha\mathbb{1}_E\|_{L_\infty(E)}^2}{t^2}\|x\|_2^2\|x\|_\infty^{p-2}dt\\&\lesssim_{\alpha,\gamma}\lim_{M\to\infty}\frac{\|\ell^\alpha\mathbb{1}_E\|_{L_\infty(E)}^2\|x\|_2^2\|x\|_\infty^{p-2}}{M}=0.\end{aligned}\tag{4.14}$$

Thus, as M tends to ∞,

$$\begin{aligned}\|(T_{\psi_t}x)_{t\ge M}\|_{L_p(\mathcal{M};\ell_\infty^c)}^p&\le\|(|T_{\psi_t}x|^2)_{t\ge M}\|_{L_{p/2}(\mathcal{M};\ell_\infty)}^{p/2}\\&\le\Big\|\Big(\int_{t\ge M}|T_{\psi_t}x|^p dt\Big)^{\frac{2}{p}}\Big\|_{p/2}^{p/2}\le\int_M^\infty\|T_{\psi_t}x\|_p^p dt\to 0.\end{aligned}$$

As a result, for any $x\in U^{-1}(C_c(\Omega;H))$, $T_{\psi_t}(x)$ converges a.u. to 0 as $t\to\infty$ according to [JX07, Lemma 6.2]. Moreover, (4.13) obviously yields that $(T_{\phi_t})_{t\in\mathbb{R}_+}$ satisfies the one-sided weak type (p,p) maximal inequality for $p\ge 2$ as in Proposition 2.9 (2). Note that the

subordinate Poisson semigroup $(P_t)_t$ is of weak type (p,p) for all $p \geq 1$ by [JX07, Remark 4.7]. Hence for any $p \geq 2$, $x \in L_p(\mathcal{M})$, we find a projection $e \in \mathcal{M}$ such that

$$\sup_t \|eP_t(x^*x)e\|_\infty \leq \alpha^2 \quad \text{and} \quad \tau(e^\perp) \lesssim_p \alpha^{-p}\|x^*x\|_{p/2}^{p/2} = \alpha^{-p}\|x\|_p^p.$$

However, since P_t is completely positive, we can use Cauchy-Schwarz inequality to get

$$\sup_t \|P_t(x)e\|_\infty = \|eP_t(x)^*P_t(x)e\|_\infty^{1/2} \leq \sup_t \|eP_t(x^*x)e\|_\infty^{1/2} \leq \alpha.$$

So $(P_t)_t$ also satisfies the one-sided weak type (p,p) maximal inequality for $p \geq 2$. Thus $(T_{m_t})_t$ also satisfies the same inequality and by Proposition 2.9 (2), we see that $T_{\psi_t}(x) = T_{m_t}(x) - x$ converges a.u. to 0 as $t \to \infty$ for all $x \in L_p(\mathcal{M})$. The case of **(A1)** can be dealt with similarly. Thus the proof of Theorem 4.2 is complete.

4.3. **Proof of Theorem 4.3 (1).** In this subsection we study the maximal inequalitities for $1 < p < 2$ in Theorem 4.3 (1). To approach this we need to develop several interpolation methods.

Lemma 4.10. *Let $(\Phi_j)_{j\in\mathbb{Z}}$ be a sequence of uniformly bounded linear maps on $\mathcal{M}$ and write $\gamma = \sup_j \|\Phi_j : \mathcal{M} \to \mathcal{M}\| < \infty$. Let $1 < p < q < 2$ and $\theta \in (0,1)$ be determined by $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$. Assume that there exist $c_1, c_2 > 0$ such that for any $s \in \mathbb{N}_+$, there is a decomposition of maps $\Phi_j = \Phi_j^{(s,1)} + \Phi_j^{(s,2)}$ where $(\Phi_j^{(s,1)})_j$ extends to a family of maps of weak type (p,p) with constant $C_p \leq s^\theta c_1$ and $(\Phi_j^{(s,2)})_j$ extends to a family of maps of weak type $(2,2)$ with constant $C_2 \leq s^{-(1-\theta)}c_2$. Then for any $x \in \mathcal{S}_\mathcal{M}$ and $\lambda > 0$, there exists a projection $e \in \mathcal{M}$ such that*

$$\sup_j \|e\Phi_j(x)e\|_\infty < \lambda \quad \textit{and} \quad \tau(e^\perp) \lesssim_\gamma (c_1^p + c_2^2)\left(\frac{\|x\|_p^{1-\theta}\|x\|_2^\theta}{\lambda}\right)^q.$$

In particular, $(\Phi_j)_{j\in\mathbb{Z}}$ is of restricted weak type (q,q) with constant

$$C_q \lesssim_\gamma (c_1^p + c_2^2)^{1/q}.$$

Proof. Let $x \in \mathcal{S}_{\mathcal{M}+}$. Consider a positive integer $s \in \mathbb{N}_+$. By the weak type estimates of $(\Phi_j^{(s,1)})_j$ and $(\Phi_j^{(s,2)})_j$, for any $\lambda > 0$, we take two projections $e_1, e_2 \in \mathcal{M}$, such that

$$\sup_j \|e_1\Phi_j^{(s,1)}(x)e_1\|_\infty \leq \lambda \quad \text{and} \quad \tau(e_1^\perp) \leq \left(c_1 s^\theta \frac{\|x\|_p}{\lambda}\right)^p,$$

$$\sup_j \|e_2\Phi_j^{(s,2)}(x)e_2\|_\infty \leq \lambda \quad \text{and} \quad \tau(e_2^\perp) \leq \left(c_2 s^{\theta-1} \frac{\|x\|_2}{\lambda}\right)^2.$$

Set $e = e_1 \wedge e_2$. Since $\Phi_j = \Phi_j^{(s,1)} + \Phi_j^{(s,2)}$ we have,

$$\|e\Phi_j(x)e\|_\infty \leq 2\lambda, \quad j \in \mathbb{Z},$$

and

$$\tau(e^\perp) \leq \tau(e_1^\perp + e_2^\perp) \leq \left(c_1 s^\theta \frac{\|x\|_p}{\lambda}\right)^p + \left(c_2 s^{\theta-1} \frac{\|x\|_2}{\lambda}\right)^2.$$

We consider $x_\lambda^\perp = x\mathbb{1}_{(\lambda,\infty)}(x)$ and $x_\lambda = x\mathbb{1}_{[0,\lambda]}(x)$. Applying the above arguments to $x_\lambda^\perp$, we can find a projection $e \in \mathcal{M}$ such that

$$\|e\Phi_j(x_\lambda^\perp)e\|_\infty \leq 2\lambda, \quad j \in \mathbb{Z}.$$

and

$$\tau(e^{\perp}) \leq \left(c_1 s^{\theta} \frac{\|x_\lambda^{\perp}\|_p}{\lambda}\right)^p + \left(c_2 s^{\theta-1} \frac{\|x_\lambda^{\perp}\|_2}{\lambda}\right)^2. \tag{4.15}$$

Since $x = x_\lambda + x_\lambda^{\perp}$ and $x_\lambda \leq \lambda \cdot 1$, for any $j \in \mathbb{Z}$, we have

$$\|e(\Phi_j x)e\|_\infty \leq \|e(\Phi_j(x_\lambda))e\|_\infty + \|e(\Phi_j(x_\lambda^{\perp}))e\|_\infty \leq (\gamma+2)\lambda.$$

Note that for $1 < p \leq 2$,

$$\big(t\mathbb{1}_{(\lambda,\infty)}(t)\big)^p = \frac{\big(t\mathbb{1}_{(\lambda,\infty)}(t)\big)^2}{t^{2-p}} \leq \frac{\big(t\mathbb{1}_{(\lambda,\infty)}(t)\big)^2}{\lambda^{2-p}}, \qquad t > 0.$$

Therefore we have $(x_\lambda^{\perp})^p \leq \frac{(x_\lambda^{\perp})^2}{\lambda^{2-p}}$ and

$$\frac{\|x_\lambda^{\perp}\|_p^p}{\lambda^p} \leq \frac{\|x_\lambda^{\perp}\|_2^2}{\lambda^2}.$$

Hence, we can choose s to be an integer satisfying

$$s \asymp \left(\frac{\lambda^p \|x_\lambda^{\perp}\|_2^2}{\lambda^2 \|x_\lambda^{\perp}\|_p^p}\right)^{\frac{q}{2p}},$$

and by (4.15) we have

$$\begin{aligned} \tau(e^{\perp}) &\lesssim \left(c_1 \left(\frac{\lambda^p \|x_\lambda^{\perp}\|_2^2}{\lambda^2 \|x_\lambda^{\perp}\|_p^p}\right)^{\frac{q-p}{p(2-p)}} \frac{\|x_\lambda^{\perp}\|_p}{\lambda}\right)^p + \left(c_2 \left(\frac{\lambda^p \|x_\lambda^{\perp}\|_2^2}{\lambda^2 \|x_\lambda^{\perp}\|_p^p}\right)^{\frac{q-2}{2(2-p)}} \left(\frac{\|x_\lambda^{\perp}\|_2}{\lambda}\right)\right)^2 \\ &\lesssim (c_1^p + c_2^2)\lambda^{-q} \|x_\lambda^{\perp}\|_p^{(1-\theta)q} \|x_\lambda^{\perp}\|_2^{\theta q}. \end{aligned} \tag{4.16}$$

Since $0 \leq x_\lambda^{\perp} \leq x$, the above inequality yields

$$\tau(e^{\perp}) \lesssim (c_1^p + c_2^2)\left(\frac{\|x\|_p^{1-\theta}\|x\|_2^{\theta}}{\lambda}\right)^q.$$

In order to obtain the restricted weak type (q,q) estimate, it suffices to take $x = f$ in the above inequality for an arbitrary projection $f \in \mathcal{S}_{\mathcal{M}+}$. Then

$$\tau(e^{\perp}) \lesssim_\gamma (c_1^p + c_2^2)\left(\frac{\tau(f^p)^{(1-\theta)/p}\tau(f^2)^{\theta/2}}{\lambda}\right)^q = (c_1^p + c_2^2)\lambda^{-q}\tau(f),$$

which implies that $(\Phi_j)_{j\in\mathbb{Z}}$ is of restricted weak type (q,q) with constant

$$C_q \lesssim_\gamma (c_1^p + c_2^2)^{1/q}$$

□

Lemma 4.11. *Assume that for all $t \in \mathbb{R}_+$, the operator S_t satisfies Rota's dialtion property. Let $s \in \mathbb{N}$, $\alpha > 0, j \in \mathbb{Z}$ and define $\Delta_{\alpha,j}^{(s)} = P_{2^{-(j+2s)/\alpha}} - P_{2^{-(j-2s)/\alpha}}$. Then for any $1 < p < 2$,*

$$\|(\Delta_{\alpha,j}^{(s)} x)_j\|_{L_p(\mathcal{M};\ell_2^{cr})} \lesssim_\alpha s(p-1)^{-6}\|x\|_p, \qquad x \in L_p(\mathcal{M}).$$

Proof. We may write

$$\Delta_{\alpha,j}^{(s)} x = \int_{2^{-(j+2s)/\alpha}}^{2^{-(j-2s)/\alpha}} \left(-\frac{\partial}{\partial t} P_t(x) \right) dt, \qquad x \in L_p(\mathcal{M}).$$

Let $x = x_1 + x_2$ for $x_1, x_2 \in L_p(\mathcal{M})$. By the convexity of the operator square function,

$$\begin{aligned} |\Delta_{\alpha,j}^{(s)} x_1|^2 &= \left| \int_{2^{-(j+2s)/\alpha}}^{2^{-(j-2s)/\alpha}} \frac{1}{\sqrt{t}} \left(-\sqrt{t} \frac{\partial}{\partial t} P_t(x_1) \right) dt \right|^2 \\ &\leq \left(\int_{2^{-(j+2s)/\alpha}}^{2^{-(j-2s)/\alpha}} t \left| \left(\frac{\partial}{\partial t} P_t(x_1) \right) \right|^2 dt \right) \left(\int_{2^{-(j-2s)/\alpha}}^{2^{-(j+2s)/\alpha}} \frac{dt}{t} \right) \\ &\lesssim_\alpha s \left(\int_{\tilde{\alpha}^{-j-2s}}^{\tilde{\alpha}^{-j+2s}} t \left| \left(\frac{\partial}{\partial t} P_t(x_1) \right) \right|^2 dt \right), \end{aligned}$$

where $\tilde{\alpha} = 2^{1/\alpha}$. Therefore,

$$\begin{aligned} \| \left(\Delta_{\alpha,j}^{(s)} x_1 \right)_j \|_{L_p(\mathcal{M};\ell_2^c)} &= \left\| \left(\sum_{j=-\infty}^{\infty} |\Delta_{\alpha,j}^{(s)} x_1|^2 \right)^{1/2} \right\| \\ &\lesssim_\alpha \sqrt{s} \left\| \left(\sum_{j=-\infty}^{\infty} \int_{\tilde{\alpha}^{-j-2s}}^{\tilde{\alpha}^{-j+2s}} t \left| \left(\frac{\partial}{\partial t} P_t(x_1) \right) \right|^2 dt \right)^{1/2} \right\|_p \\ &\lesssim_\alpha \sqrt{s} \left\| \left(\sum_{j=-\infty}^{\infty} \sum_{k=-2s}^{2s-1} \int_{\tilde{\alpha}^{-j+k}}^{\tilde{\alpha}^{-j+k+1}} t \left| \left(\frac{\partial}{\partial t} P_t(x_1) \right) \right|^2 dt \right)^{1/2} \right\|_p \\ &\lesssim_\alpha s \left\| \left(\int_0^\infty t \left| \left(\frac{\partial}{\partial t} P_t(x_1) \right) \right|^2 dt \right)^{1/2} \right\|_p \end{aligned}$$

Similarly,

$$\|(\Delta_{\alpha,j}^{(s)} x_2)_j\|_{L_p(\mathcal{M};\ell_2^r)} \lesssim_\alpha s \left\| \left(\int_0^\infty t \left| \left(\frac{\partial}{\partial t} P_t(x_2) \right)^* \right|^2 dt \right)^{1/2} \right\|_p .$$

On the other hand,

$$\|(\Delta_{\alpha,j}^{(s)} x)_j\|_{L_p(\mathcal{M};\ell_2^{cr})} \leq \inf\{ \|(\Delta_{\alpha,j}^{(s)} x_1)_j\|_{L_p(\mathcal{M};\ell_2^c)} + \|(\Delta_{\alpha,j}^{(s)} x_2)_j\|_{L_p(\mathcal{M};\ell_2^r)} \}$$

where the infimum runs over all $x_1, x_2 \in L_p(\mathcal{M})$ such that $x = x_1 + x_2$. Then the conclusion follows from Proposition 3.7. □

Now, let us prove Theorem 4.3 (1).

Proof. The case $p \geq 2$ has been already treated by Theorem 4.2. In this proof, we focus on the case $1 < p < 2$.

Fix a finite index set $J \subset \mathbb{Z}$. Denote

$$\begin{aligned} A(p,\infty) &= \|(T_{m_j})_{j\in J} : L_p(\mathcal{M};\ell_\infty) \to L_p(\mathcal{M};\ell_\infty)\|, \\ A(p,1) &= \|(T_{m_j})_{j\in J} : L_p(\mathcal{M};\ell_1) \to L_p(\mathcal{M};\ell_1)\|, \\ A(p,2) &= \|(T_{m_j})_{j\in J} : L_p(\mathcal{M};\ell_2^{cr}) \to L_p(\mathcal{M};\ell_2^{cr})\|. \end{aligned}$$

Since J is finite, all these quantities are well-defined and finite. Because the operators $(T_{m_j})_{j\in J}$ are positive maps, by Proposition 2.5 we have

$$\|(T_{m_j})_{j\in J} : L_p(\mathcal{M}) \to L_p(\mathcal{M};\ell_\infty)\| \asymp A(p,\infty). \tag{4.17}$$

Let $1 < p < 2$. It is sufficient to show that $A(p,\infty)$ is dominated by a positive constant independent of J.

Consider $1 < q_1 < q_2 < 2$ and let $\theta \in (0,1)$ be the number satisfying $\frac{1}{q_2} = \frac{1-\theta}{q_1} + \frac{\theta}{2}$. For $s \in \mathbb{N}_+$ we write $s_0 = [(1-\theta)\alpha \log_2 s] + 1$. Denote by $\Delta_j^{(s)} = P_{2^{-(j+2s_0)/2\alpha}} - P_{2^{-(j-2s_0)/2\alpha}}$ the difference introduced in Lemma 4.11 asociated with s_0 and 2α. By Proposition 3.3 and Lemma 4.11, we have

$$\begin{aligned} \|\sup_{j\in J}{}^+ T_{m_j}(\Delta_j^{(s)}x)\|_{q_1} &\le A(q_1,2)\| \left(\Delta_j^{(s)}x\right)_j \|_{L_{q_1}(\mathcal{M};\ell_2^{cr})} \\ &\lesssim_\alpha A(q_1,2)s_0(q_1-1)^{-6}\|x\|_{q_1}. \end{aligned}$$

By Proposition 2.10 and Lemma 4.11 , we have

$$\begin{aligned} \|\sup_{j\in J}{}^+ S_{2^{-j/\alpha}}(\Delta_j^{(s)}x)\|_{q_1} &\lesssim (q_1-1)^{-2}\| \left(\Delta_j^{(s)}x\right)_j \|_{L_{q_1}(\mathcal{M};\ell_2^{cr})} \\ &\lesssim_\alpha s_0(q_1-1)^{-8}\|x\|_{q_1}. \end{aligned}$$

We set $T_{\phi_j} = T_{m_j} - S_{2^{-j/\alpha}}$ with $\phi_j = m_j - e^{-\frac{\ell(\cdot)}{2^{j/\alpha}}}$. Hence,

$$\|\sup_{j\in J}{}^+ T_{\phi_j}(\Delta_j^{(s)}x)\|_{q_1} \lesssim_\alpha A(q_1,2)(q_1-1)^{-8}s_0\|x\|_{q_1}.$$

Let us assume $(1-\theta)\alpha\log_2 s \ge 1$ first. Note that for any $s > 0$ and $\delta > 0$, we have $\log_2 s \lesssim \frac{s^\delta}{\delta}$. Therefore we get that

$$s_0 \le 2(1-\theta)\alpha\log_2 s \lesssim_\alpha \frac{s^\theta}{\theta} \lesssim_\alpha (q_2-q_1)^{-1}s^\theta.$$

If $(1-\theta)\alpha\log_2 s < 1$, then $s_0 = 1 \le s^\theta \le (q_2-q_1)^{-1}s^\theta$. Hence,

$$\|\sup_{j\in J}{}^+ T_{\phi_j}(\Delta_j^{(s)}x)\|_{q_1} \lesssim_\alpha A(q_1,2)(q_1-1)^{-8}(q_2-q_1)^{-1}s^\theta\|x\|_{q_1}. \tag{4.18}$$

Let $\omega \in \Omega$ and let

$$\delta_j^{(s)}(\omega) = \exp\left(-\frac{\sqrt{\ell(\omega)}}{2^{(j+2s_0)/2\alpha}}\right) - \exp\left(-\frac{\sqrt{\ell(\omega)}}{2^{(j-2s_0)/2\alpha}}\right)$$

be the symbol of $\Delta_j^{(s)}$. Note that

$$|1-\delta_j^{(s)}(\omega)| \le \left|1-\exp\left(-\frac{\sqrt{\ell(\omega)}}{2^{(j+2s_0)/2\alpha}}\right)\right| + \left|\exp\left(-\frac{\sqrt{\ell(\omega)}}{2^{(j-2s_0)/2\alpha}}\right)\right|$$
$$\lesssim_\alpha \frac{\sqrt{\ell(\omega)}}{2^{(j+2s_0)/2\alpha}} + \frac{2^{(j-2s_0)/2\alpha}}{\sqrt{\ell(\omega)}}.$$

When $2^j \ge \ell(\omega)^\alpha$, by the above inequality we have $|1-\delta_j^{(s)}(\omega)| \lesssim_\alpha 2^{-s_0/\alpha}(\frac{2^{j/\alpha}}{\ell(\omega)})^{1/2}$, and as the computation in (4.11) we have $|\phi_j(\omega)| \lesssim_\beta \frac{\ell(\omega)}{2^{j/\alpha}}$. In particular,

$$|\phi_j(\omega)(1-\delta_j^{(s)}(\omega))| \lesssim_{\alpha,\beta} 2^{-s_0/\alpha}\left(\frac{\ell(\omega)}{2^{j/\alpha}}\right)^{1/2} \lesssim_{\alpha,\beta} 2^{-s_0/\alpha}\left(\frac{2^{j/\alpha}\ell(\omega)}{(2^{j/\alpha}+\ell(\omega))^2}\right)^{1/2}.$$

When $2^j < \ell(\omega)^\alpha$, similarly we have $|1-\delta_j^{(s)}(\omega)| \lesssim_\alpha 2^{-s_0/\alpha}(\frac{\ell(\omega)}{2^{j/\alpha}})^{1/2}$, and as the computation in (4.12) we have $|\phi_j(\omega)| \lesssim_\beta \frac{2^{j/\alpha}}{\ell(\omega)}$. Therefore

$$|\phi_j(\omega)(1-\delta_j^{(s)}(\omega))| \lesssim_{\alpha,\beta} 2^{-s_0/\alpha}\left(\frac{2^{j/\alpha}}{\ell(\omega)}\right)^{1/2} \lesssim_{\alpha,\beta} 2^{-s_0/\alpha}\left(\frac{2^{j/\alpha}\ell(\omega)}{(2^{j/\alpha}+\ell(\omega))^2}\right)^{1/2}.$$

By Proposition 4.8, we have

$$\|\sup_{j\in J}{}^+ T_{\phi_j}(1-\Delta_j^{(s)})x\|_2 \lesssim_{\alpha,\beta} 2^{-s_0/\alpha}\|x\|_2 \lesssim_{\alpha,\beta} s^{\theta-1}\|x\|_2. \tag{4.19}$$

Thus by (4.18), (4.19) and Lemma 4.10, we see that $(T_{\phi_j})_{j\in J}$ is of restricted weak type (q_2,q_2) with constant

$$C'_{q_2} \lesssim_{\alpha,\beta} \left(\left(A(q_1,2)(q_1-1)^{-8}(q_2-q_1)^{-1}\right)^{q_1}+1\right)^{1/q_2} \lesssim_{\alpha,\beta} A(q_1,2)(q_1-1)^{-8}(q_2-q_1)^{-1}. \tag{4.20}$$

Set $D = \sup_{1<u\le 2}(u-1)^{22}A(u,\infty) < \infty$. Choose an index $1 < r \le 2$ such that

$$(r-1)^{22}A(r,\infty) > \frac{D}{2}.$$

We apply the restricted weak type estimate of $(T_{\phi_j})_j$ in (4.20) to the particular case $q_1 = \frac{1}{2}(r+1)$ and $q_2 = q_1 + (r-q_1)/2$. Note that by Proposition 2.10, the semigroup $(S_t)_t$ is of strong type (q_2,q_2) with constant $c(q_2-1)^{-2}$. Recall that $T_{m_j} = T_{\phi_j} + S_{2^{-j/\alpha}}$, thus $(T_{m_j})_{j\in J}$ is also of restricted weak type (q_2,q_2) with constant

$$C_{q_2} \lesssim_{\alpha,\beta} A(q_1,2)(q_1-1)^{-8}(q_2-q_1)^{-1} + (q_2-1)^{-2}$$
$$\lesssim_{\alpha,\beta} A(\frac{q_1}{2-q_1},\infty)^{1/2}(r-1)^{-9}.$$

The last inequality above follows from Lemma 3.4 and the values of q_1 and q_2. Because $(T_{m_j})_j$ is of strong type (∞,∞) and of restricted weak type of (q_2,q_2), applying Theorem 2.6 we have

$$\|\sup_{j\in J}{}^+ T_{m_j}x\|_r \le \max\{C_{q_2},1\}(\frac{rq_2}{r-q_2}+r)^2\|x\|_r, \qquad x\in L_r(\mathcal{M}). \tag{4.21}$$

By (4.17), this means that $A(r,\infty) \lesssim C_{q_2}(r-1)^{-2}$. Therefore,

$$(r-1)^{-22}\frac{D}{2} < A(r,\infty) \lesssim_{\alpha,\beta} A(\frac{q_1}{2-q_1},\infty)^{1/2}(r-1)^{-11}. \tag{4.22}$$

Recall that $A(\frac{q_1}{2-q_1},\infty) \lesssim_{\alpha,\beta} 1$ if $\frac{q_1}{2-q_1} \geq 2$ by Theorem 4.2. Without loss of generality we assume that $1 < \frac{q_1}{2-q_1} < 2$. Then (4.22) yields

$$(r-1)^{-22}\frac{D}{2} \lesssim_{\alpha,\beta} \left(\left(\frac{q_1}{2-q_1}-1\right)^{-22} D\right)^{1/2} (r-1)^{-11}$$

Recall that $q_1 = \frac{1}{2}(r+1)$. We have

$$D \lesssim_{\alpha,\beta} 1.$$

In other words, $(p-1)^{22}A(p,\infty) \lesssim_{\alpha,\beta} 1$ for any $1 < p \leq 2$. In particular, this estimate is independent of the finite index set J. So we obtain the desired maximal inequality according to Remark 2.3.

Note that $T_{m_j}x$ converges a.u. to x as $j \to \infty$ for $x \in L_2(\mathcal{M})$ by Theorem 4.2 and that $L_2(\mathcal{M}) \cap L_p(\mathcal{M})$ is dense in $L_p(\mathcal{M})$. Applying Proposition 2.9 (1), we get the a.u. convergence of $(T_{m_j}x)_j$ for $x \in L_p(\mathcal{M})$. □

4.4. **Proof of Theorem 4.3 (2).** Our idea is reducing the desired maximal inequalities to those for lacunary subsequences already studied in the preceding subsection.

Lemma 4.12. *Assume that the family $(m_t)_{t\in\mathbb{R}_+}$ satisfies* **(A2)**. *Then for any $1 \leq q < 2$ and $q + \frac{q(2-q)}{q-1+2\eta} < p < 2$, we have*

$$\|(T_{m_t})_{t\in\mathbb{R}_+} : L_p(\mathcal{M}) \to L_p(\mathcal{M};\ell_\infty)\| \lesssim_{\beta,\eta,p,q} \sup_{1\leq\delta\leq 2} \|(T_{m_{\delta 2^j}})_{j\in\mathbb{Z}} : L_q(\mathcal{M}) \to L_q(\mathcal{M};\ell_\infty)\|^{1-\theta}$$

provided that the right hand side is finite, where θ is determined by $\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{2}$.

Proof. Our proof is based on the estimate of multi-order differences of $(m_t)_t$. For notational simplicity we denote these differences as follows: we start with setting the first order differences of the following form

$$\psi_t^{[s]} = m_{2^{2^{-s-1}}t} - m_t, \quad s \in \mathbb{N}, t \in \mathbb{R}_+,$$

and define the higher order ones inductively by

$$\psi_t^{[s_1,s_2,\ldots,s_v]} = \psi_{2^{2^{-s_v-1}}t}^{[s_1,s_2,\cdots,s_{v-1}]} - \psi_t^{[s_1,s_2,\cdots,s_{v-1}]}, \qquad s_1,\ldots,s_v \in \mathbb{N}, t \in \mathbb{R}_+, 2 \leq v \leq \eta.$$

We denote by $\Psi_t^{[s_1,s_2,\ldots,s_v]} = T_{\psi_t^{[s_1,s_2,\ldots,s_v]}}$ the associated multipliers for $1 \leq v \leq \eta$.

We will estimate the maximal norms of $(T_{m_t})_t$ by using those of $(\Psi_t^{[s_1,s_2,\ldots,s_v]})_t$. To see this, note that by Proposition 4.7, $(T_{m_t}(x))_t$ is strongly continuous on $L_p(\mathcal{M})$ for all $1 < p < \infty$. We consider the dyadic approximations with increasing index sets $I_s = \{2^{j/2^s} : j \in \mathbb{Z}\}$ for $s \in \mathbb{N}$. By Remark 2.3, we have for $x \in L_p(\mathcal{M})$,

$$\begin{aligned}\|\sup_{t\in\mathbb{R}_+}{}^+ T_{m_t}x\|_p &= \lim_{s\to\infty} \|\sup_{t\in I_s}{}^+ T_{m_t}x\|_p \\ &\leq \|\sup_{t\in I_0}{}^+ T_{m_t}x\|_p + \sum_{s=0}^{\infty}\left(\|\sup_{t\in I_{s+1}}{}^+ T_{m_t}x\|_p - \|\sup_{t\in I_s}{}^+ T_{m_t}x\|_p\right).\end{aligned}$$

Note that we have the bijection $J : I_s \to I_{s+1}\backslash I_s,\ 2^{j/2^s} \mapsto 2^{(2j+1)/2^{s+1}} = 2^{\frac{1}{2^{s+1}}}2^{j/2^s}$ and $I_{s+1} = I_s \cup J(I_s)$. Hence for

$$y_t = \begin{cases} T_{m_t}x & t \in I_s \\ T_{m_{J^{-1}(t)}}x & t \in J(I_s) \end{cases},$$

we have $\|\sup^+_{t\in I_{s+1}} y_t\| = \|\sup^+_{t\in I_s} T_{m_t}x\|$. Then applying the triangle inequality, we get

$$\|\sup_{t\in I_{s+1}}{}^+ T_{m_t}x\|_p - \|\sup_{t\in I_s}{}^+ T_{m_t}x\|_p \le \|\sup_{t\in I_s}{}^+ \Psi_t^{[s]}(x)\|_p.$$

In other words, we obtain

$$\|\sup_{t\in\mathbb{R}_+}{}^+ T_{m_t}x\|_p \le \|\sup_{t\in I_0}{}^+ T_{m_t}x\|_p + \sum_{s=0}^{\infty} \|\sup_{t\in I_s}{}^+ \Psi_t^{[s]}(x)\|_p.$$

Applying the above arguments to maps of the form $\Psi_t^{[s]}$ in place of T_{m_t}, we see that for each $s_1 \ge 1$,

$$\|\sup_{t\in I_{s_1}}{}^+ \Psi_t^{[s_1]}(x)\|_p \le \|\sup_{t\in I_0}{}^+ \Psi_t^{[s_1]}(x)\|_p + \sum_{s_2=0}^{s_1-1} \|\sup_{t\in I_{s_2}}{}^+ \Psi_t^{[s_1,s_2]}(x)\|_p.$$

Hence,

$$\|\sup_{t\in\mathbb{R}_+}{}^+ T_{m_t}x\|_p \le \|\sup_{j\in\mathbb{Z}}{}^+ T_{m_{2^j}}x\|_p + \sum_{s_1=0}^{\infty} \|\sup_{t\in I_0}{}^+ \Psi_t^{[s_1]}(x)\|_p + \sum_{s_1=1}^{\infty}\sum_{s_2=0}^{s_1-1} \|\sup_{t\in I_{s_2}}{}^+ \Psi_t^{[s_1,s_2]}(x)\|_p.$$

Repeating this process η times, we get

$$\begin{aligned} &\|\sup_{t\in\mathbb{R}_+}{}^+ T_{m_t}x\|_p \\ \le &\|\sup_{j\in\mathbb{Z}}{}^+ T_{m_{2^j}}x\|_p + \sum_{s_1=0}^{\infty} \|\sup_{t\in I_0}{}^+ \Psi_t^{[s_1]}(x)\|_p + \sum_{s_1=1}^{\infty}\sum_{s_2=0}^{s_1-1} \|\sup_{t\in I_0}{}^+ \Psi_t^{[s_1,s_2]}(x)\|_p + \cdots \\ &+ \sum_{s_1>s_2>\cdots>s_{\eta-1}} \|\sup_{t\in I_0}{}^+ \Psi_t^{[s_1,s_2,\ldots s_{\eta-1}]}(x)\|_p + \sum_{s_1>s_2>\cdots>s_\eta} \|\sup_{t\in I_{s_\eta}}{}^+ \Psi_t^{[s_1,s_2,\ldots,s_\eta]}(x)\|_p. \end{aligned} \tag{4.23}$$

It remains to study the maximal norms of $(\Psi_t^{[s_1,s_2,\ldots,s_v]})_t$ on the right hand side of the above inequality. To this end we need to estimate the derivative $\partial_t^k \psi_t^{[s_1,s_2,\ldots,s_v]}(\omega)$ by virtue of Lemma 4.9. More precisely, we will show that for any $1 \le v \le \eta$,

$$\left|\partial_t^k \left(\psi_t^{[s_1,s_2,\ldots,s_v]}\right)\right| \lesssim_\beta 2^{-(s_1+s_2\cdots+s_v)} \frac{(2k+2v)^v}{t^k} \qquad 0 \le k \le \eta - v. \tag{4.24}$$

Let us prove this inequality by induction. For notational simplicity we write $\rho_v = 2^{2^{-s_v-1}}$. Note that $1 < \rho_v < 2$ and that applying the mean value theorem to the function $x \mapsto 2^x$, we get that for any $1 \le v \le \eta$ and $k \ge 0$,

$$\rho_v^k - 1 \lesssim k2^{-s_v}. \tag{4.25}$$

Consider first $v = 1$. For any $0 \le k \le \eta - 1$,

$$|\partial_t^k \psi_t^{[s_1]}(\omega)| = |\partial_t^k m_{\rho_1 t}(\omega) - \partial_t^k m_t(\omega)| = \left|\rho_1^k \partial_\gamma^k m_\gamma(\omega)|_{\gamma=\rho_1 t} - \partial_t^k m_t(\omega)\right|$$

By **(A2)**, $\left|\partial_\gamma^k m_\gamma(\omega)|_{\gamma=\rho_1 t}\right| \lesssim_\beta \frac{1}{(\rho_1 t)^k}$. Hence

$$|\partial_t^k \psi_t^{[s_1]}(\omega)| \lesssim_\beta (\rho_1^k - 1)\frac{1}{(\rho_1 t)^k} + \left|\partial_\gamma^k m_\gamma(\omega)|_{\gamma=\rho_1 t} - \partial_t^k m_t(\omega)\right|.$$

By the mean value theorem and **(A2)**,

$$\left|\partial_\gamma^k m_\gamma(\omega)|_{\gamma=\rho_1 t} - \partial_t^k m_t(\omega)\right| \leq (\rho_1 t - t) \sup_{t\leq\gamma\leq\rho_1 t} |\partial_\gamma^{k+1} m_\gamma(\omega)| \lesssim_\beta \frac{\rho_1 t - t}{t^{k+1}}.$$

Therefore, according to (4.25), we have

$$\begin{aligned}
|\partial_t^k \psi_t^{[s_1]}(\omega)| &\lesssim_\beta (\rho_1^k - 1)\frac{1}{(\rho_1 t)^k} + \frac{\rho_1 - 1}{t^k} \\
&\lesssim_\beta \frac{k2^{-s_1}}{t^k} + \frac{2^{-s_1}}{t^k} \\
&\lesssim_\beta 2^{-s_1}\frac{k+1}{t^k}.
\end{aligned}$$

So (4.24) holds for $v = 1$. Assume that (4.24) holds for some $1 \leq v \leq \eta - 1$ and consider the case of $v + 1$. For any $0 \leq k \leq \eta - (v + 1)$, arguing as above, we have

$$\begin{aligned}
|\partial_t^k \psi_t^{[s_1,s_2,\cdots,s_{v+1}]}(\omega)| &= |\rho_{v+1}^k \partial_\gamma^k(\psi_\gamma^{[s_1,s_2,\cdots,s_v]}(\omega))|_{\gamma=\rho_{v+1}t} - \partial_t^k(\psi_t^{[s_1,s_2,\cdots,s_v]}(\omega))| \\
&\lesssim_\beta (\rho_{v+1}^k - 1)2^{-(s_1+s_2\cdots+s_v)}\frac{(2k+2v)^v}{(\rho_{v+1}t)^k} \\
&\qquad + (\rho_{v+1}t - t)\sup_{t\leq\gamma\leq\rho_{v+1}t} |\partial_\gamma^{k+1}(\psi_\gamma^{[s_1,s_2,\cdots,s_v]}(\omega))| \\
&\lesssim_\beta k2^{-s_{v+1}}2^{-(s_1+s_2\cdots+s_v)}\frac{(2k+2v)^v}{t^k} \\
&\qquad + (\rho_{v+1}t - t)2^{-(s_1+s_2\cdots+s_v)}\frac{(2k+2+2v)^v}{t^{k+1}} \\
&\lesssim_\beta 2^{-(s_1+s_2\cdots+s_{v+1})}\frac{(2(k+1+v))^{v+1}}{t^k}.
\end{aligned}$$

So (4.24) is proved. In particular, setting $k = 0$ and $k = 1$ respectively, we get for any $1 \leq v \leq \eta$

$$\left|\psi_t^{[s_1,s_2,\cdots,s_v]}(\omega)\right| \lesssim_{\beta,\eta} 2^{-(s_1+s_2\cdots+s_v)}, \tag{4.26}$$

and for any $1 \leq v \leq \eta - 1$,

$$\left|\partial_t \psi_t^{[s_1,s_2,\cdots,s_v]}(\omega)\right| \lesssim_{\beta,\eta} 2^{-(s_1+s_2\cdots+s_v)}\frac{1}{t}. \tag{4.27}$$

This also yields

$$\begin{aligned}
\left|\partial_t \psi_t^{[s_1,s_2,\cdots,s_\eta]}(\omega)\right| &\leq \left|\partial_t \psi_{\rho_\eta t}^{[s_1,s_2,\cdots,s_{\eta-1}]}(\omega)\right| + \left|\partial_t \psi_t^{[s_1,s_2,\cdots,s_{\eta-1}]}(\omega)\right| \\
&\lesssim_{\beta,\eta} 2^{-(s_1+s_2+\cdots+s_{\eta-1})}\frac{1}{t}.
\end{aligned} \tag{4.28}$$

On the other hand, by definition

$$\psi_t^{[s_1,s_2,\cdots,s_v]} = \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^v} (-1)^{\left(v+\sum_{i=1}^v \varepsilon_i\right)} m_{\boldsymbol{\rho}^{\boldsymbol{\varepsilon}} t} \tag{4.29}$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_v) \in \{0,1\}^v$ and $\boldsymbol{\rho}^{\boldsymbol{\varepsilon}} = \rho_1^{\varepsilon_1}\rho_2^{\varepsilon_2}\cdots\rho_v^{\varepsilon_v}$. Recall that $s_1 < s_2 < \cdots s_v$ and $s_v \geq v$, we have

$$1 < \boldsymbol{\rho}^{\boldsymbol{\varepsilon}} \leq 2^{2^{-s_v-1}+2^{-s_v}+\cdots+2^{-s_v+v-2}} < 2^{2^{-s_v-1+v}} < 2.$$

By **(A2)**,

$$\left|\psi_t^{[s_1,s_2,\cdots,s_v]}(\omega)\right| \leq \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^v} |m_{\boldsymbol{\rho}^{\boldsymbol{\varepsilon}}t}(\omega)| \leq \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^v} \beta\frac{\boldsymbol{\rho}^{\boldsymbol{\varepsilon}}t}{\ell(\omega)^\alpha} \leq \beta 2^{2v}\frac{t}{\ell(\omega)^\alpha},$$

$$\left|\psi_t^{[s_1,s_2,\cdots,s_v]}(\omega)\right| \leq \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^v} |1-m_{\boldsymbol{\rho}^{\boldsymbol{\varepsilon}}t}(\omega)| \leq \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^v} \beta\frac{\ell(\omega)^\alpha}{\boldsymbol{\rho}^{\boldsymbol{\varepsilon}}t} \leq 2^v\beta\frac{\ell(\omega)^\alpha}{t}.$$

Thus, setting

$$a_j^{[s_1,s_2,\cdots,s_v]} := \sup_t\left(\sup_{2^{j-2}<\frac{\ell(\omega)^\alpha}{t}\leq 2^j}\left|\psi_t^{[s_1,s_2,\cdots,s_v]}(\omega)\right|\right),$$

and

$$b_j^{[s_1,s_2,\cdots,s_v]} := \sup_t\left(\sup_{2^{j-2}<\frac{\ell(\omega)^\alpha}{t}\leq 2^j} t\left|\partial_t\psi_t^{[s_1,s_2,\cdots,s_v]}(\omega)\right|\right),$$

together with (4.26) and (4.27), we have for $1 \leq v \leq \eta-1$,

$$a_j^{[s_1,s_2,\cdots,s_v]} \lesssim_{\beta,\eta} \min\{2^{-(s_1+s_2+\cdots+s_v)}, 2^{-|j|}\}, \qquad b_j^{[s_1,s_2,\cdots,s_v]} \lesssim_{\beta,\eta} 2^{-(s_1+s_2+\cdots+s_v)}.$$

Then by Lemma 4.9, for $1 \leq v \leq \eta-1$, we have

$$\|\sup_{t\in\mathbb{R}_+}{}^+\,\Psi_t^{[s_1,s_2,\cdots,s_v]}x\|_2 \lesssim K^{[s_1,s_2,\cdots s_v]}\|x\|_2, \tag{4.30}$$

with

$$\begin{aligned}K^{[s_1,s_2,\cdots,s_v]} &= \sum_{j\in\mathbb{Z}}(a_j^{[s_1,s_2,\cdots,s_v]})^{1/2}(a_j^{[s_1,s_2,\cdots,s_v]}+b_j^{[s_1,s_2,\cdots,s_v]})^{1/2}\\ &\lesssim_{\beta,\eta}\left(\sum_{|j|\leq s_1+s_2+\cdots+s_v} 2^{-(s_1+s_2+\cdots+s_v)/2} + \sum_{|j|>s_1+s_2\cdots+s_v} 2^{-|j|/2}\right)\cdot 2^{-(s_1+s_2+\cdots+s_v)/2}\\ &\lesssim_{\beta,\eta}\frac{s_1+s_2+\cdots+s_v}{2^{(s_1+s_2+\cdots+s_v)}}.\end{aligned}$$

Similarly, for $v=\eta$, by (4.26) and (4.28),

$$\|\sup_{t\in\mathbb{R}_+}{}^+\,\Psi_t^{[s_1,s_2,\cdots,s_\eta]}x\|_2 \lesssim K^{[s_1,s_2,\cdots,s_\eta]}\|x\|_2, \tag{4.31}$$

with

$$K^{[s_1,s_2,\cdots s_\eta]} \lesssim_{\beta,\eta}\frac{s_1+s_2+\cdots+s_\eta}{2^{(s_1+s_2\cdots+s_{\eta-1})+\frac{s_\eta}{2}}}.$$

In the following we consider the case $1 \leq q < 2$. Denote

$$A_q = \sup_{1\leq\delta\leq 2}\|(T_{m_{\delta 2^j}})_{j\in\mathbb{Z}} : L_q(\mathcal{M}) \to L_q(\mathcal{M};\ell_\infty)\|.$$

For $1 \leq v \leq \eta-1$, by (4.29) we have

$$\|\sup_{t\in I_0}{}^+\Psi_t^{[s_1,s_2,\cdots,s_v]}x\|_q \leq \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^v}\|\sup_{t\in I_0}{}^+T_{m_{\boldsymbol{\rho}^{\boldsymbol{\varepsilon}}t}}x\|_q \leq 2^v A_q\|x\|_q. \tag{4.32}$$

For $v=\eta$, we decompose

$$I_{s_\eta}=\{2^{j/2^{s_\eta}}:j\in\mathbb{Z}\}=\bigcup_{l=0}^{2^{s_\eta}-1}\{2^{\frac{(2^{s_\eta})j+l}{2^{s_\eta}}}:j\in\mathbb{Z}\}=\bigcup_{l=0}^{2^{s_\eta}-1}2^{\frac{l}{2^{s_\eta}}}I_0.$$

By (4.29) and the triangle inequality, we have

$$\begin{aligned}\|\sup_{t\in I_{s_\eta}}{}^{+}\Psi_t^{[s_1,s_2,\dots,s_\eta]}(x)\|_q &\le \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^\eta}\|\sup_{t\in I_{s_\eta}}{}^{+}T_{m_{\boldsymbol{\rho}^{\boldsymbol{\varepsilon}}t}}x\|_q\\ &\le \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^\eta}\sum_{l=0}^{2^{s_\eta}-1}\|\sup_{t\in 2^{\frac{l}{2^{s_\eta}}}I_0}{}^{+}T_{m_{\boldsymbol{\rho}^{\boldsymbol{\varepsilon}}t}}x\|_q\\ &\le \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^\eta}\sum_{l=0}^{2^{s_\eta}-1}\|\sup_{t\in I_0}{}^{+}T_{m_{2^{\frac{l}{2^{s_\eta}}}\boldsymbol{\rho}^{\boldsymbol{\varepsilon}}t}}x\|_q\\ &\lesssim_\eta A_q2^{s_\eta}\|x\|_q.\end{aligned}\tag{4.33}$$

Now the conclusion follows easily from the complex interpolation. Let $1<p<2$ and $0<\theta<1$ with $\frac{1}{p}=\frac{1-\theta}{q}+\frac{\theta}{2}$. By (4.30), (4.32) and interpolation, we see that for $v\le\eta-1$,

$$\|\sup_{t\in I_0}{}^{+}\Psi_t^{[s_1,s_2,\cdots,s_v]}x\|_p\lesssim_{\beta,\eta}A_q^{1-\theta}\left(s_1+s_2\cdots+s_v\right)^\theta 2^{-\theta(s_1+s_2\cdots+s_v)}\|x\|_p.$$

By (4.31), (4.33) and interpolation, for $v=\eta$,

$$\|\sup_{t\in I_{s_\eta}}{}^{+}\Psi_t^{[s_1,s_2,\dots,s_\eta]}(x)\|_p\lesssim_{\beta,\eta}A_q^{1-\theta}2^{(1-\theta)s_\eta}\left(s_1+s_2\cdots+s_\eta\right)^\theta 2^{-\theta(s_1+s_2\cdots+\frac{s_\eta}{2})}\|x\|_p.$$

We apply the above estimate to (4.23). Note that when $v\le\eta-1$,

$$\begin{aligned}&\sum_{s_1>s_2>\cdots>s_v}\left(s_1+s_2\cdots+s_v\right)^\theta 2^{-\theta(s_1+s_2\cdots+s_v)}\\ \lesssim_\theta&\sum_{s_2>\cdots>s_v}\sum_{s_1=s_2+1}^{\infty}2^{-\theta(s_1+s_2\cdots+s_v)/2}\lesssim_\theta\sum_{s_2>\cdots>s_v}2^{-\theta s_2/2}2^{-\theta(s_2+\cdots+s_v)/2}\\ \lesssim_\theta&\sum_{s_3>\cdots>s_v}2^{-\theta s_3}2^{-\theta(s_3+\cdots+s_v)/2}\lesssim_\theta\cdots\lesssim_\theta\sum_{s_v\ge0}2^{-(v-1)\theta s_v/2}\lesssim_\theta 1.\end{aligned}$$

Applying the similar computation to $v=\eta$, we have

$$\sum_{s_1>s_2>\cdots>s_\eta}2^{(1-\theta)s_\eta}\left(s_1+s_2\cdots+s_\eta\right)^\theta 2^{-\theta(s_1+s_2\cdots+\frac{s_\eta}{2})}\lesssim_\theta\sum_{s_\eta\ge0}2^{(1-\theta)s_\eta}(s_\eta+1)^\theta 2^{-(\eta-\frac12)\theta s_\eta}.$$

Thus the above quantity is finite if $(1-\theta)<(\eta-\frac12)\theta$, i.e. $\theta>\frac{2}{2\eta+1}$, which requires that $q+\frac{q(2-q)}{q-1+2\eta}<p\le2$. Therefore, together with (4.23), if $q+\frac{q(2-q)}{q-1+2\eta}<p\le2$,

$$\|\sup_{t\in\mathbb{R}_+}{}^{+}T_{m_t}x\|_p\lesssim_{\beta,\eta,\theta}A_q^{1-\theta}\|x\|_p.$$

The proof is complete. □

Proof of Theorem 4.3 (2). For any $1 \le \delta \le 2$ and $j \in \mathbb{Z}$, set $t_j = 2^j\delta$. **(A2)** implies that

$$|1 - m_{t_j}(\omega)| \le \beta \frac{\ell(\omega)^\alpha}{2^j\delta} \le \beta \frac{\ell(\omega)^\alpha}{2^j} \quad \text{and} \quad |m_{t_j}(\omega)| \le 2\beta \frac{2^j}{\ell(\omega)^\alpha}.$$

By Thereom 4.3 (1) and Remark 4.5 (2), for any $1 < q \le 2$, we have

$$\sup_{1\le\delta\le 2} \|\sup_{j\in\mathbb{Z}}{}^+ T_{m_{t_j}} x\|_q \lesssim_{\alpha,\beta,q,\eta} \|x\|_q.$$

Note that $q + \frac{q(2-q)}{q-1+2\eta}$ tends to $1 + \frac{1}{2\eta}$ as $q \to 1$. Hence, by Lemma 4.12, for any $1 + \frac{1}{2\eta} < p \le 2$, we get

$$\|\sup_{t\in\mathbb{R}_+}{}^+ T_{m_t} x\|_p \lesssim_{\alpha,\beta,p,\eta} \|x\|_p.$$

The a.u. convergence is proved similarly as in (1). □

4.5. **The case of operator-valued multipliers and nontracial states.** Based on the previous arguments, we may extend our results to the setting of operator-valued multipliers and Haagerup's nontracial L_p-spaces. This will be particularly essential for our further study of multipliers on quantum groups in the next section. All the previous arguments for $p \ge 2$ can be transfered without difficulty into this new setting, and we will leave the details to interesting readers. However, the previous proof for the case $p < 2$ does not continue to hold for Haagerup's L_p-spaces. Based on Haagerup's reduction method, we will rather use our previous results for the tracial setting to deduce the desired properties for the nontracial ones.

4.5.1. *Operator-valued multipliers in the tracial setting.* Let us first begin with the operator-valued multipliers on tracial von Neumann algebras. Let $\mathcal{R}$ be a von Neumann algebra equipped with a semifinite normal trace. Assume that there is an isometric isomorphism

$$U : L_2(\mathcal{R}) \to \bigoplus_{i\in I} H_i$$

where I is an index set and H_i is a Hilbert space for each $i \in I$. For any bounded sequence $m := (m(i))_{i\in I}$ with $m(i) \in B(H_i)$, we can define an *operator-valued multiplier* on $\mathcal{R}$:

$$\begin{aligned} T_m : L_2(\mathcal{R}) &\to L_2(\mathcal{R}) \\ x &\mapsto U^{-1}\left(m(i)(Ux)(i)\right)_{i\in I}. \end{aligned} \tag{4.34}$$

Note that if $H_i = \mathbb{C}$ for all $i \in I$, then this goes back to our first setting in (4.1) with $\Omega = I$ equipped the counting measure. Proposition 4.8 can be adapted to this new setting.

Proposition 4.13. *Let $(T_{m_N})_{N\in\mathbb{Z}}$ be a sequence of operator-valued multipliers as above. Assume that there exist a function $f : I \to [0,\infty)$ and a positive number $a > 1$ such that*

$$\|m_N(i)\|_{B(H_i)} \le \beta \frac{a^N f(\omega)}{(a^N + f(\omega))^2} \tag{4.35}$$

Then,

$$\|(T_{m_N} x)_{N\in\mathbb{Z}}\|_{L_2(\mathcal{R};\ell_2^{cr})} \le \beta \sqrt{\frac{a^2}{a^2-1}} \|x\|_2.$$

Proof. Repeat the proof of Proposition 4.8. □

Using Proposition 4.13, Lemma 4.10, Lemma 3.4 and Lemma 4.11, we may deduce the following result. The proof is the same as that of Theorem 4.2 and Theorem 4.3 (for **(A1)**).

Theorem 4.14. *Let $\ell := (\ell(i))_{i\in I}$ be a sequence with $\ell(i) \in B(H_i)$. Assume that $(T_{e^{-t\ell}})_{t\in\mathbb{R}_+}$ is a semigroup of unital completely positive trace-preserving symmetric maps on $\mathcal{R}$. For any $N \in \mathbb{N}$, let $m_N := (m_N(i))_{i\in I}$ be a sequence with $m_N(i) \in B(H_i)$. Assume that $(T_{m_N})_{N\in\mathbb{N}}$ extends to a family of bounded maps on $\mathcal{R}$ with $\gamma := \sup_N \|T_{m_N} : \mathcal{R} \to \mathcal{R}\| < \infty$. Assume that there exist $\alpha > 0$ and $\beta > 0$ such that for all $i \in I$ we have*

$$\|\operatorname{id}_{H_i} - m_N(i)\|_{B(H_i)} \le \beta \frac{\|\ell(i)\|^{\alpha}_{B(H_i)}}{2^N}, \quad \|m_N(i)\|_{B(H_i)} \le \beta \frac{2^N}{\|\ell(i)\|^{\alpha}_{B(H_i)}}. \tag{4.36}$$

(1) *For all $2 \le p < \infty$ there is a constant $c > 0$ depending only on p, α, β, γ such that for all $x \in L_p(\mathcal{R})$, we have*

$$\|(T_{m_N}x)_N\|_{L_p(\mathcal{R};\ell_\infty)} \le c\|x\|_p, \qquad \text{and} \qquad T_{m_N}x \to x \text{ a.u. as } N \to \infty.$$

(2) *Assume additionally that the operators $(T_{m_N})_{N\in\mathbb{N}}$ extend to positive symmetric contractions on $\mathcal{R}$ and that S_t satisfies Rota's dialtion property for all $t \in \mathbb{R}_+$. Then for any $1 < p < \infty$, there is a constant c depending only on p, α, β such that for all $x \in L_p(\mathcal{R})$,*

$$\|(T_{m_N}x)_N\|_{L_p(\mathcal{R};\ell_\infty)} \le c\|x\|_p \qquad \text{and} \qquad T_{m_N}x \to x \text{ a.u. as } N \to \infty.$$

4.5.2. *Operator-valued multipliers on Haagerup noncommutative L_p spaces.* Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H. Let φ be a fixed normal semifinite faithful state on $\mathcal{M}$. Let $\sigma = (\sigma_t)_t = (\sigma^{\varphi}_t)_t$ be the modular automorphism group with respect to φ.

Let $L_p(\mathcal{M}, \varphi)$ be the Haagerup noncommutative L_p-spaces associated with $(\mathcal{M}, \varphi)$. In the following discussions we will not need the detailed information of these spaces, and we refer to [Ter81, PX03, HJX10] for a detailed presentation. We merely remind the reader that the elements in $L_p(\mathcal{M}, \varphi)$ can be realized as densely defined closed operators on $L_2(\mathbb{R}; H)$ and that $L_\infty(\mathcal{M}, \varphi)$ coincides with $\mathcal{M}$ for a certain suitable representation $\mathcal{M}$ on $L_2(\mathbb{R}; H)$. If $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$, then the associated Haagerup L_p-space $L_p(\mathcal{N}, \varphi|_{\mathcal{N}})$ can be naturally embedded as a subspace of $L_p(\mathcal{M}, \varphi)$ which preserves positivity. There is a distinguished positive element $D_\varphi \in L_1(\mathcal{M}, \varphi)_+$, usually called the *density* operator associated with φ, such that $D^{1/2p}_\varphi \mathcal{M} D^{1/2p}_\varphi$ is dense in $L_p(\mathcal{M}, \varphi)$ (see e.g. [JX03, Lemma 1.1]). The space $L_p(\mathcal{M}, \varphi)$ isometrically coincides with the usual tracial noncommutative L_p-space used previously if $\mathcal{M}$ is tracial. Indeed, if $\mathcal{M}$ is equipped with a normal faithful tracial state τ with $\varphi = \tau(\cdot\rho)$ for some $\rho \in L_0(\mathcal{M}, \tau)$, then for any $x \in \mathcal{M}$,

$$\|D^{1/2p}_\varphi x D^{1/2p}_\varphi\|_{L_p(\mathcal{M},\varphi)} = [\tau((\rho^{1/2p}x\rho^{1/2p})^p)]^{1/p}, \tag{4.37}$$

which coincides with the norm of $\rho^{1/2p}x\rho^{1/2p}$ in the tracial L_p-space $L_p(\mathcal{M}, \tau)$ in the sense of Section 2. In the sequel we will not distinguish the Haagerup L_p-spaces and the tracial L_p-spaces introduced in Section 2 if φ is tracial.

In this subsection we set $\Gamma = \bigcup_{n\ge1} 2^{-n}\mathbb{Z}$, which is regarded as a discrete subgroup of $\mathbb{R}$. We consider the crossed procuct $\mathcal{R} = \mathcal{M} \rtimes_\sigma \Gamma$. Recall that $\mathcal{R}$ is the von Neumann subalgebra generated by $\pi(\mathcal{M})$ and $\operatorname{id}_H \otimes \lambda(\Gamma)$ in $B(\ell_2(\mathbb{R}; H))$, where $\pi : \mathcal{M} \to B(\ell_2(\Gamma; H))$ is the $*$-representation given by $\pi(x) = \sum_{t\in\Gamma} \sigma_{-t}(x) \otimes e_{t,t}$ and λ is the left regular representation of Γ on $\ell_2(\Gamma)$. We will identify $\mathcal{M}$ with $\pi(\mathcal{M})$ and denote $x \rtimes \lambda(t) = \pi(x)(\operatorname{id}_H \otimes \lambda(t))$ for $x \in \mathcal{M}$ and $t \in \Gamma$. We have

$$(x \rtimes \lambda(t)) \cdot (y \rtimes \lambda(s)) = (x\sigma_t(y)) \rtimes \lambda(t + s)$$

for any $x, y \in \mathcal{M}$, $t, s \in \Gamma$. Let τ_Γ be the usual trace on $VN(\Gamma)$ given by $\tau_\Gamma(\lambda(t)) = \delta_{t=0}$. The dual state $\widehat{\varphi}$ on $\mathcal{R}$ is defined by

$$\widehat{\varphi}(x \rtimes \lambda(t)) = \varphi(x)\tau_\Gamma(\lambda(t)) \qquad x \in \mathcal{M}, \quad t \in \Gamma. \tag{4.38}$$

We set

$$a_k = -\mathrm{i}2^n \mathrm{Log}(\lambda(2^{-k})) \quad \text{and} \quad \tau_k = \widehat{\varphi}(e^{-a_k}\,\cdot)$$

with Log the principal branch of the logarithm so that $0 \leq \mathrm{Im}(\mathrm{Log}(z)) < 2\pi$. We denote by $\mathcal{R}_k$ the centralizer of τ_k in $\mathcal{R}$. We will use the following two theorems to reduce our problem to the tracial case studied previously.

Theorem 4.15 ([HJX10, Theorem 2.1, Example 5.8, Remark 6.1])**.** (1) *For each $k \geq 1$, the subalgebra $\mathcal{R}_k$ is finite and τ_k is a normal faithful tracial state on $\mathcal{R}_k$;*

(2) *$\{\mathcal{R}_k\}_{k\geq 1}$ is an increasing sequence of von Neumann subalgebras such that $\cup_{k\geq 1}\mathcal{R}_k$ is w^*-dense in $\mathcal{R}$;*

(3) *for every $k \in \mathbb{N}$, there exists a normal conditional expectation $\mathbb{E}_k$ from $\mathcal{R}$ onto $\mathcal{R}_k$ such that*

$$\widehat{\varphi} \circ \mathbb{E}_k = \widehat{\varphi} \quad \text{and} \quad \sigma_t^{\widehat{\varphi}} \circ \mathbb{E}_k = \mathbb{E}_k \circ \sigma_t^{\widehat{\varphi}}, \quad t \in \mathbb{R}.$$

For each $1 \leq p < \infty$ and any $k \in \mathbb{N}$, the map

$$\mathbb{E}_k^{(p)}(D_{\widehat{\varphi}}^{1/2p} x D_{\widehat{\varphi}}^{1/2p}) = D_{\widehat{\varphi}}^{1/2p}\mathbb{E}_k(x)D_{\widehat{\varphi}}^{1/2p}, \quad x \in \mathcal{R}$$

extends to a conditional expectation from $L_p(\mathcal{R}, \widehat{\varphi})$ onto $L_p(\mathcal{R}_k, \widehat{\varphi}|_{\mathcal{R}_k})$, and

$$\lim_k \|\mathbb{E}_k^{(p)}x - x\|_{L_p(\mathcal{R},\widehat{\varphi})} = 0, \quad x \in L_p(\mathcal{R}, \widehat{\varphi}).$$

Theorem 4.16 ([HJX10, Theorem 4.1, Proposition 4.3 and Theorem 5.1])**.** *Assume that $T : \mathcal{M} \to \mathcal{M}$ is a completely bounded normal map such that*

$$T \circ \sigma_t = \sigma_t \circ T, \qquad t \in \mathbb{R}. \tag{4.39}$$

Then T admits a unique completely bounded normal extension $\widehat{T}$ on $\mathcal{R}$ such that

$$\|\widehat{T}\|_{cb} = \|T\|_{cb} \quad \text{and} \quad \widehat{T}(x \rtimes \lambda(g)) = T(x) \rtimes \lambda(g), \qquad x \in \mathcal{M}, g \in \Gamma.$$

Moreover, $\widehat{T}$ satisfies the following properties:

(1) *$\sigma_t^{\widehat{\varphi}} \circ \widehat{T} = \widehat{T} \circ \sigma_t^{\widehat{\varphi}}, t \in \mathbb{R}$;*

(2) *$\widehat{T} \circ \mathbb{E}_k(x) = \mathbb{E}_k \circ \widehat{T}(x)$ for all $x \in \mathcal{R}$, where $(\mathbb{E}_k)_k$ are conditional expectations given in Theorem 4.15.*

Assume in addition that T is completely positive and φ-preserving. Then $\widehat{T}$ is also positive and $\tau_k \circ \widehat{T} = \tau_k$, $\widehat{\varphi} \circ \widehat{T} = \widehat{\varphi}$ where τ_k is the trace given in Theorem 4.15. Moreover, the map

$$\widehat{T}^{(p)} : D_{\widehat{\varphi}}^{1/2p} x D_{\widehat{\varphi}}^{1/2p} \mapsto D_{\widehat{\varphi}}^{1/2p}\widehat{T}(x)D_{\widehat{\varphi}}^{1/2p}, \quad x \in \mathcal{R}$$

extends to a positive bounded maps on $L_p(\mathcal{R}, \widehat{\varphi})$ for all $1 \leq p \leq \infty$.

Convention. In the sequel, for a given map $T : \mathcal{M} \to \mathcal{M}$, we will denote, by the same symbol T, all the maps $\widehat{T}^{(p)}$ and their extensions to the L_p-spaces in the above setting, whenever no confusion can occur.

Let H_φ be the GNS completion of $\mathcal{M}$ with respect to φ (we make the convention that $\|x\|_{H_\varphi}^2 = \varphi(x^*x)$ for $x \in \mathcal{M}$). Note that $x \mapsto xD_\varphi^{1/2}$ yields an isometric isomorphism from H_φ to $L_2(\mathcal{M}, \varphi)$. Assume that there is an isometric isomorphism

$$U : H_\varphi \to \bigoplus_{i\in I} H_i \tag{4.40}$$

where I is an index set and H_i is a Hilbert space for each $i \in I$. Let $m := (m(i))_i$ be a bounded sequence with $m(i) \in B(H_i)$. As in (4.34), we may define the multiplier

$$T_m : H_\varphi \to H_\varphi, \quad U(T_m x) = (m(i)(Ux)(i))_{i\in I}, \quad x \in H_\varphi. \tag{4.41}$$

Applying the reduction theorems quoted above, we obtain the maximal inequalities for the nontracial setting. The a.u. convergence can be adapted in the setting of Haagerup's L_p-spaces, usually called Jajte's *(bilaterally) almost sure* (b.a.s. and a.s. for short) convergence [Jaj91], for which we also refer to [JX07, Section 7.4].

Definition 4.17. (1) Let $x_n, x \in L_p(\mathcal{M}, \varphi)$ with $1 \le p < \infty$. The sequence (x_n) is said to converge almost surely (a.s. in short) to x if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ and a family $(a_{n,k}) \subset \mathcal{M}$ such that

$$\varphi(e^\perp) < \varepsilon \quad \text{and} \quad x_n - x = \sum_{k\ge1} a_{n,k} D^{\frac{1}{p}}, \qquad \lim_{n\to\infty} \| \sum_{k\ge1} (a_{n,k} e) \|_\infty = 0,$$

where the two series converge in norm in $L_p(\mathcal{M}, \varphi)$ and $\mathcal{M}$, respectively.

(2) Let $x_n, x \in L_p(\mathcal{M}, \varphi)$ with $1 \le p < \infty$. The sequence (x_n) is said to converge bilateral almost surely (b.a.s. in short) to x if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ and a family $(a_{n,k}) \subset \mathcal{M}$ such that

$$\varphi(e^\perp) < \varepsilon \quad \text{and} \quad x_n - x = D^{\frac{1}{2p}} \sum_{k\ge1} a_{n,k} D^{\frac{1}{2p}}, \qquad \lim_{n\to\infty} \| \sum_{k\ge1} (e a_{n,k} e) \|_\infty = 0,$$

where the two series converge in norm in $L_p(\mathcal{M}, \varphi)$ and $\mathcal{M}$, respectively.

As we mentioned at the beginning of this subsection, the space $L_p(\mathcal{M}, \varphi)$ isometrically coincides with the tracial noncommutative L_p-space if the state φ is tracial. In this case, one can easily verify that Jajte's a.s. (resp. b.a.s) convergence recovers Lance's a.u. (resp. b.a.u.) convergence defined in Definition 2.7.

We keep the notation introduced previously in this subsection. The following is our main result in this subsection, which generalizes the results for **(A1)** in Theorem 4.2 and Theorem 4.3. Those for **(A2)** can be dealt with in a similar manner, and we leave the details to interesting readers.

Theorem 4.18. *Let $\ell := (\ell(i))_{i\in I}$ be a sequence with $\ell(i) \in B(H_i)$ and write $S_t = T_{e^{-t\ell}}$ for $t \in \mathbb{R}_+$.. For any $N \in \mathbb{N}$, let $m_N := (m_N(i))_{i\in I}$ be a bounded sequence with $m_N(i) \in B(H_i)$. Assume that the following conditions hold:*

(i) *$(S_t)_{t\in\mathbb{R}_+}$ extends to a semigroup of unital completely positive φ-preserving maps on $\mathcal{M}$ and for any $t \in \mathbb{R}_+$, $r \in \mathbb{R}$,*

$$S_t \circ \sigma_r = \sigma_r \circ S_t, \qquad \varphi(S_t(x)^* y) = \varphi(x^* S_t(y)) \qquad x, y \in \mathcal{M}.$$

(ii) *$(T_{m_N})_{N\in\mathbb{N}}$ extends to a family of selfadjoint maps on $\mathcal{M}$ with*

$$\gamma := \sup_N \|T_{m_N} : \mathcal{M} \to \mathcal{M}\| < \infty.$$

(iii) *There exist $\alpha > 0$ and $\beta > 0$ such that for all $i \in I$ we have*

$$\|\operatorname{id}_{H_i} - m_N(i)\|_{B(H_i)} \le \beta \frac{\|\ell(i)\|_{B(H_i)}^{\alpha}}{2^N}, \quad \|m_N(i)\|_{B(H_i)} \le \beta \frac{2^N}{\|\ell(i)\|_{B(H_i)}^{\alpha}}. \tag{4.42}$$

Then there is a constant $c > 0$ depending only on p, α, β, γ such that for all $2 \le p < \infty$,

$$\|(T_{m_N} x)_N\|_{L_p(\mathcal{M};\ell_\infty)} \le c\|x\|_p \qquad x \in L_p(\mathcal{M}, \varphi). \tag{4.43}$$

and $T_{m_N} x \to x$ a.s. (resp. b.a.s.) as $N \to \infty$ for all $x \in L_p(\mathcal{M})$ with $2 < p < \infty$ (resp. $p = 2$).

If in addition the maps $(T_{m_N})_{N\in\mathbb{N}}$ are unital completely positive, symmetric and φ-preserving on $\mathcal{M}$ and commute with the modular automorphism group σ, i.e.

$$T_{m_N} \circ \sigma_r = \sigma_r \circ T_{m_N} \qquad r \in \mathbb{R}, N \in \mathbb{N},$$

then the above maximal inequality (4.43) also holds for all $1 < p < \infty$ and $T_{m_N} x \to x$ b.a.s. as $N \to \infty$ for all $x \in L_p(\mathcal{M})$.

The proof of (4.43) for the case of $p \ge 2$ is a *mutatis mutandis* copy of the arguments in previous subsections. It suffices to note that the proof of Proposition 4.8 and Proposition 4.13 remains valid in the setting of Haagerup's L_p-spaces if T_{m_N} is selfadjoint on $\mathcal{M}$. We leave the details to interesting readers. The reason why the previous arguments do not adapt to the case of $p < 2$ is that the weak interpolation (Theorem 2.6) fails for Haagerup's L_p-spaces. So we will provide a proof for this case using the reduction theorems. On the other hand, we will only prove the maximal inequalities. The implication from maximal inequalities to a.s. (resp. b.a.s.) convergences, in particular the analogue of Proposition 2.9 (2), remains valid on $L_p(\mathcal{M}, \varphi)$ if we replace the one sided weak type inequality (2.4) by the strong type one on $L_p(\mathcal{M}; \ell_\infty^c)$ for $p > 2$ (resp. on $L_p(\mathcal{M}; \ell_\infty)$ for $1 < p \le 2$) by using [JX07, Lemma 7.10].

Proof of (4.43) for $1 < p < 2$ and completely positive T_{m_N}. The operator U in (4.40) induces an isometry on $L_2(\mathcal{R}_k, \tau_k)$ given by

$$\begin{array}{rccc} \widehat{U}_k : & L_2(\mathcal{R}_k, \tau_k) & \to & \bigoplus_{i\in I} H_i \otimes L_2(VN(\Gamma), \tau_\Gamma(e^{-a_k}\,\cdot)) \\ & x \rtimes \lambda(g) & \mapsto & U(x) \otimes \lambda(g). \end{array}$$

Indeed, $\widehat{U}_k$ is an isometry since for any finite sum $\sum_g x_g \rtimes \lambda(g) \in \mathcal{R}$, we have

$$\begin{aligned} &\|\widehat{U}_k(\sum_g x_g \rtimes \lambda(g))\|^2_{\bigoplus_{i\in I} H_i \otimes L_2(VN(\Gamma), \tau_\Gamma(e^{-a_k}\,\cdot))} \\ &= \sum_{g,h} \varphi(x_g^* x_h) \tau_\Gamma(e^{-a_k}\lambda(h-g)) = \sum_{g,h} \varphi(\sigma_{-g}(x_g^* x_h)) \tau_\Gamma(e^{-a_k}\lambda(h-g)) \\ &= \sum_{g,h} \widehat{\varphi}\left(\sigma_{-g}(x_g^* x_h) \rtimes \lambda((h-g)e^{-a_k}\right) = \|\sum_g x_g \rtimes \lambda(g)\|^2_{L_2(\mathcal{R}_k, \tau_k)}. \end{aligned}$$

Take the Hilbert subspaces $H_i' \subset H_i \otimes L_2(VN(\Gamma), \tau_\Gamma(e^{-a_k}\,\cdot))$ so that $\operatorname{ran}(\widehat{U}_k) = \bigoplus_i H_i'$. Then $\widehat{U}_k : L_2(\mathcal{R}_k, \tau_k) \to \bigoplus_i H_i'$ becomes an isometric isomorphism. For any $x \rtimes \lambda(g) \in \mathcal{R}_k$, the element $(T_{m_N} x) \rtimes \lambda(g) = \widehat{T}_{m_N}(x \rtimes \lambda(g))$ also belongs to $\mathcal{R}_k$ since $\widehat{T}_{m_N} \circ \mathbb{E}_k = \mathbb{E}_k \circ \widehat{T}_{m_N}$ by Theorem 4.16. So $\widehat{U}_k((T_{m_N} x) \rtimes \lambda(g))$ is well-defined and moreover, by (iii), we have

$$\widehat{U}_k((T_{m_N} x) \rtimes \lambda(g)) = U(T_{m_N}(x)) \otimes \lambda(g) = (mU(x)) \otimes \lambda(g) = (m \otimes \operatorname{id})\widehat{U}_k(x \rtimes \lambda(g)).$$

In particular $m_N(i) \otimes \mathrm{id}$ sends H_i' into H_i', and $\widehat{T}_{m_N}|_{\mathcal{R}_k}$ is an operator-valued multiplier in the sense of (4.34) (recall that $(\mathcal{R}_k, \tau_k)$ is tracial). It is straightforward to verify that $\widehat{T}_{m_N}|_{\mathcal{R}_k}$ is unital completely positive τ_k-preserving on $\mathcal{R}_k$, and therefore it extends to a positive contraction on $L_p(\mathcal{R}_k, \tau_k)$. Similarly, the extension $\widehat{S}_t := \widehat{T}_{e^{-t\ell}}$ also gives rise to a semigroup of unital completely positive maps on $\mathcal{R}$. It is easy to check that $\widehat{S}_t$ is symmetric relative to $\widehat{\varphi}$. The restriction $\widehat{S}_t|_{\mathcal{R}_k}$ is τ_k-preserving and symmetric relative to τ_k since

$$\widehat{S}_t((x \rtimes \lambda(g))e^{-a_k}) = (S_t x) \rtimes \lambda(g)e^{-a_k} = (\widehat{S}_t(x \rtimes \lambda(g)))e^{-a_k}$$

for all $x \in \mathcal{M}$ and $g \in \Gamma$. Thus applying Theorem 4.14 to $\widehat{T}_{m_N}|_{\mathcal{R}_k}$, we obtain

$$\|\sup_N{}^+ \widehat{T}_{m_N}(x)\|_{L_p(\mathcal{R}_k,\tau_k)} \le c\|x\|_{L_p(\mathcal{R}_k,\tau_k)}, \quad x \in L_p(\mathcal{R}_k, \tau_k), \tag{4.44}$$

where c is a constant only depending on α, β, p.

In the following we consider $x \in L_p(\mathcal{M}, \varphi)_+$. Since $L_p(\mathcal{M}, \varphi)$ can be naturally embedded into $L_p(\mathcal{R}, \widehat{\varphi})$, we regard x as an element in $L_p(\mathcal{R}, \widehat{\varphi})_+$. By Theorem 4.16, we see that $\widehat{T}_{m_N} \circ \mathbb{E}_k = \mathbb{E}_k \circ \widehat{T}_{m_N}$ and hence $\widehat{T}^{(p)}_{m_N} \circ \mathbb{E}^{(p)}_k = \mathbb{E}^{(p)}_k \circ \widehat{T}^{(p)}_{m_N}$. By Theorem 4.15, we have

$$\lim_{k\to\infty} \widehat{T}^{(p)}_{m_N}(\mathbb{E}^{(p)}_k(x)) = \lim_{k\to\infty} \mathbb{E}^{(p)}_k(\widehat{T}^{(p)}_{m_N}(x)) = T^{(p)}_{m_N}(x) \quad \text{in} \quad L_p(\mathcal{R}, \widehat{\varphi}).$$

Thus for any $M > 0$,

$$\lim_{k\to\infty} \|\sup_{1\le N\le M}{}^+ \widehat{T}^{(p)}_{m_N}(\mathbb{E}^{(p)}_k(x))\|_{L_p(\mathcal{R}_k,\widehat{\varphi}|_{\mathcal{R}_k})} = \|\sup_{1\le N\le M}{}^+ \widehat{T}^{(p)}_{m_N}(x)\|_{L_p(\mathcal{R},\widehat{\varphi})}. \tag{4.45}$$

Without loss of generality, we assume that $x = D_{\widehat{\varphi}}^{1/2p} y D_{\widehat{\varphi}}^{1/2p}$ with some $y \in \mathcal{R}_+$. By the correspondence in (4.37), we get

$$\|\sup_{1\le N\le M}{}^+ \widehat{T}^{(p)}_{m_N}(\mathbb{E}^{(p)}_k(x))\|_{L_p(\mathcal{R}_k,\widehat{\varphi}|_{\mathcal{R}_k})} = \|\sup_{1\le N\le M}{}^+ e^{\frac{a_k}{2p}} \widehat{T}_{m_N}(\mathbb{E}_k(y)) e^{\frac{a_k}{2p}}\|_{L_p(\mathcal{R}_k,\tau_k)}.$$

Note that $e^{\frac{a_k}{2p}}$ belongs to the subalgebra generated by $1 \rtimes \lambda(\Gamma)$. For any element of the form $z = \sum_{g\in\Gamma} z(g) \rtimes \lambda(g)$ in $\mathcal{R}_k$, we have

$$e^{\frac{a_k}{2p}} \widehat{T}_{m_N}(z) e^{\frac{a_k}{2p}} = \sum_{g\in\Gamma} T_{m_N}(z(g)) \rtimes (e^{\frac{a_k}{2p}} \lambda(g) e^{\frac{a_k}{2p}}) = \widehat{T}_{m_N}(e^{\frac{a_k}{2p}} z e^{\frac{a_k}{2p}}).$$

So the previous equality reads

$$\|\sup_{1\le N\le M}{}^+ \widehat{T}^{(p)}_{m_N}(\mathbb{E}^{(p)}_k(x))\|_{L_p(\mathcal{R}_k,\widehat{\varphi}|_{\mathcal{R}_k})} = \|\sup_{1\le N\le M}{}^+ \widehat{T}_{m_N}(e^{\frac{a_k}{2p}} \mathbb{E}_k(y) e^{\frac{a_k}{2p}})\|_{L_p(\mathcal{R}_k,\tau_k)}.$$

Together with (4.44), (4.45) and (4.37) we obtain

$$\begin{aligned}\|\sup_{1\le N\le M}{}^+ \widehat{T}^{(p)}_{m_N}(x)\|_{L_p(\mathcal{R},\widehat{\varphi})} &\le \lim_{k\to\infty} \|e^{\frac{a_k}{2p}} \mathbb{E}_k(y) e^{\frac{a_k}{2p}}\|_{L_p(\mathcal{R}_k,\tau_k)} = \lim_{k\to\infty} \|D_{\widehat{\varphi}}^{1/2p} \mathbb{E}_k(y) D_{\widehat{\varphi}}^{1/2p}\|_{L_p(\mathcal{R}_k,\widehat{\varphi}|_{\mathcal{R}_k})} \\ &= \lim_{k\to\infty} \|\mathbb{E}^{(p)}_k(x)\|_{L_p(\mathcal{R}_k,\widehat{\varphi}|_{\mathcal{R}_k})} = \|x\|_{L_p(\mathcal{R},\widehat{\varphi})}.\end{aligned}$$

The proof is complete. □

Remark 4.19. Lance's notion of a.u. convergence still makes sense for $p = \infty$ in the nontracial setting. The above theorem also implies that $T_{m_N} x \to x$ a.u. as $N \to \infty$ for all $x \in \mathcal{M}$, according to [JX07, Lemma 7.13].

5. Proof of Theorem 1.3

This section will be devoted to the proof of Theorem 1.3. In other words, we will construct Fourier multipliers satisfying the pointwise convergence for groups with suitable approximation properties. As in the previous section, only the framework of the form (4.1) (or the form (4.34)) is involved in the essential part of our arguments. Since the approximation properties of discrete quantum groups have drawn wide interest in recent years, we would like to present the work in a more general setting, that is, Woronowicz's *compact quantum groups*.

We refer to [Wor87, Wor98, Tim08] for a complete description of compact quantum groups. In this paper, it suffices to recall that each compact quantum group $\mathbb{G}$ is an object corresponding to a distinguished von Neumann algebra denoted by $L_\infty(\mathbb{G})$, a unital normal $*$-homomorphism $\Delta : L_\infty(\mathbb{G}) \to L_\infty(\mathbb{G})\overline{\otimes}L_\infty(\mathbb{G})$ (usually called the *comultiplication*), and a normal faithful state $h : L_\infty(\mathbb{G}) \to \mathbb{C}$ (usually called the *Haar state*) with the following properties. First, the Haar state h is invariant in the sense that

$$(h \otimes \mathrm{id}) \circ \Delta(x) = h(x)1 = (\mathrm{id} \otimes h) \circ \Delta(x), \qquad x \in L_\infty(\mathbb{G}).$$

Second, a unitary $n \times n$ matrix $u = (u_{ij})_{i,j=1}^n$ with coefficients $u_{ij} \in L_\infty(\mathbb{G})$ is called an *n-dimensional unitary representation* of $\mathbb{G}$ if for any $1 \le i, j \le n$ we have

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$

We denote by $\mathrm{Irr}(\mathbb{G})$ the collection of unitary equivalence classes of irreducible representations of $\mathbb{G}$, and we fix a representative $u^{(\pi)}$ on a Hilbert space H_π for each class $\pi \in \mathrm{Irr}(\mathbb{G})$ and denote by d_π its dimension. In particular, we denote by $\mathbf{1} \in \mathrm{Irr}(\mathbb{G})$ the trivial representation, i.e. $u^{\mathbf{1}} = 1_{\mathbb{G}}$ with dimension 1. Then the space

$$\mathrm{Pol}(\mathbb{G}) = \mathrm{span}\{u_{ij}^{(\pi)} : u^{(\pi)} = (u_{ij}^{(\pi)})_{i,j=1}^{d_\pi}, \pi \in \mathrm{Irr}(\mathbb{G})\}$$

is a w*-dense $*$-subalgebra of $L_\infty(\mathbb{G})$. We denote by $L_p(\mathbb{G})$ the Haagerup noncommutative L_p-spaces associated with $(L_\infty(\mathbb{G}), h)$. Last, there is a linear antihomomorphism S on $\mathrm{Pol}(\mathbb{G})$, called the *antipode* of $\mathbb{G}$, determined by

$$S(u_{ij}^{(\pi)}) = (u_{ji}^{(\pi)})^* \qquad \pi \in \mathrm{Irr}(\mathbb{G}), 1 \le i, j \le d_\pi.$$

The antipode S has the polar decomposition $S = R \circ \tau_{-i/2} = \tau_{-i/2} \circ R$ where R is a $*$-antiautomorphism on $L_\infty(\mathbb{G})$ and $(\tau_t)_{t\in\mathbb{R}}$ is a one-parameter group of $*$-automorphisms on $\mathrm{Pol}(\mathbb{G})$ (called the *scaling group*). There exists a distinguished sequence of strictly positive matrices $Q_\pi \in B(H_\pi)$ with $\pi \in \mathrm{Irr}(\mathbb{G})$ implementing the scaling group $(\tau_t)_{t\in\mathbb{R}}$ and the modular automorphism group $(\sigma_t)_{t\in\mathbb{R}}$ with respect to h on $L_\infty(\mathbb{G})$, and indeed for all $z \in \mathbb{C}$ we have (see for instance the computations in [Wan17, Section 2.1.2 and Section 3]),

$$(\tau_z \otimes \mathrm{id})(u^{(\pi)}) = Q_\pi^{\mathrm{i}z} u^{(\pi)} Q_\pi^{-\mathrm{i}z}, \quad (\sigma_z \otimes \mathrm{id})(u^{(\pi)}) = Q_\pi^{\mathrm{i}z} u^{(\pi)} Q_\pi^{\mathrm{i}z}. \tag{5.1}$$

We say that $\mathbb{G}$ is of *Kac type* if $Q_\pi = \mathrm{id}_{H_\pi}$ for all $\pi \in \mathrm{Irr}(\mathbb{G})$. In other words, $\mathbb{G}$ is of Kac type if and only if the Haar state h is tracial.

Denote by $\ell_\infty(\widehat{\mathbb{G}}) = \bigoplus_{\pi\in\mathrm{Irr}(\mathbb{G})} B(H_\pi)$ the direct sum of von Neumann algebras $B(H_\pi)$ and $c_c(\widehat{\mathbb{G}})$ be the finite direct sum in $\bigoplus_{\pi\in\mathrm{Irr}(\mathbb{G})} B(H_\pi)$, i.e. $m \in c_c(\widehat{\mathbb{G}})$ if there are only finite many π such that $m(\pi) \neq 0$. The notation $\widehat{\mathbb{G}}$ used above in fact corresponds to the dual

discrete quantum group of $\mathbb{G}$ (see e.g. [VD98]). We will not involve the detailed quantum group structure of $\widehat{\mathbb{G}}$. For a linear functional φ on $\mathrm{Pol}(\mathbb{G})$, we define the *Fourier transform* by

$$\mathcal{F}(\varphi)(\pi) = (\varphi \otimes \mathrm{id})((u^{(\pi)})^*), \qquad \pi \in \mathrm{Irr}(\mathbb{G}).$$

This induces the *Fourier transform* $\mathcal{F} : \mathrm{Pol}(\mathbb{G}) \to c_c(\widehat{\mathbb{G}})$, given by

$$\mathcal{F}(x)(\pi) = (h(\cdot x) \otimes \mathrm{id}) \left((u^{(\pi)})^* \right) \in B(H_\pi), \qquad \pi \in \mathrm{Irr}(\mathbb{G}).$$

Note that $\mathcal{F} : \mathrm{Pol}(\mathbb{G}) \to c_c(\widehat{\mathbb{G}})$ is bijective. Obviously there exists a Hilbert space completion of $c_c(\widehat{\mathbb{G}})$, denoted by $\ell_2(\widehat{\mathbb{G}})$, such that $\mathcal{F}$ extends to an isometric isomorphism $\mathcal{F} : H_h \to \ell_2(\widehat{\mathbb{G}})$, where H_h denotes the GNS completion of $L_\infty(\mathbb{G})$ with respect to h as in Section 4.5.2 (see e.g. [PW90] and [Wan17, Proposition 3.2]), more precisely,

$$h(x^*x) = \sum_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathrm{Tr}(Q_\pi) \mathrm{Tr}(Q_\pi (\mathcal{F}(x)(\pi))^* \mathcal{F}(x)(\pi)), \quad x \in H_h, \tag{5.2}$$

where Tr denotes the usual (unnormalized) trace on matrices. Then $\mathcal{F}$ is consistent with our framework in Section 4.5.2. For a symbol $m = (m(\pi))_\pi \in \ell_\infty(\widehat{\mathbb{G}})$, we can define a multiplier T_m by (4.41), i.e.

$$T_m(x) = \mathcal{F}^{-1}(m \cdot \mathcal{F}(x)), \quad x \in \mathrm{Pol}(\mathbb{G}),$$

which extends to a bounded map on H_h. This coincides with the multipliers considered in [JNR09, Daw12, Wan17]. By (5.1) and the definition of T_m, we have (see e.g. [Wan17, Lemma 3.6])

$$\sigma_r \circ T_m \circ \sigma_{-r} = T_{Q^{ir} m Q^{-ir}}, \quad r \in \mathbb{R}, \quad \text{where } Q = \oplus_\pi Q_\pi. \tag{5.3}$$

Remark 5.1. Let Γ be a discrete group. We may define a comultiplication Δ on the group von Neumann algebra $VN(\Gamma)$ by

$$\Delta(\lambda(g)) = \lambda(g) \otimes \lambda(g), \qquad g \in \Gamma.$$

The triple $(VN(\Gamma), \Delta, \tau)$ carries a compact quantum group structure $\mathbb{G}$ satisfying the aformentioned properties, where we take $L_\infty(\mathbb{G}) = VN(\Gamma)$ and $h = \tau$. In this case we usually denote $\mathbb{G} = \widehat{\Gamma}$. We remark that in our language of quantum groups, Γ coincides with the dual discrete quantum group $\widehat{\mathbb{G}}$. The set of unitrary equivalence classes of irreducible representations $\mathrm{Irr}(\mathbb{G})$ can be indexed by Γ, so that for every $g \in \Gamma$ the associated representation is of dimension 1 and is given by $u^{(g)} = \lambda(g) \in VN(\Gamma)$. Therefore, the set I defined above becomes Γ, and the Fourier transform sends $\lambda(g)$ to δ_g. Hence, for any $m \in \ell_\infty(\Gamma)$, the associated multiplier is

$$T_m : \lambda(g) \mapsto m(g)\lambda(g), \quad g \in \Gamma.$$

This notion coincides with the usual one on groups mentioned in the introduction.

A straightforward computation shows the following proposition (see for instance [Wan17, Section 3]).

Proposition 5.2. *Let S be the antipode and Φ be a functional on* $\mathrm{Pol}(\mathbb{G})$.

(1) $\mathcal{F}(\Phi \circ S^{-1})(\pi) = (\Phi \otimes \mathrm{id})(u^{(\pi)})$.
(2) $\mathcal{F}(\Phi^* \circ S^{-1})(\pi) = (\mathcal{F}(\Phi)(\pi))^*$ *where* $\Phi^*(x) = \overline{\Phi(x^*)}$ *for any* $x \in \mathrm{Pol}(\mathbb{G})$.
(3) $\mathcal{F}((\mathrm{id} \otimes \Phi)\Delta(x)) = \mathcal{F}(\Phi \circ S^{-1}) \cdot \mathcal{F}(x)$.

Remark 5.3. Let Φ be a functional on $\mathrm{Pol}(\mathbb{G})$ such that $\mathcal{F}(\Phi\circ S^{-1})=m$. In other words, by Proposition 5.2 (1),

$$m(\pi)=[\Phi(u_{ij}^{(\pi)})]_{ij}.$$

By Proposition 5.2 (3), we have

$$T_m(x)=\mathcal{F}^{-1}\left(\mathcal{F}(\Phi\circ S^{-1})\cdot\mathcal{F}(x)\right)=(\mathrm{id}\otimes\Phi)\Delta(x). \tag{5.4}$$

Therefore, we have the following properties.

(1) T_m is a unital completely positive map on $\mathrm{Pol}(\mathbb{G})$ if and only if Φ is a state on $\mathrm{Pol}(\mathbb{G})$.
(2) T_m is selfadjoint if and only if $\Phi^*=\Phi$. Moreover $m(\pi)^*=m(\pi)$ if and only if $\Phi^*\circ S=\Phi$.
(3) We have

$$\begin{aligned}\|m(\pi)\|_{B(H_\pi)}=\|u^{(\pi)}m(\pi)\|_{L_\infty(\mathbb{G})\otimes B(H_\pi)}&=\left\|\left[\sum_k u_{ik}^{(\pi)}\Phi(u_{kj}^{(\pi)})\right]_{ij}\right\|_{L_\infty(\mathbb{G})\otimes B(H_\pi)}\\&=\left\|\left[T_m(u_{ij}^{(\pi)})\right]_{ij}\right\|_{L_\infty(\mathbb{G})\otimes B(H_\pi)}\le\|T_m\|_{cb}.\end{aligned}$$

In view of our study in Section 4.5.2, it is essential to consider the case where T_m commutes with the modular automorphism group σ.

Proposition 5.4. *Let Φ be a functional on $\mathrm{Pol}(\mathbb{G})$ with $\Phi^*=\Phi$ and $\varphi(\pi)=[\Phi(u_{ij}^{(\pi)})]_{i,j}$ for $\pi\in\mathrm{Irr}(\mathbb{G})$. Then the element*

$$m(\pi)=\lim_{a\to\infty}\frac{1}{2a}\int_{-a}^{a}Q_\pi^{ir}\tilde{\varphi}(\pi)Q_\pi^{-ir}dr,\quad \textit{where } \tilde{\varphi}(\pi)=[\frac{1}{2}(\Phi+\Phi\circ R)(u_{ij}^{(\pi)})]_{i,j}$$

is well-defined in $B(H_\pi)$, and satisfies:
(i) *$m(\pi)$ is a selfadjoint matrix for any $\pi\in\mathrm{Irr}(\mathbb{G})$;*
(ii) *for any $t\in\mathbb{R}$, $\sigma_t\circ T_m=T_m\circ\sigma_t$;*
(iii) *$\|m(\pi)\|_{B(H_\pi)}\le\|\varphi(\pi)\|_{B(H_\pi)}$ and $\|\mathrm{id}_{H_\pi}-m(\pi)\|_{B(H_\pi)}\le\|\mathrm{id}_{H_\pi}-\varphi(\pi)\|_{B(H_\pi)}$ for any $\pi\in\mathrm{Irr}(\mathbb{G})$;*
(iv) *If T_φ is unital completely positive on $\mathrm{Pol}(\mathbb{G})$, so is T_m.*

Proof. The construction is implicitly given in the proof of [CS15, Lemma 5.2] and [DFSW16, Proposition 7.17].

The element m is well-defined by the ergodic theorem since $B(H_\pi)$ is finite-dimensional. Note that there is a $*$-antiautomorphism $\widehat{R}$ of $\ell_\infty(\widehat{\mathbb{G}})$ with $\widehat{R}^2=\mathrm{id}$ such that $(R\otimes\widehat{R})\mathrm{U}=\mathrm{U}$ where $\mathrm{U}=\oplus_{\pi\in\mathrm{Irr}(\mathbb{G})}u^{(\pi)}$ is regarded as an element in $L_\infty(\mathbb{G})\overline{\otimes}\ell_\infty(\widehat{\mathbb{G}})$ (see [Kus01, Proposition 7.2]), and therefore for any functional Υ on $L_\infty(\mathbb{G})$, the following inequality holds

$$\begin{aligned}\|(\Upsilon\circ R\otimes\mathrm{id})(\mathrm{U})-\mathbf{1}_{\ell_\infty(\widehat{\mathbb{G}})}\|_{\ell_\infty(\widehat{\mathbb{G}})}&\le\|(\Upsilon\otimes\widehat{R})(\mathrm{U})-\widehat{R}(\mathbf{1}_{\ell_\infty(\widehat{\mathbb{G}})})\|_{\ell_\infty(\widehat{\mathbb{G}})}\\&\le\|(\Upsilon\otimes\mathrm{id})(\mathrm{U})-\mathbf{1}_{\ell_\infty(\widehat{\mathbb{G}})}\|_{\ell_\infty(\widehat{\mathbb{G}})},\end{aligned}$$

and

$$\|(\Upsilon\circ R\otimes\mathrm{id})(\mathrm{U})\|_{\ell_\infty(\widehat{\mathbb{G}})}\le\|(\Upsilon\otimes\mathrm{id})(\mathrm{U})\|_{\ell_\infty(\widehat{\mathbb{G}})}.$$

In particular, for $\pi\in\mathrm{Irr}(\mathbb{G})$, taking $\Upsilon(u_{ij}^{(\alpha)})=\delta_{\alpha\pi}\Phi(u_{ij}^{(\pi)})$ for all $\alpha\in\mathrm{Irr}(\mathbb{G})$, we get

$$\|\widetilde{\varphi}(\pi)\|_{B(H_\pi)}\le\|\varphi(\pi)\|_{B(H_\pi)},$$

and taking $\Upsilon(u_{ij}^{(\alpha)}) = \begin{cases} \Phi(u_{ij}^{(\pi)}), & \text{if } \alpha = \pi \\ \delta_{ij}, & \text{if } \alpha \neq \pi \end{cases}$, we get

$$\| \mathrm{id}_{H_\pi} - \widetilde{\varphi}(\pi)\|_{B(H_\pi)} \leq \| \mathrm{id}_{H_\pi} - \varphi(\pi)\|_{B(H_\pi)}, \quad \pi \in \mathrm{Irr}(\mathbb{G}).$$

Then (iii) follows from the definition of m. Also, by Remark 5.3 we see that $T_{\tilde{\varphi}}$ is unital completely positive if T_φ is. So in the following part we will assume without loss of generality that $\varphi = \tilde{\varphi}$ and $\Phi = \Phi \circ R$. By (5.3) we have

$$T_m = \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{a} \sigma_r \circ T_\varphi \circ \sigma_{-r} dr, \tag{5.5}$$

so we established the assertions (ii) and (iv). We consider

$$\Psi(x) = \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{a} \Phi(\tau_r(x)) dr, \quad x \in \mathrm{Pol}(\mathbb{G}).$$

Then by (5.1),

$$\begin{aligned} m(\pi) &= \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{a} Q_\pi^{ir} \varphi(\pi) Q_\pi^{-ir} dr = \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{a} Q_\pi^{ir} [(\Phi \otimes \mathrm{id})(u^{(\pi)})] Q_\pi^{-ir} dr \\ &= \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{a} (\Phi \circ \tau_r \otimes \mathrm{id})(u^{(\pi)}) dr = (\Psi \otimes \mathrm{id})(u^{(\pi)}). \end{aligned}$$

Note that Ψ is invariant under τ according to the ergodic theorem. Recall that $\Phi \circ R = \Phi$. So we have $\Psi \circ S = \Psi$. By Proposition 5.2 and $\Phi = \Phi^*$, we get

$$m(\pi) = \mathcal{F}(\Psi \circ S^{-1})(\pi) = (\mathcal{F}(\Psi)(\pi))^* = \big(\mathcal{F}(\Psi \circ S^{-1})(\pi)\big)^* = m(\pi)^*.$$

So we obtain (i). □

We recall the following approximation properties of quantum groups introduced by De Commer, Freslon and Yamashita in [DCFY14]. For simplicity of exposition, *we always assume in this paper that* $\mathrm{Irr}(\mathbb{G})$ *is countable*. The general cases can be dealt with by considering the collection of all finitely generated quantum subgroups of $\widehat{\mathbb{G}}$.

Definition 5.5. Let $\mathbb{G}$ be a compact quantum group. $\widehat{\mathbb{G}}$ is said to have the *almost completely positive approximation property* (ACPAP for short) if there are two sequences $(\varphi_s)_{s\in\mathbb{N}} \subset \ell_\infty(\widehat{\mathbb{G}})$ and $(\psi_k)_{k\in\mathbb{N}} \subset c_c(\widehat{\mathbb{G}})$ such that

(1) for any $s, k \in \mathbb{N}$, T_{φ_s} is a unital completely positive map on $L_\infty(\mathbb{G})$;

(2) for any $\pi \in \mathrm{Irr}(\mathbb{G})$, we have

$$\lim_{s\to\infty} \| \mathrm{id}_{H_\pi} - \varphi_s(\pi)\|_{B(H_\pi)} = 0 \ \text{ and } \lim_{k\to\infty} \| \mathrm{id}_{H_\pi} - \psi_k(\pi)\|_{B(H_\pi)} = 0;$$

(3) for any $s \in \mathbb{N}$ and $\varepsilon > 0$, there is a $k = k(s, \varepsilon)$ such that $\|T_{\psi_k} - T_{\varphi_s}\|_{cb} \leq \varepsilon$.

Moreover, if for any $k \in \mathbb{N}$ we may directly choose T_{ψ_k} to be unital completely positive, then $\widehat{\mathbb{G}}$ is said to be *amenable* (or to have the *completely positive approximation property*).

For notational convenience and without loss of generality, in the sequel we will always set $\varphi_0(\pi) = \psi_0(\pi) = \delta_{\mathbf{1}}(\pi)$. Note that $\varphi_s(\mathbf{1}) = 1$ for all s, therefore we will always assume that $\psi_k(\mathbf{1}) = 1$ for all k.

Remark 5.6. Assume that $\widehat{\mathbb{G}}$ has the ACPAP. Then we may indeed find two sequences $(\varphi_s)_{s\in\mathbb{N}} \subset \ell_\infty(\widehat{\mathbb{G}})$, $(\psi_k)_{k\in\mathbb{N}} \subset c_c(\widehat{\mathbb{G}})$ satisfying (1)-(3) in Definition 5.5 such that the following assertions also hold:

(4) T_{ψ_k} is selfadjoint for all $k \in \mathbb{N}$;

(5) φ_s and ψ_k are selfadjoint matrices for all $s, k \in \mathbb{N}$;

(6) for any $t \in \mathbb{R}$, $s \in \mathbb{N}$, $k \in \mathbb{N}$, $\sigma_t \circ T_{\phi_s} = T_{\phi_s} \circ \sigma_t$ and $\sigma_t \circ T_{\psi_k} = T_{\psi_k} \circ \sigma_t$.

Indeed, for any sequences $(\varphi_s)_{s\in\mathbb{N}} \subset \ell_\infty(\widehat{\mathbb{G}})$, $(\psi_k)_{k\in\mathbb{N}} \subset c_c(\widehat{\mathbb{G}})$ satisfying (1)-(3) in Definition 5.5, by Remark 5.3 we see that the map $x \mapsto (T_{\psi_k}(x^*))^*$ is a multiplier associated with the matrices $\widetilde{\psi}_k(\pi) = [\Psi_k^*(u_{ij}^{(\pi)})]_{ij}$, where Ψ_k is the functional on $\mathrm{Pol}(\mathbb{G})$ so that $\Psi_k(u_{ij}^{(\pi)}) = \psi_k(\pi)_{ij}$. Then $T_{(\psi_k+\widetilde{\psi}_k)/2}$ is selfadjoint. Note that $\lim_{k\to\infty} \| \mathrm{id}_{H_\pi} - \psi_k(\pi)\|_\infty = 0$ is equivalent to $\lim_{k\to\infty} \Psi_k(u_{ij}^{(\pi)}) \to \delta_{ij}$ pointwisely. Thus $(\psi_k + \widetilde{\psi}_k)/2$ satisfies (2). Since the operators T_{φ_s} are positive, $T_{\varphi_s}(x) = (T_{\varphi_s}(x^*))^*$. Thus $((\psi_k + \widetilde{\psi}_k)/2)_k$ satisfies (3). Therefore we may always assume that T_{ψ_k} is selfadjoint, which means that $\Psi_k = \Psi_k^*$ by Remark 5.3 (2). We construct two new sequences $(\varphi'_s)_{s\in\mathbb{N}} \subset \ell_\infty(\widehat{\mathbb{G}}), (\psi'_k)_{k\in\mathbb{N}} \subset c_c(\widehat{\mathbb{G}})$ by the formulas given in Proposition 5.4, then they immediately satisfy (3) by (5.5). It is easy to see that $(\varphi'_s)_{s\in\mathbb{N}}$ and $(\psi'_k)_{k\in\mathbb{N}}$ satisfy (1) (2) (4) (5) (6).

Remark 5.7. If $\mathbb{G}$ is of Kac type, the multipliers T_{φ_s} and T_{ψ_k} can be taken central by a simple averaging argument, that is, $\varphi_s(\pi)$ and $\psi_k(\pi)$ belong to $\mathbb{C}\,\mathrm{id}_{H_\pi}$ for all $\pi \in \mathrm{Pol}(\mathbb{G})$. We refer to [KR99, Bra17] for details.

Lemma 5.8. *For any $s \in \mathbb{N}_+$, $\varphi_s = (\varphi_s(\rho))_{\rho\in I}$ with $\varphi_s(\rho) \in B(H_\rho)$, where $(H_\rho)_\rho$ are Hilbert spaces and I is an infinite countable set. Let $(E_s)_{s\geq 1}$ be an increasing sequence of finite subsets of I with $\cup_{s\geq 1} E_s = I$. If $\lim_{s\to\infty} \|\varphi_s(\rho) - \mathrm{id}_{H_\rho}\|_{B(H_\rho)} = 0$ for any $\rho \in I$, then there is a subsequence $(s_N)_{N\geq 1}$ of $\mathbb{N}$ such that*

$$\| \mathrm{id}_{H_\rho} - \varphi_{s_{N+1}}(\rho)\|_{B(H_\rho)} \leq 2^{-N}, \qquad \rho \in E_{s_N}. \tag{5.6}$$

Proof. We will construct a sequence $(s_N)_{N\in\mathbb{N}}$ by induction. First we let $s_1 = 1$. Assume that $(s_j)_{j=0}^N$ has already been defined. For any $\rho \in E_{s_N}$, we can find an $s_{N+1}(\rho) > s_N$ large enough, such that for any $s \geq s_{N+1}(\rho)$,

$$\| \mathrm{id}_{H_\rho} - \varphi_s(\rho)\|_{B(H_\rho)} \leq 2^{-N}.$$

Since E_{s_N} is a finite set, we can set $s_{N+1} = \max\{s_{N+1}(\rho) : \rho \in E_{s_N}\}$. Therefore the proof is complete. □

Then we may construct the semigroups and multipliers satisfying the assumptions of Theorem 4.18.

Theorem 5.9. *Assume that $\widehat{\mathbb{G}}$ has the ACPAP and let $(\varphi_s)_{s\in\mathbb{N}}$ and $(\psi_k)_{k\in\mathbb{N}}$ be the corresponding sequences satisfying (1)-(6) in Definition 5.5 and Remark 5.6. Let $(k_s)_{s\in\mathbb{N}_+}$ be an increasing subsequence of $\mathbb{N}$ such that $\|T_{\varphi_s} - T_{\psi_{k_s}}\|_{cb} \leq \frac{1}{2^{s+1}}$. Let $k_0 = s_0 = 0$ and $(s_N)_{N\in\mathbb{N}_+}$ be a subsequence of $\mathbb{N}$ such that (5.6) holds with $E_s = \cup_{i=0}^s \operatorname{supp} \psi_{k_i}$. Define*

$$\ell(\pi) = \sum_{j\geq 0} \sqrt{2}^j \left(\mathrm{id}_{H_\pi} - \varphi_{s_j}(\pi)\right).$$

Then the following assertions hold.

(1) $\ell(\mathbf{1}) = 0$ *and for any* $\pi \neq \mathbf{1}$,

$$\|\ell(\pi)\|_{B(H_\pi)} \asymp \sqrt{2}^{J(\pi)}$$

where $J(\pi) := \min\{j \in \mathbb{N} : \pi \in E_{s_j}\}$.

(2) $S_t : x \mapsto \mathcal{F}^{-1}\left(e^{-t\ell(\pi)}\mathcal{F}(x)\right)$ *is a semigroup of unital completely positive* h*-preserving maps on* $L_\infty(\mathbb{G})$ *and for any* $t \geq 0$, $r \in \mathbb{R}$,

$$S_t \circ \sigma_r = \sigma_r \circ S_t \quad \text{and} \quad h(S_t(x)^* y) = h(x^* S_t(y)), \qquad x, y \in L_\infty(\mathbb{G}).$$

(3) *Denote* $m_N = \psi_{k_{s_N}}$. *Then* $(m_N)_{N\in\mathbb{N}}$ *satisfies*

$$\|\operatorname{id}_{H_\pi} - m_N(\pi)\|_{B(H_\pi)} \lesssim \frac{\|\ell(\pi)\|^2_{B(H_\pi)}}{2^N}, \qquad \|m_N(\pi)\|_{B(H_\pi)} \lesssim \frac{2^N}{\|\ell(\pi)\|^2_{B(H_\pi)}}, \qquad \pi \in \operatorname{Irr}(\mathbb{G}).$$

In particular, for any $2 \leq p < \infty$,

$$\|\sup_{N\in\mathbb{N}}{}^{+} T_{m_N} x\|_p \lesssim_p \|x\|_p, \qquad x \in L_p(\mathbb{G}).$$

For all $x \in L_p(\mathbb{G})$ *with* $2 < p < \infty$ *(resp.* $p = 2$*),* $T_{m_N}(x)$ *converges a.s. (resp. b.a.s.) to* x *as* $N \to \infty$.

Moreover, if $\widehat{\mathbb{G}}$ *is amenable, then the above results hold for all* $1 < p < \infty$ *(with the b.a.s. convergence for* $p \leq 2$*). If* $\mathbb{G}$ *is of Kac type, all the convergences above are a.u.*

Proof. By Remark 5.6 and (5.3), we have $\ell(\pi)Q_\pi = Q_\pi \ell(\pi)$ for any $\pi \in \operatorname{Irr}(\mathbb{G})$ and $S_t \circ \sigma_r = \sigma_r \circ S_t$ for any $t \in \mathbb{R}_+, r \in \mathbb{R}$.

Recall that for any N, $T_{\varphi_{s_N}}$ is unital and in particular $\varphi_{s_N}(\mathbf{1}) = 1$. As a consequence we get $\ell(\mathbf{1}) = 0$. In the following we consider $\pi \neq \mathbf{1}$ and estimate the quantity $\|\ell(\pi)\|_{B(H_\pi)}$. Recall that $k_0 = s_0 = 0$ and $\varphi_0(\pi) = \psi_0(\pi) = \delta_{\mathbf{1}}(\pi)$, so $E_0 = \{\mathbf{1}\}$, which implies $J(\pi) \geq 1$ if $\pi \neq \mathbf{1}$. By the definition of J, we have $\pi \in E_{s_{J(\pi)}} \subset E_{s_{j-1}}$ if $j \geq J(\pi) + 1$. Recall that T_{φ_s} is unital completely positive and hence $\|\varphi_s(\pi)\|_{B(H_\pi)} \leq 1$ by Remark 5.3. Therefore, by (5.6), we have

$$\begin{aligned}\|\ell(\pi)\|_{B(H_\pi)} &\leq \sum_{j=0}^{J(\pi)} \sqrt{2}^j \|\operatorname{id}_{H_\pi} - \varphi_{s_j}(\pi)\|_{B(H_\pi)} + \sum_{j\geq J(\pi)+1} \sqrt{2}^j \|\operatorname{id}_{H_\pi} - \varphi_{s_j}(\pi)\|_{B(H_\pi)} \\ &\leq \sum_{j=0}^{J(\pi)} 2\sqrt{2}^j + \sum_{j\geq J(\pi)+1} \sqrt{2}^j \cdot 2^{-j} \lesssim \sqrt{2}^{J(\pi)}.\end{aligned}$$

For $j \leq J(\pi) - 1$, we have $\pi \notin E_{s_j}$ and hence $\psi_{k_{s_j}}(\pi) = 0$. We also have $\operatorname{id}_{H_\pi} - \varphi_{s_j}(\pi) \geq 0$ for all j since $\varphi_s(\pi)$ is selfadjoint and $\|\varphi_s(\pi)\|_{B(H_\pi)} \leq 1$ as mentioned previously. Also by

Remark 5.3, we have $\|\varphi_s(\pi)-\psi_{k_s}(\pi)\|_{B(H_\pi)}\le\|T_{\varphi_s}-T_{\psi_{k_s}}\|_{cb}\le\frac{1}{2^{s+1}}$. So

$$\begin{aligned}\|\ell(\pi)\|_{B(H_\pi)}&=\|\sum_{j=0}^{J(\pi)-1}\sqrt{2}^j\left(\mathrm{id}_{H_\pi}-\varphi_{s_j}(\pi)\right)+\sum_{j\ge J(\pi)}\sqrt{2}^j(\mathrm{id}_{H_\pi}-\varphi_{s_j}(\pi))\|_{B(H_\pi)}\\&\ge\|\sum_{j=0}^{J(\pi)-1}\sqrt{2}^j\left(\mathrm{id}_{H_\pi}-\varphi_{s_j}(\pi)\right)\|_{B(H_\pi)}\\&=\|\sum_{j=0}^{J(\pi)-1}\sqrt{2}^j\left(\mathrm{id}_{H_\pi}-\varphi_{s_j}(\pi)+\psi_{k_{s_j}}(\pi)\right)\|_{B(H_\pi)}\\&\ge\sum_{j=0}^{J(\pi)-1}\sqrt{2}^j\left(\mathrm{id}_{H_\pi}-\frac{1}{2^{s_j+1}}\right)\gtrsim\sqrt{2}^{J(\pi)}.\end{aligned}$$

Hence ℓ is well defined and (1) is verified.

Recall that $\ell(\mathbf{1})=0$. Therefore $S_t(1)=e^{-t\ell(\mathbf{1})}1=1$. On the other hand, recall that $\varphi_s(\pi)$ is selfadjoint, which means that $\ell(\pi)$ is also selfadjoint. Thus by (5.2),

$$\begin{aligned}h(S_t(x)^*y)&=\sum_{\pi\in\mathrm{Irr}(\mathbb{G})}\mathrm{Tr}(Q_\pi)\,\mathrm{Tr}\left(Q_\pi\left(e^{-t\ell(\pi)}\mathcal{F}(x)(\pi)\right)^*\mathcal{F}(y)(\pi)\right)\\&=\sum_{\pi\in\mathrm{Irr}(\mathbb{G})}\mathrm{Tr}(Q_\pi)\,\mathrm{Tr}\left(Q_\pi\left(\mathcal{F}(x)(\pi)\right)^*e^{-t\ell(\pi)}\mathcal{F}(y)(\pi)\right)\\&=h(x^*S_t(y)).\end{aligned}\tag{5.7}$$

In particular,

$$h(S_t(x))=h(S_t(1)x)=h(x).$$

Now let us verify the complete positivity of S_t. We define the functionals ϵ, L and Φ_s on $\mathrm{Pol}(\mathbb{G})$ by

$$\epsilon(u_{ij}^{(\pi)})=\delta_{ij},\quad L(u_{ij}^{(\pi)})=\ell(\pi)_{ij},\quad \Phi_s(u_{ij}^{(\pi)})=\varphi_s(\pi)_{ij}.$$

Note that ϵ is a $*$-homomorphism on $\mathrm{Pol}(\mathbb{G})$, usually called the counit. By Remark 5.3, Φ_s are states and

$$L=\sum_{j\ge0}\sqrt{2}^j\left(\epsilon-\Phi_{s_j}\right)$$

with the convergence understood pointwise on $\mathrm{Pol}(\mathbb{G})$. In particular

$$L(a^*a)=-\sum_{j\ge0}\sqrt{2}^j\Phi_{s_j}(a^*a)\le0,\quad a\in\ker\epsilon.$$

This means that L is a generating functional in the sense of [DFSW16] and there is a state μ_t with $\mu_t(u_{ij}^{(\pi)})=(e^{-t\ell(\pi)})_{ij}$ for all t and π by [DFSW16, Lemma 7.14 and Equality (7.4)]. So by Remark 5.3, $S_t=T_{e^{-t\ell}}$ is completely positive.

We take $m_N=\psi_{k_{s_N}}$. (If $\widehat{\mathbb{G}}$ is amenable, we take $m_N=\psi_{k_{s_N}}=\varphi_{s_N}$.) If $\pi=\mathbf{1}$, then for any $N\in\mathbb{N}$, $1-m_N(\mathbf{1})=0$ since $T_{\psi_{k_{s_N}}}$ is unital. Note that $s_N\ge N$. If $1\le J(\pi)\le N-1$,

i.e. $\pi \in E_{s_{N-1}}$, then

$$\begin{aligned}\|\operatorname{id}_{H_\pi} - m_N(\pi)\|_{B(H_\pi)} &\le \|\operatorname{id}_{H_\pi} - \varphi_{s_N}(\pi)\|_{B(H_\pi)} + \|\varphi_{s_N}(\pi) - \psi_{k_{s_N}}(\pi)\|_{B(H_\pi)} \\ &\le (2^{-N} + 2^{-s_N-1}) \lesssim 2^{J(\pi)-N} \lesssim \frac{\|\ell(\pi)\|^2_{B(H_\pi)}}{2^N}.\end{aligned}$$

If $J(\pi) \ge N$, then

$$\|\operatorname{id}_{H_\pi} - m_N(\pi)\|_{B(H_\pi)} \le 1 \le 2^{J(\pi)-N} \lesssim \frac{\|\ell(\pi)\|^2_{B(H_\pi)}}{2^N}.$$

On the other hand,

$$\|m_N(\pi)\|_{B(H_\pi)} \le \mathbb{1}_{[0,N]}(J(\pi)) \le 2^{N-J(\pi)} \lesssim \frac{2^N}{\|\ell(\pi)\|^2_{B(H_\pi)}}, \qquad \pi \in \operatorname{Irr}(\mathbb{G}).$$

A computation similar to (5.7) yields that the map T_{m_N} is symmetric and h-preserving for any N. Applying Theorem 4.14 and Theorem 4.18, we obtain the desired maximal inequalities and pointwise convergences □

Remark 5.10. The above proof indeed shows that Theorem 5.9 also holds for any subsequence $(s_N)_{N\in\mathbb{N}_+}$ satisfying

$$\|\operatorname{id}_{H_\rho} - \varphi_{s_{N+1}}(\rho)\|_{B(H_\rho)} \le 2^{J(\rho)-N}, \qquad \rho \in E_{s_N} \tag{5.8}$$

where $(E_s)_s$ are determined increasing finite sets and $J(\rho) = \min\{j \in \mathbb{N} : \rho \in E_{s_j}\}$. This more general formulation will be useful in the next section.

In particular, we obtain Theorem 1.3 in the general setting of quantum groups.

Corollary 5.11. (1) *Assume that $\widehat{\mathbb{G}}$ has the ACPAP. Then $\widehat{\mathbb{G}}$ admits a sequence of completely contractive Fourier multipliers $(T_{m_N})_{N\in\mathbb{N}}$ on $L_\infty(\mathbb{G})$ so that m_N are finitely supported and for any $2 \le p < \infty$,*

$$\|\sup_{N\in\mathbb{N}}{}^+ T_{m_N} x\|_p \lesssim_p \|x\|_p, \qquad x \in L_p(\mathbb{G}),$$

and $T_{m_N}x$ converges to x a.s. as $N \to \infty$ for all $x \in L_p(\mathbb{G})$ with $2 < p < \infty$ and $T_{m_N}x$ converges to x b.a.s. (a.u. if $\mathbb{G}$ is of Kac type) as $N \to \infty$ for all $x \in L_2(\mathbb{G})$.

(2) *Assume that $\widehat{\mathbb{G}}$ is amenable. Then $\widehat{\mathbb{G}}$ admits a sequence of unital completely positive Fourier multipliers $(T_{m_N})_{N\in\mathbb{N}}$ on $L_\infty(\mathbb{G})$ such that m_N are finitely supported and for any $1 < p < \infty$*

$$\|\sup_{N\in\mathbb{N}}{}^+ T_{m_N} x\|_p \lesssim_p \|x\|_p, \qquad x \in L_p(\mathbb{G}),$$

and $T_{m_N}x$ converges to x a.s. as $N \to \infty$ for all $x \in L_p(\mathbb{G})$ with $2 < p < \infty$ and $T_{m_N}x$ converges to x b.a.s. (a.u. if $\mathbb{G}$ is of Kac type) as $N \to \infty$ for all $x \in L_p(\mathbb{G})$ with $1 < p \le 2$.

Moreover, we have the following a.s. convergence of Fourier series of Dirichlet type on $L_2(\mathbb{G})$. Let p_π be the projection from $\operatorname{Pol}(\mathbb{G})$ onto $\{u_{ij}^{(\pi)} : 1 \le i, j \le d_\pi\}$. It is easy to see that p_π can be extended to an orthogonal projection on $L_2(\mathbb{G})$.

Proposition 5.12. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group having the ACPAP. Then there exists an increasing sequence of finite subsets $(K_N)_N \subset \operatorname{Irr}(\mathbb{G})$ such that the maps $x \mapsto \sum_{\pi\in K_N} p_\pi(x)$ is of strong type $(2,2)$. Moreover, for all $x \in L_2(\mathbb{G})$,*

$$\sum_{\pi\in K_N} p_\pi(x) \to x \text{ b.a.s. (a.u. if } \mathbb{G} \text{ is of Kac type)) as } N \to \infty.$$

Proof. Let ℓ, $(k_s)_s$, $(s_N)_N$, $(E_s)_s$ be given by Theorem 5.9. We show that $K_N = E_{s_N}$ satisfies this proposition. Let $N \in \mathbb{N}$ and $D_N = T_{\mathbb{1}_{K_N}}$ be the multiplier associated with the characteristic function

$$\mathbb{1}_{K_N}(\pi) = \begin{cases} \mathrm{id}_{H_\pi}, & \text{if } \pi \in K_N; \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$\phi_N(\pi) = \mathbb{1}_{K_N}(\pi) - e^{-\frac{\ell(\pi)^{1/2}}{2^{N/4}}}.$$

For any $\pi \in K_N$, i.e. $J(\pi) \leq N$, we have

$$\|\phi_N(\pi)\|_{B(H_\pi)} \lesssim \frac{\|\ell(\pi)\|_{B(H_\pi)}^{1/2}}{2^{N/4}} \lesssim \frac{\|\ell(\pi)\|_{B(H_\pi)}^{1/2} 2^{N/4}}{(2^{N/4} + \|\ell(\pi)\|_{B(H_\pi)}^{1/2})^2},$$

where the last inequality follows from the fact that $\frac{\|\ell(\pi)\|_{B(H_\pi)}}{\sqrt{2}^N} \lesssim 1$. Also, for any $\pi \notin K_N$, i.e. $J(\pi) > N$, we have

$$\|\phi_N(\pi)\|_{B(H_\pi)} \lesssim \frac{2^{N/4}}{\|\ell(\pi)\|_{B(H_\pi)}^{1/2}} \lesssim \frac{\|\ell(\pi)\|_{B(H_\pi)}^{1/2} 2^{N/4}}{(2^{N/4} + \|\ell(\pi)\|_{B(H_\pi)}^{1/2})^2}.$$

As mentioned previously, Proposition 4.8 remains valid for the nontracial setting. Together with Proposition 3.3, we get

$$\|(T_{\phi_N}(x))_N\|_{L_2(\mathbb{G};\ell_\infty)} \lesssim \|x\|_2, \quad \text{and} \quad \|(T_{\phi_N}(x))_N\|_{L_2(\mathbb{G};\ell_\infty^c)} \lesssim \|x\|_2, \qquad x \in L_2(\mathbb{G}).$$

Recall that by the choice of $(E_s)_s$, for any finite subset $F \subset \mathrm{Irr}(\mathbb{G})$ there exists $M \geq 1$ with $F \subset K_N$ for all $N \geq M$. Hence for any $x \in \mathrm{Pol}(\mathbb{G})$, there is an index M large enough such that for any $N \geq M$, we have $D_N(x) = x$, and in particular $D_N(x) \to x$ a.s. Then arguing as in Subsection 4.2, by Proposition 2.9 (2) (or its nontracial analogue for $L_2(\mathbb{G};\ell_\infty)$ and $L_2(\mathbb{G};\ell_\infty^c)$ mentioned after Theorem 4.18) and the density of $\mathrm{Pol}(\mathbb{G})$, we obtain the desired pointwise convergence of $D_N(x)$ for any $x \in L_2(\mathbb{G})$ as $N \to \infty$. □

Note that we also have the corresponding a.u. convergence on $L_\infty(\mathbb{G})$ in all the previous results by Remark 4.19.

6. More concrete examples

In this last section, we apply our theorems to various explicit examples of multipliers on noncommutative L_p-spaces.

6.1. Generalized Fejér means on non-abelian groups and quantum groups.

6.1.1. *Case of nilpotent groups and amenable groups.* Let Γ be a discrete amenable group. Let $(K_N)_{N\in\mathbb{N}}$ be a Følner sequence of Γ, that is

$$\lim_{N\to\infty} \frac{|K_N \cap gK_N|}{|K_N|} = 1, \qquad g \in \Gamma.$$

For convenience we set $K_0 = \{e\}$. We define a sequence of multipliers $(m_N)_{N\in\mathbb{N}}$ by

$$m_N(g) = \frac{|K_N \cap gK_N|}{|K_N|}. \tag{6.1}$$

We remark that if $\Gamma = \mathbb{Z}^d$ and $K_N = [-N, N]^d \cap \mathbb{Z}^d$, then

$$m_N(k_1, k_2, \cdots, k_d) = \prod_{i=1}^{d} (1 - \frac{|k_i|}{2N+1}) \mathbb{1}_{[-2N,2N]}(k_i),$$

which gives the usual Fejér means on the d-dimensional tori. As a result, we regard the multipliers $(T_{m_N})_N$ as a noncommutative analogue of Fejér means.

It is easy to see that m_N is finitely supported, indeed $\operatorname{supp} m_N = K_N \cdot K_N^{-1}$. By the Følner condition, we have $m_N \to 1$ pointwise. For any $g \in \Gamma$, we have

$$m_N(g) = \langle \lambda(g) \frac{\mathbb{1}_{K_N}}{|K_N|^{1/2}}, \frac{\mathbb{1}_{K_N}}{|K_N|^{1/2}} \rangle_{\ell_2(\Gamma)}.$$

As a consequence m_N is positive definite and therefore T_{m_N} is unital completely positive on $VN(\Gamma)$ for all $N \in \mathbb{N}$ (see e.g. [BO08, Theorem 2.5.11]). Note that T_{m_N} is also τ-preserving. In particular, by Theorem 5.9, there exists a subsequence $(N_j)_{j\in\mathbb{N}}$, such that for all $1 < p < \infty$ and all $x \in L_p(VN(\Gamma))$,

$$\|\sup_{j\in\mathbb{N}}{}^{+} T_{m_{N_j}} x\|_p \lesssim_p \|x\|_p \quad \text{and} \quad T_{m_{N_j}} x \to x \text{ a.u. as } j \to \infty.$$

In the following, we would like to give a refined study in the case of nilpotent groups. First we consider a 2-step (or 1-step) nilpotent group Γ generated by a finite set S. We assume that $e \in S$ and $S = S^{-1}$. Due to [Sto98], we have the following estimates:

$$\beta^{-1} N^d \leq |S^N| \leq \beta N^d \quad \text{and} \quad \beta^{-1} N^{d-1} \leq |S^N \backslash S^{N-1}| \leq \beta N^{d-1}, \tag{6.2}$$

where $d \geq 1$ is called the *degree* of Γ, and $\beta < \infty$ is a positive constant depending only on Γ and S.

For an element $g \in \Gamma$, denote by $|g|$ the word length of g with respect to the generator set S, i.e. $|g| = \min\{k : g \in S^k\}$. Let $(m_N)_N$ be a sequence of symbols given by (6.1) with $K_N = S^N$. By (6.2), we have

$$1 - m_N(g) = \frac{|gS^N \backslash S^N|}{|S^N|} \leq \frac{|S^{N+|g|} \backslash S^N|}{|S^N|} \leq \beta \frac{\sum_{i=N+1}^{N+|g|} i^{d-1}}{N^d} \lesssim \beta \frac{|g|}{N}, \qquad g \in \Gamma. \tag{6.3}$$

In particular, this shows that $(S^N)_N$ is a Følner sequence. On the other hand,

$$|m_N(g)| \leq \mathbb{1}_{[0,2N]}(|g|) \leq 2\frac{N}{|g|}, \qquad g \in \Gamma. \tag{6.4}$$

Set $J(g) = \min\{j \in \mathbb{N} : g \in S^{2^{j+1}}\}$, i.e. $J(g)$ is the unique integer with $2^{J(g)} < |g| \leq 2^{J(g)+1}$. We have

$$|1 - m_{2^j}(g)| \leq \beta \frac{|g|}{2^j} \leq 2\beta 2^{J(g)-j}.$$

This shows that the subsequence $(2^j)_{j\in\mathbb{N}}$ satisfies the inequality in Remark 5.10. Define

$$\ell(g) = \sum_{j\geq 0} \sqrt{2}^j |1 - m_{2^j}(g)|.$$

By Theorem 5.9, for any $g \neq e$,

$$\ell(g) \asymp \sqrt{2}^{J(g)} \asymp \sqrt{|g|},$$

and $(S_t)_{t\in\mathbb{R}_+} : \lambda(g) \mapsto e^{-t\ell(g)}\lambda(g)$ is a semigroup of unital completely positive trace preserving and symmetric maps. We remark that there are other natural choices of conditionally negative

definite functions with polynomial growth (see e.g. [CS16]), but our above construction is self-contained and useful for the further purpose. Moreover, for any $t \in \mathbb{R}_+$, S_t satisfies Rota's dilation property according to Lemma 3.6. Inequalities (6.3) and (6.4) can be written as

$$|1 - m_N(g)| \lesssim \beta \frac{\ell(g)^2}{N} \quad \text{and} \quad |m_N(g)| \lesssim \beta \frac{N}{\ell(g)^2}.$$

Moreover,

$$m_N(g) = \frac{\mathbb{1}_{S^N} * \mathbb{1}_{S^N}(g)}{|S^N|}.$$

By (6.2), we obtain

$$\begin{aligned}
&|m_{N+1}(g) - m_N(g)| \\
&= \left| \frac{1}{|S^{N+1}|} \sum_{\gamma \in \Gamma} \mathbb{1}_{S^{N+1}}(\gamma) \mathbb{1}_{S^{N+1}}(g^{-1}\gamma) - \frac{1}{|S^N|} (\sum_{\gamma \in \Gamma} \mathbb{1}_{S^N}(\gamma) \mathbb{1}_{S^N}(g^{-1}\gamma)) \right| \\
&\leq \frac{1}{|S^{N+1}|} \left(\sum_{\gamma \in \Gamma} \mathbb{1}_{S^N}(\gamma)[\mathbb{1}_{S^{N+1}}(g^{-1}\gamma) - \mathbb{1}_{S^N}(g^{-1}\gamma)] + \sum_{\gamma \in \Gamma} [\mathbb{1}_{S^{N+1}}(\gamma) - \mathbb{1}_{S^N}(\gamma)] \mathbb{1}_{S^{N+1}}(g^{-1}\gamma) \right) \\
&\qquad\qquad + \left| \left(\frac{1}{|S^{N+1}|} - \frac{1}{|S_N|} \right) \sum_{\gamma \in \Gamma} \mathbb{1}_{S^N}(\gamma) \mathbb{1}_{S^N}(g^{-1}\gamma) \right| \\
&\leq 3 \frac{|S^{N+1} \backslash S^N|}{|S^{N+1}|} \lesssim \beta^2 \frac{1}{N+1}.
\end{aligned}$$

Therefore m_N satisfies (4.4). Applying Theorem 4.3, we have the following corollary.

Corollary 6.1. *Let Γ be a 2-step (or 1-step) nilpotent group generated by a finite symmetric set S. Define $m_N(g) = \frac{|S^N \cap gS^N|}{|S^N|}$. Then*

(1) *$(T_{m_{2^j}})_{j \in \mathbb{N}}$ is of strong type (p,p) for all $1 < p < \infty$. Moreover, for any $x \in L_p(VN(\Gamma))$ with $1 < p < \infty$, $T_{m_{2^j}}(x)$ converges a.u. to x as $j \to \infty$.*

(2) *$(T_{m_N})_{N \in \mathbb{N}}$ is of strong type (p,p) for all $\frac{3}{2} < p < \infty$. Moreover, for any $x \in L_p(VN(\Gamma))$ with $3/2 < p < \infty$, $T_{m_N}(x)$ converges a.u. to x as $N \to \infty$.*

Let us give some remarks on the case of general groups with polynomial growth. Indeed, it is conjectured in [Bre14] that (6.2) remains true for general groups with polynomial growth. If the conjecture has a positive answer, then the above corollary still holds in this general setting by the same arguments. Moreover a partial result was given in [BLD13] for a general r-step nilpotent group Γ generated by a finite set S. It asserts that

$$\beta^{-1} N^{d-1} \leq |S^N \backslash S^{N-1}| \leq \beta N^{d - \frac{2}{3r}}, \tag{6.5}$$

where β is a constant depending only on Γ and S. Therefore, as in the arguments for the case of 2-step nilpotent groups, we have

$$1 - m_N(g) \leq \frac{|S^{N+|g|} \backslash S^N|}{|S^N|} \lesssim \beta \frac{|g|}{N^{\frac{2}{3r}}}.$$

Let $k = k(r)$ be the minimum integer with $k \geq \frac{3r}{2}$. Then $N_j(r) = 2^{k+k^2+\cdots k^j}$ satisfies Lemma 5.8. Indeed, let $J(g) = \min\{j \in \mathbb{N} : g \in E_{N_j(r)}\}$, i.e. $2N_{J(g)-1}(r) < |g| \leq 2N_{J(g)}(r)$.

Then for any $g \in E_{N_j(r)}$, i.e. $J(g) \leq j$,

$$|1 - m_{N_{j+1}(r)}(g)| \lesssim \beta \frac{|g|}{N_{j+1}(r)^{\frac{2}{3r}}} \lesssim \beta \frac{2^{k+k^2+\cdots k^{J(g)}}}{2^{\frac{2}{3r}(k+k^2+\cdots k^{j+1})}} \lesssim \beta \frac{2^{k+k^2+\cdots k^{J(g)}}}{2^{\frac{1}{k}(k+k^2+\cdots k^{j+1})}}$$
$$\lesssim \beta 2^{-\left(k^{J(g)+1}+\cdots k^{j+1}\right)} \lesssim \beta 2^{J(g)-j}.$$

Therefore, by Theorem 5.9 and Remark 5.10, we get a conditionally negative definite function ℓ on Γ such that

$$|1 - m_{N_{j+1}(r)}(g)| \lesssim \beta \frac{\ell(g)^2}{2^j} \quad \text{and} \quad |m_{N_{j+1}(r)}(g)| \lesssim \beta \frac{2^j}{\ell(g)^2}.$$

By Theorem 4.3 (1), we have

Corollary 6.2. *Let Γ be a r-step nilpotent group generated by a finite symmetric set of elements S. Let m_N, $k(r)$ and $N_j(r)$ be defined above. Then for all $1 < p < \infty$, $(T_{m_{N_j(r)}})_{j\in\mathbb{N}}$ is of strong type (p,p) and for any $x \in L_p(VN(\Gamma))$ with $1 < p < \infty$, $T_{m_{N_j(r)}}(x)$ converges a.u. to x as $j \to \infty$.*

6.1.2. *Case of amenable discrete quantum groups.* Let $\mathbb{G}$ be a compact quantum group of Kac type. As before, we assume that $\mathrm{Irr}(\mathbb{G})$ is countable for convenience. We keep the notation introduced in Section 5. We may study the Følner sequences and the corresponding multipliers in this quantum setting as follows.

Recall that for any $\alpha, \beta \in \mathrm{Irr}(\mathbb{G})$ we have the decomposition

$$u^{(\alpha)} \boxtimes u^{(\beta)} = \oplus_{\gamma\in\mathrm{Irr}(\mathbb{G})} N^{\gamma}_{\alpha\beta} u^{(\gamma)},$$

where $u^{(\alpha)} \boxtimes u^{(\beta)}$ refers to the tensor product representation of the form $(u^{(\alpha)}_{ij} u^{(\beta)}_{kl})_{i,j,k,l}$ and $N^{\gamma}_{\alpha\beta} \in \mathbb{N}$. We have the following Frobenius reciprocity law (see [Wor87], [Kye08, Example 2.3])

$$N^{\gamma}_{\alpha\beta} = N^{\alpha}_{\gamma\overline{\beta}} = N^{\beta}_{\overline{\alpha}\gamma} \tag{6.6}$$

for all $\alpha, \beta, \gamma \in \mathrm{Irr}(\mathbb{G})$. We write $\gamma \subset \alpha\beta$ if $N^{\gamma}_{\alpha\beta} > 0$. Denote by $\overline{\pi}$ the equivalent class of the representation $((u^{(\pi)}_{ij})^*)_{ij}$. The *weighted cardinality* of a finite subset $F \subset \mathrm{Irr}(\mathbb{G})$ is defined to be

$$|F|_w = \sum_{\alpha\in F} d^2_\alpha,$$

where we recall that d_α denotes the dimension of the representation α. On the other hand, for a finite subset $F \subset \mathrm{Irr}(\mathbb{G})$ and a representation $\pi \in \mathrm{Irr}(\mathbb{G})$, *the boundary of F related to π* is defined by

$$\partial_\pi F = \{\alpha \in F : \exists \beta \in F^c \text{ such that } \beta \subset \alpha\pi\} \cup \{\beta \in F^c : \exists \alpha \in F \text{ such that } \alpha \subset \beta\pi\}.$$

Kyed [Kye08] proved that there exists a sequence of finite subsets $(K_n)_{n\in\mathbb{N}} \subset \mathrm{Irr}(\mathbb{G})$ such that for any $\pi \in \mathrm{Irr}(\mathbb{G})$,

$$\frac{|\partial_\pi K_n|_w}{|K_n|_w} \to 0 \quad \text{as} \quad n \to \infty, \tag{6.7}$$

as soon as $\mathbb{G}$ is coamenable. Note that the coamenability of $\mathbb{G}$ is nothing but a property equivalent to the amenability of the discrete quantum group $\widehat{\mathbb{G}}$ (see e.g. [Bra17]). The above sequence $(K_n)_{n\in\mathbb{B}}$ is called a *Følner sequence*. We associate a sequence of multipliers

$$\varphi_n(\pi) = \frac{\sum_{\alpha,\beta\in K_n} N^{\pi}_{\bar{\alpha}\beta} d_\alpha d_\beta}{d_\pi(\sum_{\xi\in K_n} d_\xi^2)}, \quad \pi \in \mathrm{Irr}(\mathbb{G}). \tag{6.8}$$

It is easy to see that if $\widehat{\mathbb{G}} = \Gamma$ for a discrete group Γ, then the above function coincides with the symbol introduced in (6.1).

Lemma 6.3. (1) *The maps T_{φ_n} are unital completely positive on $L_\infty(\mathbb{G})$ for all $n \in \mathbb{N}$.*
(2) *The functions φ_n converge to 1 pointwise.*

Proof. (1) It is obvious that $\varphi_n(\mathbf{1}) = 1$ and therefore T_{φ_n} is unital.

Denote by $\chi(\pi) = \sum_i u^{\pi}_{ii} \in \mathrm{Pol}(\mathbb{G})$ the *character* of π. We have for any $\alpha, \beta \in \mathrm{Irr}(\mathbb{G})$,

$$h(\chi(\bar{\beta})\chi(\alpha)) = \delta_{\alpha\beta}1, \qquad \chi(\alpha)^* = \chi(\bar{\alpha}) \quad \text{and} \quad \chi(\alpha)\chi(\beta) = \sum_{\gamma\in\mathrm{Irr}(\mathbb{G})} N^{\gamma}_{\alpha\beta}\chi(\gamma).$$

We write

$$\mathrm{Pol}_0(\mathbb{G}) = \mathrm{span}\{\chi(\pi) : \pi \in \mathrm{Irr}(\mathbb{G})\}$$

and let $\mathcal{A}_0$ be the w*-closure of $\mathrm{Pol}_0(\mathbb{G})$ in $L_\infty(\mathbb{G})$. Let $\mathbb{E} : L_\infty(\mathbb{G}) \to \mathcal{A}_0$ be the canonical conditional expectation preserving the Haar state h. Recall that we have assumed that $\mathbb{G}$ is of Kac type. It is well-known that the conditional expectation $\mathbb{E}$ can be given by the following explicit formula (see e.g. the proof of [Wan17, Lemma 6.3])

$$\mathbb{E}(u^{(\pi)}_{ij}) = \frac{\delta_{ij}}{d_\pi}\chi(\pi), \quad \pi \in \mathrm{Irr}(\mathbb{G}).$$

Set

$$\chi_n = \frac{1}{|K_n|_w^{1/2}} \sum_{\alpha\in K_n} d_\alpha \chi(\alpha) \in \mathrm{Pol}(\mathbb{G})_0.$$

Then, we have

$$T_{\varphi_n}(x) = (h\otimes \mathrm{id})\left[(\chi_n^*\otimes 1)\cdot[(\mathbb{E}\otimes\mathrm{id})\circ\Delta(x)]\cdot(\chi_n\otimes 1)\right], \quad x \in L_\infty(\mathbb{G}).$$

Indeed, by linearity and normality, we only need to check the equality for the case $x = u^{(\pi)}_{ij}$. In this case we see that

$$\begin{aligned}
&(h\otimes\mathrm{id})\left[(\chi_n\otimes 1)\cdot[(\mathbb{E}\otimes\mathrm{id})\circ\Delta(u^{(\pi)}_{ij})]\cdot(\chi_n^*\otimes 1)\right]\\
=&\frac{\sum_{\alpha,\beta\in K_n} d_\alpha d_\beta \cdot h\left(\chi(\alpha)\chi(\pi)\chi(\bar{\beta})\right)}{d_\pi|K_n|_w} u^{(\pi)}_{ij}\\
=&\frac{\sum_{\alpha,\beta\in K_n} d_\alpha d_\beta N^{\beta}_{\alpha\pi}}{d_\pi|K_n|_w} u^{(\pi)}_{ij}\\
=&T_{\varphi_n}(u^{(\pi)}_{ij}).
\end{aligned}$$

Since $\mathbb{E}$ and Δ are completely positive, so is T_{φ_n}.

(2) The support of φ_n is given by

$$\Lambda_n = \{\pi \in \mathrm{Irr}(\mathbb{G}) : \exists \alpha, \beta \in K_n \text{ such that } \pi \subset \bar{\alpha}\beta\}. \tag{6.9}$$

We denote $\Theta_\pi^n = \{\alpha \in K_n : \forall\ \beta \in K_n^c, N_{\alpha\pi}^\beta = 0\} \subset K_n$. Note that $N_{\alpha\pi}^\beta = N_{\bar{\alpha}\beta}^\pi$ and that $\alpha \in \Theta_\pi^n$ implies $\sum_{\beta\in K_n} N_{\alpha\pi}^\beta d_\beta = d_\alpha d_\pi$ by the choice of $N_{\alpha\pi}^\beta$. Then

$$\varphi_n(\pi) \geq \frac{\sum_{\alpha\in\Theta_\pi^n} d_\alpha \left(\sum_{\beta\in K_n} N_{\alpha\pi}^\beta d_\beta\right)}{d_\pi(\sum_{\xi\in K_n} d_\xi^2)} = \frac{\sum_{\alpha\in\Theta_\pi^n} d_\alpha^2}{\sum_{\xi\in K_n} d_\xi^2} = \frac{|\Theta_\pi^n|_w}{|K_n|_w}.$$

Therefore,

$$1 - \varphi_n(\pi) \leq \frac{|\{\alpha \in K_n : \exists \beta \in K_n^c \text{ such that } \beta \subset \alpha\pi\}|_w}{|K_n|_w} \leq \frac{|\partial_\pi(K_n)|_w}{|K_n|_w}.$$

By the Følner condition (6.7), $\varphi_n \to 1$ pointwise. □

Therefore, by Theorem 5.9 we get the following result.

Corollary 6.4. *Assume that $\mathbb{G}$ is of Kac type and that $\widehat{\mathbb{G}}$ is amenable. Let $(K_n)_{n\in\mathbb{N}} \subset \mathrm{Irr}(\mathbb{G})$ be a Følner sequence and φ_n be the symbols given by (6.8). Then there is a subsequence $(n_j)_{j\in\mathbb{N}}$ such that $(T_{\varphi_{n_j}})_j$ is of strong type (p,p) for any $1 < p < \infty$. Moreover for all $x \in L_p(\mathbb{G})$ with $1 < p < \infty$, $T_{\varphi_{n_j}}(x)$ converges a.u. to x as $j \to \infty$.*

6.2. **Convergence of Fourier series of L_p-functions on compact groups.** We would like to emphasize in this subsection that our work indeed brings new ideas to the analysis on classical compact groups. Recall that for a compact second countable group G, any function $f \in L_p(G)$ admits a Fourier series

$$f(x) \sim \sum_{\pi\in\mathrm{Irr}(G)} d_\pi \mathrm{Tr}(\hat{f}(\pi)\pi(x)), \quad x \in G \quad \text{with } \hat{f}(\pi) = \int_G f(x)\pi(x^{-1})dx$$

where $\mathrm{Irr}(G)$ denotes the collection of equivalence classes of irreducible unitary representations of G, and d_π denotes the dimension of π. The pointwise analysis on these Fourier series is much subtler than the abelian case. A particular difficulty is the lack of obvious order and summation methods on $\mathrm{Irr}(G)$ which is suitable for the study of Fourier series. However, from the viewpoint of quantum group theory, the set $\mathrm{Irr}(G)$ is nothing but the underlying object of an amenable discrete quantum group. So the difficulty can be overcome by transferring the method on discrete amenable groups and its quantum counterpart. The spirit is also partially inspired by the recent work [Hua16].

More precisely, the compact group G trivially gives rise to a compact quantum group. Indeed, it suffices to take the triple $(L_\infty(G), \Delta_G, \int)$ where we define for all $f \in L_\infty(G)$,

$$\Delta_G(f)(g,h) = f(gh), \quad g,h \in G.$$

Then $\widehat{G}$ is amenable since $L_\infty(G)$ is a commutative von Neumann algebra (see e.g. [Bra17]). As a result, all arguments in Section 6.1.2 work on G. In particular, there always exists an increasing sequence $(K_n)_n \subset \mathrm{Irr}(G)$ determined by the representation theory of G, such that the following finitely supported symbols

$$\varphi_n(\pi) = \frac{\sum_{\alpha,\beta\in K_n} N_{\bar{\alpha}\beta}^\pi d_\alpha d_\beta}{d_\pi(\sum_{\xi\in K_n} d_\xi^2)}, \quad \pi \in \mathrm{Irr}(G)$$

converge to 1 pointwise, where $N^{\pi}_{\bar{\alpha}\beta}$ is the unique number in the decomposition of the tensor product representation $\bar{\alpha} \boxtimes \beta = \oplus_{\pi \in \mathrm{Irr}(G)} N^{\pi}_{\bar{\alpha}\beta} \pi$. Moreover

$$T_{\varphi_n} f = \sum_{\pi \in \mathrm{Irr}(G)} d_\pi \varphi_n(\pi) \mathrm{Tr}(\hat{f}(\pi)\pi(x)), \quad x \in G, f \in L_p(G)$$

defines a unital completely positive map on $L_p(G)$. We may choose a subsequence $(n_j)_j$ inductively by the algorithm in Lemma 5.8. Set $m_j = \varphi_{n_j}$. We may rewrite Corollary 5.11 and Proposition 5.12 in this setting as follows.

Corollary 6.5. (1) *Let $1 < p < \infty$. There exists a constant $c > 0$ such that*

$$\| \sup_j |T_{m_j} f| \|_p \leq c \|f\|_p, \quad f \in L_p(G).$$

For all $f \in L_p(G)$,

$$f(x) = \lim_{j \to \infty} \sum_{\pi \in \mathrm{Irr}(G)} d_\pi m_j(\pi) \mathrm{Tr}(\hat{f}(\pi)\pi(x)), \qquad a.e.\ x \in G.$$

(2) *For all $f \in L_2(G)$,*

$$f(x) = \lim_{j \to \infty} \sum_{\pi \in K_{n_j}} d_\pi \mathrm{Tr}(\hat{f}(\pi)\pi(x)), \qquad a.e.\ x \in G.$$

As an illustration, we consider a concrete example. Let $N \geq 2$ and denote by $SU(N)$ the $N \times N$ special unitary group. The irreducible representations of $SU(N)$ can be labeled by $N-1$ non-negative integers, and we write set-theoretically $\mathrm{Irr}(SU(N)) = \mathbb{N}^{N-1}$. The representation theory of $SU(N)$ can be computed in terms of operations on Young diagrams, which yields the following fact.

Proposition 6.6 (Appendix)**.** *Let $K_n = \{0, 1, 2, \cdots, n\}^{N-1} \subset \mathbb{N}^{N-1}$. Set*

$$\varphi_n(\pi) = \frac{\sum_{\alpha, \beta \in K_n} N^{\pi}_{\bar{\alpha}\beta} d_\alpha d_\beta}{d_\pi (\sum_{\xi \in K_n} d_\xi^2)}, \qquad \pi = (t_1, \cdots, t_{N-1}) \in \mathbb{N}^{N-1}.$$

Then

$$|1 - \varphi_n(\pi)| \lesssim_N \frac{|\pi|}{n+1} \quad \text{and} \quad |\varphi_{n+1}(\pi) - \varphi_n(\pi)| \lesssim_N \frac{1}{n+1},$$

where $|\pi| = \max\{t_i : 1 \leq i \leq N-1\}$.

As in Section 6.1.1, applying Theorem 5.9 to the subsequence $(\varphi_{2^j})_j$ we obtain a function ℓ on $\mathbb{N}^{N-1}$ such that $(T_{e^{-t\ell}})_t$ is a semigroup of unital completely positive trace-preserving and symmetric maps, and for any $\pi \neq (0, 0, \cdots, 0)$,

$$\ell(\pi) \asymp \sqrt{2}^{J(\pi)} \asymp \sqrt{|\pi|}$$

where $J(\pi) = \min\{j \in \mathbb{N} : \pi \in \Lambda_{2^j}\}$ (Λ_n is given by (6.9)). Then combined with Theorem 4.3 and the proof of Proposition 5.12, the above proposition yields the following corollary.

Corollary 6.7. *Let $K_n = \{0, 1, 2, \cdots, n\}^{N-1} \subset \mathbb{N}^{N-1}$. Define*

$$\varphi_n(\pi) = \frac{\sum_{\alpha, \beta \in K_n} N^{\pi}_{\alpha\bar{\beta}} d_\alpha d_\beta}{d_\pi (\sum_{\xi \in K_n} d_\xi^2)}, \qquad \pi \in \mathbb{N}^{N-1}.$$

(1) *Let $1 < p < \infty$. There exists a constant $c > 0$ such that*

$$\|\sup_j |T_{\varphi_{2^j}} f|\|_p \leq c\|f\|_p, \quad f \in L_p(G).$$

For all $f \in L_p(G)$,

$$f(g) = \lim_{j\to\infty} \sum_{\pi\in \mathrm{Irr}(G)} d_\pi \varphi_{2^j}(\pi) \mathrm{Tr}(\hat{f}(\pi)\pi(g)), \qquad a.e.\ g \in G.$$

For all $f \in L_2(G)$,

$$f(g) = \lim_{j\to\infty} \sum_{\pi\in K_{2^j}} d_\pi \mathrm{Tr}(\hat{f}(\pi)\pi(g)), \qquad a.e.\ g \in G.$$

(2) *Let $3/2 < p < \infty$. There exists a constant $c > 0$ such that*

$$\|\sup_n |T_{\varphi_n} f|\|_p \leq c\|f\|_p, \quad f \in L_p(G).$$

For all $f \in L_p(G)$,

$$f(g) = \lim_{n\to\infty} \sum_{\pi\in \mathrm{Irr}(G)} d_\pi \varphi_n(\pi) \mathrm{Tr}(\hat{f}(\pi)\pi(g)), \qquad a.e.\ g \in G.$$

6.3. **Smooth radial multipliers on some hyperbolic groups.**

6.3.1. *Bochner-Riesz means and finitely supported completely bounded multipliers.* In this subsection we briefly discuss a noncommutative analogue of Bochner-Riesz means for the setting of hyperbolic groups. We refer to [GdlH90, Gro87] for a complete description of hyperbolic groups. We merely remind that all hyperbolic groups are weakly amenable and the completely bounded radial Fourier multipliers have been characterized in [Oza08, MdlS17]. In particular, we denote by $|\ |$ the usual word length function on a hyperbolic group Γ, then the Fourier multipliers

$$B_N^\delta(x) = \sum_{g\in\Gamma:|g|\leq N} \left(1 - \frac{|g|^2}{N^2}\right)^\delta \hat{x}(g)\lambda(g), \quad x \in VN(\Gamma)$$

define a family of completely bounded maps on $VN(\Gamma)$ with $\sup_N \|B_N^\delta\|_{cb} < \infty$ as soon as $\delta > 1$ (see [MdlS17, Example 3.4]). Let

$$b_N^\delta(g) = (1 - \frac{|g|^2}{N^2})^\delta \mathbb{1}_{[0,N]}(|g|)$$

be the corresponding symbols of the maps B_N^δ. It is easy to check that

$$|1 - b_N^\delta(g)| \leq c\frac{|g|}{N}, \quad |b_N^\delta(g)| \leq c\frac{N}{|g|}, \quad |b_{N+1}^\delta(g) - b_N^\delta(g)| \leq c\frac{1}{N}, \qquad g \in \Gamma$$

for some constant $c > 0$. We are interested in the case where the word length function $|\ |$ is conditionally negative definite. This is the case if Γ is a non-abelian free group or a hyperbolic Coxeter group. Applying Theorem 4.2, we obtain the following corollary.

Corollary 6.8. *Let Γ and B_N^δ be as above with $\delta > 1$. Assume additionally that the word length function $|\ |$ is conditionally negative definite. Let $2 \leq p < \infty$. Then there exists a constant c such that*

$$\|\sup_{N\in\mathbb{N}}{}^+ B_N^\delta(x)\|_p \leq c\|x\|_p, \qquad x \in L_p(VN(\Gamma)),$$

and for any $x \in L_p(VN(\Gamma))$, $B_N^\delta(x)$ converges a.u. to x as $N \to \infty$.

Remark 6.9. Arguing as in Step 2 of the proof of [CXY13, Theorem 6.2], we see that for all $\delta > 0$, there exists a constant $C > 0$ such that

$$\|\sup_{N\in\mathbb{N}}{}^+ B_N^\delta(x)\|_2 \le C\big(\|\sup_{N\in\mathbb{N}}{}^+ B_N^{\delta+1}(x)\|_2 + \|x\|_2\big), \quad x \in L_2(VN(\Gamma)).$$

Thus the results in the above corollary also hold for the case $\delta > 0$ and $p = 2$. We will systematically study the Bochner-Riesz means in a subsequent paper.

Remark 6.10. The pointwise convergence of the above multipliers for $1 < p < 2$ seems more delicate. However, we can still construct some finitely supported multipliers satisfying the pointwise convergence in this case. For any $n \in \mathbb{N}$, we define a multiplier p_n on $VN(\Gamma)$ by

$$p_n(x) = \sum_{|g|=n} \hat{x}(g)\lambda(g), \qquad x \in VN(\Gamma),$$

which is the projection onto the subspace $\mathrm{span}\{\lambda(g) : |g| = n\}$. Ozawa [Oza08] showed that these operators satisfy the following estimate

$$\|p_n : VN(\Gamma) \to VN(\Gamma)\| \le \beta(n+1) \tag{6.10}$$

where β is a positive constant independent of n. For any $N \in \mathbb{N}$, we set

$$m_N(g) = \mathbb{1}_{[0,N^2]}(|g|)e^{-\frac{|g|}{N}}, \qquad g \in \Gamma.$$

On the other hand, for any $t \in \mathbb{R}_+$, denote $S_t : \lambda(g) \mapsto e^{-t|g|}\lambda(g)$. By assumption $(S_t)_{t\in\mathbb{R}_+}$ is a semigroup of unital completely positive trace preserving and symmetric maps. We write

$$e^{-\frac{|g|}{N}} - m_N(g) = \mathbb{1}_{[N^2+1,\infty)}(|g|)e^{-\frac{|g|}{N}} = \sum_{r=N^2+1}^{\infty} e^{-\frac{r}{N}}\mathbb{1}_{|g|=r}.$$

Hence, by the estimate (6.10), we have

$$\|S_{1/N} - T_{m_N} : VN(\Gamma) \to VN(\Gamma)\| \le \sum_{r=N^2+1}^{\infty} \|e^{-\frac{r}{N}}p_r\| \lesssim \beta \sum_{r=N^2+1}^{\infty} \frac{N^6}{r^6}(r+1) \lesssim \beta\frac{1}{(N+1)^2}.$$

By duality and interpolation, for any $1 \le p \le \infty$, we have

$$\|S_{1/N} - T_{m_N} : L_p(VN(\Gamma)) \to L_p(VN(\Gamma))\| \lesssim \beta\frac{1}{N^2}.$$

By Proposition 2.10, $S_{1/N}$ is of strong type (p,p) for any $1 < p < \infty$. Hence for any selfadjoint element $x \in L_p(VN(\Gamma))$ with $1 < p < \infty$,

$$\begin{aligned}\|\sup_{N\in\mathbb{N}}{}^+ T_{m_N}(x)\|_p &\le \|\sup_{N\in\mathbb{N}}{}^+ S_{1/N}(x)\|_p + \|\sup_{N\in\mathbb{N}}{}^+ (S_{1/N} - T_{m_N})(x)\|_p \\ &\lesssim_{\beta,p} \|x\|_p + \sum_{N\ge 0}\frac{1}{(N+1)^2}\|x\|_p \\ &\lesssim_{\beta,p} \|x\|_p.\end{aligned}$$

Similarly, for any $2 \le p < \infty$, we have

$$\|(T_{m_N}(x))_N\|_{L_p(\mathcal{M};\ell_\infty^c)} \lesssim_{\beta,p} \|x\|_p.$$

Then by Proposition 2.9 it is easy to check that $T_{m_N}x$ converges a.u. to x as $N \to \infty$ for $x \in L_p(VN(\Gamma))$ with $2 \le p < \infty$ and converges b.a.u. for $1 < p < 2$. Note that we

may also obtain the weak type $(1,1)$ estimate and b.a.u. convergence on $L_1(VN(\Gamma))$ for multipliers $\lambda(g) \mapsto \mathbb{1}_{[0,N]}(|g|)e^{-\sqrt{|g|/N}}\lambda(g)$ by the same argument, since the subordinated Poisson semigroup $\lambda(g) \mapsto e^{-t\sqrt{|g|}}\lambda(g)$ is of weak type $(1,1)$ according to [JX07, Remark 4.7].

The above arguments work for all groups with the rapid decay property with respect to a conditionally negative definite length function.

6.3.2. *Smooth positive definite radial kernels on free groups.* Using our main result, we may provide a wide class of completely positive smooth multipliers on free groups satisfying the pointwise convergence apart from Poisson semigroups. To this end we will need the following characterization of radial positive definite functions on free groups. In the following $\mathbb{F}_d$ will denote the free group with d generators ($2 \le d \le \infty$).

Theorem 6.11 ([HK15, Theorem 1.1] and [Ver20, Theorem 1.2])**.** *Let ν be a positive Borel measure on $[-1,1]$. Define a function φ on $\mathbb{N}$ by*

$$\varphi(k) = \int_{-1}^{1} x^k d\nu(x), \qquad k \in \mathbb{N}.$$

Then $\dot{\varphi}(g) := \varphi(|g|)$ is a positive definite function on $\mathbb{F}_\infty$, where $|\ |$ is the word length function.

Then we get the following proposition. Note that if ν is the Dirac measure on 0 in this proposition, then this statement amounts to the almost uniform convergence of Poisson semigroups on $VN(\mathbb{F}_d)$ proved in [JX07].

Proposition 6.12. *Let ν be an arbitrary positive Borel measure supported on $[-1,1]$ with $\nu([-1,1]) = 1$ and write $d\nu_t(x) = d\nu(tx)$ for all $t > 0$. For any $t > 0$, set*

$$m_t(g) = \int_{\mathbb{R}} x^{|g|} d\nu_t(x - e^{-\frac{2}{t}}), \qquad g \in \mathbb{F}_d,$$

where $|\ |$ is the usual word length function. Then there exist an absolute positive number $t_0 > 0$ and a constant $c > 0$ such that for all $1 < p < \infty$ and all $x \in L_p(VN(\mathbb{F}_d))$,

$$\|(T_{m_t}x)_{t \ge t_0}\|_{L_p(\mathcal{M};\ell_\infty)} \le c\|x\|_p \qquad \text{and} \qquad T_{m_t}x \to x \text{ a.u. as } t \to \infty.$$

Proof. Using integration by substitution with $y = t(x - e^{-\frac{2}{t}})$, we have

$$m_t(g) = \int_{\mathbb{R}} x^{|g|} d\nu_t(x - e^{-\frac{2}{t}}) = \int_{-1}^{1} \left(\frac{y}{t} + e^{-\frac{2}{t}}\right)^{|g|} d\nu(y).$$

By some fundamental analysis, we can find a number t_0 large enough such that for any $t \ge t_0$,

$$e^{-\frac{2}{t}} - \frac{1}{t} \ge e^{-\frac{4}{t}} \quad \text{and} \quad \frac{1}{t} + e^{-\frac{2}{t}} < e^{-\frac{2}{3t}}.$$

Then for any $t \ge t_0$,

$$|m_t(g)| \le \left(\frac{1}{t} + e^{-\frac{2}{t}}\right)^{|g|} \le e^{-\frac{2|g|}{3t}} \lesssim \frac{t}{|g|}, \tag{6.11}$$

$$|1 - m_t(g)| \le 1 - \left(-\frac{1}{t} + e^{-\frac{2}{t}}\right)^{|g|} \le \left(1 - e^{-\frac{4|g|}{t}}\right) \lesssim \frac{|g|}{t}.$$

Moreover we note that

$$\frac{d^v}{dt^v} m_t(|g|) = \int_{-1}^{1} \frac{d^v}{dt^v} \left(\frac{y}{t} + e^{-\frac{2}{t}}\right)^{|g|} d\nu(y).$$

Set $f(t) = \frac{y}{t} + e^{-\frac{2}{t}}$. By a straightforward computation,

$$|\frac{d^v}{dt^v} f(t)| \lesssim_v \frac{1}{t^{v+1}}. \tag{6.12}$$

Recall the Faá di Bruno formula:

$$\frac{d^v}{dt^v} F(f(x)) = \sum_{P \in \mathscr{P}(v)} F^{(|P|)}(f(t)) \cdot \prod_{B \in P} f^{(|B|)}(t)$$

where $\mathscr{P}(v)$ is the set of all partitions of $\{1, \cdots, v\}$, $B \in P$ means that B is a block of the partition P, and $|B|$ denotes the size of the block B and $|P|$ means the number of blocks. Similar as (6.11), $|f(t)| \leq e^{-\frac{2}{3t}}$. Then using the Faá di Bruno formula and (6.12), we see that for any $v \geq 1$ and $y \in [-1, 1]$,

$$\begin{aligned}
\frac{d^v}{dt^v}\left(\frac{y}{t} + e^{-\frac{2}{t}}\right)^k &= \sum_{P \in \mathscr{P}(v), |P| \leq \min\{v,k\}} k(k-1) \cdot (k - |P| + 1) f(t)^{k-|P|} \cdot \prod_{B \in P} f^{(|B|)}(t) \\
&\lesssim_v \sum_{P \in \mathscr{P}(v), |P| \leq \min\{v,k\}} k(k-1) \cdot (k - |P| + 1) e^{-\frac{2(k-|P|)}{3t}} \cdot \prod_{B \in P} \frac{1}{t^{|B|+1}} \\
&\lesssim_v \sum_{P \in \mathscr{P}(v), |P| \leq \min\{v,k\}} k(k-1) \cdot (k - |P| + 1) \frac{t^{|P|}}{(k - |P|)^{|P|}} \cdot \frac{1}{t^{v+|P|}} \\
&\lesssim_v \sum_{P \in \mathscr{P}(v)} 2^{|P|} \frac{1}{t^v} \lesssim_v \frac{1}{t^v}.
\end{aligned}$$

Therefore, $(m_t)_{t \geq t_0}$ satisfies (4.3) in **(A2)**.

Note that $\mathbb{F}_d$ is a subgroup of $\mathbb{F}_\infty$, and $\nu_t(\cdot - e^{-t/2})$ is supported in $[-1, 1]$ for large $t > 0$. So m_t is positive definite on $\mathbb{F}_d$ by Theorem 6.11 for $t \geq t_0$. Also, for any $t \geq t_0$ $T_{m_t}(1) = 1$ since $\nu([-1, 1]) = 1$. Thus for any $t \geq t_0$, T_{m_t} is unital completely positive and it extends to a contraction on $L_p(VN(\mathbb{F}_d))$. Moreover, the natural length function $|\ |$ is conditionally negative definite. On the other hand, for all $1 < p < \infty$ we can always find a positive integer η depending on p such that $1 + \frac{1}{2\eta} < p$. Then the proof is complete by applying Theorem 4.3. □

We remark that the totally same argument applies to many other examples of groups acting on homogeneous trees.

6.4. **Results and problems on Euclidean spaces.** As we mentioned in Subsection 6.1.1, the definition of the symbol given by (6.1), though motivated by the geometric group theory, coincides with the Fejér means in $\mathbb{R}$. From a geometric viewpoint, even in the Euclidean spaces $\mathbb{R}^d$ it is still natural to consider other Følner sets besides cubes, such as balls and rectangles. In particular, we obtain new interesting Fourier multipliers on classical Euclidean spaces, which might be regarded as generalized Fejér means.

More precisely, let B be a symmetric convex body in $\mathbb{R}^d$ such that the interior contains 0. We define the functions φ_t associated with B as

$$\varphi_t(\xi) = \frac{\mu(B_t \cap (\xi + B_t))}{\mu(B_t)}, \tag{6.13}$$

where $B_t = \{\xi \in \mathbb{R}^d : \xi/t \in B\}$ and μ is Lebesgue measure. Then T_{φ_t} defines a completely positive Fourier multiplier on L_p-spaces. Let Φ be the inverse Fourier transform of the convolution $\mu(B)^{-1}\mathbb{1}_B * \mathbb{1}_B$, then for all $t > 0$,

$$T_{\varphi_t} f = \Phi_t * f, \quad \text{where } \Phi_t = t^{-d}\Phi(t^{-1}\cdot).$$

The following corollary is obtained similarly using the arguments in Section 6.1.1.

Corollary 6.13. *Let B and $(\varphi_t)_t$ be as above.*

(1) *$(T_{\varphi_{2^j}})_{j\in\mathbb{Z}}$ is of strong type (p,p) for any $1 < p < \infty$, and for any $f \in L_p(\mathbb{R}^d)$ with $1 < p < \infty$, $T_{\varphi_{2^j}}(f)$ converges a.e. to f as $j \to \infty$.*

(2) *$(T_{\varphi_t})_{t>0}$ is of strong type (p,p) for any $\frac{3}{2} < p < \infty$, and for any $f \in L_p(\mathbb{R}^d)$ with $3/2 < p < \infty$, $T_{\varphi_t}(f)$ converges a.e. to f as $t \to \infty$.*

Proof. For any non-zero vector $\xi \in \mathbb{R}^d$, there is a unique positive number $|\xi|_B$ such that $\xi/|\xi|_B \in \partial B$ since B is convex and its interior contains 0. We make the convention that $|0|_B = 0$. Note that $\mu(B_t) = t^d\mu(B)$. We have

$$1 - \varphi_t(\xi) \le \frac{\mu(B_{t+|\xi|_B}\backslash B_t)}{\mu(B_t)} \le \frac{(t+|\xi|_B)^d - t^d}{t^d} \lesssim_d \frac{|\xi|_B}{t}.$$

Also, we have

$$\frac{\partial\varphi_t}{\partial t} = \lim_{h\to 0}\frac{1}{h}\left(\frac{\mathbb{1}_{B_{t+h}} * \mathbb{1}_{B_{t+h}}(\xi)}{\mu(B_{t+h})} - \frac{\mathbb{1}_{B_t} * \mathbb{1}_{B_t}(\xi)}{\mu(B_t)}\right) \lesssim \lim_{h\to 0}\frac{1}{h}\frac{\mu(B_{t+h}) - \mu(B_t)}{\mu(B_t)} \lesssim_d \frac{1}{t}.$$

Then the remaining arguments are similar to those for nilpotent groups in Section 6.1.1. We skip it here. □

Remark 6.14. Let $\mathcal{N}$ be a semifinite von Neumann algebra. As in Section 6.5, we may consider the noncommutative maximal inequalities and the a.u. convergence for the sequence of multipliers $\left(T_{\varphi_t} \otimes \mathrm{id}_{L_p(\mathcal{N})}\right)_t$ on $L_p(\mathbb{R}^d; L_p(\mathcal{N}))$ with symbols $(\varphi_t)_t$ given by (6.13). These properties indeed follow from Remark 4.6, and the above corollary still holds for $(T_{\varphi_t}\otimes\mathrm{id}_{L_p(\mathcal{N})})_t$ in this noncommutative setting thanks to Lemma 3.6 (2).

The main problem left open at this stage is the following:

Problem 6.15. Let B and $(\varphi_t)_t$ be as above. Do we have

$$\|\sup_{t>0}|T_{\varphi_t}(f)|\|_{L_{1,\infty}} \le C_d\|f\|_1, \qquad f \in L_1(\mathbb{R}^d)$$

with a constant C_d depending only on d and B?

Remark 6.16. (1) The answer is well-known to be affirmative if B is a cube (where T_{φ_t} becomes the classical Fejér mean). See for example [Gra08, Section 3.3.2].

(2) If the boundary of B is sufficiently smooth and has everywhere non-vanishing Gaussian curvature, the answer is still affirmative. To see this, it suffices to note that $\Phi = (\widehat{\mathbb{1}_B})^2$, and there is a constant $C > 0$ with

$$|\widehat{\mathbb{1}_B}(\xi)| \le C(1+|\xi|)^{-(d+1)/2}, \qquad \xi \in \mathbb{R}^d. \tag{6.14}$$

See for instance [Ste93].

(3) For general symmetric convex bodies, the best estimate is due to the work [BHI03]: there is a constant $C > 0$ with

$$\int_{S^{d-1}} |\widehat{\mathbb{1}_B}(r\xi)|^2 d\xi \le Cr^{-(d+1)}$$

for large r where S^{d-1} is the unit sphere in $\mathbb{R}^d$. This estimate has important applications in the study of distributions of lattice points in convex domains and the Falconer distance problem etc. Compared with (6.14), this estimate is the spherical average of Φ. Can we get the desired maximal inequality by using this spherical average?

A positive answer to Problem 6.15 would yield the strong type (p,p) estimate for T_{φ_t} for all $p>1$ by interpolation. Recall that the constant C_p for the strong type (p,p) estimate can be taken to be independent of d in Theorem 1.9. So it is natural to raise the following question in the present setting:

Problem 6.17. Consider the best constant $C_{p,d}>0$ with

$$\|\sup_{t>0}|T_{\varphi_t}(f)|\|_p \le C_{p,d}\|f\|_p, \qquad f\in L_p(\mathbb{R}^d).$$

What is the dependence of $C_{p,d}$ on d? When can we choose $C_{p,d}$ independent of d?

6.5. **Dimension free bounds of noncommutative Hardy-Littlewood maximal operators.** Our results in particular apply to the problem of dimension free estimates for Hardy-Littlewood maximal operators. Let B be a symmetric convex body in $\mathbb{R}^d$. We consider the associated averages

$$\Phi_t(f)(x)=\frac{1}{\mu(B)}\int_B f(x-\frac{y}{t})dy, \quad f\in L_p(\mathbb{R}^d),\ x\in\mathbb{R}^d,\ t>0.$$

Let $\mathcal{N}$ be a semifinite von Neumann algebra equipped with a normal semifinite trace ν and $L_p(\mathbb{R}^d;L_p(\mathcal{N}))$ be the Bochner $L_p(\mathcal{N})$-valued L_p-spaces. Recall that we may view the space $L_p(\mathbb{R}^d;L_p(\mathcal{N}))$ as a noncommutative L_p-space associated with the von Neumann algebra $L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{N}$: for any $1\le p<\infty$,

$$L_p(L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{N},\int\otimes\nu)\cong L_p(\mathbb{R}^d;L_p(\mathcal{N})).$$

We could then extend Bourgain's results for the corresponding Hardy-Littlewood maximal operators on noncommutative L_p-spaces $L_p(\mathbb{R}^d;L_p(\mathcal{N}))$.

Proof of Theorem 1.9. Without loss of generality, we assume $\mu(B)=1$. Let

$$m_t(\xi)=t^d\widehat{\mathbb{1}_{t^{-1}B}}(\xi)=\widehat{\mathbb{1}_B}(\xi/t)$$

be the Fourier transform of the kernel of the above operator Φ_t. Then we may view Φ_t as the Fourier multiplier so that $\widehat{\Phi_t f}=m_t\hat{f}$.

Since the bound of a maximal operator is invariant under invertible linear transforms, thus by a transform, we can assume that B is in the isotropic position with isotropic constant $L=L(B)$ (see Lemma 2 in [Bou86a]) and $\widehat{\mathbb{1}_B}$ satisfies the following estimates (see the computations in Section 4 of [Bou86a]): there exists a constant c independent of d and B such that

$$|\widehat{\mathbb{1}_B}(\xi)|\le c\frac{1}{|\xi|L}, \qquad |1-\widehat{\mathbb{1}_B}(\xi)|\le cL|\xi| \qquad |\langle\nabla\widehat{\mathbb{1}_B}(\xi),\xi\rangle|\le c, \qquad \xi\in\mathbb{R}^d.$$

On the other hand, setting $\zeta=\xi/t=(\xi_l/t)_{l=1}^d$, we have

$$\frac{d}{dt}m_t(\xi)=\frac{d}{dt}\widehat{\mathbb{1}_B}(\zeta)=\sum_{l=1}^d\frac{\partial\zeta_l}{\partial t}\cdot\frac{\partial\widehat{\mathbb{1}_B}(\zeta)}{\partial\zeta_l}=-\sum_{l=1}^d\frac{\zeta_l}{t}\partial_l\widehat{\mathbb{1}_B}(\zeta)=-\frac{1}{t}\langle\nabla\widehat{\mathbb{1}_B}(\zeta),\zeta\rangle.$$

Therefore, $(m_t)_t$ satisfies **(A2)** with $\eta = 1$. Then, applying Theorem 4.3, Remark 4.5 (2), Remark 4.6, and Lemma 3.6 (2) to the modified Poisson semigroup $(P_{tL})_{t\in\mathbb{R}_+}$ on $\mathbb{R}^d$, we immediately get the desired assertions (1) and (2) of Theorem 1.9.

We use the similar arguments for higher derivatives:

$$\begin{aligned}\frac{d^2}{dt^2}\mathbb{1}_B(\zeta) &= \sum_{l=1}^d \frac{\partial\zeta_l}{\partial t}\cdot\frac{\partial}{\partial\zeta_l}\left(\sum_{l=1}^d \frac{\partial\zeta_l}{\partial t}\cdot\frac{\partial\mathbb{1}_B(\zeta)}{\partial\zeta_l}\right) = \sum_{l=1}^d \frac{\zeta_l}{t}\cdot\frac{\partial}{\partial\zeta_l}\left(\sum_{l=1}^d \frac{\zeta_l}{t}\cdot\frac{\partial\mathbb{1}_B(\zeta)}{\partial\zeta_l}\right)\\ &= \frac{1}{t^2}\left(\sum_{l=1}^d \zeta_l\partial_l\left(\mathbb{1}_B(\zeta)\right) + \sum_{l_1,l_2=1}^d \zeta_{l_1}\zeta_{l_2}\partial_{l_1}\partial_{l_2}\left(\mathbb{1}_B(\zeta)\right)\right),\end{aligned}$$

and

$$\begin{aligned}&\frac{d^3}{dt^3}\big(\mathbb{1}_B(\zeta)\big)\\ &= \frac{1}{t^3}\left(\sum_{l=1}^d \zeta_l\partial_l\big(\mathbb{1}_B(\zeta)\big) + 2\sum_{l_1,l_2=1}^d \zeta_{l_1}\zeta_{l_2}\partial_{l_1}\partial_{l_2}\big(\mathbb{1}_B(\zeta)\big) + \sum_{l_1,l_2,l_3=1}^d \zeta_{l_1}\zeta_{l_2}\zeta_{l_3}\partial_{l_1}\partial_{l_2}\partial_{l_3}\big(\mathbb{1}_B(\zeta)\big)\right).\end{aligned}$$

Repeating this process, we get

$$\frac{d^v}{dt^v}\big(\mathbb{1}_B(\zeta)\big) = \frac{1}{t^v}\sum_{k=1}^v\left(c_k\sum_{l_1,l_2,\dots,l_k=1}^d \zeta_{l_1}\zeta_{l_2}\cdots\zeta_{l_k}\partial_{l_1}\partial_{l_2}\cdots\partial_{l_k}\right)\big(\mathbb{1}_B(\zeta)\big)$$

where c_k are constants only depending on k. Recall that

$$\partial_{l_1}\partial_{l_2}\cdots\partial_{l_k}\widehat{\mathbb{1}_B}(\zeta) = \Big((-2\pi i)^k x_{l_1}x_{l_2}\cdots x_{l_k}\mathbb{1}_B\Big)\widehat{\ }(\zeta).$$

Then

$$\begin{aligned}\left|\frac{d^v}{dt^v}m_t(\xi)\right| &= \frac{1}{t^v}\left|\sum_{k=1}^v\left(c_k\sum_{l_1,l_2,\dots,l_k=1}^d \zeta_{l_1}\zeta_{l_2}\cdots\zeta_{l_k}\partial_{l_1}\partial_{l_2}\cdots\partial_{l_k}\widehat{\mathbb{1}_B}(\zeta)\right)\right|\\ &\lesssim_v \frac{1}{t^v}\left|\sum_{k=1}^v\left(\sum_{l_1,l_2\cdots,l_k=1}^d \zeta_{l_1}\zeta_{l_2}\cdots\zeta_{l_k}\Big((2\pi i)^k x_{l_1}x_{l_2}\cdots x_{l_k}\mathbb{1}_B(x)\Big)\widehat{\ }(\zeta)\right)\right|\\ &\lesssim_v \frac{1}{t^v}\sum_{k=1}^v\left|\int_{\mathbb{R}^d} e^{-2\pi i\langle x,\zeta\rangle}\langle x,\zeta\rangle^k\mathbb{1}_B(x)dx\right|.\end{aligned}$$

In particular, if B is the ℓ_q-ball with $q\in 2\mathbb{N}$, the computations in [Bou87] assert that for any $k\geq 0$ there exists a constant c'_k independent of d such that

$$\left|\int_{\mathbb{R}^d} e^{-2\pi i\langle x,\zeta\rangle}\langle x,\zeta\rangle^k\mathbb{1}_B(x)dx\right| \leq c'_k.$$

Hence for ℓ_q-ball with $q\in 2\mathbb{N}$, $(m_t)_{t\in\mathbb{R}_+}$ satisfies **(A2)** for any $\eta > 0$. For any $1 < p \leq 2$, choosing an η large enough (depending on p) and applying Theorem 4.3 (2), Remark 4.6 and Lemma 3.6 (2), we obtain Theorem 1.9 (3). □

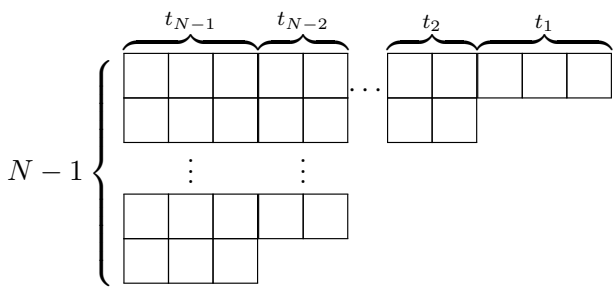
Remark 6.18. At the moment of writing, it seems that our approach is not enough to establish dimension free bounds for other ℓ_q-balls with $q \in [1,\infty] \setminus 2\mathbb{N}$ and new ideas are certainly required. The corresponding results in the classical setting are given by Müller [Mül90] (for $q \in [1,\infty) \setminus 2\mathbb{N}$) and Bourgain [Bou14] (for $q = \infty$). From Theorem 1.9, it is naturally conjectured that the noncommutative analogues of their results should still hold.

Appendix: Følner sequences in the dual of $SU(N)$

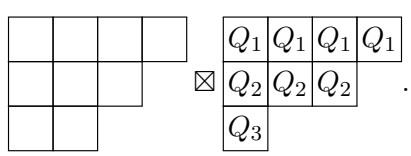
We will prove Proposition 6.6 in this section.

Fisrt, we recall briefly the representation theory of $SU(N)$. For more details, we refer to [Jon98, Chapter 8]. The irreducible representations of $SU(N)$ can be labeled by $N-1$ non-negative integers, and we write set-theoretically $\mathrm{Irr}(SU(N)) = \mathbb{N}^{N-1}$. And moreover, the decomposition of tensor products into irreducible representations can be described by Young diagrams. Young diagrams consist of boxes: we stick some boxes together so that the number of boxes in each consecutive row (from top to bottom) and each consecutive column (from left to right) does not increase.

Each irreducible representation $u^{(t_1,t_2,\cdots,t_{N-1})}$ corresponds to a Young diagrams of the following form:



Let $\alpha, \beta \in \mathrm{Irr}(SU(N))$. The family of irreducible subrepresentations of the tensor product $\alpha \boxtimes \beta$ corresponds to all the Young diagrams appeared after the following operations. We denote by X_α and X_β the corresponding Young diagrams of α and β. We are going to decompose the tensor product of Young diagrams $X_\alpha \boxtimes X_\beta$:



The prescription goes like this:

(R1) Start by filling the boxes in the top row of X_β with labels 'Q_1' and the boxes in the second row with labels 'Q_2' and etc (see the above figure).

(R2) Add boxes $\boxed{Q_1}$ $\cdots$ $\boxed{Q_1}$, $\boxed{Q_2}$ $\cdots$ $\boxed{Q_2}$, $\boxed{Q_3}$ $\cdots$ etc from X_β to X_α in order according to the following rules:

(a) Each time we add a box $\boxed{Q_i}$ from to X_β to X_α such that the augmented X_α diagram must be again a Young diagram which has at most N rows.
(b) Boxes with the same label, e.g. Q_1, must not appear in the same column.
(c) At any given box position, define n_1 to be the number of Q_1's above and to the right of it. Define n_2 for Q_2 in the similar way, etc. Then we must have $n_1 \geq n_2 \geq n_3$ etc.

(R3) If two diagrams of the same shape are produced by this process, they are counted as different only if the labels are differently distributed.

(R4) Cancel columns with N boxes.

For each $\pi \in \mathrm{Irr}(SU(N))$, we denote by X_π its corresponding Young diagram. If in the decomposition of $X_\alpha \boxtimes X_\beta$ there are at most N different operations of the above form yielding X_π, then the multiplicity $N_{\alpha\beta}^{\pi}$ equals to N.

Moreover, for an irreducible representation $u^{(t_1,t_2,\cdots,t_{N-1})}$, the dimension is given by the following formula

(A.1)

$$\begin{aligned} d(t_1,t_2,\cdots,t_{N-1}) = (t_1+1)(t_2+1)\cdots(t_{N-1}+1) \\ \cdot\left(1+\frac{t_1+t_2}{2}\right)\left(1+\frac{t_2+t_3}{2}\right)\cdots\left(1+\frac{t_{N-2}+t_{N-1}}{2}\right) \\ \cdot\left(1+\frac{t_1+t_2+t_3}{3}\right)\left(1+\frac{t_2+t_3+t_4}{3}\right)\cdots\left(1+\frac{t_{N-3}+t_{N-2}+t_{N-1}}{3}\right) \\ \vdots \\ \cdot(1+\frac{t_1+t_2+\cdots+t_{N-1}}{N-1}). \end{aligned}$$

We have the following fact.

Lemma A.1. *Let $N \geq 2$. Set $K_n = \{0,1,2,\cdots,n\}^{N-1} \subset \mathrm{Irr}(SU(N))$. Then $(K_n)_{n\in\mathbb{N}}$ satisfies the following conditions.*

(1) *We have*

$$(n+1)^{N^2-2} \lesssim_N |E_{n+1}|_w \lesssim_N (n+1)^{N^2-2} \quad \textit{where} \quad E_{n+1} = K_{n+1}\backslash K_n.$$

(2) *For any $t,n \in \mathbb{N}$ and any $\alpha \in K_n, \pi \in K_t$, all the irreducible subrepresentations of $\alpha \boxtimes \pi$ are contained in the finite set $K_{n+(N-1)t}$, i.e. $N_{\alpha\pi}^{\beta} = 0$ if $\beta \notin K_{n+(N-1)t}$.*

Proof. We write $\pi = (t_1,t_2,\cdots,t_{N-1}) \in \mathbb{N}^{N-1}$. The conjugate representation is given by $\bar{\pi} = (t_{N-1},t_{N-2},\cdots,t_1)$

(1) Denote

$$J' = \Big\{\underline{l} = (l_1,l_2,\cdots,l_{N-1}) \in \mathbb{N}^{N-1} : \forall\, 1 \leq j \leq N-1, 1 \leq l_j \leq j(N-j), \sum_{j=0}^{N-1} l_j = \frac{N(N-1)}{2}\Big\}.$$

Then we may write

$$d_\pi = \sum_{\underline{l}\in J'} C'_{\underline{l}}(t_1+1)^{l_1}(t_2+1)^{l_2}\cdots(t_{N-1}+1)^{l_{N-1}},$$

for some constants $C'_{\underline{l}}$ depending only on N, and

$$d_\pi^2 = \sum_{\underline{l}\in J} C_{\underline{l}}(t_1+1)^{l_1}(t_2+1)^{l_2}\cdots(t_{N-1}+1)^{l_{N-1}}$$

with some constants $C_{\underline{l}}$ depending only on N and

$$J=\left\{(l_1,l_2,\cdots,l_{N-1})\in\mathbb{N}^{N-1}:\forall\ 2\le l_j\le 2j(N-j),\sum_{j=0}^{N-1}l_j=N(N-1)\right\}.$$

Denote

$$E_{n+1}^j=\{(k_1,k_2,\cdots,k_{N-1})\in K_{n+1}:k_j=n+1\}.$$

By definition, $E_{n+1}\subset\cup_{j=1}^{N-1}E_{n+1}^j$. Therefore,

$$\begin{aligned}|E_{n+1}^j|_w&=\sum_{\underline{k}\in E_{n+1}^i}d_{\underline{k}}^2\\&=\sum_{\underline{k}\in E_{n+1}^i}\sum_{\underline{l}\in J}C_{\underline{l}}(k_1+1)^{l_1}(k_2+1)^{l_2}\cdots(n+2)^{l_i}\cdots(k_{N-1}+1)^{l_{N-1}}\\&\asymp\sum_{\underline{l}\in J}C_{\underline{l}}(n+2)^{l_i}(n+2)^{l_1+1}(n+2)^{l_2+1}\cdots(n+2)^{l_{i-1}+1}(n+2)^{l_{i+1}+1}\cdots(n+2)^{l_{N-1}+1}\\&=(\sum_{\underline{l}\in J}C_{\underline{l}})(n+2)^{N^2-2}=d_{\underline{0}}^2(n+2)^{N^2-2},\end{aligned}$$

where $\underline{0}=(0,0,\cdots,0)$. Note that $d_{\underline{0}}^2$ is a constant depending only on N. Thus,

$$(n+2)^{N^2-2}\lesssim_N|E_{n+1}^j|_w\le|E_n|_w\le\sum_{j=1}^{N-1}|E_{n+1}^j|_w\lesssim_N(n+2)^{N^2-2}.$$

(2) This follows directly from the rules of decomposition of tensor product of Young diagrams.

□

Proposition A.2 (Proposition 6.6)**.** *Let $(K_n)_n$ and $\pi=(t_1,\cdots,t_{N-1})$ be as above. Set*

$$\varphi_n(\pi)=\frac{\sum_{\alpha,\beta\in K_n}N_{\bar{\alpha}\beta}^{\pi}d_\alpha d_\beta}{d_\pi(\sum_{\xi\in K_n}d_\xi^2)}.$$

Then

$$1-\varphi_n(\pi)\lesssim_N\frac{|\pi|}{n+1}\quad\text{and}\quad|\varphi_{n+1}(\pi)-\varphi_n(\pi)|\lesssim_N\frac{1}{n+1},$$

where $|\pi|=\max\{t_i:1\le i\le N-1\}$.

Proof. Let $E_k=E_k\backslash E_{k-1}$ for all $k\ge1$. Denote $t=|\pi|$, i.e. $\pi\in E_t$. Let $n\ge1$. Define

$$\partial_\pi^1K_n=\{\alpha\in K_n:\exists\beta\in K_n^c,\text{ such that }\beta\subset\alpha\pi\},$$

$$\partial_\pi^2K_n=\{\beta\in K_n^c:\exists\alpha\in K_n,\text{ such that }\alpha\subset\beta\pi\}.$$

Assume first $n-(N-1)t\ge0$. By Lemma A.1 (2), we know that if $\alpha\in K_{n-(N-1)t}$, all the irreducible subrepresentations of $\alpha\boxtimes\pi$ belong to K_n, which implies $\partial_\pi^1K_n\cap K_{n-(N-1)t}=\emptyset$. Therefore

$$|\partial_\pi^1K_n|_w\le|\cup_{k=n-(N-1)t+1}^nE_k|_w\lesssim_N\sum_{k=n-(N-1)t+1}^n(k+1)^{N^2-1}\lesssim_N t(n+1)^{N^2-2}.\tag{A.2}$$

On the other hand, if $n-(N-1)t<0$, then

$$|\partial_\pi^1 K_n|_w \le |K_n|_w \lesssim_N (n+1)^{N^2-1} \lesssim_N t(n+1)^{N^2-1}.$$

Also by Lemma A.1 (2), we have $\partial_\pi^2 K_n \subset K_{n+(N-1)t}\backslash K_n$. Therefore,

$$|\partial_\pi^2 K_n|_w \le |\cup_{k=n+1}^{n+(N-1)t} E_i|_w \lesssim_N \sum_{k=n+1}^{n+(N-1)t} (k+1)^{N^2-1} \lesssim_N t(n+1)^{N^2-2}.$$

Moreover, we have $|K_n|_w = \sum_{i=1}^n |E_i|_w \gtrsim_N (n+1)^{N^2-1}$ and by Lemma A.1 (1) we have $n^{N^2-1} \lesssim_N |K_n|_w \lesssim_N n^{N^2-1}$. Therefore we obtain

$$\frac{|\partial_\pi K_n|_w}{|K_n|_w} \lesssim_N \frac{t(n+1)^{N^2-2}}{(n+1)^{N^2-1}} \lesssim_N \frac{t}{n+1}.$$

Note that for any $\alpha \in K_n\backslash\partial_\pi^1 K_n$, we have $\alpha \boxtimes \pi = \oplus_{\beta\in K_n} N_{\alpha\pi}^\beta \beta$. In particular, we see that $\sum_{\beta\in K_n} N_{\bar\alpha\beta}^\pi d_\beta = d_\pi d_\alpha$ and

$$\begin{aligned}
1-\varphi_n(\pi) &= \frac{\sum_{\alpha\in K_n} d_\alpha(d_\alpha d_\pi - \sum_{\beta\in K_n} N_{\bar\alpha\beta}^\pi d_\beta)}{d_\pi |K_n|_w} \\
&= \frac{\sum_{\alpha\in \partial_\pi^1 K_n} d_\alpha(d_\alpha d_\pi - \sum_{\beta\in K_n} N_{\bar\alpha\beta}^\pi d_\beta)}{d_\pi |K_n|_w} + 0 \\
&\le \frac{|\partial_\pi^1 K_n|_w}{|K_n|_w} \lesssim_N \frac{t}{n+1}.
\end{aligned}$$

For the second inequality, by Lemma A.1 (3) we have

$$\begin{aligned}
|\varphi_{n+1}(\pi)-\varphi_n(\pi)| &= |(1-\varphi_n(\pi)) - (1-\varphi_{n+1}(\pi))| \\
&= \left| \frac{\sum_{\alpha\in K_n} d_\alpha \left(\sum_{\beta\in K_{n+1}^c} N_{\alpha\pi}^\beta d_\beta\right)}{d_\pi |K_n|_w} - \frac{\sum_{\alpha\in K_{n+1}} d_\alpha \left(\sum_{\beta\in K_{n+1}^c} N_{\alpha\pi}^\beta d_\beta\right)}{d_\pi |K_{n+1}|_w} \right| \\
&= \left| \frac{|E_{n+1}|_w \left(\sum_{\alpha\in K_n} d_\alpha \left(\sum_{\beta\in K_{n+1}^c} N_{\alpha\pi}^\beta d_\beta\right)\right)}{d_\pi |K_n|_w |K_{n+1}|_w} \right| \\
&\quad + \left| \frac{\sum_{\alpha\in K_n} d_\alpha \left(\sum_{\beta\in K_n^c} N_{\alpha\pi}^\beta d_\beta\right) - \sum_{\alpha\in K_{n+1}} d_\alpha \left(\sum_{\beta\in K_{n+1}^c} N_{\alpha\pi}^\beta d_\beta\right)}{d_\pi |K_{n+1}|_w} \right| \\
&\le \frac{|E_{n+1}|_w \left(\sum_{\alpha\in K_n} d_\alpha d_\alpha d_\pi\right)}{d_\pi |K_n|_w |K_{n+1}|_w} \\
&\quad + \left| \frac{\sum_{\beta\in E_{n+1}} d_\beta \left(\sum_{\alpha\in K_n} N_{\beta\bar\pi}^\alpha d_\alpha\right) - \sum_{\alpha\in E_{n+1}} d_\alpha \left(\sum_{\beta\in K_{n+1}^c} N_{\alpha\pi}^\beta d_\beta\right)}{d_\pi |K_{n+1}|_w} \right| \\
&\le \frac{|E_{n+1}|_w \left(\sum_{\alpha\in K_n} d_\alpha d_\alpha d_\pi\right)}{d_\pi |K_n|_w |K_{n+1}|_w} + \frac{\sum_{\beta\in E_{n+1}} d_\beta d_\beta d_{\bar\pi} + \sum_{\alpha\in E_{n+1}} d_\alpha d_\alpha d_\pi}{d_\pi |K_{n+1}|_w} \\
&\le 3\frac{|E_{n+1}|_w}{|K_{n+1}|_w} \lesssim_N \frac{1}{n+1}.
\end{aligned}$$

□

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