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## How to choose a fuzzy similarity measure in decision-making?

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12 How to choose a fuzzy similarity measure in  
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38 August 31, 2020  
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42 **Abstract**  
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45 **Purpose:** The paper is dedicated to the analysis of fuzzy sim-  
46 ilarity measures in uncertainty analysis in general, and in economic  
47 decision-making in particular. Its purpose is to explain how a simi-  
48 larity measure can be chosen to quantify a qualitative description of  
49 similarities provided by experts of a given domain, in the case where  
50 the objects to compare are described through imprecise or linguistic  
51 attribute values represented by fuzzy sets. The case of qualitative  
52 dissimilarities is also addressed and the particular case of their repre-  
53 sentation by distances is presented.  
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8       **Design/methodology/approach:** The approach is based on  
9 measurement theory, following Tversky's well-known paradigm.

10       **Findings** A list of axioms which may or may not be satisfied  
11 by a qualitative comparative similarity between fuzzy objects is pro-  
12 posed, as extensions of axioms satisfied by similarities between crisp  
13 objects. They enable to express necessary and sufficient conditions for  
14 a numerical similarity measure to represent a comparative similarity  
15 between fuzzy objects. The representation of comparative dissimilar-  
16 ities is also addressed by means of specific functions depending on the  
17 distance between attribute values.

18       **Originality/value:** Examples of functions satisfying certain ax-  
19 ioms to represent comparative similarities are given. They are based  
20 on the choice of operators to compute intersection, union and dif-  
21 ference of fuzzy sets. A simple application of this methodology to  
22 economy is given, to show how a measure of similarity can be cho-  
23 sen to represent intuitive similarities expressed by an economist by  
24 means of a quantitative measure easily calculable. More detailed and  
25 formal results are given in Coletti and Bouchon-Meunier (2019a) for  
26 similarities and Coletti et al. (2020) for dissimilarities.

27       **Keywords:** Similarity measures, Fuzzy sets, Measurement theory,  
28 Qualitative choice, Dissimilarity measure.

## 39       **1 Introduction**

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42       The purpose of this paper is to revisit results introduced in Coletti and  
43 Bouchon-Meunier (2019a) and developed in Coletti and Bouchon-Meunier  
44 (2020) in a more intuitive approach, enabling the reader to grasp the mean-  
45 ing of the formal results based on measurement theory with the aim of making  
46 a conscious choice between several possible measures. Such a choice is neces-  
47 sary because of the number of similarity measures and their fuzzy extensions  
48 existing in the literature, whose properties cannot be directly apprehended  
49 by the user. We follow Tversky's paradigm based on measurement theory  
50 to define necessary and sufficient conditions for the existence of a particular  
51 class of fuzzy similarity measures, which we express in an intuitive formu-  
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8 lation. Similarity measures, regarded as the representation of binary rela-  
9 tions between objects expressing the idea that “ $X$  is no more similar to  $Y$   
10 than  $X'$  is to  $Y'$ ” have been extensively studied in statistics, data analysis,  
11 decision support systems, pattern recognition, clustering methods, informa-  
12 tion retrieval, case-based reasoning, similarity-based reasoning and may other  
13 fields. With the aim of helping the user to choose one of the possible mea-  
14 sures, we have presented a qualitative assessment of similarity measures in  
15 Bouchon-Meunier et al. (2009, 2010) and we have defined a more general  
16 approach of properties of similarity measures based on measurement theory  
17 in Coletti and Bouchon-Meunier (2019b, 2020). In this paper, we focus on  
18 fuzzy similarity measures, used to evaluate the similarity between two objects  
19 which are ill-defined, imprecisely or linguistically described. The concept of  
20 similarity relation has first been introduced in Zadeh (1971) to represent a  
21 gradual relation between precise elements, which can be expressed linguisti-  
22 cally by “ $x$  is rather similar to  $y$ ” or “ $x$  is very similar to  $y$ ”. The similarity  
23 between objects whose attribute values are characterised by fuzzy sets has  
24 then been explored, for instance by Costas and Nikos (1993), Lee-Kwang  
25 et al. (1994), Chen et al. (1995) Wang et al. (1995), Fan and Xie (1999).  
26 We have introduced in Bouchon-Meunier et al. (1996) a more general classi-  
27 fication of measures of comparison according to their elementary properties.  
28 In Coletti and Bouchon-Meunier (2019a) and more extensively in Coletti  
29 and Bouchon-Meunier (2020), we have proposed to discover empirically the  
30 qualitative conditions which we implicitly accept when we choose a given  
31 fuzzy similarity measure and we have highlighted the properties underlying  
32 well-known classes of fuzzy similarity measures.

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35 In Coletti et al. (2020), we have investigated the case of qualitative dis-  
36 similarities and their representation by means of quantitative dissimilarity  
37 measures, focusing on distances that are the most popular dissimilarity mea-  
38 sures.

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41 In the sequence, we first present the background of the study, based  
42 on Tversky’s model of similarities, illustrated by a paradigmatic example  
43 on Basel Accords, followed by the introduction of a comparative similarity  
44 and the main axioms that we consider. Then we present general classes

of similarities satisfies several of these axioms. We highlight independence axioms that are pointed out by Tversky (1977). We then develop these concepts on the illustrative example. We continue on the representation on comparative dissimilarities and we conclude.

## 2 Fuzzy similarity measures

### 2.1 Purpose of the study

We focus our study on the similarity between objects described by imprecise or linguistic values of attributes, represented by fuzzy membership degrees. We consider comparative similarities, regarded as binary relations on pairs of objects, meaning that “ $X$  is no more similar to  $Y$  than  $X'$  is to  $Y'$ ”. We look for a representation of such comparative similarities by means of real-valued functions defining fuzzy similarity measures, with the purpose of a numerical management of similarity that preserves the essence of qualitative comparative similarities while allowing for the automation of evaluations. To achieve this goal, we identify the requirements underlying the choice of a fuzzy similarity measure on the basis of measurement theory, in the spirit of the formalisation of classic similarities given in Tversky (1977).

### 2.2 A paradigmatic example

To clarify our path, we will consider as an example the similarity between companies applying for credit from some European financial institution in accordance with the Basel Accords, under which the bank must set aside a part of its assets according to the risk of each loan granted.

The banking and financial system considers the *rating* as the essential element to assess the financial reliability of a company. The rating is in fact a summary assessment of credit risk, it is a sort of “grade” that the banking system attributes to a company, based on the behavior that the company maintains towards the system itself. In summary, it enables to classify credit applicants into classes of risk.

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8 For the banking system, after Basel II Basel Committee on Banking Su-  
9 pervision (2001) and Basel III Basel Committee on Banking Supervision  
10 (2017), what matters is not so much the amount of a company's net debt,  
11 but rather the company's ability to pay the debts it has contracted with  
12 banks, suppliers and authorities. In other words, more value is given to the  
13 company's ability to meet its obligations on a regular basis than the level of  
14 debt exposure that weighs on the company.  
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18 To determine the rating, other parameters also concur, such as the capital  
19 solidity, the analyses related to the sector in which the company operates,  
20 the plan of medium-long term development presented by the company, the  
21 state of company liquidity and also the characteristics of the company man-  
22 agement.  
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26 The bank summarises all the information in its possession and expresses  
27 an opinion on the reliability of the company through the rating.  
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29 To date, there is no single evaluation system, in fact the rating may  
30 vary depending on the credit institution. However, the type of information  
31 analysed is similar for all banks and is:  
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- 33 – quantitative information;  
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35 – qualitative information;  
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37 – performance information.

38 Now, on the basis of the foregoing and additional information obtained  
39 from Basel II and Basel III, for the purpose of calculating the bank rating, a  
40 company can be described by the following characteristics or features:  
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- 42  $h_1$  : high quality (of the enterprise);  
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44  $h_2$  : growing sector;  
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46  $h_3$  : high budget;  
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48  $h_4$  : low cost of interest paid;  
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50  $h_5$  : compliance with the terms of repayment of the credits received;  
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52  $h_6$  : failure to exceed the credit threshold available;  
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54  $h_7$  : the breakdown of debts between short, medium and long term;  
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56  $h_8$  : high value of the level of stocks and commercial credits of the com-  
57 pany;  
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59  $h_9$  : absence of current accounts inactive or with a negative balance;  
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8  $h_{10}$  : good use of credit lines.  
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11 By their nature the characteristics  $h_i$  are not of the Boolean type and can  
12 therefore belong to a company under study to a certain degree  $x_i = \mu(h_i) \in$   
13  $[0, 1]$ .  
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15 It is natural to expect elements in one rating class to be more similar  
16 to each other than elements in different classes. This can also be used as  
17 a validity test of the rating calculation procedure done by the bank or an  
18 independent agency, However, as we will see, this depends on the choice of  
19 various factors, first of all, on the choice of the measure of similarity, or  
20 even of the class of similarity measures. Therefore this class must be chosen  
21 consciously, in the sense that it must be clear which rules are accepted when  
22 the similarity of two companies is measured through an element of the class.  
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### 29 2.3 Framework

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31 The objects we consider are described by means of a set of attributes  $\mathcal{H} =$   
32  $\{h_1, \dots, h_p\}$  in such a way that the data set  $\mathcal{Y} = [0, 1]^p$  contains all possible  
33 descriptions. For every  $X = \{x_1, \dots, x_p\} \in \mathcal{Y}$ , each  $x_i$  is defined by the degree  
34 of membership  $\mu_X(h_i) \in [0, 1]$  of  $h_i$  in object  $X$ . We consider the measure  
35 of  $X$  defined as  $m(X) = \sum_u \mu_X(u)$ . We use the classic Gödel operators  
36 on fuzzy sets, i.e. the minimum for the intersection and the maximum for  
37 the union, and we define the complement  $X^c$  of  $X$  by the classic operation  
38  $1 - x$ . Differences between fuzzy sets are defined as  $X \setminus Y = X \cap Y^c$  with  
39  $k$ -th component  $\min\{x_k, 1 - y_k\}$  and  $Y \setminus X = X^c \cap Y$  with  $k$ -th component  
40  $\min\{1 - x_k, y_k\}$ , and the symmetric difference between  $X$  and  $Y$  as  $X \Delta Y =$   
41  $(X \setminus Y) \cup (Y \setminus X)$ .  
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### 49 2.4 Comparative similarity

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51 We define a *comparative similarity* on  $\mathcal{Y}^2$  as a binary relation denoted by  
52  $\preceq$  such that, for two pairs of objects  $(X, Y)$  and  $(X', Y') \in \mathcal{Y}^2$ ,  $(X, Y) \preceq$   
53  $(X', Y')$  means that  $X$  is no more similar to  $Y$  than  $X'$  is similar to  $Y'$ . It  
54 is supposed to be provided by experts of the field, for instance economists.  
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Beyond this qualitative assessment of similarities, we propose a quantitative evaluation of similarities through a *similarity measure*  $S : \mathcal{Y}^2 \rightarrow \mathbb{R}$ . We consider that  $S$  represents  $\preceq$  if and only if for every  $(X, Y), (X', Y') \in \mathcal{Y}^2$

$$(X, Y) \preceq (X', Y') \iff S(X, Y) \leq S(X', Y')$$

The question is then to choose an appropriate similarity measure compatible with the given comparative similarity. To answer this question, we limit our research of an appropriate similarity measure to the general class introduced by Tversky (1977) in the framework of measurement theory. In this setting, the similarity  $S(X, Y)$  between two objects  $X$  and  $Y$  depends on the evaluation of elements common to  $X$  and  $Y$ , and elements pertaining to only one of them. When the values of attributes are fuzzy, it is necessary to take into account the fuzzy definition of intersection, union and differences previously introduced, and to use the indicated measure  $m$  to evaluate them.

One way to choose a similarity measure representing a given comparative similarity is to look at the main properties which are implicitly assumed by the expert of the field. To have a list of possible properties, we can for instance think of basic ones, such as reflexivity, symmetry or transitivity, since similarities are softened versions of classic equivalences. Tversky claimed that transitivity is not necessarily requested from similarities.

We recall well-known similarity measures, such as the one originally introduced in Jaccard (1908) and defined as:  $S^1(X, Y) = \frac{m(X \cap Y)}{m(X \cup Y)}$ , or the one introduced in Ochiai (1957) as  $S^2(X, Y) = \frac{m(X \cap Y)}{\sqrt{m(X)} \cdot \sqrt{m(Y)}}$ .

They satisfies a property of symmetry. Neighbouring forms are frequently used as  $S^3(X, Y) = \frac{m(X \cap Y)}{m(Y)}$  and  $S^4(X, Y) = \frac{m(X \cap Y)}{m(X)}$ . They are not based on a symmetry between  $X$  and  $Y$ , but on the comparison of an object  $Y$  with a reference  $X$ , which makes a clear difference between them,  $S^3$  being regarded as a measure of satisfiability of  $Y$  with regard to  $X$  and  $S^4$  as a measure of inclusion of  $Y$  in  $X$ , as pointed out in Bouchon-Meunier et al. (1996).

## 2.5 Axioms for comparative similarities

More generally, we present axioms that may or may not be imposed on comparative similarities, stemming from measurement theory (Luce et al. (1990)). Their formal definition is given in Coletti and Bouchon-Meunier (2019a) and developed in Coletti and Bouchon-Meunier (2020).

**Axiom FS0** (*Weak order*) is the most simple and it states that the comparative similarity  $\preceq$  is a weak order on  $\mathcal{Y}^2$ .

**Axiom FS1G** (*Boundary min-condition*) states that the similarity between two objects with no common attributes is always smaller than the similarity between objects which have some characteristics at least partly in common.

**Axiom FS2G** (*Weak boundary condition*) assumes the following: in pairs of objects such that the degree of membership of the attributes in common is null, the degree of membership of the attributes present in only one object does not influence the degree of similarity. Analogously, in pairs with no attribute present in only one object, the degree of similarity is not influenced by the presence of more or less attributes in common.

**Axiom FS3** (*Symmetry*) declares that the similarity does not depend on the order of objects.

**Axiom FS4** (*Partial attribute uniformity*) says that all the attributes play the same role with respect to the similarity degree. This means that, if in both elements of a pair, the degrees of membership of two attributes are exchanged, a pair similar to the previous one is obtained.

**Axiom FS4G** (*Partial attribute stability*) claims that the comparative degree of similarity of a pair of objects does not change if we slightly alter the values of membership of two attributes, the alteration being limited according to the difference between their values.

**Axiom FS5** (*Distinctive attribute interchangeability*) says that the degree of similarity is the same for a pair  $(X, Y)$  and a pair  $(X', Y')$  which differ by the presence and absence of an attribute present in  $X$ , but not in  $Y$ , or absent in  $X'$  and present in  $Y'$ .

**Axiom FS5G** (*Asymmetric compensation*) states that the degree of similarity between two objects remains the same when we slightly increase the membership of an attribute strongly present in one of them and we slightly decrease the membership of an attribute strongly present in the other one.

**Axiom FS6G** (*Asymmetric min-robustness*) assumes that the degree of similarity between two objects remains the same when the membership of an attribute more strongly present in one of them than in the other one is slightly changed.

**Axiom FS7<sub>1</sub>** (*Monotonicity*) considers that the greater the measure of the intersection and the smaller the measure of the difference between two objects, the higher their comparative similarity.

**Axiom FS7<sub>2</sub>** (*Weak monotonicity*) considers that the greater the measure of the intersection and the smaller the measure of the attributes present in only one of the objects, the higher their comparative similarity.

### 3 Classes of similarities

Several of these available axioms can look natural to the expert of the domain. It is therefore interesting to see if we can identify classes of similarity measures satisfying them, in order to be able to represent the qualitative comparative similarity expressed by the expert by a quantitative similarity measure. We focus on well-known general classes of similarity measures.

#### 3.1 Representation of comparative similarities

To the mentioned axioms, we add the the Debreu condition Q (Debreu (1954)), regarding the representation of any order by a real function. The following equivalence between condition (1) and condition (2) can then be proved for any comparative similarity  $\preceq$  on  $\mathcal{Y}^2$ .

- (1)  $\preceq$  satisfies *FS0*, *FS1G*, *FS2G*, *FS4*, *FS4G*, *FS5*, *FS5G*, *FS6G*, *FS7-1* and *Q*.

(2) There exists a (unique under increasing transformations) function  $S : \mathcal{Y}^2 \rightarrow [0, 1]$  representing  $\preceq$  and such that:

- a)  $S_1(X, Y) = \Phi(m(X \cap Y), m(X \Delta Y))$ ,
- b)  $\Phi(u, v)$  is increasing with respect to  $u$  and decreasing with respect to  $v$
- c)  $\Phi(0, b) = 0$ , and  $\Phi(a, 0) = 1$ , for every  $a$  and for every  $b \neq 0$ .

It means in particular that, if the expert accepts the axioms listed in condition (1), then we can represent his/her comparative similarity by means of a similarity measure of the form  $S_1$ . Moreover, if he/she refuses one of these properties, then we cannot use such a similarity measure. It can be noted that  $S^1$  is an example of similarity measure  $S_1$ .

If we consider another general class of similarity measures, we can also prove the equivalence between the following conditions (3) and (4) defined for any comparative similarity  $\preceq$  on  $\mathcal{Y}^2$ .

(3)  $\preceq$  satisfies  $FS0$ ,  $FS1G$ ,  $FS2G$ ,  $FS4$ ,  $FS4G$ ,  $FS6G$ ,  $FS7-2$  and  $Q$

(4) There exists a (unique under increasing transformations) function  $S : \mathcal{X}^2 \rightarrow [0, 1]$  representing  $\preceq$  and such that:

- a)  $S_2(X, Y) = \Psi(m(X \cap Y), m(X \cap Y) + m(X \setminus Y), m(X \cap Y) + m(Y \setminus X))$ ,
- b)  $\Psi(u, v, w)$  is increasing with respect to  $u$  and decreasing with respect to  $v$  and  $w$
- c)  $\Psi(0, b, b') = 0$ , and  $\Psi(a, 0, 0) = 1$ , for every  $a \neq 0$  and for every  $b, b'$ .

In particular, if the expert accepts the axioms listed in condition (3), his/her comparative similarity can be represented by a similarity measure of the form  $S_2$ . If he/she refuses one of them, then such a similarity measure cannot be used to represent a given comparative similarity. It can be noted that  $S^2$ ,  $S^3$ ,  $S^4$  are examples of similarity measure  $S_2$ .

### 3.2 Specific classes of similarity measures

The classes of similarity measures  $S_1$  and  $S_2$  being very general, we consider the more specific ones, defined as follows:

$$S_{f,g}(X, Y) = \begin{cases} \frac{f(m(X \cap Y))}{f(m(X \cap Y)) + g(m(X \Delta Y))} & \text{for } m(X \cap Y) + m(X \Delta Y) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where  $f$  and  $g$  are nonnegative increasing real functions with  $f(0) = g(0) = 0$ . When  $f$  and  $g$  are the identity function, we see that  $S_{f,g}$  is a fuzzified version of  $S^1$ .

$$T_{\delta,\gamma}(X, Y) = \begin{cases} \frac{\delta(m(X \cap Y))\gamma(m(X \cap Y))}{\delta(m[(X \cap Y) + m(X \setminus Y)])\gamma(m[(X \cap Y) + m(Y \setminus X)])} & \text{for } m(X \cap Y), m(X \setminus Y), m(Y \setminus X) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where  $\delta$  and  $\gamma$  are nonnegative increasing real functions with  $\delta(0) = \gamma(0) = 0$ . When  $\delta$  and  $\gamma$  are the square root function, we observe that  $T_{\delta,\gamma}$  is a fuzzified version of  $S^2$ . When  $\delta(x) = 1$  for all  $x$ , and  $\gamma$  is the identity function, then  $T_{\delta,\gamma}$  is a fuzzified version of  $S^3$ . When  $\gamma(x) = 1$  for all  $x$ , and  $\delta$  is the identity function, then  $T_{\delta,\gamma}$  is a fuzzified version of  $S^4$ .

## 4 Independence axioms

To follow Tversky's approach of similarities requests to take into account a property of independence that we can revisit in a soft manner as follows, denoting by  $\underline{0}$  the element of  $\mathcal{Y}^2$  with all components equal to 0:

**Axiom FWI** *Weak independence* states that, for every  $(X_1, Y_1), (X_2, Y_2), (Z_1, W_1), (Z_2, W_2) \in \mathcal{Y}^2$ , if one of the following conditions holds:

1.  $X_i \cap Y_i = Z_i \cap W_i (i = 1, 2)$  and  $X_1 \Delta Y_1 = X_2 \Delta Y_2 \neq \underline{0}$ ,  $Z_1 \Delta W_1 = Z_2 \Delta W_2 \neq \underline{0}$
2.  $X_i \Delta Y_i = Z_i \Delta W_i (i = 1, 2)$  and  $X_1 \cap Y_1 = X_2 \cap Y_2 \neq \underline{0}$ ,  $Z_1 \cap W_1 = Z_2 \cap W_2 \neq \underline{0}$

then  $(X_1, Y_1) \preceq (X_2, Y_2) \Leftrightarrow (Z_1, W_1) \preceq (Z_2, W_2)$ .

In the case where more such conditions are satisfied, a stronger version of independence is identified in the following axiom.

**Axiom FCI** *Cumulative independence* establishes sufficient conditions (Coletti and Bouchon-Meunier, 2019a)(Coletti and Bouchon-Meunier, 2020) for the equivalence:  $(X_1, Y_1) \preceq (Z_1, W_1) \iff (X_2, Y_2) \preceq (Z_2, W_2)$ .

We can then characterize the class of similarity measures  $S_{f,g}$  under the umbrella of independence, proving that, if a comparative similarity  $\preceq$  is represented by a similarity measure of the form  $S_{f,g}$ , then it means that  $\preceq$  satisfies *FS0*, *FS1G*, *FS2G*, *FS3*, *FS4*, *FS4G*, *FS5*, *FS5G*, *FS6G*, *FS7-1* and *FWI*.

If one of these axioms is not acceptable to the expert expressing the comparative similarity, then it is impossible to represent it by a similarity measure of the form  $S_{f,g}$ .

In an analogous manner, we can characterise the class of similarity measures  $T_{\delta,\gamma}$  by considering a property of independence. We can prove that, if a comparative similarity  $\preceq$  is represented by a similarity measure of the form  $T_{\delta,\gamma}$ , then  $\preceq$  satisfies *FS0*, *FS1G*, *FS2G*, *FS4*, *FS4G*, *FS6G*, *FS7-2*, *FCI*.

Again, if the expert refuses any of these axioms, then the comparative similarity he expresses cannot be represented by a similarity measure of the form  $T_{\delta,\gamma}$ .

## 5 Illustrative example

As an example, we report here the sketch of a procedure for finding the most appropriate measure of similarity, expressing the idea of “no more similar

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8 than” of an expert, referring to a bank, and the relevant study of the numerical  
9 measure agreeing with his/her comparative similarity. This procedure  
10 can be implemented by combining two different methodologies: presenting  
11 direct explicit queries on the simplest axioms and preparing simple fictitious  
12 pairs of bank customers applying for a loan, focusing on some axioms, and re-  
13 quiring the expert to compare these pairs in “similarity”. This last method  
14 permits to test which axioms are violated and which are accepted in the  
15 expressed similarity ordering.  
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20 To this end, the plan is to ask the bank staff of experts to prepare a  
21 database consisting in profiles of customers. The experts are required to  
22 assign degrees in  $[0, 1]$  to a set of characteristics to indicate that they are  
23 present to a certain extent. In this way, they create profiles closer to the re-  
24 ality and able to contribute in a more subtle way than a binary identification  
25 of the characteristics, to the elicitation of similarities between companies.  
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29 We refer to the example sketched in section 2.2 and we submit to a bank  
30 staff a set of profiles related to possible companies applying for a loan, char-  
31 acterised by the attributes  $h_i$ , ( $i = 1, \dots, 10$ ).  
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34 Consider now a data set consisting in the following profiles:  
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$\mathcal{H}$	$h_1$	$h_2$	$h_3$	$h_4$	$h_5$	$h_6$	$h_7$	$h_8$	$h_9$	$h_{10}$
$X_1$	0.1	0.3	0.2	0	0.1	0.01	0.2	0.5	0.2	0.1
$X_2$	0.4	0.6	0.3	0.7	0.1	0.5	0.1	0.2	0.3	0.3
$X_3$	0	0	0	1	0.8	0.9	1	0	0	0
$X_4$	0.5	0.9	1	0	0	0	0	0.2	0.5	0.4
$X_5$	0.2	0.1	0.1	0	0.2	0.3	0.01	0.2	0.1	0.5
$X_6$	0.3	0.4	0.1	0.7	0.1	0.6	0.5	0.3	0.3	0.2
$X_7$	0.5	0.4	0.1	0.6	0	0.7	0.8	0.3	0.5	0.2
$X_8$	0.5	0.4	0.05	0.6	0.05	0.9	0.6	0.3	0.5	0.2
$X_9$	0.4	0.8	0.3	0.8	0.1	0	0.1	0.4	0.1	0.6
$X_{10}$	0.4	0.6	0.3	0.7	0.1	0.5	0	0.2	0.3	0.3
$X_{11}$	0.4	0.1	0.01	0.2	0.1	0	0.2	0.5	0.2	0.1
$X_{12}$	0.4	0.1	0.3	0.4	0.2	0.6	0.1	0.2	0.3	0.5
$X_{13}$	0.1	0.6	0.2	0.7	0.9	0.01	0.2	0.7	0.1	0.8
$X_{14}$	0.6	0.3	0.3	0	0.1	0.5	0.1	0.6	0	0.9
$X_{15}$	0.4	0.6	0.01	0.2	0.1	0	0	0.7	0.1	0.8
$X_{16}$	0.6	0.1	0.3	0.7	1	0.5	0.2	0.6	0	0.9

Each expert is asked to provide a binary relation among pairs of profiles, expressing his/her opinion about the fact that the elements of a pair are no more similar than the elements of another pair.

First of all, his/her complete acceptance of the transitivity of this relation must be tested, explaining that no real function can represent a comparative similarity violating transitivity.

On the contrary the structural axioms, having essentially a syntactical role due to the representability of any relation defined on a uncountable set, and so not related to the particular meaning of the binary relation will not be submitted to the experts.

We now present just as an example a sketch of the above process, using a scheme similar the one used in Coletti and Bouchon-Meunier (2019b,a).

Questions about axioms FS1G can reveal the possible propensity of the experts to use (min, max). More easily, it will be possible to discover if some of them have a clear attitude of rejection. To achieve this, we present to the expert the pairs  $(X_1, X_2)$ ,  $(X_3, X_4)$ , to which the relation  $(X_3, X_4) \prec (X_1, X_2)$



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8 must be assigned, if one accepts axiom FS1G.

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10 In particular the experts must consciously consider two profiles having no  
11 feature in common strictly less similar than any others, independently of the  
12 degree of membership of the common feature in these last ones. If he/she  
13 consciously assesses  $(X_3, X_4) \sim (X_1, X_2)$ , it will be necessary to consider an  
14 operator different from the Gödel's one (see Coletti and Bouchon-Meunier  
15 (2019b,a)).  
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19 Now we need to test the propensity of the expert to accept the axioms  
20 necessary for the representability of a comparative similarity by any element  
21 of the two classes considered in this paper (i.e.  $S_{f,g}$  and  $T_{f,g}$ ), that is ax-  
22 ioms FS4 and FS4G. To achieve this, we first ask to the expert to order  
23 by similarity the pairs  $(X_1, X_2), (X_5, X_6)$ , which only differ with respect to  
24 a permutation of the indices, and then the pairs  $(X_7, X_9), (X_8, X_9)$ , which  
25 are such that  $X_8$  is obtained from  $X_7$  by subtracting 0.005 to the degree of  
26 belonging of the characteristic  $h_3$  and adding it to degree of belonging of  
27 the characteristic  $h_5$  and subtracting 0.2 to the degree of belonging of the  
28 characteristic  $h_7$  and adding it to degree of belonging of the characteristic  
29  $h_6$ .  
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35 If he/she does not consciously consider either the first two pairs equally  
36 similar (violating in this way axiom FS4), or the second two equally similar  
37 (violating in this way axiom FS4G), then there is no similarity measure  
38 considered in this paper representing his/her comparative similarity.  
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41 On the contrary, in the case where the judgement is  $(X_1, X_2) \sim (X_5, X_6)$ ,  
42 and  $(X_7, X_9) \sim (X_8, X_9)$ , we only need to explain that the equivalence must  
43 not be casual, but must be based on the awareness of the equal contribution  
44 of the attributes to the similarity and the comparative degree of similarity  
45 of a pair does not change if we “slightly” alter the values of membership of  
46 two attributes.  
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50 If the experts agrees, we can proceed with the process of discovering the  
51 most suitable similarity measure representing their idea of “no more similar  
52 than”.  
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54 At this point the following pairs  $\{(X_1, X_2), (X_{10}, X_{11})\}$  are submitted for a  
55 comparative evaluation in similarity, by putting in evidence that the two pairs  
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are such that the total degree of the common “presence” of characteristics (i.e.  $m(X_r \cap Y_r) = \sum_i \min(x_i, y_i)$ ) is equal, such as the total degree of the distinctive “presence” of characteristics, computed as the maximum between the two values obtained as minimum between the degree of “presence” in one company and the degree of “not presence” in the others (i.e.  $m(X_i \Delta Y_i) = \sum_i \max \{ \min(x_i, 1 - y_i), \min(y_i, 1 - x_i) \}$ ).

The fact that the experts do not consider equally similar the two pairs means that they do not accept at least one among the axioms FS5, FS6, FS5G and so no function  $\Phi$  can represent their comparative similarity. At this point it is necessary to discover which axioms are actually violated, since FS5 and FS6 are necessary for the representability by all the measures considered in the paper.

To this end, we propose to consider the three pairs  $\{(X_1, X_2), (X_{10}, X_{11}), (X_{11}, X_{12})\}$  which are such that  $m(X_{11} \cap Y_{12}) > m(X_{10} \cap Y_{11}) = m(X_1 \cap Y_2)$  and  $m(X_{11} \Delta Y_{12}) = m(X_1 \Delta Y_2) < m(X_{10} \Delta Y_{11})$  and require to order them in similarity.

If the expert provides the following ordering  $(X_{10}, X_{11}) \prec (X_1, X_2) \prec (X_{11}, X_{12})$ , his/her comparative similarity does not violate axioms FS0, FS4, FS4G, FS5, FS5G, FS7<sub>1</sub>.

In this first step, we can discuss with the expert about the following fact: accepting axioms FS0, FS4, FS4G, FS5, FS5G, FS7<sub>1</sub> is, in fact, equivalent to accepting that the similarity between two companies is evaluated through a function only depending on the fuzzy cardinality of the degree of the common “presence” of characteristics  $h_i$  (i.e.  $m(X \cap Y)$ ) and the fuzzy cardinality of the degree of the global distinctive presence of the same characteristics (i.e.  $m(X \Delta Y)$ ); vice versa, to choose as a similarity measure any function of such a large class means to accept all of the axioms mentioned above. Particular emphasis should be put on discussing axiom FS7<sub>1</sub>, clarifying that in the monotonicity the role of the intersection is greater than that of the difference.

On the contrary, if the expert proposes a different order for the above pairs, then it is necessary to stress that no similarity measure of the kind discussed before is agreeing with his/her comparative similarity.

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8 If the case of a positive agreement of the expert with the axioms, it re-  
9 mains to test which axioms determining a specific form of the function  $\Phi$  are  
10 accepted and which are not. In this regard it would be good to underline  
11 that the previous axioms fix some relations, that essentially divide the pos-  
12 sible profiles of companies into equivalence classes, but leave some degrees of  
13 freedom that are partially saturated by the axioms that identify the form of  
14 the function  $\Phi$ .

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18 To this end, we must first ask him/her if his/her order between  $(X_{13}, X_{14})$   
19 and  $(X_{15}, X_{16})$  is the same as the ordering between  $(X_1, X_2), (X_{10}, X_{11})$ ,  
20 about which he/she has already expressed the following order in similarity  
21  $(X_{10}, X_{11}) \prec (X_1, X_2)$ . A negative answer violates axiom FWI and precludes  
22 the possibility to represent the expert's comparative similarity by any ele-  
23 ment of the class  $S_{f,g}$ .

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27 If, on the contrary, the expert agrees with the proposed inequality, it  
28 is necessary to explain to him/her the actual meaning of the axiom of weak  
29 independence, by stressing the following: to refuse it underlines the idea that  
30 the characteristics  $h_i$  cannot be considered independent, but their mutual  
31 influences (more precisely their positive or negative interactions) must be  
32 taken into account in the similarity evaluation.

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36 A similar procedure can be implemented to test the agreement with the  
37 axioms related to the class of function  $S_2$  and then to the class  $T_{\delta,\gamma}$ . We note  
38 that, at the basis of this class, there is the fact that all the characteristics  $h_i$   
39 have the same importance. If this is considered not realistic it is necessary  
40 to change the class of measures, and so the set of axioms.

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## 6 Comparative dissimilarities

When similarities are used to evaluate to which extent an object is typical of a category in order to define prototypes, they are generally associated with dissimilarities, according to Rosch (1978), Rosch and Mervis (1975) and Prasetyo and Purwarianti (2014).

In this setting, an object is typical of its category if it is highly similar to other objects of this category and highly dissimilar to objects of other

categories, a prototype corresponding to the highest degrees. General classes of dissimilarities have been pointed out in (Bouchon-Meunier et al., 1996) and analysed in various papers, for instance Goshtasby (2012). It is very common to use distances to represent dissimilarities, but they are not the only possibilities. Dissimilarities should not be considered as the opposite of similarities, as they do not necessarily satisfy the same properties as similarities or their opposites. When dissimilarities are assigned by an expert in a way similar to the previous approach of similarities, we need to look at a convenient dissimilarity measure to represent it numerically (Coletti et al. (2020))

We explore the representation of a so-called *comparative dissimilarity* between objects described by means of imprecise or linguistic values of attributes, defined as a binary relation on  $\mathcal{Y}^2$  denoted by  $\preceq'$  such that, for two pairs of objects  $(X, Y)$  and  $(X', Y') \in \mathcal{Y}^2$ ,  $(X, Y) \preceq' (X', Y')$  means that  $X$  is no more dissimilar to  $Y$  than  $X'$  is dissimilar to  $Y'$ . Moreover,  $(X, Y) \prec' (X', Y')$  stands for  $(X, Y) \preceq' (X', Y')$  and not  $(X', Y') \preceq' (X, Y)$ .

We then consider that a dissimilarity measure  $D : \mathcal{Y}^2 \rightarrow R$  represents the comparative dissimilarity  $\preceq'$  if and only if, for all  $(X, Y), (X', Y') \in \mathcal{Y}^2$ , it holds that

$$\begin{aligned} (X, Y) \preceq' (X', Y') &\implies D(X, Y) \leq D(X', Y'), \\ (X, Y) \prec' (X', Y') &\implies D(X, Y) < D(X', Y'). \end{aligned}$$

With the purpose of characterising dissimilarity measures able to represent a given qualitative comparative dissimilarity, we propose a list of axioms that may or may not be satisfied by a dissimilarity measure, and that the expert providing the comparative dissimilarity can consider natural or not.

**Axiom FD0** (*Weak order*) states that  $\preceq'$  is a weak order on  $\mathcal{Y}^2$ .

**Axiom FD1** (*Incremental independence*) requires the comparative degree of dissimilarity to be independent of the common (positive or negative) increment of presence/absence of the features in the objects of a pair.

**Axiom FD2** (*Local strong symmetry*) requires that, when we exchange the values of one of the attributes in two objects, the dissimilarity between the objects is not changed.

**Axiom FD3** (*Boundary condition*) claims that an object is never less dissimilar to another object than it is to itself. An object is never more dissimilar to another object than it is to its complement if and only if it is crisp, with values of attributes in  $[0, 1]$ .

**Axiom FD4** (*Monotonicity axiom*) claims that we increase the dissimilarity between  $X$  and  $Y$  by reducing for  $X$  the value of an attribute which is less present in  $X$  than in  $Y$ , or by increasing in  $Y$  the value of the same attribute.

Considering again the Debreu condition in addition to the considered axioms, and the classic inclusion of fuzzy sets  $\subseteq$ , the following condition (5) and condition (6) can then be proven equivalent for a comparative similarity  $\preceq'$  on  $\mathcal{Y}^2$ .

(5)  $\preceq'$  satisfies axioms *FD1*, *FD2*, *FD3*, *FD4* and *D*

(6) there exists a function (unique under increasing transformations)  $\Phi : \mathcal{Y}^2 \rightarrow [0, 1]$  representing  $\preceq'$  and a function  $\varphi : \mathcal{Y} \rightarrow \mathbb{R}$  such that:

- a)  $Z \subseteq Z' \Rightarrow \varphi(Z) \leq \varphi(Z')$ , for every  $Z, Z' \in \mathcal{Y}$
- b)  $\varphi(\underline{0}) = 0$ , and  $\varphi(\underline{1}) = 1$ .
- c) for all  $X, Y \in \mathcal{Y}$ :  $\Phi(X, Y) = \Phi(|X - Y|, \underline{0}) = \varphi(|X - Y|)$ .

A classic example of dissimilarity measure is the *weighted Manhattan distance*, defined for any set of parameters  $\alpha = (\alpha_1, \dots, \alpha_p)$  with  $\alpha_k \geq 0$  and  $\sum_{k=1}^p \alpha_k = 1$ , as a function  $D_\alpha : \mathcal{Y}^2 \rightarrow \mathbb{R}$  defined, for every  $X, Y \in \mathcal{Y}$ , as

$$D_\alpha(X, Y) = \sum_{k=1}^p \alpha_k |x_k - y_k|.$$

It can be proven that such a distance satisfies axioms *FD1*, *FD2*, *FD3* and *FD4*. Therefore, if the expert does not accept any of these axioms, it is not possible to use a weighted Manhattan distance to represent his/her comparative dissimilarity. More generally, other kinds of distances can be analysed in this framework (Coletti et al. (2020)). An experiment similar to

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8 the sketch of the illustrative example presented for similarities could enable us  
9 to know how to represent the qualitative comparative dissimilarity expressed  
10 by the expert.  
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## 13 14 **7 Conclusion**

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17 The purpose of this paper is to present in a simple formulation the method-  
18 ology to represent by numerical measures the qualitative similarities and, to  
19 a certain extent the qualitative dissimilarities, expressed by experts. Such  
20 similarity or dissimilarity measures are usually chosen arbitrarily among the  
21 long list of possibilities, without checking if they correspond to the expert's  
22 interpretation of his/her spontaneous evaluations. It is therefore important  
23 to interact with him/her to find out what kind of properties he/she consid-  
24 ers to be natural in terms of the behaviour of the similarity or dissimilarity  
25 he/she expresses.  
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30 There is a large number of similarity measures in the classic case and they  
31 give rise to an even larger set of quantities in the case where the attribute  
32 values describing the objects to compare are fuzzy. We have focused on the  
33 specific Gödel operators on fuzzy sets because they are the most commonly  
34 used, but more results can be found in the authors' recent papers. Several  
35 classes of similarity and dissimilarity measures have been considered and  
36 similar work can be done on other general classes.  
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## 43 **References**

- 44  
45  
46 Basel Committee on Banking Supervision (2001). Overview of the new basel  
47 capital accord. <https://www.bis.org/publ/bcbsca02.pdf/>.  
48  
49  
50 Basel Committee on Banking Supervision (2017). Basel ii: International con-  
51 vergence of capital measurement and capital standards: a revised frame-  
52 work. <https://www.bis.org/publ/bcbs107.htm>.  
53  
54  
55 Bouchon-Meunier, B., Coletti, G., Lesot, M. J., and Rifqi, M. (2009). To-  
56 wards a conscious choice of a similarity measure: A qualitative point of  
57  
58  
59  
60

1  
2  
3  
4  
5  
6  
7  
8 view. In Sossai Claudio and Chemello Gaetano, editor, Symbolic and  
9 Quantitative Approaches to Reasoning with Uncertainty, pages 542 – 553,  
10 Berlin, Heidelberg. Springer Berlin Heidelberg.  
11  
12

13 Bouchon-Meunier, B., Coletti, G., Lesot, M.-J., and Rifqi, M. (2010). To-  
14 wards a conscious choice of a fuzzy similarity measure: A qualitative  
15 point of view. In E., H., R., K., and F., H., editors, Computational  
16 Intelligence for Knowledge-Based Systems Design, pages 1–10, Berlin, Hei-  
17 delberg. Springer Berlin Heidelberg.  
18  
19

20  
21 Bouchon-Meunier, B., Rifqi, M., and Bothorel, S. (1996). Towards general  
22 measures of comparison of objects. Fuzzy Sets and Systems, 84(2):143–153.  
23  
24

25 Chen, S. M., Yeh, M. S., and Hsiao, P. Y. (1995). A comparison of similarity  
26 measures of fuzzy values. Fuzzy Sets and Systems, 72(1):79–89.  
27  
28

29 Coletti, G. and Bouchon-Meunier, B. (2019a). Fuzzy similarity measures and  
30 measurement theory. In IEEE International Conference on Fuzzy Systems  
31 (FUZZ-IEEE), New Orleans, LA, USA, pages 1–7.  
32  
33

34 Coletti, G. and Bouchon-Meunier, B. (2019b). A study of similarity mea-  
35 sures through the paradigm of measurement theory: the classic case. Soft  
36 Computing, 23:6827– 6845.  
37  
38

39 Coletti, G. and Bouchon-Meunier, B. (2020). A study of similarity mea-  
40 sures through the paradigm of measurement theory: the fuzzy case. Soft  
41 Computing, 24:11223–11250.  
42  
43

44 Coletti, G., Petturiti, D., and Bouchon-Meunier, B. (2020). A measurement  
45 theory characterization of a class of dissimilarity measures for fuzzy de-  
46 scription profiles. In Marie-Jeanne Lesot and Susana M. Vieira and Marek  
47 Z. Reformat and João Paulo Carvalho and Anna Wilbik and Bernadette  
48 Bouchon-Meunier and Ronald R. Yager, editor, Information Processing  
49 and Management of Uncertainty in Knowledge-Based Systems - 18th  
50 International Conference, IPMU 2020, Lisbon, Portugal, June 15-19, 2020,  
51 Proceedings, Part II, volume 1238 of Communications in Computer and  
52 Information Science, pages 258–268. Springer.  
53  
54  
55  
56  
57  
58  
59  
60

- 1  
2  
3  
4  
5  
6  
7  
8 Costas, P. and Nikos, I. K. (1993). A comparative assessment of measures of  
9 similarity of fuzzy values. Fuzzy Sets and Systems, 56(2):171 – 174.  
10  
11  
12 Debreu, P. (1954). Representation of preference ordering by a numerical  
13 function. New York: Wiley.  
14  
15  
16 Fan, J. and Xie, W. (1999). Some notes on similarity measure and proximity  
17 measure. Fuzzy Sets and Systems, 101(3):403–412.  
18  
19  
20 Goshtasby, A. A. (2012). Similarity and Dissimilarity Measures, pages 7–66.  
21 Springer London.  
22  
23  
24 Jaccard, P. (1908). Nouvelles recherches sur la distribution florale. Bulletin  
25 de la Societe Vaudoise des Sciences Naturelles, 44:223–270.  
26  
27  
28 Lee-Kwang, H., Song, Y.-S., and Lee, K.-M. (1994). Similarity measure be-  
29 tween fuzzy sets and between elements. Fuzzy Sets and Systems, 62(3):291–  
30 293.  
31  
32  
33 Luce, R. D., Suppes, P., Krantz, D. H., and Tversky, A. (1990). Foundations  
34 of Measurement. Academic Press, San Diego.  
35  
36  
37 Ochiai, A. (1957). Zoogeographic studies on the soleoid fishes found in japan  
38 and its neighbouring regions. Bulletin of the Japanese Society for Science  
39 and Fisheries, 22:526–530.  
40  
41  
42 Prasetyo, H. and Purwarianti, A. (2014). Comparison of distance and dis-  
43 similarity measures for clustering data with mix attribute types. In 2014  
44 The 1st International Conference on Information Technology, Computer,  
45 and Electrical Engineering, pages 276–280.  
46  
47  
48 Rosch, E. (1978). Principles of categorization, pages 27–48. Lawrence Erl-  
49 baum associates.  
50  
51  
52 Rosch, E. and Mervis, C. (1975). Family resemblance: studies of the internal  
53 structure of categories. Cognitive psychology, 7:573–605.  
54  
55  
56  
57  
58  
59  
60



1  
2  
3  
4  
5  
6  
7  
8 Tversky, A. (1977). Features of similarity. Psychological Review, 84(4):327–  
9 352.

10  
11  
12 Wang, X., Baets, B. D., and Kerre, E. (1995). A comparative study of  
13 similarity measures. Fuzzy Sets and Systems, 73(2):259–268.

14  
15  
16 Zadeh, L. A. (1971). Similarity relations and fuzzy orderings. Information  
17 Sciences, 3(2):177–200.  
18  
19  
20  
21  
22  
23  
24  
25  
26  
27  
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