ASYMPTOTIC DECOMPOSITION OF SOLUTIONS TO RANDOM PARABOLIC OPERATORS WITH OSCILLATING COEFFICIENTS

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Abstract. We consider Cauchy problem for a divergence form second order parabolic operator with rapidly oscillating coefficients that are periodic in spatial variable and random stationary ergodic in time. As was proved in [25] and [13] in this case the homogenized operator is deterministic.

We obtain the leading terms of the asymptotic expansion of the solution, these terms being deterministic functions, and show that a properly renormalized difference between the solution and the said leading terms converges to a solution of some SPDE.

Contents

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1. Introduction

This work is devoted to obtaining an exact asymptotic development (as $\varepsilon \to 0$) of solutions to the following Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} u^\varepsilon = \operatorname{div}\Big[a\Big(\dfrac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla u^\varepsilon\Big] = \mathcal{A}^\varepsilon u^\varepsilon \qquad \text{in } \mathbb{R}^d \times (0,T] \\ u^\varepsilon(x,0) = \imath(x). \end{cases} \tag{1.1}$$

Here ε is a small positive parameter that tends to zero, $\alpha > 0$, $\alpha \neq 2$, $a(z,y)$ is periodic in z variable and $\xi = (\xi_s,\ s \geq 0)$ is a diffusion stationary ergodic process.

It is known (see [25, 13]) that this problem admits homogenization and that the homogenized operator is deterministic and has constant coefficients. The homogenized Cauchy problem takes the form

$$\begin{cases} \dfrac{\partial}{\partial t} u^0 = \operatorname{div}(\mathrm{a}^{\mathrm{eff}} \nabla u^0) \\ u^0(x,0) = \imath(x). \end{cases} \tag{1.2}$$

The formula for the effective matrix $\mathrm{a}^{\mathrm{eff}}$ is given in (2.1) in Section 2 (see also [13]).

In the existing literature there is a number of works devoted to homogenization of random parabolic problems. The results obtained in [16] and [19] for random divergence form elliptic operators also apply to the parabolic case. In the presence of large lower order terms the limit dynamics might remain random and show diffusive or even more complicated behaviour. The papers [5], [20], [15] focus on the case of time dependent parabolic operators with periodic in spatial variables and random in time coefficients. The fully random case has been studied in [21], [2], [3], [10].

One of the important aspects of homogenization theory is estimating the rate of convergence. For random operators the first estimates have been obtained in [12]. Further important progress in this direction was achieved in the recent works [9], [8].

Problem (1.1) in the case of diffusive scaling $\alpha = 2$ was studied in our previous work [14]. It was shown that, under proper mixing conditions, the difference $u^\varepsilon - u^0$ is of order ε, and that the normalized difference $\varepsilon^{-1}(u^\varepsilon - u^0)$ after subtracting an appropriate corrector, converges in law to a solution of some limit SPDE.

However, for positive $\alpha \neq 2$, the situation becomes much more intriguing. The random solution u^ε admits an asymptotic decomposition as $\varepsilon \to 0$, that is a sum of terms each of which scales as a power of ε. Our main result, Theorem 2.1 below, provides such a description; we will start by its brief description.

Heuristically speaking, this theorem can be thought of as follows. First, as $\varepsilon \to 0$, the *random* solution u^ε converges to the *deterministic* limit u^0. Considering the difference $u^\varepsilon - u^0$ and dividing it by an appropriate power

of ε, one can pass to the limit; if the limit is deterministic, we iterate this procedure until at some stage we reach a *random* limit. Returning to u^ε, we obtain its expansion being a sum of terms of increasing order of ε, with all but the last terms being deterministic, and the random term coming with the scaling factor $\varepsilon^{\alpha/2}$.

A first remark here is that the powers of ε appearing in the expansion are not all integer, but also of the form $\varepsilon^{\delta k}$, where $\delta = |\alpha - 2|$.

An important observation is that for $\alpha > 2$ the final power of ε, the one associated to the random limit, is greater than 1. And that looks very surprising (even next to impossible) due to the following handwaving argument. The solution to the Cauchy problem at some ε is naturally connected to the diffusion process on a compact at the ε^2-rescaled time t/ε^2. Now, if we were considering behaviour of the averages of the type

$$\int_0^t g(s, x_{s/\varepsilon^2}) ds,$$

where x_s is a sufficiently well-mixing ergodic process and g is a function, we would have convergence to the integral of the space average $\int_0^t \bar{g}(s)\, ds$ (where $\bar{g}(s)$ is the expectation of $g(s,\cdot)$ with respect to the stationary distribution of $x_\cdot$) with the Central Limit Theorem-governed speed $\sqrt{\varepsilon^2} = \varepsilon^1$. In our problem, the natural rescaling is $(x_{t/\varepsilon^2}, \xi_{t/\varepsilon^\alpha})$, so one would naturally expect that the randomness occurs at the scaling $\varepsilon^{1\wedge\frac{\alpha}{2}}$. However, it is not the case: for $\alpha > 2$ the randomness occurs not at the power ε^1, but still at the power $\varepsilon^{\alpha/2}$.

1.1. **Organization of the paper.** The paper is organized as follows. In Section 2 we introduce the studied problem and provide all the assumptions. Then we formulate the main result of the paper (Theorem 2.1) and distinguish how this result can be written in the different cases $\alpha < 2$, $2 < \alpha < 4$ and $\alpha \geq 4$. We also define the numerous correctors and auxiliary problems required to state the main result.

In Section 3 we give the formal expansion $\mathcal{E}^\varepsilon$ of u^ε. Formally we define the function $\mathcal{E}^\varepsilon$ such that

$$R^\varepsilon(x,t) = \varepsilon^{-\alpha/2} \left[u^\varepsilon(x,t) - \mathcal{E}^\varepsilon(x,t)\right]$$

converges in a suitable space to some non trivial and random limit q^0. The major result of this section is given by Propositions 3.1 and 3.6. To sum up, this formal expansion of u^ε gives the sequences of constants $\mathrm{a}^{k,\mathrm{eff}}$ and $\underline{\mathrm{a}}^{k,\mathrm{eff}}$ and of smooth functions v^k and u^k. Note that the functions w^k in the definition of v^k will be left as *free parameters* in this first part. The rest R^ε contains with large parameters (as ε tends to zero), both in its dynamics and, for $\alpha > 2$, in its initial condition (3.26). Therefore R^ε is split into five terms $R^\varepsilon = r^\varepsilon + \check{r}^\varepsilon + \hat{r}^\varepsilon + \widetilde{r}^\varepsilon + \rho^\varepsilon +$ such that:

- The dynamics of r^ε contains a large martingale term with a power $\varepsilon^{1-\alpha}$ if $\alpha < 2$ and ε^{-1} if $\alpha > 2$.

- $\check{r}^\varepsilon$ appears only for $\alpha < 2$ and converges to q^0 (see Proposition 3.2).
- $\hat{r}^\varepsilon$ (resp. $\widetilde{r}^\varepsilon$) converges in a weak (resp. strong) topology to zero.
- The last term ρ^ε deals with the initial condition on R^ε and contains large order terms when $\alpha > 2$.

Let us emphasize that in this section the dimension d plays no role and some terms in $\mathcal{E}^\varepsilon$ may be negligible depending on the value of α.

Section 4 focuses on the proof of the convergence of r^ε. In the dynamics of R^ε, there is a martingale term with large parameters, at least when $\alpha > 1$. The free parameters w^k are used here to obtain the weak convergence to zero if $\alpha < 2$ or to the limit q^0 if $\alpha > 2$. Roughly speaking, we need w^k to obtain a uniform bound in $H^1(\mathbb{R})$ of the indefinite integral of R^ε. Here we widely use the fact that $d = 1$.

In Section 5 the trouble comes from the initial condition on R^ε when $\alpha > 2$. Again we give a development of these terms (see Eq. 5.3) together with the properties of these expansions (Lemmata 5.2, 5.3 and 5.4). We prove that from our particular choice of the initial condition on u^k, it is possible to define the constants $\mathcal{I}_k$ such that the initial condition of R^ε does not contribute in the limit equation, that is ρ^ε converges to zero in a strong sense (Proposition 5.5). Again in this section the dimension d could be any positive integer.

To summarize, the conclusion of Theorem 2.1 follows from

- For $\alpha < 2$: Propositions 3.1, 3.2 and 4.8.
- For $\alpha > 2$: Propositions 3.6, 4.9 and 5.5.

Finally, in the Appendix we provide some straightforward but cumbersome computations.

2. Problem setup and main result

In this section we provide all the assumptions for Problem (1.1), introduce some notations and formulate the main results.

2.1. **Assumptions.** Concerning the coefficients of Equation (1.1), we assume that:

(a1) The initial condition $\imath$ belongs to space[1] $C_0^\infty(\mathbb{R})$.

(a2) Function a is periodic in z and smooth in both variables z and y. Moreover, for each $N > 0$ there exists $C_N > 0$ such that

$$\|a\|_{C^N(\mathbb{T}\times\mathbb{R}^n)} \le C_N.$$

Here and in what follows we identify periodic functions with functions on the torus $\mathbb{T}$.

(a3) Coefficient $a = a(z, y)$ satisfies the uniform ellipticity condition: there exists $\lambda > 0$ such that for any $z \in \mathbb{T}$ and any $y \in \mathbb{R}^n$:

$$\lambda^{-1} \le a(z, y) \le \lambda;$$

[1] In fact, this condition can be essentially relaxed (see Remark 2.4).

The random noise $\xi = (\xi_s,\ s \geq 0)$ is a diffusion process in $\mathbb{R}^n$ with a generator

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}[q(y)D^2] + b(y).\nabla$$

(∇ stands for the gradient, D^2 for the Hessian matrix). Moreover we suppose that matrix-function q and vector-function b possess the following properties:

(a4) The matrix $q = q(y)$ satisfies the uniform ellipticity condition: there exists $\lambda > 0$ such that

$$\lambda^{-1}|\zeta|^2 \leq q(y)\zeta\cdot\zeta \leq \lambda|\zeta|^2, \quad y,\ \zeta \in \mathbb{R}^n.$$

Moreover there exists a matrix $\sigma = \sigma(y)$ such that $q(y) = \sigma^*(y)\sigma(y)$.

(a5) The matrix function σ and vector function b are smooth, that is for each $N > 0$ there exists $C_N > 0$ such that

$$\|\sigma\|_{C^N(\mathbb{R}^n)} \leq C_N, \qquad \|b\|_{C^N(\mathbb{R}^n)} \leq C_N.$$

(a6) The following inequality holds for some $R > 0$ and $C_0 > 0$ and $p > -1$:

$$b(y)\cdot y \leq -C_0|y|^p \quad \text{for all } y \in \{y \in \mathbb{R}^n\ :\ |y| \geq R\}.$$

We say that Condition **(A)** holds if **(a1)** to **(a6)** are satisfied.

Let us recall that according to [22, 23] under conditions **(a4)** and **(a6)** a diffusion process $\xi_\cdot$ with generator $\mathcal{L}$ has an invariant measure in $\mathbb{R}^n$ that has a smooth density $p = p(y)$. For any $N > 0$ it holds

$$(1 + |y|)^N p(y) \leq C_N$$

with some constant C_N. The function p is the unique up to a multiplicative constant bounded solution of the equation $\mathcal{L}^* p = 0$; here $\mathcal{L}^*$ denotes the formally adjoint operator. We assume that the process ξ is stationary and distributed with the density p. In the rest of the paper

- $\overline{f}$ denotes the mean w.r.t. the invariant measure p;
- $\langle f\rangle$ is the mean on the torus $\mathbb{T}$.

a^ε and $\mathfrak{a}^\varepsilon$ denote the matrices:

$$a^\varepsilon = a\left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right), \quad \mathfrak{a}^\varepsilon = a\left(z, \xi_{\frac{t}{\varepsilon^{\alpha-2}}}\right).$$

From [13] under Condition **(A)**, we know that u^ε converges in probability in the space

$$V_T = L^2_w(0,T;H^1(\mathbb{R})) \cap C(0,T;L^2_w(\mathbb{R}))$$

to u^0, the solution of (1.2)

$$u^0_t = \mathrm{div}\ (\mathrm{a}^{\mathrm{eff}}\nabla u^0), \quad u^0(x,0) = \imath(x),$$

where the effective matrix $\mathrm{a}^{\mathrm{eff}}$ is defined by:

$$\mathrm{a}^{\mathrm{eff}} = \overline{\langle a + a\nabla_z\chi^0\rangle}. \tag{2.1}$$

The symbol w in the definition of V_T means that the corresponding space is endowed with its weak topology.

The corrector χ^0 is different if $\alpha < 2$ (Equation (2.2)) or $\alpha > 2$ (Equation (2.4)), thus the function u^0 is not the same for $\alpha > 2$ and $\alpha < 2$. More precisely, for $\alpha < 2$, the function $\chi^0 = \chi^0(z, y)$ is a periodic solution of the equation

$$\mathrm{div}_z\big(a(z,y)\nabla_z\chi^0(z,y)\big) = -\mathrm{div}_z a(z,y); \tag{2.2}$$

here $y \in \mathbb{R}^n$ is a parameter. We choose an additive constant in such a way that

$$\int_{\mathbb{T}} \chi^0(z,y)\,dz = 0. \tag{2.3}$$

When $\alpha > 2$, the corrector χ^0 is the solution of

$$\bar{\mathcal{A}}\chi^0 = \mathrm{div}\ \big[\bar{a}\nabla\chi^0\big] = -\mathrm{div}\ \bar{a}_i \tag{2.4}$$

where $\bar{a}$ is the mean value of a w.r.t. y:

$$\bar{a}(z) = \int_{\mathbb{R}^n} a(z,y)p(y)dy.$$

It is known that matrix $\mathrm{a}^{\mathrm{eff}}$ is positive definite in both cases (see, for instance, [5, 13]).

2.2. **Main result.** In the rest of the paper we denote

- $\delta = |\alpha - 2| > 0$,
- $J_0 = \lfloor \frac{\alpha}{2\delta} \rfloor + 1$, where $\lfloor\cdot\rfloor$ stands for the integer part,
- $J_1 = \lfloor \frac{\alpha}{2} \rfloor$.

Let us remark that: $\min(\delta + 1, J_1 + 1, \delta J_0) > \alpha/2$. For technical reasons, we also use $N_0 = 2J_0 + 2$.

For any $\alpha \neq 2$, we construct a sequence of constants $\mathrm{a}^{k,\mathrm{eff}}$, $k \geq 1$, and a sequence of functions u^j, $j \geq 1$, as solutions of problems

$$\frac{\partial}{\partial t}u^j = \mathrm{div}(\mathrm{a}^{\mathrm{eff}}\nabla u^j) + \sum_{k=1}^{j} \mathrm{a}^{k,\mathrm{eff}}\frac{\partial^2}{\partial x^2}u^{j-k} + w^j \tag{2.5}$$

with initial condition $u^j(x,0) = 0$. The definition of the sequence $\mathrm{a}^{k,\mathrm{eff}}$ depends on the sign of $\alpha - 2$ (see Eq. (2.14) and (2.31)). The functions w^j are smooth functions and defined recursively.

- For $\alpha > 2$

$$\forall k \geq 0, \quad w^{k+1}(x,t) = -\sum_{m=0}^{k} \mathcal{C}_{k,m}u^m_{xx}(x,t) - \sum_{m=1}^{k} w^m(x,t). \tag{2.6}$$

- For $\alpha \in (0,2)$, we define w^j recursively by $w^1 = 0$ and

$$\forall k \geq 0, \quad w^{k+2}(x,t) = -\sum_{m=0}^{k} \mathcal{C}_{k,m}u^m_{xx}(x,t) - \sum_{m=1}^{k} w^{m+1}(x,t). \tag{2.7}$$

The triangular array of constants $(\mathcal{C}_{k,m})_{0\le m\le k}$ is defined by (4.9). Note that these constants are not the same if $\alpha > 2$ or if $\alpha < 2$ since the correctors used in (4.9) are different. Somehow the function w^k for $\alpha > 2$ is equal to the function w^{k+1} for $\alpha < 2$; there is a shift between them.

For $\alpha > 2$, to obtain the desired convergence we need a second sequence of functions with a different scaling. We construct two other sequences of constants $(\underline{\mathrm{a}}^{k,\mathrm{eff}})_{k\ge1}$ and $(\mathcal{I}_k)_{k\ge1}$ such that we can define $v^0 = u^0$ and

$$v_t^j = \mathrm{a}^{\mathrm{eff}} v_{xx}^j + S^j, \quad v^j(x,0) = \mathcal{I}_j \partial_x^j u^0(x,0), \tag{2.8}$$

with for $j \ge 1$

$$S^j(x,t) = \sum_{k=1}^{j} \underline{\mathrm{a}}^{k,\mathrm{eff}} (\partial_x^{k+2} v^{j-k}) \ . \tag{2.9}$$

The sequence of constants $(\mathcal{I}_k)_{k\ge1}$ are such that $\mathcal{I}_0 = 1$, $\mathcal{I}_1 = 0$, and defined by (5.20) for $k \ge 0$. In the expansion of u^ε, we need to take into account the initial value of the remainder. For $\alpha < 2$, this additional term is negligible. But for $\alpha > 2$, it contains negative powers of ε and thus it should be controlled. This is the role of this sequence $\mathcal{I}_k$. Finally the correctors χ^j are defined by (2.20).

Our main result is the following.

Theorem 2.1. *Under Condition* **(A)**, *there exists a non-negative constant* Λ *(defined by* (2.16) *for* $\alpha < 2$ *and* (2.37) *for* $\alpha > 2$*), such that the normalized functions*

$$q^\varepsilon = \varepsilon^{-\alpha/2} \Bigg\{ u^\varepsilon(x,t) - u^0(x,t) - \sum_{k=1}^{J_0} \varepsilon^{k\delta} u^k(x,t) - \sum_{k=1}^{J_1} \varepsilon^k \left[v^k(x,t) + \sum_{\ell=1}^{k} \chi^{\ell-1}\left(\frac{x}{\varepsilon}\right) \partial_x^\ell v^{k-\ell}(x,t) \right] \Bigg\}$$

converge in law, as $\varepsilon \to 0$, *in* $L^2_w(\mathbb{R}\times(0,T))$ *to the unique solution of the following SPDE*

$$\begin{aligned} dq^0 &= \mathrm{div}(\mathrm{a}^{\mathrm{eff}} \nabla q^0)\, dt + (\Lambda^{1/2}) \left(\frac{\partial^2}{\partial x^2} u^0 \right) dW_t \\ q^0(x,0) &= 0; \end{aligned} \tag{2.10}$$

driven by a standard one-dimensional Brownian motion W.

Note that the values of $J_0 = \lfloor \frac{\alpha}{2\delta} \rfloor + 1$ and of α are related as follows:

- $\alpha < 2$ and
$$2 - \frac{2}{2J_0 - 1} \le \alpha < 2 - \frac{2}{2J_0 + 1};$$
- $2 < \alpha \le 4$, $J_0 \ge 2$ and
$$2 + \frac{2}{2J_0 - 1} < \alpha \le 2 + \frac{2}{2J_0 - 3};$$

- $\alpha > 4$ and $J_0 = 1$.

In other words, J_0 becomes large when α is close to 2. Let us precise a little bit what happens for q^ε in the four cases: $\alpha < 2$, $\alpha < 4$, $\alpha = 4$ and $\alpha > 4$.

- $\alpha < 2$: $J_1 = 0$ and q^ε can be written as follows:

$$q^\varepsilon = \varepsilon^{-\alpha/2} \left\{ u^\varepsilon(x,t) - u^0(x,t) - \sum_{k=1}^{J_0} \varepsilon^{k\delta} u^k(x,t) \right\}.$$

  Here the sequence v^k is not involved.
- $2 < \alpha < 4$: $J_1 = 1$ and

$$q^\varepsilon = \varepsilon^{-\alpha/2} \Bigg\{ u^\varepsilon(x,t) - u^0(x,t) - \sum_{k=1}^{J_0} \varepsilon^{k\delta} u^k(x,t) - \varepsilon \left[v^1(x,t) + \chi^0\left(\frac{x}{\varepsilon}\right) \partial_x u^0(x,t) \right] \Bigg\}$$

- $\alpha = 4$: $J_0 = 1$ and $J_1 = 2$. Thereby q^ε becomes

$$\begin{aligned} q^\varepsilon \;=\; & \varepsilon^{-2} \Bigg\{ u^\varepsilon(x,t) - u^0(x,t) - \varepsilon \left[v^1(x,t) + \chi^0\left(\frac{x}{\varepsilon}\right) \partial_x u^0(x,t) \right] \\ & - \varepsilon^2 \left[u^1(x,t) + v^2(x,t) + \chi^0\left(\frac{x}{\varepsilon}\right) \partial_x^2 u^0(x,t) + \chi^1\left(\frac{x}{\varepsilon}\right) \partial_x v^1(x,t) \right] \Bigg\}. \end{aligned}$$

  $\alpha = 4$ is a kind of critical value, since here u^1 and v^2 coexist.
- $\alpha > 4$: $J_0 = 1$ and for any $m \geq 2$

$$J_1 = m \Leftrightarrow 2m \leq \alpha \leq 2(m+1).$$

  Hence

$$q^\varepsilon = \varepsilon^{-\alpha/2} \Bigg\{ u^\varepsilon(x,t) - u^0(x,t) - \varepsilon^\delta u^1(x,t) - \sum_{k=1}^{J_1} \varepsilon^k \left[v^k(x,t) + \sum_{\ell=1}^{k} \chi^\ell\left(\frac{x}{\varepsilon}\right) \partial_x^\ell v^{k-\ell}(x,t) \right] \Bigg\}.$$

Remark 2.2 (When $J_0 = 1$)**.** For $\alpha > 4$ or $\alpha < 4/3$, we have $\delta > \alpha/2$ and $J_0 = 1$. Thus we may remove u^1 in the quantity q^ε: $\varepsilon^{\delta - \alpha/2} u^1$ tends to zero for the strong topology and thus does not contribute directly to the limit q^0 of q^ε. Nevertheless we emphasize that u^1 and w^1 are used to obtain the weak convergence of q^ε.

Remark 2.3 (When $\alpha < 1$)**.** In this range for α, we have a stronger convergence and the result can be extended to any dimension d, that is a is periodic in z with the period $[0,1]^d$ and we identify periodic functions with functions defined on the torus $\mathbb{T}^d$.

Remark 2.4 (Regularity of $\imath$)**.** The regularity assumption on $\imath$ given in condition **(a1)** can be weakened. Namely, the statement of Theorem 2.1 holds if $\imath$ is $\max(J_0+1, J_1)$ times continuously differentiable and the corresponding partial derivatives decay at infinity sufficiently fast.

Remark 2.5 (Dimension d)**.** All results in Sections 2.3 and 3 are valid if the dimension of the problem is any integer $d \geq 1$, that is if $z \in \mathbb{T}^d$. However the trick used in Section 4 is correct only in dimension 1.

2.3. **Auxiliary problems.** If **(A)** holds, since $\mathrm{a}^{\mathrm{eff}}$ is positive, the problem (1.2) is well posed, uniquely defined, smooth and satisfies the estimates

$$\left|(1+|x|)^N \frac{\partial^{\mathbf{k}} u^0(x,t)}{\partial t^{k_0} \partial x^{k_1}}\right| \leq C_{N,\mathbf{k}} \tag{2.11}$$

for all $N > 0$ and all multi index $\mathbf{k} = (k_0, k_1)$, $k_i \geq 0$.

Correctors and constants for $\alpha < 2$. We begin by considering Problem (2.2). This equation has a unique up to an additive constant vector periodic solution. By the classical elliptic estimates, under our standing assumptions we have for any $N > 0$ there exists C_N such that

$$\|\chi^0\|_{C^N(\mathbb{T}\times\mathbb{R}^n)} \leq C_N. \tag{2.12}$$

Indeed, multiplying equation (2.2) by χ^0, using the Schwarz and Poincaré inequalities and considering (2.3), the estimate follows from [7].

Higher order correctors are defined as periodic solutions of the equations

$$\mathrm{div}_z\big(a(z,y)\nabla_z\chi^j(z,y)\big) = -\mathcal{L}_y\chi^{j-1}(z,y), \quad j = 1, 2, \ldots, J_0. \tag{2.13}$$

Notice that $\int_{\mathbb{T}} \chi^{j-1}(z,y)\,dz = 0$ for all $j = 1, 2, \ldots, J^0$, thus the compatibility condition is satisfied and the equations are solvable. By the similar arguments, the solutions χ^j defined by (2.13) satisfy the same estimate as χ^0.

We introduce the real numbers for $k \geq 1$:

$$\mathrm{a}^{k,\mathrm{eff}} = \int_{\mathbb{R}^n}\int_{\mathbb{T}} \big[a(z,y)\nabla_z\chi^k(z,y) + \nabla_z\big(a(z,y)\chi^k(z,y)\big)\big]p(y)\,dzdy. \tag{2.14}$$

Arguing as for u^0, solutions u^j and w^j of problems (2.5) and (2.7) are smooth functions and they satisfy also the estimate (2.11).

Now we define:

$$\begin{aligned} \widehat{a}^0(z,y) &= a(z,y) + a(z,y)\nabla_z\chi^0(z,y) + \nabla_z\big(a(z,y)\chi^0(z,y)\big), \\ \langle a\rangle^0(y) &= \int_{\mathbb{T}^d} \big(\widehat{a}^0(z,y) - \mathrm{a}^{\mathrm{eff}}\big)dz, \end{aligned}$$

and we consider the equation

$$\mathcal{L}Q^0(y) = \langle a\rangle^0(y). \tag{2.15}$$

According to [22, Theorems 1 and 2], this equation has a unique up to an additive constant solution of at most polynomial growth. The constant Λ is defined by:

$$\Lambda = \int_{\mathbb{R}^n} \Big[\frac{\partial}{\partial y_{r_1}}(Q^0)(y)\Big] q^{r_1 r_2}(y) \Big[\frac{\partial}{\partial y_{r_2}}(Q^0)(y)\Big] p(y)\, dy. \tag{2.16}$$

Note that the matrix Λ is non-negative. Consequently its square root $\Lambda^{1/2}$ is well defined.

Correctors and constants for $\alpha > 2$. The first auxiliary problem (2.4) $\bar{\mathcal{A}}\chi^0 = -\bar{a}_z$ reads

$$\operatorname{div}\big(\bar{a}(z)\nabla\chi^0(z)\big) = -\operatorname{div}\bar{a}(z), \quad z \in \mathbb{T};$$

where

$$\bar{a}(z) = \int_{\mathbb{R}^n} a(z,y)p(y)dy.$$

It has a unique up to an additive constant periodic solution. This constant is chosen in such a way that (2.3) holds, namely $\int_{\mathbb{T}} \chi^0(z)\, dz = 0$. By the classical elliptic estimates (see [7]), under **(A)**, we have

$$\|\chi^0\|_{L^\infty(\mathbb{T})} + \|\chi^0\|_{C^k(\mathbb{T})} \le C. \tag{2.17}$$

Then we can define recursively

$$\begin{aligned} f^0(z,y) &= a - \mathrm{a}^{\mathrm{eff}} + \big(a\chi^0_z + (a\chi^0)_z\big), \\ \bar{\mathcal{A}}\chi^1(z) &= \overline{f^0}, \\ f^1(z,y) &= \chi^0(a - \mathrm{a}^{\mathrm{eff}}) + \big(a\chi^1_z + (a\chi^1)_z\big). \end{aligned}$$

Note that here we use $\langle\overline{f^0}\rangle = 0$ to obtain the periodic solution χ^1. Now we define the constant $\underline{\mathrm{a}}^{1,\mathrm{eff}}$ by:

$$\underline{\mathrm{a}}^{1,\mathrm{eff}} = \langle\overline{f^1}\rangle$$

and the corrector χ^2 by:

$$\overline{\mathcal{A}}\chi^2(z) = \overline{f^1} - \langle\overline{f^1}\rangle.$$

Let us also define by induction the following quantities for $k \ge 2$

$$f^k(z,y) = \chi^{k-1}(a - \mathrm{a}^{\mathrm{eff}}) + \Big(a\chi^k_z + (a\chi^k)_z\Big), \tag{2.18}$$

$$\underline{\mathrm{a}}^{k,\mathrm{eff}} = \langle\overline{f^k}\rangle, \tag{2.19}$$

$$\overline{\mathcal{A}}\chi^{k+1}(z) = \overline{f^k} - \langle\overline{f^k}\rangle + \left(\sum_{j=1}^{k-1} c_{k-j}\chi^{j-1}\right). \tag{2.20}$$

All functions χ^k, $k \ge 1$, are solution of an equation of the form $\overline{\mathcal{A}}v = F$, where F is a periodic with zero mean value and bounded function. Hence all functions χ^k are well defined on $\mathbb{T}$ and satisfy (2.3) and (2.17).

Finally from χ^0, we can define κ^0 as the solution of:

$$\mathcal{L}\kappa^0(z,y) = (\mathrm{div}_z a(z,y) - \mathrm{div}_z \bar{a}(z)) + \mathrm{div}_z((a(z,y)-\bar{a}(z))\nabla_z\chi^0(z)), \tag{2.21}$$

To be precised, κ^0 satisfies for any $y \in \mathbb{R}^n$

$$\mathcal{L}\kappa^0(z,y) = (\mathrm{div}_z a(z,y) - \mathrm{div}_z \bar{a}(z)) + \mathrm{div}_z((a(z,y)-\bar{a}(z))\nabla_z\chi^0(z)),$$

$z \in \mathbb{T}$ being a parameter. Since the right-hand side has a zero mean value (w.r.t. y) and is bounded, according to [22], this equation has a unique up to an additive constant (w.r.t. y) solution of at most polynomial growth:

$$|\kappa^0(z,y)| \leq C(1+|y|^p), \quad \forall (z,y) \in \mathbb{T}\times\mathbb{R}^d. \tag{2.22}$$

Moreover we can impose that

$$\int_{\mathbb{T}} \kappa^0(z,y)\, dz = 0. \tag{2.23}$$

Finally the right-hand side of (2.21) being a smooth function w.r.t. z with bounded derivatives, again according the representation of [22], $z \mapsto \kappa^0(z,y)$ is smooth. Indeed the operators $\mathcal{L}$ and derivative w.r.t. z are commutative. Then by induction, we introduce a sequence κ^k, $k \geq 0$ defined by:

$$\mathcal{L}\kappa^{k+1}(z,y) = (\mathcal{A}-\overline{\mathcal{A}})\chi^{k+1} + (f^k - \overline{f^k}). \tag{2.24}$$

for $k = 1, 2, \ldots, J_1+2$. These higher order correctors κ^k exist and satisfy Estimates (2.22) and (2.23).

We will also use the next functions or constants:

$$\overline{\mathcal{A}}\tau^0(z) = -\overline{(\mathcal{A}\kappa^0)}, \tag{2.25}$$

$$g^0(z,y) = a(\kappa^0+\tau^0)_z + (a(\kappa^0+\tau^0))_z, \tag{2.26}$$

$$\mathrm{a}^{1,\mathrm{eff}} = \langle\overline{g^0}\rangle = \overline{\langle a(\kappa^0+\tau^0)_z\rangle}, \tag{2.27}$$

Now for any $k \geq 1$

$$\overline{\mathcal{A}}\tau^k(z) = -\overline{\mathcal{A}\gamma^{k-1}}, \tag{2.28}$$

and

$$\mathcal{L}\gamma^0(z,y) = (\mathcal{A}-\overline{\mathcal{A}})\tau^0 + (\mathcal{A}\kappa^0 - \overline{\mathcal{A}\kappa^0}), \tag{2.29}$$

$$\mathcal{L}\gamma^k(z,y) = (\mathcal{A}-\overline{\mathcal{A}})\tau^k + (\mathcal{A}\gamma^{k-1} - \overline{\mathcal{A}\gamma^{k-1}}). \tag{2.30}$$

For $k \geq 2$ we put

$$\mathrm{a}^{k,\mathrm{eff}} = \overline{\langle a(\tau^{k-1}(z) + \gamma^{k-2}(z,y))_z\rangle}. \tag{2.31}$$

By the same arguments, the set of correctors τ^k and γ^k defined respectively by (2.25), (2.28), (2.29) and (2.30) verify again Estimates (2.3), (2.17), (2.22) and (2.23). Roughly speaking, they are smooth in both variables, bounded in z and in polynomial growth in y.

To obtain the result in Lemmata 3.4 and 3.5, we introduce two sequences of functions

$$g^k(z,y)=a(\tau^k+\gamma^{k-1})_z,\quad h^k(z,y)=(a(\tau^k+\gamma^{k-1}))_z \tag{2.32}$$

and $\eta^k(z)$ and $\zeta^k(z,y)$, $k\geq 1$, solutions of

$$\overline{\mathcal{A}}\eta^1 = -(\overline{\mathcal{A}\kappa^1})-(\overline{g^0}-\langle\overline{g^0}\rangle), \tag{2.33}$$

$$\mathcal{L}\zeta^1 = (\mathcal{A}-\overline{\mathcal{A}})\eta^1+(\mathcal{A}\kappa^1-\overline{\mathcal{A}\kappa^1})+(g^0-\overline{g^0}), \tag{2.34}$$

and for $k\geq 2$

$$\overline{\mathcal{A}}\eta^k = -(\overline{\mathcal{A}\zeta^{k-1}})-\overline{h^{k-1}}-(\overline{g^{k-1}}-\langle\overline{g^{k-1}}\rangle), \tag{2.35}$$

$$\begin{aligned}\mathcal{L}\zeta^k = {}&(\mathcal{A}-\overline{\mathcal{A}})\eta^k+(\mathcal{A}\zeta^{k-1}-\overline{\mathcal{A}\zeta^{k-1}})\\ &+(h^{k-1}-\overline{h^{k-1}})+(g^{k-1}-\overline{g^{k-1}}).\end{aligned} \tag{2.36}$$

Again the same arguments show that η^k and ζ^k defined by (2.33), (2.34), (2.35) and (2.36), exist and are smooth.

Finally the constant Λ is defined by

$$\Lambda=\overline{\langle\chi^0\Upsilon^0\rangle^2}>0, \tag{2.37}$$

where Υ^0 is given by:

$$\Upsilon^0(z,y)=-\kappa^0_y(z,y)q(y). \tag{2.38}$$

Here κ^0_y stands for the gradient of κ^0 w.r.t. y (this notation is used in the sequel of the paper).

3. Formal expansion for the solution

In any case $\alpha\neq 2$, we begin with a first formal expansion of u^ε, where the functions w^k in the definition of u^k will be left as *free parameters* in this first part. This development leads to a rest R^ε with large parameters (as ε tends to zero) in its dynamics and in its initial condition. Moreover this development gives the main part of q^ε.

Note that we denote by B the n-dimensional standard Brownian motion driving the process ξ.

3.1. The case $\alpha<2$.

For $k\geq 1$ we define

$$\widehat{a}^k(z,y) = a(z,y)\nabla_z\chi^k(z,y)+\nabla_z\big(a(z,y)\chi^k(z,y)\big), \tag{3.1}$$

$$\langle a\rangle^k(y) = \int_{\mathbb{T}}\big(\widehat{a}^k(z,y)-a^{k,\mathrm{eff}}\big)dz. \tag{3.2}$$

We consider the following expression:

$$\mathcal{E}^\varepsilon(x,t)=\sum_{k=0}^{J_0}\varepsilon^{k\delta}\Big(u^k(x,t)+\sum_{j=0}^{J_0-k}\varepsilon^{(j\delta+1)}\chi^j\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla u^k(x,t)\Big)$$

where $\chi^j(z,y)$ and $u^k(x,t)$ are defined in (2.13) and (2.5), respectively.

Proposition 3.1. *If we define:*

$$R^\varepsilon(x,t) = \varepsilon^{-\alpha/2}\left[u^\varepsilon(x,t) - \mathcal{E}^\varepsilon(x,t)\right],$$

then R^ε is the sum $R^\varepsilon = r^\varepsilon + \check{r}^\varepsilon + \widetilde{r}^\varepsilon + \rho^\varepsilon$ where:

- *The dynamics of r^ε contains the terms with large parameters:*

$$\begin{aligned} dr^\varepsilon = (\mathcal{A}^\varepsilon r^\varepsilon)dt - \varepsilon^{-\alpha/2}\sum_{k=1}^{J_0}\varepsilon^{k\delta}w^k(x,t)\ dt \\ -\varepsilon^{1-\alpha}\sum_{k=0}^{J_0}\sum_{j=0}^{J_0-k}\varepsilon^{(k+j)\delta}\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\nabla_y\chi^j\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla u^k(x,t)\,dB_t \end{aligned} \tag{3.3}$$

  with $r^\varepsilon(x,0)=0$.
- *The dynamics of $\check{r}^\varepsilon$ is given by:*

$$\partial_t\check{r}^\varepsilon - \mathcal{A}^\varepsilon\check{r}^\varepsilon = \varepsilon^{-\alpha/2}\sum_{j=0}^{J_0}\sum_{k=0}^{J_0-j}\varepsilon^{(k+j)\delta}\Big[\widehat{\mathrm{a}}^k\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big) - a^{k,\mathrm{eff}}\Big]\frac{\partial^2 u^j}{\partial x^2} \tag{3.4}$$

  with $\check{r}^\varepsilon(x,0)=0$.
- *The last terms $\widetilde{r}^\varepsilon$ and ρ^ε satisfy:*

$$\mathbf{E}\|\widetilde{r}^\varepsilon\|^2_{L^2(\mathbb{R}\times(0,T))} + \mathbf{E}\|\rho^\varepsilon\|^2_{L^2(\mathbb{R}\times(0,T))} \le C\varepsilon^\delta. \tag{3.5}$$

Proof. We substitute R^ε for u^ε in (1.1) using Itô's formula:

$$\begin{aligned} &dR^\varepsilon - \mathrm{div}\big[a\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\nabla R^\varepsilon\big]dt \\ &= -\varepsilon^{-\frac{\alpha}{2}}\sum_{k=0}^{J_0}\varepsilon^{k\delta}\Big[\partial_t u^k + \sum_{j=0}^{J_0-k}\varepsilon^{(j\delta+1-\alpha)}\big(\mathcal{L}_y\chi^j\big)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\nabla u^k \\ &\qquad\qquad + \sum_{j=0}^{J_0-k}\varepsilon^{(j\delta+1)}\chi^j\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\partial_t\nabla u^k\Big]dt \\ &-\sum_{k=0}^{J_0}\sum_{j=0}^{J_0-k}\varepsilon^{(1-\alpha+(k+j)\delta)}\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\nabla_y\chi^j\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla u^k(x,t)\,dB_t \\ &+\varepsilon^{-\frac{\alpha}{2}}\sum_{k=0}^{J_0}\varepsilon^{k\delta-1}\Big[(\mathrm{div}a)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big) + \sum_{j=0}^{J_0-k}\varepsilon^{j\delta}\big(\mathrm{div}(a\nabla\chi^j)\big)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\Big]\nabla u^k \\ &+\varepsilon^{-\frac{\alpha}{2}}\sum_{k=0}^{J_0}\sum_{j=0}^{J_0-k}\varepsilon^{(k+j)\delta}\ \widehat{a}^{j,im}\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\frac{\partial^2}{\partial x_i\partial x_m}u^k dt \\ &+\varepsilon^{-\frac{\alpha}{2}}\sum_{k=0}^{J_0}\sum_{j=0}^{J_0-k}\varepsilon^{(k+j)\delta+1}\ (a^{im}\chi^{j,l})\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\frac{\partial^3}{\partial x_i\partial x_m\partial x_l}u^k dt. \end{aligned}$$

Due to (2.2) and (2.13)

$$
\begin{aligned}
&-\sum_{k=0}^{J_0}\varepsilon^{k\delta}\sum_{j=0}^{J_0-k}\varepsilon^{(j\delta+1-\alpha)}\big(\mathcal{L}_y\chi^j\big)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\nabla u^k\\
&\quad+\sum_{k=0}^{J_0}\varepsilon^{k\delta-1}\Big[(\mathrm{div}a)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)+\sum_{j=0}^{J_0-k}\varepsilon^{j\delta}\big(\mathrm{div}(a\nabla\chi^j)\big)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\Big]\nabla u^k\\
&=-\varepsilon^{(J_0+1)\delta-1}\sum_{k=0}^{J_0}\big(\mathcal{L}_y\chi^{J_0-k}\big)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\nabla u^k.
\end{aligned}
$$

Considering equations (2.5) and the definitions of $a^{k,\mathrm{eff}}$ and $\widehat{\mathrm{a}}^k(z,y)$, we obtain

$$
\begin{aligned}
(3.6)\qquad &dR^\varepsilon(x,t)-\mathrm{div}\Big[\mathrm{a}\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla R^\varepsilon\Big]\,dt\\
&=\Big(\varepsilon^{-\alpha/2}\sum_{j=0}^{J_0}\sum_{k=0}^{J_0-j}\varepsilon^{(k+j)\delta}\Big[\widehat{\mathrm{a}}^k\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)-a^{k,\mathrm{eff}}\Big]^{im}\frac{\partial^2u^j}{\partial x_i\partial x_m}\Big)\,dt\\
&-\sum_{k=0}^{J_0}\sum_{j=0}^{J_0-k}\varepsilon^{(1-\alpha+(k+j)\delta)}\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\nabla_y\chi^j\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla u^k(x,t)\,dB_t\\
&-\sum_{k=1}^{J_0}\varepsilon^{k\delta-\alpha/2}w^k(x,t)\;dt\\
&+\varepsilon^{1-\alpha/2}\sum_{j=0}^{J_0}\varepsilon^{j\delta}\mathfrak{b}^j\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)\mathfrak{F}^j(x,t)\,dt,
\end{aligned}
$$

with $\mathrm{a}^{0,\mathrm{eff}}=\mathrm{a}^{\mathrm{eff}}$ and with periodic in z smooth functions $\mathfrak{b}^j=\mathfrak{b}^j(z,y)$ of at most polynomial growth in y, and $\mathfrak{F}^j$ satisfying (2.11), that is

$$
\Big|(1+|x|)^N D^{\mathbf{k}}\mathfrak{F}^j\Big|\le C_{\mathbf{k},N}.
$$

The initial condition for R^ε is given by:

$$
R^\varepsilon(x,0)=\varepsilon^{1-\alpha/2}\sum_{k=0}^{J_0}\sum_{j=0}^{J_0-k}\varepsilon^{j\delta}\chi^j\Big(\frac{x}{\varepsilon},\xi_0\Big)\nabla u^k(x,0).
$$

By the linearity of (3.6), we represent R^ε as the sum $R^\varepsilon=r^\varepsilon+\check r^\varepsilon+\widetilde r^\varepsilon+\rho^\varepsilon$ where r^ε and $\check r^\varepsilon$ are given by (3.3) and (3.4) and $\widetilde r^\varepsilon$ contains all negligible terms:

$$
\partial_t\widetilde r^\varepsilon-\mathcal{A}^\varepsilon\widetilde r=\varepsilon^{1-\alpha/2}\sum_{j=0}^{J_0}\varepsilon^{j\delta}\mathfrak{b}^j\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)\mathfrak{F}^j(x,t)=\mathcal{B}^\varepsilon(x,t)
$$

together with $\widetilde{r}^\varepsilon(x,0)=0$. We have

$$\begin{aligned}\mathbf{E}\|\mathcal{B}^\varepsilon\|^2_{L^2(\mathbb{R}\times(0,T))} &\leq C\varepsilon^{1-\alpha/2}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}^n}(1+|y|)^{N_1}(1+|x|)^{-2n}p(y)\,dydxdt\\ &\leq C\varepsilon^{1-\alpha/2}.\end{aligned}$$

Similarly, $\mathbf{E}\|\widetilde{r}^\varepsilon(\cdot,0)\|^2_{L^2(\mathbb{R}^d)}\leq C\varepsilon^{1-\alpha/2}$. Therefore, $\widetilde{r}^\varepsilon$ satisfies (3.5) and thus does not contribute in the limit. Finally ρ^ε satisfies the dynamics

$$\partial_t\rho^\varepsilon=\mathcal{A}^\varepsilon\rho^\varepsilon$$

with the initial condition $\rho^\varepsilon(x,0)=R^\varepsilon(x,0)$. Since $\alpha<2$, we deduce that $\mathbf{E}\|\rho^\varepsilon(\cdot,0)\|^2_{L^2(\mathbb{R}^d)}\leq C\varepsilon^{1-\alpha/2}$. Thereby this term also does not contribute in the limit. □

The second term $\check{r}^\varepsilon$ gives the limit in Theorem 2.1.

Proposition 3.2. *The solution $\check{r}^\varepsilon$ of Problem* (3.4) *converges in law, as ε goes to* 0*, in $L^2(\mathbb{R}\times(0,T))$ equipped with strong topology, to the solution of* (2.10).

Proof. Recall that from the very definition of $\widehat{a}_k$ (3.1) and $\langle \mathrm{a}\rangle^k$ (3.2) the problem

$$\Delta_z\zeta^k(z,y)=(\widehat{a}^k(z,y)-\langle a\rangle^k(y))$$

has a unique up to an additive constant periodic solution. Letting $\Theta^k(z,y)=\nabla\zeta^k(z,y)$, we obtain a vector functions Θ^k such that

$$\operatorname{div}\Theta^k(z,y)=(\widehat{a}^k(z,y)-\langle a\rangle^k(y)),\qquad \|\Theta^k\|^N_{C^\ell(\mathbb{T}\times\mathbb{R}^n)}\leq C_{N,\ell}.$$

It is then straightforward to check that for the functions

$$\begin{aligned}H^\varepsilon(x,t) &= \varepsilon^{-\alpha/2}\sum_{j=0}^{J_0}\sum_{k=0}^{J_0-j}\varepsilon^{(k+j)\delta}\Big[\widehat{\mathrm{a}}^k\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)-\langle \mathrm{a}\rangle^k\big(\xi_{\frac{t}{\varepsilon^\alpha}}\big)\Big]\frac{\partial^2u^j}{\partial x^2}\\ &= \varepsilon^{1-\frac{\alpha}{2}}\sum_{j=0}^{J_0}\sum_{k=0}^{J_0-j}\varepsilon^{(k+j)\delta}\Big\{\operatorname{div}\Big[\Theta^k\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\Big)\frac{\partial^2}{\partial x^2}u^j(x,t)\Big]\\ &\qquad -\Theta^k\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\Big)\nabla\Big(\frac{\partial^2}{\partial x^2}u^j(x,t)\Big)\Big\}\end{aligned}$$

the following estimate is fulfilled:

$$\mathbf{E}\|H^\varepsilon\|^2_{L^2(0,T;H^{-1}(\mathbb{R}))}\leq C\varepsilon^{2-\alpha}. \tag{3.7}$$

We split $\check{r}^\varepsilon=\check{r}^{\varepsilon,1}+\check{r}^{\varepsilon,2}$, where

- $\check{r}^{\varepsilon,1}$ solves (3.4) with H^ε on the right-hand side; it admits the estimate:

$$\mathbf{E}\|\check{r}^{\varepsilon,1}\|^2_{L^2(0,T;H^1(\mathbb{R}))}\leq C\varepsilon^\delta.$$

- $\check{r}^{\varepsilon,2}$ solves also (3.4), but with

$$\varepsilon^{-\alpha/2}\sum_{j=0}^{J_0}\sum_{k=0}^{J_0-j}\varepsilon^{(k+j)\delta}\Big[\langle \mathrm{a}\rangle^k\big(\xi_{\frac{t}{\varepsilon^\alpha}}\big)-a^{k,\mathrm{eff}}\Big]\frac{\partial^2 u^j}{\partial x^2}$$

  on the right-hand side.

According to Assumption **(a6)** and to [22, Theorem 3] (see also [11, Lemma VIII.3.102 and Theorem VIII.3.97]), the processes

$$A^k(t)=\int_0^t(\langle a\rangle^k(\xi_s)-a^{k,\mathrm{eff}})ds$$

satisfy the functional central limit theorem (invariance principle), that is the process

$$A^{\varepsilon,k}(t)=\varepsilon^{\frac{\alpha}{2}}\int_0^{\varepsilon^{-\alpha}t}(\langle a\rangle^k(\xi_s)-a^{k,\mathrm{eff}})ds$$

converges in law in $C([0,T];\mathbb{R})$ to a one-dimensional Brownian motion with covariance coefficient

$$(\Lambda_k)=\int_{\mathbb{R}^n}\Big[\frac{\partial}{\partial y_{r_1}}(Q^k)(y)\Big]q^{r_1r_2}(y)\Big[\frac{\partial}{\partial y_{r_2}}(Q^k)(y)\Big]\rho(y)\,dy.$$

with Q^0 defined in (2.15) and Q^k given by

$$\mathcal{L}Q^k(y)=\langle \mathrm{a}\rangle^k(y),\qquad k=1,\dots. \tag{3.8}$$

Denote by $\check{r}^{\varepsilon,0}$ the solution of the following problem

$$\partial_t\check{r}^{\varepsilon,0}-\mathcal{A}^\varepsilon\check{r}^{\varepsilon,0}=\varepsilon^{-\alpha/2}\Big[\langle \mathrm{a}\rangle^0\big(\xi_{\frac{t}{\varepsilon^\alpha}}\big)-\mathrm{a}^{\mathrm{eff}}\Big]\frac{\partial^2u^0}{\partial x^2}. \tag{3.9}$$

Obviously if $\check{r}^{\varepsilon,0}$ converges, then $\check{r}^{\varepsilon,2}$ also converges to the same limit. We consider an auxiliary problem

$$\begin{cases}\partial_t r^\varepsilon_{\mathrm{aux}}-\mathrm{div}\Big[\mathrm{a}^{\mathrm{eff}}\nabla r^\varepsilon_{\mathrm{aux}}\Big]=\varepsilon^{-\alpha/2}\Big[\langle \mathrm{a}\rangle^0\big(\xi_{\frac{t}{\varepsilon^\alpha}}\big)-\mathrm{a}^{\mathrm{eff}}\Big]\dfrac{\partial^2u^0}{\partial x^2}\\ r^\varepsilon_{\mathrm{aux}}(x,0)=0,\end{cases} \tag{3.10}$$

and notice that this problem admits an explicit solution

$$r^\varepsilon_{\mathrm{aux}}=\varepsilon^{\alpha/2}A^0\Big(\frac{t}{\varepsilon^\alpha}\Big)\frac{\partial^2u^0}{\partial x^2}=A^{\varepsilon,0}(t)\frac{\partial^2u^0}{\partial x^2}.$$

Since u^0 satisfies estimates (2.11), the convergence of $A^{\varepsilon,0}$ implies that $r^\varepsilon_{\mathrm{aux}}$ converges in law in $C((0,T);L^2(\mathbb{R}))$ to the solution of problem (2.10).

Next, we represent $\check{r}^{\varepsilon,0}$ as $\check{r}^{\varepsilon,0}(x,t)=\mathcal{Z}^\varepsilon(x,t)+r^\varepsilon_{\mathrm{aux}}(x,t)$. Then $\mathcal{Z}^\varepsilon$ solves the problem

$$\begin{cases}\partial_t\mathcal{Z}^\varepsilon-\mathcal{A}^\varepsilon\mathcal{Z}^\varepsilon=\mathrm{div}\Big(\Big[a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^\alpha}\Big)-\mathrm{a}^{\mathrm{eff}}\Big]\nabla r^\varepsilon_{\mathrm{aux}}(x,t)\Big)\\ \mathcal{Z}^\varepsilon(x,0)=0.\end{cases} \tag{3.11}$$

The conclusion of the proposition can be deduced from the next result. □

Lemma 3.3. *The quantity $\mathcal{Z}^\varepsilon$ goes to zero in probability in $L^2((0,T)\times\mathbb{R})$, as ε tends to 0.*

Proof. The arguments are the same as in the proof of [14, Lemma 5.1]. For completeness, they are reproduced in the appendix. □

To finish the proof of Theorem 2.1, we need to control r^ε, solution of problem (3.3).

3.2. **When $\alpha > 2$.** Contrary to the previous case, the asymptotic of u^ε is less easy to describe. The following notations will be used after for $k \geq 1$:

$$\phi^k\left(\frac{x}{\varepsilon},x,t\right) = \sum_{n=1}^{k}\chi^{n-1}\left(\frac{x}{\varepsilon}\right)\partial_x^n v^{k-n}(x,t), \tag{3.12}$$

$$\Phi^k\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right) = \sum_{n=1}^{k}\kappa^{n-1}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)\partial_x^n v^{k-n}(x,t), \tag{3.13}$$

and

$$\begin{aligned}\theta^k\left(\frac{x}{\varepsilon},x,t\right) &= \widehat{\theta}^k\left(\frac{x}{\varepsilon},x,t\right)+\chi^0\left(\frac{x}{\varepsilon}\right)u_x^k(x,t)\\ &= \left[\sum_{n=0}^{k-1}\tau^{k-1-n}\left(\frac{x}{\varepsilon}\right)u_x^n(x,t)\right]+\chi^0\left(\frac{x}{\varepsilon}\right)u_x^k(x,t),\end{aligned} \tag{3.14}$$

$$\begin{aligned}\Theta^k\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right) &= \widehat{\Theta}^k\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right)+\kappa^0\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)u_x^k(x,t)\\ &= \sum_{n=0}^{k-1}\gamma^{k-n-1}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)u_x^n(x,t)+\kappa^0\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)u_x^k(x,t).\end{aligned} \tag{3.15}$$

Remember that $v^0 = u^0$. Finally

$$\begin{aligned}\psi^k\left(\frac{x}{\varepsilon},x,t\right) &= \widehat{\psi}^k\left(\frac{x}{\varepsilon},x,t\right)+\chi^1\left(\frac{x}{\varepsilon}\right)u_{xx}^k(x,t)\\ &= \sum_{n=0}^{k-1}\eta^{k-n}\left(\frac{x}{\varepsilon}\right)u_{xx}^n(x,t)+\tau^{k-1}\left(\frac{x}{\varepsilon}\right)v_x^1(x,t)+\chi^1\left(\frac{x}{\varepsilon}\right)u_{xx}^k(x,t),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\Psi^k\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right) &= \widehat{\Psi}^k\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right)+\kappa^1\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)u_{xx}^k(x,t)\\ &= \sum_{n=0}^{k-1}\zeta^{k-n}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)u_{xx}^n+\gamma^k\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)v_x^1(x,t)\\ &\qquad+\kappa^1\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)u_{xx}^k(x,t).\end{aligned} \tag{3.17}$$

The functions ϕ^k, θ^k, ψ^k are bounded and smooth functions, whereas the random functions Φ^k, Θ^k and Ψ^k are bounded and smooth w.r.t. x with polynomial growth w.r.t. $\xi_{t/\varepsilon^\alpha}$.

As in [14], Eq. (21), we consider a first principal part of the asymptotic of u^ε of the form:

$$\begin{aligned}(3.18)\quad \mathcal{E}_1^\varepsilon(x,t) &= u^0(x,t) + \sum_{k=1}^{J_1+1} \varepsilon^k v^k(x,t) \\ &+ \sum_{k=1}^{J_1+2} \varepsilon^k \phi^k\left(\frac{x}{\varepsilon},x,t\right) + \sum_{k=1}^{J_1+2} \varepsilon^{k+\delta} \Phi^k\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right) \\ &+ \varepsilon^\delta u^1(x,t) + \varepsilon^{\delta+1}\theta^1\left(\frac{x}{\varepsilon},x,t\right) \\ &+ \varepsilon^{2\delta+1}\Theta^1\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right) + \varepsilon^{\delta+2}\psi^1\left(\frac{x}{\varepsilon},x,t\right) + \varepsilon^{2\delta+2}\Psi^1\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right).\end{aligned}$$

with ϕ^k and Φ^k defined by (3.12) and (3.13). The functions v^k are given as the solution of Eq. (2.8).

Lemma 3.4. *The decomposition of the quantity $\mathcal{S}_1^\varepsilon = (\partial_t - \mathcal{A}^\varepsilon)(\mathcal{E}_1^\varepsilon)$ is given by:*

$$\begin{aligned}(3.19)\quad \mathcal{S}_1^\varepsilon &= \left[\sum_{k=1}^{J_1+2} \varepsilon^{k+\delta-\alpha/2}\Phi_y^k + \varepsilon^{2\delta+1-\alpha/2}\Theta_y^1 + \varepsilon^{2\delta+2-\alpha/2}\Psi_y^1\right]\sigma(\xi_{t/\varepsilon^\alpha})dB_t \\ &+ \left[\varepsilon^{J_1+1}r^{a,1,\varepsilon} + \varepsilon^{\delta+1}r^{a,2,\varepsilon}\right]dt + \varepsilon^\delta w^1 dt \\ &- \left[\varepsilon^{2\delta-1}\mathcal{A}^\varepsilon\Theta^1 + \varepsilon^{2\delta}(a^\varepsilon\Theta_{xz}^1 + (a^\varepsilon\Theta_x^1)_z) + \varepsilon^{2\delta}\mathcal{A}^\varepsilon\Psi^1\right]dt\end{aligned}$$

where the two remainders $r^{a,1,\varepsilon} = r^{a,1,\varepsilon}(x,t)$ and $r^{a,2,\varepsilon} = r^{a,2,\varepsilon}(x,t)$ only contain non-negative powers of ε, and are bounded and smooth functions.

Proof. To prove this claim, we simply apply the Itô formula. Using the definitions of the objects introduced in Section 2.3, and after some computations (see Appendix), we obtain this equality (also see the beginning of the proof of Proposition 3.1). Boundedness and smoothness are consequences of the properties given in Section 2.3. □

Since we need to control the last term in (3.19), we consider a second expansion:

$$\begin{aligned}(3.20)\quad \mathcal{E}_2^\varepsilon(x,t) &= \sum_{k=2}^{N_0} \varepsilon^{k\delta} u^k(x,t) + \sum_{k=2}^{N_0} \varepsilon^{k\delta+1}\theta^k\left(\frac{x}{\varepsilon},x,t\right) \\ &+ \sum_{k=2}^{N_0} \varepsilon^{(k+1)\delta+1}\Theta^k\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right) \\ &+ \sum_{k=2}^{N_0} \varepsilon^{k\delta+2}\psi^k\left(\frac{x}{\varepsilon},x,t\right) + \sum_{k=2}^{N_0} \varepsilon^{(k+1)\delta+2}\Psi^k\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right)\end{aligned}$$

where $N_0 = 2J_0 + 2$. Adding the last term in (3.19) we define

$$\mathcal{S}_2^\varepsilon = (\partial_t - \mathcal{A}^\varepsilon)\mathcal{E}_2^\varepsilon - \left[\varepsilon^{2\delta-1}\mathcal{A}^\varepsilon\Theta^1 + \varepsilon^{2\delta}(a^\varepsilon\Theta^1_{xz} + (a^\varepsilon\Theta^1_x)_z) + \varepsilon^{2\delta}\mathcal{A}^\varepsilon\Psi^1\right]dt$$

In $\mathcal{S}_2^\varepsilon$ there is a martingale term of the from

$$\left[\sum_{k=2}^{N_0}\varepsilon^{(k+1)\delta+1-\alpha/2}\Theta^k_y + \sum_{k=2}^{N_0}\varepsilon^{(k+1)\delta+2-\alpha/2}\Psi^k_y\right]\sigma(\xi_{t/\varepsilon^\alpha})dB_t$$
$$= \frac{\varepsilon^{\alpha/2}}{\varepsilon}\sum_{k=2}^{N_0}\varepsilon^{k\delta}\Theta^k_y\sigma(\xi_{t/\varepsilon^\alpha})dB_t + \varepsilon^{\alpha/2+\delta}\sum_{k=2}^{N_0}\varepsilon^{(k-1)\delta}\Psi^k_y\sigma(\xi_{t/\varepsilon^\alpha})dB_t.$$

Lemma 3.5. *For a given sequence of smooth functions w^k, there exists a unique sequence of correctors u^k, θ^k, Θ^k, ψ^k and Ψ^k given respectively by* (2.5), (3.14), (3.15), (3.16), (3.17) *and a unique sequence of constants* $\mathrm{a}^{k,\mathrm{eff}}$ *such that*

- *The sequence* $\mathrm{a}^{k,\mathrm{eff}}$*, given by* (2.27) *and* (2.31)*, does not depend on the choice of w^k.*
- *The absolutely continuous part of $\mathcal{S}_2^\varepsilon$ is given by*

$$\sum_{k=2}^{N_0}\varepsilon^{k\delta}w^k(x,t) + \varepsilon^{\delta+1}r^{a,3,\varepsilon} + \varepsilon^{(2J_0+3)\delta-1}r^{a,4,\varepsilon}$$

where the two remainders $r^{a,3,\varepsilon} = r^{a,3,\varepsilon}(x,t)$ and $r^{a,4,\varepsilon} = r^{a,4,\varepsilon}(x,t)$ only contain non-negative powers of ε and are smooth and bounded in (x,t).

Proof. Again the proof is a long and awkward application of the Itô formula. The definition of all correctors leads to the cancellation of a lot of terms (see Appendix). □

Using the definition of J_0, J_1 and δ, the absolutely continuous term

$$\varepsilon^{-\alpha/2}\left[\varepsilon^{J_1+1}r^{a,1,\varepsilon} + \varepsilon^{\delta+1}r^{a,2,\varepsilon} + \varepsilon^{\delta+1}r^{a,3,\varepsilon} + \varepsilon^{(2J_0+3)\delta-1}r^{a,4,\varepsilon}\right]$$

will tend to zero as ε tends to zero in probability[2] in $C(0,T;L^p(\mathbb{R}^n))$ for any $p \geq 1$.

Assume again that we represent u^ε as follows:

$$u^\varepsilon = \mathcal{E}_1^\varepsilon + \mathcal{E}_2^\varepsilon + \varepsilon^{\alpha/2}R^\varepsilon$$

where $\mathcal{E}_1^\varepsilon$ is given by (3.18) and $\mathcal{E}_2^\varepsilon$ by (3.20). Recall that from the definition of δ, J_0 and J_1, we have

$$\nu = \min(J_1 + 1, J_0\delta, \delta + 1) > \alpha/2.$$

[2]Even in $\mathbb{L}^q(\Omega)$ for $q \geq 1$.

From the definition of ϕ^k given by (3.12), we obtain the development:

$$\begin{aligned} u^\varepsilon(x,t) &= u^0(x,t) + \sum_{k=1}^{J_1} \varepsilon^k \left[v^k(x,t) + \sum_{\ell=1}^{k} \chi^{\ell-1}\left(\frac{x}{\varepsilon}\right) \partial_x^\ell v^{k-\ell}(x,t) \right] \\ &+ \sum_{k=1}^{J_0} \varepsilon^{k\delta} u^k(x,t) + \varepsilon^{\alpha/2} R^\varepsilon(x,t) + \varepsilon^{\alpha/2} \widetilde{R}^\varepsilon(x,t), \end{aligned} \tag{3.21}$$

From Section 2.3, the residual $\widetilde{R}^\varepsilon(x,t)$ converges to zero when ε goes to zero (at least) in probability in $C(0,T;L^p(\mathbb{R}^n))$ for any $p \geq 1$.

Let us state our result on R^ε.

Proposition 3.6. *The discrepancy R^ε can be split in four parts:*

$$R^\varepsilon = r^\varepsilon + \hat{r}^\varepsilon + \widetilde{r}^\varepsilon + \rho^\varepsilon \tag{3.22}$$

such that

- *the dynamics of r^ε contains the terms with large parameters:*

$$\begin{aligned} dr^\varepsilon = (\mathcal{A}^\varepsilon r^\varepsilon)dt + \sum_{k=1}^{N_0} \varepsilon^{k\delta-\alpha/2} w^k(x,t)dt - \frac{1}{\varepsilon}\Bigg[\kappa_y^1\left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right) u_x^0(x,t) \\ + \sum_{k=1}^{N_0} \varepsilon^{k\delta} \Theta_y^k\left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, x, t\right)\Bigg] \sigma(\xi_{t/\varepsilon^\alpha}) dB_t, \end{aligned} \tag{3.23}$$

$$r^\varepsilon(x,0) = 0,$$

- *the dynamics of $\hat{r}^\varepsilon$ contains all other terms:*

$$\begin{aligned} d\hat{r}^\varepsilon &= (\mathcal{A}^\varepsilon \hat{r}^\varepsilon)dt - \left(\kappa_y^1 u_x^1 + \kappa_y^2 u_{xx}^0\right)\sigma(\xi_{t/\varepsilon^\alpha})dB_t \\ \hat{r}^\varepsilon(x,0) &= 0, \end{aligned} \tag{3.24}$$

- *ρ^ε satisfies*

$$d\rho^\varepsilon = (\mathcal{A}^\varepsilon \rho^\varepsilon)dt, \tag{3.25}$$

and has the initial condition $\rho^\varepsilon(x,0) = R_0^\varepsilon(x)$ with

$$\begin{aligned} \rho^\varepsilon(x,0) &= R_0^\varepsilon(x) \\ &= -\sum_{k=1}^{J_1} \varepsilon^{k-\alpha/2} \left[\mathcal{I}_k + \sum_{\ell=1}^{k} \mathcal{I}_{k-\ell} \chi^{\ell-1}\left(\frac{x}{\varepsilon}\right)\right] \partial_x^k u^0(x,0). \end{aligned} \tag{3.26}$$

- *$\widetilde{r}^\varepsilon$ contains all negligible terms and satisfies*

$$\mathbf{E}\|\widetilde{r}^\varepsilon\|^2_{L^2(\mathbb{R}\times(0,T))} \leq C\varepsilon^\nu.$$

Moreover if r^ε has a limit, then $\hat{r}^\varepsilon$ defined by (3.24) converges to zero.

Proof. Indeed, gathering Lemmas 3.4 and 3.5, the remainder R^ε satisfies:

$$\begin{aligned} dR^\varepsilon &= (\mathcal{A}^\varepsilon R^\varepsilon)dt - \mathcal{M}^\varepsilon \sigma(\xi_{t/\varepsilon^\alpha})dB_t - \sum_{k=1}^{N_0} \varepsilon^{k\delta-\alpha/2} w^k(x,t)dt \\ &- (m^\varepsilon q(\xi_{t/\varepsilon^\alpha})dW_t + r^{a,\varepsilon}dt) \end{aligned} \tag{3.27}$$

with

- a martingale term $\mathcal{M}^\varepsilon$ with "large parameters":

$$\mathcal{M}^\varepsilon = \left[\frac{1}{\varepsilon}\Phi_y^1 + \frac{1}{\varepsilon}\sum_{k=1}^{N_0}\varepsilon^{k\delta}\Theta_y^k + \Phi_y^2\right]$$

- a martingale term m^ε of order smaller than $\varepsilon^{\alpha/2}$:

$$m^\varepsilon = \left[\varepsilon\sum_{k=3}^{J_1+2}\varepsilon^{k-3}\Phi_y^k + \varepsilon^\delta\sum_{k=1}^{N_0}\varepsilon^{(k-1)\delta}\Psi_y^k\right],$$

- and a negligible term $r^{a,\varepsilon}$ of order $\mathcal{O}(\varepsilon^\nu)$ (that is, convergence to zero in strong topology).

With $\mathcal{I}_0 = 1$, $\mathcal{I}_1 = 0$, we have for any $k \geq 1$, $v^k(x,0) = \mathcal{I}_k\partial_x^k u^0(x,0)$ and thus:

$$\begin{aligned}
&\sum_{k=1}^{J_1}\varepsilon^k\left[v^k(x,0) + \sum_{\ell=1}^{k}\chi^{\ell-1}\left(\frac{x}{\varepsilon}\right)\partial_x^\ell v^{k-\ell}(x,0)\right] \\
&\quad = \sum_{k=1}^{J_1}\varepsilon^k\left[\mathcal{I}_k\partial_x^k u^0(x,0) + \sum_{\ell=1}^{k}\mathcal{I}_{k-\ell}\chi^{\ell-1}\left(\frac{x}{\varepsilon}\right)\partial_x^k u^0(x,0)\right] \\
&\quad = \sum_{k=1}^{J_1}\varepsilon^k\left[\mathcal{I}_k + \sum_{\ell=1}^{k}\mathcal{I}_{k-\ell}\chi^{\ell-1}\left(\frac{x}{\varepsilon}\right)\right]\partial_x^k u^0(x,0) = -\varepsilon^{\alpha/2}R_0^\varepsilon(x).
\end{aligned}$$

Since $u^0(x,0) = u^\varepsilon(x,0) = \imath(x)$, we deduce that R^ε satisfies the initial condition:

$$\begin{aligned}
&R^\varepsilon(x,0) = R_0^\varepsilon(x) + r_0^\varepsilon(x) \\
&= -\sum_{k=1}^{J_1}\varepsilon^{k-\alpha/2}\left[\mathcal{I}_k + \sum_{\ell=1}^{k}\mathcal{I}_{k-\ell}\chi^{\ell-1}\left(\frac{x}{\varepsilon}\right)\right]\partial_x^k u^0(x,0) + r_0^\varepsilon(x).
\end{aligned}$$

where $r_0^\varepsilon = \mathcal{O}(\varepsilon^\nu)$. By the linearity of this equation (3.27) we obtain the desired decomposition. In particular $\widetilde{r}^\varepsilon$ satisfies

$$d\widetilde{r}^\varepsilon \quad = \quad (\mathcal{A}^\varepsilon\widetilde{r}^\varepsilon)dt - (m^\varepsilon q(\xi_{t/\varepsilon^\alpha})dW_t + r^{a,\varepsilon}dt), \quad \widetilde{r}^\varepsilon(x,0) = r_0^\varepsilon(x).$$

Very classical arguments and standard parabolic estimates prove that $\widetilde{r}^\varepsilon$ goes to zero when ε goes to zero: $\mathbf{E}\|\widetilde{r}^\varepsilon\|^2_{L^2(\mathbb{R}\times(0,T))} \leq C\varepsilon^\nu$.

For the last assertion, we can apply the result concerning r^ε to $(1/\varepsilon)\hat{r}^\varepsilon$. This last quantity will converge in the same sense as r^ε. □

Note that the term R_0^ε contains negative powers of ε. Hence this term could have a priori a non trivial contribution in the behaviour of ρ^ε and thus of R^ε. Nevertheless in Section 5 we show that we can choose the constants $\mathcal{I}_k$ such that the remainder ρ^ε converges strongly in $L^2(\mathbb{R}\times(0,T))$ to zero.

4. Limit behaviour of the remainder

Let us evoke that in any case $r^\varepsilon(x,0)=0$ and that the dynamics of r^ε is given by (3.3) for $\alpha<2$ and by (3.23) for $\alpha>2$. We can summarize these equations as follows:

$$\begin{aligned} dr^\varepsilon &= (\mathcal{A}^\varepsilon r^\varepsilon)dt - \sum_{k=1}^{K_0} \varepsilon^{k\delta-\alpha/2} w^k(x,t)\,dt \\ &- \varepsilon^{\varpi-1}\sum_{k=0}^{K_0} \varepsilon^{k\delta}\Upsilon^k\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)u_x^k(x,t)\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\,dB_t \end{aligned} \tag{4.1}$$

where

- K_0 is a integer such that: $(K_0+1)\delta \geq \max(2,\alpha/2)$,
- Υ^k are defined on $\mathbb{T}\times\mathbb{R}^n$ and smooth functions satisfying (2.12) and such that $\langle \Upsilon^k\rangle = 0$,
- $\varpi = \max(2-\alpha,0)\geq 0$.

If $\varpi>1$, i.e. $\alpha<1$, we obtain a stronger convergence result (see Part 4.3).

Let $\widetilde{\Upsilon}^k$ be a function such that $\partial_z\widetilde{\Upsilon}^k = \Upsilon^k$ with zero mean value w.r.t. z. And $\widetilde{w}^k_x = w^k$. Define v^ε as the solution of

$$\begin{aligned} dv^\varepsilon &= a\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)v^\varepsilon_{xx}dt - \sum_{k=1}^{K_0}\varepsilon^{k\delta-\alpha/2}\widetilde{w}^k(x,t)dt \\ &- \varepsilon^{\varpi}\sum_{k=0}^{K_0}\varepsilon^{k\delta}\widetilde{\Upsilon}^k\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)u^k_x(x,t)\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_t. \end{aligned} \tag{4.2}$$

In the rest of this section, $\widetilde{\mathcal{G}}^k$, for $k=0,\dots,K_0$, are defined by

$$\widetilde{\mathcal{G}}^k(z,y,x,t) = \widetilde{\Upsilon}^k(z,y)\,u^k_x(x,t). \tag{4.3}$$

Then $v^\varepsilon_x = r^\varepsilon + \check{v}^\varepsilon$ where

$$d\check{v}^\varepsilon = a\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)\check{v}^\varepsilon_{xx}dt - \varepsilon^{\varpi}\sum_{k=0}^{K_0}\varepsilon^{k\delta}\widetilde{\Upsilon}^k\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)u^k_{xx}(x,t)\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_t.$$

Since $r^\varepsilon(x,0)=0$, we assume that $v^\varepsilon(x,0)=\check{v}^\varepsilon(x,0)=0$. Note that the behaviour of r^ε depends only on v^ε.

Lemma 4.1. *$\check{v}^\varepsilon$ tends to 0 in $\mathbb{L}^2(\mathbb{R}\times(0,T))$ and in probability.*

Proof. If r^ε converges in law in $\mathbb{L}^2(\mathbb{R}\times(0,T))$, then Slutsky's theorem gives the convergence for $\check{v}^\varepsilon$ to zero in probability. □

4.1. Construction of correctors.

The correctors P^k and Q^k are given by the equations

$$(\bar{a}(z)P^0(z))_{zz}=0,\quad \langle P^0\rangle = 1,\qquad \mathcal{L}Q^0 = ((\bar{a}-a)P^0)_{zz} \tag{4.4}$$

and for $k \geq 1$

$$(\bar{a}(z)P^k(z))_{zz} = -\overline{(Q^{k-1}a)_{zz}}, \quad \langle P^k \rangle = 1, \tag{4.5}$$

$$\mathcal{L}Q^k(z,y) = ((\bar{a}-a)P^k)_{zz} + \overline{(Q^{k-1}a)_{zz}} - (Q^{k-1}a)_{zz}. \tag{4.6}$$

Lemma 4.2. *The functions P^k are smooth periodic functions defined on $\mathbb{T}$ and Q^k are smooth functions on $\mathbb{R}\times\mathbb{R}^n$, bounded in z and with linear growth w.r.t. y.*

Proof. Indeed let us begin with $k = 0$ (Equation (4.4)). P_0 satisfies:

$$(\bar{a}(z)P^0)_{zz} = 0, \quad \langle P^0 \rangle = 1.$$

Hence $P^0 = 1 + \chi^1_z$ and classical computations for the dimension one show that

$$P^0(z) = \mathrm{a}^{\mathrm{eff}} \frac{1}{\bar{a}(z)}.$$

Next for Q^0 we have:

$$\mathcal{L}Q^0(z,y) = ((\bar{a}-a)P^0)_{zz}.$$

Again here z is a parameter of the equation. The right-hand side has zero mean value w.r.t. y and is a smooth bounded function of the two variables y and z. Hence we already have shown that there exists a unique solution Q^0 which is smooth w.r.t. y and z, is bounded w.r.t. z and of at most linear growth w.r.t. y. Then from (4.5) and (4.6) and by recursion we obtain the desired result. □

Let us introduce the following notations: for $k \geq 0$ and $m \geq 0$

$$\Xi^{k,m}(x,t) = \overline{\langle Q^k_y(.,.)\, \widetilde{\mathcal{G}}^m(.,.,x,t)\rangle \sigma(.)}. \tag{4.7}$$

The correctors $U^{k,\ell}$ are solution of the problem:

$$\mathcal{L}U^{k,\ell}(x,t,y) + 2\langle Q^k_y(.,y)\, \widetilde{\mathcal{G}}^\ell(.,y,x,t)\rangle \sigma(y) - \Xi^{k,\ell}(x,t) = 0. \tag{4.8}$$

Moreover

$$\widetilde{P}^k_z = P^k - \langle P^k \rangle = P^k - 1 \quad \text{and} \quad \widetilde{Q}^k_z = Q^k.$$

The correctors $\widetilde{\Upsilon}^k$ verify the inequality (2.22)

$$|\widetilde{\Upsilon}^k(z,y)| \leq C(1+|y|^p), \quad \forall (z,y) \in \mathbb{T}\times\mathbb{R}^n.$$

Thereby from the previous lemma, we deduce immediately the next result.

Lemma 4.3. *The functions U^k given by* (4.8) *are well defined and smooth and also satisfy* (2.22).

Finally we define the constant $\mathcal{C}_{k,m}$, $0 \leq k \leq m$ by

$$\mathcal{C}_{k,m} = \overline{\langle Q^\ell_y(.,.)\, \widetilde{\Upsilon}^k(.,.)\rangle \sigma(.)}. \tag{4.9}$$

4.2. **Convergence of the antiderivative v^ε.** We first prove boundedness in $H^1(\mathbb{R})$, then a tightness result and finally we identify the weak limit.

4.2.1. *Bound in $H^1(\mathbb{R})$ for v^ε.* Let us consider the quantity

$$\begin{aligned}\mathcal{V}_t^\varepsilon &= \sum_{k=0}^{K_0}\varepsilon^{k\delta}\left[\left\langle\!\!\left\langle P^k\left(\frac{\cdot}{\varepsilon}\right)v^\varepsilon(.,t),v^\varepsilon(.,t)\right\rangle\!\!\right\rangle+\varepsilon^\delta\left\langle\!\!\left\langle Q^k\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)v^\varepsilon(.,t),v^\varepsilon(.,t)\right\rangle\!\!\right\rangle\right]\\ &+ \varepsilon^\varpi\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta+\alpha/2}\left\langle\!\!\left\langle U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}),v^\varepsilon(.,t)\right\rangle\!\!\right\rangle\end{aligned}$$

where P^k, Q^k and $U^{k,\ell}$ are the correctors defined respectively by (4.4), (4.5), (4.6) and (4.8). The bracket $\langle\!\langle .,. \rangle\!\rangle$ stands for the scalar product in $L^2(\mathbb{R})$.

Again by Itô's formula and the very definition of all correctors, we deduce:

Lemma 4.4. *Then quantity $\mathcal{V}^\varepsilon$ satisfies:*

$$\begin{aligned}(4.10)\quad d\mathcal{V}_t^\varepsilon &= \mathcal{B}_t^\varepsilon dt + M_t^\varepsilon\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_t\\ &+\varepsilon^{(K_0+1)\delta-2}\left\langle\!\!\left\langle(Q^{K_0}\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)a^\varepsilon)_{zz}v^\varepsilon(.,t),v^\varepsilon(.,t)\right\rangle\!\!\right\rangle dt\\ &-2\sum_{k=0}^{K_0}\varepsilon^{k\delta}\left\langle\!\!\left\langle\left(P^k\left(\frac{\cdot}{\varepsilon}\right)+\varepsilon^\delta Q^k\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)\right)v_x^\varepsilon(.,t),a^\varepsilon v_x^\varepsilon(.,t)\right\rangle\!\!\right\rangle dt\\ &+2\sum_{k=0}^{K_0}\sum_{m=0}^{K_0-1}\varepsilon^{(k+m+1)\delta-\alpha/2}\langle\!\langle\widetilde{w}^{m+1}(.,t)+\varepsilon^\varpi\Xi^{k,m}(.,t),v^\varepsilon(.,t)\rangle\!\rangle dt\\ &+\langle\!\langle\mathcal{NT}^{1,\varepsilon}(.,t),v^\varepsilon(.,t)\rangle\!\rangle dt-\langle\!\langle\mathcal{NT}^{2,\varepsilon}(.,t),v_x^\varepsilon(.,t)\rangle\!\rangle dt,\end{aligned}$$

where:

- *M^ε stands for the integrand in the stochastic integral w.r.t. the Brownian motion B:*

$$(4.11)\quad M_t^\varepsilon=\sum_{k=0}^{K_0}\varepsilon^{(k+1)\delta}\varepsilon^{-\alpha/2}\left\langle\!\!\left\langle Q_y^k\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)v^\varepsilon(.,t),v^\varepsilon(.,t)\right\rangle\!\!\right\rangle+\varepsilon^\varpi\widetilde{M}_t^\varepsilon$$

  where in $\widetilde{M}^\varepsilon$, all powers of ε are non-negative.
- *term $\mathcal{B}^\varepsilon$ does not depend on v^ε and is bounded uniformly w.r.t. ε: for any $p\geq 1$, there exists a constant $\hat{p}\geq 1$ such that*

$$\mathbf{E}|\mathcal{B}_t^\varepsilon|^p\leq C\mathbf{E}(1+|\xi_{t/\varepsilon^\alpha}|^{\hat{p}});$$

- *there exists $\nu>0$ such that for any $N>0$ the two terms $\mathcal{NT}^{1,\varepsilon}$ and $\mathcal{NT}^{2,\varepsilon}$ (and their derivatives w.r.t. x) satisfy*

$$(4.12)\qquad \mathbf{E}|\mathcal{NT}^{\cdot,\varepsilon}(x,t)|^p\leq\frac{C\varepsilon^\nu}{(1+|x|)^N}\mathbf{E}(1+|\xi_{t/\varepsilon^\alpha}|^{\hat{p}}).$$

The constant C here does not depend on ε.

Proof. The proof is based on the Itô formula and the definitions of the correctors and is postponed in the appendix. □

The last double sum in (4.10) can be written as:

$$\begin{aligned}
&2\sum_{k=0}^{K_0}\sum_{m=0}^{K_0-1}\varepsilon^{(k+m+1)\delta-\alpha/2}\langle\!\langle \varepsilon^{\varpi}\Xi^{k,m}(.,t)+\widetilde{w}^{m+1}(.,t),v^{\varepsilon}(.,t)\rangle\!\rangle\\
&\quad=2\sum_{\ell=0}^{K_0-1}\varepsilon^{(\ell+1)\delta-\alpha/2}\sum_{m=0}^{\ell}\langle\!\langle \varepsilon^{\varpi}\Xi^{\ell-m,m}(.,t)+\widetilde{w}^{m+1}(.,t),v^{\varepsilon}(.,t)\rangle\!\rangle\\
&\quad+\langle\!\langle \mathcal{NT}^{3,\varepsilon}(.,t),v^{\varepsilon}(.,t)\rangle\!\rangle,
\end{aligned}$$

with

$$\mathcal{NT}^{3,\varepsilon}(.,t)=2\sum_{k+m\geq K_0}\varepsilon^{(k+m+1)\delta-\alpha/2}\left(\varepsilon^{\varpi}\Xi^{k,m}(.,t)+\widetilde{w}^{m+1}(.,t)\right). \tag{4.13}$$

Again since $(K_0+1)\delta>\alpha/2$, all powers of ε in (4.13) are positive and thus $\mathcal{NT}^{3,\varepsilon}$ also verifies (4.12).

If for $\ell\geq 0$

$$Z^{\ell}(x,t)=\sum_{m=0}^{\ell}\Xi^{\ell-m,m}(x,t), \tag{4.14}$$

then

$$\begin{aligned}
&2\sum_{k=0}^{K_0}\sum_{m=0}^{K_0-1}\varepsilon^{(k+m+1)\delta-\alpha/2}\langle\!\langle \varepsilon^{\varpi}\Xi^{k,m}(.,t)+\widetilde{w}^{m+1}(.,t),v^{\varepsilon}(.,t)\rangle\!\rangle\\
&\quad=2\sum_{\ell=0}^{K_0-1}\varepsilon^{(\ell+1)\delta-\alpha/2}\langle\!\langle \varepsilon^{\varpi}Z^{\ell}(.,t)+\sum_{m=0}^{\ell}\widetilde{w}^{m+1}(.,t),v^{\varepsilon}(.,t)\rangle\!\rangle\\
&\quad+\langle\!\langle \mathcal{NT}^{3,\varepsilon}(.,t),v^{\varepsilon}(.,t)\rangle\!\rangle.
\end{aligned}$$

Here we distinguish two cases.

Case $\alpha<2$.: Then $\varpi=\delta$ and choosing $\widetilde{w}^1=0$ yields to

$$\begin{aligned}
&\sum_{\ell=0}^{K_0-1}\varepsilon^{(\ell+1)\delta-\alpha/2}\langle\!\langle \varepsilon^{\varpi}Z^{\ell}(.,t)+\sum_{m=0}^{\ell}\widetilde{w}^{m+1}(.,t),v^{\varepsilon}(.,t)\rangle\!\rangle\\
&\quad=\sum_{\ell=0}^{K_0-1}\varepsilon^{(\ell+2)\delta-\alpha/2}\langle\!\langle Z^{\ell}(.,t)+\sum_{m=0}^{\ell}\widetilde{w}^{m+2}(.,t),v^{\varepsilon}(.,t)\rangle\!\rangle.
\end{aligned}$$

Let us remark that from the definition of the sequences w^k and $\mathcal{C}_{k,m}$ by (2.7) and (4.9), we have:

$$\begin{aligned}
Z^0(x,t) &= \overline{\langle Q_y^0(.,.)\,\widetilde{\Upsilon}^0(.,.)\rangle_{\sigma(.)}}\;u_x^0(x,t)\\
&= -\mathcal{C}_{0,0}u_x^0(x,t)=-\widetilde{w}^2(x,t).
\end{aligned}$$

And for $\ell = 2, \dots, K_0 - 1$

$$\begin{aligned}
\widetilde{w}^{\ell+2} &= -\sum_{m=0}^{\ell} \mathcal{C}_{\ell,m} u_x^m(x,t) - \sum_{m=2}^{\ell+1} \widetilde{w}^m \\
&= -\sum_{m=0}^{\ell} \overline{\langle Q_y^\ell(.,.)\, \widetilde{\Upsilon}^m(.,.)\rangle \sigma(.)} u_x^m(x,t) - \sum_{m=0}^{\ell-1} \widetilde{w}^{m+2} \\
&= -\sum_{m=0}^{\ell} \overline{\langle Q_y^\ell(.,.)\, \widetilde{\mathcal{G}}^m(.,.,x,t)\rangle \sigma(.)} - \sum_{m=0}^{\ell-1} \widetilde{w}^{m+2} = -Z^\ell - \sum_{m=0}^{\ell-1} \widetilde{w}^{m+2}.
\end{aligned}$$

Thereby we obtain immediately that for any $\ell = 0, \dots, K_0 - 1$

$$Z^\ell(.,t) + \sum_{k=0}^{\ell} \widetilde{w}^{k+2}(.,t) = 0.$$

Case $\alpha > 2$.: Then $\varpi = 0$ and the same arguments lead to

$$Z^\ell(.,t) + \sum_{k=0}^{\ell} \widetilde{w}^{k+1}(.,t) = 0.$$

In both cases, the equation (4.10) can be written as:

$$\begin{aligned}
(4.15)\ d\mathcal{V}_t^\varepsilon &= \mathcal{B}_t^\varepsilon dt + M_t^\varepsilon \sigma(\xi_{\frac{t}{\varepsilon^\alpha}}) dB_t \\
&+\varepsilon^{(K_0+1)\delta-2} \Big\langle\!\!\Big\langle (Q^{K_0}\left(\tfrac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right) a^\varepsilon)_{zz} v^\varepsilon(.,t), v^\varepsilon(.,t) \Big\rangle\!\!\Big\rangle dt \\
&-2\sum_{k=0}^{K_0} \varepsilon^{k\delta} \Big\langle\!\!\Big\langle \left(P^k\left(\tfrac{\cdot}{\varepsilon}\right) + \varepsilon^\delta Q^k\left(\tfrac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right)\right) v_x^\varepsilon(.,t), a^\varepsilon v_x^\varepsilon(.,t) \Big\rangle\!\!\Big\rangle dt \\
&+\langle\!\langle \mathcal{NT}^{1,\varepsilon}(.,t) + \mathcal{NT}^{3,\varepsilon}(.,t), v^\varepsilon(.,t) \rangle\!\rangle dt - \langle\!\langle \mathcal{NT}^{2,\varepsilon}(.,t), v_x^\varepsilon(.,t) \rangle\!\rangle dt.
\end{aligned}$$

Proposition 4.5. *The quantity v^ε is bounded in $L^2((0,T)\times\Omega; H^1(\mathbb{R}))$, uniformly w.r.t. ε: there exists a constant C_{H^1} independent of ε such that*

$$\mathbf{E}\int_0^T \|v^\varepsilon(.,t)\|^2_{H^1(\mathbb{R})} dt \le C_{H^1}. \tag{4.16}$$

Moreover v^ε is also bounded in $L^\infty(0,T;L^2(\mathbb{R}))$ in mean w.r.t. ω: there exists a constant C_{L^∞} again independent of ε such that

$$\mathbf{E}\left[\sup_{t\in[0,T]} \|v^\varepsilon(.,t)\|^2_{L^2(\mathbb{R})}\right] \le C_{L^\infty}. \tag{4.17}$$

Proof. Using (4.15), we have for any $s \in [0,T]$

$$\begin{aligned}
&\mathcal{V}_s^\varepsilon + 2\sum_{k=0}^{K_0}\int_0^s \varepsilon^{k\delta}\left[\langle\!\langle P^k a^\varepsilon v_x^\varepsilon, v_x^\varepsilon\rangle\!\rangle + \varepsilon^\delta \langle\!\langle Q^k v_x^\varepsilon, a^\varepsilon v_x^\varepsilon\rangle\!\rangle\right] dt \\
&\quad = \int_0^s \mathcal{B}_t^\varepsilon dt + \int_0^s M_t^\varepsilon \sigma(\xi_{\frac{t}{\varepsilon^\alpha}}) dB_t + \varepsilon^{(K_0+1)\delta-2}\int_0^s \langle\!\langle (Q^{K_0}a^\varepsilon)_{zz} v^\varepsilon, v^\varepsilon\rangle\!\rangle dt \\
&\quad + \int_0^s \langle\!\langle \mathcal{NT}^{1,\varepsilon}(.,t) + \mathcal{NT}^{3,\varepsilon}(.,t), v^\varepsilon(.,t)\rangle\!\rangle dt - \int_0^s \langle\!\langle \mathcal{NT}^{2,\varepsilon}(.,t), v_x^\varepsilon(.,t)\rangle\!\rangle dt.
\end{aligned}$$

Remember that $\mathcal{V}_0^\varepsilon = 0$ since $v^\varepsilon(.,0) = 0$. Hence

$$\begin{aligned}
&\langle\!\langle P^0 v^\varepsilon, v^\varepsilon\rangle\!\rangle + 2\int_0^s \langle\!\langle P^0 a^\varepsilon v_x^\varepsilon, v_x^\varepsilon\rangle\!\rangle dt \\
&\quad + \sum_{k=1}^{K_0} \varepsilon^{k\delta}\left[\langle\!\langle P^k v^\varepsilon, v^\varepsilon\rangle\!\rangle + 2\int_0^s \langle\!\langle P^k a^\varepsilon v_x^\varepsilon, v_x^\varepsilon\rangle\!\rangle dt\right] \\
&\quad + \sum_{k=0}^{K_0} \varepsilon^{(k+1)\delta}\left[\langle\!\langle Q^k v^\varepsilon, v^\varepsilon\rangle\!\rangle + 2\int_0^s \langle\!\langle Q^k v_x^\varepsilon, a^\varepsilon v_x^\varepsilon\rangle\!\rangle dt\right] \\
&\quad + \varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0} \varepsilon^{(k+\ell+1)\delta+\alpha/2}\langle\!\langle U^{k,\ell}, v^\varepsilon\rangle\!\rangle \\
&= \int_0^s \mathcal{B}_t^\varepsilon dt + \int_0^s M_t^\varepsilon \sigma(\xi_{\frac{t}{\varepsilon^\alpha}}) dB_t + \varepsilon^{(K_0+1)\delta-2}\int_0^s \langle\!\langle (Q^{K_0}a^\varepsilon)_{zz} v^\varepsilon, v^\varepsilon\rangle\!\rangle dt \\
&\quad + \int_0^s \langle\!\langle \mathcal{NT}^{1,\varepsilon}(.,t) + \mathcal{NT}^{3,\varepsilon}(.,t), v^\varepsilon(.,t)\rangle\!\rangle dt - \int_0^s \langle\!\langle \mathcal{NT}^{2,\varepsilon}(.,t), v_x^\varepsilon(.,t)\rangle\!\rangle dt.
\end{aligned}$$

We can take the expectation and we deduce that there exists a constant K such that for any $t \in [0,T]$

$$\mathbf{E}\left[\left\langle\!\!\left\langle P^0\left(\frac{\cdot}{\varepsilon}\right) v^\varepsilon(.,t), v^\varepsilon(.,t)\right\rangle\!\!\right\rangle + \int_0^t \left\langle\!\!\left\langle P^0\left(\frac{\cdot}{\varepsilon}\right) a^\varepsilon(.,s) v_x^\varepsilon(.,s), v_x^\varepsilon(.,s)\right\rangle\!\!\right\rangle ds\right] \le K.$$

Since a is bounded and uniformly elliptic and $P^0(z) = \frac{\mathrm{a}^{\mathrm{eff}}}{\bar{a}(z)}$, this proves that v^ε is bounded in $L^2((0,T)\times\Omega; H^1(\mathbb{R}))$.

To obtain (4.17), note that in the martingale term M^ε given by (6.11), all powers of ε are non-negative except for the first sum:

$$\sum_{k=0}^{K_0} \varepsilon^{(k+1)\delta-\alpha/2}\langle\!\langle Q_y^k v^\varepsilon, v^\varepsilon\rangle\!\rangle.$$

Nevertheless define $\widehat{Q}_z^k = Q_y^k$ (recall that $\langle Q_y^k\rangle = 0$) and make an integration by parts:

$$\sum_{k=0}^{K_0} \varepsilon^{(k+1)\delta-\alpha/2}\langle\!\langle Q_y^k v^\varepsilon, v^\varepsilon\rangle\!\rangle = -2\sum_{k=0}^{K_0} \varepsilon^{k\delta+\delta-\alpha/2+1}\langle\!\langle \widehat{Q}^k v^\varepsilon, v_x^\varepsilon\rangle\!\rangle.$$

In any case $\delta - \alpha/2 + 1 > 0$. The conclusion follows from the Burkholder-Davis-Gundy inequality. □

4.2.2. *Weak convergence of v^ε.* Here we prove that the sequence v^ε is tight in

$$V_T = L^2_w(0,T;H^1(\mathbb{R})) \cap C(0,T;L^2_w(\mathbb{R})).$$

Remenber that the index w means that the corresponding space is equipped with the weak topology. For any function $\phi \in C_0^\infty(\mathbb{R})$ we define

$$\begin{aligned}(4.18)\ \widehat{\mathcal{V}}^\varepsilon_t &= \sum_{k=0}^{K_0} \varepsilon^{k\delta}\left[\left\langle\!\!\left\langle P^k\left(\frac{\cdot}{\varepsilon}\right)\phi, v^\varepsilon(.,t)\right\rangle\!\!\right\rangle + \varepsilon^\delta\left\langle\!\!\left\langle Q^k\left(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right)\phi, v^\varepsilon(.,t)\right\rangle\!\!\right\rangle\right]\\ &+\sum_{k=0}^{K_0} \varepsilon^{k\delta+1}\left[\left\langle\!\!\left\langle \widehat{P}^k\left(\frac{\cdot}{\varepsilon}\right)\phi_x, v^\varepsilon(.,t)\right\rangle\!\!\right\rangle + \varepsilon^\delta\left\langle\!\!\left\langle \widehat{Q}^k\left(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right)\phi_x, v^\varepsilon(.,t)\right\rangle\!\!\right\rangle\right]\\ &+\varepsilon^\varpi \sum_{k=0}^{K_0}\sum_{\ell=0}^{J_0} \varepsilon^{(k+\ell+1)\delta+\alpha/2}\left\langle\!\!\left\langle U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}), \phi\right\rangle\!\!\right\rangle\end{aligned}$$

where P^k, Q^k and $U^{k,\ell}$ are defined again by (4.5), (4.6) and (4.8).

Lemma 4.6. *If $\widehat{P}^k$ and $\widehat{Q}^k$ are solutions of:*

$$(\bar{a}(z)\widehat{P}^k)_{zz} = -2(P^k\bar{a})_z,$$

and

$$\mathcal{L}\widehat{Q}^k = 2((\bar{a}-a)P^k)_z + ((\bar{a}-a)\widehat{P}^k)_{zz}$$

then $\widehat{\mathcal{V}}^\varepsilon$ has the following dynamics

$$\begin{aligned}(4.19)\qquad d\widehat{\mathcal{V}}^\varepsilon_t &= \sum_{k=0}^{K_0}\varepsilon^{k\delta}\left\langle\!\!\left\langle\left(\varepsilon^\delta Q^k\phi + \varepsilon^{\delta+1}\widehat{Q}^k\phi_x\right), a^\varepsilon v^\varepsilon_{xx}\right\rangle\!\!\right\rangle dt\\ &+\sum_{k=0}^{K_0}\varepsilon^{k\delta}\left\langle\!\!\left\langle (P^k a^\varepsilon)\phi_{xx} + 2(\widehat{P}^k a^\varepsilon)_z\phi_x + \varepsilon(\widehat{P}^k a^\varepsilon)\phi_{xxx}, v^\varepsilon\right\rangle\!\!\right\rangle dt\\ &+\langle\!\langle \mathcal{NT}^{3,\varepsilon}(.,t),\phi\rangle\!\rangle dt + \widehat{\mathcal{B}}^\varepsilon_t dt + \widehat{M}^\varepsilon_t \sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_t.\end{aligned}$$

where the terms $\mathcal{NT}^{3,\varepsilon}$ and $\widehat{\mathcal{B}}^\varepsilon$ verify (4.12). *The stochastic integrand is given by:*

$$\begin{aligned}(4.20)\quad \widehat{M}^\varepsilon_t &= \sum_{k=0}^{K_0}\varepsilon^{(k+1)\delta-\alpha/2}\left\langle\!\!\left\langle Q^k_y\phi + \varepsilon\widehat{Q}^k_y\phi_x, v^\varepsilon(.,t)\right\rangle\!\!\right\rangle\\ &+\varepsilon^\varpi\sum_{k=0}^{K_0}\sum_{m=0}^{K_0}\varepsilon^{(k+m)\delta}\left\langle\!\!\left\langle P^k\phi + \varepsilon^\delta Q^k\phi + \varepsilon\widehat{P}^k\phi_x + \varepsilon^{\delta+1}\widehat{Q}^k\phi_x, \widetilde{\mathcal{G}}^m\right\rangle\!\!\right\rangle\\ &+\varepsilon^\varpi\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta}\langle\!\langle U^{k,\ell}_y(.,t,\xi_{t/\varepsilon^\alpha}),\phi\rangle\!\rangle.\end{aligned}$$

Proof. The arguments are very similar to the proof of the lemma 4.4 and are postponed in the appendix. □

Now we get a tightness result.

Proposition 4.7. *There exist two constants $\nu > 0$ and $C > 0$ such that for any ε and any $0 \le t \le \tau \le T$,*

$$\mathbf{E}\left[\sup_{t\le s\le\tau} |\widehat{\mathcal{V}}^\varepsilon_s - \widehat{\mathcal{V}}^\varepsilon_t|\right] \le C\sqrt{|\tau - t|} + C\varepsilon^\nu. \tag{4.21}$$

Proof. Indeed the absolutely continuous terms of order ε^0 in (4.19) are

$$\langle\!\langle (P^0 a^\varepsilon)\phi_{xx}, v^\varepsilon\rangle\!\rangle dt + 2\langle\!\langle (\widehat{P}^0 a^\varepsilon)_z \phi_{xx}, v^\varepsilon\rangle\!\rangle dt.$$

And from (4.20),

$$\begin{aligned}
\widehat{M}^\varepsilon_t &= \sum_{k=0}^{K_0} \varepsilon^{(k+1)\delta-\alpha/2}\langle\!\langle Q^k_y\phi, v^\varepsilon(.,t)\rangle\!\rangle \\
&+ \varepsilon\sum_{k=0}^{K_0} \varepsilon^{(k+1)\delta-\alpha/2}\left\langle\!\!\left\langle \widehat{Q}^k_y\phi_x, v^\varepsilon(.,t)\right\rangle\!\!\right\rangle \\
&+ \varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{m=0}^{K_0} \varepsilon^{(k+m)\delta}\left\langle\!\!\left\langle P^k\phi + \varepsilon^\delta Q^k\phi + \varepsilon\widehat{P}^k\phi_x + \varepsilon^{\delta+1}\widehat{Q}^k\phi_x, \widetilde{\mathcal{G}}^m\right\rangle\!\!\right\rangle \\
&+ \varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0} \varepsilon^{(k+\ell+1)\delta}\left\langle\!\!\left\langle U^{k,\ell}_y(.,t,\xi_{t/\varepsilon^\alpha}), \phi\right\rangle\!\!\right\rangle.
\end{aligned}$$

The last three sums are multiplied by a positive power of ε, since $\delta - \alpha/2 + 1 > 0$. Note that later, for $\alpha > 2$, we have to keep the first term $\langle\!\langle P^0\phi, \widetilde{\Upsilon}^0 u^0_x\rangle\!\rangle$. For the first sum, define $\mathfrak{Q}^k_z = Q^k_y$ (recall that $\langle Q^k_y\rangle = 0$) and make an integration by parts:

$$\langle\!\langle Q^k_y\phi, v^\varepsilon\rangle\!\rangle = -\varepsilon\langle\!\langle \mathfrak{Q}^k, (\phi v^\varepsilon)_x\rangle\!\rangle.$$

Hence

$$\begin{aligned}
\widehat{M}_t^\varepsilon &= -\sum_{k=0}^{K_0}\varepsilon^{(k+1)\delta+1-\alpha/2}\langle\!\langle \mathfrak{Q}^k,(\phi v^\varepsilon(.,t))_x\rangle\!\rangle \\
&+ \sum_{k=0}^{K_0}\varepsilon^{(k+1)\delta+1-\alpha/2}\left\langle\!\!\left\langle \widehat{Q}_y^k\phi_x, v^\varepsilon(.,t)\right\rangle\!\!\right\rangle \\
&+ \varepsilon^{\varpi}\left\langle\!\!\left\langle P^0\phi, \widetilde{\Upsilon}^0 u_x^0\right\rangle\!\!\right\rangle + \varepsilon^{\varpi+\delta}\sum_{0\le k,m\le K_0;\ k+m\ge 1}\varepsilon^{(k+m-1)\delta}\left\langle\!\!\left\langle P^k\phi,\widetilde{\mathcal{G}}^m\right\rangle\!\!\right\rangle \\
&+ \varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{m=0}^{K_0}\varepsilon^{(k+m)\delta}\left\langle\!\!\left\langle \varepsilon^\delta Q^k\phi+\varepsilon\widehat{P}^k\phi_x+\varepsilon^{\delta+1}\widehat{Q}^k\phi_x,\widetilde{\mathcal{G}}^m\right\rangle\!\!\right\rangle \\
&+ \varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta}\langle\!\langle U_y^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}),\phi\rangle\!\rangle \\
&= \varepsilon^\nu\widetilde{M}_t^\varepsilon + \varepsilon^{\varpi}\left\langle\!\!\left\langle P^0\phi,\widetilde{\Upsilon}^0u_x^0\right\rangle\!\!\right\rangle
\end{aligned}$$

where $\nu=\min(\delta+1-\alpha/2,\varpi+\delta,\varpi+1)>0$. In other words for any $0\le t\le s\le T$:

$$\begin{aligned}
(4.22)\widehat{\mathcal{V}}_s^\varepsilon-\widehat{\mathcal{V}}_t^\varepsilon &= \int_t^s\left[\langle\!\langle (P^0a^\varepsilon)\phi_{xx},v^\varepsilon\rangle\!\rangle+2\langle\!\langle(\widehat{P}^0a^\varepsilon)_z\phi_{xx},v^\varepsilon\rangle\!\rangle\right]dr \\
&+\varepsilon^\nu\int_t^s\widetilde{M}_r^\varepsilon\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_r+\int_t^s\widehat{\mathcal{B}}_r^\varepsilon dr+\varepsilon^{\varpi}\int_t^s\left\langle\!\!\left\langle P^0\phi,\widetilde{\Upsilon}^0u_x^0\right\rangle\!\!\right\rangle\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_r.
\end{aligned}$$

Thereby

$$\begin{aligned}
\mathbf{E}\left[\sup_{t\le s\le\tau}|\mathcal{V}_s^\varepsilon-\mathcal{V}_t^\varepsilon|\right] &\le C\|v^\varepsilon\|_{L^2((0,T)\times\Omega;L^2(\mathbb{R}))}\times\sqrt{|\tau-t|} \\
&+ \mathbf{E}\left[\sup_{t\le s\le\tau}\left|\int_t^s\widehat{\mathcal{B}}_u^\varepsilon du\right|\right] \\
&+ \varepsilon^\nu\mathbf{E}\left[\sup_{t\le s\le\tau}\left|\int_t^s\widetilde{M}_u^\varepsilon\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_u\right|\right] \\
&+ \varepsilon^{\varpi}\mathbf{E}\left[\sup_{t\le s\le\tau}\left|\int_t^s\left\langle\!\!\left\langle P^0\phi,\widetilde{\Upsilon}^0u_x^0\right\rangle\!\!\right\rangle\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_u\right|\right].
\end{aligned}$$

Note that for $\alpha>2$, $\varpi=0$. From BDG inequality

$$\begin{aligned}
\mathbf{E}\left[\sup_{t\le s\le\tau}\left|\int_t^s\left\langle\!\!\left\langle P^0\phi,\widetilde{\Upsilon}^0u_x^0\right\rangle\!\!\right\rangle\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_u\right|\right] &\le C\mathbf{E}\left[\left(\int_t^\tau\left\langle\!\!\left\langle P^0\phi,\widetilde{\Upsilon}^0u_x^0\right\rangle\!\!\right\rangle^2du\right)^{1/2}\right] \\
&\le C\sqrt{|\tau-t|}
\end{aligned}$$

From BDG and Young's inequalities we have

$$\begin{aligned}
&\mathbf{E}\left[\sup_{t\le s\le\tau}\left|\int_t^s \widetilde{M}_u^\varepsilon\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_u\right|\right]\\
&\quad\le C\mathbf{E}\left[\left(\int_t^\tau\left(\widetilde{M}_u^\varepsilon\right)^2du\right)^{1/2}\right]\\
&\quad\le C\mathbf{E}\left[\sup_{t\le u\le\tau}\left(1+|\xi_{u/\varepsilon^\alpha}|^p\right)\right]+C\mathbf{E}\left[\int_t^\tau\|v^\varepsilon\|^2_{H^1(\mathbb{R})}du\right]
\end{aligned}$$

for some $p\ge 1$. We know that for any $\beta>0$

$$\lim_{\varepsilon\to 0}\varepsilon^\beta\mathbf{E}\left[\sup_{t\le u\le\tau}\left(1+|\xi_{u/\varepsilon^\alpha}|^p\right)\right]=0$$

(see Proposition 2.6 in [5]). Thereby since $\widehat{\mathcal{B}}^\varepsilon$ satisfies (4.12), we deduce the estimate (4.21). □

Therefore from (4.16), (4.17) and (4.21), together with Theorem 8.3 in [4] and Prokhorov criterium, the sequence v^ε is tight in V_T. Now we identify its limit as the law of the solution of a SPDE. Here we distinguish the two cases $\alpha<2$ and $\alpha>2$.

Proposition 4.8. *For $\alpha<2$, the sequence v^ε weakly converges in V_T to zero.*

Proof. Again let ϕ be a $C_0^\infty(\mathbb{R})$ test function. From the definition (4.18) of $\widehat{\mathcal{V}}^\varepsilon$, we deduce that

$$\widehat{\mathcal{V}}_t^\varepsilon=\left\langle\!\!\left\langle P^0\left(\frac{\cdot}{\varepsilon}\right)\phi,v^\varepsilon(.,t)\right\rangle\!\!\right\rangle+\varepsilon^\nu\widehat{\mathcal{V}}_t^{\varepsilon,b}$$

where $\widehat{\mathcal{V}}^{\varepsilon,b}$ is bounded in $\mathbb{L}^2((0,T)\times\Omega)$. Hence since $\langle P^0\rangle=1$, if $\mathfrak{P}^0$ is such that $\mathfrak{P}^0_z=P^0-1$, then

$$\widehat{\mathcal{V}}_t^\varepsilon=\langle\!\langle\phi,v^\varepsilon(.,t)\rangle\!\rangle+\varepsilon\left\langle\!\!\left\langle\mathfrak{P}^0_x\left(\frac{\cdot}{\varepsilon}\right)\phi,v^\varepsilon(.,t)\right\rangle\!\!\right\rangle+\varepsilon^\nu\widehat{\mathcal{V}}_t^{\varepsilon,b}$$

With an integration by parts, we deduce that:

$$\widehat{\mathcal{V}}_t^\varepsilon=\langle\!\langle\phi,v^\varepsilon(.,t)\rangle\!\rangle+\varepsilon\left\langle\!\!\left\langle\mathfrak{P}^0\left(\frac{\cdot}{\varepsilon}\right),(\phi v^\varepsilon(.,t))_x\right\rangle\!\!\right\rangle+\varepsilon^\nu\widehat{\mathcal{V}}_t^{\varepsilon,b}\tag{4.23}$$

Since v^ε is bounded in $L^2((0,T)\times\Omega,H^1(\mathbb{R}))$, the middle term converges to zero.

Now from (4.21) the sequence $\mathcal{V}^\varepsilon$ is also tight in $C(0,T;\mathbb{R})$. Recall that for $\alpha<2$, $\varpi>0$. Using (4.22) we have for some $\nu>0$ and for any $0\le t\le s\le T$

$$\begin{aligned}
\widehat{\mathcal{V}}_s^\varepsilon-\widehat{\mathcal{V}}_t^\varepsilon &= \int_t^s\left[\langle\!\langle(P^0a^\varepsilon)\phi_{xx},v^\varepsilon\rangle\!\rangle+2\langle\!\langle(\widehat{P}^0a^\varepsilon)_z\phi_{xx},v^\varepsilon\rangle\!\rangle\right]dr\\
&+ \varepsilon^\nu\int_t^s\widetilde{M}_r^\varepsilon\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_r+\varepsilon^\nu\int_t^s\widetilde{\mathcal{B}}_r^\varepsilon dr.
\end{aligned}\tag{4.24}$$

For the first integral we define

$$\widetilde{P}^0(z,y)=P^0(z)a(z,y)-\langle P^0a\rangle(y)+2(\widehat{P}^0a)_z(z,y)$$

which has zero mean value in z. We can define again $\widetilde{\mathfrak{P}}^0$ such that $\widetilde{\mathfrak{P}}^0_z=\widetilde{P}^0$ and thus

$$\begin{aligned}(4.25)\quad &\int_t^s\Big\langle\!\!\Big\langle(P^0a^\varepsilon+2(\widehat{P}^0a^\varepsilon)_z)\phi_{xx},v^\varepsilon\Big\rangle\!\!\Big\rangle dr\\ &=\int_t^s\Big\langle\!\!\Big\langle\langle P^0a\rangle(\xi_{r/\varepsilon^\alpha})\phi_{xx},v^\varepsilon\Big\rangle\!\!\Big\rangle dr+\varepsilon\int_t^s\Big\langle\!\!\Big\langle\widetilde{\mathfrak{P}}^0_x\phi_{xx},v^\varepsilon\Big\rangle\!\!\Big\rangle dr\\ &=\int_t^s\Big\langle\!\!\Big\langle\langle\overline{P^0a}\rangle\phi_{xx},v^\varepsilon\Big\rangle\!\!\Big\rangle dr+\int_t^s\Big\langle\!\!\Big\langle(\langle P^0a\rangle(\xi_{r/\varepsilon^\alpha})-\langle\overline{P^0a}\rangle)\phi_{xx},v^\varepsilon\Big\rangle\!\!\Big\rangle dr\\ &\quad+\varepsilon\int_t^s\Big\langle\!\!\Big\langle\widetilde{\mathfrak{P}}^0_x\phi_{xx},v^\varepsilon\Big\rangle\!\!\Big\rangle dr.\end{aligned}$$

For the term

$$\mathfrak{E}^\varepsilon(t)=\int_0^t\Big\langle\!\!\Big\langle(\langle P^0a\rangle(\xi_{r/\varepsilon^\alpha})-\langle\overline{P^0a}\rangle)\phi_{xx},v^\varepsilon\Big\rangle\!\!\Big\rangle dr,$$

the uniform bound (4.17), together with the mixing property implied by assumption **(A)**, lead to the convergence to zero of this term, a.s. and in $\mathbb{L}^2(\Omega)$ by the dominated convergence theorem, uniformly w.r.t. $t\in[0,T]$.

Combining (4.23), (4.24) and (4.25), we obtain for some $\nu>0$

$$\begin{aligned}F_\phi(t,v^\varepsilon)&=\langle\!\langle\phi,v^\varepsilon(.,t)\rangle\!\rangle-\int_0^t\Big\langle\!\!\Big\langle\langle\overline{P^0a}\rangle\phi_{xx},v^\varepsilon\Big\rangle\!\!\Big\rangle dr\\ &=\varepsilon^\nu\int_0^t\widetilde{M}^\varepsilon_r\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_r+\varepsilon^\nu\widehat{\mathcal{V}}^{\varepsilon,b}_t+\mathfrak{E}^\varepsilon(t).\end{aligned}$$

Let Θ^ε_s be any continuous (in the sense of the topology of V_T) and bounded functional of $\{v^\varepsilon_\tau,\ 0\le\tau\le s\}$. We have proved that for $0\le s\le t$

$$\lim_{\varepsilon\downarrow0}\mathbb{E}\,|(F_\phi(t,v^\varepsilon)-F_\phi(s,v^\varepsilon))\Theta^\varepsilon_s|=0.$$

If we compute the quadratic variation[3] of the process $\widehat{\mathcal{V}}^\varepsilon$ we have

$$\Big[\!\Big[\widehat{\mathcal{V}}^\varepsilon\Big]\!\Big]_s-\Big[\!\Big[\widehat{\mathcal{V}}^\varepsilon\Big]\!\Big]_t=\varepsilon^{2\nu}\int_t^s\left|\widetilde{M}^\varepsilon_r\right|^2\|\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\|^2dr.$$

Recall that for a vector $v\in\mathbb{R}^n$, $\|v\|^2=\mathrm{Trace}(vv^*)$ is the Euclidean norm. We deduce that

$$\lim_{\varepsilon\downarrow0}\mathbf{E}\left[(F_\phi(t,v^\varepsilon)-F_\phi(s,v^\varepsilon))^2\Theta^\varepsilon_s\right]=\lim_{\varepsilon\downarrow0}\mathbf{E}\left[\left|\Big[\!\Big[\widehat{\mathcal{V}}^\varepsilon\Big]\!\Big]_s-\Big[\!\Big[\widehat{\mathcal{V}}^\varepsilon\Big]\!\Big]_t\right|\Theta^\varepsilon_s\right]$$

is equal to zero. Passing through the limit, we deduce that $\{F_\phi(t,v^0),\ 0\le t\le T\}$ is a square integrable martingale with respect to the natural filtration

[3]Denoted by [[.]] to be distinguishable from the mean over a period or the scalar product in L^2.

of v^0, with a null quadratic variation process. In other words we proved that the sequence v^ε weakly converges in V_T to the unique solution v^0 of the PDE:

$$dv^0 = \langle \overline{P^0 a} \rangle v^0_{xx} dt$$

with initial condition zero. Hence $v^0 = 0$ and this achieves the proof of the Proposition. □

For $\alpha > 2$, the preceding result has to be modified since $\varpi = 0$, which implies that there is a zero order term in the martingale part $\widehat{M}^\varepsilon$ in (4.19).

Proposition 4.9. *If $\alpha > 2$, the sequence v^ε weakly converges in V_T to the unique solution $\widetilde{r}^0$ of the SPDE:*

$$d\widetilde{r}^0 = \langle \overline{P^0 a} \rangle \widetilde{r}^0_{xx} dt + \left(\overline{\| \langle P^0 \widetilde{\Upsilon}^0 \rangle \sigma \|^2} \right)^{1/2} u^0_x dW_t.$$

Proof. We argue almost as in the proof of Proposition 4.8. In particular the beginning of the proof is the same. But now (4.24) becomes:

$$\begin{aligned} (4.26)\quad \widehat{\mathcal{V}}^\varepsilon_s - \widehat{\mathcal{V}}^\varepsilon_t &= \int_t^s \left[\langle\!\langle (P^0 a^\varepsilon) \phi_{xx}, v^\varepsilon \rangle\!\rangle + 2 \langle\!\langle (\widehat{P}^0 a^\varepsilon)_z \phi_{xx}, v^\varepsilon \rangle\!\rangle \right] dr \\ &+ \int_t^s \langle\!\langle P^0 \phi, \widetilde{\Upsilon}^0 u^0_x \rangle\!\rangle \sigma(\xi_{\frac{t}{\varepsilon^\alpha}}) dB_r + \varepsilon^\nu \int_t^s \widetilde{M}^\varepsilon_r \sigma(\xi_{\frac{t}{\varepsilon^\alpha}}) dB_r + \int_t^s \widetilde{\mathcal{B}}^\varepsilon_r dr. \end{aligned}$$

Now we obtain for some $\nu > 0$

$$\begin{aligned} F_\phi(t, v^\varepsilon) &= \langle\!\langle \phi, v^\varepsilon(., t) \rangle\!\rangle - \int_0^t \left\langle\!\!\left\langle \langle \overline{P^0 a} \rangle \phi_{xx}, v^\varepsilon \right\rangle\!\!\right\rangle dr \\ &= \int_0^t \langle\!\langle P^0 \phi, \widetilde{\Upsilon}^0 u^0_x \rangle\!\rangle \sigma(\xi_{\frac{t}{\varepsilon^\alpha}}) dB_r + \varepsilon^\nu \int_0^t \widetilde{M}^\varepsilon_r \sigma(\xi_{\frac{t}{\varepsilon^\alpha}}) dB_r + \varepsilon^\nu \widehat{\mathcal{V}}^{\varepsilon,b}_t + \mathfrak{E}^\varepsilon(t). \end{aligned}$$

The term

$$\mathfrak{E}^\varepsilon(t) = \int_0^t \left\langle\!\!\left\langle (\langle P^0 a \rangle(\xi_{r/\varepsilon^\alpha}) - \langle \overline{P^0 a} \rangle) \phi_{xx}, v^\varepsilon \right\rangle\!\!\right\rangle dr,$$

can be handled as before and we have proved that for any continuous (in the sense of the topology of V_T) and bounded functional Θ^ε_s of $\{v^\varepsilon_\tau,\ 0 \le \tau \le s\}$ and any for $0 \le s \le t$

$$\lim_{\varepsilon \downarrow 0} \mathbf{E} \left| (F_\phi(t, \widetilde{r}^\varepsilon) - F_\phi(s, \widetilde{r}^\varepsilon)) \Theta^\varepsilon_s \right| = 0.$$

Concerning the quadratic variation of the process $\widehat{\mathcal{V}}^\varepsilon$, we have

$$\left[\!\left[\widehat{\mathcal{V}}^\varepsilon \right]\!\right]_s - \left[\!\left[\widehat{\mathcal{V}}^\varepsilon \right]\!\right]_t = \int_t^s \left(\left| \langle\!\langle P^0, \widetilde{\Upsilon}^0 \phi u^0_x \rangle\!\rangle \right|^2 + \varepsilon^{2\nu} \left| \widetilde{M}^\varepsilon_r \right|^2 \right) \| \sigma(\xi_{\frac{t}{\varepsilon^\alpha}}) \|^2 dr.$$

Recall that for a vector $v \in \mathbb{R}^n$, $\|v\|^2 = \mathrm{Trace}(vv^*)$ is the Euclidean norm. Again if we denote

$$\mathfrak{G}(y) = \langle P^0 \widetilde{\Upsilon}^0 \rangle(y), \qquad \mathfrak{Q}^0_z(z, y) = P^0(z) \widetilde{\Upsilon}^0(z, y) - \mathfrak{G}(y)$$

the mean of $P^0\widetilde{\Upsilon}^0$ w.r.t. z and the periodic antiderivative of $P^0\widetilde{\Upsilon}^0-\mathfrak{G}$, then

$$\begin{aligned}\int_t^s\left(\left|\langle\!\langle P^0,\widetilde{\Upsilon}^0\phi u_x^0\rangle\!\rangle\right|^2\right)\|\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\|^2dr&=\int_t^s\left\|\langle\!\langle P^0\widetilde{\Upsilon}^0,\phi u_x^0\rangle\!\rangle\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\right\|^2dr\\&=\int_t^s\left\|\langle\!\langle P^0\widetilde{\Upsilon}^0-\mathfrak{G}(\xi_{\frac{t}{\varepsilon^\alpha}}),\phi u_x^0\rangle\!\rangle\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\right\|^2dr\\&\quad+\int_t^s\left\|\langle\!\langle\phi,u_x^0\rangle\!\rangle\mathfrak{G}(\xi_{r/\varepsilon^\alpha})\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\right\|^2dr.\end{aligned}$$

And we have

$$\int_t^s\left\|\langle\!\langle P^0\widetilde{\Upsilon}^0-\mathfrak{G}(\xi_{\frac{t}{\varepsilon^\alpha}}),\phi u_x^0\rangle\!\rangle\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\right\|^2dr\le\varepsilon^2\int_t^s\left\|\langle\!\langle\mathfrak{Q}_x^0\phi,u_x^0\rangle\!\rangle\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\right\|^2dr.$$

Moreover again using assumption **(A)**, we obtain that

$$\int_t^s\left\|\langle\!\langle\phi,u_x^0\rangle\!\rangle\mathfrak{G}(\xi_{r/\varepsilon^\alpha})\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\right\|^2dr$$

converges a.s. and in $\mathbb{L}^2(\Omega)$ to

$$\int_t^s\overline{\left\|\mathfrak{G}(\xi_{r/\varepsilon^\alpha})\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\langle\!\langle\phi,u_x^0\rangle\!\rangle\right\|^2}dr=\int_t^s\overline{\left\|\langle P^0\widetilde{\Upsilon}^0\rangle\sigma\right\|^2}\langle\!\langle\phi,u_x^0\rangle\!\rangle^2dr.$$

We deduce that

$$\lim_{\varepsilon\downarrow0}\mathbf{E}\left[(F_\phi(t,v^\varepsilon)-F_\phi(s,v^\varepsilon))^2\Theta_s^\varepsilon\right]=\lim_{\varepsilon\downarrow0}\mathbf{E}\left(\left|\left[\left[\widehat{\mathcal{V}}^\varepsilon\right]\right]_s-\left[\left[\widehat{\mathcal{V}}^\varepsilon\right]\right]_t\right|\Theta_s^\varepsilon\right)$$

is equal to

$$\int_t^s\overline{\left\|\langle P^0\widetilde{\Upsilon}^0\rangle\sigma\right\|^2}\langle\!\langle\phi,u_x^0\rangle\!\rangle^2dr.$$

Passing through the limit, we deduce that $\{F_\phi(t,\widetilde{r}^0),\ 0\le t\le T\}$ is a square integrable martingale with respect to the natural filtration of $\widetilde{r}^0$, with the associated quadratic variation process given by

$$\overline{\|\langle P^0\widetilde{\Upsilon}^0\rangle\sigma\|^2}\langle\!\langle\phi,u_x^0\rangle\!\rangle^2t.$$

This achieves the proof of the Proposition. □

Let us remark to conclude this part that $P^0=1+\chi_z^0$, thus $\langle\overline{P^0a}\rangle=\mathrm{a}^{\mathrm{eff}}$. Moreover

$$\langle P^0\widetilde{\Upsilon}^0\rangle=-\langle\chi^0\Upsilon^0\rangle.$$

Hence

$$d\widetilde{r}^0=\mathrm{a}^{\mathrm{eff}}\widetilde{r}_{xx}^0dt+\left(\overline{\|\langle\chi^0\Upsilon^0\rangle\sigma\|^2}\right)^{1/2}u_x^0dW_t.$$

4.2.3. *Conclusion.* Now we know that there exists a constant C independent of ε such that

$$\mathbf{E}\left(\|v_x^\varepsilon\|^2_{L^2((0,T)\times\mathbb{R})}\right) \leq C.$$

By Tchebychev's inequality for any $\delta > 0$, there exists a constant K such that

$$\mathbf{P}(\|v_x^\varepsilon\|^2_{L^2((0,T)\times\mathbb{R})} \geq K) \leq C^2/K^2 \leq \delta$$

provided K is large enough. In other words, v_x^ε is tight for the weak topology on $L^2((0,T)\times\mathbb{R})$. Using the dense set of C_0^∞ functions and Propositions 4.8 and 4.9, we deduce that for $\alpha < 2$, r^ε weakly converges to the solution of:

- For $\alpha < 2$:

$$dr^0 = \mathrm{a}^{\mathrm{eff}} r^0_{xx} dt,$$

  with initial value 0, that is, $r^0 = 0$.
- For $\alpha > 2$:

$$dr^0 = \mathrm{a}^{\mathrm{eff}} r^0_{xx} dt + \left(\overline{\|\langle \chi^0 \Upsilon^0 \rangle \sigma\|^2}\right)^{1/2} u^0_{xx} dW_t$$

  again with initial value zero.

The proof of Theorem 2.1 is now complete in the case $\alpha < 2$, using Propositions 3.1 and 3.2 and the preceding results on the convergence of r^ε. For $\alpha > 2$, using Proposition 3.6, the proof will be complete after the study of ρ^ε, which is the aim of the next section 5. Before, let us consider the case $\alpha < 1$, for which an easier proof can be done.

4.3. **Case $\alpha < 1$.** Here the assumption that $d = 1$ is unnecessary for our arguments. In the problem (4.1), we now have $\varpi - 1 = 1 - \alpha > 0$. Let us take $w^k \equiv 0$ for any k :

$$\begin{aligned} dr^\varepsilon &= (\mathcal{A}^\varepsilon r^\varepsilon)dt - \sum_{k=1}^{K_0} \varepsilon^{k\delta - \alpha/2} w^k(x,t)\, dt \\ &- \varepsilon^{\varpi - 1} \sum_{k=0}^{K_0} \varepsilon^{k\delta} \Upsilon^k\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big) u^k_x(x,t) \sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\, dB_t \\ &= (\mathcal{A}^\varepsilon r^\varepsilon)dt + \varepsilon^{1-\alpha} \Theta^\varepsilon\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}, x, t\Big)\, dB_t. \end{aligned}$$

Let us define

$$v_t^\varepsilon = \int_{\mathbb{R}^d} r^\varepsilon(x,t)^2 dx = \|r^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d)}.$$

Itô's formula leads to

$$\begin{aligned} v_t^\varepsilon &= \int_0^t \int_{\mathbb{R}^d} r^\varepsilon(x,s)\mathrm{div}\Big[\mathrm{a}\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^\alpha}}\Big)\nabla r^\varepsilon(x,s)\Big]dxds \\ &+ \varepsilon^{(1-\alpha)}\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}},x,t\Big)dx\,dB_s \\ &+ \varepsilon^{2(1-\alpha)}\int_0^t\int_{\mathbb{R}^d}\Big\|\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}},x,t\Big)\Big\|^2 dx\,ds. \end{aligned}$$

An integration by part shows that

$$\begin{aligned} &v_t^\varepsilon + \int_0^t\int_{\mathbb{R}^d}\nabla r^\varepsilon(x,s)\Big[\mathrm{a}\Big(\frac{x}{\varepsilon},\xi_{\frac{s}{\varepsilon^\alpha}}\Big)\nabla r^\varepsilon(x,s)\Big]dxds \\ &\quad = \varepsilon^{(1-\alpha)}\int_0^t\int_{\mathbb{R}^d} r^\varepsilon(x,s)\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}},x,t\Big)dx\,dB_s \\ &\quad +\varepsilon^{2(1-\alpha)}\int_0^t\int_{\mathbb{R}^d}\Big\|\Theta^\varepsilon\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}},x,t\Big)\Big\|^2 dx\,ds. \end{aligned}$$

From Condition **(a3)**, taking the expectation, there exists a constant C independent of ε such that

$$\mathbf{E}\int_0^T \|\nabla r^\varepsilon(\cdot,s)\|^2_{L^2(\mathbb{R}^d)}\,ds \le C\varepsilon^{2(1-\alpha)}. \tag{4.27}$$

Moreover by Burkholder-Davis-Gundy inequality, we have

$$\mathbf{E}\left[\sup_{t\in[0,T]} v_t^\varepsilon\right] = \mathbf{E}\left[\sup_{t\in[0,T]}\|r^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d)}\right] \le C\varepsilon^{2(1-\alpha)}. \tag{4.28}$$

Hence if $\alpha<1$, the convergence of r^ε to zero holds in $\mathbb{L}^2([0,T]\times\Omega;H^1(\mathbb{R}^d))$ (and in $\mathbb{L}^\infty([0,T];L^2(\mathbb{R}^d))$ in mean w.r.t. ω).

5. Role of the initial condition in the discrepancy

Let us note that this part[4] only concerns the case $\alpha>2$ and the behavior of ρ^ε. Recall the setting concerning ρ^ε. It satisfies:

$$d\rho^\varepsilon = (\mathcal{A}^\varepsilon\rho^\varepsilon)dt$$

with initial condition (3.26):

$$\rho^\varepsilon(x,0) = -\sum_{k=1}^{J_1}\varepsilon^{k-\alpha/2}\left[\mathcal{I}_k + \sum_{\ell=1}^{k}\mathcal{I}_{k-\ell}\chi^{\ell-1}\left(\frac{x}{\varepsilon}\right)\right]\partial_x^k u^0(x,0).$$

By linearity we can write:

$$\rho^\varepsilon(x,t) = \sum_{k=1}^{J_1}\rho^{k,\varepsilon}(x,t) \tag{5.1}$$

[4]Let us emphasize that all results of this section hold in $d>1$, that is for $z\in\mathbb{T}^d$.

where the functions $\rho^{k,\varepsilon}$ have the same dynamics (3.25), $d\rho^{k,\varepsilon} = (\mathcal{A}^\varepsilon \rho^{k,\varepsilon})dt$, but with initial condition

$$\rho^{k,\varepsilon}(x,0) = -\varepsilon^{k-\alpha/2}\left[\mathcal{I}_k + \sum_{\ell=1}^{k} \mathcal{I}_{k-\ell}\chi^{\ell-1}\left(\frac{x}{\varepsilon}\right)\right]\partial_x^k u^0(x,0). \tag{5.2}$$

Recall that from (1.2), u^0 is a smooth function such that $u_t^0 = \mathrm{a}^{\mathrm{eff}} u_{xx}^0$, with initial condition $u^0(x,0) = \imath(x)$.

To lighten the notation, let us fix $k = 1, \ldots, J_1$ and define $\varrho^\varepsilon = \varrho^{k,\varepsilon}$ as the solution of (3.25) with initial condition

$$\varrho^\varepsilon(x,0) = \left[\mathcal{I}_k + \sum_{\ell=1}^{k} \mathcal{I}_{k-\ell}\chi^{\ell-1}\left(\frac{x}{\varepsilon}\right)\right]\partial_x^k u^0(x,0) = A_k\left(\frac{x}{\varepsilon}\right)\partial_x^k u^0(x,0).$$

Thus $\rho^{k,\varepsilon} = -\varepsilon^{k-\alpha/2}\varrho^\varepsilon = -\varepsilon^{k-\alpha/2}\varrho^{k,\varepsilon}$.

Lemma 5.1. *The function ϱ^ε admits the following expansion:*

$$\begin{aligned}\varrho^\varepsilon(x,t) &= \beta^{0,\varepsilon}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\partial_x^k u^0(x,t) \\ &+ \sum_{\ell=1}^{J_1-k} \varepsilon^\ell\left[\widehat{m}^{\ell-1,\varepsilon}\left(\frac{t}{\varepsilon^2}\right) + \beta^{\ell,\varepsilon}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\right]\partial_x^{k+\ell} u^0(x,t) \\ &+ \Gamma^{J_1-k,\varepsilon}(x,t).\end{aligned} \tag{5.3}$$

The functions $\beta^{0,\varepsilon}$, $\widehat{m}^{0,\varepsilon}$ and $\mu^{0,\varepsilon}$ are defined by the following equations:

$$\partial_t \beta^{0,\varepsilon}(z,t) = (\mathfrak{a}^\varepsilon \beta_z^{0,\varepsilon})_z, \qquad \beta^{0,\varepsilon}(z,0) = A^k(z) \tag{5.4}$$

$$m^{0,\varepsilon}(t) = \langle a\left(.\,, \xi_{t/\varepsilon^\delta}\right)\beta_z^{0,\varepsilon}(.\,,t)\rangle \tag{5.5}$$

$$\partial_t \widehat{m}^{0,\varepsilon}(t) = m^{0,\varepsilon}(t), \quad \widehat{m}^{0,\varepsilon}(0) = 0 \tag{5.6}$$

$$\mu^{0,\varepsilon}(z,t) = \mathfrak{a}^\varepsilon \beta_z^{0,\varepsilon}(z,t) - m^{0,\varepsilon}(t) \tag{5.7}$$

$$\partial_t \beta^{1,\varepsilon}(z,t) = (\mathfrak{a}^\varepsilon \beta_z^{1,\varepsilon})_z + (\mu^{0,\varepsilon} + (\mathfrak{a}^\varepsilon \beta^{0,\varepsilon})_z). \tag{5.8}$$

with $\beta^{1,\varepsilon}(z,0) = 0$. The other quantities are given by:

$$m^{1,\varepsilon}(t) = \langle a\left(.\,, \xi_{t/\varepsilon^\delta}\right)\beta_z^{1,\varepsilon}(.\,,t)\rangle + \langle(\mathfrak{a}^\varepsilon - \mathrm{a}^{\mathrm{eff}})\beta^{0,\varepsilon}\rangle, \tag{5.9}$$

$$\partial_t \widehat{m}^{1,\varepsilon}(t) = m^{1,\varepsilon}(t), \tag{5.10}$$

$$\mu^{1,\varepsilon}(z,t) = \mathfrak{a}^\varepsilon \beta_z^{1,\varepsilon} - m^{1,\varepsilon}(t) + (\mathfrak{a}^\varepsilon - \mathrm{a}^{\mathrm{eff}})\beta^{0,\varepsilon}. \tag{5.11}$$

And for $\ell \geq 2$, the relations are defined recursively by:

$$\begin{aligned}
(5.12)\quad \partial_t \beta^{\ell,\varepsilon} &= (\mathfrak{a}^\varepsilon \beta_z^{\ell,\varepsilon})_z + (\widehat{m}^{\ell-2,\varepsilon}(t)\mathfrak{a}_z^\varepsilon + \mu^{\ell-1,\varepsilon} + (\mathfrak{a}^\varepsilon \beta^{\ell-1,\varepsilon})_z),\\
\beta^{\ell,\varepsilon}(z,0) &= 0,\\
(5.13)\quad m^{\ell,\varepsilon}(t) &= \langle \mathfrak{a}^\varepsilon \beta_z^{\ell,\varepsilon}\rangle + \langle (\mathfrak{a}^\varepsilon - \mathrm{a}^{\mathrm{eff}})(\widehat{m}^{\ell-2,\varepsilon} + \beta^{\ell-1,\varepsilon})\rangle,\\
(5.14)\quad \partial_t \widehat{m}^{\ell,\varepsilon}(t) &= m^{\ell,\varepsilon}(t),\\
(5.15)\quad \mu^{\ell,\varepsilon}(z,t) &= \mathfrak{a}^\varepsilon \beta_z^{\ell,\varepsilon} - m^{\ell,\varepsilon}(t) + (\mathfrak{a}^\varepsilon - \mathrm{a}^{\mathrm{eff}})(\widehat{m}^{\ell-2,\varepsilon} + \beta^{\ell-1,\varepsilon}).
\end{aligned}$$

The last term in expansion (5.3) *is of order ε^ν with $\nu > \alpha/2 - k$.*

Let us emphasize here that all terms defined in this lemma depend on k.

Proof. Let us define on $\mathbb{T}\times(0,\infty)$, $\beta^{0,\varepsilon}$ by (5.4). Since A_k is periodic, $\beta^{0,\varepsilon}$ is well-defined. Let us assume that

$$\varrho^\varepsilon(x,t) = \beta^{0,\varepsilon}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\partial_x^1 u^0\,(x,t) + \Gamma^{0,\varepsilon}(x,t).$$

Then $\Gamma^{0,\varepsilon}$ satisfies: $\Gamma^{0,\varepsilon}(x,0) = 0$ and

$$\begin{aligned}
\partial_t \Gamma^{0,\varepsilon} = (\mathcal{A}^\varepsilon \Gamma^{0,\varepsilon}) &+ \frac{1}{\varepsilon}\left(a^\varepsilon \beta_z^{0,\varepsilon} + (a^\varepsilon \beta^{0,\varepsilon})_z\right)\partial_x^2 u^0\,(x,t)\\
&+ (a^\varepsilon - \mathrm{a}^{\mathrm{eff}})\beta^{0,\varepsilon}\partial_x^3 u^0\,(x,t).
\end{aligned}$$

We define $m^{0,\varepsilon}(t)$ by (5.5) as the mean value w.r.t. z of the function $\mathfrak{a}^\varepsilon \beta_z^{0,\varepsilon}$, $\widehat{m}^{0,\varepsilon}(t)$ by (5.6) and $\mu^{0,\varepsilon}$ by (5.7) such that the mean value of $\mu^{0,\varepsilon}$ w.r.t. z is zero. Hence we can define on $\mathbb{T}\times(0,\infty)$ the function $\beta^{1,\varepsilon}$ by (5.8). Now we assume that

$$\Gamma^{0,\varepsilon}(x,t) = \varepsilon\left[\widehat{m}^{0,\varepsilon}\left(\frac{t}{\varepsilon^2}\right) + \beta^{1,\varepsilon}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\right]\partial_x^2 u^0\,(x,t) + \Gamma^{1,\varepsilon}(x,t).$$

To study the behaviour of $\Gamma^{1,\varepsilon}$, let us remark first that

$$\begin{aligned}
\partial_t \Gamma^{1,\varepsilon} = (\mathcal{A}^\varepsilon \Gamma^{1,\varepsilon}) &+ \left[\widehat{m}^{0,\varepsilon}(t/\varepsilon^2)a_z^\varepsilon + \left(a^\varepsilon \beta_z^{1,\varepsilon} + (a^\varepsilon \beta^{1,\varepsilon})_z\right)\right]\partial_x^3 u^0\,(x,t)\\
&+ (a^\varepsilon - \mathrm{a}^{\mathrm{eff}})\beta^{0,\varepsilon}\partial_x^3 u^0\,(x,t)\\
&+ \varepsilon\left[\widehat{m}^{0,\varepsilon}(t/\varepsilon^2) + \beta^{1,\varepsilon}\right](a^\varepsilon - \mathrm{a}^{\mathrm{eff}})\partial_x^4 u^0\,(x,t).
\end{aligned}$$

Let us do the same trick again. Using (5.9), (5.10) and (5.11) yields to:

$$\begin{aligned}
\partial_t \Gamma^{1,\varepsilon} &= (\mathcal{A}^\varepsilon \Gamma^{1,\varepsilon}) + m^{1,\varepsilon}(t/\varepsilon^2)\partial_x^3 u^0\,(x,t)\\
&+ \left[\mu^{1,\varepsilon}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \widehat{m}^{0,\varepsilon}(t/\varepsilon^2)a_z^\varepsilon + (a^\varepsilon \beta^{1,\varepsilon})_z\right]\partial_x^3 u^0\,(x,t)\\
&+ \varepsilon\left[\widehat{m}^{0,\varepsilon}(t/\varepsilon^2) + \beta^{1,\varepsilon}\right](a^\varepsilon - \mathrm{a}^{\mathrm{eff}})\partial_x^4 u^0\,(x,t).
\end{aligned}$$

If $\beta^{2,\varepsilon}$ is the solution on $\mathbb{T}^d\times(0,\infty)$ of (5.12) and if

$$\Gamma^{1,\varepsilon}(x,t) = \varepsilon^2\left[\widehat{m}^{1,\varepsilon}\left(\frac{t}{\varepsilon^2}\right) + \beta^{2,\varepsilon}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\right]\partial_x^3 u^0\,(x,t) + \Gamma^{2,\varepsilon}(x,t),$$

then

$$\begin{aligned}\partial_t\Gamma^{2,\varepsilon} &= (\mathcal{A}^\varepsilon\Gamma^{2,\varepsilon}) + \varepsilon\left[\widehat{m}^{1,\varepsilon}(t/\varepsilon^2)a_z^\varepsilon + \left(a^\varepsilon\beta_z^{2,\varepsilon} + (a^\varepsilon\beta^{2,\varepsilon})_z\right)\right]\partial_x^4 u^0\,(x,t)\\ &+ \varepsilon\left[\widehat{m}^{0,\varepsilon}(t/\varepsilon^2) + \beta^{1,\varepsilon}\right](a^\varepsilon - \mathrm{a}^{\mathrm{eff}})\partial_x^4 u^0\,(x,t)\\ &+ \varepsilon^2\left[\widehat{m}^{1,\varepsilon}(t/\varepsilon^2) + \beta^{2,\varepsilon}\right](a^\varepsilon - \mathrm{a}^{\mathrm{eff}})\partial_x^5 u^0\,(x,t)\,.\end{aligned}$$

And

$$\begin{aligned}\varrho^\varepsilon(x,t) &= \beta^{0,\varepsilon}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\partial_x^k u^0\,(x,t)\\ &+ \varepsilon\left[\widehat{m}^{0,\varepsilon}\left(\frac{t}{\varepsilon^2}\right) + \beta^{1,\varepsilon}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\right]\partial_x^{k+1}u^0\,(x,t)\\ &+ \varepsilon^2\left[\widehat{m}^{1,\varepsilon}\left(\frac{t}{\varepsilon^2}\right) + \beta^{2,\varepsilon}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\right]\partial_x^{k+2}u^0\,(x,t) + \Gamma^{2,\varepsilon}(x,t).\end{aligned}$$

And then we iterate the arguments. For $\ell = 2,\dots,J_1 - k$, we can iterate this procedure with β^ℓ, $m^{\ell,\varepsilon}$, $\widehat{m}^{\ell,\varepsilon}$, $\mu^{\ell,\varepsilon}$ given by (5.12), (5.13), (5.14), (5.15) and

$$\Gamma^{\ell,\varepsilon} = \varepsilon^{\ell+1}\left[\widehat{m}^{\ell,\varepsilon}\left(\frac{t}{\varepsilon^2}\right) + \beta^{\ell+1,\varepsilon}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\right]\partial_x^{k+\ell+1}u^0\,(x,t) + \Gamma^{\ell+1,\varepsilon}(x,t).$$

The last term will be of the form

$$\begin{aligned}&\Gamma^{J_1-k,\varepsilon}(x,t) = \Gamma^{J_1-k+1,\varepsilon}(x,t)\\ &+\varepsilon^{J_1-k+1}\left[\widehat{m}^{J_1-k,\varepsilon}\left(\frac{t}{\varepsilon^2}\right) + \beta^{J_1-k+1,\varepsilon}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\right]\partial_x^{J_1+1}u^0\,(x,t)\end{aligned}$$

and

$$\begin{aligned}&\partial_t\Gamma^{J_1-k+1,\varepsilon} = (\mathcal{A}^\varepsilon\Gamma^{J_1-k+1,\varepsilon})\\ &+\varepsilon^{J_1-k}\left[\widehat{m}^{J_1-k,\varepsilon}(t/\varepsilon^2)a_z^\varepsilon + \left(a^\varepsilon\beta_z^{J_1-k+1,\varepsilon} + (a^\varepsilon\beta^{k,J_1-k+1})_z\right)\right]\partial_x^{J_1+2}u^0\,(x,t)\\ &+\varepsilon^{J_1-k}\left[\widehat{m}^{J_1-k-1,\varepsilon}(t/\varepsilon^2) + \beta^{J_1-k,\varepsilon}\right](a^\varepsilon - \mathrm{a}^{\mathrm{eff}})\partial_x^{n+J_1-k+2}u^0\,(x,t)\\ &+\varepsilon^{J_1-k+1}\left[\widehat{m}^{J_1-k,\varepsilon}(t/\varepsilon^2) + \beta^{J_1-k+1,\varepsilon}\right](a^\varepsilon - \mathrm{a}^{\mathrm{eff}})\partial_x^{J_1+2}u^0\,(x,t)\,.\end{aligned}$$

All powers of ε are greater than $\alpha/2 - k$. Thus the proof of the Lemma is achieved. □

Now let us precise the behaviour of the correctors $\beta^{\ell,\varepsilon}$. For $\ell = 0$, since

$$A_k(z) = \mathcal{I}_k + \sum_{m=1}^{k}\mathcal{I}_{k-m}\chi^{m-1}\,(z)\,,$$

one can easily deduce that

$$\beta^{0,\varepsilon}(z) = \mathcal{I}_k + \widetilde{\beta}^{0,\varepsilon}(z)$$

where $\widetilde{\beta}^{0,\varepsilon}$ satisfies (5.4), but with initial condition a periodic function with zero mean value. The key point in the sequel is that: $\beta_z^{0,\varepsilon} = \widetilde{\beta}_z^{0,\varepsilon}$. And in (5.5) and (5.7), only the derivative is implied. We also denote

$$\mathfrak{K}_k = \left\| \sum_{m=1}^{k} \mathcal{I}_{k-m} \chi^{m-1} \right\|^2_{L^2(\mathbb{T}^d)}.$$

The next result is an immediate consequence of Poincaré's inequality.

Lemma 5.2. *There exists a constant $\mathfrak{k}$ depending only on the uniform ellipticity constant of the matrix a, such that*

$$\forall s \geq 0, \quad \left\| \widetilde{\beta}^{0,\varepsilon}(.,s) \right\|^2_{L^2(\mathbb{T}^d)} \leq \mathfrak{K}_k e^{-\mathfrak{k}s}.$$

For simplicity for $\ell \geq 1$, let us rewrite Equations (5.8) and (5.12) as:

$$\begin{aligned} \partial_t \beta^{\ell,\varepsilon} &= (a(z, \xi_{t/\varepsilon^\delta}) \beta_z^{\ell,\varepsilon})_z + (\widehat{m}^{\ell-2,\varepsilon}(t) \mathfrak{a}_z^\varepsilon + \mu^{\ell-1,\varepsilon} + (\mathfrak{a}^\varepsilon \beta^{\ell-1,\varepsilon})_z) \\ &= (a(z, \xi_{t/\varepsilon^\delta}) \beta_z^{\ell,\varepsilon})_z + \varphi^{\ell,\varepsilon}. \end{aligned} \tag{5.16}$$

Lemma 5.3. *For $\ell = 1, \ldots, J_1 - 1$ we have:*

$$\forall s \geq 0, \quad \left\| \beta^{\ell,\varepsilon}(.,s) \right\|^2_{L^2(\mathbb{T}^d)} \leq \mathfrak{K}_k e^{-\mathfrak{k}s}.$$

Proof. Recall that λ is the ellipticity constant of a (Condition **(a4)**). Again by Poincaré's inequality, we deduce that

$$|m^{0,\varepsilon}(t)| = \left| \langle a^\varepsilon \left(., \xi_{\frac{t}{\varepsilon^\delta}} \right) \beta_z^{0,\varepsilon}(.,t) \rangle \right| \leq \frac{\mathfrak{K}_k}{\lambda} e^{-\mathfrak{k}t}. \tag{5.17}$$

And

$$\varphi^{1,\varepsilon} = \mu^{0,\varepsilon} + (\mathfrak{a}^\varepsilon \beta^{0,\varepsilon})_z = \mathfrak{a}^\varepsilon \beta_z^{0,\varepsilon} - m^{0,\varepsilon} + (\mathfrak{a}^\varepsilon \beta^{0,\varepsilon})_z$$

satisfies a similar inequality: $\left\| \varphi^{1,\varepsilon}(.,t) \right\|^2_{L^2(\mathbb{T}^d)} \leq \mathfrak{K}_k e^{-\mathfrak{k}t}$. From (5.8), we deduce that $\left\| \beta^{1,\varepsilon}(.,s) \right\|^2_{L^2(\mathbb{T}^d)} \leq \mathfrak{K}_k e^{-\mathfrak{k}s}$. By recursion, this achieves the proof of the Lemma. □

We also have to control the terms $\widehat{m}^{\ell,\varepsilon}$ for $\ell = 0, 1, \ldots, N_0$.

Lemma 5.4. *For any $\widetilde{\delta} < \delta/2$, the quantity*

$$\varepsilon^{-\widetilde{\delta}} \left| \widehat{m}^{\ell,\varepsilon}(t/\varepsilon^2) - \int_0^\infty \langle \bar{a}(.) \widehat{\beta}_z^\ell(.,s) \rangle ds \right|$$

converges in probability to zero, uniformly in time, where

$$\partial_t \widehat{\beta}^0 = (\bar{a}(z) \widehat{\beta}_z^0)_z, \quad \widehat{\beta}^0(z,0) = \sum_{j=1}^{k} \mathcal{I}_{k-j} \chi^{j-1}(z), \tag{5.18}$$

and for any $\ell \geq 1$

$$\partial_t \widehat{\beta}^\ell = (\bar{a}(z) \widehat{\beta}_z^\ell)_z + \overline{\varphi^\ell}(z,t), \quad \widehat{\beta}^\ell(z,0) = 0. \tag{5.19}$$

Proof. The function $\widehat{\beta}^0$ is well defined and do not depend on ε. Moreover it also satisfies

$$\left\|\widehat{\beta}^0(.,s)\right\|^2_{L^2(\mathbb{T}^d)} \leq \mathfrak{K}_k e^{-\mathfrak{k}s}.$$

We assume that

$$\widetilde{\beta}^{0,\varepsilon}(z,s) = \widehat{\beta}^0(z,s) + \varepsilon^\delta \Psi^{0,\varepsilon}(z,s,\xi_{s/\varepsilon^\delta}) + \mathfrak{R}^{0,\varepsilon}.$$

Then from (5.4) and (5.19) we obtain

$$\begin{aligned} d\widetilde{\beta}^{0,\varepsilon}(z,s) &= (\bar{a}(z)\widehat{\beta}^0_z)_z ds + \varepsilon^\delta \Big[\varepsilon^{-\delta}\mathcal{L}\Psi^{0,\varepsilon} ds + \varepsilon^{-\delta/2}\Psi^{0,\varepsilon}_y q(\xi_{s/\varepsilon^\delta}) dW_s \\ &\qquad + \Psi^{0,\varepsilon}_s ds\Big] + d\mathfrak{R}^{0,\varepsilon} \\ &= (a(z,\xi_{s/\varepsilon^\delta})\widehat{\beta}^0_z)_z ds + \varepsilon^\delta (a(z,\xi_{s/\varepsilon^\delta})\Psi^{0,\varepsilon}_z)_z ds + (a(z,\xi_{s/\varepsilon^\delta})\mathfrak{R}^{0,\varepsilon}_z)_z ds. \end{aligned}$$

If we define $\Psi^{0,\varepsilon}$ by:

$$\mathcal{L}\Psi^{0,\varepsilon} = ((a(z,y) - \bar{a}(z))\widehat{\beta}^0_z)_z,$$

the residual $\mathfrak{R}^{0,\varepsilon}$ satisfies the equation:

$$d\mathfrak{R}^{0,\varepsilon} = (a(z,\xi_{s/\varepsilon^\delta})\mathfrak{R}^{0,\varepsilon}_z)_z + \varepsilon^{\delta/2}\Psi^{0,\varepsilon}_y q(\xi_{s/\varepsilon^\delta}) dW_s + \varepsilon^\delta \mathfrak{B}^{0,\varepsilon}_s ds$$

where $\mathfrak{B}^{0,\varepsilon}$ is bounded. The initial condition is:

$$\mathfrak{R}^{0,\varepsilon}(z,0) = -\varepsilon^\delta \Psi^{0,\varepsilon}.$$

Coming back to (5.5) we have

$$\begin{aligned} m^{0,\varepsilon}(t) - \langle \mathfrak{a}^\varepsilon \widehat{\beta}^0_z(.,t)\rangle &= \langle \mathfrak{a}^\varepsilon \left[\beta^0_z(.,t) - \widehat{\beta}^0_z(.,t)\right]\rangle \\ &= \varepsilon^\delta \langle \mathfrak{a}^\varepsilon \Psi^{0,\varepsilon}_z(.,t,\xi_{t/\varepsilon^\delta})\rangle + \langle \mathfrak{a}^\varepsilon \mathfrak{R}^{0,\varepsilon}_z(.,t,\xi_{t/\varepsilon^\delta})\rangle \end{aligned}$$

Note that $\Psi^{0,\varepsilon}_z$ is bounded in $L^2(\mathbb{T}^d)$ by $\mathfrak{K}_k e^{-\mathfrak{k}t}$ and the quantity $\mathfrak{R}^{0,\varepsilon}_z$ is bounded in any space $\mathbb{L}^p(\Omega)$ by $\varepsilon^{\delta/2}\mathfrak{K}_k e^{-\mathfrak{k}t}$. Hence we deduce that

$$\begin{aligned} &\left|\widehat{m}^{0,\varepsilon}(t) - \int_0^t \langle a\left(.,\xi_{\frac{s}{\varepsilon^\delta}}\right)\widehat{\beta}^0_z(.,s)\rangle ds\right| \\ &\leq \int_0^t \left|m^{0,\varepsilon}(s) - \langle a\left(.,\xi_{\frac{s}{\varepsilon^\delta}}\right)\widehat{\beta}^0_z(.,s)\rangle\right| ds \\ &\leq \lambda\varepsilon^\delta \int_0^t \mathfrak{K}_k e^{-\mathfrak{k}s} ds + \lambda \int_0^t \|\mathfrak{R}^{0,\varepsilon}_z(.,s,\xi_{s/\varepsilon^\delta})\|_{\mathbb{L}^2(\mathbb{T}^d)} ds. \end{aligned}$$

Therefore for any $p \geq 1$, there exists a constant C (independent of ε) such that for any $t \geq 0$

$$\mathbb{E}\left(\left|\widehat{m}^{0,\varepsilon}(t) - \int_0^t \langle a\left(.,\xi_{\frac{s}{\varepsilon^\delta}}\right)\widehat{\beta}^0_z(.,s)\rangle ds\right|^p\right) \leq C\varepsilon^{\delta p/2}.$$

In particular the previous inequality holds when we replace t by t/ε^2. Moreover from the estimate of $\widehat{\beta}^0$, there exists a constant C such that a.s. for

any $\varepsilon > 0$ and $t > 0$

$$\left| \int_0^{t/\varepsilon^2} \langle a\left(.,\xi_{\frac{s}{\varepsilon^\delta}}\right) \widehat{\beta}_z^0(.,s)\rangle ds - \int_0^{+\infty} \langle a\left(.,\xi_{\frac{s}{\varepsilon^\delta}}\right) \widehat{\beta}_z^0(.,s)\rangle ds \right| \leq Ce^{-\mathbb{t}t/\varepsilon^2}$$

Let us consider for a fixed $T > 0$

$$\int_0^T \langle (a\left(.,\xi_{\frac{s}{\varepsilon^\delta}}\right) - \bar{a}(.))\widehat{\beta}_z^0(.,s)\rangle ds = \varepsilon^\delta \int_0^{T/\varepsilon^\delta} \langle (a(.,\xi_s) - \bar{a}(.))\widehat{\beta}_z^0\left(.,\varepsilon^\delta s\right)\rangle ds.$$

Our assumption **(A)** implies that ξ satisfies a strong mixing condition (see [24]). Thus from the ergodic theorem, this quantity converges a.s. to zero (see [18], chapter 4 or [6], chapter 1). Moreover the rate of convergence is of order $\varepsilon^{\delta/2} = \varepsilon^{\alpha/2-1}$, that is for any $\gamma > 0$ the following quantity

$$\begin{aligned} &\varepsilon^{-\delta/2+\gamma} \int_0^T \langle (a\left(.,\xi_{\frac{s}{\varepsilon^\delta}}\right) - \bar{a}(.))\widehat{\beta}_z^0(.,s)\rangle ds \\ &= \varepsilon^\gamma \left[\varepsilon^{\delta/2} \int_0^{T/\varepsilon^\delta} \langle (a(.,\xi_s) - \bar{a}(.))\widehat{\beta}_z^0\left(.,\varepsilon^\delta s\right)\rangle ds \right] \end{aligned}$$

tends to zero in probability as ε goes to zero. Indeed it is a consequence of the central limit theorem (implied by our assumption **(A)** and the mixing property, see [18], chapter 9) together with Slutsky's theorem. To finish the proof we have:

$$\begin{aligned} &\left| \int_0^\infty \langle a\left(.,\xi_{\frac{s}{\varepsilon^\delta}}\right) \widehat{\beta}_z^0(.,s)\rangle ds - \int_0^\infty \langle \bar{a}(.)\widehat{\beta}_z^0(.,s)\rangle ds \right| \\ &\leq \left| \int_0^T \langle (a\left(.,\xi_{\frac{s}{\varepsilon^\delta}}\right) - \bar{a}(.))\widehat{\beta}_z^0(.,s)\rangle ds \right| \\ &\quad + \left| \int_T^\infty \langle (a\left(.,\xi_{\frac{s}{\varepsilon^\delta}}\right) - \bar{a}(.))\widehat{\beta}_z^0(.,s)\rangle ds \right|. \end{aligned}$$

The first part converges a.s. to zero when ε tends to zero (with a rate of convergence of order $\varepsilon^{\delta/2}$ in probability) to a fixed T, whereas the second part converges to zero when T tends to $+\infty$ in any $\mathbb{L}^p(\Omega)$.

Then by recursion we can complete the proof of the lemma. □

We introduce again the constant k in all functions. Since $\mathcal{I}_1 = 0$, gathering all previous Lemmata, we deduce that the expansion (5.3) of $\varrho^{1,\varepsilon}$ can be written:

$$\begin{aligned} \varrho^{1,\varepsilon}(x,t) &= \beta^{1,0,\varepsilon}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \partial_x^1 u^0(x,t) \\ &+ \varepsilon \left[\widehat{m}^{1,0,\varepsilon}\left(\frac{t}{\varepsilon^2}\right) + \beta^{1,1,\varepsilon}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \right] \partial_x^2 u^0(x,t) + \Gamma^{1,1,\varepsilon}(x,t), \end{aligned}$$

where $\Gamma^{1,1,\varepsilon} = \mathcal{O}(\varepsilon^\nu)$ (which means ε^ν times some bounded term) with $\nu > \alpha/2 - 1$ and $\beta^{1,0,\varepsilon}$ and $\beta^{1,1,\varepsilon}$ converge exponentially fast to zero. Now

let us come again to

$$\begin{aligned}\rho^{1,\varepsilon}(x,t) &= -\varepsilon^{1-\alpha/2}\varrho^{1,\varepsilon} \\ &= -\varepsilon^{2-\alpha/2}\widehat{m}^{1,0,\varepsilon}\left(\frac{t}{\varepsilon^2}\right)\partial_x^2 u^0(x,t)+\mathcal{O}(\varepsilon^{\nu+1-\alpha/2}).\end{aligned}$$

We have proved that the remainder $\rho^{1,\varepsilon}$ converges strongly in $L^2(\mathbb{R}^d\times(0,T))$ and in probability to zero.

Now the preceding results imply that the expansion (5.3) of $\varrho^{k,\varepsilon}$

$$\begin{aligned}\varrho^{k,\varepsilon}(x,t) &= \beta^{k,0,\varepsilon}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\partial_x^k u^0(x,t) \\ &+\sum_{\ell=1}^{J_1-k}\varepsilon^\ell\left[\widehat{m}^{k,\ell-1,\varepsilon}\left(\frac{t}{\varepsilon^2}\right)+\beta^{k,\ell,\varepsilon}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\right]\partial_x^{k+\ell}u^0(x,t)+\Gamma^{J_1-k,\varepsilon}(x,t).\end{aligned}$$

can be written:

$$\begin{aligned}\varrho^{\varepsilon}(x,t) &= \mathcal{I}_k\partial_x^k u^0(x,t) \\ &+\sum_{\ell=1}^{J_1-k}\varepsilon^\ell\left[\int_0^\infty\langle\bar{a}(.)\,\widehat{\beta}_z^{k,\ell-1}(.,s)\rangle ds\right]\partial_x^{k+\ell}u^0(x,t)+\mathfrak{r}^{\varepsilon}(x,t).\end{aligned}$$

Again the remainder $\mathfrak{r}^\varepsilon$ converges in $L^2(\mathbb{R}^d\times(0,T))$ and in probability to zero. Denote by

$$\mathfrak{C}_{k,\ell-1}=\int_0^\infty\langle\bar{a}(.)\,\widehat{\beta}_z^{k,\ell-1}(.,s)\rangle ds$$

and note that $\mathfrak{C}_{k,\ell-1}$ depends only on $\mathcal{I}_0=1,\mathcal{I}_1=0,\ldots,\mathcal{I}_{k-1}$. Therefore we obtain, up to some negligible term of order $\mathcal{O}(\varepsilon^\nu)$:

$$\begin{aligned}\rho^{\varepsilon}(x,t) &= -\sum_{k=1}^{J_1}\varepsilon^{k-\alpha/2}\left[\mathcal{I}_k\partial_x^k u^0(x,t)+\sum_{\ell=1}^{J_1-k}\varepsilon^\ell\mathfrak{C}_{k,\ell-1}\partial_x^{k+\ell}u^0(x,t)\right] \\ &= -\sum_{m=2}^{J_1}\varepsilon^{m-\alpha/2}\left[\mathcal{I}_m+\sum_{\ell=1}^{m-1}\mathfrak{C}_{m-\ell,\ell-1}\right]\partial_x^m u^0(x,t).\end{aligned}$$

Then for $k\geq 2$, if we choose

$$\mathcal{I}_k=-\sum_{\ell=1}^{k-1}\mathfrak{C}_{k-\ell,\ell-1}=-\int_0^\infty\langle\bar{a}(.)\,(\sum_{\ell=1}^{k-1}\widehat{\beta}_z^{k-\ell,\ell-1})(.,s)\rangle ds \tag{5.20}$$

provided this sequence is well defined, we deduce that $\rho^\varepsilon(x,t)=\mathcal{O}(\varepsilon^\nu)$.

To complete the proof we need to show that the sequence $\mathcal{I}_k$ for $k\geq 2$ is well-defined. We have

$$\mathcal{I}_k=-\int_0^\infty\langle\bar{a}(.)\,\mathfrak{B}_z^k(.,s)\rangle ds \tag{5.21}$$

with

$$\mathfrak{B}^k = \sum_{\ell=1}^{k-1} \widehat{\beta}^{k-\ell,\ell-1}.$$

Thus $\mathcal{I}_k$ is wellposed if $\mathfrak{B}^k$ only depends on $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_{k-1}$. But for $k = 2$

$$\mathcal{C}_2 = -\mathfrak{C}_{1,0} = -\int_0^\infty \langle \bar{a}\,(.)\,\widehat{\beta}_z^{1,0}\,(.,s)\rangle ds$$

and $\widehat{\beta}^{1,0}$ depends only on χ^0. Then the function $\mathfrak{B}^k$ satisfies the equation

$$\partial_t \mathfrak{B}^k = (\bar{a}(z)\mathfrak{B}_z^k)_z + \mathcal{H}^k(z,t) \tag{5.22}$$

with initial value

$$\mathfrak{B}^k(z,0) = \widehat{\beta}^{k-1,0}(z,0) = \sum_{n=1}^{k-1} \mathcal{I}_{k-1-n}\chi^{n-1}\,(z)$$

and with

$$\mathcal{H}^k(z,t) = \sum_{\ell=1}^{k-1} \overline{\phi^{k-\ell,\ell-1}}(z,t).$$

We can prove by recursion that $\phi^{k-\ell,\ell-1}$ only depends on $\mathcal{I}_2, \dots, \mathcal{I}_{k-\ell-1}$, which leads to the well-posedness on $\mathcal{I}_k$. Finally we obtain:

Proposition 5.5. *There exists a sequence $(\mathcal{I}_k,\ k \geq 2)$ such that the residual ρ^ε converges strongly in $\mathbb{L}^2(\mathbb{R}^d \times (0,T))$ and in probability to zero.*

6. Appendix: proofs of the technical results

Proof of Lemma 3.3. We consider one more auxiliary problem that reads

$$\begin{cases} \partial_t \mathcal{Y}^\varepsilon - \mathcal{A}^\varepsilon \mathcal{Y}^\varepsilon = \operatorname{div}\Big(\Big[a\Big(\dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^\alpha}\Big) - \mathrm{a}^{\mathrm{eff}}\Big]\Xi(x,t)\Big) \\ \mathcal{Y}^\varepsilon(x,0) = 0. \end{cases} \tag{6.1}$$

If the vector function $\Xi \in L^2((0,T)\times\mathbb{R})$, then this problem has a unique solution, and, by the standard energy estimate,

$$\|\mathcal{Y}^\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}))} + \|\partial_t \mathcal{Y}^\varepsilon\|_{L^2(0,T;H^{-1}(\mathbb{R}))} \leq C\|\Xi\|_{L^2((0,T)\times\mathbb{R})}.$$

According to [17, Lemma 1.5.2] the family $\{\mathcal{Y}^\varepsilon\}$ is locally compact in $L^2((0,T)\times\mathbb{R})$. Combining this with Aronson's estimate (see [1]) we conclude that the family $\{\mathcal{Y}^\varepsilon\}$ is compact in $L^2((0,T)\times\mathbb{R})$.

Assume for a while that Ξ is smooth and satisfies estimates (2.11). Multiplying equation (6.1) by a test function of the form $\varphi(x,t)+\varepsilon\chi^0\big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\big)\nabla\varphi(x,t)$

with $\varphi \in C_0^\infty((0,T)\times\mathbb{R})$ and integrating the resulting relation yields

$$
\begin{aligned}
(6.2)\ &-\int_0^T\int_{\mathbb{R}} \mathcal{Y}^\varepsilon\Big(\partial_t\varphi + \varepsilon\chi^0\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\partial_t\nabla\varphi(x,t)\Big)dxdt\\
&-\int_0^T\int_{\mathbb{R}} \mathcal{Y}^\varepsilon\Big(\varepsilon^{1-\alpha}(\mathcal{L}_y\chi^0)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\nabla\varphi\Big)dxdt\\
&-\varepsilon^{1-\alpha/2}\int_0^T\int_{\mathbb{R}} \mathcal{Y}^\varepsilon\Big((\nabla_y\chi^0)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\nabla\varphi\Big)\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dxdB_t\\
&+\int_0^T\int_{\mathbb{R}} (\partial_x\mathcal{Y}^\varepsilon)\,a^\varepsilon\Big[\partial_x\varphi + \big(\partial_x\chi^0\big)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\partial_x\varphi + \varepsilon\chi^0\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\partial_x^2\varphi\Big]dxdt\\
&=\int_0^T\int_{\mathbb{R}} \big[a^\varepsilon - \mathrm{a}^{\mathrm{eff}}\big]\Xi\left[\partial_x\varphi + \big(\partial_x\chi^0\big)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\partial_x\varphi + \varepsilon\chi^0\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\frac{\partial^2\varphi}{\partial x^2}\right]dxdt.
\end{aligned}
$$

Considering (2.2) we obtain

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d} (\partial_x\mathcal{Y}^\varepsilon)\,a^\varepsilon\Big[\partial_x\varphi + \big(\partial_x\chi^0\big)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\partial_x\varphi\Big]dxdt\\
&\qquad=-\int_0^T\int_{\mathbb{R}^d} \mathcal{Y}^\varepsilon\left\{a^\varepsilon\big[\mathbf{I} + \big(\nabla\chi^0\big)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\big]\right\}\frac{\partial^2\varphi}{\partial x^2}dxdt
\end{aligned}
$$

and

$$
\begin{aligned}
(6.3)\ &\int_0^T\int_{\mathbb{R}^d} \big[a^\varepsilon - \mathrm{a}^{\mathrm{eff}}\big]\Xi\big[\partial_x\varphi + \big(\partial_x\chi^0\big)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\partial_x\varphi + \varepsilon\chi^0\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\frac{\partial^2\varphi}{\partial x^2}\big]dxdt\\
&\quad=\int_0^T\int_{\mathbb{R}^d}\left\{a^\varepsilon\big[\mathbf{I} + \big(\nabla\chi^0\big)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\big] - \mathrm{a}^{\mathrm{eff}}\right\}\Xi\partial_x\varphi dxdt\\
&\quad-\int_0^T\int_{\mathbb{R}^d} \mathrm{a}^{\mathrm{eff}}\Xi\big(\partial_x\chi^0\big)\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\partial_x\varphi dxdt\\
&\quad+\varepsilon\int_0^T\int_{\mathbb{R}^d} \big[a^\varepsilon - \mathrm{a}^{\mathrm{eff}}\big]\Xi\chi^0\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\big)\frac{\partial^2\varphi}{\partial x^2}dxdt.
\end{aligned}
$$

From (2.1) and since $\langle\chi^0(\cdot,y)\rangle = 0$, we deduce that all terms in (6.3) tend to zero as $\varepsilon\to 0$. Since $\langle\chi^0(\cdot,y)\rangle = 0$, we have

$$\|(\mathcal{L}_y\chi^0)(x/\varepsilon,\xi_t/\varepsilon^\alpha)\nabla\varphi\|_{L^2(0,T;H^{-1}(\mathbb{R}))} \le C\varepsilon.$$

Therefore, $\varepsilon^{1-\alpha}\int_0^T\int_{\mathbb{R}} \mathcal{Y}^\varepsilon(\mathcal{L}_y\chi^0)\big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\big)\nabla\varphi dxdt$ tends to zero, as $\varepsilon\to 0$.

Denoting by $\mathcal{Y}^0$ the limit of $\mathcal{Y}^\varepsilon$ for a subsequence, coming back to (6.2), we conclude that

$$\int_0^T \int_{\mathbb{R}} \mathcal{Y}^0 \Big(- \partial_t \varphi - \mathrm{a}^{\mathrm{eff}} \frac{\partial^2 \varphi}{\partial x^2} \Big) dx dt = 0.$$

Therefore, $\mathcal{Y}^0 = 0$, and the whole family $\mathcal{Y}^\varepsilon$ a.s. converges to 0 in $L^2((0,T)\times\mathbb{R})$. By the density argument this convergence also holds for any $\Xi \in L^2((0,T)\times\mathbb{R})$. Since $r^\varepsilon_{\mathrm{aux}}$ converges in law in $C((0,T);L^2(\mathbb{R}))$, the solution of problem (3.11) converges to zero in probability in $L^2((0,T)\times\mathbb{R})$, and the statement of the lemma follows. □

Proof of Lemma 3.4. A direct application of the Itô formula implies that

$$\mathcal{S}_1^\varepsilon = M^\varepsilon dW_t + (\mathcal{I}_1^\varepsilon + \mathcal{I}_2^\varepsilon + \mathcal{I}_3^\varepsilon)\, dt \tag{6.4}$$

where

$$\mathcal{I}_2^\varepsilon = \varepsilon^{\delta+1} r^{a,2,\varepsilon} - \varepsilon^{2\delta-1}\mathcal{A}^\varepsilon \Theta^1 - \varepsilon^{2\delta}(a^\varepsilon \Theta^1_{xz} + (a^\varepsilon \Theta^1_x)_z) - \varepsilon^{2\delta}\mathcal{A}^\varepsilon \Psi^1$$

and

$$\begin{aligned}
\mathcal{I}_1^\varepsilon &= \frac{1}{\varepsilon}\mathcal{L}\Phi^1 - \frac{1}{\varepsilon} a^\varepsilon_z u^0_x - \frac{1}{\varepsilon}\mathcal{A}^\varepsilon \phi^1 \\
&+ \sum_{k=0}^{J_1+1} \varepsilon^k v^k_t + \sum_{k=1}^{J_1+2} \varepsilon^k \phi^k_t + \sum_{k=2}^{J_1+2} \varepsilon^{k-2}\mathcal{L}\Phi^k \\
&- \sum_{k=1}^{J_1+1} \varepsilon^{k-1} a^\varepsilon_z v^k_x - \sum_{k=0}^{J_1+1} \varepsilon^k a^\varepsilon v^k_{xx} \\
&- \sum_{k=2}^{J_1+2} \varepsilon^{k-2}\mathcal{A}^\varepsilon \phi^k - \sum_{k=1}^{J_1+1} \varepsilon^{k-1} \Big(a^\varepsilon \phi^k_{xz} + (a^\varepsilon \phi^k_x)_z \Big) - \sum_{k=1}^{J_1+2} \varepsilon^k a^\varepsilon \phi^k_{xx}.
\end{aligned}$$

Thus in $\mathcal{I}_1^\varepsilon$ we have a term of order $1/\varepsilon$:

$$\mathcal{L}\Phi^1 - a^\varepsilon_z u^0_x - \mathcal{A}^\varepsilon \phi^1 = \big[(\mathcal{L}\kappa^0) - a^\varepsilon_z - (\mathcal{A}^\varepsilon \chi^0)\big]\, u^0_x.$$

From the particular choice of χ^0 and κ^0 (see Eq. (2.4) and (2.21)), we have

$$(\mathcal{L}\kappa^0) - a^\varepsilon_z - (\mathcal{A}^\varepsilon \chi^0) = 0,$$

and we cancel this term. From (3.12) and (3.13) the next term in $\mathcal{I}_1^\varepsilon$ of order ε^0 can be expressed as follows:

$$\begin{aligned}
&u^0_t + \mathcal{L}\Phi^2 - a^\varepsilon_z v^1_x - a^\varepsilon u^0_{xx} - \mathcal{A}^\varepsilon \phi^2 - \big(a^\varepsilon \phi^1_{xz} + (a^\varepsilon \phi^1_x)_z\big) \\
&\quad = u^0_t + (\mathcal{L}\kappa^1) u^0_{xx} - (a^\varepsilon + \big(a^\varepsilon \chi^0_z + (a^\varepsilon \chi^0)_z\big)) u^0_{xx} - (\mathcal{A}^\varepsilon \chi^1) u^0_{xx} \\
&\qquad + \big[(\mathcal{L}\kappa^0) - a^\varepsilon_z - (\mathcal{A}^\varepsilon \chi^0)\big]\, v^1_x \\
&\quad = u^0_t - \mathrm{a}^{\mathrm{eff}} u^0_{xx} + \Big[\mathrm{a}^{\mathrm{eff}} - (a^\varepsilon + \big(a^\varepsilon \chi^0_z + (a^\varepsilon \chi^0)_z\big))\Big]\, u^0_{xx} \\
&\qquad + \big[\mathcal{L}\kappa^1 - \mathcal{A}^\varepsilon \chi^1\big]\, u^0_{xx} + \big[(\mathcal{L}\kappa^0) - a^\varepsilon_z - (\mathcal{A}^\varepsilon \chi^0)\big]\, v^1_x \\
&\quad = u^0_t - \mathrm{a}^{\mathrm{eff}} u^0_{xx} + \big[\mathcal{L}\kappa^1 - \mathcal{A}^\varepsilon \chi^1 - f^0\big]\, u^0_{xx} + \big[(\mathcal{L}\kappa^0) - a^\varepsilon_z - (\mathcal{A}^\varepsilon \chi^0)\big]\, v^1_x.
\end{aligned}$$

But with (2.4), (2.21), and (2.24) and the definition of χ^1

$$(\mathcal{L}\kappa^0) - a_z^\varepsilon - (\mathcal{A}^\varepsilon \chi^0) = \mathcal{L}\kappa^1 - \mathcal{A}^\varepsilon \chi^1 - f^0 = 0,$$

and from (1.2) and since $\langle \overline{f^0} \rangle = 0$ this term of order ε^0 is also equal to zero.

In order to understand the recursion let us focus on the term of order ε:

$$v_t^1 + \phi_t^1 + \mathcal{L}\Phi^3 - a_z^\varepsilon v_x^2 - a^\varepsilon v_{xx}^1 - \mathcal{A}^\varepsilon \phi^3 - \left(a^\varepsilon \phi_{xz}^2 + (a^\varepsilon \phi_x^2)_z\right) - a^\varepsilon \phi_{xx}^1.$$

Once again using (3.12) and (3.13) we have:

$$\begin{aligned} & v_t^1 - \mathrm{a}^{\mathrm{eff}} v_{xx}^1 + \left[f^0 + (\mathcal{L}\kappa^1) - (\mathcal{A}^\varepsilon \chi^1)\right] v_{xx}^1 \\ & \quad + \left[(\mathcal{L}\kappa^2) + \mathrm{a}^{\mathrm{eff}} \chi^0 - (\mathcal{A}^\varepsilon \chi^2) - \left(a^\varepsilon \chi_z^1 + (a^\varepsilon \chi^1)_z\right) - a^\varepsilon \chi^0\right] \partial_x^3 u^0 \\ & \quad + \left[(\mathcal{L}\kappa^0) - a_z^\varepsilon - (\mathcal{A}^\varepsilon \chi^0)\right] v_x^2. \end{aligned}$$

From the previous choice made for $\chi^0, \chi^1, \kappa^0, \kappa^1$, we have only:

$$v_t^1 - \mathrm{a}^{\mathrm{eff}} v_{xx}^1 + \left[(\mathcal{L}\kappa^2) - (\mathcal{A}^\varepsilon \chi^2) - f^1\right] \partial_x^3 u^0,$$

with

$$f^1 = \chi^0 (a^\varepsilon - \mathrm{a}^{\mathrm{eff}}) + \left(a^\varepsilon \chi_z^1 + (a^\varepsilon \chi^1)_z\right).$$

Since v^1 satisfies (2.8) for $k = 1$:

$$u_t^1 = \mathrm{a}^{\mathrm{eff}} u_{xx}^1 + \underline{\mathrm{a}}^{1,\mathrm{eff}} \partial_x^3 u^0 = \mathrm{a}^{\mathrm{eff}} u_{xx}^1 + \langle \overline{f^1} \rangle \partial_x^3 u^0,$$

we obtain:

$$\left[(\mathcal{L}\kappa^2) - (\mathcal{A}^\varepsilon \chi^2) - f^1 + \langle \overline{f^1} \rangle\right] \partial_x^3 u^0.$$

From the very definition of χ^2 by (2.20) and κ^2 by (2.24), the term of order ε is also cancelled.

The next $J_1 - 1$ terms of order ε^k, $k = 2, \ldots, J_1$, are of the following form:

$$v_t^k + \phi_t^k + \mathcal{L}\Phi^{k+2} - a_z^\varepsilon v_x^{k+1} - a^\varepsilon v_{xx}^k - \mathcal{A}^\varepsilon \phi^{k+2} - \left(a^\varepsilon \phi_{xz}^{k+1} + (a^\varepsilon \phi_x^{k+1})_z\right) - a^\varepsilon \phi_{xx}^k.$$

Using (3.12) and (3.13) and the definition of f^k given by (2.18), we obtain the following equation:

$$\begin{aligned} & \left[(\mathcal{L}\kappa^0) - a_z^\varepsilon - (\mathcal{A}^\varepsilon \chi^0)\right] v_x^{k+1} \\ & \quad + \left[(\mathcal{L}\kappa^1) - (\mathcal{A}^\varepsilon \chi^1) - f^0\right] v_{xx}^k + v_t^k - \mathrm{a}^{\mathrm{eff}} v_{xx}^k \\ & \quad + \sum_{m=0}^{k-1} \left[(\mathcal{L}\kappa^{k+1-m}) - (\mathcal{A}^\varepsilon \chi^{k+1-m}) - f^{k-m}\right] \partial_x^{k+2-m} v^m \\ & \quad + \sum_{m=0}^{k-1} \chi^{k-m-1} \partial_x^{k-m} (v_t^m - \mathrm{a}^{\mathrm{eff}} v_{xx}^m). \end{aligned}$$

From the definition of v^m (Eq. (2.8)), the previous expression becomes:

$$\begin{aligned}
&v_t^k - \mathrm{a}^{\mathrm{eff}} v_{xx}^k \\
&\quad + \sum_{m=0}^{k-1} \left[(\mathcal{L}\kappa^{k+1-m}) - (\mathcal{A}^\varepsilon \chi^{k+1-m}) - f^{k-m} \right] \partial_x^{k+2-m} v^m \\
&\quad + \sum_{m=0}^{k-1} \chi^{k-m-1} \partial_x^{k-m} \left(\sum_{\ell=0}^{m-1} \underline{\mathrm{a}}^{m-\ell,\mathrm{eff}} (\partial_x^{m-\ell+2} v^\ell) \right) \\
&= v_t^k - \mathrm{a}^{\mathrm{eff}} v_{xx}^k \\
&\quad + \sum_{m=0}^{k-1} \left[(\mathcal{L}\kappa^{k+1-m}) - (\mathcal{A}^\varepsilon \chi^{k+1-m}) - f^{k-m} \right] \partial_x^{k+2-m} v^m \\
&\quad + \sum_{\ell=0}^{k-1} \left(\sum_{m=\ell+1}^{k-1} \chi^{k-m-1} \underline{\mathrm{a}}^{m-\ell,\mathrm{eff}} \right) \partial_x^{k+2-\ell} v^\ell \\
&= v_t^k - \mathrm{a}^{\mathrm{eff}} v_{xx}^k - \sum_{m=0}^{k-1} \underline{\mathrm{a}}^{k-m,\mathrm{eff}} (\partial_x^{k-m+2} v^m) + \sum_{m=1}^{k} \eta^{k,k-m} \partial_x^{m+2} v^{k-m}.
\end{aligned}$$

where for $m = 1, \dots, k$

$$\eta^{k,k-m} = (\mathcal{L}\kappa^{1+m}) - (\mathcal{A}^\varepsilon \chi^{1+m}) - (f^m - \langle \overline{f^m} \rangle) + \left(\sum_{j=1}^{m-1} \chi^j \underline{\mathrm{a}}^{m-j,\mathrm{eff}} \right).$$

Since $\langle \chi^k \rangle = 0$, using the expressions of $\underline{\mathrm{a}}^{k,\mathrm{eff}}$, χ^k and κ^k (Eq. (2.19), (2.20) and (2.24)), all terms $\eta^{k,k-m}$ are equal to zero. Then in $\mathcal{I}_1^\varepsilon$ it remains only a term of order ε^{J_1+1}.

Coming back to Itô's formula in (6.4), in the last part $\mathcal{I}_3^\varepsilon$, there is a term of order $\varepsilon^{\delta-1}$ given by:

$$\mathcal{L}\Theta^1 - a_z^\varepsilon u_x^1 - \mathcal{A}^\varepsilon \Phi^1 - \mathcal{A}^\varepsilon \theta^1 = \mathcal{L}\Theta^1 - a_z^\varepsilon u_x^1 - (\mathcal{A}^\varepsilon \kappa^0) u_x^0 - \mathcal{A}^\varepsilon \theta^1.$$

From (3.14) and (3.15) with $k = 1$, it disappears:

$$\begin{aligned}
\mathcal{L}\Theta^1 &= (\mathcal{L}\gamma^0) u_x^0 + (\mathcal{L}\kappa^0) u_x^1 \\
&= (\mathcal{A} - \overline{\mathcal{A}}) \tau^0 u_x^0 + (\mathcal{A}\kappa^0 - \overline{\mathcal{A}\kappa^0}) u_x^0 + (a_z - \bar{a}_z) u_x^1 + (\mathcal{A} - \bar{\mathcal{A}}) \chi^0 u_x^1 \\
&= \mathcal{A}\theta^1 + (\mathcal{A}\kappa^0) u_x^0 + a_z u_x^1.
\end{aligned}$$

Then using the definition of f^0 and (3.13) for Φ^1 and Φ^2 we have a term of order ε^δ in (6.4):

$$\begin{aligned}
u_t^1 - \mathrm{a}^{\mathrm{eff}} u_{xx}^1 - \langle \overline{g^0} \rangle u_{xx}^0 - f^0 u_{xx}^1 + \mathcal{L}\Psi^1 - \mathcal{A}^\varepsilon \psi^1 \\
-(\mathcal{A}^\varepsilon \kappa^0) u_x^1 - (\mathcal{A}^\varepsilon \kappa^1) u_{xx}^0 - (g^0 - \langle \overline{g^0} \rangle) u_{xx}^0.
\end{aligned}$$

Since

$$\mathcal{L}\zeta^1 - \mathcal{A}^\varepsilon \eta^1 - (\mathcal{A}\kappa^1) - (g^0 - \langle \overline{g^0} \rangle) = 0,$$

the particular choice (3.16) and (3.17) with $k=1$, together with the expression of u^1 given by (2.5), implies that in (3.19), there is only one absolutely continuous term of order ε^δ, namely $\varepsilon^\delta w^1$. The other terms in $r^{a,\varepsilon,2}$ come directly from the Itô formula. This achieves the proof of the lemma. □

Proof of Lemma 3.5. Again we apply Itô's formula to $\mathcal{E}_2^\varepsilon$ and we obtain a non martingale part of the form:

$$\begin{aligned}
\varepsilon^{\delta+1} r^{a,3,\varepsilon} &+ \sum_{k=2}^{N_0} \varepsilon^{k\delta} u_t^k + \sum_{k=2}^{N_0} \varepsilon^{k\delta-1} \mathcal{L}\Theta^k + \sum_{k=2}^{N_0} \varepsilon^{k\delta} \mathcal{L}\Psi^k \\
&- \sum_{k=2}^{N_0} \varepsilon^{k\delta-1} a_z^\varepsilon u_x^k - \sum_{k=2}^{N_0} \varepsilon^{k\delta} a^\varepsilon u_{xx}^k - \sum_{k=2}^{N_0} \varepsilon^{k\delta-1} \mathcal{A}^\varepsilon \theta^k \\
&- \sum_{k=2}^{N_0} \varepsilon^{k\delta} (a^\varepsilon \theta_{xz}^k + (a^\varepsilon \theta_x^k)_z) \\
&- \sum_{k=1}^{N_0} \left[\varepsilon^{(k+1)\delta-1} \mathcal{A}^\varepsilon \Theta^k + \varepsilon^{(k+1)\delta} (a^\varepsilon \Theta_{xz}^k + (a^\varepsilon \Theta_x^k)_z) \right] \\
&- \sum_{k=1}^{N_0} \varepsilon^{(k+1)\delta} \mathcal{A}^\varepsilon \Psi^k - \sum_{k=2}^{N_0} \varepsilon^{k\delta} \mathcal{A}^\varepsilon \psi^k.
\end{aligned}$$

where

$$\begin{aligned}
r^{a,3,\varepsilon} &= \sum_{k=2}^{N_0} \varepsilon^{(k-1)\delta} \theta_t^k + \sum_{k=2}^{N_0} \varepsilon^{k\delta} \Theta_t^k + \sum_{k=2}^{N_0} \varepsilon^{k\delta+1} \Psi_t^k \\
&+ \sum_{k=2}^{N_0} \varepsilon^{(k-1)\delta+1} \psi_t^k - \sum_{k=2}^{N_0} \varepsilon^{(k-1)\delta} a^\varepsilon \theta_{xx}^k - \sum_{k=2}^{N_0} \varepsilon^{k\delta} a^\varepsilon \Theta_{xx}^k \\
&- \sum_{k=2}^{N_0} \left[\varepsilon^{k\delta} (a^\varepsilon \Psi_{xz}^k + (a^\varepsilon \Psi_x^k)_z) + \varepsilon^{k\delta+1} a^\varepsilon \Psi_{xx}^k \right] \\
&- \sum_{k=2}^{N_0} \left[\varepsilon^{(k-1)\delta} (a^\varepsilon \psi_{xz}^k + (a^\varepsilon \psi_x^k)_z) + \varepsilon^{(k-1)\delta+1} a^\varepsilon \psi_{xx}^k \right].
\end{aligned} \tag{6.5}$$

If we compare the terms of the same order, we have

- for $\varepsilon^{k\delta}$, $k=2,\dots,N_0$:

$$\begin{aligned}
u_t^k + \mathcal{L}\Psi^k - a^\varepsilon u_{xx}^k - (a^\varepsilon \theta_{xz}^k + (a^\varepsilon \theta_x^k)_z) \\
-(a^\varepsilon \Theta_{xz}^{k-1} + (a^\varepsilon \Theta_x^{k-1})_z) - \mathcal{A}^\varepsilon \Psi^{k-1} - \mathcal{A}^\varepsilon \psi^k,
\end{aligned} \tag{6.6}$$

- for $\varepsilon^{k\delta-1}$, $k=2,\dots,2N_1+2$:

$$\mathcal{L}\Theta^k - a_z^\varepsilon u_x^k - \mathcal{A}^\varepsilon \theta^k - \mathcal{A}^\varepsilon \Theta^{k-1}.$$

Remember that $\mathcal{L}\kappa^0 - a_z - \mathcal{A}\chi^0 = 0$. Moreover from (3.14) and (3.15) together with (2.28) and (2.30) we obtain

$$\begin{aligned}\overline{\mathcal{A}}\hat{\theta}^k &= -\overline{\mathcal{A}\Theta^{k-1}}\\ \mathcal{L}\hat{\Theta}^k &= (\mathcal{A}-\overline{\mathcal{A}})\hat{\theta}^k + \left(\mathcal{A}\Theta^{k-1} - \overline{\mathcal{A}\Theta^{k-1}}\right)\end{aligned}$$

and therefore for any k

$$\mathcal{L}\Theta^k - a_z^\varepsilon u_x^k - \mathcal{A}^\varepsilon\theta^k - \mathcal{A}^\varepsilon\Theta^{k-1} = 0.$$

Moreover the expression (6.6) becomes:

$$\begin{aligned}(6.7)\qquad & u_t^k - \mathrm{a}^{\mathrm{eff}}u_{xx}^k - f^1 u_{xx}^k + \mathcal{L}\Psi^k - \mathcal{A}^\varepsilon\psi^k\\ & -(a^\varepsilon\hat{\theta}_{xz}^k + (a^\varepsilon\hat{\theta}_x^k)_z) - (a^\varepsilon\Theta_{xz}^{k-1} + (a^\varepsilon\Theta_x^{k-1})_z) - \mathcal{A}^\varepsilon\Psi^{k-1}.\end{aligned}$$

From (3.16) and (3.17), together with the definition of χ^1, (2.24) for $k=2$, Equation (6.7) becomes

$$\begin{aligned}(6.8)\quad & v_t^k - \mathrm{a}^{\mathrm{eff}}v_{xx}^k - \overline{\langle T^k\rangle} + \mathcal{L}\hat{\Psi}^k - \mathcal{A}^\varepsilon\hat{\psi}^k\\ & +\overline{\langle T^k\rangle} - (a^\varepsilon\hat{\theta}_{xz}^k + (a^\varepsilon\hat{\theta}_x^k)_z) - (a^\varepsilon\Theta_{xz}^{k-1} + (a^\varepsilon\Theta_x^{k-1})_z) - \mathcal{A}^\varepsilon\Psi^{k-1},\end{aligned}$$

where

$$(6.9)\qquad T^k = a\hat{\theta}_{xz}^k + a\Theta_{xz}^{k-1}.$$

Now remark that

$$\begin{aligned}a\hat{\theta}_{xz}^k + a\Theta_{xz}^{k-1} &= a\left[\tau^k(z)u_x^0 + \sum_{n=1}^{k-1}\tau^{k-n}(z)u_x^n\right]_{xz}\\ &+ a\left[\gamma^{k-1}(z,y)u_x^0 + \sum_{n=1}^{k-2}\gamma^{k-1-n}(z,y)u_x^n\right]_{xz} + \kappa_z^1(z,y)u_{xx}^{k-1}\\ &= \sum_{n=0}^{k-2} a\left[\tau^{k-n}(z) + \gamma^{k-1-n}(z,y)\right]_z u_{xx}^n + a(\tau^1+\kappa^1)_z u_{xx}^{k-1}\\ &= \sum_{n=0}^{k-2} g^{k-n}(z,y)u_{xx}^n + a(\tau^1+\kappa^1)_z u_{xx}^{k-1}\end{aligned}$$

and

$$(a\hat{\theta}_x^k + a\Theta_x^{k-1})_z = \sum_{n=0}^{k-2} h^{k-n}(z,y)u_{xx}^n + (a(\tau^1+\kappa^1))_z u_{xx}^{k-1}$$

where by convention $v^0 = u^0$ and g^k and h^k are defined by (2.32). Moreover

$$\overline{\langle a\hat{\theta}_{xz}^k + a\Theta_{xz}^{k-1}\rangle} = \mathrm{a}^{k,\mathrm{eff}}u_{xx}^0 + \sum_{n=1}^{k-1}\mathrm{a}^{k-n,\mathrm{eff}}u_{xx}^n.$$

From (2.35) and (2.36), we can check that

$$\begin{aligned}\overline{\mathcal{A}}\hat{\psi}^k &= -\overline{\mathcal{A}\Psi^{k-1}} - \overline{(a(\hat{\theta}_x^k + \Theta_x^{k-1}))_z} - (\overline{T^k} - \overline{\langle T^k\rangle})\\ \mathcal{L}\hat{\Psi}^k &= (\mathcal{A}-\overline{\mathcal{A}})\hat{\psi}^k + \mathcal{A}\Psi^{k-1} - \overline{\mathcal{A}\Psi^{k-1}}\\ &\quad +(a(\hat{\theta}_x^k + \Theta_x^{k-1}))_z - \overline{(a(\hat{\theta}_x^k + \Theta_x^{k-1}))_z} + T^k - \overline{T^k}.\end{aligned}$$

Thereby, using Equation (2.5) for u^k, Expression (6.8) is equal to w^k. There is still another remaining term of order $\varepsilon^{(N_0+1)\delta-1}$:

$$r^{a,4,\varepsilon} = -\mathcal{A}^\varepsilon\Theta^{N_0} - \varepsilon(a^\varepsilon\Theta_{xz}^{N_0} + (a^\varepsilon\Theta_x^{N_0})_z + \mathcal{A}^\varepsilon\Psi^{N_0}). \tag{6.10}$$

which achieves the proof. □

Proof of Lemma 4.4. We apply Itô's formula to $\mathcal{V}^\varepsilon$:

$$d\mathcal{V}_t^\varepsilon = (\mathcal{QF})_t^\varepsilon dt + (\mathcal{LT})_t^\varepsilon dt + \mathcal{B}_t^\varepsilon dt + M_t^\varepsilon\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_t.$$

Let us now detail the exact expression on all terms. From time to time we omit the variables in the scalar product to lighten the notations. The stochastic integrand M^ε is given by:

$$\begin{aligned}M_t^\varepsilon &= \sum_{k=0}^{K_0}\varepsilon^{(k+1)\delta}\varepsilon^{-\alpha/2}\left\langle\!\!\left\langle Q_y^k\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)v^\varepsilon(.,t),v^\varepsilon(.,t)\right\rangle\!\!\right\rangle\\ &+2\varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{m=0}^{K_0}\varepsilon^{(k+m)\delta}\left\langle\!\!\left\langle P^k\left(\frac{\cdot}{\varepsilon}\right)v^\varepsilon(.,t),\widetilde{\mathcal{G}}^m\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^\alpha},\cdot,t\right)\right\rangle\!\!\right\rangle\\ &+2\varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{m=0}^{K_0}\varepsilon^{(k+\ell+1)\delta}\left\langle\!\!\left\langle Q^k\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)v^\varepsilon(.,t),\widetilde{\mathcal{G}}^m\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^\alpha},\cdot,t\right)\right\rangle\!\!\right\rangle\\ &+\varepsilon^{2\varpi}\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta+\alpha/2}\sum_{m=0}^{K_0}\left\langle\!\!\left\langle U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}),\widetilde{\mathcal{G}}^m\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^\alpha},.,t\right)\right\rangle\!\!\right\rangle\\ &+\varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta+\alpha/2}\varepsilon^{-\alpha/2}\left\langle\!\!\left\langle U_y^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}),v^\varepsilon(.,t)\right\rangle\!\!\right\rangle.\end{aligned} \tag{6.11}$$

The term $\mathcal{B}^\varepsilon$ does not depend on v^ε:

$$
\begin{aligned}
(6.12)\quad \mathcal{B}^\varepsilon_t
&= \varepsilon^{2\varpi} \sum_{k,m,\ell=0}^{K_0} \varepsilon^{(k+m+\ell)\delta} \Big[\Big\langle\!\Big\langle P^k\left(\tfrac{\cdot}{\varepsilon}\right) \widetilde{\mathcal{G}}^m\left(\tfrac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, ., t\right), \widetilde{\mathcal{G}}^\ell\left(\tfrac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, ., t\right) \Big\rangle\!\Big\rangle \\
&\qquad + \varepsilon^\delta \Big\langle\!\Big\langle Q^k\left(\tfrac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right) \widetilde{\mathcal{G}}^m\left(\tfrac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, ., t\right), \widetilde{\mathcal{G}}^\ell\left(\tfrac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, ., t\right) \Big\rangle\!\Big\rangle \Big] \\
&\quad + \varepsilon^{2\varpi} \sum_{k,\ell,m=0}^{K_0} \varepsilon^{(k+\ell+m+1)\delta+\alpha/2} \Big\langle\!\Big\langle U^{k,\ell}(., t, \xi_{t/\varepsilon^\alpha}), \widetilde{\mathcal{G}}^m\left(\tfrac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, ., t\right) \Big\rangle\!\Big\rangle \\
&\quad + \varepsilon^{\varpi} \sum_{k,\ell=0}^{K_0} \varepsilon^{(k+\ell+1)\delta} \sum_{m=1}^{K_0} \varepsilon^{m\delta} \Big\langle\!\Big\langle U^{k,\ell}(., t, \xi_{t/\varepsilon^\alpha}), \widetilde{w}^m(., t) \Big\rangle\!\Big\rangle.
\end{aligned}
$$

This term contains only positive powers of ε. Now we detail the quadratic form $(\mathcal{Q}\mathcal{F})^\varepsilon$:

$$
\begin{aligned}
(\mathcal{Q}\mathcal{F})^\varepsilon_t &= \sum_{k=0}^{K_0} \varepsilon^{k\delta} \Big[2 \Big\langle\!\Big\langle P^k\left(\tfrac{\cdot}{\varepsilon}\right) v^\varepsilon(., t), a^\varepsilon v^\varepsilon_{xx}(., t) \Big\rangle\!\Big\rangle \\
&\qquad + \varepsilon^\delta \Big\langle\!\Big\langle \varepsilon^{-\alpha} \mathcal{L} Q^k\left(\tfrac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right) v^\varepsilon(., t), v^\varepsilon(., t) \Big\rangle\!\Big\rangle \\
&\qquad + \varepsilon^\delta 2 \Big\langle\!\Big\langle Q^k\left(\tfrac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right) v^\varepsilon(., t), a^\varepsilon v^\varepsilon_{xx}(., t) \Big\rangle\!\Big\rangle \Big].
\end{aligned}
$$

Then by integration by parts

$$
\begin{aligned}
\langle\!\langle P^k v^\varepsilon, a^\varepsilon v^\varepsilon_{xx} \rangle\!\rangle &= -\langle\!\langle P^k a^\varepsilon v^\varepsilon_x, v^\varepsilon_x \rangle\!\rangle - \langle\!\langle (P^k a^\varepsilon)_x v^\varepsilon, v^\varepsilon_x \rangle\!\rangle \\
&= -\langle\!\langle P^k a^\varepsilon v^\varepsilon_x, v^\varepsilon_x \rangle\!\rangle - \frac{1}{2} \langle\!\langle (P^k a^\varepsilon)_x, ((v^\varepsilon)^2)_x \rangle\!\rangle \\
&= -\langle\!\langle P^k a^\varepsilon v^\varepsilon_x, v^\varepsilon_x \rangle\!\rangle + \frac{1}{2} \langle\!\langle (P^k a^\varepsilon)_{xx}, (v^\varepsilon)^2 \rangle\!\rangle \\
&= -\langle\!\langle P^k a^\varepsilon v^\varepsilon_x, v^\varepsilon_x \rangle\!\rangle + \frac{1}{2\varepsilon^2} \langle\!\langle (P^k a^\varepsilon)_{zz}, (v^\varepsilon)^2 \rangle\!\rangle.
\end{aligned}
$$

The same equalities hold for Q^k. Then

$$
\begin{aligned}
(\mathcal{Q}\mathcal{F})^\varepsilon_t &= -2 \sum_{k=0}^{K_0} \varepsilon^{k\delta} \Big[\langle\!\langle P^k a^\varepsilon v^\varepsilon_x, v^\varepsilon_x \rangle\!\rangle + \varepsilon^\delta \langle\!\langle Q^k a^\varepsilon v^\varepsilon_x, v^\varepsilon_x \rangle\!\rangle \Big] \\
&+ \frac{1}{\varepsilon^2} \left[\langle\!\langle (P^0 a^\varepsilon)_{zz}, (v^\varepsilon)^2 \rangle\!\rangle + \langle\!\langle \mathcal{L} Q^0 v^\varepsilon, v^\varepsilon \rangle\!\rangle \right] + \varepsilon^{(K_0+1)\delta-2} \langle\!\langle (Q^{K_0} a^\varepsilon)_{zz}, (v^\varepsilon)^2 \rangle\!\rangle \\
&+ \frac{1}{\varepsilon^2} \sum_{k=1}^{K_0} \varepsilon^{k\delta} \Big[\langle\!\langle (P^k a^\varepsilon)_{zz}, (v^\varepsilon)^2 \rangle\!\rangle + \langle\!\langle (Q^{k-1} a^\varepsilon)_{zz}, (v^\varepsilon)^2 \rangle\!\rangle + \langle\!\langle \mathcal{L} Q^k v^\varepsilon, v^\varepsilon \rangle\!\rangle \Big].
\end{aligned}
$$

From the very definition (4.4), (4.5) and (4.6) of P^0, Q^0, P^k and Q^k, the quadratic form is reduced to

$$\begin{aligned}(\mathcal{QF})_t^\varepsilon &= -2\sum_{k=0}^{K_0}\varepsilon^{k\delta}\left[\langle\!\langle P^k a^\varepsilon v_x^\varepsilon, v_x^\varepsilon\rangle\!\rangle + \varepsilon^\delta\langle\!\langle Q^k v_x^\varepsilon, a^\varepsilon v_x^\varepsilon\rangle\!\rangle\right]\\ &+ \varepsilon^{(K_0+1)\delta-2}\langle\!\langle (Q^{K_0}a^\varepsilon)_{zz}, (v^\varepsilon)^2\rangle\!\rangle.\end{aligned}$$

Finally we study the linear part:

$$\begin{aligned}(\mathcal{LT})_t^\varepsilon &= 2\sum_{k=0}^{K_0}\varepsilon^{k\delta}\sum_{m=1}^{K_0}\varepsilon^{m\delta-\alpha/2}\langle\!\langle P^k v^\varepsilon(.,t), \widetilde{w}^m(.,t)\rangle\!\rangle\\ &+ 2\varepsilon^{\varpi}\sum_{k=0}^{J_0}\sum_{m=0}^{K_0}\varepsilon^{(k+m+1)\delta-\alpha/2}\left\langle\!\!\left\langle Q_y^k v^\varepsilon(.,t), \widetilde{\mathcal{G}}^m\left(\frac{.}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, ., t\right)\right\rangle\!\!\right\rangle\sigma(\xi_{t/\varepsilon^\alpha})\\ &+ \varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta-\alpha/2}\langle\!\langle \mathcal{L}U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}), v^\varepsilon(.,t)\rangle\!\rangle\\ &+ \sum_{k=0}^{K_0}\varepsilon^{(k+1)\delta}\sum_{m=1}^{K_0}\varepsilon^{m\delta-\alpha/2}\langle\!\langle Q^k v^\varepsilon(.,t), \widetilde{w}^m(.,t)\rangle\!\rangle\\ &+ \varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta+\alpha/2}\langle\!\langle U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}), a^\varepsilon v_{xx}^\varepsilon(.,t)\rangle\!\rangle\\ &+ \varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta+\alpha/2}\left\langle\!\!\left\langle U_t^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}), v^\varepsilon(.,t)\right\rangle\!\!\right\rangle.\end{aligned}$$

The last three double sums contains only positive powers of ε. Indeed

$$\langle\!\langle Q^k v^\varepsilon, \widetilde{w}^m(.,t)\rangle\!\rangle = \varepsilon\langle\!\langle \widetilde{Q}_x^k v^\varepsilon, \widetilde{w}^m(.,t)\rangle\!\rangle,$$

and $\delta - \alpha/2 + 1 \geq |1-\alpha/2| > 0$. For the last two sums we can integrate by parts:

$$\begin{aligned}\langle\!\langle U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}), a^\varepsilon v_{xx}^\varepsilon(.,t)\rangle\!\rangle &= -\langle\!\langle U_x^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}), a^\varepsilon v_x^\varepsilon(.,t)\rangle\!\rangle\\ &- \frac{1}{\varepsilon}\langle\!\langle U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}), a_z^\varepsilon v_x^\varepsilon(.,t)\rangle\!\rangle\end{aligned}$$

and thus

$$\begin{aligned}&\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta+\alpha/2}\left[\left\langle\!\!\left\langle U_t^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}), v^\varepsilon(.,t)\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}), a^\varepsilon v_{xx}^\varepsilon(.,t)\right\rangle\!\!\right\rangle\right]\\ &= \sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta+\alpha/2}\left\langle\!\!\left\langle U_t^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}), v^\varepsilon(.,t)\right\rangle\!\!\right\rangle\\ &\quad - \sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta+\alpha/2}\left\langle\!\!\left\langle U_x^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha})a^\varepsilon + \frac{1}{\varepsilon}U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha})a_z^\varepsilon, v_x^\varepsilon(.,t)\right\rangle\!\!\right\rangle\end{aligned}$$

with $\delta+\alpha/2-1=\delta/2>0$ for $\alpha<2$ and $\delta+\alpha/2-1=3\delta/2>0$ for $\alpha>2$. Then we have to control

$$\sum_{k=0}^{K_0}\sum_{m=0}^{K_0}2\varepsilon^{(k+m+1)\delta-\alpha/2}\Big\langle\!\Big\langle Q_y^k\widetilde{\mathcal{G}}^m\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^\alpha},.,t\right),v^\varepsilon(.,t)\Big\rangle\!\Big\rangle\sigma(\xi_{t/\varepsilon^\alpha})$$
$$+\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta-\alpha/2}\langle\!\langle\mathcal{L}U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}),v^\varepsilon(.,t)\rangle\!\rangle$$

We define

$$L^{k,m}(z,y,x,t)=Q_y^k(z,y)\,\widetilde{\mathcal{G}}^m(z,y,x,t)-\langle Q_y^k(.,y)\,\widetilde{\mathcal{G}}^m(.,y,x,t)\rangle.$$

The mean value w.r.t. z of this function is zero. Hence we can define $\widetilde{L}^{k,m}$ such that

$$\partial_z\widetilde{L}^{k,m}=L^{k,m}.$$

Thus

$$\begin{aligned}
&\Big\langle\!\Big\langle Q_y^k\widetilde{\mathcal{G}}^m\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right),v^\varepsilon\Big\rangle\!\Big\rangle\sigma(\xi_{t/\varepsilon^\alpha})\\
&=\langle\!\langle L^{k,m},v^\varepsilon\rangle\!\rangle\sigma(\xi_{t/\varepsilon^\alpha})+\Big\langle\!\Big\langle\langle Q_y^k\left(.,\xi_{t/\varepsilon^\alpha}\right)\widetilde{\mathcal{G}}^m\left(.,\xi_{t/\varepsilon^\alpha},.,t\right)\rangle,v^\varepsilon\Big\rangle\!\Big\rangle\sigma(\xi_{t/\varepsilon^\alpha})\\
&=\varepsilon\Big\langle\!\Big\langle\widetilde{L}_x^{k,m},v^\varepsilon\Big\rangle\!\Big\rangle\sigma(\xi_{t/\varepsilon^\alpha})+\Big\langle\!\Big\langle\langle Q_y^k\left(.,\xi_{t/\varepsilon^\alpha}\right)\widetilde{\mathcal{G}}^m\left(.,\xi_{t/\varepsilon^\alpha},.,t\right)\rangle\sigma(\xi_{t/\varepsilon^\alpha}),v^\varepsilon\Big\rangle\!\Big\rangle\\
&=\varepsilon\Big\langle\!\Big\langle\widetilde{L}_x^{k,m},v^\varepsilon\Big\rangle\!\Big\rangle\sigma(\xi_{t/\varepsilon^\alpha})+\langle\!\langle\Xi^{k,m}(.,t),v^\varepsilon\rangle\!\rangle\\
&\quad+\Big\langle\!\Big\langle\langle Q_y^k\left(.,\xi_{t/\varepsilon^\alpha}\right)\widetilde{\mathcal{G}}^m\left(.,\xi_{t/\varepsilon^\alpha},.,t\right)\rangle\sigma(\xi_{t/\varepsilon^\alpha})-\Xi^{k,m}(.,t),v^\varepsilon\Big\rangle\!\Big\rangle,
\end{aligned}$$

where $\Xi^{k,m}$ is defined by (4.7). From the definition (4.8) of $U^{k,\ell}$:

$$\begin{aligned}
&2\sum_{k=0}^{K_0}\sum_{m=0}^{K_0}\varepsilon^{(k+m+1)\delta-\alpha/2}\Big\langle\!\Big\langle Q_y^k\widetilde{\mathcal{G}}^m\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right),v^\varepsilon\Big\rangle\!\Big\rangle\sigma(\xi_{t/\varepsilon^\alpha})\\
&\quad+\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta-\alpha/2}\Big\langle\!\Big\langle\mathcal{L}U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}),v^\varepsilon(.,t)\Big\rangle\!\Big\rangle\\
&=2\sum_{k=0}^{K_0}\sum_{m=0}^{K_0}\varepsilon^{(k+m+1)\delta-\alpha/2+1}\Big\langle\!\Big\langle\widetilde{L}_x^{k,m},v^\varepsilon\Big\rangle\!\Big\rangle\sigma(\xi_{t/\varepsilon^\alpha})\\
&\quad+2\sum_{k=0}^{K_0}\sum_{m=0}^{K_0}\varepsilon^{(k+m+1)\delta-\alpha/2}\langle\!\langle\Xi^{k,m}(x,t),v^\varepsilon\rangle\!\rangle.
\end{aligned}$$

For the term

$$2\sum_{k=0}^{K_0}\varepsilon^{k\delta}\sum_{m=1}^{K_0}\varepsilon^{m\delta-\alpha/2}\langle\!\langle P^kv^\varepsilon,\widetilde{w}^m(.,t)\rangle\!\rangle$$

we decompose it:

$$
\begin{aligned}
&2\sum_{k=0}^{K_0}\sum_{m=1}^{K_0}\varepsilon^{(k+m)\delta-\alpha/2}\langle\!\langle (P^k-\langle P^k\rangle)v^\varepsilon,\widetilde{w}^m(.,t)\rangle\!\rangle\\
&\quad+2\sum_{k=0}^{K_0}\sum_{m=1}^{K_0}\varepsilon^{(k+m)\delta-\alpha/2}\langle\!\langle v^\varepsilon,\widetilde{w}^m(.,t)\rangle\!\rangle\\
&=2\sum_{k=0}^{K_0}\sum_{m=1}^{K_0}\varepsilon^{(k+m)\delta+1-\alpha/2}\langle\!\langle \widetilde{P}_x^k,v^\varepsilon\widetilde{w}^m\rangle\!\rangle\\
&\quad+2\sum_{k=0}^{K_0}\sum_{m=1}^{K_0}\varepsilon^{(k+m)\delta-\alpha/2}\langle\!\langle v^\varepsilon,\widetilde{w}^m(.,t)\rangle\!\rangle,
\end{aligned}
$$

where $\widetilde{P}_z^k=P^k-\langle P^k\rangle$ and recall that $\langle P^k\rangle=1$. Hence gathering all terms we obtain

$$
\begin{aligned}
(\mathcal{LT})_t^\varepsilon &= 2\sum_{k=0}^{K_0}\sum_{m=0}^{K_0-1}\varepsilon^{(k+m+1)\delta-\alpha/2}\langle\!\langle v^\varepsilon(.,t),\widetilde{w}^{m+1}(.,t)+\varepsilon^{\varpi}\Xi^{k,m}(.,t)\rangle\!\rangle\\
&+ \langle\!\langle \mathcal{NT}^{1,\varepsilon}(.,t),v^\varepsilon(.,t)\rangle\!\rangle-\langle\!\langle \mathcal{NT}^{2,\varepsilon}(.,t),v_x^\varepsilon(.,t)\rangle\!\rangle
\end{aligned}
$$

where

$$
\begin{aligned}
(6.13)\ \mathcal{NT}^{1,\varepsilon}(.,t) &= 2\varepsilon^{\varpi}\sum_{k=0}^{K_0}\varepsilon^{(k+K_0+1)\delta-\alpha/2}\Xi^{k,K_0}(.,t)\\
&+ 2\varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{m=0}^{K_0}\varepsilon^{(k+m+1)\delta-\alpha/2+1}\widetilde{L}_x^{k,m}()\,\sigma(\xi_{t/\varepsilon^\alpha})\\
&+ 2\sum_{k=0}^{K_0}\sum_{m=1}^{K_0}\varepsilon^{(k+m)\delta+1-\alpha/2}\widetilde{P}_x^k(\cdot)\,\widetilde{w}^m(.,t)\\
&+ \sum_{k=0}^{K_0}\varepsilon^{(k+1)\delta}\sum_{m=1}^{K_0}\varepsilon^{m\delta-\alpha/2+1}\widetilde{Q}_x^k\left(.,\xi_{t/\varepsilon^\alpha}\right)\widetilde{w}^m(.,t)\\
&+ \varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta+\alpha/2}U_t^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}),
\end{aligned}
$$

and

$$
\begin{aligned}
(6.14)\mathcal{NT}^{2,\varepsilon}(.,t) &= \varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0}\varepsilon^{(k+\ell+1)\delta+\alpha/2}\\
&\qquad\times\left(U_x^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha})a^\varepsilon+\frac{1}{\varepsilon}U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha})a_z^\varepsilon\right).
\end{aligned}
$$

Now let us rewrite the Itô formula as follows:

$$\begin{aligned} dV_t^\varepsilon &= \mathcal{B}_t^\varepsilon dt + M_t^\varepsilon dB_t - 2\sum_{k=0}^{K_0} \varepsilon^{k\delta} \langle\!\langle (P^k + \varepsilon^\delta Q^k) v_x^\varepsilon, a^\varepsilon v_x^\varepsilon \rangle\!\rangle dt \\ &+ 2\sum_{k=0}^{K_0}\sum_{m=0}^{K_0-1} \varepsilon^{(k+m+1)\delta-\alpha/2} \langle\!\langle v^\varepsilon(.,t), \widetilde{w}^{m+1}(.,t) + \varepsilon^\varpi \Xi^{k,m}(.,t) \rangle\!\rangle dt \\ &+ \varepsilon^{(K_0+1)\delta-2} \langle\!\langle (Q^{K_0} a^\varepsilon)_{zz}, (v^\varepsilon)^2 \rangle\!\rangle dt \\ &+ \langle\!\langle \mathcal{NT}^{1,\varepsilon}(.,t), v^\varepsilon(.,t) \rangle\!\rangle dt - \langle\!\langle \mathcal{NT}^{2,\varepsilon}(.,t), v_x^\varepsilon(.,t) \rangle\!\rangle dt. \end{aligned}$$

This achieves the proof of the Lemma. □

Proof of Lemma 4.6. Once again we apply Itô's formula

$$\begin{aligned} d\widehat{\mathcal{V}}_t^\varepsilon &= \sum_{k=0}^{K_0} \varepsilon^{k\delta} \left\langle\!\!\left\langle \left(P^k + \varepsilon^\delta Q^k)\phi + (\varepsilon \widehat{P}^k + \varepsilon^{\delta+1}\widehat{Q}^k)\phi_x \right), a^\varepsilon v_{xx}^\varepsilon \right\rangle\!\!\right\rangle dt \\ &+ \sum_{k=0}^{K_0} \varepsilon^{(k+1)\delta-\alpha} \left\langle\!\!\left\langle \left(\mathcal{L}Q^k\phi + \varepsilon \mathcal{L}\widehat{Q}^k\phi_x \right), v^\varepsilon \right\rangle\!\!\right\rangle dt \\ &+ \widehat{\mathcal{L}}_t^\varepsilon dt + \widehat{\mathcal{B}}_t^\varepsilon dt + \widehat{M}_t^\varepsilon \sigma(\xi_{\frac{t}{\varepsilon^\alpha}}) dB_t \end{aligned}$$

where the stochastic integrand $\widehat{M}^\varepsilon$ is given by (4.20) and

$$\begin{aligned} \widehat{\mathcal{L}}_t^\varepsilon &= -\sum_{k=0}^{K_0}\sum_{m=1}^{K_0} \varepsilon^{(k+m)\delta-\alpha/2} \langle\!\langle P^k\phi, \widetilde{w}^m(x,t) \rangle\!\rangle \\ &- \varepsilon^\varpi \sum_{k=0}^{K_0}\sum_{m=0}^{K_0} \varepsilon^{(k+m+1)\delta-\alpha/2} \langle\!\langle Q_y^k\phi, \widetilde{\mathcal{G}}^m \rangle\!\rangle \sigma(\xi_{t/\varepsilon^\alpha}) \\ &+ \varepsilon^\varpi \sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0} \varepsilon^{(k+\ell+1)\delta-\alpha/2} \langle\!\langle \mathcal{L}U^{k,\ell}(.,t,\xi_{t/\varepsilon^\alpha}), \phi \rangle\!\rangle. \end{aligned}$$

From the definition (4.8) of $U^{k,m}$ and (2.6) or (2.7) of w^k, arguing as in the proof of Lemma 4.4, we have

$$\mathcal{L}_t^\varepsilon = \langle\!\langle \mathcal{NT}^{3,\varepsilon}(.,t), \phi \rangle\!\rangle$$

where $\mathcal{NT}^{3,\varepsilon}$ is defined by (4.13). With $\widetilde{P}^k_z = P^k - \langle P^k \rangle$ and $\widetilde{Q}^k_z = Q^k$

$$\begin{aligned}
\widehat{\mathcal{B}}^\varepsilon_t &= \sum_{k=0}^{K_0}\sum_{m=1}^{K_0} \varepsilon^{(k+m)\delta+1-\alpha/2}\Big\langle\!\Big\langle \widehat{P}^k\phi_x + \varepsilon^\delta \widehat{Q}^k\phi_x, \widetilde{w}^m \Big\rangle\!\Big\rangle \\
&+ \sum_{k=0}^{K_0}\sum_{m=1}^{K_0} \varepsilon^{(k+m+1)\delta+1-\alpha/2}\Big\langle\!\Big\langle \widetilde{Q}^k_x\phi, \widetilde{w}^m(x,t) \Big\rangle\!\Big\rangle \\
&+ \varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{m=0}^{K_0} \varepsilon^{(k+1+m)\delta+1-\alpha/2}\Big\langle\!\Big\langle \widehat{Q}^k_y\phi_x, \widetilde{\mathcal{G}}^m \Big\rangle\!\Big\rangle \sigma(\xi_{t/\varepsilon^\alpha}) \\
&+ \varepsilon^{\varpi}\sum_{k=0}^{K_0}\sum_{\ell=0}^{K_0} \varepsilon^{(k+\ell+1)\delta+1}\Big\langle\!\Big\langle U^{k,\ell}_t(.,t,\xi_{t/\varepsilon^\alpha}), \phi \Big\rangle\!\Big\rangle.
\end{aligned}$$

Note that $\widehat{\mathcal{B}}^\varepsilon$ contains only positive powers of ε. Then an integration by parts gives:

$$\begin{aligned}
\langle\!\langle P^k\phi, a^\varepsilon v^\varepsilon_{xx}\rangle\!\rangle &= \langle\!\langle (P^k a^\varepsilon)_{xx}\phi, v^\varepsilon\rangle\!\rangle + 2\langle\!\langle (P^k a^\varepsilon)_x\phi_x, v^\varepsilon\rangle\!\rangle + \langle\!\langle (P^k a^\varepsilon)\phi_{xx}, v^\varepsilon\rangle\!\rangle \\
&= \frac{1}{\varepsilon^2}\langle\!\langle (P^k a^\varepsilon)_{zz}\phi, \widetilde{r}^\varepsilon\rangle\!\rangle + \frac{2}{\varepsilon}\langle\!\langle (P^k a^\varepsilon)_z\phi_x, \widetilde{r}^\varepsilon\rangle\!\rangle + \langle\!\langle (P^k a^\varepsilon)\phi_{xx}, \widetilde{r}^\varepsilon\rangle\!\rangle.
\end{aligned}$$

The same holds with $\widehat{P}^k\phi_x$. Thereby

$$\begin{aligned}
&\sum_{k=0}^{K_0}\varepsilon^{k\delta}\langle\!\langle P^k\phi, a^\varepsilon v^\varepsilon_{xx}\rangle\!\rangle + \sum_{k=0}^{J_0}\varepsilon^{(k+1)\delta-\alpha}\langle\!\langle \mathcal{L}Q^k\phi, v^\varepsilon\rangle\!\rangle \\
&\quad= \sum_{k=0}^{K_0}\varepsilon^{k\delta-2}\langle\!\langle (P^k a^\varepsilon)_{zz}\phi, v^\varepsilon\rangle\!\rangle + \sum_{k=0}^{K_0}\varepsilon^{k\delta-2}\langle\!\langle \mathcal{L}Q^k\phi, v^\varepsilon\rangle\!\rangle \\
&\qquad+ \sum_{k=0}^{K_0}\varepsilon^{k\delta}\left[\frac{2}{\varepsilon}\langle\!\langle (P^k a^\varepsilon)_z\phi_x, v^\varepsilon\rangle\!\rangle + \langle\!\langle (P^k a^\varepsilon)\phi_{xx}, v^\varepsilon\rangle\!\rangle\right].
\end{aligned}$$

From the definition of P^k and Q^k (Eq. (4.5) and (4.6)), the first two sums of the right-hand side disappear. Hence we obtain:

$$\begin{aligned}
d\widehat{\mathcal{V}}^\varepsilon_t &= \langle\!\langle \mathcal{NT}^{3,\varepsilon}(.,t), \phi\rangle\!\rangle dt + \widehat{\mathcal{B}}^\varepsilon_t dt + \widehat{M}^\varepsilon_t dB_t \\
&+ \sum_{k=0}^{K_0}\varepsilon^{k\delta-1}\Big\langle\!\Big\langle (\mathcal{L}\widehat{Q}^k + 2(P^k a^\varepsilon)_z + (\widehat{P}^k a^\varepsilon)_{zz})\phi_x, v^\varepsilon \Big\rangle\!\Big\rangle dt \\
&+ \sum_{k=0}^{K_0}\varepsilon^{k\delta}\Big\langle\!\Big\langle \left(\varepsilon^\delta Q^k\phi + \varepsilon^{\delta+1}\widehat{Q}^k\phi_x\right), a^\varepsilon v^\varepsilon_{xx} \Big\rangle\!\Big\rangle dt \\
&+ \sum_{k=0}^{K_0}\varepsilon^{k\delta}\Big\langle\!\Big\langle (P^k a^\varepsilon)\phi_{xx} + 2(\widehat{P}^k a^\varepsilon)_z\phi_x + \varepsilon(\widehat{P}^k a^\varepsilon)\phi_{xxx}, v^\varepsilon \Big\rangle\!\Big\rangle dt.
\end{aligned}$$

From the definition of $\widehat{P}^k$ and $\widehat{Q}^k$, the first sum is null. Hence we obtain the desired result. □

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