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# Statistical Field Theory and Networks of Spiking Neurons

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## Abstract

This paper models the dynamics of a large set of interacting neurons within the framework of statistical field theory. We use a method initially developed in the context of statistical field theory [44] and later adapted to complex systems in interaction [45][46]. Our model keeps track of individual interacting neurons dynamics but also preserves some of the features and goals of neural field dynamics, such as indexing a large number of neurons by a space variable. Thus, this paper bridges the scale of individual interacting neurons and the macro-scale modelling of neural field theory.

## 1 Introduction

Neural fields describes numerous patterns of brain activity, such as cognitive or pathologic processes. This approach models large populations of neurons as homogeneous structures in which individual neurons are indexed by some spatial coordinates. Neural fields dynamics is usually studied in the continuum limit following Wilson, Cowan and Amari ([1][2][3][4][5][6][7][8][9]). Neural activity is represented through a population-averaged firing rate, a macroscopic variable generally assumed to be deterministic and the degrees of freedom of some underlying processes are aggregated to generate an effective theory with simpler variables.

Neural fields theory, because it is a mean field approach that focuses on large scale effects, has a large range of applications, and has been extended along various lines. This approach allows for travelling wave solutions, possibly periodic (see [20][21] and references therein). Stochastic effects in firing rates or other relevant variables have been introduced [10][11][12][13][14] to model perturbations and diffusion patterns in the pulse waves dynamics. The stochastic approach has also been used to study different regimes of mean field theory and noisy transition between these regime (see [15]). Neural network topology has been studied, along with spatial configurations' effects (see [17], developments in [18], and references therein). Last but not least, the tools of Field theory have also been used to extend the Mean field approach [19][22][23][24][25][26][27]. Indeed, mean field appears as the steepest descent approximation of a Statistical Field Theory. The statistical fields involved in this formalism are directly related to the activity, i.e. the spike counts, at each point of the network. The field's action encompasses fluctuations around the mean field. In such a setting, the perturbation expansion of the effective action allows to go further than the mean field approximation, as it keeps track of covariances between neural activity at different points. However, as for mean field theory it remains at the collective level: it works with densities of activity and the field theory is built to recover the average mean field master equation plus some covariances in activity, rather than being designed on the basis of microscopic features of the network. An alternate approach, also based on field theory, computes a partition function for the whole system of neurons in which the field represents the neurons' activity (see [28][29] and references therein). The computation of the correlation functions yields results that go beyond the mean field approximation. Yet, these models use simplifying assumptions at the microscopic level, such as neglecting spatial indices and delays in interactions.

Another branch of the literature considers neural processes as an assembly of individual interacting neurons. This alternate approach allows for a more precise account of the interrelation between neurons'

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connectivity and firing rates. Such models describe a lower scale than neural fields. Usually, no spatial indices are assumed. Neurons are not positioned in a spatial structure, and the resolution of the model relies on numerical studies. These models usually allow for a precise account of neurons' cyclical dynamics and changes in oscillation regimes (for an account, see [33] and references therein). They have recently been used to study the emergence of local connectivity and account for higher scale phenomena: binding problem or polychronization ([34][35][36][37][38][39][40][41][42][43]). However, unlike mean fields, this type of models does not allow for an analytical treatment of collective effect.

This paper presents a method that bridges the gap between the assembly of interacting neurons and the macro-scale modelling of neural field theory. It is based on a method initially developed in [44] and later adapted to complex systems in interaction [45]. We develop a model of statistical field theory keeping track of individual interacting neurons dynamics while preserving some features and goals of neural field dynamics. For instance, indexes a large number of neurons by a space variable, and looks for continuous dynamic equations for the whole system. The present Field Theory differs from those used in the context of Mean Field Theory and its extensions. Actually, our fields do not directly describe any neural activity. As in Statistical Field Theory (see [44]), they are rather abstract complex valued functionals bearing microscopic information to a larger collective scale.

The field theory presented here is the result of a two-step process. In a first step, we write the dynamic equations of a large set of interacting neurons, but modify the standard formalism ([43]) to account for the dynamic nature of neurons connectivity (see [47]). This description allows to deduce the firing frequencies' dynamic equations of a large set of neurons.

In a second step, this set of dynamic equations is transformed into a second-quantized Euclidean field theory (see [45]). The description in terms of field is more compact and includes both collective and individual aspects of the system. The action functional for the field along with its associated partition function encompass the dynamics of the whole system. We use standard techniques of field theory to derive the effective action of the system. Its successive derivatives with respect to the field yield the vacuum of the theory and the correlation functions of the system. The vacuum is the background field in which the system evolves. It impacts individual neurons dynamics and depends on some internal parameters as well as external currents. The correlation functions compute the joint probabilities of transition for an arbitrary number of neurons and depend directly on the form of the vacuum. Considering the two points correlation functions, we can recover the dynamics for firing rate frequencies of individual neurons, i.e. the neural activity modelled in neural field theory. We thus obtain dynamic equations for these frequencies depending on internal and external parameters. In the linear and local approximation, we find some wave equation. This equation may present dissipative, stable or explosive patterns. Depending on the connectivity between neurons, the wave equation may display some non-linear aspects, such as position dependent coefficients. Inspecting the correlation functions for an arbitrary number of points yields an alternative and complementary description of the system. We find a joint probability density for the frequencies at different point. These probabilities depend on time, and this linear dependency reflects the undulatory behaviour of frequencies, and on the background field. Its presence ensures coordination to some extent between frequencies.

Our formalism presents several advantages. First, it highlights the influence of some internal variables on the dynamics of firing rates. Second, studying the effective action may provide a direct approach to the phenomenon of phase transition, i.e. the impact of collective patterns on individual ones. Third, our model can be generalized to include several extensions. We show for instance how to include dynamic equations for the connectivity functions. These equations are "classical" differential equations but should also be described by a field formalism. We also show that two types of neurons, inhibitory and excitatory, may be included and their interactions described by the inclusion of two interacting fields in the model.

This paper is organised as follows: section 1 describes the individual dynamics of neurons in interaction. Section 2 describes the field theoretic formulation of the model and section 3 describes the effective action of the system. In section 4, we derive the minimum of the effective action. We compute the two points correlation functions in section 5. In section 6 we find a local equation for frequencies and presents some possible variations. Section 7 discusses the implication of the results. In section 8 we derive a general form for the correlation functions in presence of strong or weak background field and present their interpretation in term of joined probabilities for frequencies at different points. Section 9 discusses possible extensions.

## 2 Individual dynamics and probability density of the system

Following [45][46], we describe a system of a large number of neurons ( $N \gg 1$ ). We define their individual equations. Then, we write a probability density for the configurations of the whole system over time.

### 2.1 Individual dynamics

We follow the description of [43] for coupled quadratic integrate-and-fire (QIF) neurons, but use the additional hypothesis that each neuron is characterized by its position in some spatial range.

Each neuron's potential  $X_i(t)$  satisfies the differential equation:

$$\dot{X}_i(t) = \gamma X_i^2(t) + J_i(t) \quad (1)$$

for  $X_i(t) < X_p$ , where  $X_p$  denotes the potential level of a spike. When  $X = X_p$ , the potential is reset to its resting value  $X_i(t) = X_r < X_p$ . For the sake of simplicity, following ([43]) we have chosen the squared form  $\gamma X_i^2(t)$  in (1). However any form  $f(X_i(t))$  could be used. The current of signals reaching cell  $i$  at time  $t$  is written  $J_i(t)$ .

Our purpose is to find the system dynamics in terms of the spikes' frequencies. First, we consider the time for the  $n$ -th spike of cell  $i$ ,  $\theta_n^{(i)}$ . This is written as a function of  $n$ ,  $\theta^{(i)}(n)$ . Then, a continuous approximation  $n \rightarrow t$  allows to write the spike time variable as  $\theta^{(i)}(t)$ . We thus have replaced:

$$\theta_n^{(i)} \rightarrow \theta^{(i)}(n) \rightarrow \theta^{(i)}(t)$$

The continuous approximation could be removed, but is convenient and simplifies the notations and computations. We assume now that the timespans between two spikes are relatively small. The time between two spikes for cell  $i$  is obtained by writing (1) as:

$$\frac{dX_i(t)}{dt} = \gamma X_i^2(t) + J_i(t)$$

and by inverting this relation to write:

$$dt = \frac{dX_i}{\gamma X_i^2 + J^{(i)}(\theta^{(i)}(n-1))}$$

Integrating the potential between two spikes thus yields:

$$\theta^{(i)}(n) - \theta^{(i)}(n-1) \simeq \int_{X_r}^{X_p} \frac{dX}{\gamma X^2 + J^{(i)}(\theta^{(i)}(n-1))}$$

Replacing  $J^{(i)}(\theta^{(i)}(n-1))$  by its average value during the small time period  $\theta^{(i)}(n) - \theta^{(i)}(n-1)$ , we can consider  $J^{(i)}(\theta^{(i)}(n-1))$  as constant in first approximation, so that:

$$\begin{aligned} \theta^{(i)}(n) - \theta^{(i)}(n-1) &\simeq \frac{\left[ \arctan \left( \sqrt{\frac{\gamma}{J^{(i)}(\theta^{(i)}(n-1))}} X \right) \right]_{X_r}^{X_p}}{\sqrt{\gamma J^{(i)}(\theta^{(i)}(n-1))}} \\ &= \frac{\left[ \arctan \left( \frac{1}{X} \sqrt{\frac{J^{(i)}(\theta^{(i)}(n-1))}{\gamma}} \right) \right]_{X_r}^{X_p}}{\sqrt{\gamma J^{(i)}(\theta^{(i)}(n-1))}} = \frac{\arctan \left( \frac{\left( \frac{1}{X_r} - \frac{1}{X_p} \right) \sqrt{\frac{J^{(i)}(\theta^{(i)}(n-1))}{\gamma}}}{1 + \frac{J^{(i)}(\theta^{(i)}(n-1))}{\gamma X_r X_p}} \right)}{\sqrt{\gamma J^{(i)}(\theta^{(i)}(n-1))}} \end{aligned}$$

For  $\gamma$  normalized to 1 along and  $\frac{J^{(n)}(\theta^{(n-1)})}{X_r X_p} \ll 1$ , this is:

$$\theta^{(i)}(n) - \theta^{(i)}(n-1) \equiv G\left(\theta^{(i)}(n-1)\right) = \frac{\arctan\left(\left(\frac{1}{X_r} - \frac{1}{X_p}\right) \sqrt{J^{(i)}\left(\theta^{(i)}(n-1)\right)}\right)}{\sqrt{J^{(i)}\left(\theta^{(i)}(n-1)\right)}} \quad (2)$$

The frequency  $\omega_i(t)$ , or firing rate at  $t$ , is defined by the inverse time span (2) between two spikes:

$$\begin{aligned} \omega_i(t) &= \frac{1}{G\left(\theta^{(i)}(n-1)\right)} \\ &\equiv F\left(\theta^{(i)}(n-1)\right) = \frac{\sqrt{J^{(i)}\left(\theta^{(i)}(n-1)\right)}}{\arctan\left(\left(\frac{1}{X_r} - \frac{1}{X_p}\right) \sqrt{J^{(i)}\left(\theta^{(i)}(n-1)\right)}\right)} \end{aligned}$$

The time interval between two spikes being considered small, we can write:

$$\theta^{(i)}(n) - \theta^{(i)}(n-1) \simeq \frac{d}{dt}\theta^{(i)}(t) - \omega_i^{-1}(t) = \varepsilon_i(t) \quad (3)$$

We added a white noise perturbation  $\varepsilon_i(t)$  for each period to account for any internal uncertainty in the timespan  $\theta^{(i)}(n) - \theta^{(i)}(n-1)$  which is independent from the instantaneous inverse frequency  $\omega_i^{-1}(t)$ . We assume these  $\varepsilon_i(t)$  to have variance  $\sigma^2$ , so that equation (3) writes:

$$\frac{d}{dt}\theta^{(i)}(t) - G\left(\theta^{(i)}(t), J^{(i)}\left(\theta^{(i)}(t)\right)\right) = \varepsilon_i(t) \quad (4)$$

The  $\omega_i(t)$  are computed by considering the overall current. Using the discrete notation, it is given by:

$$\hat{j}^{(i)}((n-1)) = J^{(i)}((n-1)) + \frac{\kappa}{N} \sum_{j,m} \frac{\omega_j(m)}{\omega_i(n-1)} \delta\left(\theta^{(i)}(n-1) - \theta^{(j)}(m) - \frac{|Z_i - Z_j|}{c}\right) T_{ij}((n-1, Z_i), (m, Z_j)) \quad (5)$$

The quantity  $J^{(i)}((n-1))$  denotes an external current. The term inside the sum is the average current sent to  $i$  by neuron  $j$  during the short time span  $\theta^{(i)}(n) - \theta^{(i)}(n-1)$ . The function  $T_{ij}((n-1, Z_i), (m, Z_j))$  is the transfer function between cells  $j$  and  $i$ . It measures the level of connectivity between  $i$  and  $j$ . We assume that:

$$T_{ij}((n-1, Z_i), (m, Z_j)) = T((n-1, Z_i), (m, Z_j))$$

The transfer function of  $Z_j$  on  $Z_i$  only depends on positions and times. It models the transfer function as an average transfer between local zones of the thread. This transfer function is typically considered as gaussian or decreasing exponentially with the distance between neurons, so that the closer the cells, the more connected they are.

The other terms arising in (5) can be justified in the following way: given the distance  $|Z_i - Z_j|$  between the two cells and the signals' velocity  $c$ , the signals arrive with a delay  $\frac{|Z_i - Z_j|}{c}$ . The spike emitted by cell  $j$  at time  $\theta^{(j)}(m)$  has thus to satisfy:

$$\theta^{(i)}(n-1) < \theta^{(j)}(m) + \frac{|Z_i - Z_j|}{c} < \theta^{(i)}(n)$$

to reach cell  $i$  during the timespan  $[\theta^{(i)}(n-1), \theta^{(i)}(n)]$ . This relation must be represented by a step function in the current formula. However given our approximations, this can be replaced by:

$$\delta\left(\theta^{(i)}(n-1) - \theta^{(j)}(m) - \frac{|Z_i - Z_j|}{c}\right)$$

as was done in (5). However, this Dirac function has to be weighted by the number of spikes emitted during the rise of the potential. This number is the ratio  $\frac{\omega_j(m)}{\omega_i(n-1)}$  that counts the number of spikes emitted by neuron  $j$  toward neuron  $i$  between the spikes  $n-1$  and  $n$  of neuron  $i$ . Again, this is valid for relatively small timespans between two spikes. For larger timespans, the frequencies should be replaced by their average over this period of time.

The sum over  $m$  and  $i$  is the overall contribution from any possible spikes of the thread to the current, provided these spikes arrive at  $i$  during the considered interval  $\theta^{(i)}(n) - \theta^{(i)}(n-1)$ . Note that the current (5) is partly an endogenous variable. It depends on signals external to  $i$ , but depends also on  $i$  through  $\omega_i(n-1)$ . This is a consequence of the intrication between the system's element.

Formula (5) shows that the dynamic equation (3) has to be coupled with the frequency equation:

$$\omega_i(t) = \frac{\sqrt{\hat{J}^{(i)}((n-1))}}{\arctan\left(\left(\frac{1}{X_r} - \frac{1}{X_p}\right)\sqrt{\hat{J}^{(i)}((n-1))}\right)} + v_i(t) \quad (6)$$

and  $J^{(i)}((n-1))$  is defined by (5). A white noise  $v_i(t)$  accounts for the possible deviations from this relation, due to some internal or external causes for each cell. We assume that the variances of  $v_i(t)$  are constant, and equal to  $\eta^2$ , such that  $\eta^2 \ll \sigma^2$ .

## 2.2 Probability density for the system

Due to the stochastic nature of equations (4) and (6), the dynamics of a single neuron can be described by the probability density for a path  $(\theta^{(i)}(t), \omega_i^{-1}(t))$  (see [45] and [46]):

$$P\left(\theta^{(i)}(t), \omega_i^{-1}(t)\right) = \exp(A_i) \quad (7)$$

where:

$$A_i = \frac{1}{\sigma^2} \int \left(\frac{d}{dt}\theta^{(i)}(t) - \omega_i^{-1}(t)\right)^2 dt + \int \frac{\left(\omega_i^{-1}(t) - G\left(\theta^{(i)}(t), J\left(\theta^{(i)}(t)\right)\right)\right)^2}{\eta^2} dt \quad (8)$$

The integral is taken between some initial and final times. This time period depends on the time scale of the interactions. Actually, the minimization of (8) imposes both (3) and (6), so that the probability density is centered around these two conditions, as expected. In other words, (3) and (6) are satisfied in mean. A probability density for the whole system is obtained by summing  $S_i$  over all agents. We thus define:

$$P\left(\left(\theta^{(i)}(t), \omega_i^{-1}(t)\right)_{i=1\dots N}\right) = \exp(-A) \quad (9)$$

with:

$$A = \sum_i A_i = \sum_i \frac{1}{\sigma^2} \int \left(\frac{d}{dt}\theta^{(i)}(t) - \omega_i^{-1}(t)\right)^2 dt + \int \frac{\left(\omega_i^{-1}(t) - G\left(\theta^{(i)}(t), J\left(\theta^{(i)}(t)\right)\right)\right)^2}{\eta^2} dt \quad (10)$$

## 3 Field theoretic description of the system

### 3.1 translation of Equation (10) in field theory

In [45][46], we show that the probabilistic description of the system (9) is equivalent to a statistical field formalism. In such a formalism, the system is collectively described by a field belonging to the Hilbert space of complex functions. The arguments of these functions are the same as those describing an individual neuron.

In our context, the field depends on the three variables  $(\theta, Z, \omega)$  and writes  $\Psi(\theta, Z, \omega)$ . The field dynamics is described by an action functional for the field and its associated partition function. This partition function,

that reflects both collective and individual aspects of the system, allows to recover correlation functions for an arbitrary number of neurons.

The field theoretic version of (8) is obtained using (10): a correspondence detailed in [45][46]) yields an action  $S(\Psi)$  for a field  $\Psi(\theta, Z, \omega)$  and a statistical weight  $\exp(-S(\Psi))$  for each configuration  $\Psi(\theta, Z, \omega)$  of this field. The functional  $S(\Psi)$  is decomposed in two parts corresponding to the two contributions in (10).

The first term of (10) is replaced by the corresponding quadratic functional in field theoretic :

$$-\frac{1}{2}\Psi^\dagger(\theta, Z, \omega)\nabla\left(\frac{\sigma^2}{2}\nabla-\omega^{-1}\right)\Psi(\theta, Z, \omega) \quad (11)$$

where  $\sigma^2$  is the variance of the errors  $\varepsilon_i$ .

The field functional corresponding to the second term of (8) is obtained by considering the statistical weight for each dynamical variable  $i$  and taking into account the current induced by all  $j$ :

$$V = \frac{1}{2\eta^2}\sum_n\sum_i(\omega_i^{-1}(n-1) - G\left(J(\theta^{(i)}(n-1), Z_i) + \frac{\kappa}{N}\sum_{j,m}\frac{\omega_j(m)T_{ij}((n-1, Z_i), m, Z_j)}{\omega_i(n-1)}\delta\left(\theta^{(i)}(n-1) - \theta^{(j)}(m) - \frac{|Z_i - Z_j|}{c}\right)\right))^2 \quad (12)$$

with  $\eta \ll 1$ . It is the constraint (6) imposed stochastically. Its continuous time equivalent is:

$$V = \frac{1}{2\eta^2}\int dt\sum_i\left(\omega_i^{-1}(t) - G\left(J(\theta^{(i)}(t), Z_i) + \frac{\kappa}{N}\int ds\sum_j\frac{\omega_j(s)T_{ij}((t, Z_i), s, Z_j)}{\omega_i(t)}\delta\left(\theta^{(i)}(t) - \theta^{(j)}(s) - \frac{|Z_i - Z_j|}{c}\right)\right)\right)^2 \quad (13)$$

The corresponding potential in field theory is obtained straightforwardly:

$$\frac{1}{2\eta^2}\int|\Psi(\theta, Z, \omega)|^2\left(\omega^{-1} - G\left(J(\theta, Z) + \int\frac{\kappa}{N}\frac{\omega_1 T\left(Z, \theta, Z_1, \theta - \frac{|Z - Z_1|}{c}\right)}{\omega}\left|\Psi\left(\theta - \frac{|Z - Z_1|}{c}, Z_1, \omega_1\right)\right|^2 dZ_1 d\omega_1\right)\right)^2 \quad (14)$$

We will write:

$$T\left(Z, \theta, Z_1, \theta - \frac{|Z - Z_1|}{c}\right) \equiv T(Z, \theta, Z_1)$$

The field action is then the sum of (11), (14):

$$S = -\frac{1}{2}\Psi^\dagger(\theta, Z, \omega)\nabla\left(\frac{\sigma_\theta^2}{2}\nabla-\omega^{-1}\right)\Psi(\theta, Z, \omega) + \frac{1}{2\eta^2}\int|\Psi(\theta, Z, \omega)|^2\left(\omega^{-1} - G\left(J(\theta, Z) + \int\frac{\kappa}{N}\frac{\omega_1}{\omega}\left|\Psi\left(\theta - \frac{|Z - Z_1|}{c}, Z_1, \omega_1\right)\right|^2 T(Z, \theta, Z_1) dZ_1 d\omega_1\right)\right)^2 \quad (15)$$

### 3.2 Projection on dependent frequency states:

We can simplify (15) using that  $\eta^2 \ll 1$ . Actually, in that case, most field configurations  $\Psi(\theta, Z, \omega)$  have negligible statistical weight. We can restrict the fields to those of the form:

$$\Psi(\theta, Z)\delta\left(\omega^{-1} - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right) \quad (16)$$

where  $\omega^{-1}(J, \theta, Z, \Psi)$  satisfies:

$$\begin{aligned}\omega^{-1}(J, \theta, Z, |\Psi|^2) &= G \left( J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega_1 T \left( Z, \theta, Z_1, \theta - \frac{|Z-Z_1|}{c} \right)}{\omega(J, \theta, Z, |\Psi|^2)} \left| \Psi \left( \theta - \frac{|Z-Z_1|}{c}, Z_1, \omega_1 \right) \right|^2 dZ_1 d\omega_1 \right) \\ &= G \left( J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega_1 T \left( Z, \theta, Z_1, \theta - \frac{|Z-Z_1|}{c} \right)}{\omega(J, \theta, Z, |\Psi|^2)} \left| \Psi \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right) \right|^2 \right. \\ &\quad \left. \times \delta \left( \omega_1^{-1} - \omega^{-1} \left( J, \theta - \frac{|Z-Z_1|}{c}, Z_1, |\Psi|^2 \right) \right) dZ_1 d\omega_1 \right)\end{aligned}$$

The last equation simplifies to yield:

$$\omega^{-1}(J, \theta, Z, |\Psi|^2) = G \left( J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega \left( J, \theta - \frac{|Z-Z_1|}{c}, Z_1, \Psi \right) T \left( Z, \theta, Z_1, \theta - \frac{|Z-Z_1|}{c} \right)}{\omega^{-1}(J, \theta, Z, |\Psi|^2)} \left| \Psi \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right) \right|^2 dZ_1 \right) \quad (17)$$

The configurations  $\Psi(\theta, Z, \omega)$  that minimize the potential (14) can thus be considered: the field  $\Psi(\theta, Z, \omega)$  is projected on the subspace (16) of functions of two variables. Therefore, we replace:

$$\omega^{-1} \rightarrow \omega^{-1}(J, \theta, Z, |\Psi|^2)$$

in (14) and the "classical" effective action becomes:

$$-\frac{1}{2} \Psi^\dagger(\theta, Z) \left( \nabla_\theta \frac{\sigma^2}{2} \nabla_\theta - \omega^{-1}(J, \theta, Z, |\Psi|^2) \right) \Psi(\theta, Z) \quad (18)$$

Eventually, a more precise form can be given to the transfer function  $T(Z, \theta, Z_1)$ . We use a simplified version of [47]. Appendix 6 shows that at the individual level and in first approximation, the transfer functions are modelled by a product of a spatial factor  $T(Z, Z_1)$  and a function  $W$  of the frequencies  $\omega \equiv \omega(J, \theta, Z, |\Psi|^2)$  and  $\omega_1 \equiv \omega \left( J, \theta - \frac{|Z-Z_1|}{c}, Z_1, |\Psi|^2 \right)$ . The function  $W$  is increasing in  $\omega$  and decreasing in  $\omega_1$ . Without loss of generality, we will consider  $W$  as an increasing function of  $\left( \frac{\omega}{\omega_1} \right)$ , so that:

$$T(Z, \theta, Z_1) = T(Z, Z_1) W \left( \frac{\omega}{\omega_1} \right) \quad (19)$$

### 3.3 Inclusion of collective stabilization potential

We ultimately modify (18) by including collective terms to stabilize the number of active connexions. Such terms correspond to some overall regulatory processes and do not appear at the individual level. In absence of "competition" between inhibitory and excitatory mechanisms, such a potential models the possibility for a system to come back to some minimal equilibrium activity.

To do so, we modify the effective action by including a potential for maintaining and activating new connections. We add to (18) the contribution:

$$\int |\Psi(\theta, Z)|^2 U_0 \left( \int \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 \right) \quad (20)$$

where  $U_0$  is a  $U$  shaped potential with  $U_0(0) = 0$ , so that  $U_0$  has a minimum for some positive value of  $\left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2$ . At this minimum, the value of  $U_0$  is negative. We write the expansion of (20) as:

$$-\zeta_1 \int \left( |\Psi(\theta, Z)|^2 \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 \right) + \sum_{n=2}^{\infty} \zeta_n \int |\Psi(\theta, Z)|^2 \left( \prod_{i=1}^{n-1} \left| \Psi \left( \theta - \frac{|Z-Z_i|}{c}, Z_i \right) \right|^2 \right) \quad (21)$$



The second term in (21) represents a global limitation to increase the overall number of connections and currents, so that we assume that:

$$\sum_{k=2}^n \zeta_k \left\langle \prod_{i=1}^{k-1} \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \right\rangle > 0$$

for  $n \geq 2$ . The bracket  $\langle \rangle$  denotes the expectation value of the product of fields.

The coefficients  $\zeta_n$  can be set to 0 for  $n > N$ , where  $N$  is an arbitrary threshold. The factor  $-\zeta_1$  accounts for a minimal number of connections maintained. It depends on external activity  $J$ . The contribution for  $n = 2$  and the one proportional to  $-\zeta_1$  can be gathered to rewrite the collective potential:

$$\sum_{n=2}^{\infty} \zeta^{(n)} \int |\Psi(\theta, Z)|^2 \left( \prod_{i=1}^{n-1} \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \right) \quad (22)$$

where  $\zeta^{(n)} = \zeta_n$  for  $n > 2$  and  $\zeta^{(2)} = \zeta_2 - \zeta_1$ . We assume that  $\zeta^{(2)} < 0$ , so that a nontrivial minimal collective state exists. The "classical" action is thus:

$$-\frac{1}{2} \Psi^\dagger(\theta, Z) \left( \nabla_\theta \frac{\sigma^2}{2} \nabla_\theta - \omega^{-1}(J, \theta, Z, |\Psi|^2) \right) \Psi(\theta, Z) + \sum_{n=2}^{\infty} \zeta^{(n)} \int |\Psi(\theta, Z)|^2 \left( \prod_{i=1}^{n-1} \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \right) \quad (23)$$

We can impose a constraint on the coefficients  $\zeta^{(n)}$  since we are interested in the relative magnitudes of the coefficients  $\sigma_\theta^2$  and quantities such as  $\omega^{-1}$ . As a consequence, we can impose, as a relative benchmark, that:

$$\begin{aligned} & \sum_{n=2}^{\infty} \zeta^{(n)} \int |\Psi(\theta, Z)|^2 \left( \prod_{i=1}^{n-1} \left\langle \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \right\rangle \right) \\ &= \int |\Psi(\theta, Z)|^2 U_0 \left( \int \left\langle \left| \Psi \left( \theta - \frac{|Z - Z'|}{c}, Z' \right) \right|^2 \right\rangle \right) \simeq 1 \end{aligned}$$

## 4 Effective action

### 4.1 Effective action at the tree order

Appendix 1.1.2 and 1.1.3 show that the graphs perturbative expansion associated to (23) can be computed using the propagator associated to the "free action":

$$-\frac{1}{2} \Psi^\dagger(\theta, Z) \left( \nabla_\theta \frac{\sigma^2}{2} \nabla_\theta - \omega^{-1}(\bar{J}, Z, \mathcal{G}_0) \right) \Psi(\theta, Z) \quad (24)$$

where  $\omega^{-1}(\bar{J}, Z, \mathcal{G}_0)$  is the static inversed frequency defined as the solution of the equation:

$$\omega^{-1}(\bar{J}, Z, \mathcal{G}_0) = G \left( \bar{J}(Z) + \int \frac{\kappa}{N} \frac{\omega(\bar{J}, Z_1, \mathcal{G}_0)}{\omega(\bar{J}, Z, \mathcal{G}_0)} \mathcal{G}_0(0, Z_1) T(Z, \theta, Z_1) dZ_1 \right) \quad (25)$$

where  $\bar{J}(Z)$  is the average over time of  $J(\theta, Z)$  and  $\mathcal{G}_0(0, Z)$  is the evaluation for  $\theta = \theta'$  of the Green function  $\mathcal{G}_0(\theta, \theta', Z)$  of the operator:

$$-\nabla_\theta \left( \frac{\sigma^2}{2} \nabla_\theta - \omega^{-1}(J, \theta, Z, 0) \right) \quad (26)$$

where  $\omega^{-1}(J, \theta, Z, 0)$  is the inverse frequency given by (17) for  $\Psi \equiv 0$ , i.e.  $\omega^{-1}(J, \theta, Z, \Psi) = G(J(\theta, Z))$ . The solution of (25) and (26) is computed in Appendix 1.1.2. We find that for an external current decomposed in a static and dynamic part  $\bar{J} + J(\theta)$ :

$$\begin{aligned} \omega^{-1}(J, \theta, Z, 0) &= G(\bar{J}(Z) + J(\theta)) \\ &\simeq G(\bar{J}(Z)) = \frac{\arctan \left( \left( \frac{1}{X_r} - \frac{1}{X_p} \right) \sqrt{\bar{J}(Z)} \right)}{\sqrt{\bar{J}(Z)}} = \frac{1}{\bar{X}_r(Z)} \end{aligned}$$

and:

$$\mathcal{G}_0(\theta, \theta', Z) = \delta(Z - Z') \frac{\exp(-\Lambda_1(Z)(\theta - \theta'))}{\Lambda(Z)} H(\theta - \theta') \quad (27)$$

where:

$$\begin{aligned} \Lambda(Z) &= \sqrt{\frac{\pi}{2}} \sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r(Z)}\right)^2 + \frac{2\alpha}{\sigma^2}} \\ \Lambda_1(Z) &= \sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r(Z)}\right)^2 + \frac{2\alpha}{\sigma^2}} - \frac{1}{\sigma^2 \bar{X}_r(Z)} \end{aligned}$$

and  $H$  is the heaviside function:

$$\begin{aligned} H(\theta - \theta') &= 0 \text{ for } \theta - \theta' < 0 \\ &= 1 \text{ for } \theta - \theta' > 0 \end{aligned}$$

so that  $\omega^{-1}(\bar{J}(Z), Z, \mathcal{G}_0(0, Z))$  solves:

$$\omega^{-1}(Z, \mathcal{G}_0) = G \left( \bar{J}(Z) + \int \frac{\kappa}{N} \frac{\omega(Z_1, \mathcal{G}_0)}{\sqrt{\frac{\pi}{2}} \sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r(Z_1)}\right)^2 + \frac{2\alpha}{\sigma^2}}} T(Z, \theta, Z_1) dZ_1 \right) \quad (28)$$

Once  $\omega^{-1}(Z, \mathcal{G}_0)$  is known, (24) implies that the effective action can be computed by considering the following action at the tree-order:

$$\begin{aligned} \Gamma(\Psi) &= -\frac{1}{2} \Psi^\dagger(\theta, Z) \nabla_\theta \left( \frac{\sigma^2}{2} \nabla_\theta - \omega^{-1}(\bar{J}(Z), Z, \mathcal{G}_0) + \alpha \right) \Psi(\theta, Z) \\ &\quad - \zeta_1 \int \left( |\Psi(\theta, Z)|^2 \left| \Psi\left(\theta - \frac{|Z - Z'|}{c}, Z'\right) \right|^2 \right) + \sum_{n=1}^{\infty} \zeta_n \int |\Psi(\theta, Z)|^2 \left( \prod_{i=1}^{n-1} \left| \Psi\left(\theta - \frac{|Z - Z_i|}{c}, Z_i\right) \right|^2 \right) \end{aligned} \quad (29)$$

In the sequel, for the sake of simplicity, the dependency in  $Z$  of  $\bar{X}_r(Z)$ ,  $\Lambda(Z)$ ,  $\Lambda_1(Z)$  will be implicit, so that we will write:

$$\bar{X}_r(Z) \equiv \bar{X}_r, \Lambda(Z) \equiv \Lambda, \Lambda_1(Z) \equiv \Lambda_1$$

## 4.2 Graph expansion and generating function for connected correlation functions

Appendix 2 computes the partition function with a source field  $\Omega(\theta^{(j)})$ . This partition function is obtained through the sum of graphs induced by (29). The computations are performed in Appendix 2.3 and yield:

$$\begin{aligned} Z(\Omega) &= \sum_{n \geq 0} \frac{1}{n!} \left( \int \Delta \Omega^\dagger(\theta_f^{(i)}) \frac{\exp(-\Lambda_1(\theta_f^{(i)} - \theta_i^{(i)}))}{\Lambda} \right. \\ &\quad \times \left. \left( 1 + \left( \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out} \bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\Lambda_1}}{\Lambda \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}} \right)} \right) \left( \exp(\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) - 1) \right) \right) \Delta \Omega(\theta_i^{(i)}) \right)^n \end{aligned}$$

Where  $\Delta \Omega(\theta^{(i)})$  is defined as:  $\Delta \Omega(\theta^{(i)}) = (\Omega(\theta^{(i)}) - \Omega_0(\theta^{(i)}))$ , where  $\Omega_0(\theta^{(i)})$  is the source term corresponding to a null expectation value of the field, i.e.  $\langle \Psi(\theta^{(i)}) \rangle = 0$ , and where the interaction vertices are:

$$\begin{aligned}
\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) &= \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) - \bar{\zeta}_n \left( \theta_f^{(i)} - \theta_i^{(i)} \right) \\
&= \sum_{l=2}^n \sum_{\{k_1, \dots, k_l\} \subset \{1, \dots, n\}, k_j \neq i} \left( \bar{\Xi}_1^{(l)} \left( Z_i, \{Z_{k_j}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) - \frac{\zeta^{(l)}}{\Lambda^l} \right)
\end{aligned} \tag{30}$$

with:

$$\begin{aligned}
\bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) &= \sum_{l=2}^n \sum_{\{k_1, \dots, k_l\} \subset \{1, \dots, n-1\}, k_j \neq i} \bar{\Xi}_1^{(l)} \left( Z_i, \{Z_{k_j}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \\
\bar{\zeta}_n \left( \theta_f^{(i)} - \theta_i^{(i)} \right) &= \sum_{l=2}^n \sum_{\{k_1, \dots, k_l\} \subset \{1, \dots, n-1\}, k_j \neq i} \frac{\zeta^{(l)}}{\Lambda^l}
\end{aligned} \tag{31}$$

and:

$$\begin{aligned}
\bar{\Xi}_1^{(l)} \left( Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i} \right) &= \sum_{\{k_1, \dots, k_l\} \subset \{1, \dots, n-1\}, k_j \neq i} \bar{\Xi}_1^{(l)} \left( Z_i, \theta^{(i)}, \{Z_{k_j}\} \right) \\
&= \frac{\sum_{\{k_1, \dots, k_{l-1}\} \subset \{1, \dots, n-1\}} \prod_{k_j} \int_{\frac{c}{|Z_i - Z_{k_j}|}}^{\theta^{(i)} - \theta_i^{(k_j)}} dl_{k_j}}{\left( \sqrt{\frac{\pi}{2}} \sqrt{\left( \frac{1}{\sigma^2 X_r} \right)^2 + \frac{2\alpha}{\sigma^2}} \right)^{l+1}} \left[ \frac{\delta^l \int \Psi^\dagger \left( \theta^{(i)}, Z_i \right) \nabla_{\theta^{(i)}} \omega^{-1} \left( J, \theta^{(i)}, Z_i, |\Psi|^2 \right) \Psi \left( \theta^{(i)}, Z_i \right) dZ_i}{\prod_{i=1}^l \delta \left| \Psi \left( \theta^{(i)} - l_{k_j}, Z_{k_j} \right) \right|^2 \delta \left| \Psi \left( \theta^{(i)}, Z_i \right) \right|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0)}
\end{aligned}$$

We also define:

$$\begin{aligned}
\bar{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) &= \lim_{n \rightarrow \infty} \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \\
\hat{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) &= \lim_{n \rightarrow \infty} \bar{\Xi}_1 \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)
\end{aligned} \tag{32}$$

Appendix 2.4 computes the generating functional for connected correlation functions:  $W(\Omega) = \ln \frac{Z(\Omega)}{Z(0)}$ . This functional is found by defining the following operators:

$$\begin{aligned}
O_{1,n} \left( \theta_i^{(i)}, \theta_f^{(i)}, \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_{j \neq i} \right) &= \left( \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out} \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda_1}}{\Lambda \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}} \right)} \left( \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1 \right) \right) \\
O_{1,\infty} \left( \theta_i^{(i)}, \theta_f^{(i)}, \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_{j \neq i} \right) &= \frac{-\bar{\zeta} + \frac{\nabla_{\theta_i^{(i)}}^{out} \bar{\Xi}_{1,\infty} \left( Z_i, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda_1}}{\Lambda \left( -\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty} \left( Z_i, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}} \right)} \left( \exp \left( \hat{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1 \right) \\
O_{1,1} \left( \theta_i^{(i)}, \theta_f^{(i)} \right) &= 1 \\
\mathcal{G}_0 \left( \theta_i^{(i)}, \theta_f^{(i)} \right) &= \frac{\exp \left( -\Lambda_1 \left( \theta_f^{(i)} - \theta_i^{(i)} \right) \right)}{\Lambda} H \left( \theta_f^{(i)} - \theta_i^{(i)} \right)
\end{aligned} \tag{33}$$

and the products:

$$\begin{aligned}
O_{1,n}^{(n)} \left( \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_i \right) &= \prod_{i=1}^n O_{1,n} \left( \theta_i^{(i)}, \theta_f^{(i)}, \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_{j \neq i} \right) \\
O_{1,\infty}^{(n)} \left( \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_i \right) &= \prod_{i=1}^n O_{1,\infty} \left( \theta_i^{(i)}, \theta_f^{(i)}, \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_{j \neq i} \right) \\
(1 + O_{1,n})^{(n)} \left( \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_i \right) &= \prod_{i=1}^n \left( 1 + O_{1,n} \left( \theta_i^{(i)}, \theta_f^{(i)}, \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_{j \neq i} \right) \right) \\
(1 + O_{1,\infty})^{(n)} \left( \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_i \right) &= \prod_{i=1}^n \left( 1 + O_{1,\infty} \left( \theta_i^{(i)}, \theta_f^{(i)}, \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_{j \neq i} \right) \right)
\end{aligned}$$

For later purpose we also define the average  $\langle 1 + O_{1,\infty} \rangle$  by:

$$\exp \langle (1 + O_{1,\infty}) \rangle = \sum_{n \geq 0} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n \quad (34)$$

To compute  $W(\Omega)$  we also need the expectations of an operator  $A$  acting on the tensor products of fields in a product of background sources  $\Delta\Omega(\theta_i^{(i)})$ :

$$\begin{aligned}
\langle A \rangle_n &= \int \left[ \prod_{i=1}^n \Delta\Omega^\dagger(\theta_f^{(i)}) \mathcal{G}_0(\theta_f^{(i)}, \theta_i^{(i)}, Z_i) \right] A \left[ \prod_{i=1}^n \Delta\Omega(\theta_i^{(i)}) \right], \quad n > 0 \\
\langle A \rangle_0 &= 1
\end{aligned}$$

For  $A$  acting on  $(\Delta\Omega(\theta_i^{(i)}))^{\otimes n}$ , the expectation  $\langle A \rangle_k$ ,  $k < n$  for  $A$  symmetric is evaluated on the  $k$  first variables and defines an operator acting on  $(\Delta\Omega(\theta_i^{(i)}))^{\otimes n-k}$ . Using these definitions, the partition function with source rewrites:

$$\begin{aligned}
Z(\Omega) &= 1 + \int \Delta\Omega^\dagger(\theta_f^{(i)}) \mathcal{G}_0(\theta_f^{(i)}, \theta_i^{(i)}, Z_i) \Delta\Omega(\theta_i^{(i)}) \\
&\quad + \int \prod_{i=1}^2 \Delta\Omega^\dagger(\theta_f^{(i)}) \mathcal{G}_0(\theta_f^{(i)}, \theta_i^{(i)}, Z_i) (1 + O_{1,2})^{(2)} \Delta\Omega(\theta_i^{(i)}) \\
&\quad + \sum_{n \geq 2} \frac{1}{n!} \int \prod_{i=1}^n \Delta\Omega^\dagger(\theta_f^{(i)}) \mathcal{G}_0(\theta_f^{(i)}, \theta_i^{(i)}, Z_i) (1 + O_{1,n})^{(n)} \Delta\Omega(\theta_i^{(i)}) \\
&= 1 + \sum_{n \geq 1} \frac{1}{n!} \langle (1 + O_{1,n})^{(n)} \rangle_n \\
&\simeq 1 + \langle 1 \rangle_1 + \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n
\end{aligned}$$

and we obtain the expression for the generating functional  $W(\Omega)$ :

$$W(\Omega) = \ln \left( 1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n \right)$$

### 4.3 Effective action

Now that  $W(\Omega)$  is known, we can derive the formal expression for the effective action. It is computed using the relation between the background field and the source field:

$$\Psi\left(\theta_1^{(i)}\right) = \frac{\delta W(\Omega)}{\delta \Omega^\dagger\left(\theta_1^{(i)}\right)} = \int^{\theta_1^{(i)}} \frac{1 + \langle(1 + O_{1,2})^{(2)}\rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle(1 + O_{1,2})^{(2)}\rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle(1 + O_{1,\infty})^{(n)}\rangle_n} \mathcal{G}_0\left(\theta_1^{(i)}, \theta^{(i)}\right) \Delta \Omega\left(\theta^{(i)}\right) d\theta^{(i)}$$

$$\Psi^\dagger\left(\theta_1^{(i)}\right) = \frac{\delta W(\Omega)}{\delta \Omega\left(\theta_1^{(i)}\right)} = \int_{\theta_1^{(i)}} \Delta \Omega^\dagger\left(\theta^{(i)}\right) \frac{1 + \langle(1 + O_{1,2})^{(2)}\rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle(1 + O_{1,2})^{(2)}\rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle(1 + O_{1,\infty})^{(n)}\rangle_n} \mathcal{G}_0\left(\theta^{(i)}, \theta_1^{(i)}\right) d\theta^{(i)}$$

where  $\mathcal{G}_0\left(\theta_1^{(i)}, \theta^{(i)}\right)$  is the free propagator defined above in (33).

To compute the effective action, we consider (168) evaluated at  $\Psi_0\left(\theta^{(i)}\right)$ , so that  $\Omega_{\Psi_0(\theta^{(i)})}\left(\theta^{(i)}\right) = 0$ :

$$\begin{aligned} - \left[ \frac{1 + \langle(1 + O_{1,2})^{(2)}\rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle(1 + O_{1,2})^{(2)}\rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle(1 + O_{1,\infty})^{(n)}\rangle_n} \right]_{\Psi_0(\theta^{(i)})} \mathcal{G}_0\left(\Omega_{\Psi(\theta^{(i)})=0}\left(\theta^{(i)}\right)\right) &= \Psi_0\left(\theta^{(i)}\right) \\ \left(\Omega_{\Psi(\theta^{(i)})=0}^\dagger\left(\theta^{(i)}\right)\right) \mathcal{G}_0\left[ \frac{1 + \langle(1 + O_{1,2})^{(2)}\rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle(1 + O_{1,2})^{(2)}\rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle(1 + O_{1,\infty})^{(n)}\rangle_n} \right]_{\Psi_0(\theta^{(i)})} &= \Psi_0^\dagger\left(\theta^{(i)}\right) \end{aligned}$$

Appendix 2.5 shows that the effective action then writes:

$$\begin{aligned} \Gamma(\Psi) &= \frac{\langle 1 \rangle_1 + \langle(1 + O_{1,2})^{(2)}\rangle_2 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_n}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle(1 + O_{1,2})^{(2)}\rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle(1 + O_{1,\infty})^{(n)}\rangle_n} \\ &\quad - \frac{1}{2} \ln \left( 1 + \langle 1 \rangle_1 + \frac{1}{2} \langle(1 + O_{1,2})^{(2)}\rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle(1 + O_{1,\infty})^{(n)}\rangle_n \right) \\ &\quad - \frac{1}{2} \Psi^\dagger\left(\theta^{(i)}\right) \mathcal{G}_0^{-1} \left[ \frac{1 + \langle(1 + O_{1,2})^{(2)}\rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle(1 + O_{1,2})^{(2)}\rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle(1 + O_{1,\infty})^{(n)}\rangle_n} \right]_{\Psi_0(\theta^{(i)})}^{-1} \Psi_0\left(\theta^{(i)}\right) + H.C. \end{aligned} \tag{35}$$

#### 4.4 Series expansion for the effective action

Appendix 4 computes the corrections to the zeroth order effective action (29) by using graphs expansion (the final formula is given in Appendix 4.3). This yields an expanded form of (29). We show that for weak values of the interaction parameters, the effective action takes the form:

$$\begin{aligned} \Gamma(\Psi) &= \tag{36} \\ &= -\frac{1}{2} \Psi^\dagger(\theta, Z) \left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1} \left( J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2 \right) \right) + \alpha \right) \Psi(\theta, Z) \\ &\quad + \sum \frac{\zeta^{(n)}}{n!} \int |\Psi(\theta^{(i)}, Z_i)|^2 \left( \mathcal{G}_0(0, Z_j) + \int_{\frac{|Z_i - Z_j|}{c}}^{\theta^{(i)} - \theta^{(j)}} dl_j \int \left| \Psi\left(\theta^{(i)} - \frac{|Z_i - Z_j|}{c}, Z_j\right) \right|^2 dZ_j \right)^n \\ &\quad + \sum \int \frac{1}{(n-1)!} \Psi^\dagger(\theta_f^{(i)}, Z_i) \hat{V}_{1,n}\left(Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)}\right) \left( \frac{\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)}{\theta_f^{(i)} - \theta_i^{(i)}} \right)^2 \Psi\left(\theta_i^{(i)}, Z_i\right) \\ &\quad \times \left[ \int \Psi^\dagger(\theta_f^{(j)}, Z_j) D\left(\theta_f^{(j)}, \theta_i^{(j)}, Z_j\right) \Psi\left(\theta_i^{(j)}, Z_j\right) dZ_j \right]^{n-1} \end{aligned}$$

where  $\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0)$  is solution of:

$$\omega^{-1}(\theta, Z) = G \left( J(\theta) + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)}{\omega(\theta, Z)} W \left( \frac{\omega(\theta, Z)}{\omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)} \right) \mathcal{G}_0(0, Z_1) dZ_1 \right)$$

and:

$$\begin{aligned} & \hat{V}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \\ = & \frac{1 + \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) - 1 \right) \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)}{\left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)^2} \\ & \times \frac{\left( -\frac{\zeta^{(n)}}{\Lambda^n} + \frac{\nabla_{\theta_i^{(i)}}^{out} \bar{\Xi}_{1,n}^{(n)} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\left( \theta_f^{(i)} - \theta_i^{(i)} \right)} \right)}{\frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \\ & + \frac{\left( \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \left( 1 - \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) + \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)^2 \right) - 1 \right)}{\left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)^2} \\ & \times \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out} \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda_1 \left( \theta_f^{(i)} - \theta_i^{(i)} \right)}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \\ & \int \Psi^\dagger \left( \theta_f^{(j)}, Z_j \right) D \left( \theta_f^{(j)}, \theta_i^{(j)}, Z_j, \right) \Psi \left( \theta_i^{(j)}, Z_j \right) dZ_j \\ = & \int \Psi^\dagger \left( \theta_f^{(j)}, Z_j \right) \left( 1 + \left( \exp \left( \hat{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \right) - 1 \right) \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1}}{\theta_f^{(j)} - \theta_i^{(j)}} \right. \right. \\ & \left. \left. \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right) \right) \Psi \left( \theta_i^{(j)}, Z_j \right) dZ_j \end{aligned}$$

In these expressions, the values of  $\bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)$  and  $\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)$  defined in (30) and (31) are approximated by:

$$\begin{aligned} \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) & \simeq \int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \sum_{l=1}^{n-1} C_{n-1}^l \left( \left[ \frac{\delta^{l-1} \omega^{-1} \left( J, \theta^{(i)}, Z_i, |\Psi|^2 \right)}{\Lambda^l \prod_{m=1}^{l-1} \delta \left| \Psi \left( \theta^{(i)} - \frac{|Z_i - \bar{Z}|}{c}, \bar{Z} \right) \right|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \right) d\theta^{(i)} \quad (37) \\ \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) & \simeq \int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \sum_{l=1}^{n-1} C_{n-1}^l \left( \left[ \frac{\delta^{l-1} \omega^{-1} \left( J, \theta^{(i)}, Z_i, |\Psi|^2 \right)}{\Lambda^l \prod_{m=1}^{l-1} \delta \left| \Psi \left( \theta^{(i)} - \frac{|Z_i - \bar{Z}|}{c}, \bar{Z} \right) \right|^2} - \frac{\zeta^{(l)}}{\Lambda^l} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \right) d\theta^{(i)} \end{aligned}$$

where  $\bar{Z}$  is the centre of the thread and depends only on  $Z_i, \theta_i^{(i)}, \theta_f^{(i)}$  and  $\bar{Z}$ .

## 5 Nontrivial minimum

Appendix 4.4 shows that, for a wide range of parameters, the effective action has a minimum. The corresponding background field decomposes into a constant part  $\Psi_0$  and a contribution depending on the external current. We show that for slowly varying currents  $J(\theta, Z_i)$ , and for  $|\zeta^{(n)}| > \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0)$  the minimum of  $\Gamma(\Psi)$  given in (36) is reached for  $\Psi(\theta, Z) = \Psi_0(\theta, Z) + \delta\Psi(\theta, Z)$  and  $\Psi^\dagger(\theta, Z) = \Psi_0^\dagger(\theta, Z) + \delta\Psi^\dagger(\theta, Z)$  where  $|\delta\Psi(\theta, Z)| \ll |\Psi_0(\theta, Z)|$  and  $|\delta\Psi^\dagger(\theta, Z)| \ll |\Psi_0^\dagger(\theta, Z)|$ . The fields  $\Psi_0(\theta, Z)$  and  $\Psi_0^\dagger(\theta, Z)$  describe the minimum of the potential:

$$\alpha \int |\Psi(\theta^{(i)}, Z_i)|^2 dZ_i + \sum \frac{\zeta^{(n)}}{n!} \left( \mathcal{G}_0(0, Z_j) + \int |\Psi(\theta^{(i)} - \frac{|Z_i - Z_j|}{c}, Z_j)|^2 dZ_j \right)^n$$

This minimum exists for  $\alpha \ll 1$  and for  $|\zeta^{(2)}|$  large. It is reached for a value  $X_0$  of  $\int |\Psi(\theta^{(i)}, Z_i)|^2 dZ_i$ . Up to an irrelevant phase,  $\Psi_0(\theta^{(i)}, Z_i) = \Psi_0^\dagger(\theta, Z) = \sqrt{\frac{X_0}{V}}$ , where  $V$  is the volume of the thread. Appendix 4.4.2 finds the expression for  $\delta\Psi(\theta, Z)$  and  $\delta\Psi^\dagger(\theta, Z)$ . They satisfy the following first order equations:

$$\begin{aligned} 0 &\simeq \frac{1}{2} \left( - \left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) \right) \right) + U''(X_0) \right) \delta\Psi(\theta, Z) \\ &\quad - \frac{1}{2} \int \delta\Psi^\dagger(\theta_1, Z_1) \sqrt{X_0} \left( \nabla_\theta \frac{\delta\omega^{-1}(J(\theta_1), \theta_1, Z_1, \mathcal{G}_0 + X_0)}{\delta|\Psi(\theta, Z)|^2} \right) \Psi_0(\theta_1, Z_1) d\theta_1 dZ_1 \\ &\quad - \frac{1}{2} (\nabla_\theta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0)) \Psi_0(\theta, Z) \\ &\quad - \frac{1}{2} \int \sqrt{X_0} \left( \nabla_\theta \frac{\delta\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0)}{\delta|\Psi(\theta_1, Z_1)|^2} \right) \Psi_0(\theta, Z) \delta\Psi(\theta_1, Z_1) d\theta_1 dZ_1 \end{aligned}$$

for  $\delta\Psi(\theta, Z)$ , and:

$$\begin{aligned} 0 &= \frac{1}{2} \delta\Psi^\dagger(\theta, Z) \left( - \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) \right) + U''(X_0) \right) \\ &\quad - \frac{1}{2} \int \delta\Psi^\dagger(\theta_1, Z_1) \sqrt{X_0} \left( \nabla_\theta \frac{\delta\omega^{-1}(J(\theta_1), \theta_1, Z_1, \mathcal{G}_0 + X_0)}{\delta|\Psi(\theta, Z)|^2} \right) \Psi_0(\theta_1, Z_1) d\theta_1 dZ_1 \end{aligned}$$

for  $\delta\Psi^\dagger(\theta_1, Z_1)$ . Appendix 4.4.2 shows that solutions are:

$$\delta\Psi^\dagger(\theta, Z) = 0$$

and:

$$\begin{aligned} \delta\Psi(\theta, Z) &= \sum_n \int (-C\bar{\Lambda})^{n+1} \prod_{l=1}^n \left\{ \frac{\exp\left(-\bar{\Lambda}\left(\theta_l - \theta_{l+1} - \frac{|Z_l - Z_{l+1}|}{c}\right)\right)}{K\Gamma\sqrt{1 + \frac{\bar{\Lambda}}{K}}} \operatorname{erf}\left(\sqrt{1 + \frac{\bar{\Lambda}}{K}} \sqrt{\theta_l - \left(\theta_{l+1} + \frac{|Z_l - Z_{l+1}|}{c}\right)}\right) \right\} \\ &\quad \times \omega(J(\theta_l), \theta_l, Z_l, \mathcal{G}_0 + X_0) \tilde{\mathcal{G}}(\theta_n, \theta_{n+1}, Z_l) (\nabla_\theta \omega^{-1}(J(\theta_{n+1}), \theta_{n+1}, Z_l, \mathcal{G}_0 + X_0)) \frac{\sqrt{X_0}}{2} \prod_{l=2}^{n+1} d\theta_l dZ_l \end{aligned}$$

with the convention that  $\theta_1 = \theta$ . The constants  $K$ ,  $\Gamma$  and  $\bar{\Lambda}$  depend on the parameters of the system. The kernel  $\tilde{\mathcal{G}}(\theta_n, \theta_{n+1})$  is computed in Appendix 4.4.2. It is given by:

$$\begin{aligned} \tilde{\mathcal{G}}(\theta, \theta') &= \frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2 X_e}\right)^2 + \frac{2U''(X_0)}{\sigma^2}} - \frac{1}{\sigma^2 X_e}\right)(\theta - \theta') - \left[\frac{\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0)}{2}\right]_{\theta'}^{\theta}\right)}{\sqrt{\frac{\pi}{2}} \sqrt{\left(\frac{1}{\sigma^2 X_e}\right)^2 + \frac{2U''(X_0)}{\sigma^2}}} H(\theta - \theta') \quad (39) \\ &\simeq \frac{\exp\left(-\left(\left(\sqrt{\left(\frac{1}{\sigma^2 X_e}\right)^2 + \frac{2U''(X_0)}{\sigma^2}}\right) + \left[\nabla_{\theta} \frac{\omega^{-1}(\bar{J}, \theta, Z, \mathcal{G}_0 + X_0)}{2}\right]\right)(\theta - \theta')\right)}{\sqrt{\frac{\pi}{2}} \sqrt{\left(\frac{1}{\sigma^2 X_e}\right)^2 + \frac{2U''(X_0)}{\sigma^2}}} H(\theta - \theta') \end{aligned}$$

where the upper bar on a quantity stands for the average computed over the period  $\theta - \theta'$ .

The inverse frequency  $\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)$  is solution of:

$$\begin{aligned} &\omega^{-1}(\theta, Z) \quad (40) \\ &= G\left(J(\theta) + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)}{\omega(\theta, Z)} W\left(\frac{\omega(\theta, Z)}{\omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)}\right) (\mathcal{G}_0(0, Z_1) + X_0) dZ_1\right) \end{aligned}$$

The field  $\Psi(\theta^{(j)}, Z_j)$  is the - phase dependent - background field. In the trivial phase, it is equal to 0, so that the effective action matches with the "classical" one. In a non-trivial phase,  $\Psi(\theta^{(j)}, Z_j)$  is not equal to zero and may be time dependent. It describes the accumulation of current, or signals, that shapes the long term dynamics of frequencies. This explains the contribution  $\bar{\Lambda}$  in the first term.

To conclude this section by expanding (40) in terms of current at the second order of approximation:

$$\begin{aligned} &\omega^{-1}(J, \theta, Z) = G[J(\theta, Z), \mathcal{G}_0 + X_0] \\ &+ \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right) - \omega(\theta, Z)}{\omega(\theta, Z)} G'[J(\theta, Z), \mathcal{G}_0] \bar{\mathcal{G}}_0(0, Z_1) dZ_1 \\ &= \frac{G[J(\theta, Z), \bar{\mathcal{G}}_0]}{1 - \frac{\kappa}{N} \int T(Z, Z_1) \left(\omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right) - \omega(\theta, Z)\right) G'[J(\theta, Z), \bar{\mathcal{G}}_0] \bar{\mathcal{G}}_0(0, Z_1) dZ_1} \\ &\simeq \frac{G[J(\theta, Z), \bar{\mathcal{G}}_0(0, Z_i)]}{1 - \int \frac{\kappa T(Z, Z_1)}{N} \left(G\left[J\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right), \bar{\mathcal{G}}_0\right] - G[J(\theta, Z), \bar{\mathcal{G}}_0]\right) G'[J(\theta, Z), \bar{\mathcal{G}}_0] \bar{\mathcal{G}}_0(0, Z_1) dZ_1} \end{aligned}$$

and:

$$\begin{aligned} &\omega(J, \theta, Z) \\ &= \frac{1}{2} \left( F[J(\theta, Z), \bar{\mathcal{G}}_0] + \left( F[J(\theta, Z), \bar{\mathcal{G}}_0] \right)^2 \right. \\ &+ 4 \left( \int \frac{\kappa}{N} T(Z, Z_1) \left( F\left[J\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right), \bar{\mathcal{G}}_0\right] - F[J(\theta, Z), \bar{\mathcal{G}}_0] \right) \bar{\mathcal{G}}_0(0, Z_1) dZ_1 \right) \\ &\left. \times F'[J(\theta, Z), \bar{\mathcal{G}}_0] \right)^{\frac{1}{2}} \end{aligned}$$

where:

$$\bar{\mathcal{G}}_0(0, Z_1) \simeq \mathcal{G}_0(0, Z_1) + X_0$$

and the constants  $C, K$  and  $\bar{\Lambda}$  that depend on the system are defined in Appendix 1.3.2.3.

Incidentally, we note that a non-trivial minimum depending on the system parameters should allow for phase transition in the system of frequencies. This question is left for further work.



## 6 Two points correlation functions

The correlation functions can be found by computing the derivatives of the effective action at the classical background field. This is done by using a graph expansion of the effective action. The two-points correlation function is defined by  $\frac{\delta \Delta \Omega(\theta_1^{(i)})}{\delta \Psi(\theta_f^{(i)})}$ . Appendix 3.1 shows that it satisfies a coupled equation with  $\frac{\delta \Delta \Omega(\theta_1^{(i)})}{\delta \Psi(\theta_f^{(i)})}$ .

Actually, defining:

$$\begin{aligned} [\Delta \Omega] &= \begin{pmatrix} \Delta \Omega \\ \Delta \Omega^\dagger \end{pmatrix} \tag{41} \\ K \left( \theta_1^{(i)}, \theta_2^{(i)} \right) &= \begin{pmatrix} K_{1,1} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) & K_{1,2} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) \\ K_{1,2}^\dagger \left( \theta_1^{(i)}, \theta_2^{(i)} \right) & K_{1,1}^\dagger \left( \theta_1^{(i)}, \theta_2^{(i)} \right) \end{pmatrix} \\ [X \left( \theta_1^{(i)}, \theta_f^{(i)} \right)] &= \begin{pmatrix} \left( \frac{1 + \langle (1+O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1+O_{1,\infty})^{(n)} \rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1+O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1+O_{1,\infty})^{(n)} \rangle_n} \mathcal{G}_0 \right)^{-1} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\ 0 \end{pmatrix} = \begin{pmatrix} X \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\ 0 \end{pmatrix} \end{aligned}$$

the vector  $[\Delta \Omega] \left( \theta_1^{(i)} \right)$  satisfies the dynamic equation:

$$\frac{\delta [\Delta \Omega] \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} = \int K \left( \theta_1^{(i)}, \theta_2^{(i)} \right) \frac{\delta [\Delta \Omega] \left( \theta_2^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} d\theta_2^{(i)} + [X \left( \theta_1^{(i)}, \theta_f^{(i)} \right)] \tag{42}$$

The operator valued coefficients:

$$K \left( \theta_1^{(i)}, \theta_2^{(i)} \right) = \begin{pmatrix} K_{1,1} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) & K_{1,2} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) \\ K_{2,1} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) & K_{2,2} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) \end{pmatrix}$$

are defined by:

$$\begin{aligned} K_{1,1} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) &= \left( - \int \Delta \Omega^\dagger \left( \left( \theta_2^{(i)} \right)' \right) \left( A \left( \theta_1^{(i)} \right) (1 + \bar{O}_{1,\infty})_{(\theta_2^{(i)})', \theta_2^{(i)}} + B \left( \theta_1^{(i)} \right) (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right) \mathcal{G}_0 d \left( \theta_2^{(i)} \right)' \right. \\ &\quad \left. + \Delta \Omega \left( \theta_1^{(i)} \right) \Psi^\dagger \left( \theta_2^{(i)} \right) \right) \\ K_{2,2} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) &= \left( - \int \mathcal{G}_0 \left( A^\dagger \left( \theta_1^{(i)} \right) (1 + \bar{O}_{1,\infty})_{\theta_2^{(i)}, (\theta_2^{(i)})'} + B^\dagger \left( \theta_1^{(i)} \right) (1 + \bar{O}_{1,2})_{\theta_2^{(i)}, (\theta_2^{(i)})'} \right) \mathcal{G}_0 \Delta \Omega \left( \left( \theta_2^{(i)} \right)' \right) d \left( \theta_2^{(i)} \right)' \right. \\ &\quad \left. + \Delta \Omega^\dagger \left( \theta_1^{(i)} \right) \Psi \left( \theta_2^{(i)} \right) \right) \\ K_{1,2} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) &= \left( - \int \left( A \left( \theta_1^{(i)} \right) (1 + \bar{O}_{1,\infty})_{(\theta_2^{(i)})', \theta_2^{(i)}} + B \left( \theta_1^{(i)} \right) (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right) \mathcal{G}_0 \Delta \Omega \left( \left( \theta_2^{(i)} \right)' \right) d \left( \theta_2^{(i)} \right)' \right. \\ &\quad \left. + \Delta \Omega \left( \theta_1^{(i)} \right) \Psi \left( \theta_2^{(i)} \right) \right) \\ K_{1,2} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) &= K_{1,2}^\dagger \left( \theta_1^{(i)}, \theta_2^{(i)} \right) \end{aligned}$$

where  $\mathcal{G}_0$  is the operator whose kernel is defined in (33), and:

$$\begin{aligned}
& A\left(\theta_1^{(i)}\right) \\
&= \mathcal{G}_0^{-1} * \frac{\sum_{n \geq 3} \frac{1}{(n-2)!} \left\langle (1 + O_{1,\infty})^{(n-2)} \right\rangle_{n-2} (1 + \bar{O}_{1,\infty})_{\theta_1^{(i)}}}{\left(1 + \left\langle (1 + O_{1,2})^{(1)} \right\rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \left\langle (1 + O_{1,\infty})^{(n-1)} \right\rangle_{n-1} (1 + \bar{O}_{1,\infty})\right)} * \mathcal{G}_0 \Delta \Omega \left(\theta_1^{(i)}\right) \\
&= A\left(\theta_1^{(i)}, \theta_f^{(i)}\right) \\
& B\left(\theta_1^{(i)}\right) \\
&= \mathcal{G}_0^{-1} * \frac{(1 + \bar{O}_{1,2})_{\theta_1^{(i)}}}{\left(1 + \left\langle (1 + O_{1,2})^{(1)} \right\rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \left\langle (1 + O_{1,\infty})^{(n-1)} \right\rangle_{n-1} (1 + \bar{O}_{1,\infty})\right)} * \mathcal{G}_0 \Delta \Omega \left(\theta_1^{(i)}\right) \\
&= B\left(\theta_1^{(i)}, \theta_f^{(i)}\right)
\end{aligned}$$

and where the operators  $\bar{O}_{1,n}$  are defined by:

$$(1 + O_{1,n}) \mathcal{G}_0 = (1 + \bar{O}_{1,n}) * \mathcal{G}_0$$

The symbol  $*$  denotes the convolution product, so that we have:

$$\bar{O}_{1,n} = \left( -\bar{\zeta}_n + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,n} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \frac{\exp \left( -\Lambda_1 \left( \theta_f^{(i)} - \theta_{i'}^{(i)} \right) \right)}{\Lambda}$$

The solution of (42) is given by:

$$\begin{aligned}
& \frac{\delta [\Delta \Omega] \left(\theta_1^{(i)}\right)}{\delta \Psi \left(\theta_f^{(i)}\right)} = \mathcal{G}_0^{-1} \frac{1 + \exp(-x) \left(-z + \frac{1}{2} (y^2 - x^2)\right)}{\left(1 + \bar{O}_{1,\infty}\right) + \exp(-x) \left(-\bar{O}_{1,\infty} + y \left(1 + \bar{O}_{1,2}\right) - x \left(1 + \bar{O}_{1,\infty}\right)\right)} \left(\theta_1^{(i)}, \theta_f^{(i)}\right) \quad (43) \\
& + \exp \left( \int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( \left(\theta^{(i)}\right) d\theta^{(i)} \right) \right) \left( \int \Psi^\dagger \left( \left(\theta_2^{(i)}\right)' \right) X \left( \left(\theta_2^{(i)}\right)', \theta_f^{(i)} \right) d \left(\theta_2^{(i)}\right)' \left(\mathcal{G}_0^{-1}\right) *_{\theta_1^{(i)}} \right. \\
& - \left. (1 - \exp(-x)) \left( \int \Psi^\dagger \left( \left(\theta_2^{(i)}\right)' \right) O_1 \mathcal{G}_0 X \left( \left(\theta_2^{(i)}\right)', \theta_f^{(i)} \right) d \left(\theta_2^{(i)}\right)' \right) \right. \\
& \times \mathcal{G}_0^{-1} \frac{\left(1 + \exp(-x) \left(-z + \frac{1}{2} (y^2 - x^2)\right)\right) \left(1 + \bar{O}_{1,\infty}\right)}{\left(\left(1 + \bar{O}_{1,\infty}\right) + \exp(-x) \left(-\bar{O}_{1,\infty} + y \left(1 + \bar{O}_{1,2}\right) - x \left(1 + \bar{O}_{1,\infty}\right)\right)\right)} *_{\theta_1^{(i)}} \\
& - \exp(-x) \left( \int \Psi^\dagger \left( \left(\theta_2^{(i)}\right)' \right) O_2 \mathcal{G}_0 X \left( \left(\theta_2^{(i)}\right)', \theta_f^{(i)} \right) d \left(\theta_2^{(i)}\right)' \right) \\
& \times \left( \mathcal{G}_0^{-1} \frac{\left(1 + \exp(-x) \left(-z + \frac{1}{2} (y^2 - x^2)\right)\right) \left(1 + \bar{O}_{1,2}\right)}{\left(\left(1 + \bar{O}_{1,\infty}\right) + \exp(-x) \left(-\bar{O}_{1,\infty} + y \left(1 + \bar{O}_{1,2}\right) - x \left(1 + \bar{O}_{1,\infty}\right)\right)\right)} *_{\theta_1^{(i)}} \right) \\
& \times \left( \frac{\left(1 + \exp(-x) \left(-z + \frac{1}{2} (y^2 - x^2)\right)\right)}{\left(\left(1 + \bar{O}_{1,\infty}\right) + \exp(-x) \left(-\bar{O}_{1,\infty} + y \left(1 + \bar{O}_{1,2}\right) - x \left(1 + \bar{O}_{1,\infty}\right)\right)\right)} \Psi \right) \left(\theta_1^{(i)}\right)
\end{aligned}$$

where  $\langle\langle 1 + O_{1,\infty} \rangle\rangle$  is given by (34):

$$\begin{aligned}
O_1 &= \frac{1}{(1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))} \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) \\
O_2 &= \frac{1}{(1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))} \mathcal{G}_0^{-1} (1 + \bar{O}_{1,2}) \\
x &= \langle 1 + O_{1,\infty} \rangle \\
y^2 &= \left\langle \langle (1 + O_{1,2})^{(2)} \rangle \right\rangle_2 \\
z &= \langle 1 + O_{1,\infty} \rangle - \langle 1 \rangle_1 = \langle O_{1,\infty} \rangle
\end{aligned}$$

and with the condition:

$$\theta_f^{(i)} < \left( \theta_2^{(i)} \right)' < \theta_1^{(i)}$$

that is implied by the Heaviside functions in the integrals defining the interaction terms. The factor  $\bar{N}((\theta_i))$  is given by:

$$\begin{aligned}
\bar{N}((\theta_i)) &\simeq \left\langle \frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right) \right)} \right\rangle^\Psi \\
&\times \left\{ 1 - \frac{\left( (1 - \exp(-x)) x \langle (1 + \bar{O}_{1,\infty}) \rangle_{\theta^{(i)}}^\Psi + y \exp(-x) \langle (1 + \bar{O}_{1,2}) \rangle_{\theta^{(i)}}^\Psi \right)}{\left( \langle 1 \rangle^\Psi \right)^2 \left( 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \right)} \right\}
\end{aligned}$$

which is also approximated locally:

$$\begin{aligned}
\frac{\delta [\Delta \Omega] \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} &= \mathcal{G}_0^{-1} \frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right) \right)} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\
&+ \exp \left( \int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( \left( \theta^{(i)} \right) d\theta^{(i)} \right) \right) \\
&\times \left( \Psi^\dagger \left( \theta_f^{(i)} \right) \mathcal{G}_0^{-1} \frac{1}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right) \right)} \left( \mathcal{G}_0^{-1} \right)_{\theta_1^{(i)}} \right. \\
&- \left. \left( \Psi^\dagger \left( \theta_f^{(i)} \right) O_1 \frac{1}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right) \right)^2} \right) \right. \\
&\times \left. \left( \mathcal{G}_0^{-1} \frac{(1 - \exp(-x)) \left( 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \right) (1 + \bar{O}_{1,\infty})}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right) \right)} \right)_{\theta_1^{(i)}} \right) \\
&- \left. \left( \Psi^\dagger \left( \theta_f^{(i)} \right) O_2 \frac{1}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right) \right)^2} \right) \right. \\
&\times \left. \left( \mathcal{G}_0^{-1} \frac{\exp(-x) \left( 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \right) (1 + \bar{O}_{1,2})}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right) \right)} \right)_{\theta_1^{(i)}} \right) \\
&\times \left. \left( \frac{\left( 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \right)^2}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right) \right)} \right) \Psi \left( \theta_1^{(i)} \right) \right)
\end{aligned}$$

This expression can be rewritten more compactly:

$$\begin{aligned}
\frac{\delta [\Delta\Omega] \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} &= \mathcal{G}_0^{-1} \frac{F}{\Theta} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\
&+ \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F}{\Theta} \Psi \right] \exp \left( \int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( \left( \theta^{(i)} \right) d\theta^{(i)} \right) \right) \left[ \int_{\theta_f^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{F}{\Theta} \right] \\
&- \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,\infty}) \Psi \right] \exp \left( \int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( \left( \theta^{(i)} \right) d\theta^{(i)} \right) \right) \\
&\times \left[ \int_{\theta_f^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{(1 - \exp(-x)) F}{\Theta^2} (1 + \bar{O}_{1,\infty}) \right] \\
&- \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + O_{1,2}) \Psi \right] \exp \left( \int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( \left( \theta^{(i)} \right) d\theta^{(i)} \right) \right) \left[ \int_{\theta_f^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{\exp(-x) F}{\Theta^2} (1 + O_{1,2}) \right]
\end{aligned}$$

where:

$$\begin{aligned}
F &= 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \\
\Theta &= ((1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})))
\end{aligned} \tag{44}$$

In the local approximation for  $\theta_1^{(i)} \simeq \theta_f^{(i)}$ , we thus obtain:

$$\begin{aligned}
\frac{\delta [\Delta\Omega] \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} &= \mathcal{G}_0^{-1} \frac{F}{\Theta} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) + \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F}{\Theta} \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{F}{\Theta} \right] \\
&- \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,\infty}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{(1 - \exp(-x)) F}{\Theta^2} (1 + \bar{O}_{1,\infty}) \right] \\
&- \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,2}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{\exp(-x) F}{\Theta^2} (1 + \bar{O}_{1,2}) \right]
\end{aligned}$$

and:

$$\begin{aligned}
\bar{N} \left( \left( \theta_i \right) \right) &\simeq \left\langle \frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{\left( \left( 1 + O_{1,\infty} \right) + \exp(-x) \left( -O_{1,\infty} + \left( y \left( 1 + O_{1,2} \right) - x \left( 1 + O_{1,\infty} \right) \right) \right)} \right\rangle^\Psi \\
&\times \left\{ 1 - \frac{\left( 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \right) \left( \left( 1 - \exp(-x) \right) x \langle \left( 1 + O_{1,\infty} \right) \rangle_{\theta^{(i)}}^\Psi + y \exp(-x) \langle \left( 1 + O_{1,2} \right) \rangle_{\theta^{(i)}}^\Psi \right)}{\left\langle \left( \left( 1 + O_{1,\infty} \right) + \exp(-x) \left( -O_{1,\infty} + \left( y \left( 1 + O_{1,2} \right) - x \left( 1 + O_{1,\infty} \right) \right) \right) \right)^2 \right\rangle^\Psi} \right\}
\end{aligned}$$

Appendix 3.1 shows that the two points correlation functions can be approximated locally by:

$$\begin{aligned} \left( \frac{\delta [\Delta\Omega] \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} \right)^{-1} &= \mathcal{G}_0^{-1} \frac{F}{\Theta} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\ &+ F^2 \left( \Psi^\dagger \mathcal{G}_0^{-1} \Psi \right) \left( \frac{(\alpha(x) - Fx \exp(-x)) \left( -\bar{\zeta} + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right)}{(x + \exp(-x)) ((1) - x + (y^2 - x^2))^4} \right) \\ &- F^2 \left( \Psi^\dagger \mathcal{G}_0^{-1} \Psi \right) \left( \frac{(\beta(x) - F \exp(-x)) y \left( -\bar{\zeta}_2 + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,2} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) + \beta(x)}{(x + \exp(-x)) ((1) - x + (y^2 - x^2))^4} \right) \end{aligned}$$

This expression will be used below to compute the effective frequencies.

## 7 Equation for frequencies:

### 7.1 General form

The frequencies can be found using the two-points correlation functions. We have found the solution for the background field. This solution is written  $\Psi_0 \left( \theta^{(j)}, Z_j \right) + \delta \Psi \left( \theta, Z \right)$ . The second derivative:

$$\frac{\delta^2 \Gamma \left( \Psi \right)}{\delta \Psi^\dagger \left( \theta_i^{(i)}, Z_i \right) \delta \Psi \left( \theta_i^{(i)}, Z_i \right)} \quad (45)$$

evaluated at  $\Psi_0 \left( \theta^{(j)}, Z_j \right) + \delta \Psi \left( \theta, Z \right)$  yields the inverse Green function. Appendix 5 computes the quadratic part  $\Psi^\dagger \left( \theta, Z \right) \nabla_{\theta N} \left[ \Psi \right] \Psi \left( \theta, Z \right)$  of (45) and identifies  $N \left[ \Psi \right]$  with the inverse effective frequency at position  $(\theta, Z)$  in time and space. It yields:

$$\begin{aligned} \omega_e^{-1} \left( J \left( \theta \right), \theta, Z \right) &\simeq \omega^{-1} \left( J \left( \theta \right), \theta, Z, \mathcal{G}_0 + \left| \Psi \left( \theta^{(j)}, Z_j \right) \right| \right) \\ &+ \omega_1^{-1} \left( J \left( \theta \right), \theta, Z, \mathcal{G}_0 + \left| \Psi \right|^2 \right) + \omega_2^{-1} \left( J \left( \theta \right), \theta, Z, \left| \Psi \right|^2 \right) \end{aligned} \quad (46)$$

where  $\omega \left( J \left( \theta^{(i)} \right), \theta^{(i)}, Z_i, \bar{\mathcal{G}}_0 + \left| \Psi \right|^2 \right)$  is the solution of:

$$\omega \left( \theta^{(i)}, Z \right) = F \left( J \left( \theta \right) + \frac{\kappa}{N} \int T \left( Z, Z_1 \right) \frac{\omega \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right)}{\omega \left( \theta, Z \right)} W \left( \frac{\omega \left( \theta, Z \right)}{\omega \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right)} \right) \left( \bar{\mathcal{G}}_0 \left( 0, Z_1 \right) + \left| \Psi \left( \theta^{(j)}, Z_1 \right) \right|^2 \right) dZ_1 \right) \quad (47)$$

The two other terms,  $\omega_1^{-1}$  and  $\omega_2^{-1}$ , are corrections due to the interactions in the system.

First, function  $\omega_1^{-1}$  is defined by its derivatives at  $\mathcal{G}_0 \left( 0, Z \right)$ :

$$\frac{\delta^n \omega_1^{-1} \left( J \left( \theta \right), \theta, Z, \mathcal{G}_0 \right)}{\delta^n \mathcal{G}_0 \left( 0, Z \right)} = \frac{\delta^n \omega^{-1} \left( J \left( \theta \right), \theta, Z, \mathcal{G}_0 \right)}{\delta^n \mathcal{G}_0 \left( 0, Z \right)} \hat{\Xi}_{1,l} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}, \left| \Psi \right|^2 \right) \quad (48)$$

where:

$$\hat{\Xi}_{1,l} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}, \left| \Psi \right|^2 \right) = \sum_{p=0}^{\infty} \frac{1}{p!} \hat{\Xi}_{1,p+l} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \times \left( \int \Psi^\dagger \left( \theta_f^{(j)}, Z_j \right) \Psi \left( \theta_i^{(j)}, Z_j \right) dZ_j \right)^p$$

Equation (48) implies that the frequency  $\omega_1 \left( J(\theta), \theta, Z, \int \mathcal{G}_0(0, Z_j) + \left| \Psi(\theta^{(j)}, Z_j) \right|^2 \right)$  satisfies an equation similar to (47) whose coefficients are modified by  $\tilde{\Xi}_{1,l} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}, |\Psi|^2 \right)$ :

$$\begin{aligned} & \omega_1^{-1} \left( J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2 \right) \\ &= \hat{G} \left( J(\theta) + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right)}{\omega(\theta, Z)} W \left( \frac{\omega(\theta, Z)}{\omega \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right)} \right) \right. \\ & \quad \left. \times \left( \mathcal{G}_0(0, Z_1) + \left| \Psi_0 + \delta \Psi \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right) \right|^2 \right) dZ_1 \right) \end{aligned} \quad (49)$$

The derivatives of  $\hat{G}$  estimated at  $\mathcal{G}_0(0, Z_1)$  are computed in Appendix 5. The function  $\hat{G}$  is of order  $\bar{\zeta}$ , the average magnitude of the coefficients  $\bar{\zeta}_n$ . For weak interaction, we show that:

$$\begin{aligned} & \hat{G}^{(n)}(\bar{J} + \bar{\mathcal{G}}_0(0, Z_i)) \\ & \simeq \left( \frac{\omega(J, \theta, Z) \bar{\Xi}_1(\bar{J}, |Z_i - Z|)}{\frac{\kappa}{N} T(Z, Z_1)} \right)^n \\ & \quad \times \left( 1 - \left( \int \frac{\kappa}{N} \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right) \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) F'[J, \omega, \theta, Z, \Psi] \right) \\ & \quad \times \left( \tilde{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}, |\Psi|^2 \right) \right) \end{aligned} \quad (50)$$

Second, frequency  $\omega_2^{-1}(J(\theta), \theta, Z, \Psi)$  is given by:

$$\begin{aligned} & \omega_2^{-1}(J(\theta), \theta, Z, \Psi) \\ &= F^2 (\Psi^\dagger \mathcal{G}_0^{-1} \Psi) \\ & \quad \times \left( \frac{(\alpha(x) - Fx \exp(-x)) \left( \bar{\Xi}_{1,\infty} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) - (\beta(x) - F \exp(-x)) y \left( \bar{\Xi}_{1,2} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right)}{(x + \exp(-x)) (\langle 1 \rangle - x + (y^2 - x^2))^4} \right) \\ & \simeq -F^2 \left( \sqrt{X_0} U''(X_0) \delta \Psi(\theta, Z) - \nabla_\theta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) X_0 \right) \\ & \quad \times \left( \frac{(\alpha(x) - Fx \exp(-x)) \left( \bar{\Xi}_{1,\infty} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) - (\beta(x) - F \exp(-x)) y \left( \bar{\Xi}_{1,2} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right)}{(x + \exp(-x)) (\langle 1 \rangle - x + (y^2 - x^2))^4} \right) \end{aligned}$$

where  $F$  is given by (44)

## 7.2 Static equilibrium

We look for a static solution of (46) for a constant background  $\Psi(\theta^{(1)}, Z_1) \simeq \Psi_0(Z_1)$  and constant current, i.e.  $J = \bar{J}$ ,  $\omega(\theta, Z) = \omega(Z)$ . For a static solution,  $(\Psi^\dagger \mathcal{G}_0^{-1} \Psi) = 0$ , or equivalently:  $\delta \Psi(\theta, Z) = \nabla_\theta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) = 0$ , and:

$$\frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \nabla_{\theta_i^{(j)}}^{out}}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{1}{\Lambda_1} \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}} \right) = 0$$

$$\omega_e^{-1}(J(\theta), \theta, Z) = \omega^{-1}(\bar{J}, Z, \mathcal{G}_0(0, Z) + X_0) + \omega_1^{-1}(\bar{J}, Z, \mathcal{G}_0(0, Z) + X_0)$$

where  $\omega(\bar{J}(Z), \mathcal{G}_0(0, Z_i) + X_0)$  is solution of:

$$\omega(Z) = F \left( \bar{J} + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega(Z_1)}{\omega(Z)} W \left( \frac{\omega(Z)}{\omega(Z_1)} \right) \bar{\mathcal{G}}_0(0, Z_i) dZ_1 \right) \quad (51)$$

and:

$$\bar{\mathcal{G}}_0(0, Z_i) \simeq \mathcal{G}_0(0, Z_i) + X_0 \quad (52)$$

Moreover, in the absence of external source, i.e. for  $J(\theta, Z) = 0$ , the solution of (51) can be written  $\omega_0(Z)$ , which satisfies:

$$\omega_0(Z) = F \left( \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega(Z_1)}{\omega(Z)} W \left( \frac{\omega(Z)}{\omega(Z_1)} \right) \bar{\mathcal{G}}_0(0, Z_i) dZ_1 \right) \quad (53)$$

where  $\frac{1}{N} \int dZ_1$  is normalized to 1.

### 7.3 Linearized differential equation for frequencies

We linearize the dynamic equation for frequencies around some constant equilibrium. We will generalize the result to a position-dependent equilibrium in the next paragraph.

A linearized equation around static equilibrium can be found by considering:

$$\Psi(\theta^{(j)}, Z_j) = \Psi_0(Z_j) + \delta\Psi(\theta^{(j)}, Z_j)$$

where:

$$|\delta\Psi(\theta^{(j)}, Z_j)| \ll |\Psi_0(Z_j)|$$

For a translation independent transfer function, i.e.  $T(Z, Z_1) = T(Z - Z_1)$ , and  $d \gg 1$  along with neglecting border effects, equation (53) simplifies and yields a constant solution:

$$\omega_0 = G \left( \frac{TW(1)}{\bar{\Lambda}} \right) \quad (54)$$

where:

$$T = \frac{\kappa}{N} \int T(Z, Z_1) dZ_1$$

We will also assume that the transfer functions are symmetric, that is:

$$T(Z, Z_1) = T(Z_1, Z) \quad (55)$$

To find the linearized equation for frequencies around the constant background (54), we first note that, given (19) and (55), one has:

$$\begin{aligned} & \frac{\partial}{\partial\omega(\theta, Z)} \left( \frac{\kappa}{N} \int W \left( \frac{\omega}{\omega_1} \right) dZ_1 \right)_{\omega(\theta, Z) = \omega_0} \\ & + \frac{\partial}{\partial\omega_1(\theta, Z)} \left( \frac{\kappa}{N} \int W \left( \frac{\omega}{\omega_1} \right) dZ_1 \right)_{\omega(\theta, Z) = \omega_0} = \frac{\kappa}{N} \int (W'(1) - W'(1)) dZ_1 = 0 \end{aligned}$$

To write the linearized equation (46) around  $\omega_0$ , we first solve (47). We start by finding a linearized equation for  $\omega \left( J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2 \right)$ , considering the other terms in the right hand side of (46) as corrections. The equation for  $\omega \left( J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2 \right)$  is:

$$\begin{aligned} & \omega \left( J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2 \right) \\ & = F \left( J(\theta) + \int \frac{\kappa T(Z, Z_1)}{N} \frac{\omega \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right)}{\omega(\theta, Z)} W \left( \frac{\omega(\theta, Z)}{\omega \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right)} \right) \right. \\ & \quad \left. \times \left( \mathcal{G}_0(0, Z_1) + \left| \Psi_0 + \delta\Psi \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right) \right|^2 \right) dZ_1 \right) \end{aligned}$$

Then, we then set  $\omega(J, \theta, Z) = \omega_0 + \Omega(\theta, Z)$ . A first approximation for small variation  $\Omega(\theta, Z)$  around  $\omega_0$  allows to rewrite (47) as a linearized expansion around the solution of (51). The function  $F$  in (47) can be expressed as:

$$\begin{aligned}
& F \left( J(\theta) + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)}{\omega(\theta, Z)} W \left( \frac{\omega(\theta, Z)}{\omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)} \right) \right. \\
& \quad \times \left. \left( \mathcal{G}_0(0, Z_1) + \left| \Psi_0 + \delta\Psi\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right) \right|^2 \right) dZ_1 \right) \\
& \simeq F \left( J(\theta) + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)}{\omega(\theta, Z)} W \left( \frac{\omega_0 + \Omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)}{\omega_0 + \Omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)} \right) \right. \\
& \quad \times \left. \left( \bar{\mathcal{G}}_0(0, Z_1) + \Psi_0 \delta\Psi\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right) dZ_1 \right) \right)
\end{aligned} \tag{56}$$

Similarly, the transfer function can be written:

$$\begin{aligned}
T(Z, Z_1, \omega, \omega_1) & \equiv T(Z, Z_1) W \left( \frac{\omega}{\omega_1} \right) \\
& \simeq T(Z, Z_1) W \left( 1 + \frac{\Omega(J, \theta, Z) - \Omega\left(J, \theta - \frac{|Z-Z_1|}{c}, Z_1\right)}{\omega_0} \right)
\end{aligned}$$

We also use the local linear approximation for  $\delta\Psi(\theta, Z)$  derived in Appendix 4.4.2:

$$\delta\Psi(\theta, Z) \simeq N_1 \nabla_{\theta} \omega(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) - N_2 \nabla_{\theta}^2 \omega(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) \tag{57}$$

The expansion of (56) for a non-static current is:

$$\Omega(\theta, Z) = \left( \hat{f}_1 + N_1 \right) \nabla_{\theta} \Omega(\theta, Z) + \hat{f}_3 \nabla_Z^2 \Omega(\theta, Z) + \left( \hat{f}_3 - N_2 \right) \nabla_{\theta}^2 \Omega(\theta, Z) + J\left(\theta^{(i)}, Z_i\right) \Gamma_0 \tag{58}$$

where we defined:

$$\begin{aligned}
\hat{f}_1 & = \frac{W'(1) - W(1)}{c} \Gamma_1, \quad \hat{f}_3 = \frac{(W(1) - W'(1)) \Gamma_2}{c^2} \\
\Gamma_1 & = \frac{\kappa}{N X_r} \int \frac{|Z - Z_1| T(Z, Z_1) dZ_1}{\omega_0} \bar{\mathcal{G}}_0(0, Z_i) \Gamma_0 \\
\Gamma_2 & = \frac{\kappa}{2N X_r} \int \frac{(Z - Z_1)^2 T(Z, Z_1) dZ_1}{\omega_0} \bar{\mathcal{G}}_0(0, Z_i) \Gamma_0 \\
\Gamma_0 & = F' \left( \frac{\kappa}{N} \int T(Z, Z_1) W(1) dZ_1 \bar{\mathcal{G}}_0(0, Z_i) \right)
\end{aligned}$$

To include the corrective terms in (46), we rewrite this equation as:

$$\begin{aligned}
& \omega_e(J(\theta), \theta, Z) \\
& \simeq \omega\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) \\
& \quad - \left( \omega_1^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) + \omega_2^{-1}\left(J(\theta), \theta, Z, \Psi\right) \right) \\
& \quad \times \left( \omega\left(\bar{J}, Z, \mathcal{G}_0 + |\Psi|^2\right) \right)^2
\end{aligned} \tag{59}$$

The corrections are proportional to two terms  $\omega_1^{-1}$  and  $\omega_2^{-1}$ :

$$\omega_1^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) \left( \omega\left(\bar{J}, Z, \mathcal{G}_0 + X_0\right) \right)^2$$



and

$$\omega_2^{-1}(J(\theta), \theta, Z, \Psi) (\omega(\bar{J}, Z, \mathcal{G}_0 + X_0))^2$$

In the linear approximation, using (47) and (49), we can combine the two first terms in (59):

$$\begin{aligned} & \omega\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) \\ & - \left(\omega_1^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right) (\omega(\bar{J}, Z, \mathcal{G}_0 + X_0))^2 \\ & \simeq \hat{\omega}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) \end{aligned}$$

where  $\hat{\omega}$  satisfies:

$$\begin{aligned} & \hat{\omega}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) \\ & = (F - \delta F) \left( J(\theta) + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)}{\omega(\theta, Z)} W \left( \frac{\omega_0 + \Omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)}{\omega_0 + \Omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)} \right) \right. \\ & \quad \left. \times \left( \bar{\mathcal{G}}_0(0, Z_i) + \Psi_0 \delta \Psi \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right) dZ_1 \right) \right) \end{aligned}$$

with  $F - \delta F = F - \hat{G}F^2$ . The factor  $\tilde{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}, |\Psi|^2\right)$  arising in the definition of  $\omega_1^{-1}$  (see 48) shows that  $\omega_1^{-1}$  is a deformation of the effective frequency due to the interaction of the oscillations with the global potential. As expected, the potential stabilizes the oscillations and thus mitigates the amplitude of the waves.

The second correction to the frequency results from the evolution of the background. Given (57),  $\omega_2^{-1}(J(\theta), \theta, Z, \Psi)$  can be written in terms of frequencies:

$$\begin{aligned} & \omega_2^{-1}(J(\theta), \theta, Z, \Psi) \\ & \simeq -F^2 \left( \left( \sqrt{X_0} U''(X_0) N_1 + \omega^2\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) X_0 \right) \nabla_\theta \omega\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) \right. \\ & \quad \left. - N_2 \nabla_\theta^2 \omega\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) \right) \\ & \quad \times \left( \frac{(\alpha(x) - Fx \exp(-x)) \left( \tilde{\Xi}_{1,\infty}\left(Z_i, \theta_i^{(i)}, \{Z_j\}_{j \neq i}\right) \right) - (\beta(x) - F \exp(-x)) y \left( \tilde{\Xi}_{1,2}\left(Z_i, \theta_i^{(i)}, \{Z_j\}_{j \neq i}\right) \right)}{(x + \exp(-x)) (\langle 1 \rangle - x + (y^2 - x^2))^4} \right) \end{aligned}$$

To sum up, in the local approximation, frequencies are described by a wave equation whose form depends on the background field. This wave is deformed by the stabilization potential and the evolution of the background itself.

## 7.4 Non constant background frequency

In the previous paragraph, we considered translation-invariant transfer functions. This hypothesis, although correct in first approximation, does not hold in general. Finite volume of the system or border conditions, for instance, may invalidate this hypothesis. We will thus consider transfer functions of the form  $T(Z, Z_1)$ . To make things simpler, we will dismiss the corrections to the frequencies due to the potential and the background field.

The derivation of the linearized expansion of (46) around  $\omega_0(Z)$  is similar to that of (58) but now yields:

$$\sigma_\theta^2 \nabla_\theta^2 \hat{\Omega}(\theta, Z) = g_0(Z) \hat{\Omega}(\theta, Z) - g_1(Z) \nabla_\theta \hat{\Omega}(\theta, Z) + g_2(Z) \nabla_Z^2 \hat{\Omega}(\theta, Z) \quad (60)$$

where we defined:

$$\begin{aligned}
\hat{\Omega}(\theta, Z) &= \frac{\Omega(\theta, Z)}{\omega_0(Z)} \\
g_1(Z) &= \frac{\Gamma'_1(Z) - \Gamma_1(Z)}{c\sigma_\theta^2 + \frac{\Gamma'_2(Z) - \Gamma_2(Z)}{c}} \\
g_2(Z) &= \frac{\Gamma'_2(Z) - \Gamma_2(Z)}{\sigma_\theta^2 + \frac{\Gamma'_2(Z) - \Gamma_2(Z)}{c^2}} \\
\Gamma_1(Z) &= \frac{\kappa}{NX_r} \int \frac{|Z - Z_1| T(Z, Z_1) \frac{\omega_0(Z_1)}{\omega_0(Z)} W\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) dZ_1}{\omega_0(Z) \sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} \Gamma_0(Z) \\
\Gamma'_1 &= \frac{\kappa}{NX_r} \int \frac{|Z - Z_1| T(Z, Z_1) \frac{\omega_0(Z_1)}{\omega_0(Z)} W'\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) dZ_1}{\omega_0(Z) \sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} \Gamma_0 \\
\Gamma_2 &= \frac{\kappa}{2NX_r} \frac{\int (Z - Z_1)^2 T(Z, Z_1) \frac{\omega_0(Z_1)}{\omega_0(Z)} W\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) dZ_1}{\omega_0(Z) \sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} \Gamma_0 \\
\Gamma'_2(Z) &= \frac{\kappa}{2NX_r} \frac{\int (Z - Z_1)^2 T(Z, Z_1) \frac{\omega_0(Z_1)}{\omega_0(Z)} W'\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) dZ_1}{\omega_0(Z) \sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} \Gamma_0(Z) \\
\Gamma_0(Z) &= G' \left( \frac{\kappa}{N} \int \frac{T(Z, Z_1) W\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) dZ_1}{\sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} \right)
\end{aligned}$$

Note that equation (60) is a wave equation in an inhomogeneous medium.

## 7.5 Arbitrary transfer functions

A straightforward generalization of (58) can be derived by considering anisotropic transfer functions. Until now we have assumed that:

$$\int (Z - Z_1)_i (Z - Z_1)_j T(Z, Z_1) \frac{\omega_0(Z_1)}{\omega_0(Z)} W\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) dZ_1 = \delta_{i,j}$$

where  $\delta_{i,j}$  is the Kronecker symbol. Relaxing this condition, we can replace  $f_2(Z) \rightarrow f_2^{ij}(Z)$ ,  $g_2(Z) \rightarrow g_2^{ij}(Z) = \frac{f_2^{ij}(Z)}{1+f_3(Z)}$ . Equation (58) becomes:

$$\nabla_\theta^2 \Omega(\theta, Z) = g_0(Z) \Omega(\theta, Z) + g_1(Z) \nabla_\theta \Omega(\theta, Z) + g_2^{ij}(Z) \nabla_{Z_i} \nabla_{Z_j} \Omega(\theta, Z)$$

for distributions:

$$\begin{aligned}
f_2(Z) &= (\omega_0 W'(1) - W(1)) \Gamma_2^{ij} \\
\Gamma_2^{ij} &= \frac{\kappa}{2NX_r} \frac{\int (Z - Z_1)_i (Z - Z_1)_j T(Z, Z_1) dZ_1}{\omega_0 \sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} \Gamma_0
\end{aligned}$$

## 8 Implications

We compute the Green functions associated to equations (58) and (60) to assess the implications of the wave equations. The Green functions allow to find the propagation of an external signal at some particular points to the all thread.

### 8.1 Green functions and external signals

The Green function of (58) and (60) are found using the usual Fourier representation. We will focus on the retarded Green functions that model the wave propagation initiated by a source.

#### 8.1.1 Constant background frequency

Let us first consider (58). For  $g_1 < 0$ , oscillations dampen over time at a rate  $|g_1|$ , and are magnified in time for  $g_1 > 0$ . Since we assumed oscillations of small magnitude around the equilibrium, this implies that our model breaks down above a certain range of amplitudes. A non-linear mechanism of regulation is probably involved at some point to drive the system back to the equilibrium. However, we are mainly interested in oscillatory patterns, and will assume  $|g_1| \ll 1$ . As a consequence, equation (58) reduces to:

$$\nabla_{\theta}^2 \Omega(\theta, Z) = g_2 \nabla_Z^2 \Omega(\theta, Z) + g_0 \Omega(\theta, Z) \quad (61)$$

which is a Klein Gordon equation. We normalize it by setting  $g_2 = 1$  and write  $g_0 = m^2$ . Using the Fourier representation of (61), the retarded Green function of (58) is given by:

$$\mathcal{G}(Z, Z', t, t') = \int dk \frac{\exp(ik \cdot (Z - Z') - i\omega_k(t - t'))}{\omega_k} H(t - t') \quad (62)$$

with  $\omega_k = \sqrt{k^2 + m^2}$ . The integral can be computed and we have:

$$\mathcal{G}(Z, Z', t, t') = H(t - t') \left( \frac{1}{2\pi} \delta(t - t') - \frac{m J_1 \left( m \sqrt{(t - t')^2 - (Z - Z')^2} \right)}{\sqrt{(t - t')^2 - (Z - Z')^2}} \right) \quad (63)$$

where  $J_1$  is the  $n = 1$  Bessel function. To inspect the implications of (63), it is sufficient to approximate (63) for small oscillations. This corresponds to  $g_0 \gg g_2$ , i.e.  $m^2 > 1$ . As a consequence, we can expand  $\sqrt{k^2 + m^2}$  at the lowest order in  $\frac{k^2}{m^2}$ , and write (62) as:

$$\mathcal{G}(Z, Z', t, t') \simeq \int dk \frac{\exp \left( ik \cdot (Z - Z') - i \left( m + \frac{k^2}{m} \right) (t - t') \right)}{m} H(t - t') \quad (64)$$

up to terms of order  $\frac{1}{m^3}$ . Computing the Fourier transform in (64), the function  $\mathcal{G}_0(Z, Z', t, t')$  can be approximated by:

$$\mathcal{G}(Z, Z', t, t') = \exp \left( i \left( \frac{m(Z - Z')^2}{2(t - t')} - m(t - t') \right) \right) H(t - t') \quad (65)$$

Equation (65) shows that the Green function  $\mathcal{G}(Z, Z', t, t')$  represents the path integral of a particle under the constant potential  $m$ .

#### 8.1.2 Non-constant background frequency

The Green function of equation (60) is a generalization of (63). It has been studied in the context of covariant quantum field theory, but (65) shows that we can produce a path integral formulation for the Green function. If  $g_2(Z)$  varies slowly with  $Z$ , the analog of (65) with non-constant coefficients is:

$$\mathcal{G}(Z, Z', t, t') = \int \exp \left( i \left( \int_{z(t')=Z'}^{z(t)=Z} \left( \frac{\sqrt{\frac{g_0(z(s))}{g_2(z(s))}}}{2} \left( \frac{dz(s)}{ds} \right)^2 - \sqrt{g_0(z(s))} \right) ds \right) \right) Dz(s) H(t-t') \quad (66)$$

where the sum is over paths  $z(s)$  starting from  $Z'$  and ending at  $Z$  in a time span of  $t-t'$ . The derivation of (66) is straightforward. Neglecting  $g_1(Z)$  as in the derivation of (61), (60) writes:

$$\sigma_\theta^2 \nabla_\theta^2 \hat{\Omega}(\theta, Z) = g_0(Z) \hat{\Omega}(\theta, Z) + g_2(Z) \nabla_Z^2 \hat{\Omega}(\theta, Z)$$

then, cutting the time span  $t-t'$  into slices  $\Delta t$  such that  $g_0(Z)$  and  $g_2(Z)$  can be considered as constant in a domain of radius  $c\Delta t$ , the Green function for a time span  $\Delta t$  is given by a formula similar to (65), except that  $g_2(Z) \neq 1$ :

$$\mathcal{G}(z(s+\Delta t), z(s), \Delta t) = \exp \left( i \left( \frac{\sqrt{\frac{g_0(z(s))}{g_2(z(s))}}}{2} \frac{(z(s+\Delta t) - z(s))^2}{\Delta t} - g_0(z(s)) \Delta t \right) \right) \quad (67)$$

The convolution of (67) over the time slices then yields (65).

## 8.2 Propagation of external signals

### 8.2.1 Constant coefficients

The Green function (65) allows to compute the diffusion of an external source along the thread by convolution. Assume an external source:

$$J(t, Z) = \exp(-i\omega_0 t) \delta(Z - Z_0) \quad (68)$$

which describes a signal located in  $Z_0$ , with frequency  $\omega_0$ . Using (65), the amplitude  $\Omega(t, Z)$  is:

$$\begin{aligned} \Omega(t, Z) &= \int \exp \left( i \left( \frac{m(Z - Z_0)^2}{2(t-t')} - \omega_0 t - (m - \omega_0)(t-t') \right) \right) H(t-t') dt' \\ &= \frac{\exp(-i\omega_0 t - i\sqrt{m}|(m - \omega_0)||Z - Z_0| + i\pi)}{\sqrt{|(m - \omega_0)|}} \end{aligned}$$

and for a signal including a whole range of frequencies:

$$\hat{f}(t, Z) = \int f(\omega_0) \exp(-i\omega_0 t) d\omega_0 \quad (69)$$

the corresponding response of the thread is:

$$\Omega(t, Z) = \int \frac{\exp(-i\omega_0 t - i\sqrt{m}|(m - \omega_0)||Z - Z_0| + i\pi)}{\sqrt{|(m - \omega_0)|}} f(\omega_0) d\omega_0$$

Assume that the range of frequencies in (69) is such that  $m - \omega_0 > 0$ . Then:

$$\begin{aligned} \Omega(t, Z) &= \int \frac{\exp(-i\omega_0 t - i\sqrt{m}(m - \omega_0)|Z - Z_0| + i\pi)}{\sqrt{|(m - \omega_0)|}} f(\omega_0) d\omega_0 \\ &= \int \frac{\exp(-i\omega_0(t - \sqrt{m}|Z - Z_0|))}{\sqrt{|(m - \omega_0)|}} f(\omega_0) d\omega_0 \frac{\exp(-i(\sqrt{m})^3|Z - Z_0| + i\pi)}{\sqrt{|(m - \omega_0)|}} \end{aligned}$$

To simplify, we also assume that the frequencies of the signal satisfy  $|\omega_0| \ll m$ , so that:

$$\Omega(t, Z) \simeq \hat{f}(t - \sqrt{m}|Z - Z_0|, Z_0) \frac{\exp(-i(\sqrt{m})^3|Z - Z_0| + i\pi)}{m} \quad (70)$$

At time  $t$ , the frequencies present the whole past history of the signal, which is thus recorded in the system of oscillations. The result (70) can be extended for several independent sources located in two points  $Z_1, Z_2$  emitting some signal  $\hat{f}_1(t)$  and  $\hat{f}_2(t)$  with frequencies below  $m$ . In that case, the response is:

$$\begin{aligned} \Omega(t, Z) \simeq & \hat{f}_1(t - \sqrt{m}|Z - Z_0|) \frac{\exp\left(-i(\sqrt{m})^3|Z - Z_0| + i\pi\right)}{\sqrt{|(m - \omega_0)|}} \\ & + \hat{f}_2(t - \sqrt{m}|Z - Z_0|) \frac{\exp\left(-i(\sqrt{m})^3|Z - Z_0| + i\pi\right)}{\sqrt{|(m - \omega_0)|}} \end{aligned} \quad (71)$$

As usual in waves dynamics, the response defined by (71) may present some interference phenomena, depending on  $\hat{f}_1$  and  $\hat{f}_2$ .

### 8.2.2 Non constant coefficients

Formula (71) may be useful to understand the implications of position-dependent coefficients in (66). Assume a thread divided in two regions, each characterized by some constant coefficients  $g_0$  and  $g_2$ . We also assume that these regions are only connected via two "entry points". This can be modelled by  $g_2 = 0$  on the border between the two regions, and  $g_2 \gg 1$  at these two points.

The path integral description (66) implies that the contributions to the Green function of the paths that do not cross the border at the points  $Z_1$  or  $Z_2$  cancel, due to large oscillations in the vicinity of the border. As a consequence, the paths contributing to the Green function have to cross at  $Z_1$  or  $Z_2$ , inducing some interference phenomenon (71) on the transmitted signal.

More generally, this dependency in  $Z$  along the paths impacts the result even for a simple signal (68). Actually, the contribution to the Green function (71) of the various paths reaching a point  $Z$  of the thread acquire a phase depending on the path and on the characteristic of the medium encountered. This may create some interference between these paths. One may conjecture that trained networks will present some particular learned features in their transfer functions, i.e. the coefficients  $g_0(Z)$  and  $g_2(Z)$ , that would produce some constructive interference for some signals, and destructive for some others.

## 9 $(l, m)$ points correlation functions and probabilistic interpretation

### 9.1 General setup

Appendices 3.2 and 3.3 show how to compute the  $(l, m)$  points correlation functions by successive derivatives of (43). Neglecting the interactions, the correlation function is given by tensor powers of  $\left(\frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger(\theta^{(i)})}\right)^{-1}$ . We give here the expressions for strong and weak background fields. Appendix 3.3.1 shows that, for strong background fields, the  $m$ -th tensor power of (43) becomes:

$$\begin{aligned} & \left(\left(\frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger(\theta^{(i)})}\right)^{-1}\right)^{\otimes m} \delta_{l,m} \\ & = \left(\left(1 + \bar{O}_{1,\infty}\right) \mathcal{G}_0 \left(1 + \exp\left(\int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N}\left(\left(\theta^{(i)}\right) d\theta^{(i)}\right)\right) \Psi\left(\theta_1^{(i)}\right) \Psi^\dagger\left(\theta_2^{(i)}\right) \nabla_{\theta\omega^{-1}}\left(J\left(\theta_2^{(i)}\right), \theta_2^{(i)}, Z, \mathcal{G}_0\right)\right)^{-1}\right)^{\otimes m} \\ & \simeq \left(\left(1 + \bar{O}_{1,\infty}\right) \mathcal{G}_0 \left(1 + \exp\left(\int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N}\left(\left(\theta^{(i)}\right) d\theta^{(i)}\right)\right) \Psi\left(\theta_1^{(i)}\right) \Psi^\dagger\left(\theta_2^{(i)}\right) \nabla_{\theta} G\left(J\left(\theta_2^{(i)}\right), \theta_2^{(i)}, Z, \mathcal{G}_0\right)\right)^{-1}\right)^{\otimes m} \end{aligned} \quad (72)$$

For weak background fields, the no-interaction part of the  $(l, m)$  points correlation function is:

$$\begin{aligned} & \left( \mathcal{G}_0^{-1} \frac{1 + \exp(-x) \left(-z + \frac{1}{2} (y^2 - x^2)\right)}{(1 + O_{1,\infty}) + \exp(-x) (-O_{1,\infty} + y(1 + O_{1,2}) - x(1 + O_{1,\infty}))} \left(\theta_1^{(i)}, \theta_f^{(i)}\right) \right)^{-\otimes m} \delta_{l,m} \quad (73) \\ & \simeq \mathcal{G}_0^{\otimes m} \delta_{l,m} = \delta(Z - Z') \frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2 X_r}\right)^2 + \frac{2\alpha}{\sigma^2}} - \frac{1}{\sigma^2 X_r}\right) (\theta - \theta')\right)}{\Lambda} H(\theta - \theta') \end{aligned}$$

These formulas show that for strong background the interacting dynamics of  $2l$  cells is mediated by the collective background, represented here by the source term  $\Delta\Omega$  that depends itself on the background field  $\Psi$ .

Appendix 3.3.1 computes the corrections to (72) and (73) due to the interactions. The general formula for the 1PI vertex  $\frac{\delta^{l+1, m+1}[\Delta\Omega](\theta_2^{(i)})}{\delta[\Psi\Psi^\dagger](\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right))}$  is recursive:

$$\begin{aligned} \frac{\delta^{l+1, m+1}[\Delta\Omega](\theta_1^{(i)})}{\delta[\Psi\Psi^\dagger](\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right))} &= M * \left(\frac{\delta}{\delta\Psi}\right)^l \left(\frac{\delta}{\delta\Psi^\dagger}\right)^m X^{1,1}(\theta_1^{(i)}, (\theta_1^{(i)\dagger})) \quad (74) \\ &+ M * \sum_{\substack{0 \leq r \leq l, 0 \leq r' \leq m \\ 1 \leq r+r'}} C_l^r C_m^{r'} \left( \int \frac{\delta^{r, r'} K(\theta_1^{(i)}, \theta_2^{(i)})}{\delta[\Psi\Psi^\dagger](\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right))} \frac{\delta^{l-r, m-r'}[\Delta\Omega](\theta_2^{(i)})}{\delta[\Psi\Psi^\dagger](\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right))} d\theta_2^{(i)} \right) \\ &= M * \left(\frac{\delta}{\delta\Psi}\right)^l \left(\frac{\delta}{\delta\Psi^\dagger}\right)^m X^{1,1}(\theta_1^{(i)}, (\theta_1^{(i)\dagger})) \\ &+ M * \sum_{\substack{r_1 + \dots + r_l = l \\ s_1 + \dots + s_m = m \\ 0 \leq r_i, 0 \leq s_i, 1 \leq r_i + s_i}} \frac{l!}{r_1! \dots r_p!} \frac{m!}{s_1! \dots s_{p'}!} \\ &\times \left[ \frac{\delta^{r_1, s_1} K}{\delta[\Psi\Psi^\dagger]} \right] * \dots * \left[ M * \frac{\delta^{r_p, s_{p'}} K(\theta_1^{(i)}, \theta_2^{(i)})}{\delta[\Psi\Psi^\dagger]} \right] * X^{1,1}(\theta_1^{(i)}, (\theta_1^{(i)\dagger})) \end{aligned}$$

where:

$$X^{1,1}(\theta_1^{(i)}, \theta_f^{(i)}) = \mathcal{G}_0^{-1} \frac{1 + \exp(-x) \left(-z + \frac{1}{2} (y^2 - x^2)\right)}{(1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))} \left(\theta_1^{(i)}, \theta_f^{(i)}\right)$$

and the kernel  $K(\theta_1^{(i)}, \theta_2^{(i)})$  has a matricial form. It is defined by:

$$\begin{aligned}
& K(\theta_1^{(i)}, \theta_2^{(i)}) \tag{75} \\
&= - \int \left( [A(\theta_1^{(i)})] [\Delta\Omega]^\dagger \left( (\theta_2^{(i)})' \right) \left[ (1 + \bar{O}_{1,\infty})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right. \\
&\quad \left. + [B(\theta_1^{(i)})] [\Delta\Omega]^\dagger \left( (\theta_2^{(i)})' \right) \left[ (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right) \mathcal{G}_0 d(\theta_2^{(i)})' \\
&\quad + [\Delta\Omega](\theta_1^{(i)}) [\Psi]^\dagger(\theta_2^{(i)}) \delta \left( (\theta_2^{(i)})' - \theta_2^{(i)} \right) \\
&= \left( - \int \left( \left[ A'(\theta_1^{(i)}, (\theta_1^{(i)})') \right] [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Delta\Omega]^\dagger \left( (\theta_2^{(i)})' \right) \left[ (1 + \bar{O}_{1,\infty})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right. \right. \\
&\quad \left. \left. + \left[ B'(\theta_1^{(i)}, (\theta_1^{(i)})') \right] [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Delta\Omega]^\dagger \left( (\theta_2^{(i)})' \right) \left[ (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right) \right) \mathcal{G}_0 d(\theta_2^{(i)})' d(\theta_1^{(i)})' \\
&\quad + [\Delta\Omega](\theta_1^{(i)}) [\Psi]^\dagger(\theta_2^{(i)}) \delta \left( (\theta_2^{(i)})' - \theta_2^{(i)} \right)
\end{aligned}$$

with  $A'(\theta_1^{(i)}, (\theta_1^{(i)})')$  and  $B'(\theta_1^{(i)}, (\theta_1^{(i)})')$  given by:

$$\begin{aligned}
A'(\theta_1^{(i)}, (\theta_1^{(i)})') &\simeq \mathcal{G}_0^{-1} \frac{(1 - \exp(-x))(1 + \bar{O}_{1,\infty})_{\theta_1^{(i)}}}{(1 + \bar{O}_{1,\infty}) + \exp(-x)(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))} \\
B'(\theta_1^{(i)}, (\theta_1^{(i)})') &\simeq \mathcal{G}_0^{-1} \frac{(1 + \bar{O}_{1,2})_{\theta_1^{(i)}, (\theta_1^{(i)})'} \exp(-x)}{(1 + \bar{O}_{1,\infty}) + \exp(-x)(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))}
\end{aligned}$$

and the matrices involved in the definition of  $K(\theta_1^{(i)}, \theta_2^{(i)})$ :

$$\begin{aligned}
[\Delta\Omega]^\dagger &= (\Delta\Omega^\dagger, \Delta\Omega), \quad [A(\theta_1^{(i)})] = \begin{pmatrix} A(\theta_1^{(i)}) \\ A^\dagger(\theta_1^{(i)}) \end{pmatrix}, \quad [B(\theta_1^{(i)})] = \begin{pmatrix} B(\theta_1^{(i)}) \\ B^\dagger(\theta_1^{(i)}) \end{pmatrix} \\
\left[ (1 + \bar{O}_{1,\infty})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] &= \begin{pmatrix} (1 + \bar{O}_{1,\infty})_{(\theta_2^{(i)})', \theta_2^{(i)}} & 0 \\ 0 & (1 + \bar{O}_{1,\infty})_{\theta_2^{(i)}, (\theta_2^{(i)})'} \end{pmatrix} \\
\left[ (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] &= \begin{pmatrix} (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} & 0 \\ 0 & (1 + \bar{O}_{1,2})_{\theta_2^{(i)}, (\theta_2^{(i)})'} \end{pmatrix} \\
\left[ A'(\theta_1^{(i)}, (\theta_1^{(i)})') \right] &= \begin{pmatrix} A'(\theta_1^{(i)}, (\theta_1^{(i)})') & 0 \\ 0 & A'((\theta_1^{(i)})', \theta_1^{(i)}) \end{pmatrix} \\
\left[ B'(\theta_1^{(i)}, (\theta_1^{(i)})') \right] &= \begin{pmatrix} B'(\theta_1^{(i)}, (\theta_1^{(i)})') & 0 \\ 0 & B'((\theta_1^{(i)})', \theta_1^{(i)}) \end{pmatrix}
\end{aligned}$$

The successive derivatives of  $K(\theta_1^{(i)}, \theta_2^{(i)})$  are given by:

$$\begin{aligned}
& \frac{\delta^{r,r'} K(\theta_1^{(i)}, \theta_2^{(i)})}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right)\right)} \\
= & [\Psi]^\dagger(\theta_2^{(i)}) \frac{\delta^{r,r'}[\Delta\Omega](\theta_1^{(i)})}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right)\right)} + \delta(\theta^{(i)\dagger} - \theta_2^{(i)}) \frac{\delta^{r,r'-1}[\Delta\Omega](\theta_1^{(i)})}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right)\right)} \\
& - \sum \int \left( \frac{\delta^{t,t'} \left[ A'(\theta_1^{(i)}, (\theta_1^{(i)})') \right]}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right)\right)} \frac{\delta^{r-s-t,r'-s'-t'}[\Delta\Omega]\left(\left(\theta^{(i)}\right)'\right)}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right)\right)} \frac{\delta^{s,s'}[\Delta\Omega]^\dagger\left(\left(\theta_2^{(i)}\right)'\right)}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right)\right)} \left[ (1 + \bar{O}_{1,\infty})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right. \\
& + \frac{\delta^{t,t'} \left[ B'(\theta_1^{(i)}, (\theta_1^{(i)})') \right]}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right)\right)} \frac{\delta^{r-s-t,r'-s'-t'}[\Delta\Omega]\left(\left(\theta^{(i)}\right)'\right)}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right)\right)} \frac{\delta^{s,s'}[\Delta\Omega]^\dagger\left(\left(\theta_2^{(i)}\right)'\right)}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right), \left(\theta^{(i)\dagger}\right)\right)} \\
& \left. \times \left[ (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right) \mathcal{G}_0 d(\theta_2^{(i)})' d(\theta_1^{(i)})'
\end{aligned} \tag{76}$$

Formula (74), (75) and (76) allow to find the 1PI  $n$ -th vertex. Appendix 3.3.2 performs the computation in the strong field and weak field approximation.

## 9.2 Strong field approximation

Appendix 3.3.2 shows that in the strong field approximation, the 1PI  $n$ -th vertex are:

$$\begin{aligned}
& \left( \int \Delta\Omega^\dagger (1 + O_{1,\infty}) \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^l \left( \int \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger(\theta^{(i)})} (1 + O_{1,\infty}) \Delta\Omega \right)^m \\
& \times \exp\left( \frac{(y^2 - x^2) \exp(-x)}{2} \left( \int [\Delta\Omega]^\dagger(\theta_2^{(i)}) [(1 + O_{1,\infty})] [\Delta\Omega] \right) \right)
\end{aligned}$$

and that the connected Green function are given by:

$$\begin{aligned}
& F_{l,m}(\Psi) \left( \int \Delta\Omega^\dagger (1 + O_{1,\infty}) \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^l \left( \int \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger(\theta^{(i)})} (1 + O_{1,\infty}) \Delta\Omega \right)^m \\
& \times \exp\left( \frac{(y^2 - x^2) \exp(-x)}{2} \left( \int [\Delta\Omega]^\dagger(\theta_2^{(i)}) [(1 + O_{1,\infty})] [\Delta\Omega] \right) \right)
\end{aligned}$$

where  $F_{l,m}(\Psi)$  is an increasing factor in the norm of the background field  $\Psi$ :

$$\begin{aligned}
F_{l,m}(\Psi) & \simeq \sum_{k=0}^{m-3} \left( \exp(-(x-\alpha)) \int \Psi^\dagger\left(\left(\theta_1^{(i)}\right)\right) \Psi\left(\left(\theta_2^{(i)}\right)\right) \right)^k \\
& \simeq \frac{1}{1 - \exp(-(x-\alpha)) \int \Psi^\dagger\left(\left(\theta_1^{(i)}\right)\right) \Psi\left(\left(\theta_2^{(i)}\right)\right)}
\end{aligned}$$



The same Appendix computes also the  $(l, m)$  correlation functions:

$$\begin{aligned}
& \left( \left( \frac{\delta \Delta \Omega (\theta_1^{(i)})}{\delta \Psi (\theta_2^{(i)})} \right)^{-1} \right)^l \delta_{l,m} + \\
& \sum_{s=0}^{\inf(l,m)} \sum_{k \geq 1} \sum_{\substack{s_1 \geq \dots \geq s_k \geq 0, t_1 \geq \dots \geq t_k \geq 0 \\ s_i + t_i \geq 1, \sum s_i = l-s, \sum t_i = m-s}} \left( \prod_{i=1}^k F_{s_i, t_i} (\Psi) \right) \left( \left( \frac{\delta \Delta \Omega (\theta_1^{(i)})}{\delta \Psi (\theta_2^{(i)})} \right)^{-1} \right)^s \\
& \times \left( \Psi^\dagger \left( (\theta_1^{(i)}) \right) \right)^{l-s-1} \left( \Psi \left( (\theta_2^{(i)}) \right) \right)^{m-s-1} \\
& \times \left( \frac{(y^2 - x^2) \exp(-x)}{2} \left( \left( \frac{\delta \Delta \Omega (\theta_1^{(i)})}{\delta \Psi (\theta_2^{(i)})} \right)^{-1} + \left( \Psi^\dagger \left( (\theta_1^{(i)}) \right) \right) \left( \Psi \left( (\theta_2^{(i)}) \right) \right) \right)^{-1} \\
& \times \exp \left( \frac{(y^2 - x^2) \exp(-x)}{2} \left( \int [\Delta \Omega]^\dagger (\theta_2^{(i)}) [(1 + O_{1,\infty})] [\Delta \Omega] \right) \right)^k
\end{aligned} \tag{77}$$

where permutations over the  $2l$  points are implicit.

### 9.3 Weak field approximation

In the weak field approximation, the correlation functions for  $l \neq m$  are negligible. Appendix 3.3.3 shows that the 1PI  $n$ -th vertex is:

$$\begin{aligned}
\frac{\delta^{l,l-1} \Delta \Omega (\theta_1^{(i)})}{\delta [\Psi \Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} & \simeq M * \left( \frac{\delta}{\delta \Psi} \right)^l \left( \frac{\delta}{\delta \Psi^\dagger} \right)^l X^{1,1} \\
& + \left[ M * \frac{\delta^{1,1} K}{\delta [\Psi \Psi^\dagger]} \right] * \dots * \left[ M * \frac{\delta^{1,1} K}{\delta [\Psi \Psi^\dagger]} \right] * X^{1,1}
\end{aligned}$$

which can be approximated by:

$$\frac{\delta^{l,l-1} \Delta \Omega (\theta_1^{(i)})}{\delta [\Psi \Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \simeq (-1)^l \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l-2} * \bar{O}_{1,\infty} * \mathcal{G}_0$$

From the vertices, the connected correlation functions can be retrieved:

$$\begin{aligned}
\mathcal{G}_C^{(l)} & \equiv \left( (-1)^l \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l-2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right) \\
& + (-1)^l \sum_{k \geq 2} \sum_{l_1 + \dots + l_k = l} \prod_{m=1}^k \sum_{*_{\mathcal{G}_0}^{\{l_m\}}} *_{\mathcal{G}_0}^{\{l_m\}} \left( \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l_m-2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right)
\end{aligned}$$

where the sum over  $*_{\mathcal{G}_0}^{\{l_m\}}$  denotes the sum over all possible convolutions between the blocks:

$$\left( \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l_m-2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right)$$

Two blocks are convoluted on at most one variable. The convolution are performed by insertion of a propagator  $\mathcal{G}_0$  between the blocks. The expression for the connected correlation functions induces the full correlation functions:

$$\begin{aligned}
& \mathcal{G}_0^{\otimes l} + \sum_{p,k} (-1)^{l-p} \sum_{l_k, \sum l_n = l-p} \prod_k \mathcal{G}_C^{(l_n)} \mathcal{G}_0^{\otimes p} \tag{78} \\
= & \mathcal{G}_0^{\otimes l} + \sum_{p,k} (-1)^{l-p} \sum_{l_n, \sum l_n = l-p} \prod_n \left( \left( (-1)^{l_n} \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l_n - 2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right) \right. \\
& \left. + (-1)^{l_n} \sum_{k \geq 2} \sum_{l_1 + \dots + l_k = l_n} \prod_{m=1}^k \sum_{\substack{\{l_m\} \\ *_{\mathcal{G}_0} \{l_m\}}} \left( \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l_m - 2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right) \right) \otimes \mathcal{G}_0^{\otimes p}
\end{aligned}$$

And this expression describes a sum of cell-to-cell interactions.

## 9.4 Interpretation: joined probabilities for frequencies

### 9.4.1 Principle

Equations (77) and (78) may be interpreted in terms of joined probabilities for frequencies at different points of the thread. To explain this point, start with the two points correlation functions. At the zeroth order in perturbation, the function  $\left( \frac{\delta[\Delta\Omega](\theta_1^{(i)})}{\delta\Psi(\theta_j^{(i)})} \right)^{-1}$  is the Green function of the operator:

$$-\nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1} (J(\theta), \theta, Z, \mathcal{G}_0) \right) + \alpha$$

which is (27):

$$\mathcal{G}_0(\theta, \theta', Z) = \delta(Z - Z') \frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}} - \frac{1}{\sigma^2 \bar{X}_r}\right)(\theta - \theta')\right)}{\Lambda} H(\theta - \theta')$$

This function is the Laplace transform of the function  $\hat{G}_{0Z}(\theta, \theta', \Delta n)$ :

$$\mathcal{G}_0(\theta, \theta', Z) = \int \hat{G}_{0Z}(\theta, \theta', \Delta n) \exp(-\alpha \Delta n) d\alpha$$

The form of  $\hat{G}_{0Z}(\theta, \theta', \Delta n)$  is not necessary here.

The function  $\hat{G}_{0Z}$  computes the probability of a time interval  $\theta - \theta'$  for  $\Delta n$  spikes of the potential at point  $Z$ . The Laplace transform  $\mathcal{G}_0(\theta, \theta', Z)$  computes the probability of a time interval  $\theta - \theta'$  for a random number of spikes  $\Delta n$  with average  $\frac{1}{\alpha}$ . Since the of spikes' frequency is  $\frac{\Delta n}{\theta - \theta'}$ ,  $\mathcal{G}_0(\theta, \theta', Z)$  computes the average probability of a frequency  $\frac{1}{\alpha(\theta - \theta')}$  of spikes. Computing the average  $\langle(\theta - \theta')\rangle$  confirms this point:

$$\begin{aligned}
\mathcal{G}_0(\theta, \theta', Z) &= \delta(Z - Z') \frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}} - \frac{1}{\sigma^2 \bar{X}_r}\right)(\theta - \theta')\right)}{\Lambda} H(\theta - \theta') \\
&\simeq \delta(Z - Z') \frac{\exp(-\alpha \bar{X}_r (\theta - \theta'))}{\Lambda} H(\theta - \theta')
\end{aligned}$$

so that  $\langle(\theta - \theta')\rangle = \frac{1}{\alpha \bar{X}_r}$ . The average inverse frequency is then  $\alpha \langle(\theta - \theta')\rangle = \frac{1}{\bar{X}_r}$ .

As a consequence, the expression of  $\mathcal{G}_0(\theta, \theta', Z)$  computed at  $\alpha = 1$  can be interpreted as the probability, at time  $\frac{\theta + \theta'}{2}$ , of a spikes' frequency equal to  $\frac{1}{\theta - \theta'}$ . The same applies for higher order correlation functions.

We show in Appendix 3 that  $\frac{\delta^{l,i} \Delta\Omega(\theta_1^{(i)}, \theta_2^{(i)})}{\delta[\Psi\Psi^\dagger](\theta^{(i)}, \theta^{(i)\dagger})}$  computes the transition probability of  $\theta^{(i)\dagger}$  to  $\theta^{(i)}$  for

$i = 1 \dots l$  for an average number of spikes of  $\frac{1}{\alpha}$ , so that:

$$\left( \frac{\delta^{l,l} \Delta \Omega \left( \theta_1^{(i)}, \theta_2^{(i)} \right)}{\delta [\Psi \Psi^\dagger] \left( \left( \theta^{(i)}, \theta^{(i)\dagger} \right) \right)} \right)_{\alpha=1} = P \left( \omega \left( Z_1, \frac{\theta^{(1)} + \theta^{(1)\dagger}}{2} \right) = \frac{1}{\theta^{(1)} - \theta^{(1)\dagger}}, \dots, \omega \left( Z_l, \frac{\theta^{(l)} + \theta^{(l)\dagger}}{2} \right) = \frac{1}{\theta^{(l)} - \theta^{(l)\dagger}} \right) \quad (79)$$

computes the joined probability for a set of  $l$  frequencies at points  $Z_1, \dots, Z_l$  and times  $\frac{\theta^{(1)} + \theta^{(1)\dagger}}{2}, \dots, \frac{\theta^{(l)} + \theta^{(l)\dagger}}{2}$ . Equation (79) can be rewritten in terms of density for the set of variables  $\theta^{(i)} = \frac{\theta^{(i)} + \theta^{(i)\dagger}}{2}$  and  $\omega^{(i)} = \frac{1}{\theta^{(1)} - \theta^{(1)\dagger}}$ :

$$P \left( \left( \omega^{(i)}, \theta^{(i)} \right)_{1 \leq i \leq l} \right) = \left( \prod_{i=1}^l \frac{1}{(\omega^{(i)})^2} \right) \left( \frac{\delta^{l,l} \Delta \Omega \left( \theta_1^{(i)}, \theta_2^{(i)} \right)}{\delta [\Psi \Psi^\dagger] \left( \left( \theta^{(i)}, \theta^{(i)\dagger} \right) \right)} \right)_{\alpha=1}$$

Using (79) we can now interpret (77) and (78): the first terms  $\left( \left( \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^{-1} \right)^m$  and  $\mathcal{G}_0^{\otimes m}$  represent an independent distribution for the frequencies at different point, and the corrective terms measure the mutual dependencies due to the interactions in the background field. Moreover, for  $l = m = 1$ , the probabilistic interpretation is an alternate description to the frequencies' local differential equation. We will examine (77) and (78) independently.

#### 9.4.2 Strong field approximation

In the strong field approximation, the fields  $\Psi^\dagger \left( \theta_1^{(i)} \right)$  and  $\Psi \left( \theta_2^{(i)} \right)$  in (77) can be written as functions of the set of variables  $\theta^{(i)}$  and  $(\omega^{(i)})^{-1}$  as  $\Psi^\dagger \left( \theta^{(i)} + \frac{(\omega^{(i)})^{-1}}{2} \right)$  and  $\Psi \left( \theta^{(i)} - \frac{(\omega^{(i)})^{-1}}{2} \right)$ . We first consider  $l = 1$  and write the individual probabilities  $P \left( \omega^{(i)}, \theta^{(i)} \right)$ , up to some normalization factor:

$$\begin{aligned} (\omega^{(i)})^2 P \left( \omega^{(i)}, \theta^{(i)} \right) &= \left( \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^{-1} \\ &= (1 + \bar{O}_{1,\infty}) * \mathcal{G}_0 \\ &\quad \times \left( 1 + \exp \left( \int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( \left( \theta^{(i)} \right) d\theta^{(i)} \right) \right) \Psi \left( \theta_1^{(i)} \right) \Psi^\dagger \left( \theta_2^{(i)} \right) \nabla_{\theta} G \left( J \left( \theta_2^{(i)} \right), \theta_2^{(i)}, Z, \mathcal{G}_0 \right) \right)^{-1} \end{aligned} \quad (80)$$

The individual probability (80) can be written explicitly. First, the propagator arising in (80) is defined by:

$$\begin{aligned} \mathcal{G}_0 \left( \theta, \theta' \right) &= \delta \left( Z - Z' \right) \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{\exp \left( - \left( \sqrt{\left( \frac{1}{\sigma^2 \bar{X}_r} \right)^2 + \frac{2\alpha}{\sigma^2} - \frac{1}{\sigma^2 \bar{X}_r}} \right) \left( \theta - \theta' \right) \right)}{\sqrt{\left( \frac{1}{\sigma^2 \bar{X}_r} \right)^2 + \frac{2\alpha}{\sigma^2}}} H \left( \theta - \theta' \right) \\ &= \delta \left( Z - Z' \right) \frac{\exp \left( - \frac{\Lambda_1}{\omega^{(i)}} \right)}{\Lambda} H \left( \theta - \theta' \right) \end{aligned} \quad (81)$$

where:

$$\begin{aligned} \Lambda_1 &= \sqrt{\left( \frac{1}{\sigma^2 \bar{X}_r} \right)^2 + \frac{2\alpha}{\sigma^2} - \frac{1}{\sigma^2 \bar{X}_r}} \\ \frac{1}{\bar{X}_r} &\equiv \frac{\arctan \left( \left( \frac{1}{\bar{X}_r} - \frac{1}{\bar{X}_p} \right) \sqrt{J(Z)} \right)}{\sqrt{J(Z)}} \end{aligned} \quad (82)$$

The factor  $\delta(Z - Z') H(\theta - \theta')$  will be skipped and reintroduced ultimately. Formula (81) is computed for an average current over some timespan. This approximation can be relaxed and we can compute the frequency at the average point  $\theta^{(i)}$ . As a consequence, (81) can be computed by setting  $\bar{J} \rightarrow J(\theta^{(i)})$  in (82). This implies that we can replace  $\Lambda_1$  in the formula (82) by:

$$\Lambda_1(\theta^{(i)}) = \sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r(\theta^{(i)})}\right)^2 + \frac{2\alpha}{\sigma^2}} - \frac{1}{\sigma^2 \bar{X}_r(\theta^{(i)})}$$

with:

$$\frac{1}{\bar{X}_r(\theta^{(i)})} \equiv \frac{\arctan\left(\left(\frac{1}{\bar{X}_r} - \frac{1}{\bar{X}_p}\right) \sqrt{J(\theta^{(i)}, Z)}\right)}{\sqrt{J(\theta^{(i)}, Z)}}$$

Ultimately the propagator is:

$$\mathcal{G}_0(\theta, \theta') = \frac{\exp\left(-\frac{\Lambda_1(\theta^{(i)})}{\omega^{(i)}}\right)}{\Lambda} \quad (83)$$

Second, the quantity  $(1 + \bar{O}_{1,\infty}) * \mathcal{G}_0 = (1 + O_{1,\infty}) \mathcal{G}_0$  in (80) is given by:

$$(1 + O_{1,\infty}) \mathcal{G}_0 = \left(1 + \frac{-\bar{\zeta} + \frac{\nabla_{\theta_i^{(j)}}^{out} \bar{\Xi}_{1,\infty}(Z_j, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}}}{\Lambda \left(-\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty}(Z_j, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}}\right)} \left(\exp\left(\hat{\Xi}_{1,\infty}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})\right) - 1\right)\right) \mathcal{G}_0 \quad (84)$$

It can be expressed in the set of variables  $\theta^{(i)}$  and  $\omega^{(i)}$ :

$$(1 + O_{1,\infty}) \mathcal{G}_0 \simeq \left(1 + \frac{-\bar{\zeta}_\infty + \frac{\nabla_{\theta_i^{(i)}}^{out} \bar{\Xi}_{1,\infty}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i})}{\theta_f^{(i)} - \theta_i^{(i)}}}{\Lambda \left(-\bar{\zeta} + \bar{\Xi}_{1,\infty}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i})\right)} \left(\exp\left(\frac{\hat{\Xi}_{1,\infty}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i})}{\omega^{(i)}}\right) - 1\right)\right) \mathcal{G}_0 \quad (85)$$

where:

$$\hat{\Xi}_{1,\infty}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i})$$

and:

$$\bar{\Xi}_{1,\infty}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i})$$

are the average of  $\hat{\Xi}_{1,\infty}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})$  and  $\bar{\Xi}_{1,\infty}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})$  on the time span  $\theta_f^{(j)} - \theta_i^{(j)}$ . In first approximation, it depends on the mid-point  $\theta^{(i)}$ . This dependency arises as a function of the external current  $J(\theta^{(i)})$ .

As a consequence of (85) and (83), we obtain:

$$(1 + \bar{O}_{1,\infty}) * \mathcal{G}_0 = \left(1 + \frac{-\bar{\zeta}_\infty + \frac{\nabla_{\theta_i^{(i)}}^{out} \bar{\Xi}_{1,\infty}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i})}{\theta_f^{(i)} - \theta_i^{(i)}}}{\Lambda \left(-\bar{\zeta} + \bar{\Xi}_{1,\infty}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i})\right)} \left(\exp\left(\frac{\hat{\Xi}_{1,\infty}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i})}{\omega^{(i)}}\right) - 1\right)\right) \times \frac{\exp\left(-\frac{\Lambda_1(\theta^{(i)})}{\omega^{(i)}}\right)}{\Lambda} \quad (86)$$

Using (64), the probability  $P(\omega^{(i)}, \theta^{(i)})$  is given by:

$$\begin{aligned}
& (\omega^{(i)})^2 P(\omega^{(i)}, \theta^{(i)}) \tag{87} \\
&= \left( 1 + \frac{-\bar{\zeta}_\infty + \frac{\nabla_{\theta^{(i)}}^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i})}{\Lambda(-\bar{\zeta} + \bar{\Xi}_{1,\infty}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}))} \left( \exp\left(\frac{\hat{\Xi}_{1,\infty}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i})}{\omega^{(i)}}\right) - 1 \right) \right) \\
&\times \exp\left(\frac{-\Lambda_1(\theta^{(i)})}{\omega^{(i)}}\right) \times \left( 1 + \exp\left(\int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N}((\theta^{(i)}) d\theta^{(i)})\right) \Psi\left(\theta^{(i)} - \frac{(\omega^{(l)})^{-1}}{2}\right) \Psi^\dagger\left(\theta^{(i)} + \frac{(\omega^{(l)})^{-1}}{2}\right) \right) \\
&\times \nabla_{\theta} G\left(J\left(\theta^{(i)} + \frac{(\omega^{(l)})^{-1}}{2}\right), \theta^{(i)} + \frac{(\omega^{(l)})^{-1}}{2}, Z, \mathcal{G}_0\right)^{-1}
\end{aligned}$$

Depending on the sign of  $\nabla_{\theta} G\left(J\left(\theta^{(i)} + \frac{(\omega^{(l)})^{-1}}{2}\right), \theta^{(i)} + \frac{(\omega^{(l)})^{-1}}{2}, Z, \mathcal{G}_0\right)$ , i.e. the variations of  $J(\theta^{(i)})$ , and the magnitude of  $\Psi$ , the denominator:

$$\begin{aligned}
& 1 + \exp\left(\int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N}((\theta^{(i)}) d\theta^{(i)})\right) \Psi\left(\theta^{(i)} - \frac{(\omega^{(l)})^{-1}}{2}\right) \Psi^\dagger\left(\theta^{(i)} + \frac{(\omega^{(l)})^{-1}}{2}\right) \\
&\times \nabla_{\theta} G\left(J\left(\theta^{(i)} + \frac{(\omega^{(l)})^{-1}}{2}\right), \theta^{(i)} + \frac{(\omega^{(l)})^{-1}}{2}, Z, \mathcal{G}_0\right)
\end{aligned}$$

cancels and  $P(\omega^{(i)}, \theta^{(i)})$  has a pole  $\omega_0^{(i)}(J(\theta^{(i)}))$ . The average frequency is switched to this pole and a discontinuity arises. The frequencies concentrate around this pole.

We now consider the general case for  $l$  correlated frequencies. Up to a normalization factor such that:

$\int P\left((\omega^{(i)}, \theta^{(i)})_{1 \leq i \leq l}\right) \prod_{i=1}^l d\omega^{(i)} = 1$ , the joint probability defined by (77) becomes:

$$\begin{aligned}
& \left(\prod_{i=1}^l (\omega^{(i)})^2\right) P\left((\omega^{(i)}, \theta^{(i)})_{1 \leq i \leq l}\right) \\
&= \prod_{i=1}^l P(\omega^{(i)}, \theta^{(i)}) + \sum_{s=0}^l \sum_{k \geq 1} \sum_{\substack{s_1 \geq \dots \geq s_k \geq 0, t_1 \geq \dots \geq t_k \geq 0 \\ s_i + t_i \geq 1, \sum s_i = l - s, \sum t_i = l - s}} \left(\prod_{i=1}^k F_{s_i, t_i}(\Psi)\right) \left(\prod_{i=1}^s P(\omega^{(i)}, \theta^{(i)})\right) \\
&\times \prod_{i=s+1}^{l-1} \Psi^\dagger\left(\theta^{(i)} + (\omega^{(i)})^{-1}\right) \prod_{i=s+1}^{l-1} \Psi\left(\theta^{(i)} - (\omega^{(i)})^{-1}\right) \\
&\times \left(\frac{(y^2 - x^2) \exp(-x)}{2} \left(P(\omega^{(l)}, \theta^{(l)}) + \Psi^\dagger\left(\theta^{(l)} + \frac{(\omega^{(l)})^{-1}}{2}\right) \Psi\left(\theta^{(l)} - \frac{(\omega^{(l)})^{-1}}{2}\right)\right)\right) \\
&\times \exp\left(\frac{(y^2 - x^2) \exp(-x)}{2} \left(\int [\Delta\Omega]^\dagger(\theta_2^{(i)}) [(1 + O_{1,\infty})] [\Delta\Omega]\right)\right)^k
\end{aligned}$$

where permutations over the variables are implicit.

In first approximation, we can replace  $\Psi^\dagger\left(\theta^{(l)} + \frac{(\omega^{(l)})^{-1}}{2}\right) \Psi\left(\theta^{(l)} - \frac{(\omega^{(l)})^{-1}}{2}\right)$  by  $\Psi_0^\dagger \Psi_0$  and for large  $\Psi_0$ , the corrective term can be approximated by a function  $F(\Psi_0^\dagger \Psi_0)$ , so that  $\left(\prod_{i=1}^l (\omega^{(i)})^2\right) P\left((\omega^{(i)}, \theta^{(i)})_{1 \leq i \leq l}\right)$

is driven towards a uniform distribution by the correlations between the points. The first order corrections due to the background field uniformize the pattern of frequencies.

### 9.4.3 Weak field approximation

In the weak field approximation, expression (78) for  $l = 1$  becomes:

$$\begin{aligned} \left(\omega^{(i)}\right)^2 P\left(\omega^{(i)}, \theta^{(i)}\right) &= \frac{(1 + O_{1,\infty}) + \exp(-x)(-O_{1,\infty} + y(1 + O_{1,2}) - x(1 + O_{1,\infty}))}{1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2))} \mathcal{G}_0\left(\theta_1^{(i)}, \theta_f^{(i)}\right) \\ &\simeq \mathcal{G}_0 = \frac{\exp\left(-\frac{\Lambda_1(\theta^{(i)})}{\omega^{(i)}}\right)}{\Lambda} \end{aligned}$$

where (83) is used. We recover the probability for the frequency in the current  $J\left(\theta^{(i)}\right)$ . In the correction factor:

$$\frac{(1 + O_{1,\infty}) + \exp(-x)(-O_{1,\infty} + y(1 + O_{1,2}) - x(1 + O_{1,\infty}))}{1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2))}$$

the contribution:

$$\frac{1 + \exp(-x)(y - x)}{1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2))}$$

can be included in the normalization factor. At the first order of approximation, we are left with:

$$\left(\omega^{(i)}\right)^2 P\left(\omega^{(i)}, \theta^{(i)}\right) \simeq \left(1 + \frac{y}{1 + y - x} O_{1,2}\right) \frac{\exp\left(-\frac{\Lambda_1(\theta^{(i)})}{\omega^{(i)}}\right)}{\Lambda}$$

Writing explicitly  $O_{1,2}$  in terms of  $\omega^{(i)}$ , we find:

$$\begin{aligned} &\left(\omega^{(i)}\right)^2 P\left(\omega^{(i)}, \theta^{(i)}\right) \tag{88} \\ &\simeq \left(1 + \frac{y}{1 + y - x} \frac{-\bar{\zeta}_2 + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,2}\left(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}\right)}{\Lambda \left(-\bar{\zeta} + \bar{\Xi}_{1,2}\left(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}\right)\right)} \left(\exp\left(\frac{\hat{\Xi}_{1,2}\left(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}\right)}{\omega^{(i)}}\right) - 1\right)\right) \\ &\times \frac{\exp\left(-\frac{\Lambda_1(\theta^{(i)})}{\omega^{(i)}}\right)}{\Lambda} \end{aligned}$$

In a background field of small magnitude, the probabilities' formula do not include any pole, and the frequencies are not shifted by an external current. They rather include the minimal interactions  $\bar{\zeta}_2$  that correspond to the minimal level of operating frequencies.

Using (78) and (88), we directly deduce the joined probability for  $l$  frequencies:

$$\begin{aligned} P\left(\left(\omega^{(i)}, \theta^{(i)}\right)_{1 \leq i \leq l}\right) &= \prod_{i=1}^l P\left(\omega^{(i)}, \theta^{(i)}\right) + \sum_{p,k} (-1)^{l-p} \\ &\times \sum_{l_n, \sum l_n = l-p} \prod_n \left( \left( (-1)^{l_n} \hat{P}\left(\left(\omega^{(i_n^{(1)})}, \theta^{(i_n^{(1)})}\right), \dots, \left(\omega^{(i_n^{(l_n)})}, \theta^{(i_n^{(l_n)})}\right)\right)\right) \right. \\ &\left. + (-1)^{l_n} \sum_{k \geq 2} \sum_{l_1 + \dots + l_k = l_n} \prod_{m=1}^k \sum_{\mathcal{G}_0^{\{l_m\}}} \hat{P}\left(\left(\omega^{(i_n^{(1)})}, \theta^{(i_n^{(1)})}\right), \dots, \left(\omega^{(i_n^{(l_m)})}, \theta^{(i_n^{(l_m)})}\right)\right)\right) \right) \mathcal{G}_0^{\otimes p} \end{aligned}$$

where the permutation over the variables  $(\omega^{(i)}, \theta^{(i)})$  are implicit, and  $\cup_m \left\{ \binom{i_n^{(k)}}{k=1, \dots, l_m} \right\} = \{1, \dots, n\}$ . The function  $\hat{P}$  is defined by:

$$\begin{aligned} \hat{P} \left( \left( \omega^{(i_n^{(1)})}, \theta^{(i_n^{(1)})} \right), \dots, \left( \omega^{(i_n^{(l_n)})}, \theta^{(i_n^{(l_n)})} \right) \right) &= P \left( \omega^{(i_n^{(1)})}, \theta^{(i_n^{(1)})} \right) \\ &\times \left( \prod_{s=2}^{l_n-1} P_1 \left( \omega^{(i_n^{(s)})}, \theta^{(i_n^{(s)})} \right) \right) P_2 \left( \omega^{(i_n^{(l_n)})}, \theta^{(i_n^{(l_n)})} \right) \end{aligned}$$

with:

$$\begin{aligned} P_1 \left( \omega^{(i)}, \theta^{(i)} \right) &= \int (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \\ &= (1 + O_{1,\infty}) \mathcal{G}_0 \\ &= P \left( \omega^{(i)}, \theta^{(i)} \right) + P_2 \left( \omega^{(i)}, \theta^{(i)} \right) \end{aligned}$$

$$\begin{aligned} P_2 \left( \omega^{(i)}, \theta^{(i)} \right) &= O_{1,\infty} \mathcal{G}_0 \\ &= \frac{-\bar{\zeta} + \frac{\nabla_{\theta_i^{(j)}}^{out} \bar{\Xi}_{1,\infty} \left( Z_j, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\Lambda_1 \frac{\theta_f^{(j)} - \theta_i^{(j)}}{\theta_f^{(j)} - \theta_i^{(j)}}}}{\Lambda \left( -\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty} \left( Z_j, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}} \right)} \left( \exp \left( \hat{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1 \right) \frac{\exp \left( -\frac{\Lambda_1 (\theta^{(i)})}{\omega} \right)}{\Lambda} \\ &\simeq \frac{-\bar{\zeta} + \frac{\nabla_{\theta_i^{(j)}}^{out} \bar{\Xi}_{1,\infty} \left( Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i} \right)}{\Lambda_1 \frac{\theta_f^{(j)} - \theta_i^{(j)}}{\theta_f^{(j)} - \theta_i^{(j)}}}}{\Lambda \left( -\bar{\zeta} + \bar{\Xi}_{1,\infty} \left( Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i} \right) \right)} \left( \exp \left( \frac{\hat{\Xi}_{1,\infty} \left( Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i} \right)}{\omega} \right) - 1 \right) \frac{\exp \left( -\frac{\Lambda_1 (\theta^{(i)})}{\omega} \right)}{\Lambda} \end{aligned}$$

## 10 Extensions

Several extensions of the formalism may be considered, the details are left for further research.

### 10.1 Excitatory vs inhibitory currents

A first possible extension is to include inhibitory currents. This is done by introducing two different types of cells, each defined by a different field. We write  $\Psi_1(\theta, Z, \omega)$  and  $\Psi_2(\theta, \tilde{Z}, \tilde{\omega})$  for excitatory and inhibitory neurons respectively. The influence of each type of cell on the other one is obtained through the actions of the induced currents. If we assume that the transfer functions are identical for both type of fields, the corresponding action terms for the frequencies are:

$$\begin{aligned} &\frac{1}{2\eta^2} \int |\Psi_1(\theta, Z, \omega)|^2 \left( \omega^{-1} - G \left( J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega_1}{\omega} \left| \Psi_1 \left( \theta - \frac{|Z - Z_1|}{c}, Z_1, \omega_1 \right) \right|^2 T(Z, \theta, Z_1) dZ_1 d\omega_1 \right. \right. \\ &\quad \left. \left. - \int \frac{\kappa}{N} \frac{\tilde{\omega}_1}{\omega} \left| \Psi_2 \left( \theta - \frac{|Z - \tilde{Z}_1|}{c}, \tilde{Z}_1, \tilde{\omega}_1 \right) \right|^2 T(Z, \theta, \tilde{Z}_1) d\tilde{Z}_1 d\tilde{\omega}_1 \right) \right)^2 \end{aligned}$$

and:

$$\frac{1}{2\eta^2} \int |\Psi_2(\theta, \tilde{Z}, \tilde{\omega})|^2 \left( \tilde{\omega}^{-1} - G \left( J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega_1}{\tilde{\omega}} \left| \Psi_1 \left( \theta - \frac{|Z - Z_1|}{c}, Z_1, \omega_1 \right) \right|^2 T(Z, \theta, Z_1) dZ_1 d\omega_1 \right) \right)^2$$

so that the action for the system writes:

$$\begin{aligned}
S &= -\frac{1}{2}\Psi_1^\dagger(\theta, Z, \omega) \nabla \left( \frac{\sigma_\theta^2}{2} \nabla - \omega^{-1} \right) \Psi_1(\theta, Z, \omega) - \frac{1}{2}\Psi_2^\dagger(\theta, \tilde{Z}, \tilde{\omega}) \nabla \left( \frac{\sigma_\theta^2}{2} \nabla - \tilde{\omega}^{-1} \right) \Psi_2(\theta, \tilde{Z}, \tilde{\omega}) \\
&\quad \frac{1}{2\eta^2} \int |\Psi_1(\theta, Z, \omega)|^2 \left( \omega^{-1} - G \left( J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega_1}{\omega} \left| \Psi_1 \left( \theta - \frac{|Z-Z_1|}{c}, Z_1, \omega_1 \right) \right|^2 T(Z, \theta, Z_1) dZ_1 d\omega_1 \right. \right. \\
&\quad \left. \left. - \int \frac{\kappa}{N} \frac{\tilde{\omega}_1}{\omega} \left| \Psi_2 \left( \theta - \frac{|Z-\tilde{Z}_1|}{c}, \tilde{Z}_1, \omega_1 \right) \right|^2 T(Z, \theta, \tilde{Z}_1) d\tilde{Z}_1 d\tilde{\omega}_1 \right) \right)^2 \\
&\quad \frac{1}{2\eta^2} \int |\Psi_2(\theta, \tilde{Z}, \tilde{\omega})|^2 \left( \tilde{\omega}^{-1} - G \left( J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega_1}{\tilde{\omega}} \left| \Psi_1 \left( \theta - \frac{|Z-Z_1|}{c}, Z_1, \omega_1 \right) \right|^2 T(Z, \theta, Z_1) dZ_1 d\omega_1 \right) \right)^2
\end{aligned} \tag{89}$$

As in the section 3.2, we can project the fields on the frequency-dependent states. This leads to the following action:

$$S = -\frac{1}{2}\Psi_1^\dagger(\theta, Z) \nabla \left( \frac{\sigma_\theta^2}{2} \nabla - \omega^{-1} \right) \Psi_1(\theta, Z) - \frac{1}{2}\Psi_2^\dagger(\theta, \tilde{Z}) \nabla \left( \frac{\sigma_\theta^2}{2} \nabla - \tilde{\omega}^{-1} \right) \Psi_2(\theta, \tilde{Z})$$

where the frequencies satisfy:

$$\begin{aligned}
\omega^{-1}(J, \theta, Z, \Psi_1, \Psi_2) &= G \left( J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega}{\omega} \left( J, \theta - \frac{|Z-Z_1|}{c}, Z_1, \Psi \right) \left| \Psi_1 \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right) \right|^2 T(Z, \theta, Z_1) dZ_1 \right. \\
&\quad \left. - \int \frac{\kappa}{N} \frac{\tilde{\omega}}{\omega} \left( J, \theta - \frac{|Z-Z_1|}{c}, \tilde{Z}_1, \Psi \right) \left| \Psi_2 \left( \theta - \frac{|Z-\tilde{Z}_1|}{c}, \tilde{Z}_1 \right) \right|^2 T(Z, \theta, \tilde{Z}_1) d\tilde{Z}_1 \right)
\end{aligned}$$

$$\begin{aligned}
\tilde{\omega}^{-1}(J, \theta, Z, \Psi_1, \Psi_2) &= G \left( J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega}{\tilde{\omega}} \left( J, \theta - \frac{|Z-Z_1|}{c}, Z_1, \Psi \right) \frac{T(Z, \theta, Z_1, \theta - \frac{|Z-Z_1|}{c})}{\tilde{\omega}} \right. \\
&\quad \left. \times \left| \Psi_2 \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right) \right|^2 dZ_1 \right)
\end{aligned}$$

As in section 3.3, a collective potential can be added:

$$\sum_{n=2}^{\infty} \zeta^{(n)} \int |\Psi(\theta, Z)|^2 \left( \prod_{i=1}^{n-1} \left| \Psi \left( \theta - \frac{|Z-Z_i|}{c}, Z_i \right) \right|^2 \right)$$

where we define:

$$|\Psi(\theta, Z)|^2 = |\Psi_1(\theta, Z)|^2 + |\Psi_2(\theta, Z)|^2$$

to model the fact that the equilibrium activity includes both excitatory and inhibitory activities.

## 10.2 Full system dynamics

Until now, we have considered the dynamics of frequencies alone. The transfer functions were considered in first approximation as depending on the frequencies. We show briefly how to generalize the model by including dynamic oscillations for the transfer function. We will work in constant background frequency to simplify the formula, but the computations can be straightforwardly generalized to a position dependent background. The computations are presented in Appendix 2.



To account for the dynamic nature of the transfer functions  $T(Z, Z_1, \omega, \omega_1)$ , equation (58) must be modified and associated with an evolution equation for  $T(Z, Z_1, \omega, \omega_1)$ . Using (56), we replace  $T(Z, Z_1, \omega, \omega_1)$  by a general function  $T(Z, Z_1, \theta)$  that is a priori independent from frequencies. Around the equilibrium defined by the background frequency  $\omega_0$ , the function  $T(Z, Z_1, \theta)$  then writes:

$$T(Z, Z_1, \theta) = T_0(Z, Z_1) + h(Z, Z_1) \hat{T}(Z, \theta, Z_1)$$

where  $T_0(Z, Z_1)$  is the transfer function in this equilibrium. The function  $\hat{T}(Z, \theta, Z_1)$  represents the fluctuations around this equilibrium. The expansion of  $G$  around  $\omega_0$  becomes:

$$\begin{aligned} G & \left( \frac{\kappa}{N} \int \frac{\omega_0 + \Omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right) - \Omega(\theta, Z)}{\omega_0 \sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2} \alpha}} T(Z, Z_1, \omega, \omega_1) dZ_1 \right) \\ & \simeq \omega_0 + \Gamma_0 \left( \int \left( \frac{\Omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right) - \Omega(\theta, Z)}{\omega_0 \sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2} \alpha}} T_0(Z, Z_1) + \frac{h(Z, Z_1) \hat{T}(Z, \theta, Z_1)}{\sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2} \alpha}} \right) \right) \end{aligned}$$

As a consequence, equation (58) is replaced by:

$$\sigma_\theta^2 \nabla_\theta^2 \Omega(\theta, Z) = \Omega(\theta, Z) + \frac{\Gamma_1}{c} \nabla_\theta \Omega(\theta, Z) - \Gamma_2 \nabla_Z^2 \Omega(\theta, Z) - \frac{\Gamma_2}{c^2} \nabla_\theta^2 \Omega(\theta, Z) - \Gamma_0 \hat{T}(Z, \theta)$$

where we defined:

$$\hat{T}(Z, \theta) = \int \frac{h(Z, Z_1) \hat{T}(Z, \theta, Z_1)}{\sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2} \alpha}}$$

and:

$$\begin{aligned} \Gamma_1 & = \frac{\kappa}{NX_r} \int \frac{|Z - Z_1| T_0(Z, Z_1) dZ_1}{\omega_0 \sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2} \alpha}} \Gamma_0 \\ \Gamma_2 & = \frac{\kappa}{2NX_r} \int \frac{(Z - Z_1)^2 T_0(Z, Z_1) dZ_1}{\omega_0 \sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2} \alpha}} \Gamma_0 \\ \Gamma_0 & = G' \left( \frac{\kappa}{N} \int \frac{T_0(Z, Z_1) dZ_1}{\sqrt{\frac{\pi}{8} \left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2} \alpha}} \right) \end{aligned}$$

Appendix 2 derives the dynamics for  $\hat{T}(Z, \theta)$  and yields a system of dynamic equations for  $(\Omega(\theta, Z), \hat{T}(Z, \theta))$ :

$$\begin{aligned} \sigma_\theta^2 \nabla_\theta^2 \Omega(\theta, Z) & = \Omega(\theta, Z) - f_1(Z) \nabla_\theta \Omega(\theta, Z) + f_2(Z) \nabla_Z^2 \Omega(\theta, Z) - f_3(Z) \nabla_\theta^2 \Omega(\theta, Z) \\ & + \Gamma_0 \nabla_Z^2 \hat{T}(Z, \theta) \end{aligned} \quad (90)$$

$$\begin{aligned} & \frac{\nabla_\theta^2 \hat{T}(Z, \theta)}{\lambda} + U_1(\omega_0) \nabla_\theta \hat{T}(Z, \theta) + U_2(\omega) \hat{T}(Z, \theta) \\ & = \left( \rho \bar{C}(Z) h'_C(\omega_0) - \frac{\rho \left( D(Z) \hat{T}_0(Z) h'_D(\omega_0) + \bar{C}_0(Z) h'_C(\omega_0) \right)}{\lambda \tau} \right) \Omega(Z, \theta) \\ & + \frac{\rho D(Z) h'_D(\omega_0) \left( \Gamma_1 \nabla_\theta \Omega(Z, \theta) - \left( \Gamma_1 \nabla_\theta^2 \Omega(Z, \theta) + \Gamma_2 \nabla_Z^2 \Omega(Z, \theta) \right) \right)}{\lambda \tau} \end{aligned} \quad (91)$$

with:

$$\begin{aligned}\bar{C}(Z) &= \frac{1}{\sqrt{\frac{\pi}{8} \left(\frac{1}{\bar{X}_r}\right)^2 + \frac{\pi}{2}\alpha}} \int h(Z, Z_1) C(Z_1) \\ \bar{C}_0(Z) &= \frac{1}{\sqrt{\frac{\pi}{8} \left(\frac{1}{\bar{X}_r}\right)^2 + \frac{\pi}{2}\alpha}} \int h(Z, Z_1) C(Z_1) \hat{T}_0(Z, Z_1) \\ \hat{T}_0(Z) &= \frac{1}{\sqrt{\frac{\pi}{8} \left(\frac{1}{\bar{X}_r}\right)^2 + \frac{\pi}{2}\alpha}} \int h(Z, Z_1) \hat{T}_0(Z, Z_1)\end{aligned}$$

Note that for a slowly varying  $\hat{T}(Z, \theta)$ , formula (91) is similar to (58).

## Appendix 1. Vertices of (18) involved in the computation of the $2n$ Green functions

To find the effective action associated to (18) and the collective term (22), we proceed in several steps. The first one is to find the vertices involved in the computation of the Green functions. To do so, we will expand the action (18) in series of field. This produces a series of an infinite series of vertices. However, given that the two points Green function are not symmetric by time reversal, we will show that only the  $2n$  first terms are involved in the computation of the  $2n$  Green functions. We will then estimate these vertices using the recursive relation (16) between frequencies depending on field. These results will be used in the next section to find the graph expansion of the system's partition function.

### 1.1 Estimation of the two points Green function

We start with the two points Green function and prove (29). To do so, we will expand the action functional in series of the field  $\Psi$ . The two points Green function will be computed by using the "free" action's propagator defined by (18) and obtained by replacing  $\omega^{-1}(J, \theta, Z, \Psi)$  by  $\omega^{-1}(J, \theta, Z, 0)$ . The free action is:

$$S_0 = -\frac{1}{2} \Psi^\dagger(\theta, Z) \nabla_\theta \left( \frac{\sigma^2}{2} \nabla_\theta - \omega^{-1}(J, \theta, Z, 0) \right) \Psi(\theta, Z) \quad (92)$$

and the series in field will be considered, as usual, as a perturbation expansion.

#### 1.1.2 "Free" action propagator.

Now, we compute the propagator associated to (92). We decompose the external current into a static and a time dependent parts  $\bar{J} + J(\theta)$  where  $\bar{J}$  can be thought as the time average of the current. We will consider that  $|\bar{J}(Z)| > |J(\theta, Z)|$ . At zeroth order in current  $J(\theta)$ , the function  $\omega^{-1}(J, \theta, Z, 0)$  satisfies:

$$\begin{aligned}\omega^{-1}(J, \theta, Z, 0) &= G(\bar{J} + J(\theta)) \\ &\simeq G(\bar{J}(Z)) = \frac{\arctan\left(\left(\frac{1}{\bar{X}_r} - \frac{1}{\bar{X}_p}\right) \sqrt{\bar{J}(Z)}\right)}{\sqrt{\bar{J}(Z)}} = \frac{1}{\bar{X}_r(Z)} \equiv \frac{1}{\bar{X}_r}\end{aligned} \quad (93)$$

where the dependence in  $Z$  of  $\bar{X}_r$  will be understood. As a consequence  $\omega(\theta, Z)$  is thus approximatively equal to  $\bar{X}_r$ . Under this approximation:

$$S_0 = -\Psi^\dagger(\theta, Z) \nabla_\theta \left( \frac{\sigma^2}{2} \nabla_\theta - \frac{1}{\bar{X}_r} \right) \Psi(\theta, Z)$$

and the Green function of the operator  $\nabla_\theta \left( \frac{\sigma^2}{2} \nabla_\theta - \frac{1}{\bar{X}_r} \right)$  is computed as:

$$\langle \Psi^\dagger(\theta, Z) \Psi(\theta', Z) \rangle \equiv \mathcal{G}_0((\theta, Z), (\theta', Z')) \equiv \mathcal{G}_0(\theta, \theta', Z) = \delta(Z - Z') \int \frac{\exp(ik(\theta - \theta'))}{\frac{\sigma^2}{2} k^2 + ik \frac{1}{\bar{X}_r} + \alpha} dk \quad (94)$$

The right hand side of (94) can be computed as:

$$\begin{aligned} \int \frac{\exp(ik(\theta - \theta'))}{\frac{\sigma^2}{2} k^2 + ik \frac{1}{\bar{X}_r} + \alpha} dk &= \exp\left(\frac{\theta - \theta'}{\sigma^2 \bar{X}_r}\right) \int \frac{\exp(ik(\theta - \theta'))}{\frac{\sigma^2}{2} k^2 + \frac{1}{2} \left(\frac{1}{\sigma \bar{X}_r}\right)^2 + \alpha} dk \\ &= \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{\exp\left(-\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}} |\theta - \theta'|\right)}{\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}}} \exp\left(\frac{\theta - \theta'}{\sigma^2 \bar{X}_r}\right) \end{aligned} \quad (95)$$

and this is quickly suppressed for  $\theta - \theta' < 0$ . This is the direct consequence of non-hermiticity of operator. In the sequel, for  $\sigma^2 \bar{X}_r \ll 1$ , we can thus consider that:

$$\mathcal{G}_0(\theta, \theta', Z) = \delta(Z - Z') \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}} - \frac{1}{\sigma^2 \bar{X}_r}\right) (\theta - \theta')\right)}{\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}}} H(\theta - \theta') \quad (96)$$

where  $H$  is the Heaviside function:

$$\begin{aligned} H(\theta - \theta') &= 0 \text{ for } \theta - \theta' < 0 \\ &= 1 \text{ for } \theta - \theta' > 0 \end{aligned}$$

This form of the propagator is sufficient to compute the graphs expansion in the next paragraphs. We can check that the corrections due to a non-static current do not modify the result. Considering:

$$G(J(\theta, Z)) = \frac{\arctan\left(\left(\frac{1}{\bar{X}_r} - \frac{1}{\bar{X}_p}\right) \sqrt{J(\theta, Z)}\right)}{\sqrt{J(\theta, Z)}}$$

For relatively high frequency firing rates, i.e., small periods of time between two spikes, we replace (94) by the Green function of:

$$\nabla_\theta \left( \frac{\sigma^2}{2} \nabla_\theta - G(J(\theta, Z)) \right) \simeq \nabla_\theta \left( \frac{\sigma^2}{2} \nabla_\theta - \frac{1}{\bar{X}_r} - J(\theta, Z) G'(\bar{J}) \right)$$

and  $\omega^{-1}(G_{0Z}(0, 0))$  by:

$$\begin{aligned} \mathcal{G}_0((\theta, Z), (\theta', Z')) &= \delta(Z - Z') \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}} - \frac{1}{\sigma^2 \bar{X}_r}\right) (\theta - \theta')\right)}{\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}}} H(\theta - \theta') \\ &\quad \times \left( 1 - \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{G'(\bar{J})}{\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}}} \int_\theta^{\theta'} J(\theta'', Z) d\theta'' \right) \end{aligned}$$

as a consequence since  $J(\theta, Z)$  is the deviation around the static part  $\bar{J}$ , the corrective term vanishes quickly as  $\theta - \theta'$  increases.

### 1.1.3 perturbation expansion and the two points Green function

Formula (96) allows to compute higher order contributions to the Green function of the action (18) by using a graph expansion. Actually, writing  $\omega^{-1}(\theta, Z)$  for  $\omega^{-1}(J, \theta, Z, \Psi)$  when no ambiguity is possible, the higher order contribution for the series expansion of  $\omega^{-1}(\theta, Z)$  in fields are obtained by solving recursively:

$$\omega^{-1}(J, \theta, Z) = G \left( J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega \left( J, \theta - \frac{|Z-Z_1|}{c}, Z_1 \right)}{\omega(J, \theta, Z)} \left| \Psi \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right) \right|^2 T(Z, \theta, Z_1) dZ_1 d\omega_1 \right) \quad (97)$$

This will be done precisely in the next paragraph. For now, it is enough to note that given (97), the recursive expansion in  $\omega^{-1}(J, \theta, Z)$  of the potential term in (18):

$$\frac{1}{2} \Psi^\dagger(\theta, Z) \nabla \left( G \left( J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega \left( J, \theta - \frac{|Z-Z_1|}{c}, Z_1 \right)}{\omega(J, \theta, Z)} \left| \Psi \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right) \right|^2 T(Z, Z_1) dZ_1 \right) \right) \Psi(\theta, Z) \quad (98)$$

induces the presence of products in the series expansion of the two points Green function:

$$\prod_{i=1}^m \int \Psi^\dagger(\theta^{(i)}, Z_i) \nabla_{\theta^{(i)}} \prod_{k=1}^{k_i} \left( \prod_{l=1}^{l_k} \prod_{\alpha(l)=1}^{n(\alpha(l))} \int \left| \Psi \left( \theta^{(i)} - \frac{|Z_i - Z_{\alpha(l)}^{(1)}| + \dots + |Z_{\alpha(l)}^{(l-1)} - Z_{\alpha(l)}^{(l)}|}{c}, Z_{\alpha(l)}^{(l)} \right) \right|^2 \right) \times dZ_{\alpha(l)}^{(1)} \dots dZ_{\alpha(l)}^{(l_k)} \Psi(\theta^{(i)}, Z_i) d\theta^{(i)} dZ_i \quad (99)$$

with  $n(\alpha(l)) \geq n(\alpha(l'))$  for  $l > l'$  and  $m \in \mathbb{N}$ . The function  $\delta(Z - Z')$  in (94) and the use of Wick's theorem imply that all closed loop subgraphs drawn from this product reduce to a product of free Green functions (96) of the following form (the gradient terms and the indices  $\alpha(l)$  are not included and do not impact the reasoning):

$$\begin{aligned} & \left( \int \prod_{i,k} \mathcal{G}_0 \left( \theta^{(i)} - \sum_{l \leq n} \frac{|Z_i - Z_1^{(l)}|}{c}, \theta^{(i+1)} - \sum_{k \leq n} \frac{|Z_{i+1} - Z_1^{(k)}|}{c}, Z_1 \right) \delta(Z_1 - Z_i) \delta(Z_1 - Z_k) dZ_i dZ_k \right) \prod_i d\theta^{(i)} \\ &= \left( \prod_i \mathcal{G}_0 \left( \theta^{(i)} - \sum_{l \leq n} \frac{|Z_i - Z_1^{(l)}|}{c}, \theta^{(i+1)} - \sum_{k \leq n} \frac{|Z_{i+1} - Z_1^{(k)}|}{c}, Z_1 \right) \right) \prod_i d\theta^{(i)} \\ &= \left( \prod_i \mathcal{G}_0(\theta^{(i)}, \theta^{(i+1)}, Z_1) \right) \prod_i d\theta^{(i)} \end{aligned} \quad (100)$$

by change of variable in the successive integrations. Moreover, the cancelation of  $\mathcal{G}_0(\theta, \theta', Z)$  for  $\theta < \theta'$  implies that this product is different from zero only for  $\theta^{(i)} < \theta^{(i+1)}$ . As a consequence, for all closed loops  $\theta_1 < \dots < \theta^{(i)} < \theta^{(i+1)} < \dots < \theta_n = \theta_1$ , the contribution for loop graphs (100) reduces to:

$$\prod_i \mathcal{G}_0(\theta_1, \theta_1, Z_1) = \prod_i \mathcal{G}_0(0, Z_1)$$

with (see (96)):

$$\mathcal{G}_0(0, Z) = \frac{1}{\sqrt{\frac{\pi}{2} \left( \frac{1}{\sigma^2 X_r} \right)^2 + \frac{2\pi\alpha}{\sigma^2}}}$$

As a consequence, the contribution of (99) to the two points Green function between an initial and final state:

$$\begin{aligned}
& \left\langle \Psi^\dagger(\theta_{in}, Z_{in}) \int \prod_{i=1}^m \Psi^\dagger(\theta^{(i)}, Z_i) \right. \\
& \times \nabla_{\theta^{(i)}} \prod_{k=1}^{k_i} \left( \left( \prod_{l=1}^{l_k} \int \left| \Psi \left( \theta^{(i)} - \frac{|Z_i - Z^{(1)}| + \dots + |Z^{(l-1)} - Z^{(l)}|}{c}, Z^{(l)} \right) \right|^2 dZ^{(1)} \dots dZ^{(l_k)} \right) \right) \\
& \left. \times \Psi(\theta^{(i)}, Z_i) d\theta^{(i)} dZ_i \Psi(\theta_{fn}, Z_{fn}) \right\rangle \tag{101}
\end{aligned}$$

reduces to sums of the type:

$$\delta(Z_{in} - Z_{fn}) \sum_p \mathcal{G}_0(\theta_{in}, \theta_1, Z_{in}) \mathcal{G}_0(\theta_1, \theta_2, Z_{in}) \dots \mathcal{G}_0(\theta_p, \theta_{fn}, Z_{in}) \left( \sum_n \sum_{\{L_1^{(p)}, \dots, L_n^{(p)}\}} \prod_{m=1}^n (\mathcal{G}_0(0, 0, Z_m))^{l(L_m^{(p)})} \right) \tag{102}$$

where  $\{L_1^{(p)}, \dots, L_n^{(p)}\}$  is the set of all  $n$ -uplet of possible closed loops that can be drawn from the remaining variables in (101) once  $p$  variables have been chosen.

The result (102) is the same as if in (98) the potential had been expanded to the second order in  $\Psi$  and in all terms of higher order,  $|\Psi(\theta, Z)|^2$  had been replaced by  $\mathcal{G}_0(0, Z)$ .

Now, writing  $\omega(J, \theta, Z, |\Psi|^2)$  for  $\omega$  and  $\omega(0) = \omega(J, \theta, Z, 0)$  (i.e. when we set  $\Psi \equiv 0$ ), this means that the 2 points Green functions are computed using the action:

$$\begin{aligned}
& -\frac{1}{2} \Psi^\dagger(\theta, Z) \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(0) \right) \Psi(\theta, Z) \tag{103} \\
& + \frac{1}{2} \Psi^\dagger(\theta, Z) \sum_{n>0} \frac{\nabla_\theta (\omega^{-1})^{(n)}(0)}{[n]!} (\mathcal{G}_0(0, Z))^n \Psi(\theta, Z) \\
& + \sum_{n>0} \left( \nabla_\theta \frac{(\omega^{-1})^{(n-1)}(0) |\Psi|^2}{[n-1]!} (\mathcal{G}_0(0, Z))^{n-1} \mathcal{G}_0(\theta, \theta', Z) \right)_{\theta'=\theta} \\
& = -\frac{1}{2} \Psi^\dagger(\theta, Z) \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(0) \right) \Psi(\theta, Z) + \frac{1}{2} \Psi^\dagger(\theta, Z) \sum_{n>0} \nabla_\theta \left( (\omega^{-1})(\mathcal{G}_0(0, Z)) - \omega^{-1}(0) \right) \Psi(\theta, Z) \\
& + \left( \nabla_\theta \left( (\omega^{-1})^{(1)} (\mathcal{G}_0(0, Z)) |\Psi|^2 \mathcal{G}_0(\theta, \theta', Z) \right) \right)_{\theta'=\theta} \\
& = -\frac{1}{2} \Psi^\dagger(\theta, Z) \left( \nabla_\theta \frac{\sigma_\theta^2}{2} \nabla_\theta \right) \Psi(\theta, Z) + \frac{1}{2} |\Psi|^2 \left[ \frac{\delta \left[ \Psi^\dagger(\theta', Z) \nabla_\theta \left( \omega^{-1}(J, \theta, Z, |\Psi|^2) \Psi(\theta, Z) \right) \right]}{\delta |\Psi|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)}
\end{aligned}$$

where  $\frac{(\omega^{-1})^{(n)}(0)}{[n]!}$  is a short notation for:

$$\sum_{l_i} \int \prod_{i=1}^n dZ_{l_i}^{(1)} \dots dZ_{l_i}^{(l_i)} \left( \frac{\delta^n \left[ \omega^{-1}(J, \theta, Z, |\Psi|^2) \right]}{\prod_{i=1}^n \delta \left( \left| \Psi \left( \theta - \frac{|Z - Z_{l_i}^{(1)}| + \dots + |Z_{l_i}^{(l-1)} - Z_{l_i}^{(l_i)}|}{c}, Z_{l_i}^{(l_i)} \right) \right|^2 \right)} \right)_{|\Psi|=0}$$

and  $\frac{(\omega^{-1})^{(n-1)}(0)|\Psi|^2}{[n-1]!}$  stands for:

$$\sum_{l_i} \int \prod_{i=1}^{n-1} dZ_{l_i}^{(1)} \dots dZ_{l_i}^{(l_i)} \left( \frac{\delta^{n-1} \left[ \omega^{-1} \left( J, \theta, Z, |\Psi|^2 \right) \right]}{\prod_i \delta \left( \left| \Psi \left( \theta - \frac{|Z - Z_{l_i}^{(1)}| + \dots + |Z_{l_i}^{(l-1)} - Z_{l_i}^{(l_i)}|}{c}, Z_{l_i}^{(l_i)} \right) \right|^2 \right)^{k_{l_i}}} \right)_{|\Psi|=0}$$

$$\times \sum_{j=1}^{n-1} \left| \Psi \left( \theta - \frac{|Z - Z_{l_j}^{(1)}| + \dots + |Z_{l_i}^{(l-1)} - Z_{l_j}^{(l_j)}|}{c}, Z_{l_j}^{(l_j)} \right) \right|^2$$

Similar notation is valid for  $\frac{(\omega^{-1})^{(n)}(\mathcal{G}_0(0,0,Z))|\Psi|^2}{[n-1]!}$ , the derivatives are evaluated at  $|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0,0,Z)$ .

We have also used  $|\Psi|^2 \left[ \frac{\delta}{\delta|\Psi|^2} \right]$  as a shorthand for:

$$\sum_l \int \left( \frac{dZ_l^{(1)} \dots dZ_l^{(l)}}{(k_l)!} \right) \left| \Psi \left( \theta - \frac{|Z - Z_l^{(1)}| + \dots + |Z_l^{(l-1)} - Z_l^{(l)}|}{c}, Z_l^{(l)} \right) \right|^2 \quad (104)$$

$$\times \frac{\delta}{\delta \left( \left| \Psi \left( \theta - \frac{|Z - Z_l^{(1)}| + \dots + |Z_l^{(l-1)} - Z_l^{(l)}|}{c}, Z_l^{(l)} \right) \right|^2 \right)}$$

## 1.2 Higher order vertices involved in the effective action

To compute the  $2n$  points Green functions, we proceed as for the two points function and consider a series expansion of the potential in powers of  $\Psi(\theta, Z)$ . In products  $\prod_{i=1}^n |\Psi(\theta_i, Z_i)|^2$ ,  $n - k$  factors  $|\Psi(\theta_i, Z_i)|^2$  are replaced by  $\mathcal{G}_0(0,0, Z_i)$  at the higher orders. A derivation similar to (103) then shows that  $2n$  Green functions are computed by using the expansion of the action:

$$-\frac{1}{2} \Psi^\dagger(\theta, Z) \left( \nabla_\theta \frac{\sigma_\theta^2}{2} \nabla_\theta \right) \Psi(\theta, Z) \quad (105)$$

$$+\frac{1}{2} \sum_{n \geq k \geq 0} |\Psi|^{2k} \left( \frac{\delta^k}{[k]! \delta^k |\Psi|^2} \left[ \Psi^\dagger(\theta', Z) \nabla_\theta \left( \omega^{-1} \left( J, \theta, Z, |\Psi|^2 \right) \Psi(\theta, Z) \right) \right] \right)_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)}$$

where  $|\Psi|^{2k} \frac{\delta^k}{[n]! \delta^k |\Psi|^2}$  generalizes (104) and stands for:

$$\sum_{l_i} \int \prod_{i=1}^k \left( dZ_{l_i}^{(1)} \dots dZ_{l_i}^{(l_i)} \right) \left| \Psi \left( \theta - \frac{|Z - Z_{l_j}^{(1)}| + \dots + |Z_{l_i}^{(l-1)} - Z_{l_j}^{(l_j)}|}{c}, Z_{l_i}^{(l_i)} \right) \right|^2$$

$$\times \left( \frac{\delta^k}{\prod_i \delta \left( \left| \Psi \left( \theta - \frac{|Z - Z_{l_i}^{(1)}| + \dots + |Z_{l_i}^{(l-1)} - Z_{l_i}^{(l_i)}|}{c}, Z_{l_i}^{(l_i)} \right) \right|^2 \right)^{k_{l_i}}} \right)$$

Equation (105) can be shown recursively. To compute the  $2n$  correlation functions, the subgraphs with  $2k$  legs,  $k < n$ , are given by (105) at order  $2k$ . For  $k = n$ , the classical action yields a vertex:

$$\frac{1}{2} \left( \frac{\delta^n}{[n]! \delta^n |\Psi|^2} \left[ \Psi^\dagger(\theta', Z) \nabla_\theta \left( \omega^{-1}(J, \theta, Z, |\Psi|^2) \Psi(\theta, Z) \right) \right] \right)_{|\Psi(\theta, Z)|^2=0} |\Psi|^{2n}$$

For  $k > n$ , a similar argument as in paragraph 1.1 in the vertex:

$$\frac{1}{2} \left( \frac{\delta^k}{[k]! \delta^k |\Psi|^2} \left[ \Psi^\dagger(\theta', Z) \nabla_\theta \left( \omega^{-1}(J, \theta, Z, |\Psi|^2) \Psi(\theta, Z) \right) \right] \right)_{|\Psi(\theta, Z)|^2=0} |\Psi|^{2k}$$

$k - n$  factor  $|\Psi(\theta, Z)|^2$  have to be replaced by  $\mathcal{G}_0(0, 0, Z)$ . Summing over  $k$ , it means that the  $2n$  vertex is computed with:

$$\frac{1}{2} \sum_{l=0}^{\infty} \left( \frac{\delta^{l+n}}{[l+n]! \delta^{l+n} |\Psi|^2} \left[ \Psi^\dagger(\theta', Z) \nabla_\theta \left( \omega^{-1}(J, \theta, Z, |\Psi|^2) \Psi(\theta, Z) \right) \right] \right)_{|\Psi(\theta, Z)|^2=0} [C_{l+n}^l] (\mathcal{G}_0(0, 0, Z))^l |\Psi|^{2n}$$

where the symbol  $[C_{l+n}^l]$  reminds that among the product  $|\Psi(\theta_1, Z_1)|^2 \dots |\Psi(\theta_{l+n}, Z_{l+n})|^2$  we sum over all the  $C_{l+n}^l$  possibilities to replace  $l$  factor  $|\Psi(\theta_j, Z_j)|^2$  by  $\mathcal{G}_0(0, 0, Z_j)$ . Summing the series, we find for the  $2n$  vertices:

$$= \frac{1}{2n!} \left( \frac{\delta^{l+n}}{[l+n]! \delta^{l+n} |\Psi|^2} \left[ \Psi^\dagger(\theta', Z) \nabla_\theta \left( \omega^{-1}(J, \theta, Z, |\Psi|^2) \Psi(\theta, Z) \right) \right] \right)_{|\Psi(\theta, Z)|^2=\mathcal{G}_0(0, Z)} |\Psi|^{2n}$$

as requested.

To compute the higher order corrections to the effective potential, it will be useful to write (105) with an other set of variables. We replace:

$$\Psi \left( \theta - \frac{|Z - Z_{l_i}^{(1)}| + \dots + |Z_{l_i}^{(l-1)} - Z_{l_i}^{(l_i)}|}{c}, Z_{l_i}^{(l_i)} \right)$$

by  $\Psi(\theta - l_i, Z_i)$  where  $l_i$  represents an arbitrary delay time. As a consequence, the  $2n$ -th vertex:

$$V_{2n} = |\Psi|^{2n} \left( \frac{\delta^n}{[n]! \delta^n |\Psi|^2} \left[ \Psi^\dagger(\theta', Z) \nabla_\theta \left( \omega^{-1}(J, \theta, Z, |\Psi|^2) \Psi(\theta, Z) \right) \right] \right)_{|\Psi(\theta, Z)|^2=\mathcal{G}_0(0, Z)}$$

becomes (where  $\omega^{-1}(J, \theta, Z)$  stands for  $\omega^{-1}(J, \theta, Z, |\Psi|^2)$  when no confusion is possible):

$$\begin{aligned} V_{2n} &= \frac{1}{2} \int \Psi^\dagger(\theta, Z) \nabla_\theta \frac{\delta^{n-1} \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^{n-1} \delta |\Psi(\theta - l_i, Z_i)|^2} \prod_{i=1}^{n-1} |\Psi(\theta - l_i, Z_i)|^2 \prod_{i=1}^{n-1} dZ_i \Psi(\theta, Z) dZ dl_i \quad (106) \\ &+ \int \mathcal{G}'_0(Z) \frac{\delta^n \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2} \prod_{i=1}^n |\Psi(\theta - l_i, Z_i)|^2 \prod_{i=1}^n dZ_i dZ dl_i \\ &+ \int \mathcal{G}_0(Z) \frac{\delta^n \nabla_\theta \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2} \prod_{i=1}^n |\Psi(\theta - l_i, Z_i)|^2 \prod_{i=1}^n dZ_i dZ dl_i \end{aligned}$$

with:

$$\mathcal{G}'_0(Z) = \left( \frac{\nabla_\theta \mathcal{G}_0(\theta, \theta', Z)}{2} \right)_{\theta=\theta'}$$

However, the two last terms in (106) come from the backreaction of the  $n$  vertices on the whole system, and can be neglected in first approximation. Actually in a neighborhood of the permanent regime, we have:

$$\mathcal{G}_0(Z) \frac{\delta^n \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2} \ll \frac{\delta^{n-1} \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^{n-1} \delta |\Psi(\theta - l_i, Z_i)|^2}$$

The neglected terms will be reintroduced later. We can thus consider that:

$$V_{2n} = \frac{1}{2} \int \Psi^\dagger(\theta, Z) \nabla_\theta \frac{\delta^{n-1} \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^{n-1} \delta |\Psi(\theta - l_i, Z_i)|^2} \prod_{i=1}^{n-1} |\Psi(\theta - l_i, Z_i)|^2 \prod_{i=1}^{n-1} dZ_i \Psi(\theta, Z) dZ dl_i \quad (107)$$

These terms are the coefficients obtained by the expansion of  $\omega^{-1}(J, \theta, Z)$  in powers of  $\Psi^\dagger(\theta, Z) \Psi(\theta, Z)$ . It is valid for  $|\Psi(\theta, Z)| < 1$ . For  $|\Psi(\theta, Z)| > 1$ , we can expand  $\omega^{-1}(J, \theta, Z)$  in powers of  $\frac{1}{|\Psi(\theta, Z)|}$ . Given the form of  $F$  and since  $\arctan(x) = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right)$ , the expansion is obtained by replacing the derivatives of  $F$  by those of  $-x^2 F$  and by replacing  $\omega$  with  $\omega^{-1}$ .

Formula (107) yields the vertices  $V_{2n}$ ,  $n \leq N$ , intervening in the computation of the  $2N$  correlation functions. We have to estimate the derivatives arising in (107), before computing the effective action.

### 1.3 Estimation of the derivatives involved in (107)

To compute the  $2n$  Green functions with vertices (107) and the graph expansion of the effective action, we first need to estimate the derivatives  $\frac{\delta^n \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2}$  appearing in (107). These derivatives can be computed

recursively. To do so, we will need to approximate the results around some static solution. We define  $\bar{\omega}$  as solution of:

$$\begin{aligned} \bar{\omega}^{-1}(J, Z) &= G \left( \bar{J}(Z) + \int \frac{\kappa}{N} \frac{\bar{\omega}(\bar{J}, Z_1)}{\bar{\omega}(\bar{J}, Z)} \mathcal{G}_0(0, 0, Z_1) T(Z, Z_1) dZ_1 \right) \\ &= G \left( \bar{J}(Z) + \int \frac{\kappa}{N} \frac{\bar{\omega}(\bar{J}, Z_1)}{\bar{\omega}(\bar{J}, Z) \sqrt{\frac{\pi}{2} \left( \frac{1}{\sigma^2 X_r} \right)^2 + \frac{2\pi\alpha}{\sigma^2}}} T(Z, Z_1) dZ_1 \right) \end{aligned} \quad (108)$$

where  $\bar{J}(Z)$  is the average of  $J(\theta, Z)$  over the full timespan. We also define:

$$G'_0(J, Z) \equiv G' \left( \bar{J}(Z) + \int \frac{\kappa}{N} \frac{\bar{\omega}(\bar{J}, Z_1)}{\bar{\omega}(\bar{J}, \theta, Z)} G_{0Z_1}(0, 0) T(Z, Z_1) dZ_1 \right) \quad (109)$$

These quantities will be useful below. Now, we will find a recursive expansion for  $\frac{\delta^n \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2}$ .

#### 1.3.1 Computation of the first order derivatives in (107)

Using the recursive definition of  $\omega^{-1}(J, \theta, Z)$ :

$$\omega^{-1}(J, \theta, Z) = G \left( J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega \left( J, \theta - \frac{|Z - Z_1|}{c}, Z_1 \right)}{\omega(J, \theta, Z)} \left| \Psi \left( \theta - \frac{|Z - Z_1|}{c}, Z_1 \right) \right|^2 T(Z, Z_1) dZ_1 \right) \quad (110)$$



we first compute  $\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}$ :

$$\begin{aligned}
& \frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} \\
&= \frac{\delta G \left( J(\theta,Z) + \int \frac{\kappa}{N} \frac{\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\omega(J,\theta,Z)} \left| \Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right) \right|^2 T(Z,Z') dZ' \right)}{\delta|\Psi(\theta-l_1,Z_1)|^2} \\
&= \frac{\frac{\kappa}{N} \frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega(J,\theta,Z)} T(Z,Z_1) G'[J,\omega,\theta,Z,\Psi] \delta\left(l_1-\frac{|Z-Z_1|}{c}\right)}{1 - \left( \int \frac{\kappa}{N} \omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right) \left| \Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right) \right|^2 T(Z,Z') dZ' \right) G'[J,\omega,\theta,Z,\Psi]} \\
&\quad + \frac{\frac{1}{\omega(J,\theta,Z)} \int \frac{\kappa}{N} \frac{\delta\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\delta|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)|^2} \left| \Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right) \right|^2 T(Z,Z') dZ' G'[J,\omega,\theta,Z,\Psi]}{1 - \left( \int \frac{\kappa}{N} \omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right) \left| \Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right) \right|^2 T(Z,Z') dZ' \right) G'[J,\omega,\theta,Z,\Psi]} \\
&= \omega(J,\theta-l_1,Z_1) \hat{T}_1(\theta,Z,Z_1,\omega,\Psi) \delta\left(l_1-\frac{|Z-Z_1|}{c}\right) \\
&\quad + \int \frac{\delta\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\delta|\Psi(\theta-l_1,Z_1)|^2} \left| \Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right) \right|^2 \hat{T}_1(\theta,Z,Z',\omega,\Psi) dZ'
\end{aligned} \tag{111}$$

where we defined:

$$\begin{aligned}
& \hat{T}_1(\theta,Z,Z_1,\omega,\Psi) \\
&= \frac{1}{\omega(J,\theta,Z)} \frac{\frac{\kappa}{N} T(Z,Z_1) G'[J,\omega,\theta,Z,\Psi]}{1 - \left( \int \frac{\kappa}{N} \omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right) \left| \Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right) \right|^2 T(Z,Z') dZ' \right) G'[J,\omega,\theta,Z,\Psi]}
\end{aligned} \tag{112}$$

Equation (111) shows that we also need  $\frac{\delta\omega(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}$  to compute  $\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}$ . This is obtained by:

$$\begin{aligned}
\frac{\delta\omega(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} &= \frac{\delta F \left( J(\theta,Z) + \int \frac{\kappa}{N} \frac{\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\omega(J,\theta,Z)} \left| \Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right) \right|^2 T(Z,Z') dZ' \right)}{\delta|\Psi(\theta-l_1,Z_1)|^2} \\
&= \omega(J,\theta-l_1,Z_1) \hat{T}(\theta,Z,Z_1,\omega,\Psi) \delta\left(l_1-\frac{|Z-Z_1|}{c}\right) \\
&\quad + \int \frac{\delta\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\delta|\Psi(\theta-l_1,Z_1)|^2} \left| \Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right) \right|^2 \hat{T}(\theta,Z,Z',\omega,\Psi) dZ'
\end{aligned} \tag{113}$$

with:

$$\hat{T}(\theta,Z,Z_1,\omega,\Psi) = \frac{\frac{\kappa}{N} \omega(J,\theta,Z) T(Z,Z_1) F'[J,\omega,\theta,Z,\Psi]}{\omega^2(J,\theta,Z) + \left( \int \frac{\kappa}{N} \omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right) \left| \Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right) \right|^2 T(Z,Z') dZ' \right) F'[J,\omega,\theta,Z,\Psi]}
\tag{114}$$

Equation (113) and (114) define  $\frac{\delta\omega(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}$  recursively. Actually, writing:

$$\begin{aligned} & \frac{\delta\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\delta|\Psi(\theta-l_1,Z_1)|^2} \\ = & \int \omega\left(J,\theta-\frac{|Z-Z'|}{c}-\frac{|Z'-Z''|}{c},Z''\right)\hat{T}\left(\theta-\frac{|Z-Z'|}{c},Z',Z'',\omega,\Psi\right)\delta\left(\frac{|Z-Z'|}{c}+\frac{|Z'-Z''|}{c}-l_1\right)dZ'' \\ & + \int \frac{\delta\omega\left(J,\theta-\frac{|Z-Z'|}{c}-\frac{|Z'-Z''|}{c},Z''\right)}{\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c}-\frac{|Z'-Z''|}{c},Z''\right)\right|^2\hat{T}\left(\theta-\frac{|Z-Z'|}{c},Z',Z'',\omega,\Psi\right)dZ'' \end{aligned}$$

we have:

$$\begin{aligned} & \frac{\delta\omega(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} \\ = & \int \omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)\hat{T}(\theta,Z,Z_1,\omega,\Psi)\delta\left(\frac{|Z-Z'|}{c}-l_1\right)dZ' \\ & + \int \omega\left(J,\theta-\frac{|Z-Z'|}{c}-\frac{|Z'-Z''|}{c},Z''\right)\hat{T}\left(\theta-\frac{|Z-Z'|}{c},Z',Z'',\omega,\Psi\right) \\ & \times \hat{T}(\theta,Z,Z',\omega,\Psi)\delta\left(\frac{|Z-Z'|}{c}+\frac{|Z'-Z''|}{c}-l_1\right)dZ'dZ'' \\ & + \int \frac{\delta\omega\left(J,\theta-\frac{|Z-Z'|}{c}-\frac{|Z'-Z''|}{c},Z''\right)}{\delta|\Psi(\theta-l_1,Z_1)|^2}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c}-\frac{|Z'-Z''|}{c},Z''\right)\right|^2 \\ & \times \hat{T}\left(\theta-\frac{|Z-Z'|}{c},Z',Z'',\omega,\Psi\right)\hat{T}(\theta,Z,Z',\omega,\Psi)dZ'dZ'' \end{aligned}$$

which yields the series expansion:

$$\begin{aligned} \frac{\delta\omega(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} & = \sum_{n=1}^{\infty} \int \omega\left(J,\theta-\sum_{l=1}^n \frac{|Z^{(l-1)}-Z^{(l)}|}{c},Z_1\right) \prod_{l=1}^n \hat{T}\left(\theta-\sum_{j=1}^{l-1} \frac{|Z^{(j-1)}-Z^{(j)}|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right) \\ & \times \delta\left(l_1-\sum_{l=1}^n \frac{|Z^{(l-1)}-Z^{(l)}|}{c}\right) \prod_{l=1}^{n-1} dZ^{(l)} \end{aligned} \quad (115)$$

and:

$$\begin{aligned} \frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} & = \sum_{n=1}^{\infty} \int \omega\left(J,\theta-\sum_{l=1}^n \frac{|Z^{(l-1)}-Z^{(l)}|}{c},Z_1\right) \hat{T}_1(\theta,Z,Z^{(1)},\omega,\Psi) \\ & \times \prod_{l=2}^n \hat{T}\left(\theta-\sum_{j=1}^{l-1} \frac{|Z^{(j-1)}-Z^{(j)}|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right) \delta\left(l_1-\sum_{l=1}^n \frac{|Z^{(l-1)}-Z^{(l)}|}{c}\right) \prod_{l=1}^{n-1} dZ^{(l)} \end{aligned} \quad (116)$$

with the convention that  $Z^{(0)} = Z$  and  $Z^{(n)} = Z_1$ . We now use the static approximations (108) and (109). Actually, the values of  $\hat{T}_1(\theta,Z,Z_1\omega,\Psi)$  and  $\hat{T}(\theta,Z,Z_1\omega,\Psi)$  can be estimated for  $\bar{\omega}^{-1}(\bar{J},Z)$ . Moreover, in the limit of small fluctuations,  $\bar{\omega}^{-1}(\bar{J},Z)$ ,  $F'[J,\bar{\omega},Z,\Psi]$  and  $G'[J,\bar{\omega},Z,\Psi]$  can be approximated by their average over  $Z$ , denoted  $\bar{\omega}^{-1}$ ,  $\bar{F}'$  and  $\bar{G}'$ . Moreover for  $\bar{\omega}$ , both  $\hat{T}_1$  and  $\hat{T}$  can be considered independent of  $\theta$ :

$$\begin{aligned} \hat{T}_1(\theta,Z,Z_1\bar{\omega},\Psi) & \simeq \hat{T}_1(Z,Z_1,\bar{\omega}) \\ & = \frac{\frac{\kappa}{N}\bar{\omega}^{-1}T(Z,Z_1)\bar{G}'}{1-\frac{\bar{G}'\bar{\omega}\int\frac{\kappa}{N}T(Z,Z')dZ'}{\sqrt{\frac{\pi}{2}\left(\frac{1}{\sigma^2\bar{\chi}_r}\right)^2+\frac{2\pi\alpha}{\sigma^2}}}} \end{aligned}$$

$$\begin{aligned}
\hat{T}(\theta, Z, Z_1, \omega, \Psi) &\simeq \hat{T}(Z, Z_1, \bar{\omega}) \\
&= \frac{\frac{\kappa}{N} T(Z, Z_1) \bar{F}'}{\bar{\omega} + \frac{\bar{F}' \int \frac{\kappa}{N} T(Z, Z') dZ'}{\sqrt{\frac{\pi}{2} \left( \frac{1}{\sigma^2 \bar{\kappa}_r} \right)^2 + \frac{2\pi\alpha}{\sigma^2}}}}
\end{aligned}$$

as a consequence  $\hat{T}_1(Z, Z_1, \bar{\omega})$  and  $\hat{T}(Z, Z_1, \bar{\omega})$  are functions of  $|Z - Z_1|$  denoted  $\hat{T}_1(|Z - Z_1|)$ . As a consequence (116) can be estimated by:

$$\begin{aligned}
\frac{\delta\omega^{-1}(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} &= \sum_{n=1}^{\infty} \int \omega(J, \theta - l_1, Z_1) \hat{T}_1(|Z - Z^{(1)}|) \\
&\quad \times \prod_{l=2}^n \hat{T}(|Z^{(l-1)} - Z^{(l)}|) \delta\left(l_1 - \sum_{l=1}^n \frac{|Z^{(l-1)} - Z^{(l)}|}{c}\right) \\
&\quad \times \delta\left(Z - Z_1 - \sum_{l=1}^n (Z^{(l-1)} - Z^{(l)})\right) \prod_{l=1}^{n-1} dZ^{(l)}
\end{aligned} \tag{117}$$

and (115) is:

$$\begin{aligned}
\frac{\delta\omega(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} &= \sum_{n=1}^{\infty} \int \omega\left(J, \theta - \sum_{l=1}^n \frac{|Z^{(l-1)} - Z^{(l)}|}{c}, Z_1\right) \prod_{l=1}^n \hat{T}(|Z^{(l-1)} - Z^{(l)}|) \\
&\quad \times \delta\left(l_1 - \sum_{l=1}^n \frac{|Z^{(l-1)} - Z^{(l)}|}{c}\right) \times \delta\left(Z - Z_1 - \sum_{l=1}^n (Z^{(l-1)} - Z^{(l)})\right) \prod_{l=1}^{n-1} dZ^{(l)}
\end{aligned} \tag{118}$$

### 1.3.2 Estimation of (117) and (115) close to the permanent regime

**1.3.2.1 General formula** The series (117) can be computed by using the Fourier transform of the Dirac functions:

$$\begin{aligned}
\frac{\delta\omega^{-1}(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} &= \sum_{n=1}^{\infty} \int \omega(J, \theta - l_1, Z_1) \times \hat{T}_1(|Z - Z^{(1)}|) \prod_{l=2}^n \hat{T}(|Z^{(l-1)} - Z^{(l)}|) \\
&\quad \times \exp\left(i\lambda\left(cl_1 - \sum_{l=1}^n |Z^{(l-1)} - Z^{(l)}|\right)\right) \\
&\quad \times \exp\left(i\lambda_1 \cdot \left(Z - Z_1 - \sum_{l=1}^n (Z^{(l-1)} - Z^{(l)})\right)\right) d\lambda d\lambda_1 \prod_{l=1}^n |Z^{(l-1)} - Z^{(l)}|^2 d|Z^{(l-1)} - Z^{(l)}| dv_l
\end{aligned}$$

where the unit vectors  $v_l$  are defined such that:

$$Z^{(l-1)} - Z^{(l)} = v_l |Z^{(l-1)} - Z^{(l)}|$$

We also define

$$\begin{aligned}
\lambda_1 \cdot (Z - Z_1) &= |\lambda_1| |Z - Z_1| \cos(\theta_1) \\
\lambda_1 \cdot v_l &= |\lambda_1| \cos(\theta_l)
\end{aligned}$$

The angles  $\theta_l$  are computed in the plane  $(\lambda_1, Z - Z_1)$  between the projection of  $v_l$  and  $Z - Z_1$ . The angles  $\varphi_l$  are defined as the angle between  $v_l$  and the plane  $\lambda_1, Z - Z_1$ . As a consequence:

$$\begin{aligned}
& \frac{\delta\omega^{-1}(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} \\
&= \sum_{n=1}^{\infty} \int \omega(J, \theta - l_1, Z_1) \times T_1''(\lambda + \lambda_1 \cdot v_1) dv_1 \prod_{l=2}^n \int T''(\lambda + \lambda_1 \cdot v_l) dv_l \exp(i\lambda cl_1 + i\lambda_1 \cdot (Z - Z_1)) d\lambda d\lambda_1 \\
&= \delta(|Z_1 - Z| - cl_1) \hat{T}_1 \left( |Z - Z^{(1)}| \right) \omega(J, \theta - l_1, Z_1) \\
&\quad + (-1)^n \int \omega(J, \theta - l_1, Z_1) \times \frac{T_1''(\lambda + \lambda_1 \cdot v_1)}{2} dv_1 \prod_{l=2}^n \int \frac{T''(\lambda + \lambda_1 \cdot v_l)}{2} dv_l \exp(i\lambda cl_1 + i\lambda_1 \cdot (Z - Z_1)) d\lambda d\lambda_1
\end{aligned} \tag{119}$$

With the convention that for  $n = 1$ , the product  $\prod_{l=2}^n$  is set to be equal to 1. The functions  $T_1$  and  $T$  are the fourier transform of  $\hat{T}_1 H$  and  $\hat{T} H$  respectively, and  $H$  is the heaviside function. Remark that the first term of (119) expresses the Dirac function  $\delta(|Z_1 - Z| - cl_1)$  as a Fourier transform:

$$\begin{aligned}
& \exp \left( i\lambda \left( cl_1 - \sum_{l=1}^n |Z^{(0)} - Z^{(1)}| \right) \right) \\
& \times \exp \left( i\lambda_1 \cdot \left( Z - Z_1 - \sum_{l=1}^n (Z^{(0)} - Z^{(1)}) \right) \right) d\lambda d\lambda_1 |Z^{(0)} - Z^{(1)}|^2 d|Z^{(0)} - Z^{(1)}| dv_l
\end{aligned}$$

Some terms of (119) can be written in a useful form for the sequel:

$$\begin{aligned}
\frac{1}{2} \int T''(\lambda + \lambda_1 \cdot v_l) dv_l &= \pi \int_0^\pi T''(\lambda + |\lambda_1| \cos(\theta_l)) \sin(\theta_l) d\theta_l \\
&= \pi \int_{-1}^1 T''(\lambda + |\lambda_1| u) du \\
&= \frac{2\pi (T'(\lambda + |\lambda_1|) - T'(\lambda - |\lambda_1|))}{2|\lambda_1|} \\
&\equiv \bar{T}(\lambda, \lambda_1)
\end{aligned} \tag{120}$$

$$\begin{aligned}
\int T_1''(\lambda + \lambda_1 \cdot v_l) dv_l &= \frac{2\pi (T_1'(\lambda + |\lambda_1|) - T_1'(\lambda - |\lambda_1|))}{2|\lambda_1|} \\
&\equiv \bar{T}_1(\lambda, |\lambda_1|)
\end{aligned} \tag{121}$$

$$\begin{aligned}
\exp(i\lambda_1 \cdot (Z - Z_1)) d\lambda_1 &= \exp(i \cos(\theta_1) |\lambda_1| |Z - Z_1| \sin(\theta_1) |\lambda_1|^2 d|\lambda_1| d\theta_1) \\
&= \exp(iu |\lambda_1| |Z - Z_1| |\lambda_1|^2 d|\lambda_1| du)
\end{aligned} \tag{122}$$

**1.3.2.2 Estimation of (119)** Using (120), (121) and (122), equation (119) becomes:

$$\begin{aligned}
& \frac{\delta\omega^{-1}(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} \\
&= \sum_{n=1}^{\infty} (-1)^n \int \omega(J, \theta - l_1, Z_1) \times T_1(\lambda + \lambda_1 \cdot v_1) dv_1 \prod_{l=2}^n \int T(\lambda + \lambda_1 \cdot v_l) dv_l \exp(i\lambda cl_1 + i\lambda_1 \cdot (Z - Z_1)) d\lambda d\lambda_1 \\
&= - \int \omega(J, \theta - l_1, Z_1) \times \frac{\bar{T}_1(\lambda, |\lambda_1|)}{1 + \bar{T}(\lambda, |\lambda_1|)} \exp(i\lambda cl_1) \int_{-1}^1 \exp(iu |\lambda_1| |Z - Z_1| |\lambda_1|^2 d|\lambda_1| dud\lambda \\
&= - \int \omega(J, \theta - l_1, Z_1) \times \frac{\bar{T}_1(\lambda, |\lambda_1|)}{1 + \bar{T}(\lambda, |\lambda_1|)} \exp(i\lambda cl_1) \left( 2 \frac{\sin(|\lambda_1| |Z - Z_1|)}{|Z - Z_1|} |\lambda_1| \right) d|\lambda_1| d\lambda
\end{aligned} \tag{123}$$

We remark that for even functions  $f$ , the following identity holds:

$$\begin{aligned}
& \int_0^{+\infty} f(|\lambda_1|) 2 \frac{\sin(|\lambda_1| |Z - Z_1|)}{|Z - Z_1|} |\lambda_1| d|\lambda_1| \\
&= \int_0^{+\infty} f(x) \frac{\exp(ix|Z - Z_1|) - \exp(-ix|Z - Z_1|)}{i|Z - Z_1|} x dx \\
&= \int_0^{+\infty} f(x) \frac{\exp(ix|Z - Z_1|)}{i|Z - Z_1|} x dx + \int_{-\infty}^0 f(-x) \frac{\exp(ix|Z - Z_1|)}{i|Z - Z_1|} x dx \\
&= \int_{-\infty}^{+\infty} f(x) \frac{\exp(ix|Z - Z_1|)}{i|Z - Z_1|} x dx
\end{aligned}$$

so that (123) becomes:

$$\begin{aligned}
& \frac{\delta\omega^{-1}(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} \\
&= - \int \omega(J, \theta - l_1, Z_1) \times \frac{\bar{T}_1(\lambda, \lambda_1)}{1 + \bar{T}(\lambda, \lambda_1)} \frac{\lambda_1}{i|Z - Z_1|} \exp(i\lambda cl_1 + i\lambda_1 |Z - Z_1|) d\lambda_1 d\lambda \\
&= - \int \omega(J, \theta - l_1, Z_1) \times \frac{\pi(T'_1(\lambda + \lambda_1) - T'_1(\lambda - \lambda_1))}{\lambda_1 + \pi(T'(\lambda + \lambda_1) - T'(\lambda - \lambda_1))} \frac{\lambda_1}{i|Z - Z_1|} \\
&\quad \times \exp(i\lambda cl_1 + i\lambda_1 |Z - Z_1|) d\lambda_1 d\lambda
\end{aligned} \tag{124}$$

Another simplification follows if we write  $T_1$  as a function of  $T$ :

$$T'_1(\lambda + \lambda_1) - T'_1(\lambda - \lambda_1) = \frac{A_1}{A} (T'(\lambda + \lambda_1) - T'(\lambda - \lambda_1))$$

Thus, setting:

$$\begin{aligned}
u &= \lambda + \lambda_1 \\
v &= \lambda - \lambda_1
\end{aligned}$$

equation (124) becomes:

$$\begin{aligned}
\frac{\delta\omega^{-1}(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} &= - \frac{A_1}{iA|Z - Z_1|} \int \omega(J, \theta - l_1, Z_1) \times \frac{\pi(T'(u) - T'(v))}{1 + 2\pi \frac{(T'(u) - T'(v))}{u-v}} \\
&\quad \times \exp\left(i\frac{u}{2}(cl_1 + |Z - Z_1|) + i\frac{v}{2}(cl_1 - |Z - Z_1|)\right) d\lambda_1 d\lambda
\end{aligned} \tag{125}$$

To compute (125), we study its two components independently:

$$\begin{aligned}
& - \frac{A_1}{iA|Z - Z_1|} \int \omega(J, \theta - l_1, Z_1) \times \frac{\pi T'(u)}{1 + 2\pi \frac{(T'(u) - T'(v))}{u-v}} \\
&\quad \times \exp\left(i\frac{u}{2}(cl_1 + |Z - Z_1|) + i\frac{v}{2}(cl_1 - |Z - Z_1|)\right) dudv
\end{aligned} \tag{126}$$

and:

$$\begin{aligned}
& \frac{A_1}{iA|Z - Z_1|} \int \omega(J, \theta - l_1, Z_1) \times \frac{\pi T'(v)}{1 + 2\pi \frac{(T'(u) - T'(v))}{u-v}} \\
&\quad \times \exp\left(i\frac{u}{2}(cl_1 + |Z - Z_1|) + i\frac{v}{2}(cl_1 - |Z - Z_1|)\right) dudv
\end{aligned} \tag{127}$$

In the integral (126), we first estimate the  $v$  integral using the residues theorem. The poles are solutions of:

$$1 + 2\pi \frac{T'(u) - T'(v)}{u - v} = 0$$

That is:

$$v + 2\pi T'(v) = u + 2\pi T'(u) \quad (128)$$

with  $v \neq u$ . In the gaussian approximation for the transfer functions,  $T(\lambda)$  has the form:

$$T(\lambda) = A \exp\left(-\nu \frac{\lambda^2}{4}\right) (1 - \operatorname{erf}(i\sqrt{\nu}\lambda)) \quad (129)$$

and its derivative satisfies:

$$T'(\lambda) = -\nu \frac{\lambda}{2} T(\lambda) - i\sqrt{\nu}$$

As a consequence of these two identities, the solutions of (128) are given by:

$$v(1 - \pi\eta T(v)) = z \quad (130)$$

with:

$$z = u(1 - \pi\eta T(u))$$

To solve (130) it will be useful to expand  $T(\lambda)$  as a series expansion. In first approximation, one has (see Abramovitz stegun):

$$\operatorname{Im} \operatorname{erf}(i\sqrt{\nu}\lambda) \simeq \frac{2}{\pi} \sum_{k=1}^{+\infty} \frac{\exp(-k^2) \sinh k\sqrt{\nu}\lambda}{k} \simeq \sqrt{\frac{\nu}{\pi}} \lambda$$

and:

$$\operatorname{Im} T(\lambda) \simeq A\sqrt{\nu} \left( \frac{1}{2\sqrt{\pi}} \exp\left(-\nu \frac{\lambda^2}{4}\right) \nu \lambda^2 \right) > 0$$

as a consequence  $\operatorname{Im} z > 0$  and asymptotically, equation (130) reduces to:

$$v(\pi\eta T(v)) = -z$$

that is:

$$(A\pi\eta v)^2 \exp\left(-\eta \frac{v^2}{2}\right) (1 - \operatorname{erf}(i\sqrt{\eta}v))^2 = z^2$$

for  $\eta \ll 1$

$$\left(\frac{A\pi\eta v}{2}\right)^2 \exp\left(-\eta \frac{v^2}{2}\right) = z^2$$

and the poles arising in (126) are given by the Whittaker functions  $W_k$ :

$$v = \sqrt{-\frac{2}{\eta} W_k\left(\frac{-2z^2}{(A\pi)^2 \eta}\right)}$$

for  $k > 0$ . They are approximatively equal to:

$$v \simeq \pm i \sqrt{\frac{2}{\eta} \left( \ln\left(\frac{2u^2}{(A\pi)^2 \eta}\right) + i(2k+1)\pi \right)}$$

The terms involved in (126) can thus be evaluated at the poles. First, for  $(A\pi)^2 \ll 1$ :

$$\begin{aligned} \pi\eta |T(v)| &\simeq A\pi\eta \left| \exp\left(-\eta \frac{v^2}{4}\right) \right| \\ &\simeq A\pi\eta \exp\left(\ln\left(\frac{\sqrt{2}u^2}{(A\pi)\sqrt{\eta}}\right)\right) = \sqrt{2\eta}u \end{aligned}$$

Asymptotically, for  $\sqrt{2\eta}u \gg 1$ , this formula justifies our previous approximation  $v(1 - \pi\eta T(v)) \simeq -v\pi\eta T(v)$ . For  $\sqrt{2\eta}u \ll 1$ , the solution is  $v = u$  and there is no pole. Second, we have:

$$\begin{aligned} & \left( 1 + 2\pi \frac{(T'(u) - T'(v))}{u - v} \right)' \\ &= -2\pi \frac{T''(v)}{u - v} + 2\pi \frac{(T'(u) - T'(v))}{(u - v)^2} \\ &= -\frac{1 + 2\pi T''(v)}{u - v} \end{aligned}$$

and (126) becomes:

$$\begin{aligned} & \sum_{k \neq 0} \frac{2\pi A_1}{A|Z - Z_1|} \int \omega(J, \theta - l_1, Z_1) \times \frac{(u - v) T'(u)}{2(1 + T''(v))} \\ & \times \exp \left( i \frac{u}{2} (cl_1 + |Z - Z_1|) - \sqrt{\frac{2}{\eta} \left( \ln \left( \frac{2u^2}{(A\pi)^2 \eta} \right) + i(2k + 1)\pi \right)} |cl_1 - |Z - Z_1|| \right) du \end{aligned}$$

Note that for  $(A\pi)^2 \eta \ll 1$ , we recover  $\delta(cl_1 - |Z - Z_1|)$  as needed in the lowest order approximation. The second integral (127) is obtained by inverting the role of  $u$  and  $v$ . It yields:

$$\begin{aligned} & - \sum_{k \neq 0} \frac{2\pi A_1}{A|Z - Z_1|} \int \omega(J, \theta - l_1, Z_1) \times \frac{(u - v) T'(v)}{2(1 + T''(u))} \\ & \times \exp \left( - \sqrt{\frac{2}{\eta} \left( \ln \left( \frac{2u^2}{(A\pi)^2 \eta} \right) + i(2k + 1)\pi \right)} (cl_1 + |Z - Z_1|) + i \frac{v}{2} (cl_1 - |Z - Z_1|) \right) du \end{aligned}$$

and this can be neglected, since  $cl_1 + |Z - Z_1| > 0$  and for  $(A\pi)^2 \eta \ll 1$  this becomes  $\delta(cl_1 + |Z - Z_1|) = 0$ .

Gathering the results for (126) and (127), we are left with:

$$\begin{aligned} & \frac{\delta\omega^{-1}(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} \\ &= \sum_{k \neq 0} \frac{2\pi A_1}{A|Z - Z_1|} \int \omega(J, \theta - l_1, Z_1) \times \frac{(u - v) T'(u)}{2(1 + T''(v))} \\ & \times \exp \left( i \frac{u}{2} (cl_1 + |Z - Z_1|) - \sqrt{\frac{2}{\eta} \left( \ln \left( \frac{2u^2}{(A\pi)^2 \eta} \right) + i(2k + 1)\pi \right)} |cl_1 - |Z - Z_1|| \right) du \end{aligned} \tag{131}$$

Ultimately, some simplifications can be performed on (131). Actually, we have the following identities for  $T$ :

$$T''(\lambda) = -\frac{\nu}{2} T(\lambda) + \left( \nu \frac{\lambda}{2} \right)^2 T(\lambda) + Ai(\sqrt{\nu})^3 \frac{\lambda}{2}$$

$$\begin{aligned} v(1 - \pi\eta T(v)) &= u(1 - \pi\eta T(u)) \simeq u \\ \pi\eta T(v) &\simeq v - u \end{aligned}$$

and these two equations imply that, for  $A(\sqrt{\eta})^3 \ll 1$ :

$$\begin{aligned} & 1 + 2\pi T''(v) \\ &= 1 - \pi\eta T(v) + 2\pi \left( \eta \frac{v}{2} \right)^2 T(v) + 2\pi i A (\sqrt{\eta})^3 \frac{v}{2} \\ &\simeq (v - u)(-1 + \pi\eta v) \simeq i(v - u) \sqrt{\frac{2}{\eta}} C \end{aligned} \tag{132}$$

where  $C = \sqrt{\left(\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right) + i(2k+1)\pi\right)}$ . A consequence of (132) is that:

$$-\frac{1+2\pi T''(v)}{u-v} \simeq \frac{1}{i\pi\sqrt{2\eta}C}$$

Moreover, for  $(A\pi)^2\eta \ll 1$ , the function  $T'(u)$  can be replaced by the multiplication by  $i\frac{cl_1+|Z-Z_1|}{2}$ . We are thus led to rewrite (131):

$$\begin{aligned} & \frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} \\ = & \frac{A_1}{A} \sum_{k \neq 0} \frac{(cl_1+|Z-Z_1|)}{2|Z-Z_1|} \int \omega(J,\theta-l_1,Z_1) \times \frac{T(u)}{\sqrt{2\eta}C} \\ & \times \exp\left(i\frac{u}{2}(cl_1+|Z-Z_1|) - \sqrt{\frac{2}{\eta}\left(\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right) + i(2k+1)\pi\right)}|cl_1-|Z-Z_1||\right) du \\ \equiv & \frac{A_1}{A} \Xi(|Z_1-Z|, l_1, \bar{\omega}) \omega(J,\theta-l_1,Z_1) \end{aligned} \tag{133}$$

Remark that, for  $(A\pi)^2\eta \ll 1$ :

$$\begin{aligned} & \sqrt{\frac{2}{\eta}\left(\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right) + i(2k+1)\pi\right)} \\ & \times \exp\left(-\sqrt{\frac{2}{\eta}\left(\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right) + i(2k+1)\pi\right)}|cl_1-|Z-Z_1||\right) \\ & \simeq \delta(cl_1-|Z-Z_1|) \end{aligned}$$

so that one recovers the first order term.

For  $(A\pi)^2\eta \ll 1$ ,  $\Xi(|Z_1-Z|, l_1, \bar{\omega})$  is a function of  $|Z_1-Z|$  written  $\Xi(|Z_1-Z|, \bar{\omega})$ .

Finally, the sum in (133) can be estimated in the following way:

$$\begin{aligned} & \sum_{k \neq 0} \exp\left(-\sqrt{\frac{2}{\eta}\left(\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right) + i(2k+1)\pi\right)}|cl_1-|Z-Z_1||\right) \\ = & \sum_{k \neq 0} \exp\left(-\sqrt{\frac{2}{\eta}\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right)}\sqrt{1+i\frac{(2k+1)}{\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right)}\pi}|cl_1-|Z-Z_1||\right) \\ \simeq & \frac{C^2}{\pi} \operatorname{Re} \int_0^1 \exp\left(-C\sqrt{\frac{2}{\eta}}\sqrt{1+ix}|cl_1-|Z-Z_1||\right) dx \\ = & \frac{C^2}{\pi} \operatorname{Re} \int_0^1 \exp\left(-C\sqrt{\frac{2}{\eta}}(1+x^2)^{\frac{1}{4}} \exp\left(\frac{i}{2}\arctan(x)\right)|cl_1-|Z-Z_1||\right) dx \end{aligned}$$

with:

$$C = \sqrt{\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right)}$$

The upper bound of the integral is set to 1, in agreement with our approximation  $\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right) \gg 1$ . It amounts to neglect the poles for  $k \gg 1$ , whose contributions are decreasing quicly with  $k$  as given by oscillatory integrals of frequencies proportional to  $k$ .



By a change of variable, the last integral is also given by:

$$\frac{2C^2}{\pi} \operatorname{Re} \int_0^1 \exp \left( -C \sqrt{\frac{2}{\eta}} \left( \sqrt{1+v^2} + iv \right) |cl_1 - |Z - Z_1|| \right) \left( \sqrt{1+v^2} + \frac{v^2}{\sqrt{1+v^2}} \right) dv$$

and we are left with the estimation for the first vertex:

$$\begin{aligned} & \frac{\delta \omega^{-1}(J, \theta, Z)}{\delta |\Psi(\theta - l_1, Z_1)|^2} \\ &= \frac{A_1}{A} \sum_{k \neq 0} \frac{(cl_1 + |Z - Z_1|)}{2|Z - Z_1|} \int \omega(J, \theta - l_1, Z_1) \times \frac{T(u)}{\sqrt{2\eta}C} du \\ & \frac{2C^2}{\pi} \operatorname{Re} \int_0^1 \exp \left( -C \sqrt{\frac{2}{\eta}} \left( \sqrt{1+v^2} + iv \right) |cl_1 - |Z - Z_1|| \right) \left( \sqrt{1+v^2} + \frac{v^2}{\sqrt{1+v^2}} \right) dv \end{aligned} \quad (134)$$

**1.3.2.3 Gaussian approximation** We can estimate the integral  $\int_0^1 dv$  in (134) by integrating between 0 and  $+\infty$ .

$$\begin{aligned} & \frac{2C^2}{\pi} \operatorname{Re} \int_0^{+\infty} \exp \left( -C \sqrt{\frac{2}{\eta}} \left( \sqrt{1+v^2} + iv \right) |cl_1 - |Z - Z_1|| \right) \left( 2\sqrt{1+v^2} - \frac{1}{\sqrt{1+v^2}} \right) dv \\ &= \frac{2C^2}{\pi} \operatorname{Re} \int_0^{+\infty} \exp \left( -C \sqrt{\frac{2}{\eta}} \left( \sqrt{1+v^2}a + ivb \right) \right) \left( 2\sqrt{1+v^2} - \frac{1}{\sqrt{1+v^2}} \right) dv \end{aligned}$$

with  $a = b = |cl_1 - |Z - Z_1||$ . The last integral can be rewritten:

$$\begin{aligned} & \left( -\frac{2\partial_a}{C\sqrt{\frac{2}{\eta}}} + C\sqrt{\frac{2}{\eta}} \int da \right) \frac{2C^2}{\pi} \operatorname{Re} \int_0^{+\infty} \exp \left( -C \sqrt{\frac{2}{\eta}} \left( \sqrt{1+v^2}a + ivb \right) \right) dv \\ &= \left( -\frac{2\partial_a}{C\sqrt{\frac{2}{\eta}}} + C\sqrt{\frac{2}{\eta}} \int da \right) \frac{2C^2}{\pi b} \operatorname{Re} \int_0^{+\infty} \exp \left( -C \sqrt{\frac{2}{\eta}} \left( \sqrt{b^2 + v^2} \frac{a}{b} + iv \right) \right) dv \\ &\simeq \left( -\frac{2\partial_a}{C\sqrt{\frac{2}{\eta}}} + C\sqrt{\frac{2}{\eta}} \int da \right) \frac{2C^2}{\pi b} \operatorname{Re} \int_0^{+\infty} \exp \left( -C \sqrt{\frac{2}{\eta}} \left( -\left( \frac{a}{b} + i \right) v + \frac{ab}{2v} \right) \right) dv \end{aligned} \quad (135)$$

for  $a \simeq b \ll 1$ . We use that:

$$\begin{aligned} & \left( -\frac{2\partial_a}{C\sqrt{\frac{2}{\eta}}} + C\sqrt{\frac{2}{\eta}} \int da \right) \frac{2C^2}{\pi} \operatorname{Re} \int_0^{+\infty} \exp \left( -C \sqrt{\frac{2}{\eta}} \left( -\left( (a+ib)v + \frac{a}{2v} \right) \right) \right) dv \\ &= \left( -\frac{2\partial_a}{C\sqrt{\frac{2}{\eta}}} + C\sqrt{\frac{2}{\eta}} \int da \right) \frac{2C^2}{\pi} \operatorname{Re} \sqrt{\frac{2a}{a+ib}} K_1 \left( 2C \sqrt{\frac{a(a+ib)}{\eta}} \right) \end{aligned} \quad (136)$$

where  $K_1$  is a modified Bessel function, and that the following identity holds for  $K_1$ :

$$\sqrt{\frac{2a}{a+ib}} K_1 \left( 2C \sqrt{\frac{a(a+ib)}{\eta}} \right) \simeq \sqrt{\frac{2a}{a+ib}} \sqrt{\frac{\pi}{4C\sqrt{\frac{a(a+ib)}{\eta}}}} \exp \left( -2C \sqrt{\frac{a(a+ib)}{\eta}} \right)$$

for  $C \gg 1$ . Then computing the integral  $\int da$  in (136) yields:

$$\begin{aligned}
& C\sqrt{\frac{2}{\eta}} \int da \left( \frac{\sqrt{\frac{2a}{a+ib}} \sqrt{\frac{\pi}{4C\sqrt{\frac{a(a+ib)}{\eta}}}}}{-\frac{C}{\eta} \frac{2a+ib}{\sqrt{\frac{a}{\eta}(a+ib)}}} \left( -\frac{C}{\eta} \frac{2a+ib}{\sqrt{\frac{a}{\eta}(a+ib)}} \right) \exp \left( -2C\sqrt{\frac{a(a+ib)}{\eta}} \right) \right) \\
& \simeq C\sqrt{\frac{2}{\eta}} \frac{\sqrt{\frac{2a}{a+ib}} \sqrt{\frac{\pi}{4C\sqrt{\frac{a(a+ib)}{\eta}}}}}{-\frac{C}{\eta} \frac{2a+ib}{\sqrt{\frac{a}{\eta}(a+ib)}}} \exp \left( -2C\sqrt{\frac{a(a+ib)}{\eta}} \right) \\
& = -2\sqrt{\pi}a \frac{\sqrt{\frac{1}{4C\sqrt{\frac{a}{\eta}(a+ib)}}}}{2a+ib} \exp \left( -2C\sqrt{\frac{a(a+ib)}{\eta}} \right)
\end{aligned}$$

for  $C \gg 1$ . For  $a = b$ , this identity reduces to:

$$-\sqrt{\pi} \frac{2}{5} - \frac{1}{5}i \sqrt{\frac{\sqrt{\eta}}{Ca}} \exp \left( -2\frac{Ca}{\sqrt{\eta}} \sqrt{1+i} \right) \quad (137)$$

The derivative arising in (136) can be estimated by:

$$\begin{aligned}
& -\frac{2\partial_a}{C\sqrt{\frac{2}{\eta}}} \left( \sqrt{\frac{2a}{a+ib}} \sqrt{\frac{\pi}{4C\sqrt{\frac{a(a+ib)}{\eta}}}} \exp \left( -2C\sqrt{\frac{a(a+ib)}{\eta}} \right) \right) \\
& = -\frac{1}{8} \sqrt{\left(\frac{1}{2} - \frac{1}{2}i\right)} \frac{\sqrt{\pi}}{C^2 a^3 \eta} \frac{\exp \left( -2\sqrt{(1+i)}C\sqrt{\frac{a^2}{\eta}} \right)}{\left(\frac{1}{\eta}\right)^{\frac{3}{2}} \sqrt{\frac{1}{C\sqrt{\frac{a^2}{\eta}}}}} \left( (1+2i)((1+i))^{\frac{3}{4}} \eta \sqrt{\frac{a^2}{\eta}} - (12-4i) \sqrt[4]{(1+i)}Ca^2 \right) \\
& \simeq \frac{1}{8} \sqrt{\left(\frac{1}{2} - \frac{1}{2}i\right)} \frac{\sqrt{\pi}}{C^2 a^3 \eta} \frac{\exp \left( -2\sqrt{(1+i)}C\sqrt{\frac{a^2}{\eta}} \right)}{\left(\frac{1}{\eta}\right)^{\frac{3}{2}} \sqrt{\frac{1}{C\sqrt{\frac{a^2}{\eta}}}}} \left( (12-4i) \sqrt[4]{(1+i)}Ca^2 \right) \\
& = \frac{\sqrt{\pi}}{8} \sqrt{\left(\frac{1}{2} - \frac{1}{2}i\right)} \frac{\exp \left( -2\sqrt{\frac{(1+i)}{\eta}}Ca \right)}{\sqrt{\frac{Ca}{\sqrt{\eta}}}} \left( (12-4i) \sqrt[4]{(1+i)} \right) \quad (138)
\end{aligned}$$

Gathering (137) and (138), we find that for  $C \gg 1$ , For  $a = b = |cl_1 - |Z - Z_1||$ , we find for (136):

$$\begin{aligned}
& C^2 \frac{\sqrt{130}}{5\sqrt{\pi}} \left( \frac{1}{\sqrt{2}} \right)^{\frac{1}{4}} \frac{\exp \left( -\frac{2^{\frac{3}{8}} \cos\left(\frac{\pi}{8}\right)C}{\sqrt{\eta}} |cl_1 - |Z - Z_1|| \right)}{\sqrt{\frac{C|cl_1 - |Z - Z_1||}{\sqrt{\eta}}}} \cos \left( \frac{2^{\frac{3}{8}} \cos\left(\frac{\pi}{8}\right)C}{\sqrt{\eta}} |cl_1 - |Z - Z_1|| \right) \\
& = C^2 \frac{\sqrt{65\sqrt{2}}}{5\sqrt{\pi}} \frac{\exp \left( -\frac{2^{\frac{3}{8}} \sqrt{\sqrt{2}+1}C}{\sqrt{\eta}} |cl_1 - |Z - Z_1|| \right)}{\sqrt{\frac{C|cl_1 - |Z - Z_1||}{\sqrt{\eta}}}} \cos \left( \frac{2^{\frac{3}{8}} \sqrt{\sqrt{2}+1}C}{\sqrt{\eta}} |cl_1 - |Z - Z_1|| \right)
\end{aligned}$$

In the sequel, for  $(A\pi)^2 \eta \ll 1$ , we approximate:

$$C = \sqrt{\ln \left( \frac{2u^2}{(A\pi)^2 \eta} \right)} \simeq \sqrt{\ln \left( \frac{2}{(A\pi)^2 \eta} \right)}$$

Finally, the integral over  $u$  in (134) is:

$$\int \frac{T(u)}{\sqrt{2\eta}C} \exp\left(i\frac{u}{2}(cl_1 + |Z - Z_1|)\right) du = \frac{1}{\sqrt{2\eta}C} \hat{T}\left(\frac{cl_1 + |Z - Z_1|}{2}\right) du$$

so that, using that

$$\hat{T}_1\left(\frac{cl_1 + |Z - Z_1|}{2}\right) = \frac{A_1}{A} \hat{T}\left(\frac{cl_1 + |Z - Z_1|}{2}\right)$$

The result for (134) is:

$$\begin{aligned} \frac{\delta\omega^{-1}(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} &\simeq \frac{\sqrt{65}}{5\sqrt{\pi}2^{\frac{3}{8}}(\sqrt{2}+1)^{\frac{1}{4}}} D \frac{\exp(-D|cl_1 - |Z - Z_1||)}{\sqrt{\frac{C|cl_1 - |Z - Z_1||}{\sqrt{\eta}}}} \cos(D|cl_1 - |Z - Z_1||) \\ &\times \frac{(cl_1 + |Z - Z_1|)}{2|Z - Z_1|} \hat{T}_1\left(\frac{cl_1 + |Z - Z_1|}{2}\right) \end{aligned}$$

where:

$$D = \frac{2^{\frac{3}{8}}\sqrt{\sqrt{2}+1}C}{\sqrt{\eta}}$$

We also write this result in a more compact form:

$$\frac{\delta\omega^{-1}(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} = \frac{A_1}{A} \Xi(|Z_1 - Z|, l_1, \bar{\omega}) \omega(J, \theta - l_1, Z_1) \quad (139)$$

for  $\frac{|Z_1 - Z|}{cl_1} < 1$ , and 0 otherwise, with:

$$\begin{aligned} \Xi(|Z_1 - Z|, l_1, \bar{\omega}) &= \frac{\sqrt{65}}{5\sqrt{\pi}2^{\frac{3}{8}}(\sqrt{2}+1)^{\frac{1}{4}}} D \frac{\exp(-D|cl_1 - |Z - Z_1||)}{\sqrt{D|cl_1 - |Z - Z_1||}} \cos(D|cl_1 - |Z - Z_1||) \\ &\times \frac{(cl_1 + |Z - Z_1|)}{2|Z - Z_1|} \hat{T}\left(\frac{cl_1 + |Z - Z_1|}{2}\right) \end{aligned} \quad (140)$$

Similarly:

$$\frac{\delta\omega(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} = \Xi(|Z_1 - Z|, l_1, \bar{\omega}) \omega(J, \theta - l_1, Z_1) \quad (141)$$

The appearance of the  $\cos(D|cl_1 - |Z - Z_1||)$  in (140) is a consequence of our approximation computing the integral between 0 and  $+\infty$ . This approximation breaks down when the cos function becomes negative. As a consequence for  $D|cl_1 - |Z - Z_1|| > \frac{\pi}{2}$ , we can set  $\Xi(|Z_1 - Z|, l_1, \bar{\omega}) \simeq 0$ .

## 1.4 Computation of the $2n$ -th Vertices in (107)

### 1.4.1 Second order vertex

The two previous paragraphs computed the first order derivatives  $\frac{\delta\omega^{-1}(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2}$ . This can be used to compute higher order terms involved in (107). We first compute the 2-th vertex is given by:

$$\begin{aligned} &-\frac{1}{2} \left( \nabla_{\theta} \frac{\sigma_{\theta}^2}{2} \nabla_{\theta} \right) + \frac{1}{2} \left[ \frac{\delta \left[ \Psi^{\dagger}(\theta', Z) \nabla_{\theta} \left( \omega^{-1}(J, \theta, |\Psi|^2) \Psi(\theta, Z) \right) \right]}{\delta |\Psi|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \\ &= -\frac{1}{2} \left( \nabla_{\theta} \frac{\sigma_{\theta}^2}{2} \nabla_{\theta} \right) + \frac{1}{2} \left[ \nabla_{\theta} \left( \omega^{-1}(J, \theta, \mathcal{G}_0(0, Z)) \right) + \omega^{-1}(J, \theta, \mathcal{G}_0(0, Z)) \nabla_{\theta} \right] \\ &+ \left( \frac{\nabla_{\theta} \mathcal{G}_0(\theta, \theta', Z)}{2} \right)_{\theta = \theta'} \left[ \frac{\delta \left[ \nabla_{\theta} \left( \omega^{-1}(J, \theta, |\Psi|^2) \right) \right]}{\delta |\Psi|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \end{aligned}$$

and given the results of the previous section, it writes:

$$\begin{aligned}
&= -\frac{1}{2} \left( \nabla_\theta \frac{\sigma_\theta^2}{2} \nabla_\theta \right) + \frac{1}{2} [\nabla_\theta \omega^{-1}(J, \theta, Z, \mathcal{G}_0)] \\
&\quad + \left( \sqrt{\left( \frac{1}{\sigma^2 \bar{X}_r} \right)^2 + \frac{2\alpha}{\sigma^2} - \frac{1}{\sigma^2 \bar{X}_r}} \right) \mathcal{G}_0(0, Z) \int \frac{A_1}{A} \Xi(|Z_1 - Z|, l_1, \bar{\omega}) \omega(J, \theta - l_1, Z_1) dZ_1 dl_1 \\
&\simeq -\frac{1}{2} \left( \nabla_\theta \frac{\sigma_\theta^2}{2} \nabla_\theta \right) + \frac{1}{2} [\nabla_\theta \omega^{-1}(J, \theta, Z, \mathcal{G}_0) \nabla_\theta]
\end{aligned}$$

at the lowest order in perturbation theory. The inverse frequency  $\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z))$  is solution of:

$$\begin{aligned}
\omega^{-1}(J, \theta, Z) &= G \left( J(\theta) + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega\left(J, \theta - \frac{|Z-Z_1|}{c}, Z_1\right)}{\omega(J, \theta, Z)} W \left( \frac{\omega(J, \theta, Z)}{\omega\left(J, \theta - \frac{|Z-Z_1|}{c}, Z_1\right)} \right) \mathcal{G}_0(0, Z_1) dZ_1 \right) \\
&\simeq G \left( J(\theta) + \frac{\kappa}{N} \int \frac{T(Z, Z_1) \omega\left(J, \theta - \frac{|Z-Z_1|}{c}, Z_1\right) W \left( \frac{\omega(J, \theta, Z)}{\omega\left(J, \theta - \frac{|Z-Z_1|}{c}, Z_1\right)} \right) dZ_1}{\sqrt{\frac{\pi}{8} \left( \frac{1}{\bar{X}_r} \right)^2 + \frac{\pi}{2} \alpha \omega(J, \theta, Z)}} \right) \quad (142)
\end{aligned}$$

where we used (96). Recall that in first approximation, that will be used to compute higher order vertices, the solution is the constant:

$$\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z)) \simeq \bar{X}_r = G \left( \bar{J} + \frac{\kappa}{N} \int \frac{T(Z, Z_1) W(1) dZ_1}{\sqrt{\frac{\pi}{8} \left( \frac{1}{\bar{X}_r} \right)^2 + \frac{\pi}{2} \alpha}} \right)$$

where  $\bar{J}$  is the average of the external current. In this approximation, the 2-th vertex writes:

$$-\frac{1}{2} \left( \nabla_\theta \frac{\sigma_\theta^2}{2} \nabla_\theta \right) + \frac{1}{2} [\omega^{-1}(G_{0Z}(0, 0)) \nabla_\theta] = -\frac{1}{2} \left( \nabla_\theta \frac{\sigma_\theta^2}{2} \nabla_\theta \right) + \frac{1}{2} \bar{X}_r \nabla_\theta$$

and the associated Green function is (96):

$$\mathcal{G}_{\bar{X}_r} = \frac{\exp\left(-\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}} L\right)}{\sqrt{\frac{\pi}{2}} \sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}}} \exp\left(\frac{\theta_f - \theta_i}{\sigma^2 \bar{X}_r}\right)$$

We conclude by computing the derivative of the functionals  $\hat{T}_1(\theta, Z, Z_1\omega, \Psi)$  and  $\hat{T}(\theta, Z, Z_1\omega, \Psi)$  (112) and (114) arising in (113) and (114). They will be useful to obtain the higher order vertices. We find:

$$\begin{aligned}
& \frac{\delta \hat{T}_1(\theta, Z, Z_1\omega, |\Psi|^2)}{\delta |\Psi(\theta - l_2, Z_2)|^2} \\
&= \frac{1}{\omega(J, \theta, Z)} \frac{\frac{\kappa}{N} T(Z, Z_1) G' [J, \omega, \theta, Z, |\Psi|^2]}{1 - \left( \int \frac{\kappa}{N} \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right) \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) G' [J, \omega, \theta, Z, |\Psi|^2]} \\
&= \left( \frac{\delta \ln \omega^{-1}(J, \theta, Z)}{\delta |\Psi(\theta - l_2, Z_2)|^2} \hat{T}_1(\theta, Z, Z_1\omega, |\Psi|^2) \right) \\
&+ \frac{\frac{\delta \ln G' [J, \omega, \theta, Z, |\Psi|^2]}{\delta |\Psi(\theta - l_2, Z_2)|^2} \hat{T}_1(\theta, Z, Z_1\omega, |\Psi|^2)}{1 - \left( \int \frac{\kappa}{N} \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right) \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) G' [J, \omega, \theta, Z, |\Psi|^2]} \\
&+ \hat{T}_1(\theta, Z, Z_1, \omega, |\Psi|^2) \omega(J, \theta, Z) \hat{T}_1(\theta, Z, Z_2, \omega, |\Psi|^2) \omega(J, \theta - l_2, Z_2) \delta \left( l_2 - \frac{|Z - Z_2|}{c} \right) \\
&+ \hat{T}_1(\theta, Z, Z_1, \omega, |\Psi|^2) \omega(J, \theta, Z) \int \frac{\delta \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right)}{\delta |\Psi(\theta - l_2, Z_2)|^2} \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 \hat{T}(\theta, Z, Z', \omega, |\Psi|^2) dZ' \\
& \\
& \frac{\delta \hat{T}(\theta, Z, Z_1\omega, |\Psi|^2)}{\delta |\Psi(\theta - l_2, Z_2)|^2} \\
&= \hat{T}_1(\theta, Z, Z_1, \omega, \Psi) \left( \frac{\delta \ln \omega^{-1}(J, \theta, Z)}{\delta |\Psi(\theta - l_2, Z_2)|^2} \right. \\
& \left. + \frac{\frac{\delta \ln G' [J, \omega, \theta, Z, |\Psi|^2]}{\delta |\Psi(\theta - l_2, Z_2)|^2}}{1 - \left( \int \frac{\kappa}{N} \frac{\omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right)}{\omega(J, \theta, Z)} \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) G' [J, \omega, \theta, Z, |\Psi|^2]} \right) \\
&+ \hat{T}_1(\theta, Z, Z_1, \omega, \Psi) \frac{\delta \ln \omega^{-1}(J, \theta, Z)}{\delta |\Psi(\theta - l_2, Z_2)|^2}
\end{aligned}$$

In the limit of small fluctuations around an equilibrium frequency  $\omega_0$ ,  $\frac{\delta \ln G' [J, \omega, \theta, Z, |\Psi|^2]}{\delta |\Psi(\theta - l_2, Z_2)|^2} \ll G' [J, \omega, \theta, Z, |\Psi|^2]$

$$\frac{\delta \hat{T}_1(\theta, Z, Z_1\omega, |\Psi|^2)}{\delta |\Psi(\theta - l_2, Z_2)|^2} \simeq 2\hat{T}_1(\theta, Z, Z_1, \omega, |\Psi|^2) \frac{\delta \ln \omega^{-1}(J, \theta, Z)}{\delta |\Psi(\theta - l_2, Z_2)|^2} \quad (143)$$

with:

$$\begin{aligned}
& \frac{\delta \hat{T}(\theta, Z, Z_1 \omega, |\Psi|^2)}{\delta |\Psi(\theta - l_2, Z_2)|^2} \\
= & \frac{\frac{\kappa}{N} \omega(J, \theta, Z) T(Z, Z_1) F' [J, \omega, \theta, Z, |\Psi|^2]}{\omega^2(J, \theta, Z) + \left( \int \frac{\kappa}{N} \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right) \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) F' [J, \omega, \theta, Z, |\Psi|^2]} \\
= & \left( \frac{\delta \ln \omega(J, \theta, Z)}{\delta |\Psi(\theta - l_2, Z_2)|^2} \hat{T}(\theta, Z, Z_1 \omega, |\Psi|^2) \right) \\
& \times \frac{\left( \int \frac{\kappa}{N} \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right) \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) F' [J, \omega, \theta, Z, |\Psi|^2] - \omega^2(J, \theta, Z)}{\omega^2(J, \theta, Z) + \left( \int \frac{\kappa}{N} \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right) \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) F' [J, \omega, \theta, Z, |\Psi|^2]} \\
& + \frac{\omega^2(J, \theta, Z) \frac{\delta \ln F' [J, \omega, \theta, Z, \Psi]}{\delta |\Psi(\theta - l_2, Z_2)|^2} \hat{T}(\theta, Z, Z_1 \omega, \Psi)}{\omega^2(J, \theta, Z) + \left( \int \frac{\kappa}{N} \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right) \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) F' [J, \omega, \theta, Z, \Psi]} \\
& + \hat{T}(\theta, Z, Z_1, \omega, |\Psi|^2) \omega^{-1}(J, \theta, Z) \hat{T}(\theta, Z, Z_2, \omega, \Psi) \omega(J, \theta - l_2, Z_2) \delta \left( l_2 - \frac{|Z - Z_2|}{c} \right) \\
& + \hat{T}(\theta, Z, Z_1, \omega, |\Psi|^2) \omega^{-1}(J, \theta, Z) \int \frac{\delta \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right)}{\delta |\Psi(\theta - l_2, Z_2)|^2} \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 \hat{T}(\theta, Z, Z', \omega, |\Psi|^2) dZ' \\
= & 2 \left( \frac{\delta \ln \omega(J, \theta, Z)}{\delta |\Psi(\theta - l_2, Z_2)|^2} \hat{T}(\theta, Z, Z_1 \omega, |\Psi|^2) \right) \\
& \times \frac{\left( \int \frac{\kappa}{N} \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right) \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) F' [J, \omega, \theta, Z, |\Psi|^2]}{\omega^2(J, \theta, Z) + \left( \int \frac{\kappa}{N} \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right) \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) F' [J, \omega, \theta, Z, |\Psi|^2]} \\
& + \frac{\omega^2(J, \theta, Z) \frac{\delta \ln F' [J, \omega, \theta, Z, |\Psi|^2]}{\delta |\Psi(\theta - l_2, Z_2)|^2} \hat{T}(\theta, Z, Z_1 \omega, |\Psi|^2)}{\omega^2(J, \theta, Z) + \left( \int \frac{\kappa}{N} \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right) \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) F' [J, \omega, \theta, Z, |\Psi|^2]}
\end{aligned}$$

For relatively high frequency,  $\omega^2(J, \theta, Z) > 1$ ,  $F' [J, \omega, \theta, Z, \Psi] \ll 1$ ,  $\frac{\delta \ln G' [J, \omega, \theta, Z, \Psi]}{\delta |\Psi(\theta - l_2, Z_2)|^2} \ll 1$  and  $\frac{\delta \hat{T}(\theta, Z, Z_1 \omega, \Psi)}{\delta |\Psi(\theta - l_2, Z_2)|^2}$  can be discarded in first approximation.

## 1.4.2 Computation of the $2n$ -th vertex

### 1.4.2.1 Expression close to the permanent regime

Close to the permanent regime, the  $2n$  point vertex contribution  $\frac{\delta^n \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2}$  can be computed using

(116) and neglecting the derivatives of  $\hat{T}(\theta, Z, Z_1, \omega, \Psi)$ . We assume that we have ranked the derivatives such that  $l_1 < l_2 < \dots < l_n$ . An other simplification arises. Considering the vertex:

$$\begin{aligned}
V_{2n} &= \frac{1}{2(n)!} \left[ \int \Psi^\dagger(\theta, Z) \frac{\delta^n [\int \Psi^\dagger(\theta, Z) \nabla_\theta \omega^{-1}(J, \theta, Z) \Psi(\theta, Z) dZ d\theta]}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2} \right. \\
&\quad \left. \times \prod_{i=1}^{n-1} |\Psi(\theta - l_i, Z_i)|^2 dl_i \Psi(\theta, Z) dZ d\theta \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \quad (144) \\
&= \frac{1}{2(n-1)!} \int \Psi^\dagger(\theta, Z) \nabla_\theta \left[ \frac{\delta^{n-1} \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^{n-1} \delta |\Psi(\theta - l_i, Z_i)|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \prod_{i=1}^{n-1} |\Psi(\theta - l_i, Z_i)|^2 dZ_i \Psi(\theta, Z) dZ d\theta dl_i \\
&\quad + \frac{1}{2n!} \int \nabla_\theta \left[ \frac{\delta^n \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \prod_{i=1}^n |\Psi(\theta - l_i, Z_i)|^2 \prod_{i=1}^n dZ_i dZ d\theta dl_i
\end{aligned}$$

As for (107), we can neglect the second term, so that  $V_{2n}$  reduces to:

$$V_{2n} = \frac{1}{2(n-1)!} \int \Psi^\dagger(\theta, Z) \nabla_\theta \frac{\delta^{n-1} \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^{n-1} \delta |\Psi(\theta - l_i, Z_i)|^2} \prod_{i=1}^{n-1} |\Psi(\theta - l_i, Z_i)|^2 \prod_{i=1}^{n-1} dZ_i dl_i \Psi(\theta, Z) d\theta dZ \quad (145)$$

The neglected contributions will be reintroduced in Appendix 4.

#### 1.4.2.2 Estimation of (145)

The  $2n$  vertex contribution  $\frac{\delta^n \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2}$  can then be computed recursively. Assuming  $l_1 < l_2 < \dots < l_n$  and using (139) and (141), one has:

$$\begin{aligned}
&\frac{\delta^n \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2} \\
&= \sum_{k=1}^n \left( \frac{4}{3} \pi^3 \bar{\omega} l_1^2 \right)^k \sum_{i_1 < i_2 < \dots < i_k \leq n} \sum_{m_p} \sum_{\substack{i_{p,1} < \dots < i_{p,m_p}, \\ \cup \{i_p, i_{p,1}, \dots, i_{p,m_p}\} = \{1, \dots, n\}}} \\
&\quad \times \prod_{p=1}^k \Xi_1(|Z_{i_p} - Z|, l_{i_p}, \bar{\omega}) \Xi(|Z_{i_{p,1}} - Z_{i_p}|, l_{i_{p,1}} - l_{i_p}, \bar{\omega}) \dots \\
&\quad \times \Xi(|Z_{i_{p,m_p}} - Z_{i_{p,m_p-1}}|, l_{i_{p,m_p}} - l_{i_{p,m_p-1}}, \bar{\omega}) \omega(J, \theta - l_{i_{p,m_p}}, Z_2)
\end{aligned}$$

The  $\frac{4}{3} \pi^3$  factor amounts for spherical symmetry in  $3D$ ,  $\bar{\omega}$  is the average of  $\omega(J, \theta - l_i, Z_i)$  over variable  $Z_i$  and  $l_1^2$  comes from the integration on delay  $l_1$ .

Then, the  $2n$  vertex writes:

$$\begin{aligned}
V_{2n} &\simeq \Psi^\dagger(\theta, Z) \nabla_\theta \prod_{i=1}^{n-1} |\Psi(\theta - l_i, Z_i)|^2 \sum_{k=1}^n \bar{\omega}^k \sum_{\cup_i \{l_j^{(i)}\} = \{l_1, \dots, l_{n-1}\}} \prod_{i=1}^k \Xi \left( J, \theta, l_1^{(i)}, |Z_1^{(i)} - Z| \right) \Xi \left( J, \theta, l_2^{(i)} - l_1^{(i)}, |Z_2^{(i)} - Z| \right) \\
&\quad \times \dots \Xi \left( J, \theta, l_{p_i}^{(i)} - l_{p_i-1}^{(i)}, |Z_{p_i}^{(i)} - Z_{p_i-1}^{(i)}| \right) \omega \left( J, \theta - l_{p_i}^{(i)}, Z_{p_i}^{(i)} \right) \Psi(\theta, Z) dZ d\theta \\
&\equiv \Psi^\dagger(\theta, Z) \prod_{i=1}^{n-1} |\Psi(\theta - l_i, Z_i)|^2 V_{2n}(l_i, Z_i) \Psi(\theta, Z) dZ d\theta
\end{aligned} \tag{146}$$

The dominant term in the sum is:

$$\begin{aligned}
&\frac{1}{2} \Psi^\dagger(\theta, Z) \nabla_\theta \left( \Psi(\theta, Z) \prod_{i=1}^{n-1} |\Psi(\theta - l_i, Z_i)|^2 V_{2n}(l_i, Z_i) \right) \\
&= \frac{1}{2} \Psi^\dagger(\theta, Z) \nabla_\theta \left( \Psi(\theta, Z) \prod_{i=1}^{n-1} |\Psi(\theta - l_i, Z_i)|^2 \Xi_1(J, \theta, l_i, |Z_i - Z|) \omega(J, \theta - l_i, Z_i) \right) \\
&\simeq \frac{1}{2} \Psi^\dagger(\theta, Z) \nabla_\theta \left( \Psi(\theta, Z) \prod_{i=1}^{n-1} |\Psi(\theta - l_i, Z_i)|^2 \Xi_1(J, \theta, l_i, |Z_i - Z|) \bar{\omega} \right) \\
&\equiv \frac{1}{2} \Psi^\dagger(\theta, Z) \nabla_\theta \left( \Psi(\theta, Z) \prod_{i=1}^{n-1} |\Psi(\theta - l_i, Z_i)|^2 \bar{\Xi}_1(J, \theta, l_i, |Z_i - Z|) \right)
\end{aligned} \tag{147}$$

where  $\bar{\omega}$  is the average of  $\omega(J, \theta - l_i, Z_i)$  on variable  $Z_i$ . This represents the vertex of valence  $n$  issued from  $Z$ .

## Appendix 2 Computation of the graphs expansion and generating functional for correlation functions

We compute the sum of graphs involved in the partition function with source term, i.e. the graphs deduced from the interaction terms (147). This is done in several steps. We first give the general form of these graphs. Then, we compute the factors arising from the vertices (147). This allows to find the full sum of graphs, that is, the generating function for correlation functions, then the sum of connected graphs, and ultimately the one particle irreducible graphs, which yields the effective action. The computation of the graphs is first performed without inertia coefficients, i.e. without considering the potential and setting  $\zeta^{(k)} = 0$ . These coefficients are included in the computation subsequently.

### 2.1 Case of null inertia coefficients $\zeta_l$

#### 2.1.1 General form of the graphs

The  $2n$ -th point vertex  $\Gamma_{2n}$  contribution to the effective action is obtained by considering any  $1PI$  graph made of arbitrary number  $m_i$  of vertices  $V_{2k_i}(l_i, Z_i)$  defined in (146), where  $k_i \leq n$ . Those graphs have no loop drawn between two legs of any of the vertices (these contributions are already taken into account by the expansion around  $\mathcal{G}_0(0, Z)$ ). They have  $\sum_i m_i k_i - n$  segments.

The absence of internal loops implies that the graphs associated to  $\Gamma_{2n}$  are made of  $n$  paths  $P_i$  with  $l_{P_i}$  segment and:  $\sum l_{P_i} = \sum_i m_i k_i - n$ . The segments are connected by the vertices  $V_{2k_i}(l_i, Z_i)$ . The contributions associated to these paths are products of 2 points Green functions. In the approximation of constant  $\mathcal{G}_0(0, Z) = \bar{\mathcal{G}}_0(0, 0)$ , the contribution of a path with  $k$  segments and of total length  $L$  between two



points  $\theta_i$  and  $\theta_f$  is:

$$\frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}} - \frac{1}{\sigma^2 \bar{X}_r}\right)L\right)}{\left(\sqrt{\frac{\pi}{2}}\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}}\right)^k}$$

where:

$$\bar{X}_r = \frac{\arctan\left(\left(\frac{1}{\bar{X}_r} - \frac{1}{\bar{X}_r}\right)\sqrt{\bar{J} + \bar{\mathcal{G}}_0(0,0)}\right)}{\sqrt{\bar{J} + \bar{\mathcal{G}}_0(0,0)}}$$

The quantities  $\bar{J}$ ,  $\bar{X}_r$ ,  $\bar{\mathcal{G}}_0(0,0)$  are the averages along the paths, since we sum over  $\theta$  along the paths. These paths are connected through the vertices  $V_{2k_i}(l_i, Z_i)$  of valence  $k_i \leq n$ . These vertices connect one path at some time  $\theta$  and  $k$  others at time  $\theta - l_i$  with  $i = 1, \dots, k-1$ . To compute the sum of connected graphs, we have to add the graphs associated to all possible repartitions of vertices  $V_{2k_i}(l_i, Z_i)$ ,  $k_i \leq n$  between the paths  $P_i$ . Each of this graph has to be summed over the time of insertion (i.e.  $\theta, \theta - l_i$ ) for each vertex. To perform this computation, we consider the graph made of  $n$  paths for the points  $Z_i$  with  $k_l^{(i)}$  vertices of valence  $2l$  with  $l = 1, \dots, n$  issued from the  $i$ -th path.

This means that the total number of segments of this graph is:

$$\sum_{l=1}^n l \sum_{i=1}^n k_l^{(i)} - n$$

### 2.1.2 Factors due to the vertices

Equation (147) shows that the vertices issued from  $Z_i$  induce factors of the form  $\bar{\Xi}_1(J, \theta, l_i, |Z_i - Z_k|) + \frac{\nabla_\theta \bar{\Xi}_1(J, \theta, l_i, |Z_i - Z_k|)}{\bar{\mathcal{G}}'_0}$ , and a factor:

$$\mathcal{G}'_0 = \left(\frac{\nabla_\theta \mathcal{G}_0(\theta, \theta', Z)}{2}\right)_{\theta=\theta'} = -\left(\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}} - \frac{1}{\sigma^2 \bar{X}_r}\right) \mathcal{G}_0(0, Z)$$

due to the insertion of  $\frac{1}{2}\Psi^\dagger(\theta, Z)\nabla_\theta\Psi(\theta, Z)$  except at the final points of the graphs. Indeed, in:

$$\frac{1}{2}\Psi^\dagger(\theta, Z)\nabla_\theta\left(\Psi(\theta, Z)\prod_{i=1}^{n-1}|\Psi(\theta - l_i, Z_i)|^2\bar{\Xi}_1(J, \theta, l_i, |Z_i - Z|)\right)$$

the gradient  $\nabla_\theta|\Psi(\theta - l_i, Z_i)|^2$  induces a nul contribution in the Green functions since:

$$\nabla_\theta(\mathcal{G}_0(\theta_1, \theta - l_i, Z)\mathcal{G}_0(\theta - l_i, \theta_2, Z)) = 0$$

The insertion  $\frac{1}{2}\Psi^\dagger(\theta, Z)\nabla_\theta\Psi(\theta, Z)$  inserted between two points  $\theta_1$  and  $\theta_2$  yields terms of the form:

$$(\nabla_\theta\mathcal{G}_0(\theta_1, \theta, Z))\mathcal{G}_0(\theta, \theta_2, Z)$$

that results in the presence of the term  $-\mathcal{G}'_0(Z)\bar{\Xi}_1(J, \theta, l_i, |Z_i - Z|)$ , where  $\mathcal{G}'_0(Z) = \nabla_\theta(\mathcal{G}_0(\theta_1, \theta, Z))_{\theta_1=\theta}$ . On the other hand, the term  $\mathcal{G}_0(Z)\nabla_\theta\bar{\Xi}_1(J, \theta, l_i, |Z_i - Z|)$  with  $\mathcal{G}_0(Z) = \mathcal{G}_0(\theta, \theta, Z)$ , has to be added and the overall vertice yields a factor

$$-\mathcal{G}'_0(Z)\bar{\Xi}_1(J, \theta, l_i, |Z_i - Z_k|) + \mathcal{G}_0(Z)\nabla_\theta\bar{\Xi}_1(J, \theta, l_i, |Z_i - Z_k|) \equiv \nabla_\theta\mathcal{G}_0(Z)\bar{\Xi}_1(J, \theta, l_i, |Z_i - Z_k|) \quad (148)$$

In the sequel it is approximated by its value in the permanent regime  $-\mathcal{G}'_0(Z)\bar{\Xi}_1(J, \theta, l_i, |Z_i - Z_k|)$  for the sake of simplicity.

We consider the contributions of graphs without external legs. These ones are reintroduced ultimately.

At a final point of graph  $\theta_f^{(i)}$ , the insertion of  $\frac{1}{2}\Psi^\dagger(\theta, Z)\nabla_\theta\Psi(\theta, Z)$  induces a term  $\nabla_{\theta_i^{(i)}}$ . There is an overall factor of  $(-1)$  for each vertex, that can be accounted for by inserting rather  $G'_0$  at each vertices, except the initial one that includes a factor  $-\nabla_{\theta_i^{(i)}}$ .

The  $\sum_i k_l^{(i)}$  vertices of valence  $l$  are associated to a factor  $\frac{1}{(\sum_i k_l^{(i)})!}$  due to the development of the exponential. There is  $\frac{(\sum_i k_l^{(i)})!}{\prod_i k_l^{(i)!}}$  ways to distribute these vertices between the  $n$  points. Once  $k_l^{(i)}$  vertices,  $l = 1; \dots, n$  are attributed to a point  $i$ , there are  $(\sum_{l=1}^n k_l^{(i)})!$  ways to order in time these vertices.

The factor associated to a vertex of valence  $l$  issued from  $i$  at time  $\theta^{(i)}$  is:

$$\begin{aligned} & \bar{\Xi}_1^{(l)}\left(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}\right) \tag{149} \\ &= \frac{\sum_{\substack{\{k_1, \dots, k_{l-1}\} \\ \subset \{1, \dots, n-1\}}} \prod_{k_j} \int_{\frac{|Z_i - Z_{k_j}|}{c}}^{\theta^{(i)} - \theta_i^{(k_j)}} dl_{k_j}}{\left(\sqrt{\frac{\pi}{2}} \sqrt{\left(\frac{1}{\sigma^2 X_r}\right)^2 + \frac{2\alpha}{\sigma^2}}\right)^{l-1}} \left[ \int \Psi^\dagger(\theta^{(i)}, Z_i) \nabla_{\theta^{(i)}} \frac{\delta^{l-1} \omega^{-1}(J, \theta^{(i)}, Z_i)}{\prod_{i=1}^{l-1} \delta \left| \Psi(\theta^{(i)} - l_{k_j}, Z_{k_j}) \right|^2} \Psi(\theta^{(i)}, Z_i) dZ_i \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \\ &\simeq \frac{\sum_{\substack{\{k_1, \dots, k_{l-1}\} \\ \subset \{1, \dots, n-1\}}} \prod_{k_j} \int_{\frac{|Z_i - Z_{k_j}|}{c}}^{\theta^{(i)} - \theta_i^{(k_j)}} dl_{k_j}}{\left(\sqrt{\frac{\pi}{2}} \sqrt{\left(\frac{1}{\sigma^2 X_r}\right)^2 + \frac{2\alpha}{\sigma^2}}\right)^l} \left[ \nabla_{\theta^{(i)}} \mathcal{G}_0(Z) \frac{\delta^{l-1} \int \omega^{-1}(J, \theta^{(i)}, Z_i) dZ_i}{\mathcal{G}_0(Z) \prod_{i=1}^{l-1} \delta \left| \Psi(\theta^{(i)} - l_{k_j}, Z_{k_j}) \right|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \end{aligned}$$

with the convention  $\bar{\Xi}_1^{(l)}\left(Z_i, \theta_q^{(i)}, \{Z_j\}_{j \neq i}\right) = 0$  for  $n = 1$ . We also use the same convention as for (148): for any quantity  $X$ ,  $\nabla_{\theta^{(i)}} \mathcal{G}_0(Z) X = \mathcal{G}_0(Z) \nabla_{\theta^{(i)}} X + \mathcal{G}'_0(Z) X$ .

The propagators induced by the vertices has been included in the definition of  $\bar{\Xi}_1^{(l)}\left(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}\right)$ . The functions depend implicitly on the border of the timespans  $[\theta_f^{(j)}, \theta_i^{(j)}]$ . Actually, the integrations  $\int_{\frac{|Z_i - Z_{k_j}|}{c}}^{\theta^{(i)} - \theta_i^{(k_j)}} dl_{k_j}$  induces the presence of products of Heaviside functions  $H\left(\theta_f^{(i)} - \theta_i^{(k_j)} - \frac{|Z_i - Z_{k_j}|}{c}\right)$ .

### 2.1.3 Sum of graphs

We first start by computing the full sum of graphs without external legs and arising from all combination of vertices between  $n$  initial points and  $n$  final points. The factors associated to each vertices have been found in the previous paragraph. The  $\sum_{l=1}^n k_l^{(i)}$  vertices implies the integration over  $\theta_i^{(i)} < \theta_1^{(i)} < \dots < \theta_{\sum_{l=1}^n k_l^{(i)}}^{(i)} < \theta_f^{(i)}$  of the product of terms  $\bar{\Xi}_1^{(l_q)}\left(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}\right)$  for  $q = 1$  to  $\sum_{l=1}^n k_l^{(i)}$ . The number of  $l_q$  equal to  $l$  is  $k_l^{(i)}$ . Once an order  $l_1, l_2, \dots$  is chosen, there are  $\prod k_l^{(i)}!$  ways to order the vertices satisfying this order. Then, summing over the various orders  $l_1, l_2, \dots$  and over the  $k_l^{(i)}$  such that  $\sum_{l=1}^n k_l^{(i)} = m$  is fixed, the global factor associated to the vertices is:

$$\int_{\theta_i^{(i)} < \theta_1^{(i)} < \dots < \theta_m^{(i)} < \theta_f^{(i)}} \prod_{q=1}^m \left( \sum_l \bar{\Xi}_1^{(l)}\left(Z_i, \theta_q^{(i)}, \{Z_j\}_{j \neq i}\right) \right) \delta\left(\theta_1^{(i)} - \theta_i^{(i)}\right) \delta\left(\theta_m^{(i)} - \theta_f^{(i)}\right) d\theta_q^{(i)} \tag{150}$$

The delta functions accounts for the fact that without external legs, two vertices are set at the borders of the interval. If we approximate  $\sum_l \bar{\Xi}_1^{(l)}\left(Z_i, \theta_q^{(i)}, \{Z_j\}_{j \neq i}\right)$  by its average on the interval  $[\theta_i^{(i)}, \theta_f^{(i)}]$ , that is:

$$\frac{\int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \sum_l \bar{\Xi}_1^{(l)}\left(Z_i, \theta_q^{(i)}, \{Z_j\}_{j \neq i}\right) d\theta_q^{(i)}}{\theta_f^{(i)} - \theta_i^{(i)}}$$

the sum of vertices for a pair of external points, denoted  $i$  becomes:

$$\left( \frac{\int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \sum_l \Xi_1^{(l)} (Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}) d\theta^{(i)}}{\theta_f^{(i)} - \theta_i^{(i)}} \right)^m \int_{\theta_i^{(i)} < \theta_1^{(i)} < \dots < \theta_m^{(i)} < \theta_f^{(i)}} \delta(\theta_1^{(i)} - \theta_i^{(i)}) \delta(\theta_m^{(i)} - \theta_f^{(i)}) \prod_{q=1}^m d\theta_q^{(i)}$$

The contribution of  $i$  to the graphs is obtained by convoluting this quantity with the free propagator on the left and on the right and adding the free propagator. The convolution by the propagator on the right and on the left is:

$$\begin{aligned} & \int \mathcal{G}_0(\theta_f^{(i)}, \theta_{f'}^{(i)}, Z_i) \left( \frac{\int_{\theta_{i'}^{(i)}}^{\theta_f^{(i)}} \sum_l \Xi_1^{(l)} (Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}) d\theta^{(i)}}{\theta_f^{(i)} - \theta_{i'}^{(i)}} \right)^m \\ & \times \int_{\theta_{i'}^{(i)} < \theta_1^{(i)} < \dots < \theta_m^{(i)} < \theta_{f'}^{(i)}} \delta(\theta_1^{(i)} - \theta_{i'}^{(i)}) \delta(\theta_m^{(i)} - \theta_{f'}^{(i)}) \prod_{q=1}^m d\theta_q^{(i)} \mathcal{G}_0(\theta_{i'}^{(i)}, \theta_i^{(i)}, Z_i) d\theta_{i'}^{(i)} d\theta_{f'}^{(i)} \\ & = \int \mathcal{G}_0(\theta_f^{(i)}, \theta_m^{(i)}, Z_i) \int_{\theta_i^{(i)} < \theta_1^{(i)} < \dots < \theta_m^{(i)} < \theta_f^{(i)}} \left( \frac{\int_{\theta_1^{(i)}}^{\theta_m^{(i)}} \sum_l \Xi_1^{(l)} (Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}) d\theta^{(i)}}{\theta_m^{(i)} - \theta_1^{(i)}} \right)^m \prod_{q=1}^m d\theta_q^{(i)} \mathcal{G}_0(\theta_1^{(i)}, \theta_i^{(i)}, Z_i) \end{aligned}$$

Replacing  $\left( \frac{\int_{\theta_1^{(i)}}^{\theta_m^{(i)}} \sum_l \Xi_1^{(l)} (Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}) d\theta^{(i)}}{\theta_m^{(i)} - \theta_1^{(i)}} \right)$  by its average over  $[\theta_i^{(i)}, \theta_f^{(i)}]$  it becomes:

$$\begin{aligned} & \int \left( \frac{\int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \sum_l \Xi_1^{(l)} (Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}) d\theta^{(i)}}{\theta_f^{(i)} - \theta_i^{(i)}} \right)^m \int_{\theta_1^{(i)} < \theta_1^{(i)} < \dots < \theta_m^{(i)} < \theta_m^{(i)}} \prod_{q=1}^m d\theta_q^{(i)} \frac{\exp(-\Lambda_1(\theta_f^{(i)} - \theta_i^{(i)}))}{\Lambda^2} \\ & = \frac{1}{m!} \left( \int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \sum_l \Xi_1^{(l)} (Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}) d\theta^{(i)} \right)^m \frac{\exp(-\Lambda_1(\theta_f^{(i)} - \theta_i^{(i)}))}{\Lambda^2} \end{aligned}$$

where we write:

$$\begin{aligned} \int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \sum_{l=2}^n \Xi_1^{(l)} (Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}) & = \Xi_{1,n} (Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \\ & = \sum_{l=2}^n \sum_{\{k_1, \dots, k_l\} \subset \{1, \dots, n\}, k_j \neq i} \Xi_1^{(l)} (Z_i, \{Z_{k_j}\}, \theta_i^{(i)}, \theta_f^{(i)}) \end{aligned} \quad (151)$$

Then summing over  $m$ , adding the free propagator and performing the product over  $i = 1, \dots, n$  yields the sum of graphs for the  $n$  paths between  $n$  initial points  $\theta_i^{(j)}, Z^{(j)}$  and  $n$  final points  $\theta_f^{(j)}, Z^{(j)}$ :

$$\begin{aligned} & \prod_i \left( \mathcal{G}_0(\theta_f^{(i)}, \theta_i^{(i)}, Z_i) + \left( \frac{\nabla_{\theta_i^{(i)}}^{\text{out}}}{G_0'} \sum_{m>0} \frac{1}{m!} \left( \Xi_{1,n} (Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \right)^m \right) \right) \\ & = \prod_i \left( 1 + \left( \frac{\nabla_{\theta_i^{(i)}}^{\text{out}}}{\Lambda_1} \left( \exp(\Xi_{1,n} (Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})) - 1 \right) \right) \frac{1}{\Lambda} \right) \\ & \quad \times \frac{\exp(-\Lambda_1(\sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)}))}{\Lambda^n} \end{aligned}$$

where  $\nabla_{\theta_f^{(i)}}^{out}$  are understood to act outside the terms in the expression and with:

$$\begin{aligned}\Lambda &= \sqrt{\frac{\pi}{2}} \sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}} \\ \Lambda_1 &= \sqrt{\left(\frac{1}{\sigma^2 \bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}} - \frac{1}{\sigma^2 \bar{X}_r}\end{aligned}$$

and the full sum of graphs without inertia coefficients and an arbitrary number of external points is:

$$\begin{aligned}& \sum_n \prod_i \left( \mathcal{G}_0 \left( \theta_f^{(i)}, \theta_i^{(i)}, Z_i \right) + \left( \frac{\nabla_{\theta_i^{(i)}}^{out}}{G'_0} \sum_{m>0} \frac{1}{m!} \left( \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)^m \right) \right) \quad (152) \\ &= \prod_i \left( 1 + \left( \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1} \left( \exp \left( \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1 \right) \right) \frac{1}{\Lambda} \right) \\ & \quad \times \frac{\exp \left( -\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right) \right)}{\Lambda^n}\end{aligned}$$

For later purposes, we can rewrite (152) in two alternative manners. First, we can keep a propagator in factor. In that case, the sum of graphs becomes, for  $m \geq 2$ :

$$\begin{aligned}& \int \mathcal{G}_0 \left( \theta_f^{(i)}, \theta_{f'}^{(i)}, Z_i \right) \left( \frac{\int_{\theta_{i'}^{(i)}}^{\theta_{f'}^{(i)}} \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i} \right) d\theta^{(i)}}{\theta_{f'}^{(i)} - \theta_{i'}^{(i)}} \right)^m \\ & \quad \times \int_{\theta_{i'}^{(i)} < \theta_1^{(i)} < \dots < \theta_m^{(i)} < \theta_{f'}^{(i)}} \delta \left( \theta_1^{(i)} - \theta_{i'}^{(i)} \right) \delta \left( \theta_m^{(i)} - \theta_{f'}^{(i)} \right) \prod_{q=1}^m d\theta_q^{(i)} d\theta_{f'}^{(i)} \mathcal{G}_0 \left( \theta_{i'}^{(i)}, \theta_i^{(i)}, Z_i \right) d\theta_{i'}^{(i)} \\ &= \int \mathcal{G}_0 \left( \theta_f^{(i)}, \theta_m^{(i)}, Z_i \right) \int_{\theta_{i'}^{(i)} < \theta_2^{(i)} < \dots < \theta_m^{(i)} < \theta_f^{(i)}} \left( \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \left( \frac{\int_{\theta_{i'}^{(i)}}^{\theta_m^{(i)}} \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i} \right) d\theta^{(i)}}{\theta_m^{(i)} - \theta_{i'}^{(i)}} \right)^{m-1} \\ & \quad \times \prod_{q=2}^m d\theta_q^{(i)} \mathcal{G}_0 \left( \theta_{i'}^{(i)}, \theta_i^{(i)}, Z_i \right) d\theta_{i'}^{(i)} \\ & \simeq \int \left( \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \left( \frac{\int_{\theta_{i'}^{(i)}}^{\theta_f^{(i)}} \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i} \right) d\theta^{(i)}}{\theta_f^{(i)} - \theta_{i'}^{(i)}} \right)^{m-1} \\ & \quad \times \int_{\theta_{i'}^{(i)} < \theta_2^{(i)} < \dots < \theta_m^{(i)} < \theta_f^{(i)}} \prod_{q=2}^m d\theta_q^{(i)} \frac{\exp \left( -\Lambda_1 \left( \theta_f^{(i)} - \theta_{i'}^{(i)} \right) \right)}{\Lambda} \mathcal{G}_0 \left( \theta_{i'}^{(i)}, \theta_i^{(i)}, Z_i \right) d\theta_{i'}^{(i)}\end{aligned}$$

and for  $m = 1$ :

$$\int \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \frac{\exp \left( -\Lambda_1 \left( \theta_f^{(i)} - \theta_{i'}^{(i)} \right) \right)}{\Lambda} \mathcal{G}_0 \left( \theta_{i'}^{(i)}, \theta_i^{(i)}, Z_i \right) d\theta_{i'}^{(i)}$$

Adding the free propagator yields for the sum of graphs:

$$\sum_n \prod_i \int \left( \delta \left( \theta_f^{(i)} - \theta_{i'}^{(i)} \right) + \frac{\nabla_{\theta_{i'}^{(i)}}^{out}}{G'_0} \left( \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \exp \left( \frac{\int_{\theta_{i'}^{(i)}}^{\theta_f^{(i)}} \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i} \right) d\theta^{(i)}}{\theta_f^{(i)} - \theta_{i'}^{(i)}} \right) \right) \exp \left( -\Lambda_1 \left( \theta_f^{(i)} - \theta_{i'}^{(i)} \right) \right) \frac{1}{\Lambda} \mathcal{G}_0 \left( \theta_{i'}^{(i)}, \theta_i^{(i)}, Z_i \right) d\theta_{i'}^{(i)} \quad (153)$$

Second, we can also factor two propagator, on the left and on the right, and this yields the following form for the sum:

$$\sum_n \prod_i \int \mathcal{G}_0 \left( \theta_f^{(i)}, \theta_{f'}^{(i)}, Z_i \right) \left( \mathcal{G}^{-1} \left( \theta_{f'}^{(i)}, \theta_{i'}^{(i)} \right) + \frac{\nabla_{\theta_{i'}^{(i)}}^{out}}{G'_0} \left( \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \right) \delta \left( \theta_{f'}^{(i)} - \theta_{i'}^{(i)} \right) \left( \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \left( \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta_f^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \exp \left( \frac{\int_{\theta_{i'}^{(i)}}^{\theta_f^{(i)}} \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i} \right) d\theta^{(i)}}{\theta_f^{(i)} - \theta_{i'}^{(i)}} \right) \mathcal{G}_0 \left( \theta_{i'}^{(i)}, \theta_i^{(i)}, Z_i \right) d\theta_{i'}^{(i)} d\theta_f^{(i)} \quad (154)$$

## 2.2 Inclusion of inertia coefficients $\zeta_l$

### 2.2.1 Series expansion of classical action

As presented in the text, the effective action may be modified by including both inertia in frequency change, through a potential for maintaining and activating of new connections:

$$\begin{aligned} & \int \Psi^\dagger(\theta, Z) \left( \nabla_\theta \left( \omega^{-1} \left( J(\theta), \theta, Z, \mathcal{G}_0 \right) \right) \right)^2 \Psi(\theta, Z) - \frac{\zeta_1}{2} \int \left( |\Psi(\theta, Z)|^2 \left| \Psi \left( \theta - \frac{|Z - Z'|}{c}, Z' \right) \right|^2 \right) \\ & + \sum_{n=1}^{\infty} \frac{\zeta_n}{2n!} \int |\Psi(\theta, Z)|^2 \left( \prod_{i=1}^{n-1} \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \right) \quad (155) \\ & = -\frac{1}{2} \Psi^\dagger(\theta, Z) \left( \nabla_\theta \frac{\sigma^2}{2} \nabla_\theta - \omega^{-1} \left( J, \theta, Z, |\Psi|^2 \right) \right) \Psi(\theta, Z) + \sum_{n=2}^{\infty} \frac{1}{2n!} \zeta^{(n)} \int |\Psi(\theta, Z)|^2 \left( \prod_{i=1}^{n-1} \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \right) \end{aligned}$$

with:

$$\begin{aligned} \zeta^{(l)} &= \zeta_l, \quad l > 2 \\ \zeta^{(2)} &= \zeta_2 - \zeta_1 \\ \zeta^{(1)} &= 0 \end{aligned}$$

The second term represents the limitation in increasing the number of connections. This amounts to shift the vertices by  $+\zeta_2$ . The factor  $-\zeta_1$  accounts for a minimal number of connections maintained. It depends on external activity  $J$ .

The first terms modify the 4-th vertices by  $-\zeta_1$ . We write  $\bar{\Xi}_1^{(l)}(Z_i)$  for  $\bar{\Xi}_1^{(l)}(Z_i, \theta_q^{(i)}, \{Z_j\}_{j \neq i})$ . The 2 points propagator is modified by replacing  $\alpha$  with:

$$\alpha + \sum_{k \geq 2} \frac{1}{k!} C_k^1 \frac{\zeta^{(k)}}{\Lambda^{k-1}} = \sum_{k \geq l} \frac{1}{(k-1)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}}$$

which modifies the values of  $\Lambda$  and  $\Lambda_1$ . To obtain the contribution of the potential to the the vertices, we proceed as for the frequency. The vertices involved in the  $2n$  points correlation function are given by an expansion of:

$$\sum_{k=2}^{\infty} \frac{1}{2k!} \zeta^{(k)} \int |\Psi(\theta, Z)|^2 \left( \prod_{i=1}^{k-1} \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \right)$$

of order  $2n$ . The  $2n$  vertex is then

$$\left[ \frac{\delta^n}{\prod_{j=1}^n \delta |\Psi(\theta_j, Z_j)|^2} \left( \sum_{k=n}^{\infty} \frac{1}{2k!} \zeta^{(k)} \int |\Psi(\theta, Z)|^2 \left( \prod_{i=1}^{k-1} \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \right) \right) \right]_{|\Psi(\theta_j, Z_j)|^2 = \mathcal{G}_0(0, Z_j)}$$

Similarly to the derivation of (144), we decompose the vertex in two parts. One comes from direct interactions, while the other one corresponds to the backreaction of the  $n$  points on the system.

$$\begin{aligned} & \left[ \sum_{k=n}^{\infty} \frac{k}{2k!} \zeta^{(k)} \int |\Psi(\theta, Z)|^2 \frac{\delta^{n-1}}{\prod_{j=1}^{n-1} \delta |\Psi(\theta_j, Z_j)|^2} \left( \prod_{i=1}^{k-1} \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \right) \right]_{\substack{|\Psi(\theta_j, Z_j)|^2 \\ = \mathcal{G}_0(0, Z_j)}} \prod_{j=1}^{n-1} |\Psi(\theta_j, Z_j)|^2 \\ & + \left[ \sum_{k=n}^{\infty} \frac{1}{2k!} \zeta^{(k)} \int \mathcal{G}_0(\theta, \theta, Z) \frac{\delta^n}{\prod_{j=1}^n \delta |\Psi(\theta_j, Z_j)|^2} \left( \prod_{i=1}^{k-1} \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \right) \right]_{\substack{|\Psi(\theta_j, Z_j)|^2 \\ = \mathcal{G}_0(0, Z_j)}} \prod_{j=1}^n |\Psi(\theta_j, Z_j)|^2 \\ & = \sum_{k=n}^{\infty} \frac{k}{2k!} C_{k-1}^{n-1} \frac{\zeta^{(k)}}{\Lambda^{k-n}} \int |\Psi(\theta, Z)|^2 \prod_{i=1}^{n-1} \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \\ & + \sum_{k=n}^{\infty} \frac{1}{2k!} \zeta^{(k)} \int \mathcal{G}_0(\theta, \theta, Z) C_{k-1}^n \prod_{i=1}^n \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \\ & = \frac{1}{2(n-1)!} \sum_{k=n}^{\infty} \frac{1}{(k-n)!} \frac{\zeta^{(k)}}{\Lambda^{k-n}} \int |\Psi(\theta, Z)|^2 \prod_{i=1}^{n-1} \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \\ & + \frac{1}{2n!} \sum_{k=n}^{\infty} \int \frac{\mathcal{G}_0(\theta, \theta, Z)}{k(k-n-1)!} \frac{\zeta^{(k)}}{\Lambda^{k-n}} \prod_{i=1}^n \left| \Psi \left( \theta - \frac{|Z - Z_i|}{c}, Z_i \right) \right|^2 \end{aligned} \quad (156)$$

The inclusion of the first term of (156) in the graphs expansion is straightforward. In terms of vertices, and given the overall  $-1$  factor for each vertices, the introduction of the  $\zeta^{(l)}$  amounts to replace:

$$\Xi_1^{(l)}(Z_i) \rightarrow \bar{\Xi}_1^{(l)}(Z_i) - \frac{\zeta^{(l)}}{\Lambda^{l-1}}$$

With respect to the computation of graphs with  $n$  vertices, the terms  $\zeta^{(l)}$ ,  $l \leq n$  have to be replaced with:

$$\zeta_e^{(l)} = \sum_{k \geq l} \frac{l!}{k!} C_k^l \frac{\zeta^{(k)}}{\Lambda^{k-l}} = \sum_{k \geq l} \frac{1}{(k-l)!} \frac{\zeta^{(k)}}{\Lambda^{k-l}}$$

For  $\zeta^{(k)}$  slowly varying and  $\Lambda \gg 1$ , this is approximatively equal to  $\zeta^{(l)}$ . We keep the notation  $\zeta_e^{(l)} \rightarrow \zeta^{(l)}$ .

The second term of (156) corresponds to the backreaction term that can be neglected at first. It will be reintroduced later.

We note that if we write the potential as:

$$\int |\Psi(\theta, Z)|^2 V \left( \int |\Psi|^2 \right)$$

the second term of (156) writes:

$$\int \mathcal{G}_0(\theta, \theta, Z) \frac{\delta^n}{\prod_{j=1}^n \left| \Psi \left( \theta - \frac{|Z-Z_i|}{c}, Z_i \right) \right|^2} V \left( \int |\Psi|^2 \right) \quad (157)$$

### 2.2.2 Computation of full sum of graphs

The computation of the sum of graphs without external legs is the same as before, the vertices are  $\frac{-\zeta^{(l)}}{\Lambda^{l-1}} + \bar{\Xi}_1^{(l)}(Z_i)$ , except that the last vertex among the  $m = \sum_{l=1}^n k_l^{(i)}$  is replaced by  $\frac{-\zeta^{(l)}}{\Lambda^{l-1}} + \bar{\Xi}_1^{(l)}(Z_i) \frac{\nabla_{\theta_f}^{out}}{\Lambda_1}$ . For a particular  $l$ , this induces an additional factor  $k_l^{(i)}$ . The  $\sum_{l=1}^n k_l^{(i)} - 1$  remaining vertices are time ordered as previously, and we find the contributions for the graphs with  $2n$  external legs:

$$\prod_i \left( 1 + \left( \sum_{m>0} \frac{1}{m!} \int_{\theta_i^{(i)} < \theta_1^{(i)} < \dots < \theta_m^{(i)} < \theta_f^{(i)}} \prod_{q=1}^{m-1} \left( \sum_l \left( \frac{-\zeta^{(l)}}{\Lambda^{l-1}} + \bar{\Xi}_1^{(l)}(Z_i, \theta_q^{(i)}, \{Z_j\}_{j \neq i}) \right) \right) d\theta_q^{(i)} \right) \right) \quad (158)$$

$$\times \left( \sum_l \left( -\frac{\zeta^{(l)}}{\Lambda^{l-1}} + \frac{\theta_i^{(i)}}{\Lambda_1} \bar{\Xi}_1^{(l)}(Z_i, \theta_m, \{Z_j\}_{j \neq i}) \right) \right) d\theta_m$$

$$\times \Lambda^n \exp \left( -\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right) \right)$$

We define:

$$\begin{aligned} \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) &= \bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) - \bar{\zeta}_n(\theta_f^{(i)} - \theta_i^{(i)}) \\ &= \sum_{l=2}^n \sum_{\{k_1, \dots, k_{l-1}\} \subset \{1, \dots, n-1\}, k_j \neq i} \left( \bar{\Xi}_1^{(l)}(Z_i, \{Z_{k_j}\}, \theta_i^{(i)}, \theta_f^{(i)}) - \frac{\zeta^{(l)}(\theta_f^{(i)} - \theta_i^{(i)})}{\Lambda^l} \right) \end{aligned}$$

and:

$$\bar{\zeta}_n = \sum_{l=2}^n \sum_{\{k_1, \dots, k_{l-1}\} \subset \{1, \dots, n-1\}, k_j \neq i} \frac{\zeta^{(l)}}{\Lambda^l} = \sum_{l=1}^n C_{n-1}^{l-1} \frac{\zeta^{(l)}}{\Lambda^l}$$

For example:

$$\bar{\zeta}_2 = \frac{\zeta^{(2)}}{\Lambda}, \quad \bar{\zeta}_3 = \frac{\zeta^{(3)}}{\Lambda^2} + 3 \frac{\zeta^{(2)}}{\Lambda}$$

If we express  $\bar{\zeta}_n$  as a function of the initial set of variables  $\zeta^{(l)}$ , we have:

$$\bar{\zeta}_n = \sum_{l=1}^n C_n^l \frac{\sum_{k \geq l} \frac{1}{(k-l)!} \frac{\zeta^{(k)}}{\Lambda^{k-l}}}{\Lambda^{l-1}} = \sum_{l=1}^n C_n^l \sum_{k \geq l} \frac{1}{(k-l)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}}$$

so that:

$$\bar{\zeta}_2 = \sum_{k \geq 2} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} = \frac{\zeta^{(2)}}{\Lambda} + \sum_{k \geq 3} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}}$$

and:

$$\begin{aligned}
\bar{\zeta}_3 &= \sum_{k \geq 3} \frac{1}{(k-3)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} + \sum_{k \geq 2} \frac{3}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} \\
&= 3 \frac{\zeta^{(2)}}{\Lambda} + 3 \sum_{k \geq 3} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} + \sum_{k \geq 3} \frac{1}{(k-3)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} \\
&= 3 \frac{\zeta^{(2)}}{\Lambda} + 3 \sum_{k \geq 3} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} + \sum_{k \geq 3} \frac{Sup(1, (k-3)) \zeta^{(k)}}{(k-2)! \Lambda^{k-1}}
\end{aligned}$$

We will assume that  $\bar{\zeta}_2 < 0$  and  $\bar{\zeta}_n > 0$  for  $n > 2$ . This is possible under the conditions:

$$\sum_{k \geq 3} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} < \frac{|\zeta^{(2)}|}{\Lambda} < \sum_{k \geq 3} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} + \sum_{k \geq 3} \frac{Sup(1, (k-2)) \zeta^{(k)}}{3(k-2)! \Lambda^{k-1}}$$

that are satisfied for a certain range of the parameters, since:

$$\sum_{k \geq 3} \frac{Sup(1, (k-2)) \zeta^{(k)}}{3(k-2)! \Lambda^{k-1}} - \sum_{k \geq 3} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} = \sum_{k \geq 3} \frac{(Sup(1, (k-2)) - 3) \zeta^{(k)}}{3(k-2)! \Lambda^{k-1}}$$

is positive for  $\zeta^{(k)}$  large enough for  $k > 5$ . Replacing the terms  $\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})$  by their average over the timespan  $\theta_f^{(j)} - \theta_i^{(j)}$ :

$$\begin{aligned}
\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}) &\equiv \frac{\langle \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \rangle_{\theta_f^{(i)} - \theta_i^{(i)}}}{\theta_f^{(i)} - \theta_i^{(i)}} \\
&= \frac{\int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \left( \sum_l \frac{\zeta^{(l)}}{\Lambda^{l-1}} + \bar{\Xi}_1^{(l)}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}) \right) d\theta^{(i)}}{\theta_f^{(i)} - \theta_i^{(i)}}
\end{aligned}$$

expression (158) writes:

$$\begin{aligned}
&\prod_i \left( 1 + \sum_{m > 0} \frac{1}{m!} \left( \left( \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \right)^{m-1} \left( -\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1} \bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \right) \right) \right) \\
&\times \frac{\exp\left(-\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right)\right)}{\Lambda^n} \\
&= \prod_i \left( 1 + \sum_{m > 0} \frac{1}{m!} \left( \left( \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \right)^m \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1} \bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \right) \quad (159) \\
&\times \frac{\exp\left(-\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right)\right)}{\Lambda^n}
\end{aligned}$$

so that the sum of graphs for an arbitrary number of external points becomes:

$$\begin{aligned}
&\sum_n \prod_{i=1}^n \left( 1 + \frac{1}{\Lambda} \left( \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1} \bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}} \left( \exp\left(\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})\right) - 1 \right) \right) \right) \\
&\times \frac{\exp\left(-\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right)\right)}{\Lambda^n} \quad (160)
\end{aligned}$$



Given that  $\frac{1}{\Lambda} \simeq \mathcal{G}_0(0, Z)$ , the hypothesis of the text, we may assume that  $\bar{\zeta}_n > 0$ .

As before, we can find alternate descriptions for the sum of graphs by factorization with one propagator, or two, one on each side. We find:

$$\sum_n \prod_{i=1}^n \left( \int \delta(\theta_f^{(i)} - \theta_{i'}^{(i)}) + \left( -\bar{\zeta}_n + \frac{\nabla_{\theta_{i'}^{(i)}}^{out}}{\Lambda_1} \bar{\Xi}_{1,n}(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}) \right) \exp\left(\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)}, \theta_f^{(i)})\right) \frac{\exp\left(-\Lambda_1(\theta_f^{(i)} - \theta_{i'}^{(i)})\right)}{\Lambda} \right) \mathcal{G}_0(\theta_{i'}^{(i)}, \theta_i^{(i)}, Z_i)$$

and:

$$\sum_n \prod_{i=1}^n \left\{ \int \mathcal{G}_0(\theta_f^{(i)}, \theta_{f'}^{(i)}, Z_i) \left( \mathcal{G}^{-1}(\theta_{f'}^{(i)}, \theta_{i'}^{(i)}, Z_i) + \left( -\bar{\zeta}_n + \frac{\nabla_{\theta_{i'}^{(i)}}^{out}}{\Lambda_1} \bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)}) \right) \delta(\theta_{f'}^{(i)} - \theta_{i'}^{(i)}) \right. \right. \quad (162)$$

$$\left. \left. + \left( -\bar{\zeta}_n + \frac{\nabla_{\theta_{i'}^{(i)}}^{out}}{\Lambda_1} \bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)}) \right) \left( \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_{f'}^{(i)}) \right) \exp\left(\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)}, \theta_{f'}^{(i)})\right) \right) \right\}$$

$$\times \mathcal{G}_0(\theta_{i'}^{(i)}, \theta_i^{(i)}, Z_i) d\theta_{i'}^{(i)} d\theta_{f'}^{(i)}$$

This formula will be used to compute the correlation functions and the effective action respectively.

### 2.3 Generating functional for correlation functions

The generating functional is obtained by including a source term  $\int (\Omega^\dagger(\theta^{(i)}) \Psi(\theta^{(i)}) + \Psi^\dagger(\theta^{(i)}) \Omega(\theta^{(i)}))$  in the action (155)

$$\int \Psi^\dagger(\theta, Z) (\nabla_\theta(\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z))))^2 \Psi(\theta, Z) \quad (163)$$

$$- \frac{\zeta_1}{2} \int \left( |\Psi(\theta, Z)|^2 \left| \Psi\left(\theta - \frac{|Z - Z'|}{c}, Z'\right) \right|^2 \right)$$

$$+ \sum_{n=1}^{\infty} \frac{\zeta_n}{(2n)!} \int |\Psi(\theta, Z)|^2 \left( \prod_{i=1}^{n-1} \left| \Psi\left(\theta - \frac{|Z - Z_i|}{c}, Z_i\right) \right|^2 \right) + \int (\Omega^\dagger(\theta^{(i)}) \Psi(\theta^{(i)}) + \Psi^\dagger(\theta^{(i)}) \Omega(\theta^{(i)}))$$

The partition function can thus be expanded as a series expansion:

$$Z(\Omega) = 1 + \sum_{n \geq 1} \frac{1}{n!} \prod_{i=1}^n \int \Omega^\dagger(\theta_f^{(i)}) \mathcal{G}_n(\theta_f^{(1)}, \theta_i^{(1)}, \dots, \theta_f^{(n)}, \theta_i^{(n)}) \Omega(\theta_i^{(i)})$$

The graphs  $\mathcal{G}_n(\theta_f^{(1)}, \theta_i^{(1)}, \dots, \theta_f^{(n)}, \theta_i^{(n)})$  are not the graphs computed in the previous paragraph. They are rather computed around  $\Omega(\theta_i^{(i)}) = 0$ . This value of the source term corresponds, after legendre transform, to the minimum of the effective action, i.e. a possible non nul expectation for the field  $\Psi_0(\theta^{(i)}) = \langle \Psi(\theta^{(i)}) \rangle$ . The graphs computed in the previous paragraph are on the contrary computed for a null expectation, i.e.

$\langle \Psi(\theta^{(i)}) \rangle = 0$ . This value of the field corresponds to a source term  $\Omega_0(\theta^{(i)})$ . Decomposing (163) as:

$$\begin{aligned} & \int \Psi^\dagger(\theta, Z) (\nabla_\theta (\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z))))^2 \Psi(\theta, Z) - \frac{\zeta_1}{2} \int \left( |\Psi(\theta, Z)|^2 \left| \Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right) \right|^2 \right) \\ & + \sum_{n=1}^{\infty} \frac{\zeta_n}{(2n)!} \int |\Psi(\theta, Z)|^2 \left( \prod_{i=1}^{n-1} \left| \Psi\left(\theta - \frac{|Z-Z_i|}{c}, Z_i\right) \right|^2 \right) + \int (\Omega_0^\dagger(\theta^{(i)}) (\theta^{(i)}) \Psi(\theta^{(i)}) + \Psi^\dagger(\theta^{(i)}) \Omega_0(\theta^{(i)})) \\ & + \int \left( (\Omega(\theta^{(i)}) - \Omega_0(\theta^{(i)}))^\dagger (\theta^{(i)}) \Psi(\theta^{(i)}) + \Psi^\dagger(\theta^{(i)}) (\Omega(\theta^{(i)}) - \Omega_0(\theta^{(i)})) \right) \end{aligned} \quad (164)$$

Expanding in series of  $(\Omega(\theta^{(i)}) - \Omega_0(\theta^{(i)}))$  and  $(\Omega(\theta^{(i)}) - \Omega_0(\theta^{(i)}))^t$  yields a series of expectations  $\langle \prod \Psi(\theta^{(i)}) \Psi^\dagger(\theta^{(i)}) \rangle$  computed with  $\langle \Psi(\theta^{(i)}) \rangle$  which are precisely the graphs computed previously: This series  $Z(\Omega)$  is thus obtained from (160) by multiplication by source terms and summing over  $n$ . Actually, the expansion under the condition that  $\langle \Psi(\theta, Z) \rangle = \langle \Psi^\dagger(\theta, Z) \rangle = 0$ , the sum of graphs including a sub tadpole graph, i.e. the graphs that can be factored by  $\Omega_0(\theta^{(i)})$  or  $\Omega_0^\dagger(\theta^{(i)})$  at one end, cancel. Moreover, the sum of non connected graphs including factors  $\Omega_0^\dagger(\theta^{(i)})$  and  $\Omega_0(\theta^{(i)})$  at each end can be factored in the partition function, and can thus be discarded.

We define  $\Delta\Omega(\theta^{(i)}) = (\Omega(\theta^{(i)}) - \Omega_0(\theta^{(i)}))$ , so that:

$$\begin{aligned} Z(\Omega) &= 1 + \int \Delta\Omega^\dagger(\theta_f^{(i)}) \mathcal{G}_0(\theta_f^{(i)}, \theta_i^{(i)}, Z_i) \Delta\Omega(\theta_i^{(i)}) + \sum_{n \geq 2} \frac{1}{n!} \prod_{i=1}^n \int \Delta\Omega^\dagger(\theta_f^{(i)}) \\ & \times \left( 1 + \frac{1}{\Lambda} \left( \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta^{(i)}}^{out} \bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}} \left( \exp(\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})) - 1 \right) \right) \right) \Delta\Omega(\theta_i^{(i)}) \\ & \times \frac{\exp(-\Lambda_1 (\sum_{j=1}^n \theta_f^{(i)} - \sum_{j=1}^n \theta_i^{(i)}))}{\Lambda^n} \end{aligned}$$

we can define  $\hat{\Xi}_{1,1}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) = 0$  and thus:

$$\int \Delta\Omega^\dagger(\theta_f^{(i)}) \left( \frac{\exp(-\Lambda_1 (\theta_f^{(i)} - \theta_i^{(i)}))}{\Lambda} \right) \Delta\Omega(\theta_i^{(i)}) = \int \Delta\Omega^\dagger(\theta_f^{(i)}) \mathcal{G}_0(\theta_f^{(i)}, \theta_i^{(i)}, Z_i) \Delta\Omega(\theta_i^{(i)})$$

so that  $Z(\Omega)$  can be written:

$$\begin{aligned} Z(\Omega) &= \sum_{n \geq 0} \frac{1}{n!} \int \prod_{i=1}^n \Delta\Omega^\dagger(\theta_f^{(i)}) \mathcal{G}_0(\theta_f^{(i)}, \theta_i^{(i)}, Z_i) \\ & \times \left( 1 + \frac{1}{\Lambda} \left( \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta^{(i)}}^{out} \bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}} \left( \exp(\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})) - 1 \right) \right) \right) \Delta\Omega(\theta_i^{(i)}) \end{aligned}$$

that is:

$$Z(\Omega) = \sum_{n \geq 0} \frac{1}{n!} \left( \int \Delta \Omega^\dagger \left( \theta_f^{(i)} \right) \frac{\exp \left( -\Lambda_1 \left( \theta_f^{(i)} - \theta_i^{(i)} \right) \right)}{\Lambda} \right. \\ \left. \times \left( 1 + \left( \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out} \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda_1 \left( \theta_f^{(i)} - \theta_i^{(i)} \right)} \right)}{\Lambda \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}} \right)} \right) \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1 \right) \Delta \Omega \left( \theta_i^{(i)} \right) \right)^n$$

## 2.4 Generating functional for connected correlations

$$\begin{aligned} & \bar{\Xi}_1^{(l)} \left( Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i} \right) \\ & \simeq \frac{G'_0}{\left( \sqrt{\frac{\pi}{2}} \sqrt{\left( \frac{1}{\sigma^2 X_r} \right)^2 + \frac{2\alpha}{\sigma^2}} \right)^l} \sum_{\{k_1, \dots, k_{l-1}\} \subset \{1, \dots, n-1\}, k_j \neq i} \prod_{k_j \neq i} \int_{\theta_i^{(k_j)}}^{\theta^{(i)} - \frac{|Z_i - Z_{k_j}|}{c}} dl_{k_j} \bar{\Xi}_1 \left( J, \theta^{(i)}, l_{k_j}, |Z_i - Z_{k_j}| \right) \\ & = \sum_{\{k_1, \dots, k_{l-1}\} \subset \{1, \dots, n-1\}, k_j \neq i} \bar{\Xi}_1^{(l)} \left( Z_i, \theta^{(i)}, \{Z_{k_j}\} \right) \\ & \simeq \frac{\sum_{\{k_1, \dots, k_{l-1}\} \subset \{1, \dots, n-1\}} \prod_{k_j} \int_{\frac{|Z_i - Z_{k_j}|}{c}}^{\theta^{(i)} - \theta_i^{(k_j)}} dl_{k_j}}{\left( \sqrt{\frac{\pi}{2}} \sqrt{\left( \frac{1}{\sigma^2 X_r} \right)^2 + \frac{2\alpha}{\sigma^2}} \right)^{l-1}} \left[ \nabla_{\theta^{(i)}} \mathcal{G}_0(Z) \frac{\delta^{l-1} \int \omega^{-1} \left( J, \theta^{(i)}, Z_i \right) \Psi \left( \theta^{(i)}, Z_i \right) dZ_i}{\prod_{i=1}^{l-1} \delta \left| \Psi \left( \theta^{(i)} - l_{k_j}, Z_{k_j} \right) \right|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \end{aligned}$$

The generating functional for connected correlation functions is given by:

$$W(\Omega) = \ln \frac{Z(\Omega)}{Z(0)}$$

We assume that  $-\bar{\zeta}_n$  and  $\bar{\Xi}_{1,n}$  grow approximatively at the same rate, so that  $\frac{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}}}$

depends weakly on  $n$  and can be replaced by its limit for  $n \rightarrow \infty$  for  $n \geq 3$ . We also replace  $\hat{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)$  and  $\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)$  by their limit  $\hat{\Xi}_{1,\infty} \left( Z_i, \theta_i^{(j)}, \theta_f^{(j)} \right)$  and  $\bar{\Xi}_{1,\infty} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)$ . As a consequence, for  $n \geq 3$ :

$$\frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(j)}}^{out} \bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\Lambda_1 \left( \theta_f^{(j)} - \theta_i^{(j)} \right)}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}}} \simeq \frac{-\bar{\zeta} + \frac{\nabla_{\theta_i^{(j)}}^{out} \bar{\Xi}_{1,\infty} \left( Z_j, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\Lambda_1 \left( \theta_f^{(j)} - \theta_i^{(j)} \right)}}{-\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty} \left( Z_j, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}}}$$

Define:

$$\begin{aligned}
O_{1,n} \left( \theta_i^{(i)}, \theta_f^{(i)}, \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_{j \neq i} \right) &= \left( \frac{-\bar{\zeta}_n + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}} \right)} \left( \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1 \right) \right) \\
O_{1,\infty} \left( \theta_i^{(i)}, \theta_f^{(i)}, \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_{j \neq i} \right) &= \frac{-\bar{\zeta} + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty} \left( Z_j, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\Lambda \left( -\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty} \left( Z_j, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}} \right)} \left( \exp \left( \hat{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1 \right) \\
O_{1,1} \left( \theta_i^{(i)}, \theta_f^{(i)} \right) &= 1 \\
\mathcal{G} &= \frac{\exp \left( -\Lambda_1 \left( \theta_f^{(i)} - \theta_i^{(i)} \right) \right)}{\Lambda} H \left( \theta_f^{(i)} - \theta_i^{(i)} \right)
\end{aligned}$$

Remark that these operators depend on all variables  $\theta_i^{(j)}, \theta_f^{(j)}$  for  $i \neq j$ , mainly through Heaviside functions. We thus define:

$$\begin{aligned}
O_{1,n}^{(n)} \left( \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_i \right) &= \prod_{i=1}^n O_{1,n} \left( \theta_i^{(i)}, \theta_f^{(i)}, \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_{j \neq i} \right) \\
O_{1,\infty}^{(n)} \left( \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_i \right) &= \prod_{i=1}^n O_{1,\infty} \left( \theta_i^{(i)}, \theta_f^{(i)}, \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_{j \neq i} \right) \\
(1 + O_{1,n})^{(n)} \left( \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_i \right) &= \prod_{i=1}^n \left( 1 + O_{1,n} \left( \theta_i^{(i)}, \theta_f^{(i)}, \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_{j \neq i} \right) \right) \\
(1 + O_{1,\infty})^{(n)} \left( \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_i \right) &= \prod_{i=1}^n \left( 1 + O_{1,\infty} \left( \theta_i^{(i)}, \theta_f^{(i)}, \left( \theta_i^{(i)}, \theta_f^{(i)} \right)_{j \neq i} \right) \right)
\end{aligned}$$

and for any operator  $A$ :

$$\begin{aligned}
\langle A \rangle_n &= \int \left[ \prod_{i=1}^n \Delta \Omega^\dagger \left( \theta_f^{(i)} \right) \mathcal{G}_0 \left( \theta_f^{(i)}, \theta_i^{(i)}, Z_i \right) \right] A \left[ \prod_{i=1}^n \Delta \Omega \left( \theta_i^{(i)} \right) \right], \quad n > 0 \\
\langle A \rangle_0 &= 1
\end{aligned}$$

For  $A$  acting on  $\left( \Delta \Omega \left( \theta_i^{(i)} \right) \right)^{\otimes n}$ , the expectation  $\langle A \rangle_k$ ,  $k < n$  for  $A$  symmetric is evaluated on the  $k$  first variables and defines an operator acting on  $\left( \Delta \Omega \left( \theta_i^{(i)} \right) \right)^{\otimes n-k}$ .

With these assumptions:

$$\begin{aligned}
Z(\Omega) &= 1 + \int \Delta \Omega^\dagger \left( \theta_f^{(i)} \right) \mathcal{G}_0 \left( \theta_f^{(i)}, \theta_i^{(i)}, Z_i \right) \Delta \Omega \left( \theta_i^{(i)} \right) \\
&\quad + \int \prod_{i=1}^2 \Delta \Omega^\dagger \left( \theta_f^{(i)} \right) \mathcal{G}_0 \left( \theta_f^{(i)}, \theta_i^{(i)}, Z_i \right) (1 + O_{1,2})^{(2)} \Delta \Omega \left( \theta_i^{(i)} \right) \\
&\quad + \sum_{n \geq 2} \frac{1}{n!} \int \prod_{i=1}^n \Delta \Omega^\dagger \left( \theta_f^{(i)} \right) \mathcal{G}_0 \left( \theta_f^{(i)}, \theta_i^{(i)}, Z_i \right) (1 + O_{1,n})^{(n)} \Delta \Omega \left( \theta_i^{(i)} \right) \\
&= 1 + \sum_{n \geq 1} \frac{1}{n!} \left\langle (1 + O_{1,n})^{(n)} \right\rangle_n \\
&\simeq 1 + \langle 1 \rangle_1 + \left\langle (1 + O_{1,2})^{(2)} \right\rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \left\langle (1 + O_{1,\infty})^{(n)} \right\rangle_n
\end{aligned}$$

and we obtain the expression for the generating functional  $W(\Omega)$ :

$$W(\Omega) = \ln \left( 1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n \right)$$

## 2.5 Effective action

### 2.5.1 General form

The derivative of  $W(\Omega)$  with respect to  $\Omega^\dagger(\theta^{(i)})$  and  $\Omega(\theta^{(i)})$  defines the background field:

$$\frac{\delta W(\Omega)}{\delta \Omega^\dagger(\theta^{(i)})} = \int_{\theta_1^{(i)}}^{\theta^{(i)}} \frac{1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n} \mathcal{G}_0 \Delta \Omega(\theta^{(i)}) d\theta^{(i)} \quad (166)$$

and:

$$\frac{\delta W(\Omega)}{\delta \Omega(\theta_1^{(i)})} = \int_{\theta_1^{(i)}}^{\theta^{(i)}} \Delta \Omega^\dagger(\theta^{(i)}) \frac{1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n} \mathcal{G}_0 d\theta^{(i)} \quad (167)$$

and this yields the following equation for the classical field  $\Psi(\theta^{(i)})$ :

$$\int_{\theta_1^{(i)}}^{\theta^{(i)}} \frac{1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n} \mathcal{G}_0(\theta_1^{(i)}, \theta^{(i)}) \Delta \Omega(\theta^{(i)}) d\theta^{(i)} = \Psi(\theta_1^{(i)}) \quad (168)$$

along with:

$$\int_{\theta_1^{(i)}}^{\theta^{(i)}} \Delta \Omega^\dagger(\theta^{(i)}) \frac{1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n} \mathcal{G}_0(\theta^{(i)}, \theta_1^{(i)}) d\theta^{(i)} = \Psi^\dagger(\theta_1^{(i)})$$

To compute the effective action, we consider (168) evaluated at  $\Psi_0(\theta^{(i)})$ , so that  $\Omega_{\Psi_0(\theta^{(i)})}(\theta^{(i)}) = 0$ :

$$- \left[ \frac{1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n} \right]_{\Psi_0(\theta^{(i)})} \mathcal{G}_0(\Omega_{\Psi(\theta^{(i)})=0}(\theta^{(i)})) = \Psi_0(\theta^{(i)})$$

Then, we have:

$$\begin{aligned}
& \int \Omega^\dagger(\theta^{(i)}) \Psi(\theta^{(i)}) + \int \Omega(\theta^{(i)}) \Psi^\dagger(\theta^{(i)}) \\
= & \left( \Delta\Omega(\theta^{(i)}) \right)^\dagger \left( \frac{1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n} \right) \mathcal{G}_0(\Delta\Omega(\theta^{(i)})) + H.C. \\
& + \Psi^\dagger(\theta^{(i)}) \Omega_{\Psi(\theta^{(i)})=0}(\theta^{(i)}) + H.C. \\
= & 2 \frac{\langle 1 \rangle_1 + \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n} \\
& - \Psi^\dagger(\theta^{(i)}) \left[ \frac{1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n} \mathcal{G}_0 \right]_{\Psi_0(\theta^{(i)})}^{-1} \Psi_0(\theta^{(i)}) + H.C.
\end{aligned}$$

and the effective action writes:

$$\begin{aligned}
\Gamma(\Psi) = & \frac{\langle 1 \rangle_1 + \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n} \tag{169} \\
& - \frac{1}{2} \ln \left( 1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n \right) \\
& - \frac{1}{2} \Psi^\dagger(\theta^{(i)}) \left[ \frac{1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n} \mathcal{G}_0 \right]_{\Psi_0(\theta^{(i)})}^{-1} \Psi_0(\theta^{(i)}) + H.C.
\end{aligned}$$

### 2.5.2 Estimation of the expectations in (169)

The expectations appearing in (169) are functions of  $\Psi$  through (168). They can be estimated by using averages quantities. To so we will rewrite (168) in a compact form. Given our definitions, we have:

$$\begin{aligned}
& \int (\Delta\Omega(\theta^{(i)}))^\dagger \langle (1 + O_{1,2})^{(2)} \rangle_1 \mathcal{G}_0(\Delta\Omega\theta^{(i)}) = \langle (1 + O_{1,2})^{(2)} \rangle_2 \\
& \int (\Delta\Omega(\theta^{(i)}))^\dagger \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1} (\Delta\Omega\theta^{(i)}) = \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n
\end{aligned}$$

We also define the average  $\langle 1 + O_{1,\infty} \rangle$  by:

$$\exp(\langle 1 + O_{1,\infty} \rangle) = \sum_{n \geq 0} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n$$

so that, for a background field of relatively small amplitude:

$$\begin{aligned}
\sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n & \simeq \langle 1 + O_{1,\infty} \rangle \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n-1)} \rangle_{n-1} \\
& \simeq \langle 1 + O_{1,\infty} \rangle (\exp(\langle 1 + O_{1,\infty} \rangle) - 1 - \langle 1 + O_{1,\infty} \rangle)
\end{aligned}$$

and:

$$\sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n \simeq \exp(\langle 1 + O_{1,\infty} \rangle) - 1 - \langle 1 + O_{1,\infty} \rangle - \frac{1}{2} \langle 1 + O_{1,\infty} \rangle^2$$

In average, we also have

$$\left\langle (1 + O_{1,2})^{(2)} \right\rangle_1 \simeq (1 + O_{1,2}) \sqrt{\left\langle (1 + O_{1,2})^{(2)} \right\rangle_2}$$

and:

$$\sum_{n \geq 3} \frac{1}{(n-1)!} \left\langle (1 + O_{1,\infty})^{(n)} \right\rangle_{n-1} \simeq (1 + O_{1,\infty}) (\exp(\langle 1 + O_{1,\infty} \rangle) - 1 - \langle 1 + O_{1,\infty} \rangle)$$

where the operators  $(1 + O_{1,2})$  and  $(1 + O_{1,\infty})$  are understood as operators acting only the  $i$ -th variable, obtained by averaging over  $Z_j$  and  $\theta^{(j)}$  where  $j \neq i$ . We define:

$$\begin{aligned} x &= \langle 1 + O_{1,\infty} \rangle \\ y^2 &= \left\langle (1 + O_{1,2})^{(2)} \right\rangle_2 \\ z &= \langle 1 + O_{1,\infty} \rangle - \langle 1 \rangle_1 = \langle O_{1,\infty} \rangle \end{aligned}$$

and (168) becomes:

$$\frac{(1 + O_{1,\infty}) + \exp(-x) (-O_{1,\infty} + y(1 + O_{1,2}) - x(1 + O_{1,\infty}))}{1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))} \mathcal{G}_0(\theta_1^{(i)}, \theta^{(i)}) \Delta \Omega(\theta^{(i)}) = \Psi(\theta_1^{(i)}) \quad (170)$$

Equation (170) can be used to find defining equations for  $x, y, z$ . Actually, inverting (170):

$$\Delta \Omega(\theta^{(i)}) = \left( \left( \frac{(1 + O_{1,\infty}) + \exp(-x) (-O_{1,\infty} + y(1 + O_{1,2}) - x(1 + O_{1,\infty}))}{1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))} \right) \mathcal{G}_0 \right)^{-1} \Psi(\theta_1^{(i)}) \quad (171)$$

The inverse of the operator is found by using (161) to write:

$$(1 + O_{1,n}) \mathcal{G}_0 = (1 + \bar{O}_{1,n}) * \mathcal{G}_0$$

where  $*$  denotes the convolution product and:

$$\bar{O}_{1,n} = \left( -\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1} \bar{\Xi}_{1,n}(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}) \right) \exp\left(\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})\right) \frac{\exp(-\Lambda_1(\theta_f^{(i)} - \theta_{i'}^{(i)}))}{\Lambda}$$

As a consequence, we write:

$$\begin{aligned} & \left( \left( \frac{(1 + O_{1,\infty}) + \exp(-x) (-O_{1,\infty} + y(1 + O_{1,2}) - x(1 + O_{1,\infty}))}{1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))} \right) \mathcal{G}_0 \right)^{-1} \\ &= \mathcal{G}_0^{-1} * \left( \frac{1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))}{(1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))} \right) \end{aligned}$$

Computing the expectations of  $(1 + O_{1,\infty})$ ,  $(1 + O_{1,2})$  and  $O_{1,\infty}$  in state (171). yields:

$$\begin{aligned} x &= \left\langle \mathcal{G}_0^{-1} \left[ 1 + \exp(-x) \left( -z + \frac{1}{2}(y^2 - x^2) \right) \right] A^{-1} \Psi \middle| (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \middle| \mathcal{G}_0^{-1} \left[ 1 + \exp(-x) \left( -z + \frac{1}{2}(y^2 - x^2) \right) \right] A^{-1} \Psi \right\rangle \\ y &= \left\langle \mathcal{G}_0^{-1} \left[ 1 + \exp(-x) \left( -z + \frac{1}{2}(y^2 - x^2) \right) \right] A^{-1} \Psi \middle| (1 + \bar{O}_{1,2}) \mathcal{G}_0 \middle| \mathcal{G}_0^{-1} \left[ 1 + \exp(-x) \left( -z + \frac{1}{2}(y^2 - x^2) \right) \right] A^{-1} \Psi \right\rangle \\ z &= \left\langle \mathcal{G}_0^{-1} \left[ 1 + \exp(-x) \left( -z + \frac{1}{2}(y^2 - x^2) \right) \right] A^{-1} \Psi \middle| \bar{O}_{1,\infty} \mathcal{G}_0 \middle| \mathcal{G}_0^{-1} \left[ 1 + \exp(-x) \left( -z + \frac{1}{2}(y^2 - x^2) \right) \right] A^{-1} \Psi \right\rangle \quad (172) \end{aligned}$$

with:

$$A^{-1} \Psi = [(1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))]^{-1} \Psi$$

Expression (172) can be rewritten by using that the operators  $\bar{O}$  (standing for  $1+\bar{O}_{1,2}$ ,  $1+\bar{O}_{1,\infty}$ ...) decompose as  $\bar{O} = \bar{O}^{(C)} + \bar{O}^{(D)} \nabla$   $|\bar{O}_D| < \bar{O}_C$  and  $\bar{O}^{-1} \simeq \frac{(\bar{O}^{(C)} - \bar{O}^{(D)} \nabla)}{(\bar{O}^{(C)})^2} \equiv \frac{\bar{O}^\dagger}{(\bar{O}^{(C)})^2}$  and  $[A]^{-1}$  can be obtained by:

$$\begin{aligned} [A]^{-1} &= [(1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))]^{-1} \\ &= \frac{(1 + \bar{O}_{1,\infty}^\dagger) + \exp(-x) (-\bar{O}_{1,\infty}^\dagger + y(1 + \bar{O}_{1,2}^\dagger) - x(1 + \bar{O}_{1,\infty}^\dagger))}{(1 + \bar{O}_{1,\infty}^{(C)} + \exp(-x) (-\bar{O}_{1,\infty}^{(C)} + y(1 + \bar{O}_{1,2}^{(C)}) - x(1 + \bar{O}_{1,\infty}^{(C)}))} \end{aligned}$$

and the expressions for the expectations are:

$$\begin{aligned} x &= \left( \frac{1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))}{(1 + \bar{O}_{1,\infty}^{(C)} + \exp(-x) (-\bar{O}_{1,\infty}^{(C)} + y(1 + \bar{O}_{1,2}^{(C)}) - x(1 + \bar{O}_{1,\infty}^{(C)}))} \right)^2 \langle \Psi_0 | \mathcal{G}_0^{-1} B^\dagger (1 + \bar{O}_{1,\infty}) B | \Psi_0 \rangle \\ y &= \left( \frac{1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))}{(1 + \bar{O}_{1,\infty}^{(C)} + \exp(-x) (-\bar{O}_{1,\infty}^{(C)} + y(1 + \bar{O}_{1,2}^{(C)}) - x(1 + \bar{O}_{1,\infty}^{(C)}))} \right)^2 \langle \Psi_0 | \mathcal{G}_0^{-1} B^\dagger (1 + \bar{O}_{1,2}) B | \Psi_0 \rangle \\ z &= \left( \frac{1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))}{(1 + \bar{O}_{1,\infty}^{(C)} + \exp(-x) (-\bar{O}_{1,\infty}^{(C)} + y(1 + \bar{O}_{1,2}^{(C)}) - x(1 + \bar{O}_{1,\infty}^{(C)}))} \right)^2 \langle \Psi_0 | \mathcal{G}_0^{-1} B^\dagger \bar{O}_{1,\infty} B | \Psi_0 \rangle \end{aligned}$$

with:

$$\begin{aligned} B &= (1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})) \\ &= (1 + \bar{O}_{1,\infty}) (1 - x \exp(-x)) - \exp(-x) \bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) \exp(-x) \end{aligned}$$

Now, use that for slowly varying field, we can replace:

$$\begin{aligned} &\left( (1 + \bar{O}_{1,\infty}^\dagger) (1 - x \exp(-x)) + \exp(-x) (-\bar{O}_{1,\infty}^\dagger + y(1 + \bar{O}_{1,2}^\dagger)) \right) (1 + \bar{O}_{1,\infty}) \\ &\times \left( (1 - x \exp(-x)) + \exp(-x) (-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2})) \right) \\ &\simeq \left( 1 + \bar{O}_{1,\infty}^{(C)} + \exp(-x) (-\bar{O}_{1,\infty}^{(C)} + y(1 + \bar{O}_{1,2}^{(C)}) - x(1 + \bar{O}_{1,\infty}^{(C)})) \right)^2 \end{aligned}$$

since the omitted terms are of second order in derivatives. As a consequence:

$$\begin{aligned} x &\simeq \left( \frac{1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))}{1 + \bar{O}_{1,\infty}^{(C)} + \exp(-x) (-\bar{O}_{1,\infty}^{(C)} + y(1 + \bar{O}_{1,2}^{(C)}) - x(1 + \bar{O}_{1,\infty}^{(C)})} \right)^2 \langle \Psi_0 | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) | \Psi_0 \rangle \\ y &\simeq \left( \frac{1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))}{1 + \bar{O}_{1,\infty}^{(C)} + \exp(-x) (-\bar{O}_{1,\infty}^{(C)} + y(1 + \bar{O}_{1,2}^{(C)}) - x(1 + \bar{O}_{1,\infty}^{(C)})} \right)^2 \langle \Psi_0 | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,2}) | \Psi_0 \rangle \\ z &\simeq \left( \frac{1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))}{1 + \bar{O}_{1,\infty}^{(C)} + \exp(-x) (-\bar{O}_{1,\infty}^{(C)} + y(1 + \bar{O}_{1,2}^{(C)}) - x(1 + \bar{O}_{1,\infty}^{(C)})} \right)^2 \langle \Psi_0 | \mathcal{G}_0^{-1} \bar{O}_{1,\infty} | \Psi_0 \rangle \end{aligned}$$

These expressions allows a further simplification. We define:

$$X = \left( \frac{1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))}{1 + \bar{O}_{1,\infty}^{(C)} + \exp(-x) (-\bar{O}_{1,\infty}^{(C)} + y(1 + \bar{O}_{1,2}^{(C)}) - x(1 + \bar{O}_{1,\infty}^{(C)})} \right)^2$$



so that:

$$\begin{aligned}
x &= X \langle \Psi | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) | \Psi \rangle \equiv rX \\
y &= X \langle \Psi | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,2}) | \Psi \rangle \equiv sX \\
z &= X \langle \Psi | \mathcal{G}_0^{-1} \bar{O}_{1,\infty} | \Psi \rangle \equiv tX
\end{aligned} \tag{173}$$

The variable  $X$  satisfies the equation:

$$X = \left( \frac{1 + \exp(-rX) \left( -tX + \frac{1}{2} (s^2 - r^2) X^2 \right)}{1 + \bar{O}_{1,\infty}^{(C)} + \exp(-rX) \left( -\bar{O}_{1,\infty}^{(C)} + sX \left( 1 + \bar{O}_{1,2}^{(C)} \right) - rX \left( 1 + \bar{O}_{1,\infty}^{(C)} \right) \right)} \right)^2$$

whose solution is approximatively:

$$\begin{aligned}
X &\simeq \left( \frac{1 + \exp\left(-\frac{r}{(1+\bar{O}_{1,\infty}^{(C)})^2}\right) \left( -\frac{t}{(1+\bar{O}_{1,\infty}^{(C)})^2} + \frac{1}{2} \frac{(s^2-r^2)}{(1+\bar{O}_{1,\infty}^{(C)})^4} \right)}{1 + \bar{O}_{1,\infty}^{(C)} + \exp\left(-\frac{r}{(1+\bar{O}_{1,\infty}^{(C)})^2}\right) \left( -\bar{O}_{1,\infty}^{(C)} + s \frac{(1+\bar{O}_{1,2}^{(C)})}{(1+\bar{O}_{1,\infty}^{(C)})^2} - \frac{r}{(1+\bar{O}_{1,\infty}^{(C)})} \right)} \right)^2 \\
&= \left( \frac{\left( 1 + \bar{O}_{1,\infty}^{(C)} \right)^2 + \exp\left(-\frac{\langle \Psi | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) | \Psi \rangle}{(1+\bar{O}_{1,\infty}^{(C)})^2}\right) N}{\left( 1 + \bar{O}_{1,\infty}^{(C)} \right)^3 + \exp\left(-\frac{\langle \Psi | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) | \Psi \rangle}{(1+\bar{O}_{1,\infty}^{(C)})^2}\right) D} \right)^2
\end{aligned} \tag{174}$$

where:

$$\begin{aligned}
N &= \left( -\langle \Psi | \mathcal{G}_0^{-1} \bar{O}_{1,\infty} | \Psi \rangle + \frac{1}{2} \frac{\left( \langle \Psi | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,2}) | \Psi \rangle^2 - \langle \Psi | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) | \Psi \rangle^2 \right)}{(1 + \bar{O}_{1,\infty}^{(C)})^2} \right) \\
D &= \left( -\bar{O}_{1,\infty}^{(C)} \left( 1 + \bar{O}_{1,\infty}^{(C)} \right)^2 + \left( 1 + \bar{O}_{1,2}^{(C)} \right) \langle \Psi | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,2}) | \Psi \rangle - \left( 1 + \bar{O}_{1,\infty}^{(C)} \right) \langle \Psi | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) | \Psi \rangle \right)
\end{aligned}$$

In first approximation, this reduces to:

$$X \simeq \left( \frac{\left( 1 + \bar{O}_{1,\infty}^{(C)} \right)^2 + \exp\left(-\frac{\langle \Psi | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) | \Psi \rangle}{(1+\bar{O}_{1,\infty}^{(C)})^2}\right) \left( -\langle \Psi | \mathcal{G}_0^{-1} \bar{O}_{1,\infty} | \Psi \rangle + \frac{1}{2} \frac{\left( \langle \Psi | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,2}) | \Psi \rangle^2 - \langle \Psi | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) | \Psi \rangle^2 \right)}{(1+\bar{O}_{1,\infty}^{(C)})^2} \right)}{\left( 1 + \bar{O}_{1,\infty}^{(C)} \right)^3 + \exp\left(-\frac{\langle \Psi | \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) | \Psi \rangle}{(1+\bar{O}_{1,\infty}^{(C)})^2}\right) \bar{O}_{1,\infty}^{(C)} \left( 1 + \bar{O}_{1,\infty}^{(C)} \right)^2} \right)^2$$

### Appendix 3. Correlation functions

The correlation functions can be computed by successive derivatives of the effective action (169) estimated at the background field. The first derivative (169) yields  $\Delta\Omega\left(\theta^{(i)}\right)$  as usual, apart from a constant term:

$$\begin{aligned}
\frac{\partial\Gamma(\Psi)}{\partial\Psi^\dagger(\theta^{(i)}, Z_i)} &= \frac{\partial\Delta\Omega^\dagger(\theta^{(i)}, Z_i)}{\partial\Psi^\dagger(\theta^{(i)}, Z_i)} \frac{\partial\Gamma(\Psi)}{\partial\Omega^\dagger(\theta^{(i)}, Z_i)} \\
&= \frac{1}{2}\Delta\Omega(\theta^{(i)}) - \frac{1}{2} \left[ \frac{1 + \langle(1 + O_{1,2})^{(2)}\rangle_1 + \sum_{n\geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle(1 + O_{1,2})^{(2)}\rangle_2 + \sum_{n\geq 3} \frac{1}{n!} \langle(1 + O_{1,\infty})^{(n)}\rangle_n} \mathcal{G}_0 \right]_{\Psi_0(\theta^{(i)})}^{-1} \Psi_0(\theta^{(i)}, Z_i) \\
&= \frac{1}{2} \left[ \frac{(1 + O_{1,\infty}) + \exp(-x)(-O_{1,\infty} + y(1 + O_{1,2}) - x(1 + O_{1,\infty}))}{1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2))} \mathcal{G}_0 \right]^{-1} \Psi(\theta^{(i)}, Z_i) \\
&\quad - \frac{1}{2} \left[ \frac{(1 + O_{1,\infty}) + \exp(-x)(-O_{1,\infty} + y(1 + O_{1,2}) - x(1 + O_{1,\infty}))}{1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2))} \mathcal{G}_0 \right]_{\Psi_0(\theta^{(i)})}^{-1} \Psi_0(\theta^{(i)}, Z_i)
\end{aligned} \tag{175}$$

This formula does not allow to compute the vacuum of the system, since (175) is identically nul for the vacuum  $\Psi_0(\theta^{(i)}, Z_i)$ . We will derive  $\Psi_0(\theta^{(i)}, Z_i)$  below through a graph expansion, but differentiating (175) yields the 2 points effective vertex, and successive derivatives will compute the higher order correlation functions.

### 3.1 Two points correlation functions

#### 3.1.1 Equations for two points correlation functions

We will need  $\frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta_f^{(i)})}$  and  $\frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger(\theta_f^{(i)})}$ , the derivatives  $\frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi^\dagger(\theta_f^{(i)})}$  and  $\frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi(\theta_f^{(i)})}$  being obtained by hermitian conjugation. We have:

$$\begin{aligned}
\frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta_f^{(i)})} &= \left( \left( \frac{(1 + O_{1,\infty}) + \exp(-x)(-O_{1,\infty} + y(1 + O_{1,2}) - x(1 + O_{1,\infty}))}{1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2))} \right) \mathcal{G}_0 \right)^{-1} (\theta_1^{(i)}, \theta_f^{(i)}) \tag{176} \\
&\quad + \left( \frac{\delta\Delta\Omega(\theta^{(i)})}{\delta\Psi(\theta_f^{(i)})} \Psi^\dagger(\theta^{(i)}) + \frac{\delta\Delta\Omega^\dagger(\theta^{(i)})}{\delta\Psi(\theta_f^{(i)})} \Psi(\theta^{(i)}) \right) \Delta\Omega(\theta_1^{(i)}) \\
&\quad - \frac{\delta\Delta\Omega(\theta^{(i)})}{\delta\Psi(\theta_f^{(i)})} \frac{\frac{\delta}{\delta\Omega(\theta^{(i)})} \left( 1 + \langle(1 + O_{1,2})^{(2)}\rangle_1 + \sum_{n\geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_{n-1} \right)}{1 + \langle(1 + O_{1,2})^{(2)}\rangle_1 + \sum_{n\geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_{n-1}} \Delta\Omega(\theta_1^{(i)}) \\
&\quad - \frac{\delta\Delta\Omega^\dagger(\theta^{(i)})}{\delta\Psi(\theta_f^{(i)})} \frac{\frac{\delta}{\delta\Omega^\dagger(\theta^{(i)})} \left( 1 + \langle(1 + O_{1,2})^{(2)}\rangle_1 + \sum_{n\geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_{n-1} \right)}{1 + \langle(1 + O_{1,2})^{(2)}\rangle_1 + \sum_{n\geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_{n-1}} \Delta\Omega(\theta_1^{(i)})
\end{aligned}$$

where:

$$\frac{\frac{\delta}{\delta\Omega(\theta^{(i)})} \left( 1 + \langle(1 + O_{1,2})^{(2)}\rangle_1 + \sum_{n\geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_{n-1} \right)}{1 + \langle(1 + O_{1,2})^{(2)}\rangle_1 + \sum_{n\geq 3} \frac{1}{(n-1)!} \langle(1 + O_{1,\infty})^{(n)}\rangle_{n-1}} \Delta\Omega(\theta_1^{(i)})$$

is a notation for the convolution:

$$\left[ \left[ 1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1} \right] * \left[ \frac{\delta}{\delta\Omega(\theta^{(i)})} \left( 1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1} \right) \right] * \Delta\Omega \right] (\theta_1^{(i)})$$

and:

$$\begin{aligned} \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi(\theta_2^{(i)})} &= \Delta\Omega^\dagger(\theta_1^{(i)}) \left( \frac{\delta\Delta\Omega(\theta^{(i)})}{\delta\Psi(\theta_f^{(i)})} \Psi^\dagger(\theta^{(i)}) + \frac{\delta\Delta\Omega^\dagger(\theta^{(i)})}{\delta\Psi(\theta_f^{(i)})} \Psi(\theta^{(i)}) \right) \\ &\quad - \frac{\delta\Delta\Omega(\theta^{(i)})}{\delta\Psi(\theta_f^{(i)})} \frac{\frac{\delta}{\delta\Omega(\theta^{(i)})} \left( 1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1} \right)}{1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1}} \Delta\Omega^\dagger(\theta_1^{(i)}) \\ &\quad - \frac{\delta\Delta\Omega^\dagger(\theta^{(i)})}{\delta\Psi(\theta_f^{(i)})} \frac{\frac{\delta}{\delta\Omega^\dagger(\theta^{(i)})} \left( 1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1} \right)}{1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1}} \Delta\Omega^\dagger(\theta_1^{(i)}) \end{aligned} \quad (177)$$

These expression can be rewritten by using that the operators  $O$  (standing for  $1+O_{1,2}, 1+O_{1,\infty}\dots$ ) decompose as  $O = O^{(C)} + O^{(D)}\nabla |O_D| < O_C$  and  $O^{-1} \simeq \frac{(O^{(C)} - O^{(D)}\nabla)}{(O^{(C)})^2} \equiv \frac{O^\dagger}{(O^{(C)})^2}$ . We have:

$$\begin{aligned} &\frac{\frac{\delta}{\delta\Omega(\theta^{(i)})} \left( 1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1} \right)}{1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1}} \Delta\Omega(\theta_1^{(i)}) \\ &= - \frac{\frac{\delta}{\delta\Omega(\theta^{(i)})} \left( 1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1} \right)}{\left( 1 + \langle (1 + O_{1,2})^{(1)} \rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n-1)} \rangle_{n-1} (1 + \bar{O}_{1,\infty}) \right) * \mathcal{G}_0} \Delta\Omega(\theta_1^{(i)}) \\ &\simeq \Delta\Omega^\dagger(\theta^{(i)}) \\ &\quad \times \frac{\left( (1 + \bar{O}_{1,2}) * \mathcal{G}_0 \right) \left( (1 + \bar{O}_{1,2})_{\theta_1^{(i)}} * \mathcal{G}_0 \right) + \left( (1 + \bar{O}_{1,\infty}) * \mathcal{G}_0 \right) \sum_{n \geq 3} \frac{\langle (1 + O_{1,\infty})^{(n-2)} \rangle_{n-2}}{(n-2)!} \left( (1 + \bar{O}_{1,\infty})_{\theta_1^{(i)}} * \mathcal{G}_0 \right)}{\left( 1 + \langle (1 + O_{1,2})^{(1)} \rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n-1)} \rangle_{n-1} (1 + \bar{O}_{1,\infty}) \right) * \mathcal{G}_0} \Delta\Omega(\theta_1^{(i)}) \\ &= \Delta\Omega^\dagger(\theta^{(i)}) \mathcal{G}_0^{-1} \\ &\quad * \frac{\left( (1 + \bar{O}_{1,2}) * \mathcal{G}_0 \right) (1 + \bar{O}_{1,2})_{\theta_1^{(i)}} + \left( (1 + \bar{O}_{1,\infty}) * \mathcal{G}_0 \right) \sum_{n \geq 3} \frac{\langle (1 + O_{1,\infty})^{(n-2)} \rangle_{n-2}}{(n-2)!} (1 + \bar{O}_{1,\infty})_{\theta_1^{(i)}}}{1 + \langle (1 + O_{1,2})^{(1)} \rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n-1)} \rangle_{n-1} (1 + \bar{O}_{1,\infty})} * \mathcal{G}_0 \Delta\Omega(\theta_1^{(i)}) \end{aligned}$$

where the convolutions are understood. Thus, (176) and (177) write:

$$\begin{aligned}
\frac{\delta\Delta\Omega\left(\theta_1^{(i)}\right)}{\delta\Psi\left(\theta_f^{(i)}\right)} &= \int \left( -\Delta\Omega^\dagger\left(\left(\theta_2^{(i)}\right)'\right) \left( A\left(\theta_1^{(i)}\right) \left(1 + \bar{O}_{1,\infty}\right)_{\left(\theta_2^{(i)}\right)',\theta_2^{(i)}} + B\left(\theta_1^{(i)}\right) \left(1 + \bar{O}_{1,2}\right)_{\left(\theta_2^{(i)}\right)',\theta_2^{(i)}} \right) \mathcal{G}_0 \right. \\
&\quad \left. + \Delta\Omega\left(\theta_1^{(i)}\right) \Psi^\dagger\left(\left(\theta_2^{(i)}\right)'\right) \delta\left(\left(\theta_2^{(i)}\right)' - \theta_2^{(i)}\right) \right) \\
&\quad \times \frac{\delta\Delta\Omega\left(\theta_2^{(i)}\right)}{\delta\Psi\left(\theta_f^{(i)}\right)} d\theta_2^{(i)} d\left(\theta_2^{(i)}\right)' \\
&= \int \left( - \int \Delta\Omega^\dagger\left(\left(\theta_2^{(i)}\right)'\right) \left( A\left(\theta_1^{(i)}\right) \left(1 + \bar{O}_{1,\infty}\right)_{\left(\theta_2^{(i)}\right)',\theta_2^{(i)}} + B\left(\theta_1^{(i)}\right) \left(1 + \bar{O}_{1,2}\right)_{\left(\theta_2^{(i)}\right)',\theta_2^{(i)}} \right) \mathcal{G}_0 d\left(\theta_2^{(i)}\right)' \right. \\
&\quad \left. + \Delta\Omega\left(\theta_1^{(i)}\right) \Psi^\dagger\left(\theta_2^{(i)}\right) \frac{\delta\Delta\Omega\left(\theta_2^{(i)}\right)}{\delta\Psi\left(\theta_f^{(i)}\right)} \right) d\theta_2^{(i)} \\
&= \int K_{1,1}\left(\theta_1^{(i)},\theta_2^{(i)}\right) \frac{\delta\Delta\Omega\left(\theta_2^{(i)}\right)}{\delta\Psi\left(\theta_f^{(i)}\right)} d\theta_2^{(i)}
\end{aligned}$$

where:

$$\begin{aligned}
&A\left(\theta_1^{(i)}\right) \\
&= \mathcal{G}_0^{-1} * \frac{\sum_{n \geq 3} \frac{1}{(n-2)!} \left\langle (1 + O_{1,\infty})^{(n-2)} \right\rangle_{n-2} (1 + \bar{O}_{1,\infty})_{\theta_1^{(i)}}}{\left( 1 + \left\langle (1 + O_{1,2})^{(1)} \right\rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \left\langle (1 + O_{1,\infty})^{(n-1)} \right\rangle_{n-1} (1 + \bar{O}_{1,\infty}) \right)} * \mathcal{G}_0 \Delta\Omega\left(\theta_1^{(i)}\right) \\
&= A\left(\theta_1^{(i)},\theta_f^{(i)}\right) \\
&B\left(\theta_1^{(i)}\right) \\
&= \mathcal{G}_0^{-1} * \frac{(1 + \bar{O}_{1,2})_{\theta_1^{(i)}}}{\left( 1 + \left\langle (1 + O_{1,2})^{(1)} \right\rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \left\langle (1 + O_{1,\infty})^{(n-1)} \right\rangle_{n-1} (1 + \bar{O}_{1,\infty}) \right)} * \mathcal{G}_0 \Delta\Omega\left(\theta_1^{(i)}\right) \\
&= B\left(\theta_1^{(i)},\theta_f^{(i)}\right)
\end{aligned}$$

Similar formula applies for the derivatives of  $\Delta\Omega^\dagger\left(\theta_2^{(i)}\right)$ :

$$\begin{aligned}
&\int \frac{\delta\Delta\Omega^\dagger\left(\theta_2^{(i)}\right)}{\delta\Psi\left(\theta_f^{(i)}\right)} \left( - \left( A\left(\theta_1^{(i)}\right) \left(1 + \bar{O}_{1,\infty}\right)_{\left(\theta_2^{(i)}\right)',\theta_2^{(i)}} + B\left(\theta_1^{(i)}\right) \left(1 + \bar{O}_{1,2}\right)_{\left(\theta_2^{(i)}\right)',\theta_2^{(i)}} \right) \mathcal{G}_0 \Delta\Omega\left(\left(\theta_2^{(i)}\right)'\right) \right. \\
&\quad \left. + \Delta\Omega\left(\theta_1^{(i)}\right) \Psi\left(\theta_2^{(i)}\right) \delta\left(\left(\theta_2^{(i)}\right)' - \theta_2^{(i)}\right) \right) d\theta_2^{(i)} d\left(\theta_2^{(i)}\right)' \\
&= \int \frac{\delta\Delta\Omega^\dagger\left(\theta_2^{(i)}\right)}{\delta\Psi\left(\theta_f^{(i)}\right)} \left( - \int \left( A\left(\theta_1^{(i)}\right) \left(1 + \bar{O}_{1,\infty}\right)_{\left(\theta_2^{(i)}\right)',\theta_2^{(i)}} + B\left(\theta_1^{(i)}\right) \left(1 + \bar{O}_{1,2}\right)_{\left(\theta_2^{(i)}\right)',\theta_2^{(i)}} \right) \mathcal{G}_0 \Delta\Omega\left(\left(\theta_2^{(i)}\right)'\right) d\left(\theta_2^{(i)}\right)' \right. \\
&\quad \left. + \Delta\Omega\left(\theta_1^{(i)}\right) \Psi\left(\theta_2^{(i)}\right) \right) d\theta_2^{(i)} \\
&= \int K_{1,2}\left(\theta_1^{(i)},\theta_2^{(i)}\right) \frac{\delta\Delta\Omega^\dagger\left(\theta_2^{(i)}\right)}{\delta\Psi\left(\theta_f^{(i)}\right)} d\theta_2^{(i)}
\end{aligned}$$

$$\begin{aligned}
& \int \frac{\delta \Delta \Omega^\dagger \left( \theta_2^{(i)} \right)}{\delta \Psi^\dagger \left( \theta_f^{(i)} \right)} \left( - \left( A^\dagger \left( \theta_1^{(i)} \right) \left( 1 + \bar{O}_{1,\infty} \right)_{\theta_2^{(i)}, \left( \theta_2^{(i)} \right)'} + B^\dagger \left( \theta_1^{(i)} \right) \left( 1 + \bar{O}_{1,2} \right)_{\theta_2^{(i)}, \left( \theta_2^{(i)} \right)'} \right) \mathcal{G}_0 \Delta \Omega \left( \left( \theta_2^{(i)} \right)' \right) \right. \\
& \quad \left. + \Delta \Omega^\dagger \left( \theta_1^{(i)} \right) \Psi \left( \theta_2^{(i)} \right) \delta \left( \left( \theta_2^{(i)} \right)' - \theta_2^{(i)} \right) \right) d\theta_2^{(i)} d \left( \theta_2^{(i)} \right)' \\
&= \int \frac{\delta \Delta \Omega^\dagger \left( \theta_2^{(i)} \right)}{\delta \Psi^\dagger \left( \theta_f^{(i)} \right)} \left( - \int \mathcal{G}_0 \left( A^\dagger \left( \theta_1^{(i)} \right) \left( 1 + \bar{O}_{1,\infty} \right)_{\theta_2^{(i)}, \left( \theta_2^{(i)} \right)'} + B^\dagger \left( \theta_1^{(i)} \right) \left( 1 + \bar{O}_{1,2} \right)_{\theta_2^{(i)}, \left( \theta_2^{(i)} \right)'} \right) \mathcal{G}_0 \Delta \Omega \left( \left( \theta_2^{(i)} \right)' \right) d \left( \theta_2^{(i)} \right)' \right. \\
& \quad \left. + \Delta \Omega^\dagger \left( \theta_1^{(i)} \right) \Psi \left( \theta_2^{(i)} \right) \right) d\theta_2^{(i)} \\
&= \int K_{2,2} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) \frac{\delta \Delta \Omega^\dagger \left( \theta_2^{(i)} \right)}{\delta \Psi^\dagger \left( \theta_f^{(i)} \right)} d\theta_2^{(i)}
\end{aligned}$$

Where  $\mathcal{G}_0 \Psi \left( \theta_2^{(i)} \right)$  denotes the result of the action of  $\mathcal{G}_0$  on  $\Psi$ :

$$\mathcal{G}_0 \Psi \left( \theta_2^{(i)} \right) \equiv \left( \mathcal{G}_0 \Psi \right) \left( \theta_2^{(i)} \right)$$

As a consequence, defining:

$$\begin{aligned}
[\Delta \Omega] &= \begin{pmatrix} \Delta \Omega \\ \Delta \Omega^\dagger \end{pmatrix} \tag{178} \\
K \left( \theta_1^{(i)}, \theta_2^{(i)} \right) &= \begin{pmatrix} K_{1,1} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) & K_{1,2} \left( \theta_1^{(i)}, \theta_2^{(i)} \right) \\ K_{1,2}^\dagger \left( \theta_1^{(i)}, \theta_2^{(i)} \right) & K_{1,1}^\dagger \left( \theta_1^{(i)}, \theta_2^{(i)} \right) \end{pmatrix} \\
[X \left( \theta_1^{(i)}, \theta_f^{(i)} \right)] &= \begin{pmatrix} \left( \frac{1 + \langle (1 + O_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + O_{1,\infty})^{(n)} \rangle_{n-1}}{1 + \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n} \mathcal{G}_0 \right)^{-1} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\ 0 \end{pmatrix} = \begin{pmatrix} X \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\ 0 \end{pmatrix}
\end{aligned}$$

Equations (176) and (177) can be written in the compact form of a dynamic equation:

$$\frac{\delta [\Delta \Omega] \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} = \int K \left( \theta_1^{(i)}, \theta_2^{(i)} \right) \frac{\delta [\Delta \Omega] \left( \theta_2^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} d\theta_2^{(i)} + [X \left( \theta_1^{(i)}, \theta_f^{(i)} \right)] \tag{179}$$

### 3.1.2 Solution of equation (179)

The solution of (179) is given by a series expansion:

$$\frac{\delta [\Delta \Omega] \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} = \sum_{n \geq 0} \int \left( \prod_{k=1}^{n-1} K \left( \theta_k^{(i)}, \theta_{k+1}^{(i)} \right) \right) [X \left( \theta_n^{(i)}, \theta_f^{(i)} \right)] \prod_{k=2}^n d\theta_k^{(i)} \tag{180}$$

We can check recursively that the coefficients:

$$\begin{pmatrix} K_{1,1}^{(n)} & K_{1,2}^{(n)} \\ K_{2,1}^{(n)} & K_{2,2}^{(n)} \end{pmatrix}$$

have the form:

$$\begin{aligned}
K_{1,1}^{(n)} &= \int \Delta\Omega^\dagger \left( \left( \theta_n^{(i)} \right)' \right) \left( - \left( A_n \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) (1 + \bar{O}_{1,\infty})_{(\theta_n^{(i)})', \theta_n^{(i)}} + B_n \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) (1 + \bar{O}_{1,2})_{(\theta_n^{(i)})', \theta_n^{(i)}} \right) \right) \mathcal{G}_0 \\
&\quad + C_n \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) \Psi^\dagger \left( \theta_n^{(i)} \right) \\
K_{2,2}^{(n)} &= - \int \left( A_n^\dagger \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) (1 + \bar{O}_{1,\infty})_{\theta_n^{(i)}, (\theta_n^{(i)})'} + B_n^\dagger \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) (1 + \bar{O}_{1,2})_{\theta_n^{(i)}, (\theta_n^{(i)})'} \right) \\
&\quad \times \mathcal{G}_0 \Delta\Omega \left( \left( \theta_n^{(i)} \right)' \right) d \left( \theta_n^{(i)} \right)' + C_n^\dagger \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) \Psi \left( \theta_n^{(i)} \right) \\
K_{1,2}^{(n)} &= - \int \left( A_n \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) (1 + \bar{O}_{1,\infty})_{\theta_n^{(i)}, (\theta_n^{(i)})'} + B_n \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) (1 + \bar{O}_{1,2})_{\theta_n^{(i)}, (\theta_n^{(i)})'} \right) \\
&\quad \times \mathcal{G}_0 \Delta\Omega \left( \left( \theta_n^{(i)} \right)' \right) d \left( \theta_n^{(i)} \right)' + C_n \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) \Psi \left( \theta_n^{(i)} \right) \\
K_{2,1}^{(n)} &= \int \Delta\Omega^\dagger \left( \left( \theta_n^{(i)} \right)' \right) \left( - \left( A_n^\dagger \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) (1 + \bar{O}_{1,\infty})_{(\theta_n^{(i)})', \theta_n^{(i)}} + B_n^\dagger \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) (1 + \bar{O}_{1,2})_{(\theta_n^{(i)})', \theta_n^{(i)}} \right) \right) \mathcal{G}_0 \\
&\quad + C_n^\dagger \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) \Psi^\dagger \left( \theta_n^{(i)} \right)
\end{aligned}$$

where the coefficients  $A_n$ ,  $B_n$  and  $C_n$  are found recursively. To find these coefficients, we use that:

$$\begin{pmatrix} K_{1,1}^{(n)} & K_{1,2}^{(n)} \\ K_{2,1}^{(n)} & K_{2,2}^{(n)} \end{pmatrix} \begin{pmatrix} K_{1,1} & K_{1,2} \\ K_{2,1} & K_{2,2} \end{pmatrix} = \begin{pmatrix} K_{1,1}^{(n)} K_{1,1} + K_{1,2}^{(n)} K_{2,1} & K_{1,1}^{(n)} K_{1,2} + K_{1,2}^{(n)} K_{2,2} \\ K_{2,2}^{(n)} K_{2,1} + K_{2,1}^{(n)} K_{1,1} & K_{2,2}^{(n)} K_{2,2} + K_{2,1}^{(n)} K_{1,2} \end{pmatrix}$$

and note that the first coefficients  $A_1$  and  $B_1$  are defined by:

$$\begin{aligned}
A_1 \left( \theta_1^{(i)}, \left( \theta_1^{(i)} \right)' \right) \Delta\Omega \left( \left( \theta_1^{(i)} \right)' \right) &= A \left( \theta_1^{(i)} \right) \\
B_1 \left( \theta_1^{(i)}, \left( \theta_1^{(i)} \right)' \right) \Delta\Omega \left( \left( \theta_1^{(i)} \right)' \right) &= B \left( \theta_1^{(i)} \right)
\end{aligned}$$

The recursive equations for  $A_k$  and  $B_k$  are:

$$\begin{aligned}
& A_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right) \tag{181} \\
= & - \int \mathcal{G}_0 \left( \Delta \Omega^\dagger \left( \left( \theta_k^{(i)} \right)' \right) A_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) (1 + O_{1,\infty})_{\left( \theta_k^{(i)} \right)', \theta_k^{(i)}} A \left( \theta_k^{(i)} \right) \right. \\
& + B_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) (1 + O_{1,2})_{\left( \theta_k^{(i)} \right)', \theta_k^{(i)}} A \left( \theta_k^{(i)} \right) \left. + C_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \Psi^\dagger \left( \theta_k^{(i)} \right) A \left( \theta_k^{(i)} \right) \right. \\
& - \int \mathcal{G}_0 \left( A_k \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) A^\dagger \left( \theta_k^{(i)} \right) (1 + O_{1,\infty})_{\theta_k^{(i)}, \left( \theta_k^{(i)} \right)'} \right. \\
& + B_k \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) A^\dagger \left( \theta_k^{(i)} \right) (1 + O_{1,2})_{\theta_k^{(i)}, \left( \theta_k^{(i)} \right)'} \left. \Delta \Omega \left( \left( \theta_k^{(i)} \right)' \right) + C_k \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) A^\dagger \left( \theta_k^{(i)} \right) \Psi \left( \theta_k^{(i)} \right) \right) \\
= & -A_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \left( \langle (1 + O_{1,\infty}) A_1 \rangle_{>\theta_k^{(i)}} + \langle A_1 (1 + O_{1,\infty}) \rangle_{<\theta_k^{(i)}} \right) \\
& -B_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \left( \langle (1 + O_{1,2}) A_1 \rangle_{>\theta_k^{(i)}} + \langle A_1 (1 + O_{1,2}) \rangle_{<\theta_k^{(i)}} \right) + C_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \left( \langle A_1 \rangle_{>\theta_k^{(i)}}^{\Omega \Psi} + \langle A_1 \rangle_{<\theta_k^{(i)}}^{\Omega \Psi} \right) \\
& B_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right) \tag{182} \\
= & -A_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \left( \langle (1 + O_{1,\infty}) B_1 \rangle_{>\theta_k^{(i)}} + \langle B_1 (1 + O_{1,\infty}) \rangle_{<\theta_k^{(i)}} \right) \\
& -B_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \left( \langle (1 + O_{1,2}) B_1 \rangle_{>\theta_k^{(i)}} + \langle B_1 (1 + O_{1,2}) \rangle_{<\theta_k^{(i)}} \right) + C_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \left( \langle B_1 \rangle_{>\theta_k^{(i)}}^{\Omega \Psi} + \langle B_1 \rangle_{<\theta_k^{(i)}}^{\Omega \Psi} \right)
\end{aligned}$$

$$\begin{aligned}
C_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right) & = -A_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \left( \langle (1 + O_{1,\infty}) \rangle_{>\theta_k^{(i)}} + \langle (1 + O_{1,\infty}) \rangle_{<\theta_k^{(i)}} \right) \tag{183} \\
& -B_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \left( \langle (1 + O_{1,2}) \rangle_{>\theta_k^{(i)}} + \langle (1 + O_{1,2}) \rangle_{<\theta_k^{(i)}} \right) \\
& + C_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \left( \langle 1 \rangle_{>\theta_k^{(i)}}^{\Omega \Psi} + \langle 1 \rangle_{<\theta_k^{(i)}}^{\Omega \Psi} \right)
\end{aligned}$$

with:

$$\begin{aligned}
\langle A_1 \rangle_{>\theta_k^{(i)}}^{\Omega \Psi} & = \int \Psi^\dagger \left( \theta_k^{(i)} \right) A \left( \theta_k^{(i)} \right) \\
\langle A_1 \rangle_{<\theta_k^{(i)}}^{\Omega \Psi} & = \int A^\dagger \left( \theta_k^{(i)} \right) \Psi \left( \theta_k^{(i)} \right) \\
\langle B_1 \rangle_{>\theta_k^{(i)}}^{\Omega \Psi} & = \int \Psi^\dagger \left( \theta_k^{(i)} \right) B \left( \theta_k^{(i)} \right) \\
\langle B_1 \rangle_{<\theta_k^{(i)}}^{\Omega \Psi} & = \int B^\dagger \left( \theta_k^{(i)} \right) \Psi \left( \theta_k^{(i)} \right)
\end{aligned}$$

To compute the various brackets involved in (181), (182) and (183), we define the following  $\Delta \Omega$  average values for an arbitrary operator  $M$ :

$$\begin{aligned}
\langle M \rangle_{>\theta_k^{(i)}} & = \int \Delta \Omega^\dagger \left( \left( \theta_k^{(i)} \right)' \right) M_{\left( \theta_k^{(i)} \right)', \theta_k^{(i)}} \mathcal{G}_0 \Delta \Omega \left( \theta_k^{(i)} \right) \tag{184} \\
\langle M \rangle_{<\theta_k^{(i)}} & = \int \Delta \Omega^\dagger \left( \left( \theta_k^{(i)} \right)' \right) M_{\left( \theta_k^{(i)} \right)', \theta_k^{(i)}}^\dagger \mathcal{G}_0 \Delta \Omega \left( \theta_k^{(i)} \right)
\end{aligned}$$

and the  $\Psi$  average values:

$$\begin{aligned}\langle M \rangle_{>\theta_k^{(i)}}^\Psi &= \int \Psi^\dagger \left( \left( \theta_k^{(i)} \right)' \right) \mathcal{G}_0^{-1} M_{(\theta_k^{(i)})', \theta_k^{(i)}} \Psi \left( \theta_k^{(i)} \right) \\ \langle M \rangle_{<\theta_k^{(i)}}^\Psi &= \int \Psi^\dagger \left( \left( \theta_k^{(i)} \right)' \right) \mathcal{G}_0^{-1} M_{(\theta_k^{(i)})', \theta_k^{(i)}}^\dagger \Psi \left( \theta_k^{(i)} \right)\end{aligned}\quad (185)$$

Using (171):

$$\begin{aligned}\Delta\Omega \left( \theta^{(i)} \right) &= \left( \left( \frac{(1 + O_{1,\infty}) + \exp(-x)(-O_{1,\infty} + y(1 + O_{1,2}) - x(1 + O_{1,\infty}))}{1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2))} \right) \mathcal{G}_0 \right)^{-1} \Psi \left( \theta_1^{(i)} \right) \\ &= \mathcal{G}_0^{-1} * \left( \frac{1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2))}{(1 + \bar{O}_{1,\infty}) + \exp(-x)(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))} \right) \Psi \left( \theta_1^{(i)} \right)\end{aligned}$$

we can compute the  $\Delta\Omega$  averages (184) of  $M$  in terms of its  $\Psi$  averages (185) up to fourth order in derivatives:

$$\begin{aligned}\langle M \rangle_{>\theta_k^{(i)}} &\simeq \left( \frac{1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2))}{(1 + \bar{O}_{1,\infty}^C) + \exp(-x)(-\bar{O}_{1,\infty}^C + (y(1 + \bar{O}_{1,2}^C) - x(1 + \bar{O}_{1,\infty}^C)))} \right)^2 \langle \bar{M} \rangle_{>\theta_k^{(i)}}^\Psi \\ &\simeq \frac{1}{\langle 1 \rangle^\Psi} \left\langle \frac{(1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2)))^2}{((1 + \bar{O}_{1,\infty}) + \exp(-x)(-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))))^2} \right\rangle^\Psi \langle \bar{M} \rangle_{>\theta_k^{(i)}}^\Psi \\ \langle M \rangle_{<\theta_k^{(i)}} &\simeq \left( \frac{1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2))}{(1 + \bar{O}_{1,\infty}^C) + \exp(-x)(-\bar{O}_{1,\infty}^C + (y(1 + \bar{O}_{1,2}^C) - x(1 + \bar{O}_{1,\infty}^C)))} \right)^2 \langle \bar{M} \rangle_{<\theta_k^{(i)}}^\Psi \\ &\simeq \frac{1}{\langle 1 \rangle^\Psi} \left\langle \frac{(1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2)))^2}{((1 + \bar{O}_{1,\infty}) + \exp(-x)(-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))))^2} \right\rangle^\Psi \langle \bar{M} \rangle_{<\theta_k^{(i)}}^\Psi\end{aligned}$$

with  $\bar{M}$  given by:

$$M\mathcal{G}_0 = \bar{M} * \mathcal{G}_0$$

The previous formula imply that several quantities arising in (181), (182) and (183) can be computed:

$$\begin{aligned}\langle (1 + O_{1,\infty}) A_1 \rangle_{>\theta_k^{(i)}}^\Psi &= \frac{1}{\langle 1 \rangle^\Psi} \left\langle \frac{(1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2)))^2}{((1 + \bar{O}_{1,\infty}) + \exp(-x)(-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))))^2} \right\rangle^\Psi \\ &\quad \times \langle (1 + \bar{O}_{1,\infty}) \bar{A}_1 \rangle_{>\theta_k^{(i)}}^\Psi \\ \langle (1 + O_{1,\infty}) B_1 \rangle_{>\theta_k^{(i)}}^\Psi &= \frac{1}{\langle 1 \rangle^\Psi} \left\langle \frac{(1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2)))^2}{((1 + \bar{O}_{1,\infty}) + \exp(-x)(-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))))^2} \right\rangle^\Psi \\ &\quad \times \langle (1 + \bar{O}_{1,\infty}) \bar{B}_1 \rangle_{>\theta_k^{(i)}}^\Psi\end{aligned}$$



where:

$$\begin{aligned}
\bar{A}_1(\theta_1^{(i)}) &= \frac{\sum_{n \geq 3} \frac{1}{(n-2)!} \left\langle (1 + O_{1,\infty})^{(n-2)} \right\rangle_{n-2} (1 + \bar{O}_{1,\infty})_{\theta_1^{(i)}}}{\left(1 + \left\langle (1 + O_{1,2})^{(1)} \right\rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \left\langle (1 + O_{1,\infty})^{(n-1)} \right\rangle_{n-1} (1 + \bar{O}_{1,\infty})\right)} \\
&\simeq \frac{(1 - \exp(-x)) (1 + \bar{O}_{1,\infty})_{\theta_1^{(i)}}}{(1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})))} \\
\bar{B}_1(\theta_1^{(i)}) &= \frac{(1 + \bar{O}_{1,2})_{\theta_1^{(i)}}}{\left(1 + \left\langle (1 + O_{1,2})^{(1)} \right\rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \left\langle (1 + O_{1,\infty})^{(n-1)} \right\rangle_{n-1} (1 + \bar{O}_{1,\infty})\right)} \\
&\simeq \frac{(1 + \bar{O}_{1,2})_{\theta_1^{(i)}}}{(1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})))}
\end{aligned}$$

We also define  $\langle M \rangle_{\theta_k^{(i)}}^\Psi = \langle M \rangle_{>\theta_k^{(i)}}^\Psi + \langle M \rangle_{<\theta_k^{(i)}}^\Psi$ . We can replace in average  $\langle M \rangle_{>\theta_k^{(i)}}^\Psi$  and  $\langle M \rangle_{<\theta_k^{(i)}}^\Psi$  by  $\frac{1}{2} \langle M \rangle_{\theta_k^{(i)}}^\Psi$  in (181), (182) and (183). Moreover:

$$\begin{aligned}
&\Psi^\dagger(\theta_k^{(i)}) \Delta\Omega(\theta_k^{(i)}) \\
&\simeq \frac{(1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))) \left\langle \left( (1 + \bar{O}_{1,\infty}^\dagger) + \exp(-x) (-\bar{O}_{1,\infty}^\dagger + y(1 + \bar{O}_{1,2}^\dagger) - x(1 + \bar{O}_{1,\infty}^\dagger)) \right) \right\rangle_{>\theta_k^{(i)}}^\Psi}{\left( (1 + \bar{O}_{1,\infty}^C) + \exp(-x) (-\bar{O}_{1,\infty}^C + (y(1 + \bar{O}_{1,2}^C) - x(1 + \bar{O}_{1,\infty}^C))) \right)^2} \\
&= \left( 1 + \exp(-x) \left( -z + \frac{1}{2}(y^2 - x^2) \right) \right) \left\langle \frac{1}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))) \right)} \right\rangle_{>\theta_k^{(i)}}^\Psi \\
&\Delta\Omega^\dagger(\theta_k^{(i)}) \Psi(\theta_k^{(i)}) \\
&= \frac{(1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))) \left\langle \left( (1 + \bar{O}_{1,\infty}^\dagger) + \exp(-x) (-\bar{O}_{1,\infty}^\dagger + y(1 + \bar{O}_{1,2}^\dagger) - x(1 + \bar{O}_{1,\infty}^\dagger)) \right) \right\rangle_{<\theta_k^{(i)}}^\Psi}{\left( (1 + \bar{O}_{1,\infty}^C) + \exp(-x) (-\bar{O}_{1,\infty}^C + (y(1 + \bar{O}_{1,2}^C) - x(1 + \bar{O}_{1,\infty}^C))) \right)^2} \\
&= \left\langle \frac{1 + \exp(-x) (-z + \frac{1}{2}(y^2 - x^2))}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))) \right)} \right\rangle_{<\theta_k^{(i)}}^\Psi
\end{aligned}$$

and, for the coefficients  $A(\theta_1^{(i)})$ ,  $B(\theta_1^{(i)})$ :

$$\begin{aligned}
&A(\theta_1^{(i)}) \tag{186} \\
&= \mathcal{G}_0^{-1} * \frac{\sum_{n \geq 3} \frac{1}{(n-2)!} \left\langle (1 + O_{1,\infty})^{(n-2)} \right\rangle_{n-2} (1 + \bar{O}_{1,\infty})_{\theta_1^{(i)}}}{\left(1 + \left\langle (1 + O_{1,2})^{(1)} \right\rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \left\langle (1 + O_{1,\infty})^{(n-1)} \right\rangle_{n-1} (1 + \bar{O}_{1,\infty})\right)} * \mathcal{G}_0 \Delta\Omega(\theta_1^{(i)}) \\
&\simeq \mathcal{G}_0^{-1} * \frac{(1 - \exp(-x)) (1 + O_{1,\infty})_{\theta_1^{(i)}}}{(1 + O_{1,\infty}) + \exp(-x) (-O_{1,\infty} + (y(1 + O_{1,2}) - x(1 + O_{1,\infty})))} * \mathcal{G}_0 \Delta\Omega(\theta_1^{(i)}) \\
&B(\theta_1^{(i)}) \\
&= \mathcal{G}_0^{-1} * \frac{(1 + \bar{O}_{1,2})_{\theta_1^{(i)}}}{\left(1 + \left\langle (1 + O_{1,2})^{(1)} \right\rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \left\langle (1 + O_{1,\infty})^{(n-1)} \right\rangle_{n-1} (1 + \bar{O}_{1,\infty})\right)} * \mathcal{G}_0 \Delta\Omega(\theta_1^{(i)}) \\
&\simeq \mathcal{G}_0^{-1} * \frac{\exp(-x) (1 + O_{1,2})_{\theta_1^{(i)}}}{(1 + O_{1,\infty}) + \exp(-x) (-O_{1,\infty} + (y(1 + O_{1,2}) - x(1 + O_{1,\infty})))} * G \Delta\Omega(\theta_1^{(i)})
\end{aligned}$$

Ultimately, we will need the following identities:

$$\begin{aligned}
& \frac{\sum_{n \geq 3} \frac{1}{(n-2)!} \left\langle (1 + O_{1,\infty})^{(n-2)} \right\rangle_{n-2} (1 + O_{1,\infty})_{\theta_1^{(i)}}}{\left(1 + \left\langle (1 + O_{1,2})^{(1)} \right\rangle_1 (1 + O_{1,2}^C) + \sum_{n \geq 3} \frac{1}{(n-1)!} \left\langle (1 + O_{1,\infty})^{(n-1)} \right\rangle_{n-1} (1 + O_{1,\infty}^C)\right)} \\
& \simeq \frac{\exp(-x) \sum_{n \geq 3} \frac{\langle (1 + O_{1,\infty})^{(n-2)} \rangle_{n-2} (1 + O_{1,\infty})_{\theta_1^{(i)}} \left( (1 + O_{1,\infty}^\dagger) + \exp(-x) \left( -O_{1,\infty}^\dagger + y \left( 1 + O_{1,2}^\dagger \right) - x \left( 1 + O_{1,\infty}^\dagger \right) \right) \right)}{\left( (1 + O_{1,\infty}^C) + \exp(-x) \left( -O_{1,\infty}^C + \left( y \left( 1 + O_{1,2}^C \right) - x \left( 1 + O_{1,\infty}^C \right) \right) \right) \right)^2} \\
& \simeq \frac{(1 - \exp(-x)) (1 + O_{1,\infty})_{\theta_1^{(i)}} \left( (1 + O_{1,\infty}^\dagger) + \exp(-x) \left( -O_{1,\infty}^\dagger + y \left( 1 + O_{1,2}^\dagger \right) - x \left( 1 + O_{1,\infty}^\dagger \right) \right) \right)}{\left( (1 + O_{1,\infty}^C) + \exp(-x) \left( -O_{1,\infty}^C + \left( y \left( 1 + O_{1,2}^C \right) - x \left( 1 + O_{1,\infty}^C \right) \right) \right) \right)^2}
\end{aligned} \tag{187}$$

As a consequence, the coefficients  $A_{k+1}$ ,  $B_{k+1}$ ,  $C_{k+1}$  can be written as a function of a coefficient  $N_{k+1}$ :

$$\begin{aligned}
& A_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right) \\
& \simeq \frac{1}{\langle 1 \rangle^\Psi} \left\langle \frac{\sum_{n \geq 3} \frac{1}{(n-2)!} \left\langle (1 + O_{1,\infty})^{(n-2)} \right\rangle_{n-2} (1 + \bar{O}_{1,\infty})_{\theta_1^{(i)}}}{\left(1 + \left\langle (1 + O_{1,2})^{(1)} \right\rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \left\langle (1 + O_{1,\infty})^{(n-1)} \right\rangle_{n-1} (1 + \bar{O}_{1,\infty})\right)} \right\rangle^\Psi N_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right) \\
& \simeq \frac{1}{\langle 1 \rangle^\Psi} \left\langle \frac{(1 - \exp(-x)) (1 + \bar{O}_{1,\infty})_{\theta_1^{(i)}}}{(1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + \left( y \left( 1 + \bar{O}_{1,2} \right) - x \left( 1 + \bar{O}_{1,\infty} \right) \right) \right)} \right\rangle^\Psi N_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right) \\
& B_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right) \\
& \simeq \frac{1}{\langle 1 \rangle^\Psi} \left\langle \frac{\exp(-x) (1 + \bar{O}_{1,2})_{\theta_1^{(i)}}}{\left(1 + \left\langle (1 + O_{1,2})^{(1)} \right\rangle_1 (1 + \bar{O}_{1,2}) + \sum_{n \geq 3} \frac{1}{(n-1)!} \left\langle (1 + O_{1,\infty})^{(n-1)} \right\rangle_{n-1} (1 + \bar{O}_{1,\infty})\right)} \right\rangle^\Psi N_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right) \\
& \simeq \frac{1}{\langle 1 \rangle^\Psi} \left\langle \frac{\exp(-x) (1 + \bar{O}_{1,2})_{\theta_1^{(i)}}}{(1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + \left( y \left( 1 + \bar{O}_{1,2} \right) - x \left( 1 + \bar{O}_{1,\infty} \right) \right) \right)} \right\rangle^\Psi N_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right) \\
& C_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right) \simeq N_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right)
\end{aligned}$$

Inserting this relation in (183) yields the recursive relation for  $N_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right)$ :

$$\begin{aligned}
N_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right) &\simeq -\frac{1}{\langle 1 \rangle^\Psi} \left\langle \frac{1}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))) \right)} \right\rangle^\Psi \\
&\times \left\{ \frac{1}{\langle 1 \rangle^\Psi} \left\langle \frac{\left( 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \right)^2}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))) \right)^2} \right\rangle^\Psi \right\} \\
&\times \left( (1 - \exp(-x)) \frac{\langle (1 + \bar{O}_{1,\infty}) \rangle^\Psi}{\langle 1 \rangle^\Psi} \langle (1 + \bar{O}_{1,\infty}) \rangle_{\theta_k^{(i)}}^\Psi + \frac{\langle (1 + \bar{O}_{1,2}) \rangle^\Psi}{\langle 1 \rangle^\Psi} \exp(-x) \langle (1 + \bar{O}_{1,2}) \rangle_{\theta_k^{(i)}}^\Psi \right) \left\} N_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \\
&+ \left\langle \frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))) \right)} \right\rangle^\Psi N_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \\
= &-\frac{1}{\langle 1 \rangle^\Psi} \left\langle \frac{1}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))) \right)} \right\rangle^\Psi \\
&\times \left( (1 - \exp(-x)) \frac{x}{\langle 1 \rangle^\Psi} \langle (1 + \bar{O}_{1,\infty}) \rangle_{\theta_k^{(i)}}^\Psi + \frac{y}{\langle 1 \rangle^\Psi} \exp(-x) \langle (1 + \bar{O}_{1,2}) \rangle_{\theta_k^{(i)}}^\Psi \right) N_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right) \\
&+ \left\langle \frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))) \right)} \right\rangle^\Psi N_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right)
\end{aligned}$$

and this formula factors as:

$$\begin{aligned}
N_{k+1} \left( \left( \theta_l^{(i)} \right)_{l \leq k} \right) &\simeq \left\langle \frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))) \right)} \right\rangle^\Psi \\
&\times \left\{ 1 - \frac{\left( (1 - \exp(-x)) x \langle (1 + \bar{O}_{1,\infty}) \rangle_{\theta_k^{(i)}}^\Psi + y \exp(-x) \langle (1 + \bar{O}_{1,2}) \rangle_{\theta_k^{(i)}}^\Psi \right)}{\left( \langle 1 \rangle^\Psi \right)^2 \left( 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \right)} \right\} N_k \left( \left( \theta_l^{(i)} \right)_{l \leq k-1} \right)
\end{aligned}$$

whose solution is:

$$\begin{aligned}
N_n \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) &\simeq N_2 \left( \left( \theta_1^{(i)} \right) \right) \prod_{k=2}^{n-1} \left\langle \frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))) \right)} \right\rangle^\Psi \\
&\times \left\{ 1 - \frac{\left( (1 - \exp(-x)) x \langle (1 + \bar{O}_{1,\infty}) \rangle_{\theta_k^{(i)}}^\Psi + y \exp(-x) \langle (1 + \bar{O}_{1,2}) \rangle_{\theta_k^{(i)}}^\Psi \right)}{\left( \langle 1 \rangle^\Psi \right)^2 \left( 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \right)} \right\}
\end{aligned}$$

This yields ultimately the solution of (179):

$$\begin{aligned}
& \frac{\delta [\Delta\Omega] \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} \\
&= \int K_{1,1}^{(n)} \left( \theta_1^{(i)}, \theta_n^{(i)} \right) X \left( \theta_n^{(i)}, \theta_f^{(i)} \right) \\
&= \sum_{n \geq 0} \int \left( \prod_{k=1}^{n-1} K \left( \theta_k^{(i)}, \theta_{k+1}^{(i)} \right) \right) \left[ X \left( \theta_n^{(i)}, \theta_f^{(i)} \right) \right] \prod_{k=2}^n d\theta_k^{(i)} \\
&= X \left( \theta_1^{(i)}, \theta_f^{(i)} \right) + \sum_{n \geq 1} \int_{\theta_1^{(i)} > \theta_k^{(i)} \dots > \theta_n^{(i)}} \bar{N}_n \left( \left( \theta_l^{(i)} \right)_{l \leq n-1} \right) \left( \Delta\Omega \left( \left( \theta_1^{(i)} \right) \right) \Psi^\dagger \left( \theta_n^{(i)} \right) \right. \\
&\quad \left. - \left( A \left( \theta_1^{(i)} \right) \int_{\theta_2^{(i)}} \Delta\Omega^\dagger \left( \left( \theta_2^{(i)} \right)' \right) \mathcal{G}_0 \left( 1 + O_{1,\infty} \right) + B \left( \theta_1^{(i)} \right) \int_{\theta_2^{(i)}} \Delta\Omega^\dagger \left( \left( \theta_2^{(i)} \right)' \right) \mathcal{G}_0 \left( 1 + O_{1,2} \right) \right) \right) \\
&\quad \times X \left( \theta_n^{(i)}, \theta_f^{(i)} \right) \prod_{k=2}^n d\theta_k^{(i)}
\end{aligned}$$

that can be approximated by:

$$\begin{aligned}
& \frac{\delta [\Delta\Omega] \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} \\
&\simeq X \left( \theta_1^{(i)}, \theta_f^{(i)} \right) + \int d\theta_2^{(i)} \exp \left( \int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( \left( \theta^{(i)} \right) d\theta^{(i)} \right) \right) \left( \Delta\Omega \left( \left( \theta_1^{(i)} \right) \right) \Psi^\dagger \left( \theta_2^{(i)} \right) X \left( \theta_2^{(i)}, \theta_f^{(i)} \right) \right. \\
&\quad \left. - A \left( \theta_1^{(i)} \right) \int_{\theta_2^{(i)}} \Delta\Omega^\dagger \left( \left( \theta_2^{(i)} \right)' \right) \left( 1 + \bar{O}_{1,\infty} \right)_{\left( \theta_2^{(i)} \right)', \theta_2^{(i)}} \mathcal{G}_0 X \left( \left( \theta_2^{(i)} \right)', \theta_f^{(i)} \right) d \left( \theta_2^{(i)} \right)' \right. \\
&\quad \left. - B \left( \theta_1^{(i)} \right) \int_{\theta_2^{(i)}} \Delta\Omega^\dagger \left( \left( \theta_2^{(i)} \right)' \right) \left( 1 + \bar{O}_{1,2} \right)_{\left( \theta_2^{(i)} \right)', \theta_2^{(i)}} \mathcal{G}_0 X \left( \left( \theta_2^{(i)} \right)', \theta_f^{(i)} \right) d \left( \theta_2^{(i)} \right)' \right)
\end{aligned} \tag{188}$$

where the factor  $\bar{N} \left( \left( \theta_i \right) \right)$  is given by:

$$\begin{aligned}
\bar{N} \left( \left( \theta_i \right) \right) &\simeq \left\langle \frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{\left( \left( 1 + \bar{O}_{1,\infty} \right) + \exp(-x) \left( -\bar{O}_{1,\infty} + \left( y \left( 1 + \bar{O}_{1,2} \right) - x \left( 1 + \bar{O}_{1,\infty} \right) \right) \right) \right)} \right\rangle^\Psi \\
&\quad \times \left\{ 1 - \frac{\left( \left( 1 - \exp(-x) \right) x \langle \left( 1 + \bar{O}_{1,\infty} \right) \rangle_{\theta^{(i)}}^\Psi + y \exp(-x) \langle \left( 1 + \bar{O}_{1,2} \right) \rangle_{\theta^{(i)}}^\Psi \right)}{\langle 1 \rangle^\Psi \left( 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \right)} \right\}
\end{aligned}$$

and with condition:

$$\theta_f^{(i)} < \left( \theta_2^{(i)} \right)' < \theta_1^{(i)}$$

due to the Heaviside functions in the integrals defining the interaction terms.

Using (186) and (187), the coefficients involved in (188) write:

$$\begin{aligned}
\Delta\Omega\left(\left(\theta_1^{(i)}\right)\right) &= \left(\int^{\theta_1^{(i)}} \left(\left(\frac{(1+O_{1,\infty})+\exp(-x)(-O_{1,\infty}+y(1+O_{1,2})-x(1+O_{1,\infty}))}{1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2))}\right)\mathcal{G}_0\right)^{-1}\right)\Psi\left(\left(\theta_1^{(i)}\right)'\right) \\
&= \left(\int^{\theta_1^{(i)}} \mathcal{G}_0^{-1}\left(\frac{1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2))}{(1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+(y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty})))}\right)\right)\Psi\left(\left(\theta_1^{(i)}\right)'\right) \\
A\left(\theta_1^{(i)}\right) &= \left(\int^{\theta_1^{(i)}} \mathcal{G}_0^{-1}\frac{(1-\exp(-x))(1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2)))(1+\bar{O}_{1,\infty})\theta_1^{(i)}}{((1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+(y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty}))))^2}\right)\Psi\left(\left(\theta_1^{(i)}\right)'\right) \\
B\left(\theta_1^{(i)}\right) &= \left(\int^{\theta_1^{(i)}} \mathcal{G}_0^{-1}\frac{\exp(-x)(1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2)))(1+\bar{O}_{1,2})\theta_1^{(i)}}{((1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+(y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty}))))^2}\right)\Psi\left(\left(\theta_1^{(i)}\right)'\right)
\end{aligned}$$

and  $\mathcal{G}_0 X\left(\theta_n^{(i)}, \theta_f^{(i)}\right)$  is equal to:

$$\begin{aligned}
\mathcal{G}_0 X\left(\theta_n^{(i)}, \theta_f^{(i)}\right) &= \mathcal{G}_0 \left(\frac{1+\langle(1+O_{1,2})^{(2)}\rangle_1+\sum_{n\geq 3}\frac{1}{(n-1)!}\langle(1+O_{1,\infty})^{(n)}\rangle_{n-1}}{1+\langle 1\rangle_1+\frac{1}{2}\langle(1+O_{1,2})^{(2)}\rangle_2+\sum_{n\geq 3}\frac{1}{n!}\langle(1+O_{1,\infty})^{(n)}\rangle_n}\right)^{-1}\left(\theta_1^{(i)}, \theta_f^{(i)}\right) \\
&\simeq \frac{1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2))}{(1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty}))}\left(\theta_n^{(i)}, \theta_f^{(i)}\right)
\end{aligned}$$

In the local approximation  $X\left(\theta_2^{(i)}, \theta_f^{(i)}\right) \propto \delta\left(\theta_2^{(i)}-\theta_f^{(i)}\right)$ . Defining:

$$\begin{aligned}
O_1 &= \frac{1}{(1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty}))}\mathcal{G}_0^{-1}(1+\bar{O}_{1,\infty}) \quad (189) \\
O_2 &= \frac{1}{(1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty}))}\mathcal{G}_0^{-1}(1+\bar{O}_{1,2})
\end{aligned}$$

this yields the expanded form for the two points correlation functions (188):

$$\begin{aligned}
\frac{\delta[\Delta\Omega]\left(\theta_1^{(i)}\right)}{\delta\Psi\left(\theta_f^{(i)}\right)} &= \mathcal{G}_0^{-1}\frac{1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2))}{(1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty}))}\left(\theta_1^{(i)}, \theta_f^{(i)}\right) \\
&+ \exp\left(\int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \bar{N}\left(\left(\theta^{(i)}\right) d\theta^{(i)}\right)\right)\left(\int\Psi^\dagger\left(\left(\theta_2^{(i)}\right)'\right)X\left(\left(\theta_2^{(i)}\right)', \theta_f^{(i)}\right)d\left(\theta_2^{(i)}\right)'(\mathcal{G}_0^{-1})^*_{\theta_1^{(i)}}\right. \\
&- (1-\exp(-x))\left(\int\Psi^\dagger\left(\left(\theta_2^{(i)}\right)'\right)O_1\mathcal{G}_0 X\left(\left(\theta_2^{(i)}\right)', \theta_f^{(i)}\right)d\left(\theta_2^{(i)}\right)'\right) \\
&\times\left(\mathcal{G}_0^{-1}\frac{(1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2)))(1+\bar{O}_{1,\infty})}{((1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty})))}\right)^*_{\theta_1^{(i)}} \\
&- \exp(-x)\left(\int\Psi^\dagger\left(\left(\theta_2^{(i)}\right)'\right)O_2\mathcal{G}_0 X\left(\left(\theta_2^{(i)}\right)', \theta_f^{(i)}\right)d\left(\theta_2^{(i)}\right)'\right) \\
&\left(\mathcal{G}_0^{-1}\frac{(1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2)))(1+\bar{O}_{1,2})}{((1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty})))}\right)^*_{\theta_1^{(i)}} \\
&\times\left(\frac{(1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2))}{(1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty}))}\right)\Psi\left(\theta_1^{(i)}\right)
\end{aligned}$$

that can be further simplified:

$$\begin{aligned}
\frac{\delta [\Delta\Omega] \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} &= \mathcal{G}_0^{-1} \frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{\left( 1 + \bar{O}_{1,\infty} \right) + \exp(-x) \left( -\bar{O}_{1,\infty} + y \left( 1 + \bar{O}_{1,2} \right) - x \left( 1 + \bar{O}_{1,\infty} \right) \right)} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\
&+ \exp \left( \int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( \left( \theta^{(i)} \right) d\theta^{(i)} \right) \right) \left( \Psi^\dagger \left( \theta_f^{(i)} \right) \mathcal{G}_0^{-1} \frac{1}{\left( 1 + \bar{O}_{1,\infty} \right) + \exp(-x) \left( -\bar{O}_{1,\infty} + y \left( 1 + \bar{O}_{1,2} \right) - x \left( 1 + \bar{O}_{1,\infty} \right) \right)} \right. \\
&- \left. \left( \Psi^\dagger \left( \theta_f^{(i)} \right) O_1 \frac{1}{\left( 1 + \bar{O}_{1,\infty} \right) + \exp(-x) \left( -\bar{O}_{1,\infty} + y \left( 1 + \bar{O}_{1,2} \right) - x \left( 1 + \bar{O}_{1,\infty} \right) \right)} \right) \right) \\
&\times \mathcal{G}_0^{-1} \frac{\left( 1 - \exp(-x) \right) \left( 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \right) \left( 1 + \bar{O}_{1,\infty} \right) \theta_1^{(i)}}{\left( \left( 1 + \bar{O}_{1,\infty} \right) + \exp(-x) \left( -\bar{O}_{1,\infty} + y \left( 1 + \bar{O}_{1,2} \right) - x \left( 1 + \bar{O}_{1,\infty} \right) \right) \right)} \\
&- \left( \Psi^\dagger \left( \theta_f^{(i)} \right) O_2 \frac{1}{\left( 1 + \bar{O}_{1,\infty} \right) + \exp(-x) \left( -\bar{O}_{1,\infty} + y \left( 1 + \bar{O}_{1,2} \right) - x \left( 1 + \bar{O}_{1,\infty} \right) \right)} \right) \\
&\times \mathcal{G}_0^{-1} \frac{\exp(-x) \left( 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \right) \left( 1 + \bar{O}_{1,2} \right) \theta_1^{(i)}}{\left( \left( 1 + \bar{O}_{1,\infty} \right) + \exp(-x) \left( -\bar{O}_{1,\infty} + y \left( 1 + \bar{O}_{1,2} \right) - x \left( 1 + \bar{O}_{1,\infty} \right) \right) \right)} \\
&\times \left( \frac{\left( 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \right)^2}{\left( 1 + \bar{O}_{1,\infty} \right) + \exp(-x) \left( -\bar{O}_{1,\infty} + y \left( 1 + \bar{O}_{1,2} \right) - x \left( 1 + \bar{O}_{1,\infty} \right) \right)} \right) \Psi \left( \theta_1^{(i)} \right)
\end{aligned}$$

### 3.1.3 Compact formulation

This previous expression for the two points correlation function can be rewritten more compactly:

$$\begin{aligned}
\frac{\delta [\Delta\Omega] \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta_f^{(i)} \right)} &= \mathcal{G}_0^{-1} \frac{F}{\Theta} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\
&+ \left[ \int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F}{\Theta} \Psi \right] \exp \left( \int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( \left( \theta^{(i)} \right) d\theta^{(i)} \right) \right) \left[ \int_{\theta_f^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{F}{\Theta} \right] \\
&- \left[ \int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} \left( 1 + \bar{O}_{1,\infty} \right) \Psi \right] \exp \left( \int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( \left( \theta^{(i)} \right) d\theta^{(i)} \right) \right) \\
&\times \left[ \int_{\theta_f^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{\left( 1 - \exp(-x) \right) F}{\Theta^2} \left( 1 + \bar{O}_{1,\infty} \right) \right] \\
&- \left[ \int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} \left( 1 + O_{1,2} \right) \Psi \right] \exp \left( \int_{\theta_f^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( \left( \theta^{(i)} \right) d\theta^{(i)} \right) \right) \left[ \int_{\theta_f^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{\exp(-x) F}{\Theta^2} \left( 1 + O_{1,2} \right) \right]
\end{aligned}$$

where:

$$\begin{aligned}
F &= 1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right) \\
\Theta &= \left( \left( 1 + \bar{O}_{1,\infty} \right) + \exp(-x) \left( -\bar{O}_{1,\infty} + y \left( 1 + \bar{O}_{1,2} \right) - x \left( 1 + \bar{O}_{1,\infty} \right) \right) \right)
\end{aligned} \tag{190}$$

In the local approximation for  $\theta_1^{(i)} \simeq \theta_f^{(i)}$ :

$$\begin{aligned} \frac{\delta[\Delta\Omega](\theta_1^{(i)})}{\delta\Psi(\theta_f^{(i)})} &= \mathcal{G}_0^{-1} \frac{F}{\Theta}(\theta_1^{(i)}, \theta_f^{(i)}) + \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F}{\Theta} \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{F}{\Theta} \right] \\ &- \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,\infty}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{(1 - \exp(-x)) F}{\Theta^2} (1 + \bar{O}_{1,\infty}) \right] \\ &- \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,2}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{\exp(-x) F}{\Theta^2} (1 + \bar{O}_{1,2}) \right] \end{aligned} \quad (191)$$

The contributions into brackets can be neglected in first approximation since they include a factor  $\exp(-x) \times \text{polynomial}(x, y)$ .

Actually decomposing:

$$\begin{aligned} &\left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,\infty}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{(1 - \exp(-x)) F}{\Theta^2} (1 + \bar{O}_{1,\infty}) \right] \\ &= \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,\infty}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{F}{\Theta^2} (1 + \bar{O}_{1,\infty}) \right] \\ &+ \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,\infty}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{(-\exp(-x)) F}{\Theta^2} (1 + \bar{O}_{1,\infty}) \right] \end{aligned} \quad (192)$$

we can regroup the second term of (191) and the first term of the right hand side of (192):

$$\left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F}{\Theta} \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{F}{\Theta} \right] - \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,\infty}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{F}{\Theta^2} (1 + \bar{O}_{1,\infty}) \right]$$

In average:

$$\frac{(1 + \bar{O}_{1,\infty})}{\Theta} = \left( \frac{x}{(x + \exp(-x)) (\langle 1 \rangle - x + (y^2 - x^2))} \right)$$

So that:

$$\begin{aligned} &\left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F}{\Theta} \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{F}{\Theta} \right] - \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,\infty}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{F}{\Theta^2} (1 + \bar{O}_{1,\infty}) \right] \\ &\simeq \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F}{\Theta} \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{F}{\Theta} \right] \\ &\times \left( 1 - \frac{x (1 + \bar{O}_{1,\infty}) (1 + \exp(-x) (1 - x + \frac{1}{2} (y^2 - x^2)))}{(x + \exp(-x)) (\langle 1 \rangle - x + (y^2 - x^2)) ((1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y (1 + \bar{O}_{1,2}) - x (1 + \bar{O}_{1,\infty})))} \right) \end{aligned}$$

In average this is equal to:

$$\simeq \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F}{\Theta} \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{F}{\Theta} \right] \left( 1 - \left( \frac{x}{(x + \exp(-x)) (1 - x + (y^2 - x^2))} \right)^2 \left( 1 + \exp(-x) \left( 1 - x + \frac{1}{2} (y^2 - x^2) \right) \right) \right)$$

which has the stated form. This corrective term can be given a more precise form:

$$\begin{aligned} &\left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F}{\Theta} \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \right] F \left( \frac{\alpha(x) (1 + \bar{O}_{1,\infty}) + \beta(x) (1 + y (1 + \bar{O}_{1,2}))}{(x + \exp(-x)) (\langle 1 \rangle - x + (y^2 - x^2))^3} \right) \\ &\simeq \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \right] F^2 \langle 1 \rangle^3 \left( \frac{\alpha(x) (1 + \bar{O}_{1,\infty}) + \beta(x) (1 + y (1 + \bar{O}_{1,2}))}{(x + \exp(-x)) (\langle 1 \rangle - x + (y^2 - x^2))^4} \right) \end{aligned}$$

$$\begin{aligned}\alpha(x) &= (x + \exp(-x) (\langle 1 \rangle - x + (y^2 - x^2))) (1 - (x + 1) \exp(-x)) - x \left( 1 + \exp(-x) \left( 1 - x + \frac{1}{2} (y^2 - x^2) \right) \right) \\ \beta(x) &= (x + \exp(-x) (\langle 1 \rangle - x + (y^2 - x^2))) \exp(-x)\end{aligned}$$

Moreover, gathering the second term of (192) and the last term of (191) yields in average:

$$\begin{aligned}& - \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,\infty}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{(-\exp(-x)) F}{\Theta^2} (1 + \bar{O}_{1,\infty}) \right] \\ & - \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,2}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{\exp(-x) F}{\Theta^2} (1 + \bar{O}_{1,2}) \right] \\ \simeq & - \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,\infty}) \Psi \right] \left( \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{(-\exp(-x)) F}{\Theta^2} (1 + \bar{O}_{1,\infty}) + \frac{y}{x} \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{\exp(-x) F}{\Theta^2} (1 + \bar{O}_{1,2}) \right) \\ = & - \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + \bar{O}_{1,\infty}) \Psi \right] \left( \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} \frac{\exp(-x) (y (1 + \bar{O}_{1,2}) - x (1 + \bar{O}_{1,\infty}))}{x \Theta^2} F \right) \\ \simeq & - \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \Psi \right] \left( \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} F^3 \langle 1 \rangle^3 \frac{\exp(-x) (y (1 + \bar{O}_{1,2}) - x (1 + \bar{O}_{1,\infty}))}{(x + \exp(-x) (\langle 1 \rangle - x + (y^2 - x^2)))^4} \right) \\ \simeq & - \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \Psi \right] \left( \int_{\theta_1^{(i)}} \Psi^\dagger \mathcal{G}_0^{-1} F^3 \langle 1 \rangle^3 \frac{\exp(-x) (y^2 - x^2)}{(x + \exp(-x) (\langle 1 \rangle - x + (y^2 - x^2)))^4} \right)\end{aligned}$$

which has again the expected form. Using that  $\Psi^\dagger$  is constant, we can write the corrective terms as:

$$F^2 (\Psi^\dagger \mathcal{G}_0^{-1} \Psi) \mathcal{G}_0^{-1} \left( \langle 1 \rangle^3 \frac{(\alpha(x) + Fx \exp(-x)) (1 + \bar{O}_{1,\infty}) + (\beta(x) - F \exp(-x)) y (1 + \bar{O}_{1,2}) + \beta(x)}{(x + \exp(-x) (\langle 1 \rangle - x + (y^2 - x^2)))^4} \right) \quad (193)$$

The two points correlation function (191) is thus:

$$\begin{aligned}& \mathcal{G}_0^{-1} \frac{F}{\Theta} (\theta_1^{(i)}, \theta_f^{(i)}) \\ & + F^2 (\Psi^\dagger \mathcal{G}_0^{-1} \Psi) \mathcal{G}_0^{-1} \left( \langle 1 \rangle^3 \frac{(\alpha(x) + Fx \exp(-x)) (1 + \bar{O}_{1,\infty}) + (\beta(x) - F \exp(-x)) y (1 + \bar{O}_{1,2}) + \beta(x)}{(x + \exp(-x) (\langle 1 \rangle - x + (y^2 - x^2)))^4} \right) \quad (194)\end{aligned}$$

The various terms arising in (194) can be found by using (161) which shows that for  $\theta_1^{(i)} \simeq \theta_f^{(i)}$ :

$$\begin{aligned}(1 + \bar{O}_{1,n}) &= \delta(\theta_f^{(i)} - \theta_{i'}^{(i)}) + \left( -\bar{\zeta}_n + \frac{\nabla_{i'}^{out}}{\Lambda_1} \bar{\Xi}_{1,n} (Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}) \right) \\ &\quad \times \exp \left( \bar{\Xi}_{1,n} (Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \right) \frac{\exp(-\Lambda_1 (\theta_f^{(i)} - \theta_{i'}^{(i)}))}{\Lambda}\end{aligned}$$

so that:

$$\mathcal{G}_0^{-1} (1 + \bar{O}_{1,n}) = \mathcal{G}_0^{-1} + \left( -\bar{\zeta}_n + \frac{\nabla_{i'}^{out}}{\Lambda_1} \bar{\Xi}_{1,n} (Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}) \right) \delta(\theta_f^{(i)} - \theta_{i'}^{(i)}) \quad (195)$$

and:

$$\mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) = \mathcal{G}_0^{-1} + \left( -\bar{\zeta} + \frac{\nabla_{i'}^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty} (Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}) \right) \delta(\theta_f^{(i)} - \theta_{i'}^{(i)}) \quad (196)$$



As a consequence, (194) rewrites:

$$\begin{aligned}
& \mathcal{G}_0^{-1} \frac{F}{\Theta} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\
& + F^2 \left( \Psi^\dagger \mathcal{G}_0^{-1} \Psi \right) \left( \frac{(\alpha(x) - Fx \exp(-x)) \left( -\bar{\zeta} + \frac{\nabla_{i'}^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right)}{(x + \exp(-x)) ((1) - x + (y^2 - x^2))^4} \right) \\
& - F^2 \left( \Psi^\dagger \mathcal{G}_0^{-1} \Psi \right) \left( \frac{(\beta(x) - F \exp(-x)) y \left( -\bar{\zeta}_2 + \frac{\nabla_{i'}^{out}}{\Lambda_1} \bar{\Xi}_{1,2} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) + \beta(x)}{(x + \exp(-x)) ((1) - x + (y^2 - x^2))^4} \right) \quad (197)
\end{aligned}$$

### 3.2 Correlation functions of order $(l, m)$ at different points without interaction

The expressions for correlation functions at  $m$  different points are directly given by:

$$\begin{aligned}
& \left[ \mathcal{G}_0^{-1} \frac{1 + \exp(-x) (-z + \frac{1}{2} (y^2 - x^2))}{(1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y (1 + \bar{O}_{1,2}) - x (1 + \bar{O}_{1,\infty}))} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \right. \\
& + \int d\theta_2^{(i)} \exp \left( \int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( \left( \theta^{(i)} \right) d\theta^{(i)} \right) \right) \left( \Psi^\dagger \left( \theta_2^{(i)} \right) X \left( \theta_2^{(i)}, \theta_f^{(i)} \right) \right) \\
& - (1 - \exp(-x)) \left( \int \Psi^\dagger \left( \theta_2^{(i)} \right) (O_1)_{(\theta_2^{(i)})', \theta_2^{(i)}} \mathcal{G}_0 X \left( \left( \theta_2^{(i)} \right)', \theta_f^{(i)} \right) \right) \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty})_{\theta_1^{(i)}} \\
& - \exp(-x) \left( \int \Psi^\dagger \left( \theta_2^{(i)} \right) (O_2)_{(\theta_2^{(i)})', \theta_2^{(i)}} \mathcal{G}_0 X \left( \left( \theta_2^{(i)} \right)', \theta_f^{(i)} \right) \right) \mathcal{G}_0^{-1} (1 + \bar{O}_{1,2})_{\theta_1^{(i)}} \\
& \left. \left( \frac{1 + \exp(-x) (-z + \frac{1}{2} (y^2 - x^2))}{(1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y (1 + \bar{O}_{1,2}) - x (1 + \bar{O}_{1,\infty}))} \right) \Psi \left( \theta_1^{(i)} \right) \right]^{-\otimes m}
\end{aligned}$$

Where the notation  $-\otimes m$  of an expression:

$$[X_i]^{-\otimes m}$$

represents the  $m$  tensor products:

$$\prod_{i=1}^m [X_i]^{-1}$$

For weak background fields this reduces to:

$$\left[ \mathcal{G}_0^{-1} \frac{1 + \exp(-x) (-z + \frac{1}{2} (y^2 - x^2))}{(1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y (1 + \bar{O}_{1,2}) - x (1 + \bar{O}_{1,\infty}))} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \right]^{-\otimes m}$$

whereas, for strong fields, we have:

$$\begin{aligned}
& \left( \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger(\theta_f^{(i)})} \right)^{-1} \tag{198} \\
&= \left[ \left( \mathcal{G}_0^{-1} \frac{1}{1+\bar{O}_{1,\infty}} \right)_{\theta_1^{(i)}, \theta_2^{(i)}} + \exp \left( \int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( (\theta^{(i)}) d\theta^{(i)} \right) \right) \left( \int \Psi^\dagger(\theta_2^{(i)}) X(\theta_2^{(i)}, \theta_f^{(i)}) d\theta_2^{(i)} \right. \right. \\
&\quad \left. \left. - \left( \int \Psi^\dagger(\theta_2^{(i)}) (O_1)_{(\theta_2^{(i)})', \theta_2^{(i)}} \mathcal{G}_0 X \left( (\theta_2^{(i)})', \theta_f^{(i)} \right) \right) \mathcal{G}_0^{-1} (1+\bar{O}_{1,\infty})_{\theta_1^{(i)}} \mathcal{G}_0^{-1} (1+\bar{O}_{1,\infty})_{\theta_1^{(i)}} \right. \right. \\
&\quad \left. \left. \times \left( \frac{(1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2)))}{(1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty}))} \right) \Psi(\theta_1^{(i)}) \right]^{-\otimes m}
\end{aligned}$$

In this expression, the individual elements interact directly with field and indirectly with other elements. Expression (198) can be written more explicitly, using (189). Actually, in strong background fields approximation, we have:

$$\begin{aligned}
O_1 &\simeq \frac{1}{(1+\bar{O}_{1,\infty})} \mathcal{G}_0^{-1} (1+\bar{O}_{1,\infty}) \\
O_2 &\simeq \frac{1}{(1+\bar{O}_{1,\infty})} \mathcal{G}_0^{-1} (1+\bar{O}_{1,2}) \\
X &\simeq \mathcal{G}_0^{-1} \left( \frac{1}{1+\bar{O}_{1,\infty}} \right)
\end{aligned}$$

and the  $m$ -th tensor power of (198) becomes:

$$\begin{aligned}
& \left( \left( \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger(\theta_2^{(i)})} \right)^{-1} \right)^{\otimes m} \delta_{l,m} = \left[ \left( \mathcal{G}_0^{-1} \frac{1}{1+\bar{O}_{1,\infty}} \right)_{\theta_1^{(i)}, \theta_2^{(i)}} + \exp \left( \int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( (\theta^{(i)}) d\theta^{(i)} \right) \right) \right] \tag{199} \\
&\times \left( \int_{\theta_2^{(i)}} \Psi^\dagger \left( (\theta_2^{(i)})' \right) \mathcal{G}_0^{-1} \frac{1}{(1+\bar{O}_{1,\infty})} d(\theta_2^{(i)})' - \int_{\theta_2^{(i)}} \Psi^\dagger \left( (\theta_2^{(i)})' \right) \frac{1}{(1+\bar{O}_{1,\infty})} \mathcal{G}_0^{-1} d(\theta_2^{(i)})' \right) \\
&\left( \mathcal{G}_0^{-1} \left( \frac{1}{1+\bar{O}_{1,\infty}} \right) \Psi \right) (\theta_1^{(i)}) \Big]^{-\otimes m} \delta_{l,m} \\
&\simeq \left[ \left( \mathcal{G}_0^{-1} \frac{1}{1+\bar{O}_{1,\infty}} \right)_{\theta_1^{(i)}, \theta_2^{(i)}} - \exp \left( \int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( (\theta^{(i)}) d\theta^{(i)} \right) \right) \right. \\
&\quad \left. \times \left( \int \Psi^\dagger \left( (\theta_2^{(i)})' \right) \frac{1}{(1+\bar{O}_{1,\infty})} \mathcal{G}_0^{-1} d(\theta_2^{(i)})' \right) \left( \mathcal{G}_0^{-1} \frac{1}{1+\bar{O}_{1,\infty}} \Psi \right) (\theta_1^{(i)}) \right]^{-\otimes m} \delta_{l,m}
\end{aligned}$$

given that  $\Psi^\dagger \left( (\theta_2^{(i)})' \right) \mathcal{G}_0^{-1} \simeq 0$ . Factoring by  $\mathcal{G}_0^{-1} \left( \frac{1}{1+\bar{O}_{1,\infty}} \right)$ , we can rewrite:

$$\begin{aligned}
& \left( \mathcal{G}_0^{-1} \left( \frac{1}{1+\bar{O}_{1,\infty}} \right) - \exp \left( \int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( (\theta^{(i)}) d\theta^{(i)} \right) \right) \right. \\
&\quad \left. \times \left( \int \Psi^\dagger \left( (\theta_2^{(i)})' \right) \frac{1}{(1+\bar{O}_{1,\infty})} \mathcal{G}_0^{-1} d(\theta_2^{(i)})' \right) \left( \mathcal{G}_0^{-1} \left( \frac{1}{1+\bar{O}_{1,\infty}} \right) \Psi \right) (\theta_1^{(i)}) \right)^{-1} \\
&= (1+\bar{O}_{1,\infty}) \mathcal{G}_0 \left( 1 - \exp \left( \int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( (\theta^{(i)}) d\theta^{(i)} \right) \right) \Psi(\theta_1^{(i)}) \left( \int \Psi^\dagger \left( (\theta_2^{(i)})' \right) \frac{1}{(1+\bar{O}_{1,\infty})} \mathcal{G}_0^{-1} d(\theta_2^{(i)})' \right) \right)^{-1}
\end{aligned}$$

and using the following expression for  $\mathcal{G}_0^{-1}$ :

$$\mathcal{G}_0^{-1} = -\nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1} (J(\theta), \theta, Z, \mathcal{G}_0) + X_0 \right)$$

we have, for  $\sigma_\theta^2 \ll 1$ :

$$\begin{aligned} \int \Psi^\dagger \left( (\theta_2^{(i)})' \right) \frac{1}{(1 + \bar{O}_{1,\infty})} \mathcal{G}_0^{-1} d(\theta_2^{(i)})' &\simeq -\Psi^\dagger \left( \theta_2^{(i)} \right) \frac{1}{(1 + \bar{O}_{1,\infty})} \nabla_\theta \omega^{-1} \left( J \left( \theta_2^{(i)} \right), \theta_2^{(i)}, Z, \mathcal{G}_0 \right) \\ &= -\Psi^\dagger \left( \theta_2^{(i)} \right) \nabla_\theta \omega^{-1} \left( J \left( \theta_2^{(i)} \right), \theta_2^{(i)}, Z, \mathcal{G}_0 \right) \end{aligned}$$

and the  $n$  points vertices for strong background field become:

$$\begin{aligned} &\left( \left( \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( \theta^{(i)} \right)} \right)^{-1} \right)^{\otimes m} \delta_{l,m} \tag{200} \\ &= \left( (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \left( 1 + \exp \left( \int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( (\theta^{(i)}) d\theta^{(i)} \right) \right) \Psi \left( \theta_1^{(i)} \right) \Psi^\dagger \left( \theta_2^{(i)} \right) \nabla_\theta \omega^{-1} \left( J \left( \theta_2^{(i)} \right), \theta_2^{(i)}, Z, \mathcal{G}_0 \right) \right)^{-1} \right)^{\otimes m} \\ &\simeq \left( (1 + \bar{O}_{1,\infty}) \mathcal{G} \left( 1 + \exp \left( \int_{\theta_2^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( (\theta^{(i)}) d\theta^{(i)} \right) \right) \Psi \left( \theta_1^{(i)} \right) \Psi^\dagger \left( \theta_2^{(i)} \right) \nabla_\theta G \left( J \left( \theta_2^{(i)} \right), \theta_2^{(i)}, Z, \mathcal{G}_0 \right) \right)^{-1} \right)^{\otimes m} \end{aligned}$$

### 3.3 Interaction Corrections to Correlation functions of order $(l, m)$ at different points

#### 3.3.1 General set up

To find the correlation functions including the interaction corrections, we first define the compact notation:

$$\begin{aligned} \frac{\delta^{l,m-1} \Delta \Omega \left( \theta^{(i_1)} \right)}{\delta \Psi^\dagger \left( \theta^{(i_2)} \right) \dots \delta \Psi^\dagger \left( \theta^{(i_m)} \right) \delta \Psi \left( \theta^{(i_{m+k})\dagger} \right) \dots \delta \Psi \left( \theta^{(i_{m+l})\dagger} \right)} &= \frac{\delta^{l,m-1} \Delta \Omega \left( \theta^{(i_1)} \right)}{\delta \left[ \Psi \Psi^\dagger \left( \left( \theta^{(i)^\dagger} \right), \left( \theta^{(i)} \right) \right) \right]} \\ &\equiv \frac{\delta^{l,m} \Delta \Omega \left( \theta^{(i_1)} \right)}{\delta \left[ \Psi \Psi^\dagger \right]} \end{aligned}$$

the number of variables is implicitly  $(l, m-1)$  in  $\left( \left( \theta^{(i)^\dagger} \right), \left( \theta^{(i)} \right) \right)$ . The notation  $\Psi \left( \theta^{(i_k)^\dagger} \right)$  and  $\Psi^\dagger \left( \theta^{(i_k)} \right)$  represent  $\Psi \left( \theta^{(i_k)^\dagger}, Z_{i_k} \right)$  and  $\Psi^\dagger \left( \theta^{(i_k)}, Z_{i_k} \right)$ .

Similarly:

$$\begin{aligned} \frac{\delta^{l,m-1} \Delta \Omega^\dagger \left( \theta^{(i_1)} \right)}{\delta \Psi^\dagger \left( \theta^{(i_2)} \right) \dots \delta \Psi^\dagger \left( \theta^{(i_m)} \right) \delta \Psi \left( \theta^{(i_{m+k})\dagger} \right) \dots \delta \Psi \left( \theta^{(i_{m+l})\dagger} \right)} &= \frac{\delta^{l,m-1} \Delta \Omega^\dagger \left( \theta^{(i_1)} \right)}{\delta \left[ \Psi \Psi^\dagger \left( \left( \theta^{(i)^\dagger} \right), \left( \theta^{(i)} \right) \right) \right]} \\ &\equiv \frac{\delta^{l,m} \Delta \Omega^\dagger \left( \theta^{(i_1)} \right)}{\delta \left[ \Psi \Psi^\dagger \right]} \end{aligned}$$

We will show recursively that  $\frac{\delta^{l,m} [\Delta \Omega] \left( \theta_1^{(i)} \right)}{\delta \left[ \Psi \Psi^\dagger \left( \left( \theta^{(i)} \right), \left( \theta^{(i)^\dagger} \right) \right) \right]}$  satisfies the relation (the index  $m-1$  is shifted to  $m$  for the sake of simplicity):

$$\frac{\delta^{l,m} [\Delta \Omega] \left( \theta^{(i_1)} \right)}{\delta \left[ \Psi \Psi^\dagger \left( \left( \theta^{(i)} \right), \left( \theta^{(i)^\dagger} \right) \right) \right]} = \int K^{l,m} \left( \left( \theta^{(i_1)} \right), \theta_1^{(i)} \right) \frac{\delta^{l,m} [\Delta \Omega] \left( \theta_1^{(i)} \right)}{\delta \left[ \Psi \Psi^\dagger \right] \left( \left( \theta^{(i)} \right), \left( \theta^{(i)^\dagger} \right) \right)} d\theta_1^{(i)} + \left[ X^{l,m} \left( \theta^{(i_1)} \right), \left( \left( \theta^{(i)} \right), \left( \theta^{(i)^\dagger} \right) \right) \right] \tag{201}$$

where the matrix  $K^{l,m}(\theta^{(i_1)}, \underline{\theta}_1^{(i)})$  and the vector  $[X^{l,m}(\theta^{(i_1)}, ((\theta^{(i)}), (\theta^{(i)\dagger})))]$  have to be determined. The derivative of this equation with respect to  $\Psi$  allows to find recursively the matrix  $K^{l,m}(\theta^{(i_1)}, \underline{\theta}_1^{(i)})$  and the vector  $[X^{l,m}(\theta^{(i_1)}, ((\theta^{(i)}), (\theta^{(i)\dagger})))]$ :

$$\begin{aligned} \frac{\delta^{l+1,m}[\Delta\Omega](\theta^{(i_1)})}{\delta[\Psi\Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} &= \int K^{l,m}(\theta^{(i_1)}, \underline{\theta}_1^{(i)}) \frac{\delta^{l+1,m}[\Delta\Omega](\underline{\theta}_1^{(i)})}{\delta[\Psi\Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} d\underline{\theta}_1^{(i)} \\ &+ \int \frac{\delta K^{l,m}(\theta^{(i_1)}, \underline{\theta}_1^{(i)})}{\delta\Psi(\theta^{(i_{l+1}\dagger})\dagger)} \frac{\delta^{l,m}[\Delta\Omega](\underline{\theta}_1^{(i)})}{\delta[\Psi\Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} d\underline{\theta}_1^{(i)} + \frac{\delta X^{l,m}(\theta^{(i_1)}, ((\theta^{(i)}), (\theta^{(i)\dagger})))}{\delta\Psi(\theta^{(i_{l+1}\dagger})\dagger)} \end{aligned} \quad (202)$$

As a consequence,  $K^{l+1,m}(\theta^{(i_1)}, \underline{\theta}_1^{(i)}) = K^{l,m}(\theta^{(i_1)}, \underline{\theta}_1^{(i)})$ . Using the results for the two points correlation function, we have:

$$K^{l,m}(\theta^{(i_1)}, \underline{\theta}_1^{(i)}) = K(\theta^{(i_1)}, \underline{\theta}_1^{(i)}) \quad (203)$$

The recursive relation for  $[X^{l,m}(\theta^{(i_1)}, ((\theta^{(i)}), (\theta^{(i)\dagger})))]$  is also obtained from (202):

$$\begin{aligned} &X^{l+1,m}(\theta^{(i_1)}, ((\theta^{(i)}), (\theta^{(i)\dagger}))) \\ &= \int \frac{\delta K(\theta^{(i_1)}, \underline{\theta}_1^{(i)})}{\delta\Psi(\theta^{(i_{l+1}\dagger})\dagger)} \frac{\delta^{l,m}[\Delta\Omega](\underline{\theta}_1^{(i)})}{\delta[\Psi\Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} d\underline{\theta}_1^{(i)} + \frac{\delta X^{l,m}(\theta^{(i_1)}, ((\theta^{(i)}), (\theta^{(i)\dagger})))}{\delta\Psi(\theta^{(i_{l+1}\dagger})\dagger)} \\ &= \int \frac{\delta K(\theta^{(i_1)}, \underline{\theta}_1^{(i)})}{\delta\Psi(\theta^{(i_{l+1}\dagger})\dagger)} \frac{\delta^{l,m}[\Delta\Omega](\underline{\theta}_1^{(i)})}{\delta[\Psi\Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} d\underline{\theta}_1^{(i)} \\ &+ \frac{\delta}{\delta\Psi(\theta^{(i_{l+1}\dagger})\dagger)} \left( \frac{\delta X^{l-1,m}(\theta^{(i_1)}, ((\theta^{(i)}), (\theta^{(i)\dagger})))}{\delta\Psi(\theta^{(i)}\dagger)} + \int \frac{\delta K(\theta^{(i_1)}, \underline{\theta}_1^{(i)})}{\delta\Psi(\theta^{(i)}\dagger)} \frac{\delta^{l-1,m}[\Delta\Omega](\underline{\theta}_1^{(i)})}{\delta[\Psi\Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} d\underline{\theta}_1^{(i)} \right) \\ &= 2 \int \frac{\delta K(\theta^{(i_1)}, \underline{\theta}_1^{(i)})}{\delta\Psi(\theta^{(i_{l+1}\dagger})\dagger)} \frac{\delta^{l,m}[\Delta\Omega](\underline{\theta}_1^{(i)})}{\delta[\Psi\Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} d\underline{\theta}_1^{(i)} + \int \frac{\delta^2 K(\theta^{(i_1)}, \underline{\theta}_1^{(i)})}{\delta\Psi(\theta^{(i_{l+1}\dagger})\dagger) \delta\Psi(\theta^{(i)}\dagger)} \frac{\delta^{l-1,m}[\Delta\Omega](\underline{\theta}_1^{(i)})}{\delta[\Psi\Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} d\underline{\theta}_1^{(i)} \\ &+ \frac{\delta}{\delta\Psi(\theta^{(i_{l+1}\dagger})\dagger)} \frac{\delta}{\delta\Psi(\theta^{(i)}\dagger)} X^{l-1,m}(\theta^{(i_1)}, ((\theta^{(i)}), (\theta^{(i)\dagger}))) \\ &= \sum_{r \leq l} C_{l+1}^{r+1} \left( \int \left( \left( \frac{\delta}{\delta\Psi} \right)^r \frac{\delta K(\theta^{(i_1)}, \underline{\theta}_1^{(i)})}{\delta\Psi(\theta^{(i_{l+1}\dagger})\dagger)} \right) \frac{\delta^{l-r,m}[\Delta\Omega](\underline{\theta}_1^{(i)})}{\delta[\Psi\Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} d\underline{\theta}_1^{(i)} \right) + \left( \frac{\delta}{\delta\Psi} \right)^l X^{1,m}(\theta^{(i_1)}, ((\theta^{(i)}), (\theta^{(i)\dagger}))) \end{aligned}$$

And we obtain recursively:

$$\begin{aligned} &X^{l+1,m}(\theta^{(i_1)}, ((\theta^{(i)}), (\theta^{(i)\dagger}))) \\ &= \sum_{r \leq l} C_{l+1}^{r+1} \left( \int \left( \left( \frac{\delta}{\delta\Psi} \right)^r \frac{\delta K(\theta^{(i_1)}, \underline{\theta}_1^{(i)})}{\delta\Psi(\theta^{(i_{l+1}\dagger})\dagger)} \right) \frac{\delta^{l-r,m}[\Delta\Omega](\underline{\theta}_1^{(i)})}{\delta[\Psi\Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} d\underline{\theta}_1^{(i)} \right) + \left( \frac{\delta}{\delta\Psi} \right)^l X^{1,m}(\theta^{(i_1)}, ((\theta^{(i)}), (\theta^{(i)\dagger}))) \end{aligned}$$

Similarly, differentiating with respect to  $\Psi^\dagger(\theta^{(i)\dagger})$  yields:

$$\begin{aligned}
& X^{l+1,m+1} \left( \theta^{(i_1)}, \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right) \right) \tag{204} \\
&= \sum_{1 \leq r \leq l} C_l^r \left( \int \left( \left( \frac{\delta}{\delta \Psi} \right)^{r-1} \frac{\delta K \left( \theta^{(i_1)}, \underline{\theta}_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right) \frac{\delta^{l-r,m} [\Delta \Omega] \left( \underline{\theta}_1^{(i)} \right)}{\delta [\Psi \Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} d\underline{\theta}_1^{(i)} \right) \\
&\quad + \left( \frac{\delta}{\delta \Psi} \right)^l X^{1,m} \left( \theta^{(i_1)}, \left( (\theta^{(i)}), \theta^{(i_{m+1})\dagger} \right) \right) \\
&= \sum_{1 \leq r \leq l} C_l^r \left( \int \left( \left( \frac{\delta}{\delta \Psi} \right)^{r-1} \frac{\delta K \left( \theta^{(i_1)}, \underline{\theta}_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right) \frac{\delta^{l-r,m} [\Delta \Omega] \left( \underline{\theta}_1^{(i)} \right)}{\delta [\Psi \Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} d\underline{\theta}_1^{(i)} \right) \\
&\quad + \left( \frac{\delta}{\delta \Psi} \right)^l \sum_{1 \leq r' \leq m} C_m^{r'} \left( \int \left( \left( \frac{\delta}{\delta \Psi^\dagger} \right)^{r'-1} \frac{\delta K \left( \theta^{(i_1)}, \underline{\theta}_1^{(i)} \right)}{\delta \Psi^\dagger \left( \theta^{(i)} \right)} \right) \frac{\delta^{0,m-r'} [\Delta \Omega] \left( \underline{\theta}_1^{(i)} \right)}{\delta [\Psi \Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} d\underline{\theta}_1^{(i)} \right) \\
&\quad + \left( \frac{\delta}{\delta \Psi} \right)^l \left( \frac{\delta}{\delta \Psi^\dagger} \right)^m X^{1,1} \left( \theta^{(i_1)}, \left( \theta^{(i_{m+1})\dagger} \right) \right) \\
&= \sum_{\substack{0 \leq r \leq l, 0 \leq r' \leq m \\ 1 \leq r+r' \leq m}} C_l^r C_m^{r'} \left( \int \frac{\delta^{r,r'} K \left( \theta^{(i_1)}, \underline{\theta}_1^{(i)} \right)}{\delta [\Psi \Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \frac{\delta^{l-r,m-r'} [\Delta \Omega] \left( \underline{\theta}_1^{(i)} \right)}{\delta [\Psi \Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} d\underline{\theta}_1^{(i)} \right) \\
&\quad + \left( \frac{\delta}{\delta \Psi} \right)^l \left( \frac{\delta}{\delta \Psi^\dagger} \right)^m X^{1,1} \left( \theta^{(i_1)}, \left( \theta^{(i_{m+1})\dagger} \right) \right)
\end{aligned}$$

Using (203) and (201), we can write that:

$$\frac{\delta^{l-r,m-r'} [\Delta \Omega] \left( \underline{\theta}_1^{(i)} \right)}{\delta [\Psi \Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} = M * X^{l-r,m-r'} \left( \underline{\theta}_1^{(i)}, \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right) \right) \tag{205}$$

where  $M$  is the matrix defined in (188):

$$\begin{aligned}
& M \left( \theta^{(i_1)}, \underline{\theta}_1^{(i)} \right) \tag{206} \\
&= \delta \left( \theta^{(i_1)} - \underline{\theta}_1^{(i)} \right) + \exp \left( \int_{\underline{\theta}_1^{(i)}}^{\theta^{(i_1)}} \bar{N} \left( (\theta^{(i)}) d\theta^{(i)} \right) \right) \left( \Delta \Omega \left( (\theta^{(i_1)}) \right) \Psi^\dagger \left( \underline{\theta}_1^{(i)} \right) \right. \\
&\quad \left. - \left( A \left( \theta^{(i_1)} \right) \int_{\underline{\theta}_1^{(i)}} \Delta \Omega^\dagger \left( \left( \underline{\theta}_1^{(i)} \right)' \right) (1 + \bar{O}_{1,\infty})_{(\underline{\theta}_1^{(i)})', \underline{\theta}_1^{(i)}} \mathcal{G} + B \left( \theta^{(i_1)} \right) \int_{\underline{\theta}_1^{(i)}} \Delta \Omega^\dagger \left( \left( \underline{\theta}_1^{(i)} \right)' \right) (1 + \bar{O}_{1,2})_{(\underline{\theta}_1^{(i)})', \underline{\theta}_1^{(i)}} \mathcal{G} \right) \right)
\end{aligned}$$

Expression (204) writes:

$$\begin{aligned}
& X^{l+1,m+1} \left( \theta^{(i_1)}, \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right) \right) \\
&= \left( \frac{\delta}{\delta \Psi} \right)^l \left( \frac{\delta}{\delta \Psi^\dagger} \right)^m X^{1,1} \left( \theta^{(i_1)}, \left( \theta^{(i_{m+1})\dagger} \right) \right) \\
&\quad + \sum_{\substack{r_1 + \dots + r_l = l \\ s_1 + \dots + s_m = m \\ 0 \leq r_i, 0 \leq s_i, 1 \leq r_i + s_i}} \frac{l!}{r_1! \dots r_p!} \frac{m!}{s_1! \dots s_{p'}!} \\
&\quad \times \left[ \frac{\delta^{r_1, s_1} K}{\delta [\Psi \Psi^\dagger]} \right] * \dots * \left[ M * \frac{\delta^{r_p, s_{p'}} K}{\delta [\Psi \Psi^\dagger]} \right] * X^{1,1} \left( \theta^{(i_1)}, \left( \theta^{(i_{m+1})\dagger} \right) \right)
\end{aligned}$$

and using again (205), we obtain the  $(l+1, m+1)$  correlations:

$$\begin{aligned}
& \frac{\delta^{l+1, m+1} [\Delta\Omega] \left( \underline{\theta}_1^{(i)} \right)}{\delta [\Psi\Psi^\dagger] \left( \left( \theta^{(i)} \right), \left( \theta^{(i)\dagger} \right) \right)} \\
= & M * \left( \frac{\delta}{\delta\Psi} \right)^l \left( \frac{\delta}{\delta\Psi^\dagger} \right)^m X^{1,1} \left( \theta^{(i_1)}, \left( \theta^{(i_{m+1})\dagger} \right) \right) \\
& + M * \sum_{\substack{0 \leq r \leq l, 0 \leq r' \leq m \\ 1 \leq r+r'}} C_l^r C_m^{r'} \left( \int \frac{\delta^{r, r'} K \left( \theta_1^{(i)}, \underline{\theta}_1^{(i)} \right)}{\delta [\Psi\Psi^\dagger] \left( \left( \theta^{(i)} \right), \left( \theta^{(i)\dagger} \right) \right)} \frac{\delta^{l-r, m-r'} [\Delta\Omega] \left( \underline{\theta}_1^{(i)} \right)}{\delta [\Psi\Psi^\dagger] \left( \left( \theta^{(i)} \right), \left( \theta^{(i)\dagger} \right) \right)} d\underline{\theta}_1^{(i)} \right) \\
= & M * \left( \frac{\delta}{\delta\Psi} \right)^l \left( \frac{\delta}{\delta\Psi^\dagger} \right)^m X^{1,1} \left( \theta^{(i_1)}, \left( \theta^{(i_{m+1})\dagger} \right) \right) \\
& + M * \sum_{\substack{r_1 + \dots + r_l = l \\ s_1 + \dots + s_m = m \\ 0 \leq r_i, 0 \leq s_i, 1 \leq r_i + s_i}} \frac{l!}{r_1! \dots r_p!} \frac{m!}{s_1! \dots s_{p'}!} \\
& \times \left[ \frac{\delta^{r_1, s_1} K}{\delta [\Psi\Psi^\dagger]} \right] * \dots * \left[ M * \frac{\delta^{r_p, s_{p'}} K}{\delta [\Psi\Psi^\dagger]} \right] * X^{1,1} \left( \theta^{(i_1)}, \left( \theta^{(i_{m+1})\dagger} \right) \right)
\end{aligned} \tag{207}$$

For the sequel, we will rewrite the kernel  $K \left( \theta_1^{(i)}, \underline{\theta}_1^{(i)} \right)$  in a matricial form. We use that:

$$\begin{aligned}
A' \left( \theta_1^{(i)}, \left( \theta_1^{(i)} \right)' \right) & \simeq \mathcal{G}^{-1} \frac{(1 - \exp(-x)) (1 + \bar{O}_{1, \infty})_{\theta_1^{(i)}, \left( \theta_1^{(i)} \right)'}}{(1 + \bar{O}_{1, \infty}) + \exp(-x) (-\bar{O}_{1, \infty} + y(1 + \bar{O}_{1, 2}) - x(1 + \bar{O}_{1, \infty}))} \\
B' \left( \theta_1^{(i)}, \left( \theta_1^{(i)} \right)' \right) & \simeq \mathcal{G}^{-1} \frac{(1 + \bar{O}_{1, 2})_{\theta_1^{(i)}, \left( \theta_1^{(i)} \right)' } \exp(-x)}{(1 + \bar{O}_{1, \infty}) + \exp(-x) (-\bar{O}_{1, \infty} + y(1 + \bar{O}_{1, 2}) - x(1 + \bar{O}_{1, \infty}))}
\end{aligned}$$

and define the matrices:

$$\begin{aligned}
[\Delta\Omega]^\dagger & = (\Delta\Omega^\dagger, \Delta\Omega), \quad [A \left( \theta_1^{(i)} \right)] = \begin{pmatrix} A \left( \theta_1^{(i)} \right) \\ A^\dagger \left( \theta_1^{(i)} \right) \end{pmatrix}, \quad [B \left( \theta_1^{(i)} \right)] = \begin{pmatrix} B \left( \theta_1^{(i)} \right) \\ B^\dagger \left( \theta_1^{(i)} \right) \end{pmatrix} \\
\left[ (1 + \bar{O}_{1, \infty})_{\left( \theta_2^{(i)} \right)', \theta_2^{(i)}} \right] & = \begin{pmatrix} (1 + \bar{O}_{1, \infty})_{\left( \theta_2^{(i)} \right)', \theta_2^{(i)}} & 0 \\ 0 & (1 + \bar{O}_{1, \infty})_{\theta_2^{(i)}, \left( \theta_2^{(i)} \right)'} \end{pmatrix} \\
\left[ (1 + \bar{O}_{1, 2})_{\left( \theta_2^{(i)} \right)', \theta_2^{(i)}} \right] & = \begin{pmatrix} (1 + \bar{O}_{1, 2})_{\left( \theta_2^{(i)} \right)', \theta_2^{(i)}} & 0 \\ 0 & (1 + \bar{O}_{1, 2})_{\theta_2^{(i)}, \left( \theta_2^{(i)} \right)'} \end{pmatrix} \\
\left[ A' \left( \theta_1^{(i)}, \left( \theta_1^{(i)} \right)' \right) \right] & = \begin{pmatrix} A' \left( \theta_1^{(i)}, \left( \theta_1^{(i)} \right)' \right) & 0 \\ 0 & A' \left( \left( \theta_1^{(i)} \right)', \theta_1^{(i)} \right) \end{pmatrix} \\
\left[ B' \left( \theta_1^{(i)}, \left( \theta_1^{(i)} \right)' \right) \right] & = \begin{pmatrix} B' \left( \theta_1^{(i)}, \left( \theta_1^{(i)} \right)' \right) & 0 \\ 0 & B' \left( \left( \theta_1^{(i)} \right)', \theta_1^{(i)} \right) \end{pmatrix}
\end{aligned}$$

to write the kernel  $K(\theta_1^{(i)}, \theta_2^{(i)})$  (for two arbitrary times  $(\theta_1^{(i)}, \theta_2^{(i)})$  and two associated points  $Z_{i_1} Z_{i_2}$ ) as:

$$\begin{aligned}
K(\theta_1^{(i)}, \theta_2^{(i)}) &= \int \left( - \left( [A(\theta_1^{(i)})] [\Delta\Omega]^\dagger \left( (\theta_2^{(i)})' \right) \left[ (1 + \bar{O}_{1,\infty})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right. \right. \\
&\quad \left. \left. + [B(\theta_1^{(i)})] [\Delta\Omega]^\dagger \left( (\theta_2^{(i)})' \right) \left[ (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right) \mathcal{G} \right. \\
&\quad \left. + [\Delta\Omega] (\theta_1^{(i)}) [\Psi]^\dagger (\theta_2^{(i)}) \delta \left( (\theta_2^{(i)})' - \theta_2^{(i)} \right) \right) d(\theta_2^{(i)})' \\
&= \left( - \int \left( [A'(\theta_1^{(i)}, (\theta_1^{(i)})')] [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Delta\Omega]^\dagger \left( (\theta_2^{(i)})' \right) \left[ (1 + \bar{O}_{1,\infty})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right. \right. \\
&\quad \left. \left. + [B'(\theta_1^{(i)}, (\theta_1^{(i)})')] [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Delta\Omega]^\dagger \left( (\theta_2^{(i)})' \right) \left[ (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right) \mathcal{G} d(\theta_2^{(i)})' \right. \\
&\quad \left. + [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Psi]^\dagger (\theta_2^{(i)}) \delta \left( (\theta_1^{(i)})' - \theta_1^{(i)} \right) \right) d(\theta_1^{(i)})'
\end{aligned} \tag{208}$$

so that its successive derivatives become:

$$\begin{aligned}
\frac{\delta^{r,r'} K(\theta_1^{(i)}, \theta_2^{(i)})}{\delta [\Psi\Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} &= [\Psi]^\dagger (\theta_2^{(i)}) \frac{\delta^{r,r'} [\Delta\Omega] (\theta_1^{(i)})}{\delta [\Psi\Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} + \delta (\theta^{(i)\dagger} - \theta_2^{(i)}) \frac{\delta^{r,r'-1} [\Delta\Omega] (\theta_1^{(i)})}{\delta [\Psi\Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \\
&\quad - \sum \int \left( \frac{\delta^{t,t'} [A'(\theta_1^{(i)}, (\theta_1^{(i)})')] [\Delta\Omega] \left( (\theta_1^{(i)})' \right)}{\delta [\Psi\Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \frac{\delta^{r-s-t, r'-s'-t'} [\Delta\Omega] \left( (\theta_1^{(i)})' \right)}{\delta [\Psi\Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \right. \\
&\quad \times \frac{\delta^{s,s'} [\Delta\Omega]^\dagger \left( (\theta_2^{(i)})' \right)}{\delta [\Psi\Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \left[ (1 + \bar{O}_{1,\infty})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \\
&\quad \left. + \frac{\delta^{t,t'} [B'(\theta_1^{(i)}, (\theta_1^{(i)})')] [\Delta\Omega] \left( (\theta_1^{(i)})' \right)}{\delta [\Psi\Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \frac{\delta^{r-s-t, r'-s'-t'} [\Delta\Omega] \left( (\theta_1^{(i)})' \right)}{\delta [\Psi\Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \right. \\
&\quad \left. \frac{\delta^{s,s'} [\Delta\Omega]^\dagger \left( (\theta_2^{(i)})' \right)}{\delta [\Psi\Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \left[ (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right) \mathcal{G} d(\theta_2^{(i)})' d(\theta_1^{(i)})'
\end{aligned}$$

### 3.3.2 Strong Background field limit

All the derivatives  $\frac{\delta^{r_1, s_1} K(\theta_1^{(i)}, \theta_1^{(i)})}{\delta [\Psi\Psi^\dagger]}$  are proportional to  $\exp(-x)$ . For background fields of magnitude greater than 1, the dominant term is given by the term:

$$M * \left( \frac{\delta}{\delta\Psi} \right)^l \left( \frac{\delta}{\delta\Psi^\dagger} \right)^m X^{1,1}(\theta^{(i_1)}, (\theta^{(i_{m+1})\dagger})) + \left[ M * \frac{\delta^{l,m} K}{\delta [\Psi\Psi^\dagger]} \right] X^{1,1}(\theta^{(i_1)}, (\theta^{(i_{m+1})\dagger})) \tag{209}$$

with:

$$\begin{aligned}
X^{1,1} \left( \theta^{(i_1)}, \left( \theta^{(i_{m+1})\dagger} \right) \right) &= \left( \mathcal{G}^{-1} \left( \frac{1 + \langle (1 + \bar{O}_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + \bar{O}_{1,\infty})^{(n)} \rangle_{n-1}}{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + \bar{O}_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + \bar{O}_{1,\infty})^{(n)} \rangle_n} \right)^{-1} \left( \theta^{(i_1)}, \left( \theta^{(i_{m+1})\dagger} \right) \right) \right) \\
&\simeq \left( \mathcal{G}^{-1} \left( \frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{(1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right)} \right) \left( \theta^{(i_1)}, \left( \theta^{(i_{m+1})\dagger} \right) \right) \right)
\end{aligned}$$

We can skip the variables  $\left( \theta^{(i_1)}, \left( \theta^{(i_{m+1})\dagger} \right) \right)$  since  $X^{1,1} \left( \theta^{(i_1)}, \left( \theta^{(i_{m+1})\dagger} \right) \right)$  is the kernel of an operator  $X^{1,1}$ .

To compute  $\left( \frac{\delta}{\delta \Psi} \right)^l \left( \frac{\delta}{\delta \Psi^\dagger} \right)^m X^{1,1}$  in (209), we write the first order expansion of  $X^{1,1}$  in  $\exp(-x)$  :

$$\begin{aligned}
&\frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{(1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right)} \\
&\simeq \frac{1}{\bar{O}_{1,\infty} + 1} - \exp(-x) \left( \frac{1}{\bar{O}_{1,\infty} + 1} \left( \frac{1}{2} x^2 - \frac{1}{2} y^2 + z \right) - \frac{1}{(\bar{O}_{1,\infty} + 1)^2} \left( x(\bar{O}_{1,\infty} + 1) - y(\bar{O}_{1,2} + 1) + \bar{O}_{1,\infty} \right) \right)
\end{aligned}$$

so that:

$$\begin{aligned}
&\left( \frac{\delta}{\delta \Psi} \right)^l \left( \frac{\delta}{\delta \Psi^\dagger} \right)^m X^{1,1} \\
&\simeq \mathcal{G}_0^{-1} \left( \frac{\delta}{\delta \Psi} \right)^l \left( \frac{\delta}{\delta \Psi^\dagger} \right)^m \left( -\exp(-x) \left( \frac{1}{\bar{O}_{1,\infty} + 1} \left( \frac{1}{2} x^2 - \frac{1}{2} y^2 + z \right) - \frac{x(\bar{O}_{1,\infty} + 1) - y(\bar{O}_{1,2} + 1) + \bar{O}_{1,\infty}}{(\bar{O}_{1,\infty} + 1)^2} \right) \right)
\end{aligned}$$

The dominant term in field is then:

$$\begin{aligned}
&\left( \frac{\delta}{\delta \Psi} \right)^l \left( \frac{\delta}{\delta \Psi^\dagger} \right)^m X^{1,1} \\
&\simeq \left( - \left( \frac{\delta}{\delta \Psi} \right)^l \left( \frac{\delta}{\delta \Psi^\dagger} \right)^m \exp(-x) \right) \mathcal{G}_0^{-1} \\
&\quad \times \left( \left( \frac{1}{2} x^2 - \frac{1}{2} y^2 + z \right) (\bar{O}_{1,\infty} + 1)^{-1} - \left( x(\bar{O}_{1,\infty} + 1) - y(\bar{O}_{1,2} + 1) + \bar{O}_{1,\infty} \right) (\bar{O}_{1,\infty} + 1)^{-2} \right)
\end{aligned}$$

Using that:

$$\begin{aligned}
&\left( \frac{\delta}{\delta \Psi} \right)^r \left( \frac{\delta}{\delta \Psi^\dagger} \right)^s \exp(-x) \\
&\simeq (-1)^r \left( \frac{\delta}{\delta \Psi^\dagger} \right)^s \left( \left( \int \Delta \Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^r \exp(-x) \right) \\
&\simeq (-1)^{r+s} \sum_{\sigma} \left( \int \Delta \Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^r \left( \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( \theta^{(i)} \right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta \Omega \right)^s \exp(-x) \\
&\quad + \text{terms with powers of } \Psi \text{ lowered by 2}
\end{aligned}$$



and we obtain that:

$$\begin{aligned}
& \left( \frac{\delta}{\delta\Psi} \right)^l \left( \frac{\delta}{\delta\Psi^\dagger} \right)^m X^{1,1} \tag{210} \\
& \simeq (-1)^{l+m+1} \sum_{\sigma} \mathcal{G}_0^{-1} \left( \int \Delta\Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^l \left( \int \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger(\theta^{(i)})} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta\Omega \right)^m \exp(-x) \\
& \quad \times \left( \left( \frac{1}{2}x^2 - \frac{1}{2}y^2 + z \right) (1 + \bar{O}_{1,\infty})^{-1} - (x(1 + \bar{O}_{1,\infty}) - y(\bar{O}_{1,2} + 1) + \bar{O}_{1,\infty})(1 + \bar{O}_{1,\infty})^{-2} \right) \\
& \simeq (-1)^{l+m} \mathcal{G}_0^{-1} \left( \int \Delta\Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^l \left( \int \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger(\theta^{(i)})} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta\Omega \right)^m \exp(-x) \\
& \quad \times \left( \frac{1}{2}y^2 - \frac{1}{2}x^2 - z \right) (1 + \bar{O}_{1,\infty})^{-1}
\end{aligned}$$

We also rewrite  $K(\theta_1^{(i)}, \theta_2^{(i)})$  as:

$$\begin{aligned}
& \int [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Psi]^\dagger(\theta_2^{(i)}) \delta \left( (\theta_1^{(i)})' - \theta_1^{(i)} \right) d(\theta_1^{(i)})' \\
& - \left( \int \left[ A'(\theta_1^{(i)}, (\theta_1^{(i)})') \right] [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Psi]^\dagger \left( (\theta_2^{(i)})' \right) \right. \\
& \quad \times \left. \left[ \left( \frac{(1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2)))}{(1 + \bar{O}_{1,\infty}) + \exp(-x)(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))} \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) \right)_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right. \\
& \quad \left. + \left[ B'(\theta_1^{(i)}, (\theta_1^{(i)})') \right] [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Delta\Omega]^\dagger \left( (\theta_2^{(i)})' \right) \left[ (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right) \mathcal{G}_0 d(\theta_2^{(i)})' \right) d(\theta_1^{(i)})'
\end{aligned}$$

At the lowest order in  $\exp(-x)$  and in perturbation:

$$\begin{aligned}
0 & \simeq \frac{\delta^{l,m} \left( [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Psi]^\dagger(\theta_2^{(i)}) \delta \left( (\theta_1^{(i)})' - \theta_1^{(i)} \right) \right)}{\delta[\Psi\Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \\
& - \sum \int \left[ A'(\theta_1^{(i)}, (\theta_1^{(i)})') \right] \frac{\delta^{l,m} [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Psi]^\dagger \left( (\theta_2^{(i)})' \right)}{\delta[\Psi\Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \\
& \quad \times \left( \frac{(1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2)))}{(1 + \bar{O}_{1,\infty}) + \exp(-x)(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))} \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) \right)_{(\theta_2^{(i)})', \theta_2^{(i)}} \mathcal{G}_0 d(\theta_2^{(i)})' d(\theta_1^{(i)})'
\end{aligned}$$

As a consequence, the dominant part in powers of  $\exp(-x)$  of the kernel's derivatives is:

$$\begin{aligned}
& \frac{\delta^{l,m} K(\theta_1^{(i)}, \theta_2^{(i)})}{\delta[\Psi\Psi^\dagger]\left((\theta^{(i)}), (\theta^{(i)\dagger})\right)} \tag{211} \\
& \simeq -\sum \int \left( \frac{\delta^{l,m} \left[ A'(\theta_1^{(i)}, (\theta_1^{(i)})') \right]}{\delta[\Psi\Psi^\dagger]\left((\theta^{(i)}), (\theta^{(i)\dagger})\right)} [\Delta\Omega]\left((\theta_1^{(i)})'\right) [\Delta\Omega]^\dagger\left((\theta_2^{(i)})'\right) \left[ (1 + O_{1,\infty})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right. \\
& \quad \left. + \frac{\delta^{l,m} \left[ B'(\theta_1^{(i)}, (\theta_1^{(i)})') \right]}{\delta[\Psi\Psi^\dagger]\left((\theta^{(i)}), (\theta^{(i)\dagger})\right)} [\Delta\Omega]\left((\theta_1^{(i)})'\right) [\Delta\Omega]^\dagger\left((\theta_2^{(i)})'\right) \left[ (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \right) \mathcal{G}_0 d(\theta_2^{(i)})' d(\theta_1^{(i)})' \\
& \quad - \sum \int \left[ A'(\theta_1^{(i)}, (\theta_1^{(i)})') \right] [\Delta\Omega]\left((\theta_1^{(i)})'\right) [\Psi]^\dagger\left((\theta_2^{(i)})'\right) \\
& \quad \times \left[ \frac{\delta^{l,m} \left( \frac{(1 + \exp(-x)(-z + \frac{1}{2}(y^2 - x^2)))}{(1 + \bar{O}_{1,\infty}) + \exp(-x)(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))} \mathcal{G}_0^{-1} (1 + \bar{O}_{1,\infty}) \right)_{(\theta_2^{(i)})', \theta_2^{(i)}}}{\delta[\Psi\Psi^\dagger]\left((\theta^{(i)}), (\theta^{(i)\dagger})\right)} \mathcal{G}_0 d(\theta_2^{(i)})' d(\theta_1^{(i)})' \right]
\end{aligned}$$

with:

$$\begin{aligned}
& \frac{\delta^{l,m} A' \left( \theta_1^{(i)}, (\theta_1^{(i)})' \right)}{\delta [\Psi \Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \\
& \simeq (-1)^{l+m+1} \left( \int \Delta \Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^l \left( \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( (\theta^{(i)\dagger}) \right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta \Omega \right)^m \\
& \quad \times \mathcal{G}_0^{-1} \frac{\exp(-x) (1 + \bar{O}_{1,\infty})_{\theta_1^{(i)}, (\theta_1^{(i)})'} (1 + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})))}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))) \right)^2} \mathcal{G}_0 \\
& \simeq (-1)^{l+m+1} \left( \int \Delta \Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^l \left( \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( (\theta^{(i)\dagger}) \right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta \Omega \right)^m \mathcal{G}_0^{-1} \\
& \quad \times \frac{\exp(-x) (1 + (y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})))}{(1 + \bar{O}_{1,\infty})} \mathcal{G}_0 \\
& \frac{\delta^{l,m} B' \left( \theta_1^{(i)}, (\theta_1^{(i)})' \right)}{\delta [\Psi \Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \\
& \simeq (-1)^{l+m} \left( \int \Delta \Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^l \left( \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( (\theta^{(i)\dagger}) \right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta \Omega \right)^m B' \left( \theta_1^{(i)}, (\theta_1^{(i)})' \right) \\
& \simeq (-1)^{l+m} \left( \int \Delta \Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^l \left( \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( (\theta^{(i)\dagger}) \right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta \Omega \right)^m B' \left( \theta_1^{(i)}, (\theta_1^{(i)})' \right) \\
& \simeq (-1)^{l+m} \left( \int \Delta \Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^l \left( \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( (\theta^{(i)\dagger}) \right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta \Omega \right)^m \\
& \quad \times \mathcal{G}_0^{-1} \frac{(1 + \bar{O}_{1,2})_{\theta_1^{(i)}, (\theta_1^{(i)})'} \exp(-x)}{(1 + \bar{O}_{1,\infty})} \mathcal{G}_0
\end{aligned}$$

and:

$$\begin{aligned}
& \frac{\delta^{l,m}}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right),\left(\theta^{(i)\dagger}\right)\right)}\left(\frac{(1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2)))}{(1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty}))}\mathcal{G}_0^{-1}(1+\bar{O}_{1,\infty})\right)_{\left(\underline{\theta}_1^{(i)}\right)',\underline{\theta}_1^{(i)}} \\
& \simeq (-1)^{l+m}\frac{1}{2}(y^2-x^2)\left(\int\Delta\Omega^\dagger(1+\bar{O}_{1,\infty})\mathcal{G}_0\frac{\delta\Delta\Omega\left(\theta_1^{(i)}\right)}{\delta\Psi\left(\theta^{(i)}\right)}\right)^l\left(\int\frac{\delta\Delta\Omega^\dagger\left(\theta_1^{(i)}\right)}{\delta\Psi^\dagger\left(\left(\theta^{(i)\dagger}\right)\right)}(1+\bar{O}_{1,\infty})\mathcal{G}_0\Delta\Omega\right)^m \\
& \quad \times\left(\frac{\exp(-x)}{(1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty}))}\mathcal{G}_0^{-1}(1+\bar{O}_{1,\infty})\right)_{\left(\underline{\theta}_1^{(i)}\right)',\underline{\theta}_1^{(i)}} \\
& \simeq (-1)^{l+m}\frac{1}{2}(y^2-x^2)\exp(-x)\left(\int\Delta\Omega^\dagger(1+\bar{O}_{1,\infty})\mathcal{G}_0\frac{\delta\Delta\Omega\left(\theta_1^{(i)}\right)}{\delta\Psi\left(\theta^{(i)}\right)}\right)^l\left(\int\frac{\delta\Delta\Omega^\dagger\left(\theta_1^{(i)}\right)}{\delta\Psi^\dagger\left(\left(\theta^{(i)\dagger}\right)\right)}(1+\bar{O}_{1,\infty})\mathcal{G}_0\Delta\Omega\right)^m \\
& \quad \times\left(\frac{1}{(1+\bar{O}_{1,\infty})}\mathcal{G}_0^{-1}(1+\bar{O}_{1,\infty})\right)_{\left(\underline{\theta}_1^{(i)}\right)',\underline{\theta}_1^{(i)}}
\end{aligned}$$

so that:

$$\begin{aligned}
& [\Psi]^\dagger\left(\left(\underline{\theta}_1^{(i)}\right)'\right)\left[\frac{\delta^{l,m}\left(\frac{(1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2)))}{(1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty}))}\mathcal{G}_0^{-1}(1+\bar{O}_{1,\infty})\right)_{\left(\underline{\theta}_1^{(i)}\right)',\underline{\theta}_1^{(i)}}}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right),\left(\theta^{(i)\dagger}\right)\right)}\right] \\
& \simeq (-1)^{l+m}\frac{1}{2}(y^2-x^2)\exp(-x)[\Delta\Omega]^\dagger\left(\left(\underline{\theta}_1^{(i)}\right)'\right)\left[(1+\bar{O}_{1,\infty})_{\left(\underline{\theta}_1^{(i)}\right)',\underline{\theta}_1^{(i)}}\right] \\
& \quad \times\left(\int\Delta\Omega^\dagger(1+\bar{O}_{1,\infty})\mathcal{G}_0\frac{\delta\Delta\Omega\left(\theta_1^{(i)}\right)}{\delta\Psi\left(\theta^{(i)}\right)}\right)^l\left(\int\frac{\delta\Delta\Omega^\dagger\left(\theta_1^{(i)}\right)}{\delta\Psi^\dagger\left(\left(\theta^{(i)\dagger}\right)\right)}(1+\bar{O}_{1,\infty})\mathcal{G}_0\Delta\Omega\right)^m
\end{aligned}$$

As a consequence, for background fields of large magnitude, the dominant term of (211) is:

$$\begin{aligned}
& \frac{\delta^{l,m}K\left(\theta_1^{(i)},\underline{\theta}_1^{(i)}\right)}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right),\left(\theta^{(i)\dagger}\right)\right)} \\
& \simeq -\sum\int\left(\frac{\delta^{l,m}\left[A'\left(\theta_1^{(i)},\left(\theta_1^{(i)}\right)'\right)\right]}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right),\left(\theta^{(i)\dagger}\right)\right)}[\Delta\Omega]\left(\left(\theta_1^{(i)}\right)'\right)[\Delta\Omega]^\dagger\left(\left(\underline{\theta}_1^{(i)}\right)'\right)\left[(1+\bar{O}_{1,\infty})_{\left(\underline{\theta}_1^{(i)}\right)',\underline{\theta}_1^{(i)}}\right]\right)\mathcal{G}_0d\left(\underline{\theta}_1^{(i)}\right)'d\left(\theta_1^{(i)}\right)' \\
& -\sum\int\left[A'\left(\theta_1^{(i)},\left(\theta_1^{(i)}\right)'\right)\right][\Delta\Omega]\left(\left(\theta_1^{(i)}\right)'\right)[\Psi]^\dagger\left(\left(\underline{\theta}_1^{(i)}\right)'\right) \\
& \quad \times\left[\frac{\delta^{l,m}\left(\frac{(1+\exp(-x)(-z+\frac{1}{2}(y^2-x^2)))}{(1+\bar{O}_{1,\infty})+\exp(-x)(-\bar{O}_{1,\infty}+y(1+\bar{O}_{1,2})-x(1+\bar{O}_{1,\infty}))}\mathcal{G}_0^{-1}(1+\bar{O}_{1,\infty})\right)_{\left(\underline{\theta}_1^{(i)}\right)',\underline{\theta}_1^{(i)}}}{\delta[\Psi\Psi^\dagger]\left(\left(\theta^{(i)}\right),\left(\theta^{(i)\dagger}\right)\right)}\right]\mathcal{G}_0d\left(\underline{\theta}_1^{(i)}\right)'d\left(\theta_1^{(i)}\right)'
\end{aligned}$$

Since  $A' \left( \theta_1^{(i)}, \theta_1^{(i)}, (\theta_1^{(i)})' \right) \simeq 1$  at the lowest order in  $\exp(-x)$ , this leads to the following expression:

$$\begin{aligned}
& \frac{\delta^{l,m} K \left( \theta_1^{(i)}, \theta_1^{(i)} \right)}{\delta [\Psi \Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \\
& \simeq -\exp(-x) \sum \left( \int \Delta \Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^l \left( \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( (\theta^{(i)\dagger}) \right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta \Omega \right)^m \\
& \times \left( \left[ \mathcal{G}_0^{-1} \frac{(y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))}{(1 + \bar{O}_{1,\infty})} \mathcal{G}_0 \right] [\Delta \Omega] \left( (\theta_1^{(i)})' \right) [\Delta \Omega]^\dagger \left( (\theta_1^{(i)})' \right) \left[ (1 + \bar{O}_{1,\infty}) (\theta_1^{(i)})', \theta_1^{(i)} \right] \right. \\
& \left. + \frac{1}{2} (y^2 - x^2) [\Delta \Omega] \left( (\theta_1^{(i)})' \right) [\Delta \Omega]^\dagger \left( (\theta_1^{(i)})' \right) \left[ (1 + \bar{O}_{1,\infty}) (\theta_1^{(i)})', \theta_1^{(i)} \right] \right) \mathcal{G}_0 d \left( \theta_1^{(i)} \right)' d \left( \theta_1^{(i)} \right)'
\end{aligned} \tag{212}$$

To find  $M * \frac{\delta^{l,m} K \left( \theta_1^{(i)}, \theta_1^{(i)} \right)}{\delta [\Psi \Psi^\dagger]} X^{1,1} \left( \theta_1^{(i)}, (\theta^{(i)\dagger}) \right)$  arising in (209), we compute the convolution between  $M$  and the last factor in (212). For fields of large magnitude:

$$\begin{aligned}
A' \left( \theta_1^{(i)}, \theta_1^{(i)}, (\theta_1^{(i)})' \right) & \simeq 1 \\
B' \left( \theta_1^{(i)}, \theta_1^{(i)}, (\theta_1^{(i)})' \right) & < < 1
\end{aligned}$$

$$\begin{aligned}
M & = \delta \left( \theta_1^{(i)} - \theta_1^{(i)} \right) + \exp \left( \int_{\theta_1^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( (\theta^{(i)}) d\theta^{(i)} \right) \right) \left( \Delta \Omega \left( (\theta_1^{(i)}) \right) \Psi^\dagger \left( \theta_1^{(i)} \right) \right. \\
& \quad \left. - \left( A \left( \theta_1^{(i)} \right) \int_{\theta_1^{(i)}} \Delta \Omega^\dagger \left( (\theta_1^{(i)})' \right) (1 + \bar{O}_{1,\infty}) (\theta_1^{(i)})', \theta_1^{(i)} \mathcal{G}_0 + B \left( \theta_1^{(i)} \right) \int_{\theta_1^{(i)}} \Delta \Omega^\dagger \left( (\theta_1^{(i)})' \right) (1 + \bar{O}_{1,2}) (\theta_1^{(i)})', \theta_1^{(i)} \mathcal{G}_0 \right) \right) \\
& \simeq \delta \left( \theta_1^{(i)} - \theta_1^{(i)} \right) + \exp \left( \int_{\theta_1^{(i)}}^{\theta_1^{(i)}} \bar{N} \left( (\theta^{(i)}) d\theta^{(i)} \right) \right) \\
& \quad \times \left( \Delta \Omega \left( (\theta_1^{(i)}) \right) \Psi^\dagger \left( \theta_1^{(i)} \right) - \Delta \Omega \left( (\theta_1^{(i)}) \right) \int_{\theta_1^{(i)}} \Delta \Omega^\dagger \left( (\theta_1^{(i)})' \right) (1 + \bar{O}_{1,\infty}) (\theta_1^{(i)})', \theta_1^{(i)} \mathcal{G}_0 \right) \\
& \simeq \delta \left( \theta_1^{(i)} - \theta_1^{(i)} \right)
\end{aligned}$$

since:

$$\Delta \Omega \left( (\theta_1^{(i)}) \right) \int_{\theta_1^{(i)}} \Delta \Omega^\dagger \left( (\theta_1^{(i)})' \right) (1 + \bar{O}_{1,\infty}) (\theta_1^{(i)})', \theta_1^{(i)} \mathcal{G}_0 \simeq \Delta \Omega \left( (\theta_1^{(i)}) \right) \Psi^\dagger \left( \theta_1^{(i)} \right)$$

and:

$$\left( \Delta \Omega \left( (\theta_1^{(i)}) \right) \Psi^\dagger \left( \theta_1^{(i)} \right) - \Delta \Omega \left( (\theta_1^{(i)}) \right) \int_{\theta_1^{(i)}} \Delta \Omega^\dagger \left( (\theta_1^{(i)})' \right) (1 + \bar{O}_{1,\infty}) (\theta_1^{(i)})', \theta_1^{(i)} \mathcal{G}_0 \right)$$

is of order  $\exp(-x)$ . As a consequence:

$$\begin{aligned}
& M * \left[ \frac{\delta^{l,m} K(\theta_1^{(i)}, \underline{\theta}_1^{(i)})}{\delta[\Psi\Psi^\dagger]\left((\theta^{(i)}), (\theta^{(i)\dagger})\right)} \right] X^{1,1}\left(\theta_2^{(i)}, (\theta^{(i)\dagger})\right) \\
& \simeq \left[ \frac{\delta^{l,m} K(\theta_1^{(i)}, \underline{\theta}_1^{(i)})}{\delta[\Psi\Psi^\dagger]\left((\theta^{(i)}), (\theta^{(i)\dagger})\right)} \right] \frac{\delta\Delta\Omega(\underline{\theta}_1^{(i)})}{\delta\Psi(\theta^{(i)})} \\
& \simeq (-1)^{l+m} \exp(-x) \sum \left( \int \Delta\Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^l \left( \int \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger\left((\theta^{(i)\dagger})\right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta\Omega \right)^m \\
& \quad \times \left( \left[ \mathcal{G}_0^{-1} \frac{(y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))}{(1 + \bar{O}_{1,\infty})} \mathcal{G}_0 \right] [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Delta\Omega]^\dagger \left( (\underline{\theta}_1^{(i)})' \right) \left[ (1 + \bar{O}_{1,\infty}) (\underline{\theta}_1^{(i)})', \underline{\theta}_1^{(i)} \right] \right. \\
& \quad \left. + \frac{1}{2} (y^2 - x^2) [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Delta\Omega]^\dagger \left( (\theta_1^{(i)})' \right) \left[ (1 + \bar{O}_{1,\infty}) (\underline{\theta}_1^{(i)})', \underline{\theta}_1^{(i)} \right] \right) \mathcal{G}_0 \frac{\delta\Delta\Omega(\underline{\theta}_1^{(i)})}{\delta\Psi(\theta^{(i)})} d(\underline{\theta}_1^{(i)})' d(\theta_1^{(i)})' \\
& \simeq \frac{1}{2} (y^2 - x^2) (-1)^{l+m} \exp(-x) \sum \left( \int \Delta\Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^l \left( \int \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger\left((\theta^{(i)\dagger})\right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta\Omega \right)^m \\
& \quad \times \int \left( 1 + \frac{1}{x} \right) [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Delta\Omega]^\dagger (\underline{\theta}_1^{(i)}) \\
& \simeq \frac{1}{2} (y^2 - x^2) (-1)^{l+m} \exp(-x) \sum \left( \int \Delta\Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^l \left( \int \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger\left((\theta^{(i)\dagger})\right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta\Omega \right)^m \\
& \quad \times \int [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Delta\Omega]^\dagger (\underline{\theta}_1^{(i)}) \tag{213}
\end{aligned}$$

and (212) can be written:

$$\begin{aligned}
& M * \frac{\delta^{l,m} K(\theta_1^{(i)}, \underline{\theta}_1^{(i)})}{\delta[\Psi\Psi^\dagger]\left((\theta^{(i)}), (\theta^{(i)\dagger})\right)} X^{1,1}\left(\theta_1^{(i)}, (\theta^{(i)\dagger})\right) \tag{214} \\
& \simeq \exp(-x) \left( \int \Delta\Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^l \left( \int \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger\left((\theta^{(i)\dagger})\right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta\Omega \right)^m \\
& \quad \times \frac{1}{2} (y^2 - x^2) \int [\Delta\Omega] \left( (\theta_1^{(i)})' \right) [\Delta\Omega]^\dagger \left( (\underline{\theta}_1^{(i)})' \right)
\end{aligned}$$

Gathering (210) and (213) leads to the strong background field approximation of (209):

$$\begin{aligned}
& \frac{\delta^{l,m} \Delta \Omega^\alpha \left( \theta_1^{(i)} \right)}{\delta \left[ \Psi \Psi^\dagger \left( \left( \theta^{(i)} \right), \left( \theta^{(i)\dagger} \right) \right) \right]} \\
&= M * \left( \frac{\delta}{\delta \Psi} \right)^{l-1} \left( \frac{\delta}{\delta \Psi^\dagger} \right)^{m-1} X^{1,1} \left( \theta_1^{(i)}, \left( \theta^{(i)\dagger} \right) \right) + \left[ M * \frac{\delta^{l-1,m-1} K \left( \theta_1^{(i)}, \theta_1^{(i)} \right)}{\delta \left[ \Psi \Psi^\dagger \right]} \right] X^{1,1} \left( \theta_1^{(i)}, \left( \theta^{(i)\dagger} \right) \right) \\
&\simeq \frac{1}{2} (y^2 - x^2) \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \left( \int \Delta \Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^{l-1} \left( \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( \theta^{(i)} \right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta \Omega \right)^{m-1} \exp(-x) \\
&+ \frac{1}{2} (y^2 - x^2) \exp(-x) \left( \int \Delta \Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^{l-1} \left( \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( \left( \theta^{(i)\dagger} \right) \right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta \Omega \right)^{m-1} \\
&\times \int \Delta \Omega^\dagger \int \Delta \Omega
\end{aligned} \tag{215}$$

Additional contributions to (207) can be considered. We consider  $r$  products of (214). Using (215), each term of this product is given by:

$$\begin{aligned}
& M * \frac{\delta^{r,s} K \left( \theta_1^{(i)}, \theta_2^{(i)} \right)}{\delta \left[ \Psi \Psi^\dagger \right] \left( \left( \theta^{(i)} \right), \left( \theta^{(i)\dagger} \right) \right)} \\
&\simeq \exp(-x) \left( \int \Delta \Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^{l_i} \left( \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( \left( \theta^{(i)\dagger} \right) \right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta \Omega \right)^{m_i} \\
&\times \frac{1}{2} (y^2 - x^2) \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( \left( \theta^{(i)\dagger} \right) \right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 [\Delta \Omega] \left( \theta_1^{(i)} \right) \int [\Delta \Omega]^\dagger \left( \theta_2^{(i)} \right) (1 + O_{1,\infty}) \\
&\simeq \exp(-x) \int \mathcal{G}_0 \left( \int \Delta \Omega^\dagger (1 + O_{1,\infty}) \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^r \left( \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( \left( \theta^{(i)\dagger} \right) \right)} (1 + O_{1,\infty}) \Delta \Omega \right)^s \\
&\times \frac{1}{2} (y^2 - x^2) [\Delta \Omega] \left( \theta_1^{(i)} \right) [\Delta \Omega]^\dagger \left( \theta_2^{(i)} \right) (1 + O_{1,\infty})
\end{aligned}$$

where  $\sum l_i = l$  and  $\sum m_i = m$ .

$$\begin{aligned}
& \frac{1}{2} (y^2 - x^2) \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( \left( \theta^{(i)\dagger} \right) \right)} (1 + O_{1,\infty}) [\Delta \Omega] \left( \theta_1^{(i)} \right) \int [\Delta \Omega]^\dagger \left( \theta_2^{(i)} \right) (1 + O_{1,\infty}) \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \\
&\simeq \frac{1}{2} (y^2 - x^2) [\Delta \Omega] \left( \theta_1^{(i)} \right) [\Delta \Omega]^\dagger \left( \theta_2^{(i)} \right)
\end{aligned}$$

where  $\sum l_i = l$  and  $\sum m_i = m$ . The successive convolutions multiplied by  $X^{1,1} \left( \theta_1^{(i)}, \left( \theta^{(i)\dagger} \right) \right)$  yield:

$$\begin{aligned}
& \left( \frac{1}{2} (y^2 - x^2) \right)^r \exp(-rx) \left( \int \Delta \Omega^\dagger (1 + O_{1,\infty}) \frac{\delta \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right)^l \left( \int \frac{\delta \Delta \Omega^\dagger \left( \theta_1^{(i)} \right)}{\delta \Psi^\dagger \left( \theta^{(i)} \right)} (1 + O_{1,\infty}) \Delta \Omega \right)^m \\
&\times \prod_{m=1}^r \left[ \int \left( [\Delta \Omega]^\dagger \left( \theta'_m \right) [(1 + O_{1,\infty}) [\Delta \Omega \left( \theta_m \right)]] \right) \right]
\end{aligned}$$

the integral of the product being on the  $2r$  dimensional simplex  $\theta'_m < \theta_{m+1}$ . In first approximation the sum of these expression can be replaced by:

$$\begin{aligned} & \left( \int \Delta\Omega^\dagger (1 + O_{1,\infty}) \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^l \left( \int \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger(\theta^{(i)})} (1 + O_{1,\infty}) \Delta\Omega \right)^m \\ & \times \exp\left(\frac{(y^2 - x^2) \exp(-x)}{2} \left( \int [\Delta\Omega]^\dagger(\theta_2^{(i)}) [(1 + O_{1,\infty})] [\Delta\Omega] \right)\right) \end{aligned}$$

That is, the expression (214) for  $\frac{\delta^{l,m} \Delta\Omega^\alpha(\theta_1^{(i)})}{\delta[\Psi\Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger})]}$  is multiplied by the exponential. At the lowest order in  $\exp(-x)$ , the  $(l, m)$  points connected correlation functions are obtained by the convolution of this expression with  $(1, 1)$  propagators  $\left(\frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})}\right)^{-1}$ . At the lowest order in  $\exp(-x)$ , this leads to:

$$\begin{aligned} & \frac{1}{2} (y^2 - x^2) \left( \left( \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^{-1} \left( \int \Delta\Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \right)^{l-1} \left( \int (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta\Omega \right)^{m-1} \right. \\ & \left. + \left( \int \Delta\Omega^\dagger (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \right)^l \left( \int (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \Delta\Omega \right)^m \right) \\ & \times \exp\left(\frac{(y^2 - x^2) \exp(-x)}{2} \left( \int [\Delta\Omega]^\dagger(\theta_2^{(i)}) [(1 + O_{1,\infty})] [\Delta\Omega] \right)\right) \\ & \simeq \frac{1}{2} (y^2 - x^2) \left( \left( \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^{-1} \left( \Psi^\dagger((\theta_1^{(i)})) \right)^{l-1} \left( \Psi((\theta_2^{(i)})) \right)^{m-1} \right. \\ & \left. + \left( \Psi^\dagger((\theta_1^{(i)})) \right)^l \left( \Psi((\theta_2^{(i)})) \right)^m \right) \\ & \times \exp\left(\frac{(y^2 - x^2) \exp(-x)}{2} \left( \int [\Delta\Omega]^\dagger(\theta_2^{(i)}) [(1 + O_{1,\infty})] [\Delta\Omega] \right)\right) \\ & \simeq \frac{1}{2} (y^2 - x^2) \left( \left( \Psi^\dagger((\theta_1^{(i)})) \right)^l \left( \Psi((\theta_2^{(i)})) \right)^m \right) \\ & \times \exp\left(\frac{(y^2 - x^2) \exp(-x)}{2} \left( \int [\Delta\Omega]^\dagger(\theta_2^{(i)}) [(1 + O_{1,\infty})] [\Delta\Omega] \right)\right) \end{aligned}$$

Additional contributions are obtained by convoluting the connected vertices of lowest order. Each of this convolution adds an additional factor:

$$\exp(-x) \int \Psi^\dagger((\theta_1^{(i)})) \Psi((\theta_2^{(i)}))$$

There are up to  $m - 3$  factors contributing to the connected correlations. The contribution of each term including the factor:

$$\left( \exp(-x) \int \Psi^\dagger((\theta_1^{(i)})) \Psi((\theta_2^{(i)})) \right)^p$$

are identical. The number of such contributions, written  $A_{l,m,p}$ , satisfies:

$$A_{l,m,p} = \frac{p}{2} A_{l-1,m,p} + \left( l + \frac{p-1}{2} \right) A_{l-1,m,p-1} \quad (216)$$



Actually, differentiation with respect to  $l$  of a connected graphs with  $l-1, m$  external lines and  $p-1$  internal lines and  $p$  factors of the type:  $X_{l_i, m_i} = \left( \left( \frac{\delta \Delta \Omega(\theta_1^{(i)})}{\delta \Psi(\theta^{(i)})} \right)^{-1} \right)^{l_i} \frac{\delta^{l_1, m_1} \Delta \Omega^\alpha(\theta_1^{(i)})}{\delta [\Psi \Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} \left( \left( \frac{\delta \Delta \Omega(\theta_1^{(i)})}{\delta \Psi(\theta^{(i)})} \right)^{-1} \right)^{m_1}$  yields two types of contributions to the graphs with  $l, m$  external lines. First, due to the heaviside functions in the propagators, it yields in average  $\frac{p}{2}$  graphs with  $p-1$  internal lines. Each of them is built by adding an external line to one of the  $X_{l_i, m_i}$ .

The second type of contribution has  $l+m$  external lines and  $p$  internal lines,  $p$  factors of the type:

$$X_{l_i, m_i} = \left( \left( \frac{\delta \Delta \Omega(\theta_1^{(i)})}{\delta \Psi(\theta^{(i)})} \right)^{-1} \right)^{l_i} \frac{\delta^{l_1, m_1} \Delta \Omega^\alpha(\theta_1^{(i)})}{\delta [\Psi \Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} \left( \left( \frac{\delta \Delta \Omega(\theta_1^{(i)})}{\delta \Psi(\theta^{(i)})} \right)^{-1} \right)^{m_1}$$

and one three points vertex:

$$\left( \left( \frac{\delta \Delta \Omega(\theta_1^{(i)})}{\delta \Psi(\theta^{(i)})} \right)^{-1} \right)^{l_i} \frac{\delta^{l', m'} \Delta \Omega^\alpha(\theta_1^{(i)})}{\delta [\Psi \Psi^\dagger((\theta^{(i)}), (\theta^{(i)\dagger}))]} \left( \left( \frac{\delta \Delta \Omega(\theta_1^{(i)})}{\delta \Psi(\theta^{(i)})} \right)^{-1} \right)^{m_1}$$

with  $l' + m' = 3$ . The insertion of this three points vertex arises in average at one of the  $l + \frac{p-1}{2}$  lines at the left of the initial graphs.

Similarly:

$$A_{l, m, p} = \frac{p}{2} A_{l, m-1, p} + \left( m + \frac{p-1}{2} \right) A_{l, m-1, p-1} \quad (217)$$

Asymptotically, (216) and (217) yield that

$$A_{l+m, p} \simeq \left( m + \frac{p-1}{2} \right) \left( l + \frac{p-2}{2} \right) A_{l-1, m-1, p-2}$$

and recursively:

$$A_{l+m, p} \simeq \frac{(m + \frac{p-1}{2})! (l + \frac{p-1}{2})!}{m! l!}$$

Given that the sum over the permutations of points involved in the  $(l, m)$  correlation functions are  $m! l!$ , this yields for the connected correlation functions:

$$F_{l, m}(\Psi) \left( \frac{1}{2} (y^2 - x^2) \left( (\Psi^\dagger((\theta_1^{(i)})))^l (\Psi((\theta_2^{(i)})))^m \right) \exp(-x) \right) \quad (218)$$

with:

$$F_{l, m}(\Psi) = \sum_{k=0}^{m-3} \frac{(m + \frac{p-1}{2})! (l + \frac{p-1}{2})!}{(l! m!)^2} \left( \exp(-x) \int \Psi^\dagger((\theta_1^{(i)})) \Psi((\theta_2^{(i)})) \right)^p$$

and where a sum over the permutations of the sets of  $l$  variables and  $m$  variables respectively is understood. For  $p \rightarrow m+l$ ,  $\frac{(m + \frac{p-1}{2})! (l + \frac{p-1}{2})!}{(l! m!)^2}$  behaves as  $\exp(\alpha p)$  and:

$$\begin{aligned} F_{l, m}(\Psi) &\simeq \sum_{k=0}^{m-3} \left( \exp(-(x-\alpha)) \int \Psi^\dagger((\theta_1^{(i)})) \Psi((\theta_2^{(i)})) \right)^p \\ &\simeq \frac{1}{1 - \exp(-(x-\alpha)) \int \Psi^\dagger((\theta_1^{(i)})) \Psi((\theta_2^{(i)}))} \end{aligned}$$

Including the corrective factor  $F_{l,m}(\Psi)$  leads to the connected correlation functions:

$$F_{l,m}(\Psi) \left( \int \Delta\Omega^\dagger (1 + O_{1,\infty}) \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^l \left( \int \frac{\delta\Delta\Omega^\dagger(\theta_1^{(i)})}{\delta\Psi^\dagger(\theta^{(i)})} (1 + O_{1,\infty}) \Delta\Omega \right)^m \\ \times \exp\left(\frac{(y^2 - x^2) \exp(-x)}{2} \left( \int [\Delta\Omega]^\dagger(\theta_2^{(i)}) [(1 + O_{1,\infty})] [\Delta\Omega] \right)\right)$$

Ultimately, we can derive from (218) the  $(l, m)$  points correlation functions:

$$\left( \left( \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^{-1} \right)^m \delta_{l,m} + \sum_{s=0}^{\inf(l,m)} \sum_{k \geq 1} \sum_{\substack{s_1 \geq \dots \geq s_k \geq 0, t_1 \geq \dots \geq t_k \geq 0 \\ s_i + t_i \geq 1, \sum s_i = l - s, \sum t_i = m - s}} \left( \prod_{i=1}^k F_{s_i, t_i}(\Psi) \right) \left( \left( \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^{-1} \right)^s \\ \times \left( \Psi^\dagger(\theta_1^{(i)}) \right)^{l-s-1} \left( \Psi(\theta_2^{(i)}) \right)^{m-s-1} \\ \times \left( \frac{(y^2 - x^2) \exp(-x)}{2} \left( \left( \frac{\delta\Delta\Omega(\theta_1^{(i)})}{\delta\Psi(\theta^{(i)})} \right)^{-1} + \left( \Psi^\dagger(\theta_1^{(i)}) \right) \left( \Psi(\theta_2^{(i)}) \right) \right) \right)^{-1} \\ \times \exp\left(\frac{(y^2 - x^2) \exp(-x)}{2} \left( \int [\Delta\Omega]^\dagger(\theta_2^{(i)}) [(1 + O_{1,\infty})] [\Delta\Omega] \right)\right)^k \quad (219)$$

### 3.3.3 Weak Background Field

For background of small magnitude, the correlation functions for  $l \neq m$  are negligible and the dominant contribution while computing:

$$\left( \frac{\delta}{\delta\Psi} \right)^l \left( \frac{\delta}{\delta\Psi^\dagger} \right)^l \frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{(1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right)} \quad (220)$$

is obtained by replacing each derivative  $\frac{\delta}{\delta\Psi} \frac{\delta}{\delta\Psi^\dagger}$  of  $\exp(-x)$  by  $1 + \bar{O}_{1,\infty} = \nu$ . Thus writing  $\frac{d}{dx}$  for  $\frac{\delta}{\delta\Psi} \frac{\delta}{\delta\Psi^\dagger}$ , the operator  $\left( \frac{\delta}{\delta\Psi} \frac{\delta}{\delta\Psi^\dagger} \right)^l$  can be replaced by  $\left( \exp(-\nu x) \frac{\partial}{\partial U} + \frac{\partial}{\partial x} \right)^l$  where  $U = \exp(-x)$ . All references to the  $2l$  points arising in the functional derivatives can be skipped and reintroduced at the end of the calculus. We thus compute:

$$\left( \left( \exp(-\nu x) \frac{\partial}{\partial U} + \frac{\partial}{\partial x} \right)^l \frac{1 + U \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{(1 + \bar{O}_{1,\infty}) + U \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right)} \right)_{U=\exp(-\nu x)}$$

The operator  $\left( \exp(-\nu x) \frac{\partial}{\partial U} + \frac{\partial}{\partial x} \right)^l$  can be found recursively:

$$\left( -\nu \exp(-\nu x) \frac{\partial}{\partial U} + \frac{\partial}{\partial x} \right)^{n+1} \\ = \left( \exp(-x) \frac{\partial}{\partial U} + \frac{\partial}{\partial x} \right) \left( \sum C_n^k (-\nu)^k \exp(-k\nu x) \frac{\partial^k}{\partial U^k} \frac{\partial^{n-k}}{\partial x^{n-k}} + \sum_p a_{n,k}^p (-\nu)^k \exp(-k\nu x) \frac{\partial^k}{\partial U^k} \frac{\partial^{n-k-p}}{\partial x^{n-k-p}} \right) \\ = \sum C_{n+1}^k (-\nu)^k \exp(-k\nu x) \frac{\partial^k}{\partial U^k} \frac{\partial^{n+1-k}}{\partial x^{n+1-k}} \\ + \sum_p a_{n,k}^p (-\nu)^{k+1} \exp(-(k+1)\nu x) \frac{\partial^{k+1}}{\partial U^{k+1}} \frac{\partial^{n-k-p}}{\partial x^{n-k-p}} \\ + a_{n,k}^p (-\nu)^k \exp(-k\nu x) \frac{\partial^k}{\partial U^k} \frac{\partial^{n+1-k-p}}{\partial x^{n+1-k-p}} - k a_{n,k}^p (-\nu)^k \exp(-k\nu x) \frac{\partial^k}{\partial U^k} \frac{\partial^{n-k-p}}{\partial x^{n-k-p}} \\ - \sum C_n^k k (-\nu)^{k+1} \exp(-k\nu x) \frac{\partial^k}{\partial U^k} \frac{\partial^{n-k}}{\partial x^{n-k}}$$

and the coefficients  $a_{n,k}^p$  satisfy:

$$a_{n+1,k+1}^p = a_{n,k}^p + a_{n,k+1}^p - (k+1)a_{n,k+1}^{p-1} - (k+1)C_n^{k+1}\delta_{p,1}$$

Remark that the expression to differentiate in (220) rewrites:

$$\begin{aligned} & \frac{1 + \exp(-x) \left(-z + \frac{1}{2}(y^2 - x^2)\right)}{(1 + \bar{O}_{1,\infty}) + \exp(-x) \left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)} \\ = & \frac{1 + U \left(-z + \frac{1}{2}(y^2 - x^2)\right)}{(1 + \bar{O}_{1,\infty}) + U \left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)} \\ = & \frac{\left(-z + \frac{1}{2}(y^2 - x^2)\right)}{\left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)} + \frac{1 - \frac{(1 + \bar{O}_{1,\infty}) \left(-z + \frac{1}{2}(y^2 - x^2)\right)}{\left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)}}{(1 + \bar{O}_{1,\infty}) + U \left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)} \end{aligned}$$

and its  $k$  th derivative with respect to  $U$  becomes:

$$\frac{1 - \frac{(1 + \bar{O}_{1,\infty}) \left(-z + \frac{1}{2}(y^2 - x^2)\right)}{\left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)} \left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)^k}{\left(1 + \bar{O}_{1,\infty}\right) + U \left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)^{k+1}}$$

This is of order  $k$  in perturbation: it involves terms of order  $\frac{1}{\Lambda^k}$ . The dominant term in perturbation is thus for  $k = 0, 1$ . As a consequence the operator  $(-\nu \exp(-\nu x) \frac{\partial}{\partial U} + \frac{\partial}{\partial x})^l$  reduces to:

$$\begin{aligned} & \sum C_l^k (-\nu)^k \exp(-k\nu x) \frac{\partial^k}{\partial U^k} \frac{\partial^{l-k}}{\partial x^{l-k}} + \sum_p a_{l,k}^p (-\nu)^{k+p} \exp(-k\nu x) \frac{\partial^k}{\partial U^k} \frac{\partial^{l-k-p}}{\partial x^{l-k-p}} \\ \simeq & \frac{\partial^n}{\partial x^n} - l\nu \exp(-\nu x) \frac{\partial}{\partial U} \frac{\partial^{l-1}}{\partial x^{l-1}} - \nu \sum_p \exp(-\nu x) \frac{\partial}{\partial U} \frac{\partial^{l-1-p}}{\partial x^{l-1-p}} \sum_{q \geq p} C_q^p \\ = & \frac{\partial^n}{\partial x^n} - l\nu \exp(-\nu x) \frac{\partial}{\partial U} \frac{\partial^{l-1}}{\partial x^{l-1}} - \nu \sum_p (-\nu)^p C_l^{p+1} \exp(-\nu x) \frac{\partial}{\partial U} \frac{\partial^{l-1-p}}{\partial x^{l-1-p}} \end{aligned}$$

But, we note that in the expression:

$$\begin{aligned} & \frac{\partial^n}{\partial x^n} \left( \frac{1 + \left(-z + \frac{1}{2}(y^2 - x^2)\right)}{(1 + \bar{O}_{1,\infty}) + \left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)} \right) \\ & - l\nu \exp(-\nu x) \frac{\partial^{l-1}}{\partial x^{l-1}} \frac{\left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right) - (1 + \bar{O}_{1,\infty}) \left(-z + \frac{1}{2}(y^2 - x^2)\right)}{\left(1 + \bar{O}_{1,\infty}\right) + \left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)^2} \\ & - \nu \sum_{p=1}^{l-1} (-\nu)^p C_l^{p+1} \exp(-\nu x) \frac{\partial^{l-1-p}}{\partial x^{l-1-p}} \frac{\left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right) - (1 + \bar{O}_{1,\infty}) \left(-z + \frac{1}{2}(y^2 - x^2)\right)}{\left(1 + \bar{O}_{1,\infty}\right) + \left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)^2} \\ & \frac{\partial^n}{\partial x^n} \left( \frac{1 + \left(-z + \frac{1}{2}(y^2 - x^2)\right)}{(1 + \bar{O}_{1,\infty}) + \left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)} \right) \\ & - \nu \sum_{p=0}^{l-1} (-\nu)^p C_l^{p+1} \exp(-\nu x) \frac{\partial^{l-1-p}}{\partial x^{l-1-p}} \frac{\left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right) - (1 + \bar{O}_{1,\infty}) \left(-z + \frac{1}{2}(y^2 - x^2)\right)}{\left(1 + \bar{O}_{1,\infty}\right) + U \left(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})\right)^2} \end{aligned}$$

and:

$$\begin{aligned}
& \frac{\partial^{l-1-p}}{\partial x^{l-1-p}} \frac{\bar{O}_{1,\infty}}{\left( (1 + \bar{O}_{1,\infty}) + -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right)^2} \\
& + \frac{\partial^{l-1-p}}{\partial x^{l-1-p}} \frac{y(1 + \bar{O}_{1,2}) + (z-x)(1 + \bar{O}_{1,\infty})}{\left( (1 + \bar{O}_{1,\infty}) + -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right)^2} \\
& - \frac{1}{2} \frac{\partial^{l-1-p}}{\partial x^{l-1-p}} \frac{(1 + \bar{O}_{1,\infty})(y^2 - x^2)}{\left( (1 + \bar{O}_{1,\infty}) + (-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})) \right)^2} \\
\approx & \frac{\partial^{l-1-p}}{\partial x^{l-1-p}} \frac{y(1 + \bar{O}_{1,2}) + (z-x)(1 + \bar{O}_{1,\infty})}{\left( (1 + \bar{O}_{1,\infty}) + -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right)^2} \\
& - \frac{1}{2} \frac{\partial^{l-1-p}}{\partial x^{l-1-p}} \frac{(1 + \bar{O}_{1,\infty})(y^2 - x^2)}{\left( (1 + \bar{O}_{1,\infty}) + -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right)^2} \\
\approx & \frac{\left( (1 + \bar{O}_{1,2})^2 - (1 - 2\bar{O}_{1,\infty})(1 + \bar{O}_{1,\infty}) \right) \left( (1 + \bar{O}_{1,2})^2 - (1 + \bar{O}_{1,\infty})^2 \right)^{l-2-p}}{\left( (1 + \bar{O}_{1,\infty}) + -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right)^{l+1-p}} \\
& - \frac{1}{2} \frac{(l-1-p)(l-2-p)}{2} \frac{(1 + \bar{O}_{1,\infty}) \left( (1 + \bar{O}_{1,2})^2 - (1 + \bar{O}_{1,\infty})^2 \right)^{l-1-p}}{\left( (1 + \bar{O}_{1,\infty}) + -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right)^{l-p}}
\end{aligned}$$

As a consequence, the dominant contribution is for  $p = l - 1$ :

$$\begin{aligned}
& \left( \frac{\delta}{\delta\Psi} \right)^l \left( \frac{\delta}{\delta\Psi^\dagger} \right)^l \frac{1 + \exp(-x) \left( -z + \frac{1}{2}(y^2 - x^2) \right)}{\left( (1 + \bar{O}_{1,\infty}) + \exp(-x) \left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right) \right)} \\
\approx & (-\nu)^l \exp(-\nu x) \frac{\left( -\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}) \right) - (1 + \bar{O}_{1,\infty}) \left( -z + \frac{1}{2}(y^2 - x^2) \right)}{\left( (1 + \bar{O}_{1,\infty}) + (-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})) \right)^2}
\end{aligned}$$

In our order of approximation  $A(\theta_1^{(i)}) \ll 1$ :

$$\begin{aligned}
& K(\theta_1^{(i)}, \theta_2^{(i)}) \\
\approx & - \int \left( \left( [B(\theta_1^{(i)})] [\Delta\Omega]^\dagger \left( (\theta_2^{(i)})' \right) \left[ (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \mathcal{G}_0 \right) d(\theta_1^{(i)})' d(\theta_2^{(i)})' \right. \\
& \left. + [\Delta\Omega](\theta_1^{(i)}) [\Psi]^\dagger(\theta_2^{(i)}) \delta \left( (\theta_2^{(i)})' - \theta_2^{(i)} \right) \right) \\
\approx & \int \left( \mathcal{G}_0^{-1} [\Psi](\theta_1^{(i)}) [\Psi]^\dagger(\theta_2^{(i)}) \right. \\
& \left. - \int \mathcal{G}_0^{-1} \left[ (1 + \bar{O}_{1,2})_{\theta_1^{(i)}, (\theta_1^{(i)})'} \right] [\Psi] \left( (\theta_1^{(i)})' \right) [\Psi]^\dagger \left( (\theta_2^{(i)})' \right) \mathcal{G}_0^{-1} \left[ (1 + \bar{O}_{1,2})_{(\theta_2^{(i)})', \theta_2^{(i)}} \right] \mathcal{G}_0 \right) d(\theta_2^{(i)})' d(\theta_2^{(i)})'
\end{aligned}$$

and in first approximation (207) becomes:

$$\begin{aligned}
& \frac{\delta^{l,l} \Delta\Omega(\theta_1^{(i)}, \theta_2^{(i)})}{\delta[\Psi\Psi^\dagger] \left( (\theta^{(i)}), (\theta^{(i)\dagger}) \right)} \\
\approx & M * \left( \frac{\delta}{\delta\Psi} \right)^l \left( \frac{\delta}{\delta\Psi^\dagger} \right)^l X^{1,1}(\theta_1^{(i)}, (\theta^{(i)\dagger})) \\
& + \left[ M * \frac{\delta^{1,1} K}{\delta[\Psi\Psi^\dagger]} \right] * \dots * \left[ M * \frac{\delta^{1,1} K}{\delta[\Psi\Psi^\dagger]} \right] * X^{1,1}(\theta_1^{(i)}, (\theta_1^{(i)\dagger}))
\end{aligned}$$

For weak fields  $M \simeq 1$  and  $X^{1,1} \simeq \mathcal{G}_0^{-1}$ . As a consequence:

$$\begin{aligned}
& \frac{\delta^{l+1,l} \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta [\Psi \Psi^\dagger] \left( \left( \theta^{(i)} \right), \left( \theta^{(i)\dagger} \right) \right)} \\
& \simeq \left( \frac{\delta}{\delta \Psi} \right)^l \left( \frac{\delta}{\delta \Psi^\dagger} \right)^l X^{1,1} \left( \theta_1^{(i)}, \left( \theta_1^{(i)\dagger} \right) \right) + \left[ \frac{\delta^{1,1} K}{\delta [\Psi \Psi^\dagger]} \right] * \dots * \left[ \frac{\delta^{1,1} K}{\delta [\Psi \Psi^\dagger]} \right] * X^{1,1} \left( \theta_1^{(i)}, \left( \theta_1^{(i)\dagger} \right) \right) \\
& \simeq (-1)^l \sum_p \sum_{u_p+v_p=l} \prod_p \left( \int \frac{\delta^{u_p, u_p} \Delta \Omega^\dagger \left( \theta_2^{(i)} \right)'}{\delta [\Psi \Psi^\dagger] \left( \left( \theta^{(i)} \right), \left( \theta^{(i)\dagger} \right) \right)} (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \frac{\delta^{v_p, v_p} \Delta \Omega \left( \theta_1^{(i)} \right)}{\delta \Psi \left( \theta^{(i)} \right)} \right) \\
& \times \exp(-x) \frac{(-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty})) - (1 + \bar{O}_{1,\infty})(-z + \frac{1}{2}(y^2 - x^2))}{(1 + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))^2} \\
& + \prod_{i=1}^l \left( \mathcal{G}_0^{-1} \left( \theta_1^{(i)}, \left( \theta_1^{(i)} \right) \right) \otimes 1_{\left( \theta^{(i)\dagger}, \left( \theta^{(i)} \right) \right)} - \mathcal{G}_0^{-1} (1 + \bar{O}_{1,2})_{\left( \theta_1^{(i)}, \left( \theta^{(i)} \right) \right)} \otimes \mathcal{G}_0^{-1} (1 + \bar{O}_{1,2})_{\left( \theta^{(i)\dagger}, \left( \theta_2^{(i)} \right) \right)} \mathcal{G}_0 \right)^l d \left( \theta_1^{(i)} \right) d \left( \theta_2^{(i)} \right) \\
& \simeq (-1)^{l+1} \left( \mathcal{G}_0^{-1} * (1 + \bar{O}_{1,\infty}) \right)^{\otimes l} \mathcal{G}_0^{-1} \bar{O}_{1,\infty}
\end{aligned}$$

where the symmetrisation over the variables is understood. To find the connected correlation functions, we proceed as for the strong field case. The connected vertices are convoluted with the propagators  $\mathcal{G}_0$ . The term of lowest order in perturbation for the  $(l, l)$  correlation functions is:

$$\left( (-1)^l \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l-2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right)$$

and including successive corrections:

$$\begin{aligned}
& \left( (-1)^l \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l-2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right) \\
& + \sum_{l_1+l_2=l} \left( (-1)^{l_1} \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l_1-2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right) \\
& * \mathcal{G}_0 * \left( (-1)^{l_2} \mathcal{G}_0 \left( \int (1 + \bar{O}_{1,\infty}) \mathcal{G}_0 \right)^{\otimes l_2-1} \bar{O}_{1,\infty} \mathcal{G}_0 \right) \\
& + \dots
\end{aligned}$$

The convolution of the two blocks with  $\mathcal{G}_0$  being performed on any external point of these blocks. The previous expression can be rewritten:

$$\begin{aligned}
& \left( (-1)^l \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l-2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right) \\
& + (-1)^l \sum_{k \geq 2} \sum_{l_1 + \dots + l_k = l} \prod_{m=1}^k \sum_{*_{\mathcal{G}_0}^{\{l_m\}}} *_{\mathcal{G}_0}^{\{l_m\}} \left( \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l_m-2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right) \\
& \equiv \mathcal{G}_C^{(l)}
\end{aligned}$$

where the sum over  $*_{\mathcal{G}_0}^{\{l_m\}}$  denotes the sum over all possible convolutions between the different blocks:

$$\left( \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l_m-2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right)$$

Two blocks are convoluted on at most one variable. The convolution are performed by insertion of a propagator  $\mathcal{G}_0$  between the blocks. The expression for the connected correlation functions induces the full correlation functions:

$$\begin{aligned}
& \mathcal{G}_0^{\otimes n} + \sum_{p,k} (-1)^{l-p} \sum_{l_k, \sum l_n = l-p} \prod_k \mathcal{G}_C^{(l_n)} \mathcal{G}_0^{\otimes p} \\
= & \mathcal{G}_0^{\otimes n} + \sum_{p,k} (-1)^{l-p} \sum_{l_n, \sum l_n = l-p} \prod_n \left( \left( (-1)^{l_n} \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l_n - 2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right) \right. \\
& \left. + (-1)^{l_n} \sum_{k \geq 2} \sum_{l_1 + \dots + l_k = l_n} \prod_{m=1}^k \sum_{\substack{\{l_m\} \\ *_{\mathcal{G}_0}}} *_{\mathcal{G}_0}^{\{l_m\}} \left( \mathcal{G}_0 \otimes ((1 + \bar{O}_{1,\infty}) * \mathcal{G}_0)^{\otimes l_m - 2} \otimes \bar{O}_{1,\infty} * \mathcal{G}_0 \right) \right) \otimes \mathcal{G}_0^{\otimes p}
\end{aligned}$$

## Appendix 4. Estimation of 1PI graphs contributions and minimum of the effective action

The previous computations gave the form of the correlation functions once the minimum of the effective action is known. To find this minimum we have to write directly the sum of 1PI graphs, and to find its minimum. As before we first compute the sum without inertia coefficients, and then with the inclusion of these coefficients.

### 4.1 Without inertia coefficients

We write the sum of 1PI graphs. These are the graphs that cannot be factored into product of subgraphs between  $2k$  and  $2n - 2k$  points respectively, with  $k > 0$  after cutting one edge and that are amputated from the external 2 points graphs. In this section we will consider non-amputated graphs, removing ultimately the external legs while computing the effective action.

We define the minimal graphs as 1PI graphs decomposed as a product of vertices:

$$\bar{\Xi}_{\min} \left( \left( Z_i, l_1^{(i)}, \{Z_{k_{j,i}^{(1)}}\}, \dots, l_{p_i}^{(i)}, \{Z_{k_{j,i}^{(p_i)}}\} \right)_{i=1, \dots, n} \right) = \prod_i \bar{\Xi}_1^{(l_1)} \left( Z_i, \{Z_{k_{j,i}^{(1)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \dots \bar{\Xi}_1^{(l_{p_i})} \left( Z_i, \{Z_{k_{j,i}^{(p_i)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right)$$

that are connected, and such that removing one of the vertices turns the graph in a not 1PI graph. The 1PI graphs are those that can be factored by a minimal graph. The factorization is defined here by a convolution product in the  $2n$  time variables between graphs with  $2n$  external vertices. The precise form of the minimal graphs will be given below, but it is enough here to state that for  $p_i = 1$ , the valence  $l_1$  is equal to  $n - 1$ . For  $p_i > 1$ , one has  $\sum l_1^{(i)} = n$ . The factorization by a minimal graph amounts, in the computation of:

$$\frac{1}{m!} \left( \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)^m = \frac{1}{m!} \left( \int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \sum_l \bar{\Xi}_1^{(l)} \left( Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i} \right) d\theta^{(i)} \right)^m$$

to replace one of the factor  $\bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)$  by  $\bar{\Xi}_1^{(l_1)} \left( Z_i, \{Z_{k_{j,i}^{(1)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right)$ . There are  $m$  possibilities to do so. Thus in the computation of the connected graphs, the term:

$$\frac{1}{m!} \left( \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)^m$$

is replaced by:

$$\frac{1}{(m-1)!} \bar{\Xi}_1^{(l_1)} \left( Z_i, \{Z_{k_{j,i}^{(1)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \left( \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)^{m-1} \quad (221)$$

and similarly, the introduction of:

$$\bar{\Xi}_1^{(l_1)} \left( Z_i, \{Z_{k_{j,i}^{(1)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \dots \bar{\Xi}_1^{(l_{p_i})} \left( Z_i, \{Z_{k_{j,i}^{(p_i)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right)$$

implies a contribution:

$$\frac{1}{(m-p_i)!} \bar{\Xi}_1^{(l_1)} \left( Z_i, \{Z_{k_{j,i}^{(1)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \dots \bar{\Xi}_1^{(l_{p_i})} \left( Z_i, \{Z_{k_{j,i}^{(p_i)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \left( \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta \right) \right)^{m-p_i}$$

Ultimately, we have to take into account  $n$  external propagators. The dominant contribution at the lowest order in perturbation is obtained by factoring these graphs:

$$\begin{aligned} & \Gamma_{\text{1PI}} \left( \left( Z_i, \left( \theta_i^{(i)}, \theta_f^{(i)} \right) \right)_n \right) \\ &= \sum_{p_1, \dots, p_n} \sum_{\substack{l_1^{(i)}, \{Z_{k_{j,i}^{(1)}}\}, \dots, l_{p_i}^{(i)}, \{Z_{k_{j,i}^{(p_i)}}\} \\ \sum_i (l_1^{(i)} + \dots + l_{p_i}^{(i)}) = n}} \\ & \times \prod_{i, p_i \neq 0} \left( \left( \exp \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \prod_{k=1}^{p_i} \left( \bar{\Xi}_1^{(l_k)} \left( Z_i, \{Z_{k_{j,i}^{(k)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \frac{\nabla_{\theta_i^{(i)}}^{\text{out}}}{\Lambda_1} \right) \right) \\ & \times \prod_{i, p_i=0} \left( 1 + \left( \left( \exp \left( \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1 \right) \frac{\nabla_{\theta_i^{(i)}}^{\text{out}}}{\Lambda_1} \right) \right) \times \frac{\exp \left( -\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right) \right)}{\Lambda^n} \end{aligned}$$

Including the other perturbative contributions is straightforward. The sum of 1PI graphs becomes:

$$\begin{aligned} & \Gamma_{\text{1PI}} \left( \left( Z_i, \left( \theta_i^{(i)}, \theta_f^{(i)} \right) \right)_n \right) \tag{222} \\ &= \sum_i \left( \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \exp \left( \bar{\Xi}_1^{(n)} \left( Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \frac{\nabla_{\theta_i^{(i)}}^{\text{out}}}{\Lambda_1} \right) \right) \\ & \times \prod_{j \neq i} \left( 1 + \left( \left( \exp \left( \bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \right) - 1 \right) \frac{\nabla_{\theta_i^{(j)}}^{\text{out}}}{\Lambda_1} \right) \right) \times \Lambda^n \exp \left( -\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right) \right) \\ & + \sum_{p_1, \dots, p_n} \sum_{\substack{l_1^{(i)}, \{Z_{k_{j,i}^{(1)}}\}, \dots, l_{p_i}^{(i)}, \{Z_{k_{j,i}^{(p_i)}}\} \\ \sum_i (l_1^{(i)} + \dots + l_{p_i}^{(i)}) = n}} \\ & \prod_{i, p_i \neq 0} \left( \left( \exp \left( \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta \right) \right) \right) \prod_{k=1}^{p_i} \bar{\Xi}_1^{(l_k)} \left( Z_i, \{Z_{k_{j,i}^{(k)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \frac{\nabla_{\theta_i^{(i)}}^{\text{out}}}{\Lambda_1} \right) \\ & \times \prod_{i, p_i=0} \left( 1 + \left( \left( \exp \left( \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1 \right) \frac{\nabla_{\theta_i^{(i)}}^{\text{out}}}{\Lambda_1} \right) \right) \times \frac{\exp \left( -\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right) \right)}{\Lambda^n} \end{aligned}$$

The last sum is implicitly constrained by imposing that every propagator for  $Z_j$  between  $\theta_i^{(j)}$  and  $\theta_f^{(j)}$  should be connected to at least one other points  $Z_{k_1}$ .

## 4.2 inclusion of inertia coefficients

The connected graphs can be approximated as before apart from some modification in the sum of connected graphs . By a similar reasoning that led to (221), the sum (159) is replaced by:

$$\begin{aligned} & \sum_{m>0} \frac{(m-1)}{m!} \hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)^{m-1} \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out} \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \\ & + \sum_{m>0} \frac{1}{(m)!} \hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)^{m-1} \frac{-\zeta^{(n)} + \frac{\nabla_{\theta_i^{(i)}}^{out} \hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\zeta^{(n)} + \frac{\hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \end{aligned}$$

The  $(m-1)$  factor in the first term holds for dispatching the insertion  $\hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right)$  at the  $m-1$  first positions, leaving the last one for derivative factor. The second term accounts for inserting  $\hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right)$  (with derivative) in last position. This leads to the following expression for the sum of vertices:

$$\begin{aligned} & \left( -\frac{\zeta^{(n)}}{\Lambda^n} \left( \theta_f^{(i)} - \theta_i^{(i)} \right) + \frac{\nabla_{\theta_i^{(i)}}^{out} \hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda_1} \right) \frac{\exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1}{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)} \\ & + \hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \left( \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - \frac{\exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1}{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)} \right) \\ & \times \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out} \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \end{aligned}$$

and this expression can be also written for later purposes:

$$\begin{aligned} & \left( -\frac{\zeta^{(n)}}{\Lambda^n} \left( \theta_f^{(i)} - \theta_i^{(i)} \right) + \frac{\nabla_{\theta_i^{(i)}}^{out} \hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda_1} \right) \\ & + \left( -\frac{\zeta^{(n)}}{\Lambda^n} \left( \theta_f^{(i)} - \theta_i^{(i)} \right) + \frac{\nabla_{\theta_i^{(i)}}^{out} \hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda_1} \right) \\ & \times \frac{\exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1 - \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)} \\ & + \hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \\ & \times \left( \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - \frac{\exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1}{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)} \right) \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out} \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \end{aligned}$$



For  $\bar{\zeta}_n \gg \hat{\Xi}_1^{(n)} \left( Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)} \right)$ , this expression reduces to

$$\hat{\Xi}_1^{(n)} \left( Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out} \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}}$$

and we will use this expression to estimate the various contributions, coming back later to the full expression for the vertices in order to find a precise expression for the effective action. The sum of one particle irreducible graphs becomes thus:

$$\begin{aligned} & \Gamma_{1PI} \left( \left( Z_i, \left( \theta_i^{(i)}, \theta_f^{(i)} \right) \right)_n \right) \tag{223} \\ &= \sum_i \left( \hat{\Xi}_1^{(n)} \left( Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \frac{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}} \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \\ & \times \prod_{j \neq i} \left( 1 + \left( \exp \left( \hat{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \right) - 1 \right) \frac{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}}} \right) \\ & \times \frac{\exp \left( -\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right) \right)}{\Lambda^n} \\ & + \sum_{p_1, \dots, p_n} \sum_{l_1^{(i)}, \{Z_{k_j^{(1)}}, \dots, l_{p_i}^{(i)}, \{Z_{k_j^{(p_i)}}\}}_{\sum_i (l_1^{(i)} + \dots + l_{p_i}^{(i)}) = n}} \prod_{i, p_i \neq 0} \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \\ & \times \left( \prod_{k=1}^{p_i} \hat{\Xi}_1^{(l_k^{(i)})} \left( Z_i, \{Z_{k_j^{(k)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \left( \frac{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}} \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \\ & \times \prod_{i, p_i=0} \left( 1 + \left( \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) - 1 \right) \frac{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}} \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \\ & \times \frac{\exp \left( -\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right) \right)}{\Lambda^n} \end{aligned}$$

with:

$$\begin{aligned} & \hat{\Xi}_1^{(l_k^{(i)})} \left( Z_i, \{Z_{k_j^{(k)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) = \hat{\Xi}_1^{(l_k^{(i)})} \left( Z_i, \{Z_{k_j^{(k)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) - \frac{\zeta^{(l)}}{\Lambda^{l-1}} \left( \theta_f^{(i)} - \theta_i^{(i)} \right) \\ & \hat{\Xi}_{\min} \left( \left( Z_i, l_1^{(i)}, \{Z_{k_j^{(1)}}\}, \dots, l_{p_i}^{(i)}, \{Z_{k_j^{(p_i)}}\} \right)_{i=1, \dots, n} \right) = \prod_i \hat{\Xi}^{(l_1)} \left( Z_i, \{Z_{k_j^{(1)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \dots \hat{\Xi}_1^{(l_{p_i})} \left( Z_i, \{Z_{k_j^{(p_i)}}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \end{aligned}$$

The sum of graphs can thus be decomposed in a sum of two terms:

$$\Gamma_{1PI} \left( \left( Z_i, \left( \theta_i^{(i)}, \theta_f^{(i)} \right) \right)_n \right) = T_1^{(n)} + T_2^{(n)} \tag{224}$$

### 4.3 Effective action

Having determined the sum of 1PI graphs we can write a series expansion of the effective action for an arbitrary field  $\Psi$ . Actually, the  $2n$ -th order contribution to the generating functional is obtained directly from (222):

$$\begin{aligned} & V_n \left( \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right), \Psi \left( \theta_i^{(i)}, Z_i \right) \right) \\ &= -\frac{1}{n!} \prod_{i=1}^n \int \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right) \mathcal{G}_0^{-1} \Gamma_{\text{1PI}} \left( \left( Z_i, \left( \theta_i^{(i)}, \theta_f^{(i)} \right) \right)_n \right) \\ & \quad \times \frac{\exp \left( -\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right) \right)}{\Lambda^n} \mathcal{G}_0^{-1} \Psi \left( \theta_i^{(i)}, Z_i \right) d\theta_f^{(i)} d\theta_i^{(i)} dZ_i \end{aligned} \quad (225)$$

where  $\mathcal{G}_0^{-1}$  is the inverse propagator whose role is to remove external legs of the graphs. In this paragraph, we change the variable  $\mathcal{G}_0^{-1} \Psi \left( \theta_i^{(i)}, Z_i \right) \rightarrow \Psi \left( \theta_i^{(i)}, Z_i \right)$  and  $\Psi^\dagger \left( \theta_f^{(i)}, Z_i \right) \mathcal{G}_0^{-1} \rightarrow \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right)$ . The impact of the factors  $\mathcal{G}_0^{-1}$  will be computed once an expression for the effective action will be found.

Using (224), we have  $\Gamma_{\text{1PI}} \left( \left( Z_i, \left( \theta_i^{(i)}, \theta_f^{(i)} \right) \right)_n \right) = \sum_{n \geq 2} \left( T_1^{(n)} + T_2^{(n)} \right)$ , so that:

$$V_n \left( \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right), \Psi \left( \theta_i^{(i)}, Z_i \right) \right) = S_1 + S_2$$

where:

$$S_i = \sum_{n \geq 2} \int \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right) T_i^{(n)} \frac{1}{n!} \frac{\exp \left( -\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(i)} - \sum_{j=1}^n \theta_i^{(i)} \right) \right)}{\Lambda^n} \Psi \left( \theta_i^{(i)}, Z_i \right) d\theta_f^{(i)} d\theta_i^{(i)} \quad (226)$$

The generating functional is obtained by summing the lowest order contribution (222) and the terms (225). It has the form:

$$\begin{aligned} & -\frac{1}{2} \Psi^\dagger \left( \theta, Z \right) \left( \nabla_\theta \frac{\sigma_\theta^2}{2} \nabla_\theta \right) \Psi \left( \theta, Z \right) + \frac{1}{2} |\Psi|^2 \left[ \frac{\delta \left[ \Psi^\dagger \left( \theta', Z \right) \nabla_\theta \left( \omega^{-1} \left( |\Psi \left( \theta, Z \right)|^2 \right) \Psi \left( \theta, Z \right) \right) \right]}{\delta |\Psi|^2} \right]_{|\Psi \left( \theta, Z \right)|^2 = \mathcal{G}_0 \left( 0, Z \right)} \\ & + \alpha \int \left| \Psi \left( \theta^{(i)}, Z_i \right) \right|^2 + \sum_{n \geq 2} V_n \left( \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right), \Psi \left( \theta_i^{(i)}, Z_i \right) \right) \end{aligned} \quad (227)$$

As computed before:

$$\begin{aligned} & -\frac{1}{2} \left( \nabla_\theta \frac{\sigma_\theta^2}{2} \nabla_\theta \right) + \frac{1}{2} \left[ \frac{\delta \left[ \Psi^\dagger \left( \theta', Z \right) \nabla_\theta \left( \omega^{-1} \left( |\Psi \left( \theta, Z \right)|^2 \right) \Psi \left( \theta, Z \right) \right) \right]}{\delta |\Psi|^2} \right]_{|\Psi \left( \theta, Z \right)|^2 = \mathcal{G}_0 \left( 0, Z \right)} \\ &= -\frac{1}{2} \left( \nabla_\theta \frac{\sigma_\theta^2}{2} \nabla_\theta \right) + \frac{1}{2} \left[ \nabla_\theta \omega^{-1} \left( J \left( \theta \right), \theta, Z, \mathcal{G}_0 \left( 0, Z \right) \right) \right] \\ & \quad + \frac{1}{2} \left[ \Psi^\dagger \left( \theta', Z \right) \frac{\delta \left[ \nabla_\theta \left( \omega^{-1} \left( |\Psi \left( \theta, Z \right)|^2 \right) \right) \right]}{\delta |\Psi|^2} \Psi \left( \theta, Z \right) \right]_{|\Psi \left( \theta, Z \right)|^2 = \mathcal{G}_0 \left( 0, Z \right)} \\ & \simeq -\frac{1}{2} \left( \nabla_\theta \frac{\sigma_\theta^2}{2} \nabla_\theta \right) + \frac{1}{2} \left[ \nabla_\theta \omega^{-1} \left( J \left( \theta \right), \theta, Z, \mathcal{G}_0 \left( 0, Z \right) \right) \right] \end{aligned}$$

where  $\omega^{-1} \left( J \left( \theta \right), \theta, Z, \mathcal{G}_0 \left( 0, Z \right) \right)$  is solution of:

$$\omega^{-1} \left( \theta, Z \right) = \mathcal{G}_0 \left( J \left( \theta \right) + \frac{\kappa}{N} \int \frac{T \left( Z, Z_1 \right) \omega \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right) W \left( \frac{\omega \left( \theta, Z \right)}{\omega \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right)} \right) dZ_1}{\sqrt{\frac{\pi}{8} \left( \frac{1}{X_r} \right)^2 + \frac{\pi}{2} \alpha \omega \left( Z \right)}} \right)$$

The potential  $\sum_{n \geq 2} V_n \left( \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right), \Psi \left( \theta_i^{(i)}, Z_i \right) \right)$  is the sum of 1PI graphs  $\Gamma_{1\text{PI}} \left( \left( Z_i, \left( \theta_i^{(i)}, \theta_f^{(i)} \right) \right)_n \right) = \sum_{n \geq 2} \left( T_1^{(n)} + T_2^{(n)} \right)$  (see (224)) multiplied by:

$$\times \frac{\Lambda^n}{n!} \exp \left( -\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right) \right) |\Psi|^{2n}$$

As a consequence, the effective action as a sum of the classical action and two higher order terms:

$$\Gamma(\Psi) = -\frac{1}{2} \Psi^\dagger(\theta, Z) \left( \nabla_\theta \frac{\sigma_\theta^2}{2} (\nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z))) \right) \Psi(\theta, Z) + \alpha \int \left| \Psi \left( \theta^{(i)}, Z_i \right) \right|^2 + S_1 + S_2 \quad (228)$$

We will see that for  $\frac{\zeta}{\Lambda}$  of order 1, the term  $S_1$  dominates. For this term, only the contributions for small  $n$  are relevant, the others are dampened. For  $\frac{\zeta}{\Lambda} < 1$ , both terms  $S_1$  and  $S_2$  a priori contribute, but the contributions involving the derivatives are dominant in  $S_1$  since they are of order  $\hat{\Xi}_1^{(n-1)}$ , while those coming from  $S_2$  are of order  $\hat{\Xi}_1^{(n)}$ . We show below that  $S_2$  can be neglected with respect to  $S_1$  to find the saddle point.

#### 4.3.1 Estimation of $S_1$ for an arbitrary field

Using the previous form (228) of the effective action, the first term  $S_1$  writes :

$$\begin{aligned} S_1 &= \sum_{n \geq 2} \frac{1}{n!} \left( \prod_i \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right) \right) \frac{\exp \left( -\Lambda_1 \left( \sum_{j=1}^n \theta_f^{(j)} - \sum_{j=1}^n \theta_i^{(j)} \right) \right)}{\Lambda^n} \\ &\times \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \times \hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \frac{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \nabla_{\theta_i^{(i)}}^{out}}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \\ &\times \prod_{j \neq i} \left( 1 + \left( \exp \left( \hat{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \right) - 1 \right) \frac{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \nabla_{\theta_i^{(j)}}^{out}}{\theta_f^{(j)} - \theta_i^{(j)}}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}}} \right) \prod_i \Psi \left( \theta_i^{(i)}, Z_i \right) \end{aligned} \quad (229)$$

The coefficients  $\hat{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right)$  include product of heaviside functions  $H \left( \theta^{(i)} - \theta^{(j)} - \frac{|Z_i - Z_j|}{c} \right)$ .

We want to approximate the products of terms for  $j \neq i$ . We assume that  $-\bar{\zeta}_n$  and  $\bar{\Xi}_{1,n}$  grow approxima-

tively at the same rate, so that  $\frac{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \nabla_{\theta_i^{(j)}}^{out}}{\theta_f^{(j)} - \theta_i^{(j)}}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}}}$  depends weakly on  $n$  and can be replaced

by its limit for  $n \rightarrow \infty$ . We also replace in the product the terms  $\hat{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)$  and  $\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)$  by their limit  $\hat{\Xi}_{1,\infty} \left( Z_i, \theta_i^{(j)}, \theta_f^{(j)} \right)$  and  $\bar{\Xi}_{1,\infty} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)$ . As a consequence:

$$\frac{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \nabla_{\theta_i^{(j)}}^{out}}{\theta_f^{(j)} - \theta_i^{(j)}}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}}} \simeq \frac{-\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty} \left( Z_j, \theta_i^{(j)}, \theta_f^{(j)} \right) \nabla_{\theta_i^{(j)}}^{out}}{\theta_f^{(j)} - \theta_i^{(j)}}}{-\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty} \left( Z_j, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}}}$$

This allows to rewrite:

$$\begin{aligned}
& \int \Psi^\dagger(\theta_f^{(j)}, Z_j) \left( 1 + \left( \left( \exp\left(\hat{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})\right) - 1 \right) \frac{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{\text{out}}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}}} \right) \right) \\
& \times \exp\left(-\Lambda_1(\theta_f^{(j)} - \theta_i^{(j)})\right) \Psi(\theta_i^{(j)}, Z_j) dZ_j \\
& \simeq \int \Psi^\dagger(\theta_f^{(j)}, Z_j) \left( 1 + \left( \left( \exp\left(\hat{\Xi}_{1,\infty}(Z_i, \theta_i^{(j)}, \theta_f^{(j)})\right) - 1 \right) \frac{-\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty}(Z_j, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{\text{out}}}{\Lambda_1}}{-\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty}(Z_j, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}}} \right) \right) \\
& \times \exp\left(-\Lambda_1(\theta_f^{(j)} - \theta_i^{(j)})\right) \Psi(\theta_i^{(j)}, Z_j) dZ_j \\
& \equiv \int \Psi^\dagger(\theta_f^{(j)}, Z_j) \left( H_1(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) + H_2(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) \frac{\nabla_{\theta_i^{(j)}}^{\text{out}}}{\Lambda_1} \right) \Psi(\theta_i^{(j)}, Z_j) dZ_j \tag{230}
\end{aligned}$$

where:

$$\begin{aligned}
H_1(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) &= \left( 1 + \left( \frac{-\bar{\zeta} \left( \exp\left(\hat{\Xi}_{1,\infty}(Z_j, \theta_i^{(j)}, \theta_f^{(j)})\right) - 1 \right)}{-\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty}(Z_j, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}}} \right) \right) \exp\left(-\Lambda_1(\theta_f^{(j)} - \theta_i^{(j)})\right) \\
H_2(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) &= \frac{\left( \exp\left( \left( -\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty}(Z_j, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \right) (\theta_f^{(j)} - \theta_i^{(j)}) \right) - 1 \right) \frac{\bar{\Xi}_{1,\infty}(Z_j, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}}}{-\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty}(Z_j, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}}} \exp\left(-\Lambda_1(\theta_f^{(j)} - \theta_i^{(j)})\right)
\end{aligned}$$

The function  $H(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)})$  represents the average dependence of  $Z_j$  in the whole system. We define:

$$\begin{aligned}
\hat{\Xi}_{1,\infty}(Z_i, \{Z_j\}_{j \neq i}) &= \frac{\hat{\Xi}_{1,\infty}(Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(j)} - \theta_i^{(j)}} \tag{231} \\
\bar{\Xi}_{1,\infty}(Z_i, \{Z_j\}_{j \neq i}) &= \frac{\bar{\Xi}_{1,\infty}(Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(j)} - \theta_i^{(j)}}
\end{aligned}$$

In (231) the quantities without time are averaged over  $\theta_f^{(i)} - \theta_i^{(i)}$ . In addition to that, we can also consider the averages over the  $\{Z_j\}_{j \neq i}$  and define:

$$\begin{aligned}
\hat{\Xi}_{1,\infty}(Z_i) &= \left\langle \frac{\hat{\Xi}_{1,\infty}(Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(j)} - \theta_i^{(j)}} \right\rangle_{\{Z_j, j \neq i\}} \tag{232} \\
\bar{\Xi}_{1,\infty}(Z_i) &= \left\langle \frac{\bar{\Xi}_{1,\infty}(Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(j)} - \theta_i^{(j)}} \right\rangle_{\{Z_j, j \neq i\}}
\end{aligned}$$

As explained before the term  $\hat{\Xi}_1^{(n)}(Z_i, \{Z_{j,j \neq i}\})$  includes product of heaviside functions. Quantities  $\hat{\Xi}_1^{(n)}(Z_i, \{Z_{j,j \neq i}\})$ ,  $\hat{\Xi}_{1,n}(Z_i, \{Z_{j,j \neq i}\})$  depend implicitly on  $\theta^{(i)}$  and  $\theta^{(j)}$ . Then, (229) writes:

$$\begin{aligned}
S_1 &= \sum_{n \geq 2} \sum_{i,j} \frac{1}{n!} \int \Psi^\dagger(\theta_f^{(i)}, Z_i) \\
&\times \hat{\Xi}_1^{(n)}(Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)}) \exp\left(\left(-\Lambda_1 + \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i})\right) (\theta_f^{(i)} - \theta_i^{(i)})\right) \\
&\times \left( \frac{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}} \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \Psi(\theta_i^{(i)}, Z_i) \\
&\times \prod_{j \neq i} \int \Psi^\dagger(\theta_f^{(j)}, Z_j) \left( H_1(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) + H_2(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \right) \Psi(\theta_i^{(j)}, Z_j) dZ_j
\end{aligned} \tag{233}$$

it allows to rewrite (229) by taking into account that there are  $n$  possibilities to attribute  $\hat{\Xi}_1^{(n)}$  to a point:

$$\begin{aligned}
S_1 &\simeq \sum_{n \geq 2} \frac{1}{(n-1)!} \int \Psi^\dagger(\theta_f^{(i)}, Z_i) \exp\left(\left(-\Lambda_1 + \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i})\right) (\theta_f^{(i)} - \theta_i^{(i)})\right) \\
&\times \left( \hat{\Xi}_1^{(n)}(Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)}) \frac{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}} \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \Psi(\theta_i^{(i)}, Z_i) \\
&\times \left( \int \Psi^\dagger(\theta_f^{(i)}, Z_i) \left( H_1(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) + \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} H_2(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) \right) \Psi(\theta_i^{(i)}, Z_i) dZ_j \right)^{n-1}
\end{aligned} \tag{234}$$

#### 4.3.2 Estimation of $S_2$ for an arbitrary field

The contribution of  $S_2$  can be computed similarly to (234) under the same assumptions and approximations. The difference comes from the insertion of vertices at different points in (222). To each factor:

$$\left( \left( \exp\left(\left(\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})\right)\right) \prod_{k=1}^{p_i} \left( \hat{\Xi}_1^{(l_k^{(i)})}(Z_i, \{Z_{k_j^{(k)}, i}\}, \theta_i^{(i)}, \theta_f^{(i)}) \right) \right) \right)$$

is associated a contribution:

$$\begin{aligned}
&\int \Psi^\dagger(\theta_f^{(i)}, Z_i) \exp\left(\left(-\Lambda_1 + \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i})\right) (\theta_f^{(i)} - \theta_i^{(i)})\right) \left( \prod_{k=1}^{p_i} \hat{\Xi}_1^{(l_k^{(i)})}(Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)}) \right) \\
&\times \left( \frac{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}} \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \Psi(\theta_i^{(i)}, Z_i)
\end{aligned}$$

We first compute the contribution for  $p_i = 1$ . Assuming  $p$  vertices  $\hat{\Xi}_1^{(l^{(i)})}(Z_i, \{Z_{k_j^{(k)}, i}\}, \theta_i^{(i)}, \theta_f^{(i)})$ , the lowest part in perturbation theory is obtained for  $\sum l^{(i)} = n$ . by sharing the  $n$  points among  $p$  vertices. Once the  $p$  vertices are given, the connected contributions are obtained first by distributing the remaining  $n - p$  points.

We attach  $l^{(1)} - l_{(1)}^{(1)}$  of these points to  $Z_1, \dots$   $l^{(p)} - l_{(1)}^{(p)}$  to  $Z_p$ . We have  $\sum (l^{(i)} - l_{(1)}^{(i)}) = n - p$ . There is:

$$C_{n-p}^{l^{(1)} - l_{(1)}^{(1)}} C_{n-p - (l^{(1)} - l_{(1)}^{(1)})}^{l^{(2)} - l_{(1)}^{(2)}} \dots C_{n-p - ((l^{(1)} - l_{(1)}^{(1)}) + \dots + l^{(p-1)} - l_{(1)}^{(p-1)})}^{l^{(p)} - l_{(1)}^{(p)}} = \frac{(n-p)!}{(l^{(1)} - l_{(1)}^{(1)})! \dots (l^{(p)} - l_{(1)}^{(p)})!}$$

possibilities. Then, we have  $p$  blocks each having vertices with valence  $l_{(1)}^{(i)}$  and  $\sum l_{(1)}^{(i)} = p$ . Each block contains  $l^{(i)} - l_{(1)}^{(i)} + 1$  lines. To compute 1PI graphs, one has to consider  $l_{(1)}^{(i)} \leq 2$ . The number of vertices is  $\sum_{i=1}^p \left(1 - \delta_{0, l_{(1)}^{(i)}}\right) = p_1$ . Among these vertices  $p - p_1$  have valence 2 and  $p - p_1$  have valence 0. The last  $2p_1 - p$  have valence 1. As a consequence,  $p_1 > \frac{p}{2}$ . The blocks have to be linked to form a loop of length  $p$  including all blocks. The blocks of valence 1 are grouped in  $k$  blocks of valence 1 for a factor

$$\begin{aligned} \sum_{r_i} C_{2p_1-p}^{r_1} \cdots C_{2p_1-p-r_1-\dots-r_{k-1}}^{r_k} r_1! \dots r_k! &= (2p_1 - p)! \sum_{r_i} 1 \\ &= (2p_1 - p)! P(2p_1 - p, k) \\ &\simeq (2p_1 - p)! \frac{(2p_1 - p - k)^{k-1}}{(k-1)!} \end{aligned}$$

These blocks are linked to  $k$  blocks of valence 0. This yields a factor  $C_{p-p_1}^k$  and a factor:

$$(2p_1 - p)! \sum_k C_{p-p_1}^k \frac{(2p_1 - p - k)^{k-1}}{(k-1)!}$$

This produces  $p-p_1$  blocks of valence 0 and  $p-p_1$  blocks of valence 2. There are  $(p-p_1-1)!(p-p_1)!$  possible loops. Each block of valence 0 can be attached in  $l^{(i)}+1$  ways, each block of valence 1 in  $l^{(i)}$  ways. The blocks of valence 0 are attached twice. This yields a factor:  $\prod_{i=1}^p \left(1 + \delta_{1, l_{(1)}^{(i)}} \left(l^{(i)} - l_{(1)}^{(i)} - 1\right) + \delta_{0, l_{(1)}^{(i)}} \left(l^{(i)} - l_{(1)}^{(i)}\right)\right)^{1+\delta_{0, l_{(1)}^{(i)}}}$ .

The factor is:

$$(p-p_1-1)!(p-p_1)!(2p_1-p)! \sum_k C_{p-p_1}^k \frac{(2p_1-p-k)^{k-1}}{(k-1)!} \prod_{i=1}^p \left(1 + \delta_{1, l_{(1)}^{(i)}} \left(l^{(i)} - l_{(1)}^{(i)} - 1\right) + \delta_{0, l_{(1)}^{(i)}} \left(l^{(i)} - l_{(1)}^{(i)}\right)\right)^{1+\delta_{0, l_{(1)}^{(i)}}}$$

$$\begin{aligned} N\left(n, p, \left(l^{(i)}\right)_{i=1, \dots, p}\right) &\simeq \sum_{0 \leq l_{(1)}^{(i)} \leq (2, l^{(1)})} \frac{(n-p)!}{\left(l^{(1)} - l_{(1)}^{(1)}\right)! \dots \left(l^{(p)} - l_{(1)}^{(p)}\right)!} \sum_{p_1} \delta\left(p_1 - \sum_{i=1}^p \left(1 - \delta_{0, l_{(1)}^{(i)}}\right)\right) \\ &\times (p-p_1-1)!(p-p_1)!(2p_1-p)! \sum_k C_{p-p_1}^k \frac{(2p_1-p-k)^{k-1}}{(k-1)!} \\ &\times \prod_{i=1}^p \left(1 + \delta_{1, l_{(1)}^{(i)}} \left(l^{(i)} - l_{(1)}^{(i)} - 1\right) + \delta_{0, l_{(1)}^{(i)}} \left(l^{(i)} - l_{(1)}^{(i)}\right)\right)^{1+\delta_{0, l_{(1)}^{(i)}}} \end{aligned} \quad (235)$$

it can be approximated by the average for  $l_{(1)}^{(1)} = 1$  and a number of terms such that  $p-p_1$  vertices have valence 2,  $p-p_1$  have valence 0 and the last  $2p_1-p$  have valence 1., that is  $\frac{p!}{(p-p_1)!(p-p_1)!(2p_1-p)!}$  terms. We can also approximate:

$$\begin{aligned} \frac{\prod_{i=1}^p \left(1 + \delta_{1, l_{(1)}^{(i)}} \left(l^{(i)} - l_{(1)}^{(i)} - 1\right) + \delta_{0, l_{(1)}^{(i)}} \left(l^{(i)} - l_{(1)}^{(i)}\right)\right)}{\left(l^{(1)} - l_{(1)}^{(1)}\right)! \dots \left(l^{(p)} - l_{(1)}^{(p)}\right)!} &\simeq \frac{\prod_{l_{(1)}^{(i)}=2} \left(l^{(i)} - 1\right) \prod_{l_{(1)}^{(i)}=1} l^{(i)} \prod_{l_{(1)}^{(i)}=0} \frac{\left(l^{(i)}+1\right)^2}{l^{(i)}}}{\left(l^{(1)} - 1\right)! \dots \left(l^{(p)} - 1\right)!} \\ &\simeq \frac{\prod l^{(i)}}{\left(l^{(1)} - 1\right)! \dots \left(l^{(p)} - 1\right)!} \end{aligned}$$

$$N\left(n, p, \left(l^{(i)}\right)_{i=1, \dots, p}\right) \simeq \frac{(n-p)! \prod l^{(i)}}{\left(l^{(1)} - 1\right)! \dots \left(l^{(p)} - 1\right)!} \sum_{p_1} \frac{p!}{(p-p_1-1)} \sum_k C_{p-p_1}^k \frac{(2p_1-p-k)^{k-1}}{(k-1)!}$$

The sum over  $k$  can be replaced by its maximal value. Numerically, this corresponds to  $k \simeq \frac{p-p_1+1}{2}$  so that:

$$\begin{aligned} & \sum_k C_{p-p_1}^k \frac{(2p_1 - p - k)^{k-1}}{(k-1)!} \\ & \simeq \frac{(p-p_1)!}{(p-p_1-1) \left(\frac{p-p_1-1}{2}\right)! \left(\frac{p-p_1+1}{2}\right)!} \frac{\left(\frac{5p_1-3p-1}{2}\right)^{\frac{p-p_1-1}{2}}}{\left(\frac{p-p_1-1}{2}\right)!} \end{aligned}$$

Similarly, the sum over  $p_1 > \frac{p}{2}$  can be replaced by its maximal value, for  $p-p_1 \simeq \frac{p}{4}$  and:

$$\begin{aligned} \sum_{p_1} \frac{(p-p_1)!}{(p-p_1-1) \left(\frac{p-p_1-1}{2}\right)! \left(\frac{p-p_1+1}{2}\right)!} \frac{\left(\frac{2p-5(p-p_1)-1}{2}\right)^{\frac{p-p_1-1}{2}}}{\left(\frac{p-p_1-1}{2}\right)!} & \simeq \frac{\Gamma\left(\frac{p}{4}+1\right)}{\left(\Gamma\left(\frac{p}{8}+1\right)\right)^3} \left(\frac{3p}{8}\right)^{\frac{p}{8}} \\ & \simeq \exp\left(\frac{p}{8}(2\ln 2 + \ln 3 + 1)\right) \\ & \simeq \exp\left(\frac{p}{2}\right) \end{aligned}$$

and  $N\left(n, p, (l^{(i)})_{i=1, \dots, p}\right)$  can be written:

$$N\left(n, p, (l^{(i)})_{i=1, \dots, p}\right) \simeq \frac{(n-p)! p! \exp\left(\frac{p}{2}\right)}{(l^{(1)}-1)! \dots (l^{(p)}-1)!}$$

The contribution for  $p_i > 1$  and  $\sum_{i, k_i} l_{k_i}^{(i)} = n$ , is obtained by a similar computation. The only difference is that the blocks  $l_{k(1)}^{(i)}$  are attached and to produce 1PI graphs, one has to assume that  $\sum_k l_{k(1)}^{(i)} = 0, 1, 2$  in (235). The number of blocks is thus  $p_1 - (\sum (p_i - 1))$ . The computation yields thus the same approximation if we replace the factor associated to the blocks:

$$(p-p_1-1)! (p-p_1)! 2^{p-p_1} (2p_1-p)! \sum_k C_{p-p_1}^k \frac{(2p_1-p-k)^k}{k!}$$

by the following expression:

$$(p-p'_1-1)! (p-p'_1)! 2^{p-p'_1} (2p'_1-p)! \sum_k C_{p-p'_1}^k \frac{(2p'_1-p-k)^k}{k!}$$

where  $p'_1 = p_1 - (\sum (p_i - 1))$ . This factor diminishes quickly when  $\sum (p_i - 1) \gg 1$ . We can thus discard the contributions with  $p_i > 1$ .

The contribution becomes:

$$\begin{aligned} S_2^{(0)} & \simeq \sum_n \sum_{\{Z_{k_i}\}} \frac{1}{n!} \int \Psi^\dagger\left(\theta_f^{(i)}, Z_i\right) \exp\left(\left(-\Lambda_1 + \hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}\right)\right) \left(\theta_f^{(j)} - \theta_i^{(j)}\right)\right) \times (n-p)! p! \exp\left(\frac{p}{2}\right) X_n^{(0)}(\{Z_{k_i}\}) \\ & \times \left( \frac{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right) \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1}}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \Psi\left(\theta_i^{(i)}, Z_i\right) \\ & \times \left( \int \Psi^\dagger\left(\theta_f^{(i)}, Z_i\right) \left( H_1\left(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}\right) + H_2\left(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}\right) \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \right) \Psi\left(\theta_i^{(i)}, Z_i\right) dZ_j \right)^{n-p} \end{aligned}$$

where:

$$X_n^{(0)}(\{Z_{k_i}\}) = \sum_{l^{(i)}} \prod_{l^{(i)}} \frac{\left\langle \hat{\Xi}_1^{(l^{(i)})}\left(Z_i, \theta^{(i)}, \{Z_{k_j}\}\right) \right\rangle}{(l^{(i)}-1)! C_{n-1}^{l^{(i)}}}$$

$$\left\langle \frac{\Xi_1^{l^{(i)}}(Z_i, \theta^{(i)}, \{Z_{k_j}\})}{C_{n-1}^{l^{(i)}}} \right\rangle = \hat{\Xi}_1^{l_{k_i}^{(i)}}(Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)})$$

The contributions for  $\sum_{i,k_i} l_{k_i}^{(i)} = n + m$  add a factor  $\prod_{\epsilon_{k_i}^{(i)}} C_{n-1}^{l_{k_i}^{(i)}}$  where  $\sum \epsilon_{k_i}^{(i)} = m$ , and, as before, the contributions for  $p_i > 1$  can be neglected. Among the  $l_{k_i}^{(i)}$ ,  $n$  are selected to produce the factor (235) The contributions becomes:

$$\sum_{\epsilon_1 + \dots + \epsilon_p = m} C_{n-1}^{\epsilon_i} \frac{(n-p)! p! \exp\left(\frac{p}{2}\right)}{(l^{(1)} - \epsilon_1 - 1)! \dots (l^{(p)} - \epsilon_p - 1)!}$$

Replacing  $m = \sum_i l^{(i)} - n$ , and estimating the sum by its maximal value for the  $\epsilon_i$  all equal to  $\frac{\sum_i l^{(i)} - n}{p}$ , we get:

$$\frac{C_{n-1}^{\frac{\sum_i l^{(i)} - n}{p}} (n-p)! p! \exp\left(\frac{p}{2}\right)}{\left(l^{(1)} - \frac{\sum_i l^{(i)} - n}{p} - 1\right)! \dots \left(l^{(p)} - \frac{\sum_i l^{(i)} - n}{p} - 1\right)!}$$

The contribution becomes:

$$\begin{aligned} S_2^{(m)} &\simeq \sum_n \sum_{l^{(i)}, \{Z_{k_i}\}} \frac{1}{n!} \int \Psi^\dagger(\theta_f^{(i)}, Z_i) \exp\left(\left(-\Lambda_1 + \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i})\right) (\theta_f^{(j)} - \theta_i^{(j)})\right) \\ &\times (n-p)! p! \exp\left(\frac{p}{2}\right) X_n^{(m)}(\{Z_{k_i}\}) \\ &\times \left( \frac{-\bar{\zeta}_n + \frac{\Xi_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1}}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\Xi_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \Psi(\theta_i^{(i)}, Z_i) \\ &\times \left( \int \Psi^\dagger(\theta_f^{(i)}, Z_i) \left( H_1(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) + H_2(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \right) \Psi(\theta_i^{(i)}, Z_i) dZ_j \right)^{n-p} \end{aligned}$$

with:

$$\begin{aligned} X_n^{(m)}(\{Z_{k_i}\}) &= \sum_{\sum_{i=1}^p l^{(i)} = n+m} \prod_{l^{(i)}} \frac{C_{n-1}^{\frac{\sum_i l^{(i)} - n}{p}} \left\langle \Xi_1^{l^{(i)}}(Z_i, \theta^{(i)}, \{Z_{k_j}\}) \right\rangle}{\left(l^{(i)} - \frac{\sum_i l^{(i)} - n}{p} - 1\right)! C_{n-1}^{l^{(i)}}} \\ &= \sum_{\sum_{i=1}^p l^{(i)} = n+m} \prod_{l^{(i)}} \frac{C_{n-1}^{\frac{m}{p}} \left\langle \Xi_1^{l^{(i)}}(Z_i, \theta^{(i)}, \{Z_{k_j}\}) \right\rangle}{\left(l^{(i)} - \frac{m}{p} - 1\right)! C_{n-1}^{l^{(i)}}} \end{aligned}$$

The contributions of  $S_2^{(m)}$  to the 1PI graphs are obtained by removing double counting. Defining:

$$Y_n^{(m)}(\{Z_{k_i}\}) = X_n^{(m)}(\{Z_{k_i}\}) \left( 1 - \sum_{k=0}^{m-1} Y_n^{(k)}(\{Z_{k_i}\}) \right)$$



the contribution to be considered is obtained by replacing  $X_n^{(m)}(\{Z_{k_i}\})$  by  $Y_n^{(m)}(\{Z_{k_i}\})$  in  $S_2^{(m)}$ . Rewriting:

$$\begin{aligned}
Y_n^{(m)}(\{Z_{k_i}\}) &= X_n^{(m)}(\{Z_{k_i}\}) \left( 1 - \sum_{k=0}^{m-1} Y_n^{(k)}(\{Z_{k_i}\}) \right) \\
&= X_n^{(m)}(\{Z_{k_i}\}) \left( 1 - \sum_{k=0}^{m-1} X_n^{(k)}(\{Z_{k_i}\}) + \sum_{\substack{k_1=0 \\ k_2=0,}}^{m-1} X_n^{(k_1)}(\{Z_{k_i}\}) X_n^{(k_2)}(\{Z_{k_i}\}) - \dots \right) \\
&= X_n^{(m)}(\{Z_{k_i}\}) \prod_{k=0}^{m-1} \left( 1 - X_n^{(k)}(\{Z_{k_i}\}) \right)
\end{aligned}$$

The quantity  $X_n^{(m)}(\{Z_{k_i}\})$  can be evaluated at the extremum  $l^{(i)} = \frac{n+m}{p}$  this yields that  $X_n^{(m)}(\{Z_{k_i}\})$  is of order:

$$\left( \frac{(n-1-\frac{m}{p})! (\frac{m}{p})!}{(n-1-\frac{n+m}{p})! (\frac{n+m}{p})!} \right)^p \left\langle \hat{\Xi}_1^{(l^{(i)})} (Z_i, \theta^{(i)}, \{Z_{k_j}\}) \right\rangle$$

The factor  $\left( \frac{(n-1-\frac{m}{p})! (\frac{m}{p})!}{(n-1-\frac{n+m}{p})! (\frac{n+m}{p})!} \right)$  is maximal for  $p \rightarrow 2$  and  $m = n-1$  and a value  $\frac{((\frac{n-1}{2})!)^2}{(\frac{n-1}{2})!} < 1$  for  $n > 1$ .

As a consequence  $|X_n^{(m)}(\{Z_{k_i}\})| \ll 1$  in the perturbative regime, so that:

$$X_n^{(m)}(\{Z_{k_i}\}) \prod_{k=0}^{m-1} \left( 1 - X_n^{(k)}(\{Z_{k_i}\}) \right) \simeq X_n^{(m)}(\{Z_{k_i}\}) \exp \left( - \sum_{k=1}^{m-1} X_n^{(k)}(\{Z_{k_i}\}) \right)$$

The factors dampen quickly for  $m$  increasing, so that in first approximation, we can keep the contribution for  $m = 0$ . Since  $C_n^p \frac{(n-p)!}{n!} = \frac{1}{p!}$ , we have the contribution:

$$\begin{aligned}
S_2 &\simeq \sum_n \sum_{\substack{l^{(i)}, \{Z_{k_i}\} \\ \sum_{i=1}^p l^{(i)} = n}} \frac{1}{n!} \int \Psi^\dagger(\theta_f^{(i)}, Z_i) \exp \left( \left( -\Lambda_1 + \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}) \right) (\theta_f^{(j)} - \theta_i^{(j)}) \right) \\
&\times \prod_{l^{(i)}} \hat{\Xi}_1^{(l^{(i)})} (Z_i, \{Z_{j, j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)}) \\
&\times \frac{(n-p)! p! \exp(\frac{p}{2})}{\prod_{l^{(i)}} (l^{(i)} - 1)!} \times \left( \frac{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \nabla_{\theta_i^{(i)}}^{out}}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \Psi(\theta_i^{(i)}, Z_i) \\
&\times \left( \int \Psi^\dagger(\theta_f^{(i)}, Z_i) \left( H_1(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) + H_2(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \right) \Psi(\theta_i^{(i)}, Z_i) dZ_j \right)^{n-p}
\end{aligned}$$

and using that:

$$\begin{aligned} & \hat{\Xi}_1^{(l_k^{(i)})} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \\ &= \left[ -\zeta^{(l_k^{(i)})+1} \left( \theta_f^{(i)} - \theta_i^{(i)} \right) + \int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \left[ \frac{\delta^{l_k^{(i)}+1} \Psi^\dagger \left( \theta^{(i)}, Z_i \right) \nabla_{\theta_i^{(i)}} \omega^{-1} \left( J, \theta^{(i)}, Z_i, \mathcal{G}_0 \left( Z_i \right) \right) \Psi \left( \theta^{(i)}, Z_i \right)}{\prod_{i=1}^{l_k^{(i)}+1} \delta \left| \Psi \left( \theta^{(i)} - \frac{|Z_i - Z_j|}{c}, Z_j \right) \right|^2} \right]_{\substack{|\Psi(\theta, Z)|^2 \\ = \mathcal{G}_0(0, Z)}} dZ_i d\theta^{(i)} \end{aligned}$$

this can be rewritten as:

$$\begin{aligned} S_2 &\simeq \sum_p \frac{\exp\left(\frac{p}{2}\right)}{p!} \left\{ \int \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right) \sum_{l^{(i)}} \frac{\exp\left(\left(-\Lambda_1 + \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i} \right)\right) \left( \theta_f^{(j)} - \theta_i^{(j)} \right)\right)}{(l^{(i)} - 1)!} \right. \\ &\quad \times \left[ \zeta^{(l^{(i)})+1} + \nabla_{\theta_i^{(i)}} \int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \left[ \frac{\delta^{l^{(i)}+1} \Psi^\dagger \left( \theta^{(i)}, Z_i \right) \omega^{-1} \left( J, \theta^{(i)}, Z_i, \mathcal{G}_0 \left( Z_i \right) \right) \Psi \left( \theta^{(i)}, Z_i \right)}{\prod_{i=1}^{l^{(i)}+1} \delta \left| \Psi \left( \theta^{(i)} - \frac{|Z_i - Z_j|}{c}, Z_j \right) \right|^2} \right]_{\substack{|\Psi(\theta, Z)|^2 \\ = \mathcal{G}_0(0, Z)}} dZ_i d\theta^{(i)} \\ &\quad \times \left( \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out} \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda_1} \frac{\theta_f^{(i)} - \theta_i^{(i)}}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \Psi \left( \theta_i^{(i)}, Z_i \right) \\ &\quad \times \left. \left( \int \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right) \left( H_1 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) + H_2 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \right) \Psi \left( \theta_i^{(i)}, Z_i \right) dZ_j \right)^{l^{(i)}-1} \right\}^p \\ &= \exp \left( \sqrt{e} \int \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right) \exp \left( \left(-\Lambda_1 + \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i} \right)\right) \left( \theta_f^{(j)} - \theta_i^{(j)} \right)\right) \right. \\ &\quad \times \int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \frac{\delta}{\Lambda \delta |\bar{\Psi}(\theta, Z)|^2} \left[ U \left( |\bar{\Psi}(\theta, Z)|^2 \right) + \nabla_{\theta_i^{(i)}} \omega^{-1} \left( J, \theta^{(i)}, Z_i, |\bar{\Psi}(\theta, Z)|^2 \right) \right] dZ_i d\theta^{(i)} \\ &\quad \times \left. \left( \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out} \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\Lambda_1} \frac{\theta_f^{(i)} - \theta_i^{(i)}}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \Psi \left( \theta_i^{(i)}, Z_i \right) \right) \end{aligned}$$

where:

$$|\bar{\Psi}(\theta, Z)|^2 = \mathcal{G}_0(0, Z) + \int \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right) \left( H_1 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) + H_2 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \right) \Psi \left( \theta_i^{(i)}, Z_i \right) dZ_j$$

As a consequence, at the lowest order in perturbation theory, or  $\Lambda \gg 1$ , the contribution of  $S_2$  can be neglected compared to  $S_1$  to compute the saddle point field.

### 4.3.3 Effective action

Having shown that the contribution  $S_2$  can be neglected with respect to  $S_1$ , we can sum the expressions (234) for  $n > 1$  with the zeroth order contribution, i.e. the 2-th vertex (142). We use the initial formulation (150) and reintroduce the inverse propagator  $\mathcal{G}_0^{-1}$  which leads thus to the following expression for (229):

$$\begin{aligned} \Gamma(\Psi) \simeq & -\frac{1}{2}\Psi^\dagger(\theta, Z) \left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1} (J(\theta), \theta, Z, \mathcal{G}_0(0, Z)) \right) \right) \Psi(\theta, Z) + \alpha \int \left| \Psi(\theta^{(i)}, Z_i) \right|^2 \\ & + \sum_{n \geq 2} \frac{1}{(n-1)! \Lambda^n} \int \Psi^\dagger(\theta_f^{(i)}, Z_i) \mathcal{G}_0^{-1}(\bar{V}_1^{(n)} + \bar{V}_{1,n}) \mathcal{G}_0^{-1} \Psi(\theta_i^{(i)}, Z_i) \\ & \times \left( \int \Psi^\dagger(\theta_f^{(i)}, Z_i) \mathcal{G}_0^{-1} \left( H_1(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) + \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} H_2(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) \right) \mathcal{G}_0^{-1} \Psi(\theta_i^{(i)}, Z_i) dZ_j \right)^{n-1} \end{aligned} \quad (236)$$

where:

$$\begin{aligned} \bar{V}_1^{(n)} &= \left( -\frac{\zeta^{(n)}}{\Lambda^n} (\theta_f^{(i)} - \theta_i^{(i)}) + \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1} \hat{\Xi}_1^{(n)}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \right) \exp(-\Lambda_1 (\theta_f^{(i)} - \theta_i^{(i)})) \\ \bar{V}_{1,n} &= \left( -\frac{\zeta^{(n)}}{\Lambda^n} (\theta_f^{(i)} - \theta_i^{(i)}) + \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1} \hat{\Xi}_1^{(n)}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \right) \\ & \times \frac{\exp(\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})) - 1 - \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})} \exp(-\Lambda_1 (\theta_f^{(i)} - \theta_i^{(i)})) \\ & + \hat{\Xi}_1^{(n)}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \left( \exp(\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})) - \frac{\exp(\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})) - 1}{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})} \right) \\ & \times \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out}}{\Lambda_1} \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}}} \exp(-\Lambda_1 (\theta_f^{(i)} - \theta_i^{(i)})) \end{aligned}$$

The effective action is the sum of two terms according to the decomposition of the vertex into  $\bar{V}_1^{(n)} + \bar{V}_{1,n}$ . We estimate the two contributions independently.

**Estimation of the term proportional to  $\bar{V}_1^{(n)}$**  Recall from (150) and equations below that the expression:

$$\hat{\Xi}_1^{(n)}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \exp(-\Lambda_1 (\theta_f^{(i)} - \theta_i^{(i)}))$$

stands for:

$$\begin{aligned} & \int_{\theta_i^{(i)} < \theta^{(i)} < \theta_f^{(i)}} \hat{\Xi}_1^{(n)}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}) \exp(-\Lambda_1 (\theta_f^{(i)} - \theta_i^{(i)})) \\ &= \Lambda^2 \int_{\theta_i^{(i)} < \theta^{(i)} < \theta_f^{(i)}} \frac{\exp(-\Lambda_1 (\theta_f^{(i)} - \theta^{(i)}))}{\Lambda} \hat{\Xi}_1^{(n)}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}) \frac{\exp(-\Lambda_1 (\theta^{(i)} - \theta_i^{(i)}))}{\Lambda} \\ &= \Lambda^2 \mathcal{G}_0 * \hat{\Xi}_1^{(n)}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}) * \mathcal{G}_0 \end{aligned} \quad (237)$$

where:

$$\begin{aligned} & \hat{\Xi}_1^{(n)} \left( Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i} \right) \\ & \simeq \frac{\prod_{k_j=1}^{n-1} \int_{\frac{|Z_i - Z_{k_j}|}{c}}^{\theta^{(i)} - \theta^{(k_j)}} dl_{k_j}}{\left( \sqrt{\frac{\pi}{2}} \sqrt{\left( \frac{1}{\sigma^2 X_r} \right)^2 + \frac{2\alpha}{\sigma^2}} \right)^n} \left[ -\zeta^{(n)} + \nabla_{\theta^{(i)}} \mathcal{G}_0(Z) \frac{\delta^{n-1} \int \omega^{-1} \left( J, \theta^{(i)}, Z_i \right) dZ_i}{\mathcal{G}_0(Z) \prod_{i=1}^{n-1} \delta \left| \Psi \left( \theta^{(i)} - l_{k_j}, Z_{k_j} \right) \right|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \end{aligned}$$

The convolution of  $\bar{V}_1^{(n)}$  with  $\mathcal{G}_0^{-1}$  on both side is obtained in (154) and (161) and (162) with inertia coefficients included. The convolution cancels the propagator  $\frac{\exp(-\Lambda_1(\theta_f^{(j)} - \theta_i^{(j)}))}{\Lambda}$ , removes the integral and introduces a  $\delta$  function for the contribution proportional to  $\bar{V}_1^{(n)}$ :

$$\begin{aligned} & \mathcal{G}_0^{-1} \bar{V}_1^{(n)} \mathcal{G}_0^{-1} \\ & \int_{\theta_i^{(i)} < \theta^{(i)} < \theta_f^{(i)}} \mathcal{G}_0^{-1} \left( -\frac{\zeta^{(n)}}{\Lambda^n} \left( \theta_f^{(i)} - \theta_i^{(i)} \right) + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_1^{(n)} \left( Z_i, \{Z_j, j \neq i\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \exp \left( -\Lambda_1 \left( \theta_f^{(j)} - \theta_i^{(j)} \right) \right) \mathcal{G}_0^{-1} \\ & = \Lambda^2 \left( -\frac{\zeta^{(n)}}{\Lambda^n} + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_1^{(n)} \left( Z_i, \theta_f^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \delta \left( \theta_f^{(i)} - \theta_i^{(i)} \right) \end{aligned} \quad (238)$$

The presence of the delta function also localizes the terms in  $\Psi \left( \theta_i^{(j)}, Z_j \right)$  that interact with  $\Psi \left( \theta_i^{(i)}, Z_i \right)$  in the factor:

$$\int \Psi^\dagger \left( \theta_f^{(j)}, Z_j \right) \mathcal{G}_0^{-1} \left( H_1 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) + \frac{\nabla^{out}}{\Lambda_1} H_2 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) \right) \mathcal{G}_0^{-1} \Psi \left( \theta_i^{(j)}, Z_j \right) dZ_j$$

Actually, we have:

$$\begin{aligned} & H_1 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) + \frac{\nabla^{out}}{\Lambda_1} H_2 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) \\ & = \left( 1 + \left( \left( \exp \left( \hat{\Xi}_{1,\infty} \left( Z_i, \theta_i^{(j)}, \theta_f^{(j)} \right) \right) - 1 \right) \frac{-\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty} \left( Z_j, \theta_i^{(j)}, \theta_f^{(j)} \right) \frac{\nabla^{out}}{\Lambda_1}}{\theta_f^{(j)} - \theta_i^{(j)}}}{-\bar{\zeta} + \frac{\bar{\Xi}_{1,\infty} \left( Z_j, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}}} \right) \right) \exp \left( -\Lambda_1 \left( \theta_f^{(j)} - \theta_i^{(j)} \right) \right) \end{aligned}$$

and an expansion similar to (150) yields:

$$\begin{aligned} & H_1 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) + \frac{\nabla^{out}}{\Lambda_1} H_2 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) \\ & = \sum_n \prod_{i=1}^n \left\{ \int \mathcal{G}_0 \left( \theta_f^{(i)}, \theta_{f'}^{(i)}, Z_i \right) \left( \Lambda^2 \delta \left( \theta_{f'}^{(i)} - \theta_{i'}^{(i)} \right) + \Lambda^2 \left( -\bar{\zeta} + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)} \right) \right) \delta \left( \theta_{f'}^{(i)} - \theta_{i'}^{(i)} \right) \right. \right. \\ & \quad \left. \left. + \left( -\bar{\zeta} + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)} \right) \right) \right) \right. \\ & \quad \left. \times \left( \hat{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{f'}^{(i)} \right) \right) \exp \left( \hat{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_{f'}^{(i)} \right) \right) \mathcal{G}_0 \left( \theta_{i'}^{(i)}, \theta_i^{(i)}, Z_i \right) d\theta_{i'}^{(i)} d\theta_{f'}^{(i)} \right\} \end{aligned}$$

which implies that:

$$\begin{aligned}
& \mathcal{G}_0^{-1} \left( H_1 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) + \frac{\nabla^{out}}{\Lambda_1} H_2 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) \right) \mathcal{G}_0^{-1} \\
&= \Lambda^2 \delta \left( \theta_{f'}^{(i)} - \theta_{i'}^{(i)} \right) + \Lambda^2 \left( -\bar{\zeta} + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)} \right) \right) \delta \left( \theta_{f'}^{(i)} - \theta_{i'}^{(i)} \right) \\
&+ \Lambda^2 \left( -\bar{\zeta} + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)} \right) \right) \left( \hat{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{f'}^{(i)} \right) \right) \exp \left( \hat{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)
\end{aligned} \tag{239}$$

The localization of (239) due to the interaction with  $\mathcal{G}_0^{-1} \bar{V}_1^{(n)} \mathcal{G}_0^{-1}$  in (238) induces in first approximation a smeared distribution of Dirac function  $\int_{\frac{|Z_i - Z_j|}{c}}^{\theta_i^{(i)} - \theta_i^{(j)}} dl_j \delta \left( \theta_i^{(i)} - l_j - \frac{|Z_i - Z_j|}{c}, Z_j \right)$  in the two first terms of the expansion of (239). It leads to replace (239) by the contribution:

$$\begin{aligned}
& \Lambda^{2n-2} \int_{\frac{|Z_i - Z_j|}{c}}^{\theta_i^{(i)} - \theta_i^{(j)}} dl_j \int \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right) \left( \left( 1 + \left( -\bar{\zeta} + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)} \right) \right) \right) \right) \\
& \times \delta \left( \theta_i^{(i)} - l_j - \frac{|Z_i - Z_j|}{c}, Z_j \right) \delta \left( \theta_f^{(j)} - \theta_i^{(j)} \right) + \left( -\bar{\zeta} + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)} \right) \right) \\
& \times \left( \hat{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{f'}^{(i)} \right) \right) \exp \left( \hat{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \Psi \left( \theta_i^{(i)}, Z_i \right) dZ_j^{n-1}
\end{aligned} \tag{240}$$

At the lowest order in perturbation, (240) is approximatively equal to:

$$\Lambda^{2n-2} \left( \int_{\frac{|Z_i - Z_j|}{c}}^{\theta_i^{(i)} - \theta_i^{(j)}} dl_j \int \left| \Psi \left( \theta_i^{(i)} - l_j - \frac{|Z_i - Z_j|}{c}, Z_j \right) \right|^2 dZ_j \right)^{n-1}$$

and the product with (238) yields the part of the effective action involving  $\bar{V}_1^{(n)}$ :

$$\begin{aligned}
& \Psi^\dagger \left( \theta_i^{(i)}, Z_i \right) \mathcal{G}_0^{-1} \exp \left( -\Lambda_1 \left( \theta_f^{(j)} - \theta_i^{(j)} \right) \right) \left( -\frac{\zeta^{(n)}}{\Lambda^n} \left( \theta_f^{(i)} - \theta_i^{(i)} \right) + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_1^{(n)} \left( Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \\
& \times \mathcal{G}_0^{-1} \Psi \left( \theta_i^{(i)}, Z_i \right) \\
& \times \left( \int \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right) \mathcal{G}_0^{-1} \left( H_1 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) + \frac{\nabla^{out}}{\Lambda_1} H_2 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) \right) \mathcal{G}_0^{-1} \Psi \left( \theta_i^{(i)}, Z_i \right) dZ_j \right)^{n-1} \\
& = \frac{\prod_{k_j=1}^{n-1} \int_{\frac{|Z_i - Z_{k_j}|}{c}}^{\theta^{(i)} - \theta_i^{(k_j)}} dl_{k_j}}{\Lambda^{2n-2}} \left[ -\zeta^{(n)} + \frac{\delta^{n-1} \int \Psi^\dagger \left( \theta^{(i)}, Z_i \right) \nabla_{\theta^{(i)}} \omega^{-1} \left( J, \theta^{(i)}, Z_i \right) \Psi \left( \theta^{(i)}, Z_i \right) dZ_i}{\prod_{i=1}^{n-1} \delta \left| \Psi \left( \theta^{(i)} - l_{k_j}, Z_{k_j} \right) \right|^2 \delta \left| \Psi \left( \theta^{(i)}, Z_i \right) \right|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \\
& \times \left( \int \Psi^\dagger \left( \theta_f^{(i)}, Z_i \right) \left( \mathcal{G}_0^{-1} \delta \left( \theta_i^{(i)} - l_j - \frac{|Z_i - Z_j|}{c}, Z_j \right) \delta \left( \theta_f^{(j)} - \theta_i^{(j)} \right) \right. \right. \\
& \left. \left. + \mathcal{G}_0^{-1} \left( \left( H_1 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) - 1 \right) + \frac{\nabla^{out}}{\Lambda_1} H_2 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) \right) \mathcal{G}_0^{-1} \right) \Psi \left( \theta_i^{(i)}, Z_i \right) dZ_j \right)^{n-1} \\
& \simeq \frac{\prod_{k_j=1}^{n-1} \int_{\frac{|Z_i - Z_{k_j}|}{c}}^{\theta_q^{(i)} - \theta_i^{(k_j)}} dl_{k_j}}{\Lambda^{2n-2}} \left[ -\zeta^{(n)} + \frac{\delta^n \int \Psi^\dagger \left( \theta^{(i)}, Z_i \right) \nabla_{\theta^{(i)}} \omega^{-1} \left( J, \theta^{(i)}, Z_i \right) \Psi \left( \theta^{(i)}, Z_i \right) dZ_i}{\prod_{i=1}^n \delta \left| \Psi \left( \theta^{(i)} - l_{k_j}, Z_{k_j} \right) \right|^2 \delta \left| \Psi \left( \theta^{(i)}, Z_i \right) \right|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \\
& \times \left( \int \left| \Psi \left( \theta_i^{(i)} - l_j - \frac{|Z_i - Z_j|}{c}, Z_j \right) \right|^2 dZ_j \right)^{n-1}
\end{aligned}$$

**Estimation of the term proportional to  $\bar{V}_{1,n}$**  For the second contribution proportional to  $\bar{V}_{1,n}$ , we use the fact that the series expansion of this term is of second order in interaction. As for the contribution proportional to  $\bar{V}_1^{(n)}$ , the convolution with  $\mathcal{G}_0^{-1}$  on the right replaces the terms:

$$\left( -\frac{\zeta^{(n)}}{\Lambda^n} \left( \theta_f^{(i)} - \theta_i^{(i)} \right) + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_1^{(n)} \left( Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \quad (241)$$

and:

$$-\bar{\zeta}_n \left( \theta_f^{(i)} - \theta_i^{(i)} \right) + \frac{\nabla^{out}}{\Lambda_1} \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \quad (242)$$

by their derivatives evaluated at  $\theta_f^{(i)}$ . These derivative are equal to their average in first approximation, so that (241) and (242) can be replaced by:

$$\left( -\frac{\zeta^{(n)}}{\Lambda^n} + \frac{\nabla^{out}}{\Lambda_1} \frac{\bar{\Xi}_1^{(n)} \left( Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\left( \theta_f^{(i)} - \theta_i^{(i)} \right)} \right)$$

and:

$$-\bar{\zeta}_n + \frac{\nabla^{out}}{\Lambda_1} \frac{\bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\left( \theta_f^{(i)} - \theta_i^{(i)} \right)}$$

Moreover, the left convolution by  $\mathcal{G}_0^{-1}$  replaces one term in the series expansion of:

$$\frac{\exp\left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right) - 1 - \hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)}{\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)}$$

and the series expansion of:

$$\left(\exp\left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right) - \frac{\exp\left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right) - 1}{\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)}\right)$$

by its derivative at  $\theta_i^{(i)}$  or approximatively by its average. This amounts to replace those terms by their derivative multiplied by the averaged term, that is:

$$\frac{\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right) \left(1 + \left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right) - 1\right) \exp\left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right)\right)}{\theta_f^{(i)} - \theta_i^{(i)}} \frac{1}{\left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right)^2}$$

and:

$$\frac{\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)}{\theta_f^{(i)} - \theta_i^{(i)}} \frac{\exp\left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right) \left(1 - \hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right) + \left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right)^2\right) - 1}{\left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right)^2}$$

As consequence, gathering the different terms,  $\bar{V}_{1,n}$  is replaced by:

$$\hat{V}_{1,n}\left(Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)}\right) \left(\frac{\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)}{\theta_f^{(i)} - \theta_i^{(i)}}\right)^2$$

with:

$$\begin{aligned} & \hat{V}_{1,n}\left(Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)}\right) \\ = & \frac{1 + \left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right) - 1\right) \exp\left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right)}{\left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right)^2} \frac{\left(-\frac{\zeta^{(n)}}{\Lambda^n} + \frac{\nabla_{\theta_i^{(i)}}^{out} \bar{\Xi}_1^{(n)}\left(Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)}\right)}{\Lambda_1 \left(\theta_f^{(i)} - \theta_i^{(i)}\right)}\right)}{\frac{\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \\ & + \left(\frac{\exp\left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right) \left(1 - \hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right) + \left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right)^2\right) - 1}{\left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)\right)^2}\right) \\ & \times \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(i)}}^{out} \bar{\Xi}_1^{(n)}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)}{\Lambda_1 \left(\theta_f^{(i)} - \theta_i^{(i)}\right)}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_1^{(n)}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \end{aligned}$$

The corresponding factors multiplying  $\bar{V}_{1,n}$ :

$$\left( \int \Psi^\dagger(\theta_f^{(j)}, Z_j) \mathcal{G}_0^{-1} \left( 1 + \left( \exp\left(\hat{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})\right) - 1 \right) \frac{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}}} \right) \right) \\ \times \exp\left(-\Lambda_1(\theta_f^{(j)} - \theta_i^{(j)})\right) \mathcal{G}_0^{-1} \Psi(\theta_i^{(j)}, Z_j) dZ_j \Big)^{n-1}$$

are localized, depending on the number of graphs issued from  $i$  reach  $j$ . For 1 link only, issued from  $\bar{\Xi}_1^{(n)}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})$ , the factor 1 is replaced by  $\delta(\theta_f^{(j)} - \theta_i^{(j)})$ . For 2 links ore more it remains unchanged. The factor:

$$\left( \exp\left(\hat{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})\right) - 1 \right) \frac{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}}}$$

is replaced, as in the previous paragraph, by:

$$\left( \exp\left(\hat{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})\right) - 1 \right) \frac{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}}} \\ \times \left( \bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \right)^2 \\ = \left( \exp\left(\hat{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})\right) - 1 \right) \\ \times \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \right) \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \right)$$

In the first approximation we have thus:

$$\left( \int \Psi^\dagger(\theta_f^{(j)}, Z_j) \mathcal{G}_0^{-1} \left( 1 + \left( \exp\left(\hat{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})\right) - 1 \right) \frac{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}}} \right) \right) \\ \times \exp\left(-\Lambda_1(\theta_f^{(j)} - \theta_i^{(j)})\right) \mathcal{G}_0^{-1} \Psi(\theta_i^{(j)}, Z_j) dZ_j \Big)^{n-1} \\ \simeq \left( \int \Psi^\dagger(\theta_f^{(j)}, Z_j) \left( 1 + \left( \exp\left(\hat{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})\right) - 1 \right) \right. \right. \\ \left. \left. \times \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \right) \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right) \Psi(\theta_i^{(j)}, Z_j) dZ_j \right)^{n-1}$$



**Effective action** Gathering the several contributions in the two previous paragraphs yields  $\Gamma(\Psi)$ :

$$\begin{aligned}
& \Gamma(\Psi) \\
&= -\frac{1}{2}\Psi^\dagger(\theta, Z) \left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z)) \right) \right) \Psi(\theta, Z) + \alpha \int \left| \Psi(\theta^{(i)}, Z_i) \right|^2 \\
&\quad - \sum \frac{1}{(n-1)!} \int \Psi^\dagger(\theta, Z) \left( -\zeta^{(n)} + \bar{\Xi}_1^{(n)}(Z_i, \theta^{(i)}, \{Z_j\}_{j \neq i}) \right) \\
&\quad \times \left( \int \frac{\theta_i^{(i)} - \theta_i^{(j)}}{|Z_i - Z_j|} dl_j \int \left| \Psi\left(\theta_i^{(i)} - \frac{|Z_i - Z_j|}{c}, Z_j\right) \right|^2 dZ_j \right)^{n-1} \Psi(\theta, Z) \\
&\quad + \sum \frac{1}{(n-1)!} \int \Psi^\dagger(\theta_f^{(i)}, Z_i) \hat{V}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \left( \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}} \right)^2 \Psi(\theta_i^{(i)}, Z_i) \\
&\quad \times \left( \int \Psi^\dagger(\theta_f^{(j)}, Z_j) \left( 1 + \left( \exp\left(\hat{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})\right) - 1 \right) \right. \right. \\
&\quad \times \left. \left. \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \right) \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right) \Psi(\theta_i^{(j)}, Z_j) dZ_j \right)^{n-1}
\end{aligned}$$

or resumming the series expansion:

$$\begin{aligned}
& \Gamma(\Psi) \tag{243} \\
&= -\frac{1}{2}\Psi^\dagger(\theta, Z) \left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1} \left( J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + \int \left| \Psi(\theta^{(j)}, Z_j) \right|^2 \right) \right) \right) \Psi(\theta, Z) \\
&\quad + \alpha \int \left| \Psi(\theta^{(i)}, Z_i) \right|^2 + \sum \frac{\zeta^{(n)}}{n!} \int \left| \Psi(\theta^{(i)}, Z_i) \right|^2 \left( \mathcal{G}_0(0, Z) + \int \frac{\theta_i^{(i)} - \theta_i^{(j)}}{|Z_i - Z_j|} dl_j \int \left| \Psi\left(\theta_i^{(i)} - \frac{|Z_i - Z_j|}{c}, Z_j\right) \right|^2 dZ_j \right)^n \\
&\quad + \sum \int \frac{1}{(n-1)!} \Psi^\dagger(\theta_f^{(i)}, Z_i) \hat{V}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \left( \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}} \right)^2 \Psi(\theta_i^{(i)}, Z_i) \\
&\quad \times \left( \int \Psi^\dagger(\theta_f^{(j)}, Z_j) \left( 1 + \left( \exp\left(\hat{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})\right) - 1 \right) \right. \right. \\
&\quad \times \left. \left. \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \right) \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right) \Psi(\theta_i^{(j)}, Z_j) dZ_j \right)^{n-1}
\end{aligned}$$

where  $\omega^{-1}(J, \theta^{(j)}, \mathcal{G}_0(Z))$  is solution of:

$$\omega^{-1}(\theta^{(i)}, Z) = \mathcal{G}_0 \left( J(\theta) + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right)}{\omega(\theta, Z)} W \left( \frac{\omega(\theta, Z)}{\omega\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right)} \right) \bar{\mathcal{G}}_0(0, Z_i) dZ_1 \right)$$

and with  $\hat{V}_{1,n} \left( Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)} \right)$  defined by:

$$\begin{aligned}
& \hat{V}_{1,n} \left( Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \\
&= \frac{1 + \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) - 1 \right) \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)}{\left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)^2} \\
& \times \frac{\left( -\frac{\zeta^{(n)}}{\Lambda^n} + \frac{\nabla^{out}}{\Lambda_1} \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\left( \theta_f^{(i)} - \theta_i^{(i)} \right)} \right)}{\frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \\
& + \frac{\left( \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \left( 1 - \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) + \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)^2 \right) - 1 \right)}{\left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)^2} \\
& \times \frac{-\bar{\zeta}_n + \frac{\nabla^{out}}{\Lambda_1} \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}}
\end{aligned}$$

**Inclusion of backreaction terms** Once the effective action has been found, we can include the neglected part of the vertex (145) :

$$\frac{1}{2n!} \int \nabla_\theta \left[ \mathcal{G}_0 \frac{\delta^n \omega^{-1} (J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \prod_{i=1}^n |\Psi(\theta - l_i, Z_i)|^2 \prod_{i=1}^n dZ_i dZ d\theta dl_i$$

There is also a contribution to the potential given

The computation of the corresponding graphs is identical to that of the previous paragraphs. The dominant contributions of these vertices to the effective action modify  $S_1$  in (233) and (234) by a term:

$$\begin{aligned}
\delta S_1 &= \sum_{n \geq 2} \frac{1}{n!} \int \mathcal{G}_0(Z) dZ \times \int \prod_{j=1}^n \nabla_\theta \left[ \mathcal{G}_0 \frac{\delta^n \omega^{-1} (J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \\
& \times \left( \int \Psi^\dagger \left( \theta_f^{(j)}, Z_j \right) \left( H_1 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) + H_2 \left( Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)} \right) \frac{\nabla^{out}}{\Lambda_1} \right) \Psi \left( \theta_i^{(j)}, Z_j \right) dZ_j \right)
\end{aligned} \tag{244}$$

Actually, compared to (233), the term  $\Psi^\dagger \left( \theta_f^{(i)}, Z_i \right) \Psi \left( \theta_i^{(i)}, Z_i \right)$  and the factor  $n$  corresponding to the  $n$  possibilities to choose the point  $i$ , are replaced by  $\int \mathcal{G}_0(Z) dZ$ . These terms correspond to the back reaction of the  $n$  points, including  $i$ , on the whole system. The action of the  $n-1$  points  $j$  on  $i$  in (233), i.e. the term

$$\begin{aligned}
& \hat{\Xi}_{1,n}^{(n)} \left( Z_i, \{Z_{j,j \neq i}\}, \theta_i^{(i)}, \theta_f^{(i)} \right) \left( \frac{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}} \frac{\nabla^{out}}{\Lambda_1}}{-\bar{\zeta}_n + \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right)}{\theta_f^{(i)} - \theta_i^{(i)}}} \right) \\
& \times \exp \left( \left( -\Lambda_1 + \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i} \right) \right) \left( \theta_f^{(i)} - \theta_i^{(i)} \right) \right)
\end{aligned}$$

is thus replaced by:

$$\nabla_{\theta} \left[ \mathcal{G}_0 \frac{\delta^n \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2} - \int \mathcal{G}_0 \frac{\delta^n V(f |\Psi|^2)}{\prod_{j=1}^n \delta |\Psi(\theta - l_j, Z_j)|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)}$$

The second term being the contribution of the potential given in (157). The contribution to the effective action is thus:

$$\begin{aligned} \delta S_1 = & \sum_{n \geq 2} \frac{1}{n!} \int \mathcal{G}_0(Z) dZ \times \int \nabla_{\theta} \left[ \mathcal{G}_0 \frac{\delta^n \omega^{-1}(J, \theta, Z)}{\prod_{i=1}^n \delta |\Psi(\theta - l_i, Z_i)|^2} - \int \mathcal{G}_0 \frac{\delta^n V(f |\Psi|^2)}{\prod_{j=1}^n \delta |\Psi(\theta - l_j, Z_j)|^2} \right]_{|\Psi(\theta, Z)|^2 = \mathcal{G}_0(0, Z)} \\ & \times \left( \int \Psi^{\dagger}(\theta_f^{(j)}, Z_j) \left( H_1(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) + H_2(Z_j, \bar{Z}, \theta_f^{(j)}, \theta_i^{(j)}) \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \right) \Psi(\theta_i^{(j)}, Z_j) dZ_j \right)^n \end{aligned} \quad (245)$$

The contribution of (245) to the effective action (243) is equivalent to shift  $|\Psi(\theta, Z)|^2$  and  $|\Psi(\theta_i^{(i)}, Z_i)|^2$  by  $\mathcal{G}_0(0, Z)$ .

We will see below that this shift does not modify fundamentally the form of the vacuum. As a consequence it is convenient to work with (243) in the sequel.

## 4.4 First order condition and non trivial vacuum

### 4.4.1 First order condition

$$\begin{aligned} & \Gamma(\Psi) \\ = & -\frac{1}{2} \Psi^{\dagger}(\theta, Z) \left( \nabla_{\theta} \left( \frac{\sigma_{\theta}^2}{2} \nabla_{\theta} - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2) \right) \right) \Psi(\theta, Z) \\ & + \alpha \int |\Psi(\theta^{(i)}, Z_i)|^2 + \sum \frac{\zeta^{(n)}}{n!} \left( \mathcal{G}_0(0, Z) + \int \left| \Psi \left( \theta_i^{(i)} - \frac{|Z_i - Z_j|}{c}, Z_j \right) \right|^2 dZ_j \right)^n \\ & + \sum \int \frac{1}{(n-1)!} \Psi^{\dagger}(\theta_f^{(i)}, Z_i) \hat{V}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}) \left( \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}} \right)^2 \Psi(\theta_i^{(i)}, Z_i) \\ & \times \left( \int \Psi^{\dagger}(\theta_f^{(j)}, Z_j) \left( 1 + \left( \exp \left( \hat{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)}) \right) - 1 \right) \right. \right. \\ & \left. \left. \times \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \right) \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right) \Psi(\theta_i^{(j)}, Z_j) dZ_j \right)^{n-1} \end{aligned}$$

We consider the lowest order in perturbation saddle point equation. Given our assumptions  $\bar{\zeta}_{n+1} > 0$  for all  $n \geq 2$ ,  $\zeta^{(n+1)} > 0$ ,  $\zeta^{(2)} < 0$ , the potential:

$$\alpha \int |\Psi(\theta^{(i)}, Z_i)|^2 dZ_i + \sum \frac{\zeta^{(n)}}{n!} \left( \mathcal{G}_0(0, Z) + \int \left| \Psi \left( \theta_i^{(i)} - \frac{|Z_i - Z_j|}{c}, Z_j \right) \right|^2 dZ_j \right)^n$$

has a minimum for  $\alpha \ll 1$  and for  $|\zeta^{(2)}|$  large. This minimum is reached for a value  $X_0$  of  $\int |\Psi(\theta^{(i)}, Z_i)|^2 dZ_i$ .

Up to an irrelevant phase,  $\Psi(\theta^{(i)}, Z_i) = \Psi^{\dagger}(\theta, Z) = \sqrt{\frac{X_0}{V}}$  where  $V$  is the volume of the thread.

Moreover the operator  $O = \nabla_\theta \frac{\sigma_\theta^2}{2} (\nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0))$  has positive eigenvalues. Developing  $\Psi(\theta, Z) = \sum a_n \Psi_n(\theta, Z)$  where  $\Psi_n(\theta, Z)$  are the eigenstates of  $O$ , the definition of  $\Psi^\dagger(\theta, Z)$  (see [45] and [46]) is given by:

$$\sum \bar{a}_n \Psi_n^\dagger(\theta, Z)$$

where  $\Psi_n^\dagger(\theta, Z)$  are the eigenstates of the adjoint operator of  $O$ . As a consequence

$$\int -\frac{1}{2} \Psi^\dagger(\theta, Z) \left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0) \right) \right) \Psi(\theta, Z)$$

is positive, and null for constant  $\Psi^\dagger(\theta, Z)$  and  $\Psi(\theta, Z)$ . As a consequence, for  $|\zeta^{(n)}| > \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0)$  the minimum of  $\Gamma(\Psi)$  is reached for  $\Psi(\theta, Z) = \Psi_0(\theta, Z) + \delta\Psi(\theta, Z)$  and  $\Psi^\dagger(\theta, Z) = \Psi_0^\dagger(\theta, Z) + \delta\Psi^\dagger(\theta, Z)$  where  $|\delta\Psi(\theta, Z)| \ll |\Psi_0(\theta, Z)|$  and  $|\delta\Psi^\dagger(\theta, Z)| \ll |\Psi_0^\dagger(\theta, Z)|$ .

Expanding the potential around  $\Psi_0(\theta, Z)$  yields at the first order:

$$\begin{aligned} \Gamma(\Psi) &= -\frac{1}{2} \delta\Psi^\dagger(\theta, Z) \left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0 + \sqrt{X_0}(\delta(\Psi^\dagger + \delta\Psi))) \right) \right) \Psi_0(\theta, Z) \\ &\quad -\frac{1}{2} \delta\Psi^\dagger(\theta, Z) \left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2) \right) \right) \delta\Psi(\theta, Z) \\ &\quad +\frac{1}{2} \delta\Psi^\dagger(\theta, Z) U''(X_0) \delta\Psi(\theta, Z) \\ &\simeq -\frac{1}{2} \delta\Psi^\dagger(\theta, Z) \left( \nabla_\theta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0 + \sqrt{X_0}(\delta(\Psi^\dagger + \delta\Psi))) \right) \Psi_0(\theta, Z) \\ &\quad -\frac{1}{2} \delta\Psi^\dagger(\theta, Z) \left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) \right) \right) \delta\Psi(\theta, Z) \\ &\quad +\frac{1}{2} \delta\Psi^\dagger(\theta, Z) U''(X_0) \delta\Psi(\theta, Z) \end{aligned}$$

and this leads to the first order condition for  $\delta\Psi(\theta, Z)$ :

$$\begin{aligned} 0 &= \frac{1}{2} \left( -\left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) \right) \right) + U''(X_0) \right) \delta\Psi(\theta, Z) \\ &\quad -\frac{1}{2} \left( \nabla_\theta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0 + \sqrt{X_0}(\delta(\Psi^\dagger + \delta\Psi))) \right) \Psi_0(\theta, Z) \\ &\quad -\frac{1}{2} \delta\Psi^\dagger(\theta_1, Z_1) \sqrt{X_0} \left( \nabla_\theta \frac{\delta\omega^{-1}(J(\theta_1), \theta_1, Z_1, \mathcal{G}_0 + X_0)}{\delta|\Psi(\theta, Z)|^2} \right) \Psi_0(\theta_1, Z_1) \\ &\simeq \frac{1}{2} \left( -\left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) \right) \right) + U''(X_0) \right) \delta\Psi(\theta, Z) \\ &\quad -\frac{1}{2} \int \delta\Psi^\dagger(\theta_1, Z_1) \sqrt{X_0} \left( \nabla_\theta \frac{\delta\omega^{-1}(J(\theta_1), \theta_1, Z_1, \mathcal{G}_0 + X_0)}{\delta|\Psi(\theta, Z)|^2} \right) \Psi_0(\theta_1, Z_1) d\theta_1 dZ_1 \\ &\quad -\frac{1}{2} \left( \nabla_\theta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) \right) \Psi_0(\theta, Z) \\ &\quad -\frac{1}{2} \int \sqrt{X_0} \nabla_\theta \left( \frac{\delta\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0)}{\delta|\Psi(\theta_1, Z_1)|^2} \delta\Psi(\theta_1, Z_1) \right) \Psi_0(\theta, Z) d\theta_1 dZ_1 \end{aligned}$$

and for  $\delta\Psi^\dagger(\theta_1, Z_1)$ :

$$\begin{aligned} 0 &= \frac{1}{2} \delta\Psi^\dagger(\theta, Z) \left( -\nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) \right) + U''(X_0) \right) \\ &\quad -\frac{1}{2} \int \delta\Psi^\dagger(\theta_1, Z_1) \sqrt{X_0} \left( \nabla_\theta \frac{\delta\omega^{-1}(J(\theta_1), \theta_1, Z_1, \mathcal{G}_0 + X_0)}{\delta|\Psi(\theta, Z)|^2} \right) \Psi_0(\theta_1, Z_1) d\theta_1 dZ_1 \end{aligned}$$

#### 4.4.2 Non-trivial vacuum

The solution for  $\delta\Psi^\dagger(\theta, Z)$  is:

$$\delta\Psi^\dagger(\theta, Z) = 0 \quad (246)$$

This translates that there is no backward propagation of the signals. As a consequence, the equation for  $\delta\Psi(\theta, Z)$  rewrites:

$$\begin{aligned} 0 \simeq & \frac{1}{2} \left( - \left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) \right) \right) + U''(X_0) \right) \delta\Psi(\theta, Z) \\ & - \frac{1}{2} (\nabla_\theta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)) \Psi_0(\theta, Z) \\ & - \frac{1}{2} \int \sqrt{X_0} \nabla_\theta \left( \frac{\delta\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0)}{\delta|\Psi(\theta_1, Z_1)|^2} \delta\Psi(\theta_1, Z_1) \right) \Psi_0(\theta, Z) d\theta_1 dZ_1 \end{aligned} \quad (247)$$

Solving (247) amounts to find the Green function of the operator:

$$\begin{aligned} \bar{\mathcal{G}}^{-1}(Z, \theta, Z_1, \theta_1) &= \frac{1}{2} \left( - \left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) \right) \right) + U''(X_0) \right) \delta(\theta - \theta_1) \\ & - \frac{1}{2} \int \sqrt{X_0} \nabla_\theta \frac{\delta\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0)}{\delta|\Psi(\theta_1, Z_1)|^2} \Psi_0(\theta, Z) \\ & \equiv \tilde{\mathcal{G}}_Z^{-1}(\theta, \theta_1) \delta(Z - Z_1) - \frac{1}{2} \int \sqrt{X_0} \nabla_\theta \frac{\delta\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0)}{\delta|\Psi(\theta_1, Z_1)|^2} \Psi_0(\theta, Z) \end{aligned} \quad (248)$$

and the vacuum is given by:

$$\frac{1}{2} \int \bar{\mathcal{G}}^{-1}(Z, \theta, Z_1, \theta_1) (\nabla_\theta \omega^{-1}(J(\theta_1), \theta_1, Z_1, \mathcal{G}_0 + X_0)) \Psi_0(\theta_1, Z_1)$$

The computation of  $\bar{\mathcal{G}}^{-1}(Z, \theta, Z_1, \theta_1)$  is done in two steps:

**4.4.2.1 Green function of  $\tilde{\mathcal{G}}_Z^{-1}(\theta, \theta_1)$**  We first find the Green function of the operator:

$$\tilde{\mathcal{G}}_Z^{-1}(\theta, \theta_1) = \frac{1}{2} \left( - \left( \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) \right) \right) + U''(X_0) \right) \delta(\theta - \theta_1) \quad (249)$$

To do so, we note that  $\tilde{\mathcal{G}}_Z^{-1}(\theta, \theta_1)$  has the form  $-\frac{d}{d\theta} \left( \frac{a}{2} \frac{d}{d\theta} - b(\theta) \right)$  where  $a = \sigma_\theta^2$  and  $b(\theta) = \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)$ . For any function  $d(\theta)$ , the following change of basis holds:

$$\begin{aligned} & - \exp(-d(\theta)) \frac{d}{d\theta} \left( \frac{a}{2} \frac{d}{d\theta} - b(\theta) \right) \exp(d(\theta)) \\ &= - \exp(-d(\theta)) \frac{d}{d\theta} \left( \exp(d(\theta)) \left( \frac{a}{2} \frac{d}{d\theta} + \frac{a}{2} d'(\theta) - b(\theta) \right) \right) \\ &= - \left( \left( \frac{d}{d\theta} + d'(\theta) \right) \left( \left( \frac{a}{2} \frac{d}{d\theta} + \frac{a}{2} d'(\theta) - b(\theta) \right) \right) \right) \\ &= - \left( \left( \frac{d}{d\theta} \left( \left( \frac{a}{2} \frac{d}{d\theta} + a d'(\theta) - b(\theta) \right) \right) - \frac{a}{2} d''(\theta) + \left( \frac{a}{2} d'(\theta) - b(\theta) \right) d'(\theta) \right) \right) \end{aligned}$$

Defining  $\bar{b}$  as the average of  $b(\theta)$  over some time span, and setting:

$$d'(\theta) = \frac{b(\theta) - \bar{b}}{a}$$

we have:

$$\begin{aligned} & \exp(-d(\theta)) \left( -\frac{d}{d\theta} \left( \frac{a}{2} \frac{d}{d\theta} - b(\theta) \right) \right) \exp(d(\theta)) \\ &= -\frac{d}{d\theta} \left( \left( \frac{a}{2} \frac{d}{d\theta} - \bar{b} \right) \right) - \frac{b'(\theta)}{2} - \frac{(b(\theta) + \bar{b})(b(\theta) - \bar{b})}{2a} \end{aligned}$$

As a consequence, the change of basis allows to decompose the initial operator under the form:

$$-\frac{d}{d\theta} \left( \left( \frac{a}{2} \frac{d}{d\theta} - \bar{b} \right) \right)$$

for some constant  $\bar{b}$  plus an additional term. This allows to find the inverse of the initial operator by a series expansion:

$$\begin{aligned} & \left( -\frac{d}{d\theta} \left( \frac{a}{2} \frac{d}{d\theta} - b(\theta) \right) \right)^{-1} (\theta', \theta) \\ &= \exp(-d(\theta')) \left( -\frac{d}{d\theta} \left( \left( \frac{a}{2} \frac{d}{d\theta} - \bar{b} \right) \right) - \frac{b'(\theta)}{2} - \frac{(b(\theta) + \bar{b})(b(\theta) - \bar{b})}{2a} \right)^{-1} (\theta', \theta) \exp(d(\theta)) \\ &= \exp(-d(\theta')) \sum_{n=0}^{\infty} \left( \mathcal{G}_{\bar{b}} \left( \left( \frac{b'(\theta)}{2} + \frac{(b(\theta) + \bar{b})(b(\theta) - \bar{b})}{2a} \right) \mathcal{G}_{\bar{b}} \right)^n \right) (\theta', \theta) \exp(d(\theta)) \end{aligned}$$

where  $\mathcal{G}_{\bar{b}}$  is the Green function of  $-\frac{d}{d\theta} \left( \left( \frac{a}{2} \frac{d}{d\theta} - \bar{b} \right) \right)$  and the products are products of convolution. For  $(b(\theta) - \bar{b})$  having periods shorter than  $\frac{1}{a}$  the integrals of  $\frac{(b(\theta) + \bar{b})(b(\theta) - \bar{b})}{2a}$  cancel. As a consequence:

$$\left( -\frac{d}{d\theta} \left( \frac{a}{2} \frac{d}{d\theta} - b(\theta) \right) \right)^{-1} (\theta', \theta) \simeq \exp(-d(\theta')) \sum_{n=0}^{\infty} \left( \mathcal{G}_{\bar{b}} \left( \left( \frac{b'(\theta)}{2} \right) \mathcal{G}_{\bar{b}} \right)^n \right) (\theta', \theta) \exp(d(\theta)) \quad (250)$$

We apply this result to the operator arising in our problem. We define  $\bar{b} = \omega^{-1}(\bar{J}, \theta, Z, \mathcal{G}_0(0, Z) + X_0)$  so that:

$$\mathcal{G}_{\bar{b}}(\theta, \theta') = \left( \frac{1}{2} \left( - \left( \nabla_{\theta} \left( \frac{\sigma_{\theta}^2}{2} \nabla_{\theta} - \omega^{-1}(\bar{J}, \theta, Z, \mathcal{G}_0(0, Z) + X_0) \right) \right) + U''(X_0) \right) \right)^{-1} = \mathcal{G}(\theta, \theta')$$

where  $\bar{J}$  is the average current. The function  $\mathcal{G}(\theta, \theta')$  is given by:

$$\begin{aligned} \mathcal{G}(\theta, \theta') &= \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{\exp \left( -\sqrt{\left( \frac{1}{\sigma^2 \bar{X}_e} \right)^2 + \frac{2U''(X_0)}{\sigma^2}} |\theta - \theta'| \right)}{\sqrt{\left( \frac{1}{\sigma^2 \bar{X}_e} \right)^2 + \frac{2U''(X_0)}{\sigma^2}}} \exp \left( \frac{\theta - \theta'}{\sigma^2 \bar{X}_e} \right) \\ &\simeq \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{\exp \left( - \left( \sqrt{\left( \frac{1}{\sigma^2 \bar{X}_e} \right)^2 + \frac{2U''(X_0)}{\sigma^2}} - \frac{1}{\sigma^2 \bar{X}_e} \right) (\theta - \theta') \right)}{\sqrt{\left( \frac{1}{\sigma^2 \bar{X}_e} \right)^2 + \frac{2U''(X_0)}{\sigma^2}}} H(\theta - \theta') \end{aligned}$$

with:

$$\frac{1}{\bar{X}_e(Z)} = \omega^{-1}(\bar{J}(Z), \theta, Z, \mathcal{G}_0(0, Z) + X_0) = \frac{\arctan \left( \left( \frac{1}{\bar{X}_r} - \frac{1}{\bar{X}_p} \right) \sqrt{\bar{J}(Z) + \mathcal{G}_0(0, Z) + X_0} \right)}{\sqrt{\bar{J}(Z) + \mathcal{G}_0(0, Z) + X_0}}$$

and (250) yields the Green function of (249):

$$\begin{aligned}
\tilde{\mathcal{G}}(\theta, \theta', Z) &= \left( - \left( \nabla_{\theta} \left( \frac{\sigma_{\theta}^2}{2} \nabla_{\theta} - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) \right) \right) + U''(X_0) \right)^{-1} \\
&= \sum_{n=0}^{\infty} (-1)^n \left( \mathcal{G} \left( \left( \frac{\nabla_{\theta} \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{2} \right) \mathcal{G} \right)^n \right) (\theta', \theta) \\
&\quad \times \exp \left( \frac{\int_{\theta}^{\theta'} (\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) - \omega^{-1}(\bar{J}, \theta, Z, \mathcal{G}_0(0, Z) + X_0))}{\sigma_{\theta}^2} \right)
\end{aligned} \tag{251}$$

or in expanded form:

$$\begin{aligned}
\tilde{\mathcal{G}}(\theta, \theta', Z) &= \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{\exp \left( - \left( \sqrt{\left( \frac{1}{\sigma^2 X_e} \right)^2 + \frac{2U''(X_0)}{\sigma^2}} - \frac{1}{\sigma^2 X_e} \right) (\theta - \theta') \right)}{\sqrt{\left( \frac{1}{\sigma^2 X_e} \right)^2 + \frac{2U''(X_0)}{\sigma^2}}} \\
&\quad \times H(\theta - \theta') \left( 1 + \sum_{n=1}^{\infty} (-1)^n \int_{\theta < \theta_1 < \dots < \theta'} \frac{\nabla_{\theta} \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{2} \right) \\
&\quad \times \exp \left( \frac{\int_{\theta}^{\theta'} (\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) - \omega^{-1}(\bar{J}, \theta, Z, \mathcal{G}_0(0, Z) + X_0))}{\sigma_{\theta}^2} \right) \\
&= \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{\exp \left( - \left( \sqrt{\left( \frac{1}{\sigma^2 X_e} \right)^2 + \frac{2U''(X_0)}{\sigma^2}} - \frac{1}{\sigma^2 X_e} \right) (\theta - \theta') - \left[ \frac{\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{2} \right]_{\theta'}^{\theta} \right)}{\sqrt{\left( \frac{1}{\sigma^2 X_e} \right)^2 + \frac{2U''(X_0)}{\sigma^2}}} H(\theta - \theta') \\
&\quad \times \exp \left( \frac{\int_{\theta}^{\theta'} (\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) - \omega^{-1}(\bar{J}, \theta, Z, \mathcal{G}_0(0, Z) + X_0))}{\sigma_{\theta}^2} \right)
\end{aligned}$$

This can be simplified by using that in average:

$$\int_{\theta}^{\theta'} (\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) - \omega^{-1}(\bar{J}, \theta, Z, \mathcal{G}_0(0, Z) + X_0)) \simeq 0$$

As a consequence:

$$\begin{aligned}
\tilde{\mathcal{G}}(\theta, \theta', Z) &= \frac{\exp \left( - \left( \sqrt{\left( \frac{1}{\sigma^2 X_e} \right)^2 + \frac{2U''(X_0)}{\sigma^2}} - \frac{1}{\sigma^2 X_e} \right) (\theta - \theta') - \left[ \frac{\omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{2} \right]_{\theta'}^{\theta} \right)}{\sqrt{\frac{\pi}{2}} \sqrt{\left( \frac{1}{\sigma^2 X_e} \right)^2 + \frac{2U''(X_0)}{\sigma^2}}} H(\theta - \theta') \\
&\simeq \frac{\exp \left( - \left( \left( \sqrt{\left( \frac{1}{\sigma^2 X_e} \right)^2 + \frac{2U''(X_0)}{\sigma^2}} - \frac{1}{\sigma^2 X_e} \right) + \left[ \nabla_{\theta} \frac{\omega^{-1}(\bar{J}, \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{2} \right] \right) (\theta - \theta') \right)}{\sqrt{\frac{\pi}{2}} \sqrt{\left( \frac{1}{\sigma^2 X_e} \right)^2 + \frac{2U''(X_0)}{\sigma^2}}} H(\theta - \theta')
\end{aligned} \tag{252}$$

where the upper bar on a quantity stands for the average computed over the period  $\theta - \theta'$ .

**4.4.2.2 Green function of  $\bar{\mathcal{G}}^{-1}(\theta, \theta_1)$**  Using (249), equation (247) rewrites:

$$\begin{aligned}
0 &\simeq \tilde{\mathcal{G}}^{-1} \delta \Psi(\theta, Z) \\
&\quad - \frac{1}{2} \nabla_{\theta} \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) \sqrt{X_0} \\
&\quad - \frac{X_0}{2} \int \nabla_{\theta} \left( \left( \frac{\delta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{\delta |\Psi(\theta_1, Z_1)|^2} \right) \delta \Psi(\theta_1, Z_1) \right) d\theta_1 dZ_1 \\
&= \bar{\mathcal{G}}^{-1} \delta \Psi(\theta, Z) - \frac{1}{2} \nabla_{\theta} \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) \sqrt{X_0}
\end{aligned} \tag{253}$$

The Green function of the operator  $\bar{\mathcal{G}}^{-1}$  defined in (248) is given by a series expansion:

$$\begin{aligned}
\bar{\mathcal{G}} &= \sum_{n=0}^{\infty} \left( \frac{X_0}{2} \right)^n \left( \tilde{\mathcal{G}} * \nabla_{\theta} \left( \left( \frac{\delta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{\delta |\Psi(\theta_1, Z_1)|^2} \right) \right) \right)^n \tilde{\mathcal{G}} \\
&= \sum_{n=0}^{\infty} \left( \frac{X_0}{2} \right)^n \left( \nabla_{\theta} \tilde{\mathcal{G}} * \left( \left( \frac{\delta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{\delta |\Psi(\theta_1, Z_1)|^2} \right) \right) \right)^n \tilde{\mathcal{G}}
\end{aligned} \tag{254}$$

The convolution:

$$\nabla_{\theta} \tilde{\mathcal{G}} * \left( \left( \frac{\delta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{\delta |\Psi(\theta_1, Z_1)|^2} \right) \right)$$

is computed using (252) and the expression derived previously for the kernel  $\frac{\delta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{\delta |\Psi(\theta_1, Z_1)|^2}$ :

$$\frac{\delta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{\delta |\Psi(\theta_1, Z_1)|^2} = C \frac{\exp\left(-K \left| (\theta_i - \theta_{i+1}) - \frac{|Z_i - Z_{i+1}|}{c} \right| \right)}{2\sqrt{|c(\theta_i - \theta_{i+1}) - |Z_i - Z_{i+1}||}}$$

where  $C$  and  $K$  are some parameters depending on the system. The convolution is thus:

$$\begin{aligned}
&\nabla_{\theta} \tilde{\mathcal{G}} * \left( \left( \frac{\delta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{\delta |\Psi(\theta_1, Z_1)|^2} \right) \right) (\theta_{i-1} - \theta_{i+1}) \\
&= \int \nabla_{\theta_{i-1}} \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{\exp\left(-\left(\left(\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_e}\right)^2 + \frac{2U''(X_0)}{\sigma^2} - \frac{1}{\sigma^2 \bar{X}_e}\right)} + \left[\nabla_{\theta} \frac{\omega^{-1}(\bar{J}, \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{2}\right]\right)}{\sqrt{\left(\frac{1}{\sigma^2 \bar{X}_e}\right)^2 + \frac{2U''(X_0)}{\sigma^2}}} (\theta_{i-1} - \theta_i) \\
&\quad \times \delta(Z_i - Z_{i-1}) C \frac{\exp\left(-K \left| (\theta_i - \theta_{i+1}) - \frac{|Z_i - Z_{i+1}|}{c} \right| \right)}{2\sqrt{K \left| (\theta_i - \theta_{i+1}) - \frac{|Z_i - Z_{i+1}|}{c} \right|}} d\theta_i dZ_i \\
&= \int \nabla_{\theta_{i-1}} \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{\exp(-\bar{\Lambda}(\theta_{i-1} - \theta_i))}{\Gamma} \delta(Z_i - Z_{i-1}) C \frac{\exp\left(-K \left| (\theta_i - \theta_{i+1}) - \frac{|Z_i - Z_{i+1}|}{c} \right| \right)}{2\sqrt{K \left| (\theta_i - \theta_{i+1}) - \frac{|Z_i - Z_{i+1}|}{c} \right|}} d\theta_i dZ_i
\end{aligned} \tag{255}$$

where:

$$\begin{aligned}
\bar{\Lambda} &= \left( \sqrt{\left(\frac{1}{\sigma^2 \bar{X}_e}\right)^2 + \frac{2U''(X_0)}{\sigma^2} - \frac{1}{\sigma^2 \bar{X}_e}} \right) + \left[ \nabla_{\theta} \frac{\omega^{-1}(\bar{J}, \theta, Z, \mathcal{G}_0(0, Z) + X_0)}{2} \right] \\
\Gamma &= \sqrt{\frac{\pi}{2}} \sqrt{\left(\frac{1}{\sigma^2 \bar{X}_e}\right)^2 + \frac{2U''(X_0)}{\sigma^2}}
\end{aligned}$$



The computation of the integral yields:

$$\begin{aligned}
& \int_{\theta_{i+1} + \frac{|Z_i - Z_{i+1}|}{c}}^{\theta_{i-1}} C\bar{\Lambda} \frac{\exp\left(-\bar{\Lambda}(\theta_{i-1} - \theta_i) - K\left|\theta_i - \theta_{i+1} - \frac{|Z_{i-1} - Z_{i+1}|}{c}\right|\right)}{2\Gamma\sqrt{K\left|\theta_i - \theta_{i+1} - \frac{|Z_{i-1} - Z_{i+1}|}{c}\right|}} d\theta_i \\
&= \int_0^{\sqrt{\theta_{i-1} - \left(\theta_{i+1} + \frac{|Z_i - Z_{i+1}|}{c}\right)}} C\bar{\Lambda} \frac{\exp\left(-\bar{\Lambda}\left(\theta_{i-1} - \left(\frac{u^2}{c} + \theta_{i+1} + \frac{|Z_{i-1} - Z_{i+1}|}{c}\right)\right) - u^2\right)}{K\Gamma} du \\
&= \int_0^{\sqrt{\theta_{i-1} - \left(\theta_{i+1} + \frac{|Z_i - Z_{i+1}|}{c}\right)}} C\bar{\Lambda} \frac{\exp\left(-\bar{\Lambda}\left(\theta_{i-1} - \theta_{i+1} - \frac{|Z_{i-1} - Z_{i+1}|}{c}\right) - \left(1 + \frac{\bar{\Lambda}}{K}\right)u^2\right)}{K\Gamma} du \\
&= C\bar{\Lambda} \frac{\exp\left(-\bar{\Lambda}\left(\theta_{i-1} - \theta_{i+1} - \frac{|Z_{i-1} - Z_{i+1}|}{c}\right)\right)}{K\Gamma\sqrt{1 + \frac{\bar{\Lambda}}{K}}} \operatorname{erf}\left(\sqrt{1 + \frac{\bar{\Lambda}}{K}}\sqrt{\theta_{i-1} - \left(\theta_{i+1} + \frac{|Z_{i-1} - Z_{i+1}|}{c}\right)}\right)
\end{aligned}$$

and (254) becomes:

$$\begin{aligned}
& \bar{\mathcal{G}}(Z_1, \theta_1, Z_{n+1}, \theta_{n+1}) \\
&= \sum_n \int \prod_{l=1}^n (-C\bar{\Lambda})^{n+1} \left\{ \frac{X_0}{2} \frac{\exp\left(-\bar{\Lambda}\left(\theta_l - \theta_{l+1} - \frac{|Z_l - Z_{l+1}|}{c}\right)\right)}{K\Gamma\sqrt{1 + \frac{\bar{\Lambda}}{K}}} \operatorname{erf}\left(\sqrt{1 + \frac{\bar{\Lambda}}{K}}\sqrt{\theta_l - \left(\theta_{l+1} + \frac{|Z_l - Z_{l+1}|}{c}\right)}\right) \right\} \\
& \quad \times \omega(J(\theta_l), \theta_l, Z_l, \mathcal{G}_0 + X_0) \tilde{\mathcal{G}}(\theta_n, \theta_{n+1}, Z_l) \prod_{l=2}^n d\theta_l dZ_l
\end{aligned}$$

with:

$$\omega^{-1}(\theta^{(i)}, Z) = G \left( J(\theta) + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right)}{\omega(\theta, Z)} W \left( \frac{\omega(\theta, Z)}{\omega\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right)} \right) (\mathcal{G}_0 + X_0) dZ_1 \right) \quad (256)$$

Formula (256) can be expanded in terms of current, at the second order of approximation:

$$\begin{aligned}
& \omega^{-1}(J, \theta, Z) = G[J(\theta, Z), \mathcal{G}_0 + X_0] \quad (257) \\
& + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right) - \omega(\theta, Z)}{\omega(\theta, Z)} G' [J(\theta, Z), \bar{\mathcal{G}}_0(0, Z_i)] \bar{\mathcal{G}}_0(0, Z_1) dZ_1 \\
&= \frac{G[J(\theta, Z), \bar{\mathcal{G}}_0(0, Z_i)]}{1 - \frac{\kappa}{N} \int T(Z, Z_1) \left(\omega\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right) - \omega(\theta, Z)\right) G' [J(\theta, Z), \bar{\mathcal{G}}_0(0, Z_i)] \bar{\mathcal{G}}_0(0, Z_1) dZ_1} \\
&\simeq \frac{G[J(\theta, Z), \bar{\mathcal{G}}_0(0, Z_i)]}{1 - \int \frac{\kappa T(Z, Z_1)}{N} \left(G\left[J\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right), \bar{\mathcal{G}}_0(0, Z_i)\right] - G[J(\theta, Z), \bar{\mathcal{G}}_0(0, Z_i)]\right) G' [J(\theta, Z), \bar{\mathcal{G}}_0(0, Z_i)] \bar{\mathcal{G}}_0(0, Z_1) dZ_1}
\end{aligned}$$

and:

$$\begin{aligned}
& \omega(J, \theta, Z) \quad (258) \\
&= \frac{1}{2} \left( F [J(\theta, Z), \bar{\mathcal{G}}_0(0, Z_i)] + \left( (F [J(\theta, Z), \bar{\mathcal{G}}_0(0, Z_i)])^2 \right. \right. \\
& \quad \left. \left. + 4 \left( \int \frac{\kappa}{N} T(Z, Z_1) \left( F \left[ J \left( \theta - \frac{|Z - Z_1|}{c}, Z_1 \right), \bar{\mathcal{G}}_0(0, Z_i) \right] - F [J(\theta, Z), \bar{\mathcal{G}}_0(0, Z_i)] \right) \bar{\mathcal{G}}_0(0, Z_1) dZ_1 \right) \right. \right. \\
& \quad \left. \left. F' [J(\theta, Z), \bar{\mathcal{G}}_0(0, Z_i)] \right)^{\frac{1}{2}} \right)
\end{aligned}$$

where:

$$\bar{\mathcal{G}}_0(0, Z_i) \simeq \mathcal{G}_0(0, Z_i) + X_0$$

The solution of (253) writes:

$$\begin{aligned}
& \delta\Psi(\theta, Z) \tag{259} \\
&= \sum_n \int (-C\bar{\Lambda})^{n+1} \prod_{l=1}^n \left\{ \frac{\exp\left(-\bar{\Lambda}\left(\theta_l - \theta_{l+1} - \frac{|Z_i - Z_{i+1}|}{c}\right)\right)}{K\Gamma\sqrt{1 + \frac{\bar{\Lambda}}{K}}} \operatorname{erf}\left(\sqrt{1 + \frac{\bar{\Lambda}}{K}}\sqrt{\theta_l - \left(\theta_{l+1} + \frac{|Z_i - Z_{i+1}|}{c}\right)}\right) \right. \\
&\quad \left. \times \omega(J(\theta_l), \theta_l, Z, \mathcal{G}_0 + X_0)\right\} \tilde{\mathcal{G}}(\theta_n, \theta_{n+1}) (\nabla_{\theta}\omega^{-1}(J(\theta_{n+1}), \theta_{n+1}, Z, \mathcal{G}_0 + X_0)) \frac{\sqrt{X_0}}{2} \prod_{l=2}^{n+1} d\theta_l
\end{aligned}$$

with the convention that  $\theta_1 = \theta$ .

For  $\omega(J(\theta_l), \theta_l, Z, \mathcal{G}_0(0, Z) + X_0)$  constant in first approximation, the series can be computed. The Fourier transform of (255) is:

$$C \frac{ik}{\frac{\sigma^2}{2}k^2 + ik\frac{1}{X_e} + U''(X_0)} \frac{\sqrt{K + \sqrt{K^2 + k^2}}}{\sqrt{K^2 + k^2}} \exp\left(-ik\left(\frac{|Z_i - Z_{i+1}|}{c}\right)\right)$$

and  $\bar{\mathcal{G}}$  rewrites:

$$\bar{\mathcal{G}} = \sum_n \left( \frac{CX_0}{2} \frac{ik}{\frac{\sigma^2}{2}k^2 + ik\frac{1}{X_e} + U''(X_0)} \frac{\sqrt{K + \sqrt{K^2 + k^2}}}{\sqrt{K^2 + k^2}} \omega(\bar{J}, \mathcal{G}_0 + X_0) \right)^n \left( \prod_{l=1}^n \exp\left(-ik\left(\frac{|Z_i - Z_{i+1}|}{c}\right)\right) \right) \tilde{\mathcal{G}}$$

The integrals:

$$\int \prod_{l=1}^n \exp\left(-ik\left(\frac{|Z_i - Z_{i+1}|}{c}\right)\right) dZ_i$$

with fixed endpoints  $Z_0$  and  $Z_{n+1}$  can be written:

$$\begin{aligned}
& \int \exp\left(-ik\left(\sum_{l=1}^n \frac{|Z_i - Z_{i+1}|}{c}\right)\right) \delta\left(Z_0 - Z_{n+1} - \sum_{l=1}^n (Z_i - Z_{i+1})\right) dZ_i \\
&= \int \exp\left(-ik\left(\sum_{l=1}^n \frac{|Z_i - Z_{i+1}|}{c}\right)\right) \exp\left(i\lambda\left(Z_0 - Z_{n+1} - \sum_{l=1}^n (Z_i - Z_{i+1})\right)\right) dZ_i \\
&= \int \exp\left(-ik\left(\sum_{l=1}^n \frac{|Z_i - Z_{i+1}|}{c}\right)\right) \exp\left(i\lambda\left(Z_0 - Z_{n+1} - \sum_{l=1}^n (Z_i - Z_{i+1})\right)\right) \\
&\quad \times d\lambda \prod_{i=1}^n \left|Z^{(i-1)} - Z^{(i)}\right|^2 d\left|Z^{(i-1)} - Z^{(i)}\right| dv_i
\end{aligned}$$

where the unit vectors  $v_i$  are defined such that:

$$Z^{(i-1)} - Z^{(i)} = v_i \left|Z^{(i-1)} - Z^{(i)}\right|$$

We also define:

$$\begin{aligned}
\lambda \cdot (Z_0 - Z_{n+1}) &= |\lambda| |Z_0 - Z_{n+1}| \cos(\theta_0) \\
\lambda \cdot v_i &= |\lambda| \cos(\theta_i)
\end{aligned}$$

The angles  $\theta_l$  are computed in the plane  $(\lambda, Z - Z_1)$  between the projection of  $v_l$  and  $Z - Z_1$ .

$$\begin{aligned}
& \pi^{n+1} \int_0^\pi \int \exp\left(-ik \left(\sum_{l=1}^n \frac{|Z_l - Z_{l+1}|}{c}\right)\right) \exp\left(i|\lambda| \left(|Z_0 - Z_{n+1}| \cos(\theta_0) - \sum_{l=1}^n |Z_l - Z_{l+1}| \cos(\theta_l)\right)\right) \\
& \times d|\lambda| \prod_{i=1}^n \left|Z^{(i-1)} - Z^{(i)}\right|^2 d\left|Z^{(i-1)} - Z^{(i)}\right| \sin(\theta_i) d\theta_i \sin(\theta_0) d\theta_0 \\
& = 2(\pi)^{n+1} \int \frac{\sin(|\lambda| |Z_0 - Z_{n+1}|)}{|\lambda| |Z_0 - Z_{n+1}|} \prod_{i=1}^n \frac{\exp\left(i\left(|\lambda| - \frac{k}{c}\right) |Z^{(i-1)} - Z^{(i)}|\right) - \exp\left(-i\left(|\lambda| + \frac{k}{c}\right) |Z^{(i-1)} - Z^{(i)}|\right)}{i|\lambda|} \\
& \times \left|Z^{(i-1)} - Z^{(i)}\right| d\left|Z^{(i-1)} - Z^{(i)}\right| d|\lambda| \\
& = 2(\pi)^{n+1} \int \frac{\sin(|\lambda| |Z_0 - Z_{n+1}|)}{2|\lambda|^{n+1} |Z_0 - Z_{n+1}|} \\
& \times \prod_{i=1}^n \left(-\frac{d}{d|\lambda|} \int \left(\exp\left(i\left(|\lambda| - \frac{k}{c}\right) |Z^{(i-1)} - Z^{(i)}|\right) + \exp\left(-i\left(|\lambda| + \frac{k}{c}\right) |Z^{(i-1)} - Z^{(i)}|\right)\right) d\left|Z^{(i-1)} - Z^{(i)}\right| d|\lambda|\right) \\
& \simeq 2(\pi)^{n+1} \int \frac{\sin(|\lambda| |Z_0 - Z_{n+1}|)}{2|\lambda|^{n+1} |Z_0 - Z_{n+1}|} \prod_{i=1}^n \left(-\frac{d}{d|\lambda|} \frac{1}{2} \left(\left(\delta\left(|\lambda| - \frac{k}{c}\right) - \frac{i}{\pi\left(|\lambda| - \frac{k}{c}\right)} + \delta\left(|\lambda| + \frac{k}{c}\right) + \frac{i}{\pi\left(|\lambda| + \frac{k}{c}\right)}\right)\right) d|\lambda| \\
& = 2(\pi)^{n+1} \int \frac{\sin(|\lambda| |Z_0 - Z_{n+1}|)}{2|\lambda| |Z_0 - Z_{n+1}|} \left(\frac{d}{|\lambda| d|\lambda|} \left(\frac{\exp\left(-i\left(|\lambda| + \frac{k}{c}\right) L\right) - 1}{i\left(|\lambda| + \frac{k}{c}\right)} - \frac{\exp\left(i\left(|\lambda| - \frac{k}{c}\right) L\right) - 1}{i\left(|\lambda| - \frac{k}{c}\right)}\right)\right)^n d|\lambda| \\
& 2\pi \int \frac{\sin(|\lambda| |Z - Z'|)}{2|\lambda| |Z - Z'|} \sum_n \left(\frac{ik\pi C}{\frac{\sigma^2}{2}k^2 + ik\frac{1}{X_e} + U''(X_0)} \frac{\sqrt{K + \sqrt{K^2 + k^2}}}{\sqrt{K^2 + k^2}} \frac{X_0}{2} \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0)\right)^n \\
& \times \left(\frac{d}{|\lambda| d|\lambda|} \left(\frac{\exp\left(-i\left(|\lambda| + \frac{k}{c}\right) L\right) - 1}{i\left(|\lambda| + \frac{k}{c}\right)} - \frac{\exp\left(i\left(|\lambda| - \frac{k}{c}\right) L\right) - 1}{i\left(|\lambda| - \frac{k}{c}\right)}\right)\right)^n d|\lambda| \frac{ik}{\frac{\sigma^2}{2}k^2 + ik\frac{1}{X_e} + U''(X_0)} \\
& = 2\pi \left(\int \frac{\frac{\sin(|\lambda| |Z - Z'|)}{2|\lambda| |Z - Z'|}}{1 - \frac{ik\pi C \frac{d}{|\lambda| d|\lambda|} \left(\frac{\exp\left(-i\left(|\lambda| + \frac{k}{c}\right) L\right) - 1}{i\left(|\lambda| + \frac{k}{c}\right)} - \frac{\exp\left(i\left(|\lambda| - \frac{k}{c}\right) L\right) - 1}{i\left(|\lambda| - \frac{k}{c}\right)}\right)}{\frac{\sigma^2}{2}k^2 + ik\frac{1}{X_e} + U''(X_0)} \frac{\sqrt{K + \sqrt{K^2 + k^2}}}{\sqrt{K^2 + k^2}} \frac{X_0}{2} \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0)} d|\lambda|\right) \\
& \times \frac{ik}{\frac{\sigma^2}{2}k^2 + ik\frac{1}{X_e} + U''(X_0)}
\end{aligned}$$

which can be written:

$$\begin{aligned}
& \pi \int \frac{\frac{\sin(|\lambda| |Z - Z'|)}{|\lambda| |Z - Z'|} ik d|\lambda| dk}{\left(\frac{\sigma^2}{2}k^2 + ik\frac{1}{X_e} + U''(X_0) - \frac{ik\pi C \frac{X_0}{2} d}{|\lambda| d|\lambda|} \left(\frac{\exp\left(-i\left(|\lambda| + \frac{k}{c}\right) L\right) - 1}{i\left(|\lambda| + \frac{k}{c}\right)} - \frac{\exp\left(i\left(|\lambda| - \frac{k}{c}\right) L\right) - 1}{i\left(|\lambda| - \frac{k}{c}\right)}\right) \frac{\sqrt{K + \sqrt{K^2 + k^2}}}{\sqrt{K^2 + k^2}} \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0)\right)} \\
& \frac{d}{|\lambda| d|\lambda|} \left(\frac{\exp\left(-i\left(|\lambda| + \frac{k}{c}\right) L\right) - 1}{i\left(|\lambda| + \frac{k}{c}\right)} - \frac{\exp\left(i\left(|\lambda| - \frac{k}{c}\right) L\right) - 1}{i\left(|\lambda| - \frac{k}{c}\right)}\right) \\
& = -\left(\left(\frac{\exp\left(-i\left(|\lambda| + \frac{k}{c}\right) L\right)}{|\lambda| \left(|\lambda| + \frac{k}{c}\right)} + \frac{\exp\left(i\left(|\lambda| - \frac{k}{c}\right) L\right)}{|\lambda| \left(|\lambda| - \frac{k}{c}\right)}\right) L + \left(\frac{\exp\left(-i\left(|\lambda| + \frac{k}{c}\right) L\right) - 1}{i|\lambda| \left(|\lambda| + \frac{k}{c}\right)^2} - \frac{\exp\left(i\left(|\lambda| - \frac{k}{c}\right) L\right) - 1}{i|\lambda| \left(|\lambda| - \frac{k}{c}\right)^2}\right)\right) \\
& 2\pi \int \frac{\sin(|\lambda| |Z - Z'|) ik \exp(ik(\theta - \theta'))}{2|\lambda| |Z - Z'| D} d|\lambda| dk
\end{aligned}$$

with:

$$\begin{aligned}
D &= \frac{\sigma^2}{2} k^2 + ik \frac{1}{\bar{X}_e} + U''(X_0) \\
&+ ik\pi C \frac{X_0}{2} \left( \left( \frac{\exp(-i(|\lambda| + \frac{k}{c})L)}{|\lambda|(|\lambda| + \frac{k}{c})} + \frac{\exp(i(|\lambda| - \frac{k}{c})L)}{|\lambda|(|\lambda| - \frac{k}{c})} \right) L \right. \\
&+ \left. \left( \frac{(\exp(-i(|\lambda| + \frac{k}{c})L) - 1)}{i|\lambda|(|\lambda| + \frac{k}{c})^2} - \frac{(\exp(i(|\lambda| - \frac{k}{c})L) - 1)}{i|\lambda|(|\lambda| - \frac{k}{c})^2} \right) \right) \\
&\times \frac{\sqrt{K + \sqrt{K^2 + k^2}}}{\sqrt{K^2 + k^2}} \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0)
\end{aligned}$$

The integral over  $k$  presents a pole such that  $ik$  is real and negative. Due to the exponential in the denominator,  $|k| \ll 1$  for this pole. It satisfies:

$$\begin{aligned}
0 &\simeq U''(X_0) + ik\pi C \frac{X_0}{2} \left( \left( \frac{\exp(-i(|\lambda| + \frac{k}{c})L)}{|\lambda|(|\lambda| + \frac{k}{c})} + \frac{\exp(i(|\lambda| - \frac{k}{c})L)}{|\lambda|(|\lambda| - \frac{k}{c})} \right) L \right) \sqrt{\frac{2}{K}} \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0) \\
&\simeq U''(X_0) + ik\pi \frac{CX_0}{|\lambda|^2} \cos(|\lambda|L) \left( \exp\left(-i\frac{k}{c}L\right) \right) L \sqrt{\frac{2}{K}} \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0)
\end{aligned}$$

so that:

$$ik \simeq - \frac{|\lambda|^2 U''(X_0)}{L \sqrt{\frac{2}{K}} C X_0 \pi \cos(|\lambda|L) \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0)}$$

and the integral over  $|\lambda|$  is such that  $\cos(|\lambda|L) > 0$ .

$$\begin{aligned}
&- \int \frac{\sin(|\lambda||Z - Z'|) \frac{|\lambda|^2 U''(X_0)}{L \sqrt{\frac{2}{K}} C X_0 \pi \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0) \cos(|\lambda|L)} \exp\left(-\frac{|\lambda|^2 U''(X_0)}{L \sqrt{\frac{2}{K}} C X_0 \pi \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0) \cos(|\lambda|L)} (\theta - \theta')\right) |\lambda|^2}{2 C X_0 \pi |\lambda| |Z - Z'| \cos(|\lambda|L) L \sqrt{\frac{2}{K}} \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0)} d|\lambda| \\
&= - \int \frac{\sin(|\lambda||Z - Z'|) |\lambda| U''(X_0) \exp\left(-\frac{|\lambda|^2 U''(X_0)}{L \sqrt{\frac{2}{K}} C X_0 \pi \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0) \cos(|\lambda|L)} (\theta - \theta')\right) |\lambda|^2}{2 \left(L \sqrt{\frac{2}{K}} C X_0 \pi \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0) \cos(|\lambda|L)\right)^2 |Z - Z'|} d|\lambda|
\end{aligned}$$

and:

$$\begin{aligned}
\delta\Psi(\theta, Z) &= - \int \frac{\sin(|\lambda||Z - Z'|) |\lambda| U''(X_0) \exp\left(-\frac{|\lambda|^2 U''(X_0)}{L \sqrt{\frac{2}{K}} C X_0 \pi \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0) \cos(|\lambda|L)} (\theta - \theta')\right) |\lambda|^2}{2 \left(L \sqrt{\frac{2}{K}} C X_0 \pi \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0) \cos(|\lambda|L)\right)^2 |Z - Z'|} \\
&\quad \times d|\lambda| (\nabla_{\theta} \omega^{-1}(J(\theta'), \theta', Z', \mathcal{G}_0(0, Z) + X_0)) \frac{\sqrt{X_0}}{2} d\theta' \\
&\simeq \int \frac{K \sin(|\lambda||Z - Z'|) |\lambda| U''(X_0) \exp\left(-\frac{|\lambda|^2 U''(X_0)}{L \sqrt{\frac{2}{K}} C X_0 \pi \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0) \cos(|\lambda|L)} (\theta - \theta')\right) |\lambda|^2}{8 (L C X_0 \pi \cos(|\lambda|L))^2 (\omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0))^4 |Z - Z'|} \\
&\quad \times d|\lambda| (\nabla_{\theta} \omega(J(\theta'), \theta', Z', \mathcal{G}_0(0, Z) + X_0)) \sqrt{X_0} d\theta'
\end{aligned}$$

and in the local approximation, it reduces to:

$$\begin{aligned}
\delta\Psi(\theta, Z) &\simeq \int \frac{K \sin(|\lambda| |Z - Z'|) |\lambda| U''(X_0) \exp\left(-\frac{|\lambda|^2 U''(X_0)}{L\sqrt{\frac{2}{K}} C X_0 \pi \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0) \cos(|\lambda| L)} (\theta - \theta')\right) |\lambda|^2}{8 (L C X_0 \pi \cos(|\lambda| L))^2 (\omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0))^4 |Z - Z'|} \\
&\quad \times d|\lambda| (\nabla_{\theta} \omega(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)) \sqrt{X_0} d\theta' \\
&\quad - \int \frac{K |\lambda|^4 U''(X_0) \exp\left(-\frac{|\lambda|^2 U''(X_0)}{L\sqrt{\frac{2}{K}} C X_0 \pi \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0) \cos(|\lambda| L)} (\theta - \theta')\right) (\theta - \theta')}{8 (L C X_0 \pi \cos(|\lambda| L))^2 (\omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0))^4} \\
&\quad \times d|\lambda| d\theta' (\nabla_{\theta}^2 \omega(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)) \sqrt{X_0} \\
&= \frac{K \sin(|\lambda| |Z - Z'|) |\lambda| U''(X_0) L \sqrt{\frac{2}{K}} C X_0 \pi \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0) \cos(|\lambda| L) |\lambda|^2}{8 |\lambda|^2 U''(X_0) (L C X_0 \pi \cos(|\lambda| L))^2 (\omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0))^4 |Z - Z'|} \\
&\quad \times \nabla_{\theta} \omega(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) \sqrt{X_0} \\
&\quad - \int \frac{K |\lambda|^4 U''(X_0) \left(L \sqrt{\frac{2}{K}} C X_0 \pi \omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0) \cos(|\lambda| L)\right)^2}{8 (L C X_0 \pi \cos(|\lambda| L))^2 (\omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0))^4 \left(|\lambda|^2 U''(X_0)\right)^2 d|\lambda|} \\
&\quad \times (\nabla_{\theta}^2 \omega(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)) \sqrt{X_0} \\
&\simeq \int \frac{|\lambda|^2 \sqrt{K}}{4\sqrt{2} (L C X_0 \pi \cos(|\lambda| L)) (\omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0))^3} \nabla_{\theta} \omega(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) \sqrt{X_0} \\
&\quad - \int \frac{\sqrt{X_0}}{4 (\omega(\bar{J}, \mathcal{G}_0(0, Z) + X_0))^2 U''(X_0)} d|\lambda| \nabla_{\theta}^2 \omega(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) \\
&\equiv N_1 \nabla_{\theta} \omega(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0) - N_2 \nabla_{\theta}^2 \omega(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + X_0)
\end{aligned}$$

where we used that the integral over  $|\lambda|$  is constrained on a finite interval.

**Corrections to (259) due to (245)** We saw in (245), that the backreaction terms, shift  $|\Psi(\theta, Z)|^2$  by  $\mathcal{G}_0(0, Z)$ .

This shift can be absorbed by a redefinition of  $\Psi(\theta, Z)$ . Actually, let:

$$\begin{aligned}
\delta\Psi(\theta, Z) &= \delta\tilde{\Psi}(\theta, Z) - \frac{\mathcal{G}_0(0, Z)}{\sqrt{X_0}} \\
\delta\Psi^\dagger(\theta, Z) &= \delta\tilde{\Psi}^\dagger(\theta, Z) - \frac{\mathcal{G}_0(0, Z)}{\sqrt{X_0}}
\end{aligned}$$

For  $\frac{\mathcal{G}_0(0, Z)}{\sqrt{X_0}} \ll 1$ , it yields at the first order:

$$\begin{aligned}
|\Psi(\theta, Z)|^2 + \mathcal{G}_0(0, Z) &= \left(\sqrt{X_0} + \delta\Psi^\dagger(\theta, Z)\right) \left(\sqrt{X_0} + \delta\Psi(\theta, Z)\right) + \mathcal{G}_0(0, Z) \\
&= \left|\tilde{\Psi}(\theta, Z)\right|^2
\end{aligned}$$

As a consequence, the computations leading to the minimum are the same as in the previous paragraphs if we replace  $\Psi(\theta, Z) \rightarrow \tilde{\Psi}(\theta, Z)$ . Coming back ultimately to the variable  $\Psi(\theta, Z)$ , the minimum (259) for the action becomes:

$$\delta\Psi^\dagger(\theta, Z) = -\frac{\mathcal{G}_0(0, Z)}{\sqrt{X_0}} \tag{260}$$

$$\begin{aligned}
& \delta\Psi(\theta, Z) \tag{261} \\
&= \sum_n \int (-C\bar{\Lambda})^{n+1} \prod_{l=1}^n \left\{ \frac{\exp\left(-\bar{\Lambda}\left(\theta_l - \theta_{l+1} - \frac{|Z_l - Z_{l+1}|}{c}\right)\right)}{K\Gamma\sqrt{1 + \frac{\bar{\Lambda}}{K}}} \operatorname{erf}\left(\sqrt{1 + \frac{\bar{\Lambda}}{K}}\sqrt{\theta_l - \left(\theta_{l+1} + \frac{|Z_l - Z_{l+1}|}{c}\right)}\right) \right. \\
&\quad \times \omega(J(\theta_l), \theta_l, Z, \mathcal{G}_0(0, Z) + X_0) \} \tilde{\mathcal{G}}(\theta_n, \theta_{n+1}) (\nabla_{\theta}\omega^{-1}(J(\theta_{n+1}), \theta_{n+1}, Z, \mathcal{G}_0(0, Z) + X_0)) \frac{\sqrt{X_0}}{2} \prod_{l=2}^{n+1} d\theta_l \\
&\quad - \frac{\mathcal{G}_0(0, Z)}{\sqrt{X_0}}
\end{aligned}$$

and these shifts do not impact the main conclusions of our results.

#### 4.4.3 Correction for the potential

At lowest order in perturbation, the correction to the action is:

$$\begin{aligned}
& \frac{1}{2} \sum \int \frac{1}{(n-1)!} \Psi^\dagger(\theta_f^{(i)}, Z_i) \left( \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}} \right)^2 \Psi(\theta_i^{(i)}, Z_i) \tag{262} \\
& \times \left( \int \Psi^\dagger(\theta_f^{(j)}, Z_j) \right. \\
& \times \left. \left( 1 + \left( \left( \exp\left(\hat{\Xi}_{1,n}(Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)})\right) - 1 \right) \left( \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}} \right)^2 \right) \right) \Psi(\theta_i^{(j)}, Z_j) dZ_j \right)^{n-1}
\end{aligned}$$

and this shifts the minimum of the potential on the left by a term proportional to  $\left( \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}} \right)^2$ ,

so that  $X_0 \rightarrow X_0 - \varrho \left( \frac{\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)})}{\theta_f^{(i)} - \theta_i^{(i)}} \right)^2$  where  $\varrho < 0$  depends on the parameters of the potential.

## Appendix 5. Frequency equation

The potential terms (21) along with its corrections (262) evaluated at the background field  $\Psi_0$  yield a constant  $U(\Psi_0)$ . This constitutes a modification to the scale  $\alpha$  which is replaced by  $\alpha + U''(\Psi_0)$ . This modifies accordingly the 2 points correlation function. Since  $U(\Psi_0) < 0$  for a non-trivial vacuum, the phase corresponding to is characterized by  $\alpha + U(\Psi_0) < \alpha$ . This implies a longer average interaction time.

The effective frequencies of the system are obtained from the second derivative of the effective action:

$$\begin{aligned}
& \frac{\delta^2\Gamma(\Psi)}{\delta\Psi^\dagger(\theta^{(j)}, Z_j) \Psi(\theta^{(j)}, Z_j)} = \mathcal{G}_0^{-1} \frac{F}{\Theta}(\theta_1^{(i)}, \theta_f^{(i)}) + \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F}{\Theta} \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \frac{F}{\Theta} \mathcal{G}_0^{-1} \right] \tag{263} \\
& - \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + O_{1,\infty}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \frac{(1 - \exp(-x)) F}{\Theta^2} (1 + O_{1,\infty}) \mathcal{G}_0^{-1} \right] \\
& - \left[ \int^{\theta_1^{(i)}} \mathcal{G}_0^{-1} \frac{F^2}{\Theta^2} (1 + O_{1,2}) \Psi \right] \left[ \int_{\theta_1^{(i)}} \Psi^\dagger \frac{\exp(-x) F}{\Theta^2} (1 + O_{1,2}) \mathcal{G}_0^{-1} \right]
\end{aligned}$$

with  $F$  and  $\Theta$  given by (190).

We showed that it can be approximated by:

$$\begin{aligned}
& \frac{\delta^2 \Gamma(\Psi)}{\delta \Psi^\dagger(\theta^{(j)}, Z_j) \Psi(\theta^{(j)}, Z_j)} \\
= & \mathcal{G}_0^{-1} \frac{F}{\Theta} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\
& + F^2 (\Psi^\dagger \mathcal{G}_0^{-1} \Psi) \langle 1 \rangle^3 \left( \frac{\left( (\alpha(x) + Fx \exp(-x)) \left( -\bar{\zeta} + \frac{\nabla_{\theta_{i'}}^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \right)}{(x + \exp(-x) (\langle 1 \rangle - x + (y^2 - x^2)))^4} \right) \\
& + F^2 (\Psi^\dagger \mathcal{G}_0^{-1} \Psi) \langle 1 \rangle^3 \left( \frac{\left( (\beta(x) - F \exp(-x)) y \left( -\bar{\zeta}_2 + \frac{\nabla_{\theta_{i'}}^{out}}{\Lambda_1} \bar{\Xi}_{1,2} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) + \beta(x) \right)}{(x + \exp(-x) (\langle 1 \rangle - x + (y^2 - x^2)))^4} \right)
\end{aligned} \tag{264}$$

In the local approximation, the dominant term of (264):

$$\mathcal{G}_0^{-1} \frac{F}{\Theta} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) = \mathcal{G}_0^{-1} \frac{1 + \exp(-x) \left( -z + \frac{1}{2} (y^2 - x^2) \right)}{(1 + \bar{O}_{1,\infty}) + \exp(-x) (-\bar{O}_{1,\infty} + y(1 + \bar{O}_{1,2}) - x(1 + \bar{O}_{1,\infty}))} \left( \theta_1^{(i)}, \theta_f^{(i)} \right)$$

can be rewritten in its expanded form:

$$\begin{aligned}
& \mathcal{G}_0^{-1} \frac{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n}{1 + \langle (1 + \bar{O}_{1,2})^{(2)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} \langle (1 + \bar{O}_{1,n})^{(n)} \rangle_{n-1}} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\
= & \mathcal{G}_0^{-1} \frac{1 + \langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n}{1 + (1 + \bar{O}_{1,2}) \langle (1 + \bar{O}_{1,2})^{(1)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} (1 + \bar{O}_{1,n}) \langle (1 + \bar{O}_{1,n})^{(n-1)} \rangle_{n-1}} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\
= & \mathcal{G}_0^{-1} + \frac{\langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n}{1 + (1 + \bar{O}_{1,2}) \langle (1 + \bar{O}_{1,2})^{(1)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} (1 + \bar{O}_{1,n}) \langle (1 + \bar{O}_{1,n})^{(n-1)} \rangle_{n-1}} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\
& - \frac{\left( (1 + \bar{O}_{1,2}) \langle (1 + \bar{O}_{1,2})^{(1)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} (1 + \bar{O}_{1,n}) \langle (1 + \bar{O}_{1,n})^{(n-1)} \rangle_{n-1} \right)}{1 + (1 + \bar{O}_{1,2}) \langle (1 + \bar{O}_{1,2})^{(1)} \rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} (1 + \bar{O}_{1,n}) \langle (1 + \bar{O}_{1,n})^{(n-1)} \rangle_{n-1}} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\
= & \mathcal{G}_0^{-1} \frac{\langle 1 \rangle_1 + \frac{1}{2} \langle (1 + O_{1,2})^{(2)} \rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \langle (1 + O_{1,\infty})^{(n)} \rangle_n}{1 + (1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,2} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,2})^{(1)} \rangle_1 + \sum_{n \geq 3} \frac{(1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,n} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,n})^{(n-1)} \rangle_{n-1}}{(n-1)!}} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\
& - \mathcal{G}_0^{-1} \frac{(1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,2} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,2})^{(1)} \rangle_1}{1 + (1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,2} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,2})^{(1)} \rangle_1 + \sum_{n \geq 3} \frac{(1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,n} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,n})^{(n-1)} \rangle_{n-1}}{(n-1)!}} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\
& - \mathcal{G}_0^{-1} \frac{\sum_{n \geq 3} \frac{1}{(n-1)!} (1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,n} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,n})^{(n-1)} \rangle_{n-1}}{1 + (1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,2} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,2})^{(1)} \rangle_1 + \sum_{n \geq 3} \frac{(1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,n} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,n})^{(n-1)} \rangle_{n-1}}{(n-1)!}} \left( \theta_1^{(i)}, \theta_f^{(i)} \right)
\end{aligned} \tag{265}$$

where:

$$\begin{aligned}
\bar{O}_{1,n} &= \left( -\bar{\zeta}_n + \frac{\nabla^{out}_{\theta_{i'}^{(i)}}}{\Lambda_1} \bar{\Xi}_{1,n} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \\
&\quad \times \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \frac{\exp \left( -\Lambda_1 \left( \theta_f^{(i)} - \theta_{i'}^{(i)} \right) \right)}{\Lambda} \right) \\
\bar{O}_{1,\infty} &= \left( -\bar{\zeta} + \frac{\nabla^{out}_{\theta_{i'}^{(i)}}}{\Lambda_1} \bar{\Xi}_{1,\infty} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \exp \left( \hat{\Xi}_{1,\infty} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \frac{\exp \left( -\Lambda_1 \left( \theta_f^{(i)} - \theta_{i'}^{(i)} \right) \right)}{\Lambda} \right)
\end{aligned} \tag{266}$$

and:

$$\begin{aligned}
&\mathcal{G}_0^{-1} \bar{O}_{1,n} \\
&= \mathcal{G}_0^{-1} O_{1,n} \mathcal{G}_0^{-1} \\
&= \left( -\bar{\zeta}_n + \frac{\nabla^{out}_{\theta_{i'}^{(i)}}}{\Lambda_1} \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)} \right) \right) \delta \left( \theta_{f'}^{(i)} - \theta_{i'}^{(i)} \right) \\
&\quad + \left( -\bar{\zeta}_n + \frac{\nabla^{out}_{\theta_{i'}^{(i)}}}{\Lambda_1} \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)} \right) \right) \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{f'}^{(i)} \right) \right) \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right)
\end{aligned} \tag{267}$$

In the local approximation (267) becomes:

$$\begin{aligned}
\mathcal{G}_0^{-1} \bar{O}_{1,n} &= \mathcal{G}_0^{-1} O_{1,n} \mathcal{G}_0^{-1} \\
&= \left( -\bar{\zeta}_n + \frac{\nabla^{out}_{\theta_{i'}^{(i)}}}{\Lambda_1} \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)} \right) \right) \delta \left( \theta_{f'}^{(i)} - \theta_{i'}^{(i)} \right) \\
&\quad + \left( -\bar{\zeta}_n + \frac{\nabla^{out}_{\theta_{i'}^{(i)}}}{\Lambda_1} \bar{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{i'}^{(i)} \right) \right) \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_{f'}^{(i)} \right) \right)
\end{aligned} \tag{268}$$

Moreover, in (265), the sum:

$$O = (1 + \bar{O}_{1,2}) \left\langle (1 + \bar{O}_{1,2})^{(1)} \right\rangle_1 + \sum_{n \geq 3} \frac{1}{(n-1)!} (1 + \bar{O}_{1,n}) \left\langle (1 + \bar{O}_{1,n})^{(n-1)} \right\rangle_{n-1}$$

is the sum of graphs contributing to the two points correlation, and:

$$S = \langle 1 \rangle_1 + \frac{1}{2} \left\langle (1 + O_{1,2})^{(2)} \right\rangle_2 + \sum_{n \geq 3} \frac{1}{n!} \left\langle (1 + O_{1,\infty})^{(n)} \right\rangle_n$$

computes the sum of all graphs in the background state  $\Psi$ . As a consequence, rewriting (265) as:

$$\mathcal{G}_0^{-1} + \frac{S - O}{1 + O} = \mathcal{G}_0^{-1} + \sum_{n=1}^{\infty} (-1)^{n-1} (O^n - O^{n-1} S)$$

The subtraction by the second term removes all the graphs contributing to the inverse two points correlation function that can be factored by any  $n$ . Moreover, the alternate series removes the graph that can be factored as a convolution of two graphs through one external leg. As a consequence, the series computes the 1PI graphs.



We can thus write:

$$\begin{aligned} & \mathcal{G}_0^{-1} \frac{F}{\Theta} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \\ = & \left( \mathcal{G}_0^{-1} \frac{1}{1 + (1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,2} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,2})^{(1)} \rangle_1 + \sum_{n \geq 3} \frac{(1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,n} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,n})^{(n-1)} \rangle_{n-1}}{(n-1)!}} \right) \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \Big|_{\text{1PI}} \end{aligned}$$

where the upper script recalls that only 1PI part of the series expansion is kept. In an expanded form, we have:

$$\begin{aligned} & \mathcal{G}_0^{-1} \frac{F}{\Theta} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \tag{269} \\ = & \left( \mathcal{G}_0^{-1} + \sum_{n=0}^{\infty} (-1)^n \mathcal{G}_0^{-1} \left\{ (1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,2} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,2})^{(1)} \rangle_1 \right. \right. \\ & \left. \left. + \sum_{n \geq 3} \frac{(1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,n} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,n})^{(n-1)} \rangle_{n-1}}{(n-1)!} \right\}^n \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \right) \Big|_{\text{1PI}} \end{aligned}$$

In particular, at the lowest order of the series expansion, a graph:

$$(1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O_{1,n} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,n})^{(n-1)} \rangle_{n-1}$$

can be replaced by:

$$(1 + \mathcal{G}_0 (\mathcal{G}_0^{-1} O'_{1,n} \mathcal{G}_0^{-1})) \langle (1 + \bar{O}_{1,n})^{(n-1)} \rangle_{n-1} \tag{270}$$

where:

$$\bar{O}'_{1,n} = \left( -\zeta^{(n)} + \frac{\nabla_{i'}^{out}}{\Lambda_1} \bar{\Xi}_1^{(n)} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \exp \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \right) \frac{\exp \left( -\Lambda_1 \left( \theta_f^{(i)} - \theta_{i'}^{(i)} \right) \right)}{\Lambda}$$

The term (270) keeps the 1PI part of the initial graph in the state  $\Psi$ . The computation of the series (269)

is similar to (243) and gives in the local approximation:

$$\begin{aligned}
& \mathcal{G}_0^{-1} \frac{F}{\Theta} \left( \theta_1^{(i)}, \theta_f^{(i)} \right) \tag{271} \\
&= -\frac{1}{2} \nabla_\theta \left( \frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1} \left( J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2 \right) \right) + \alpha \\
&+ \sum \frac{\zeta^{(n)}}{n!} \left( \mathcal{G}_0(0, Z) + \int_{\frac{|z_i - z_j|}{c}}^{\theta^{(i)} - \theta^{(j)}} dl_j \int \left| \Psi \left( \theta_i^{(i)} - \frac{|Z_i - Z_j|}{c}, Z_j \right) \right|^2 dZ_j \right)^n \\
&+ \sum \left( \frac{-\bar{\zeta}_n + \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1} \bar{\Xi}_{1,n} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) - \zeta^{(n)} + \frac{\nabla_{\theta_{i'}^{(j)}}^{out}}{\Lambda_1} \bar{\Xi}_1^{(n)} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right)}{2} \right) \frac{\hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)} \right)}{(n-1)!} \\
&\times \left( \int \Psi^\dagger \left( \theta_f^{(j)}, Z_j \right) \left( 1 + \left( \exp \left( \hat{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \right) - 1 \right) \right. \right. \\
&\times \left. \left. \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1}}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right) \right. \\
&\times \left. \left. \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right) \right) \Psi \left( \theta_i^{(j)}, Z_j \right) dZ_j \Big)^{n-1}
\end{aligned}$$

where  $\omega^{-1} \left( J, \theta^{(j)}, \bar{\mathcal{G}}_0(Z) \right)$  is solution of:

$$\omega^{-1} \left( \theta^{(i)}, Z \right) = G \left( J(\theta) + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega \left( \theta - \frac{|Z - Z_1|}{c}, Z_1 \right)}{\omega(\theta, Z)} W \left( \frac{\omega(\theta, Z)}{\omega \left( \theta - \frac{|Z - Z_1|}{c}, Z_1 \right)} \right) \bar{\mathcal{G}}_0(0, Z_i) dZ_1 \right)$$

We can now identify the various contributions in (264). First, the expansion (271) of  $\mathcal{G}_0^{-1} \frac{F}{\Theta} \left( \theta_1^{(i)}, \theta_f^{(i)} \right)$  includes a potential term which, evaluated at the state  $\Psi$ , defines an effective value for  $\alpha$ :

$$\begin{aligned}
& \alpha + \sum \frac{\zeta^{(n)}}{n!} \left( \mathcal{G}_0(0, Z) + \int_{\frac{|z_i - z_j|}{c}}^{\theta^{(i)} - \theta^{(j)}} dl_j \int \left| \Psi \left( \theta_i^{(i)} - \frac{|Z_i - Z_j|}{c}, Z_j \right) \right|^2 dZ_j \right)^n \\
&+ \sum \frac{\zeta^{(n)}}{n!} \left( \mathcal{G}_0(0, Z) + \int_{\frac{|z_i - z_j|}{c}}^{\theta^{(i)} - \theta^{(j)}} dl_j \int \left| \Psi \left( \theta_i^{(i)} - \frac{|Z_i - Z_j|}{c}, Z_j \right) \right|^2 dZ_j \right)^n \\
&- \sum \frac{1}{(n-1)!} \left( \frac{\bar{\zeta}_n + \zeta^{(n)}}{2} \right) \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)} \right) \right) \\
&\times \left( \int \Psi^\dagger \left( \theta_f^{(j)}, Z_j \right) \left( 1 + \left( \exp \left( \hat{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \right) - 1 \right) \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1}}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right. \right. \\
&\times \left. \left. \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right) \right) \Psi \left( \theta_i^{(j)}, Z_j \right) dZ_j \Big)^{n-1}
\end{aligned}$$

The other terms of (264) also includes a potential term, contributing to the effective value of  $\alpha$ :

$$-F^2 \left( \Psi^\dagger \mathcal{G}_0^{-1} \Psi \right) \left( \frac{(\alpha(x) - Fx \exp(-x)) \bar{\zeta} - (\beta(x) - F \exp(-x)) y \bar{\zeta}_2 - \beta(x)}{(x + \exp(-x)(1 - x + (y^2 - x^2)))^4} \right)$$

Second, the effective frequency is given by a contribution of (271):

$$\begin{aligned}
& \omega^{-1} \left( J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2 \right) \\
& + \sum \frac{1}{(n-1)!} \left( \frac{\left( \bar{\Xi}_{1,n} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) + \bar{\Xi}_1^{(n)} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right)}{2} \right) \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)} \right) \right) \\
& \times \left( \int \Psi^\dagger \left( \theta_f^{(j)}, Z_j \right) \left( 1 + \left( \exp \left( \hat{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \right) - 1 \right) \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1}}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right. \\
& \left. \times \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right) \Psi \left( \theta_i^{(j)}, Z_j \right) dZ_j \Big)^{n-1}
\end{aligned}$$

and a contribution due to the correction terms in (264):

$$\begin{aligned}
& F^2 \left( \Psi^\dagger \mathcal{G}_0^{-1} \Psi \right) \left( \frac{\left( \alpha(x) - Fx \exp(-x) \right) \left( \frac{\nabla_{\theta_{i'}^{(i)}}^{out}}{\Lambda_1} \bar{\Xi}_{1,\infty} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right)}{\left( x + \exp(-x) (\langle 1 \rangle - x + (y^2 - x^2)) \right)^4} \right. \\
& \left. - (\beta(x) - F \exp(-x)) y \left( \frac{\nabla_{\theta_{i'}^{(i)}}^{out}}{\Lambda_1} \bar{\Xi}_{1,2} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) \right)
\end{aligned}$$

Gathering these two contributions leads to the effective frequency:

$$\begin{aligned}
& \omega_e^{-1} \left( J(\theta), \theta, Z \right) = \omega^{-1} \left( J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2 \right) \\
& + \sum \frac{1}{(n-1)!} \left( \frac{\left( \bar{\Xi}_{1,n} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) + \bar{\Xi}_1^{(n)} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right)}{2} \right) \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)} \right) \right) \times \\
& \left( \int \Psi^\dagger \left( \theta_f^{(j)}, Z_j \right) \left( 1 + \left( \exp \left( \hat{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \right) - 1 \right) \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right) \frac{\nabla_{\theta_i^{(j)}}^{out}}{\Lambda_1}}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right. \\
& \left. \times \left( -\bar{\zeta}_n + \frac{\bar{\Xi}_{1,n} \left( Z_j, \{Z_m\}_{m \neq j}, \theta_i^{(j)}, \theta_f^{(j)} \right)}{\theta_f^{(j)} - \theta_i^{(j)}} \right) \right) \Psi \left( \theta_i^{(j)}, Z_j \right) dZ_j \Big)^{n-1} \\
& + F^2 \left( \Psi^\dagger \mathcal{G}_0^{-1} \Psi \right) \left( \frac{\left( \alpha(x) - Fx \exp(-x) \right) \left( \bar{\Xi}_{1,\infty} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right) - (\beta(x) - F \exp(-x)) y \left( \bar{\Xi}_{1,2} \left( Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i} \right) \right)}{\left( x + \exp(-x) (\langle 1 \rangle - x + (y^2 - x^2)) \right)^4} \right)
\end{aligned}$$

$$\begin{aligned}
& \omega_e^{-1}(J(\theta), \theta, Z) \simeq \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0(0, Z) + |\Psi|^2\right) \\
& + \sum \frac{1}{(n-1)!} \left( \frac{\left(\bar{\Xi}_{1,n}\left(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}\right) + \bar{\Xi}_1^{(n)}\left(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}\right)\right)}{2} \right) \\
& \times \left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}\right)\right) \left(\int \Psi^\dagger\left(\theta_f^{(j)}, Z_j\right) \Psi\left(\theta_i^{(j)}, Z_j\right) dZ_j\right)^{n-1} \\
& + F^2\left(\Psi^\dagger \mathcal{G}_0^{-1} \Psi\right) \left(\frac{(\alpha(x) - Fx \exp(-x))\left(\bar{\Xi}_{1,\infty}\left(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}\right)\right)}{(x + \exp(-x))(\langle 1 \rangle - x + (y^2 - x^2))^4}\right) \\
& - F^2\left(\Psi^\dagger \mathcal{G}_0^{-1} \Psi\right) \left(\frac{(\beta(x) - F \exp(-x))y\left(\bar{\Xi}_{1,2}\left(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}\right)\right)}{(x + \exp(-x))(\langle 1 \rangle - x + (y^2 - x^2))^4}\right)
\end{aligned}$$

$$\begin{aligned}
\omega_e^{-1}(J(\theta), \theta, Z) & \simeq \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) \\
& + \omega_1^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) + \omega_2^{-1}\left(J(\theta), \theta, Z, |\Psi|^2\right)
\end{aligned}$$

The function  $\omega_1^{-1}$  is defined by:

$$\begin{aligned}
\omega_1^{-1}(J(\theta), \theta, Z, \mathcal{G}_0) & = \sum \frac{\left(\bar{\Xi}_{1,n}\left(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}\right) + \bar{\Xi}_1^{(n)}\left(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}\right)\right)}{2(n-1)!} \\
& \times \left(\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}\right)\right) \times \left(\int \Psi^\dagger\left(\theta_f^{(j)}, Z_j\right) \Psi\left(\theta_i^{(j)}, Z_j\right) dZ_j\right)^{n-1}
\end{aligned} \tag{272}$$

Given (32) and (38):

$$\begin{aligned}
\bar{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right) & \simeq \int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \sum_{l=1}^n C_{n-1}^{l-1} \left( \left[ \frac{\delta^{l-1} \omega^{-1}\left(J, \theta^{(i)}, Z_i, |\Psi|^2\right)}{\Lambda^l \prod_{m=1}^{l-1} \delta \left| \Psi\left(\theta^{(i)} - \frac{|Z_i - \bar{Z}|}{c}, \bar{Z}\right) \right|^2} \right]_{\substack{|\Psi(\theta, Z)|^2 \\ = \mathcal{G}_0(0, Z)}} \right) d\theta^{(i)} \\
\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right) & \simeq \int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \sum_{l=1}^{n-1} C_{n-1}^{l-1} \left( \left[ \frac{\delta^{l-1} \omega^{-1}\left(J, \theta^{(i)}, Z_i, |\Psi|^2\right)}{\Lambda^l \prod_{m=1}^{l-1} \delta \left| \Psi\left(\theta^{(i)} - \frac{|Z_i - \bar{Z}|}{c}, \bar{Z}\right) \right|^2} - \frac{\zeta^{(l)}}{\Lambda^l} \right]_{\substack{|\Psi(\theta, Z)|^2 \\ = \mathcal{G}_0(0, Z)}} \right) d\theta^{(i)}
\end{aligned}$$

We can approximate:

$$\hat{\Xi}_{1,n}\left(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}\right)$$

by the constant  $-\int_{\theta_i^{(i)}}^{\theta_f^{(i)}} \sum_{l=1}^n C_{n-1}^{l-1} \frac{\zeta^{(l)}}{\Lambda^l} d\theta^{(i)}$  and:

$$\frac{\left(\bar{\Xi}_{1,n}\left(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}\right) + \bar{\Xi}_1^{(n)}\left(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}\right)\right)}{2}$$

by  $\frac{\hat{\Xi}_{1,n}(Z_i, \theta_i^{(i)}, \{Z_j\}_{j \neq i})}{2}$ . Using (32), we can compute the coefficient of  $\frac{\delta^{l-1} \omega^{-1}(J, \theta^{(i)}, Z_i, |\Psi|^2)}{\prod_{m=1}^{l-1} \delta \left| \Psi \left( \theta^{(i)} - \frac{|Z_i - \bar{Z}|}{c}, \bar{Z} \right) \right|^2}$  (272). It is

given by:

$$\begin{aligned} & \frac{1}{2} \sum_{n=l}^{\infty} \frac{1}{(n-1)!} C_{n-1}^{l-1} \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \times \left( \int \Psi^\dagger \left( \theta_f^{(j)}, Z_j \right) \Psi \left( \theta_i^{(j)}, Z_j \right) dZ_j \right)^{n-l} \\ &= \frac{1}{2!} \sum_{p=0}^{\infty} \frac{1}{p!} \hat{\Xi}_{1,p+l} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)} \right) \times \left( \int \Psi^\dagger \left( \theta_f^{(j)}, Z_j \right) \Psi \left( \theta_i^{(j)}, Z_j \right) dZ_j \right)^p \\ &\equiv \frac{1}{2!} \hat{\Xi}_{1,l} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}, |\Psi|^2 \right) \end{aligned} \quad (273)$$

Thus,  $\omega_1^{-1}(J(\theta), \theta, Z, \mathcal{G}_0(0, Z))$  can be defined by its derivatives at  $\mathcal{G}_0(0, Z)$ :

$$\frac{\delta^n \omega_1^{-1}(J(\theta), \theta, Z, \mathcal{G}_0)}{\delta^n \mathcal{G}_0(0, Z)} = \frac{\delta^n \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0)}{\delta^n \mathcal{G}_0(0, Z)} \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}, |\Psi|^2 \right) \quad (274)$$

It defines a function similar to  $\omega^{-1}(\theta^{(i)}, Z)$  which satisfies:

$$\omega_1^{-1}(J(\theta), \theta, Z, \mathcal{G}_0) = \hat{G} \left( J(\theta) + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega_1 \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right)}{\omega_1(\theta, Z)} W \left( \frac{\omega(\theta, Z)}{\omega \left( \theta - \frac{|Z-Z_1|}{c}, Z_1 \right)} \right) \bar{\mathcal{G}}_0(0, Z_i) dZ_1 \right)$$

for some function  $\hat{G}$ . The derivatives of  $\hat{G}$  computed at  $\bar{J} + \bar{\mathcal{G}}_0(0, Z_i)$  are such that (274) is satisfied. For weak interactions  $\hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}) \simeq \bar{\zeta}_n \ll 1$ , a solution of (274) is given by:

$$\begin{aligned} & \hat{G}^{(n)}(\bar{J} + \bar{\mathcal{G}}_0(0, Z_i)) \\ & \simeq \left( \frac{\omega(J, \theta, Z) \hat{\Xi}_1(\bar{J}, |Z_i - Z|)}{\frac{\kappa}{N} T(Z, Z_1)} \right)^n \\ & \times \left( 1 - \left( \int \frac{\kappa}{N} \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right) \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) F' [J, \omega, \theta, Z, \Psi] \right) \\ & \times \left( \hat{\Xi}_{1,n} \left( Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}, \theta_f^{(i)}, |\Psi|^2 \right) \right) \end{aligned} \quad (275)$$

where  $\omega(J, \theta, Z)$  is the static solution and denotes the average over time and space of  $\hat{\Xi}_1(J, \theta, l_i, |Z_i - Z|)$ . In that case, equation (147) computing the successive derivatives  $\frac{\delta^n \omega_1^{-1}(J(\theta), \theta, Z, \mathcal{G}_0)}{\delta^n \mathcal{G}_0(0, Z)}$  is no more valid. Actually, given (275), the successive products of  $\hat{G}'(\bar{J} + \bar{\mathcal{G}}_0(0, Z_i))$  arising in (146) are negligible. On the contrary, the successive derivatives of  $\hat{T}(\theta, Z, Z_1, \omega, \Psi)$  that were neglected in the derivation of (147), involve contributions proportional to  $\hat{G}^{(n)}(\bar{J} + \bar{\mathcal{G}}_0(0, Z_i))$  that become dominant. The successive derivatives in this approximation are:

$$\begin{aligned} & \frac{\delta^{(n)} \hat{T}(\theta, Z, Z_1 \omega, |\Psi|^2)}{\delta \mathcal{G}_0(0, Z)} \\ & \simeq \frac{\left( \frac{\kappa}{N} T(Z, Z_1) \right)^n \hat{F}^{(n)} [J, \omega, \theta, Z, |\Psi|^2]}{\omega^2(J, \theta, Z) + \left( \int \frac{\kappa}{N} \omega \left( J, \theta - \frac{|Z-Z'|}{c}, Z' \right) \left| \Psi \left( \theta - \frac{|Z-Z'|}{c}, Z' \right) \right|^2 T(Z, Z') dZ' \right) F' [J, \omega, \theta, Z, |\Psi|^2]} \end{aligned}$$

and:

$$\begin{aligned} & \frac{\delta^n \hat{T}_1(\theta, Z, Z_1 \omega, \Psi)}{\delta^n \mathcal{G}_0(0, Z)} \\ & \simeq \frac{1}{\omega(J, \theta, Z)} \frac{(\frac{\kappa}{N} T(Z, Z_1))^n \hat{G}^{(n)}[J, \omega, \theta, Z, |\Psi|^2]}{1 - \left( \int \frac{\kappa}{N} \omega\left(J, \theta - \frac{|Z-Z'|}{c}, Z'\right) \left| \Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right) \right|^2 T(Z, Z') dZ' \right) G'[J, \omega, \theta, Z, |\Psi|^2]} \end{aligned}$$

and (274) becomes:

$$\begin{aligned} & \frac{\delta^{n-1} \omega_1^{-1}(J(\theta), \theta, Z, \mathcal{G}_0)}{\delta^n \mathcal{G}_0(0, Z)} \\ & \simeq \int \frac{\omega\left(J, \theta - \sum_{l=1}^n \frac{|Z^{(l-1)} - Z^{(l)}|}{c}, Z_1\right)}{\omega(J, \theta, Z)} \\ & \quad \times \frac{(\frac{\kappa}{N} T(Z, Z_1))^n \hat{G}^{(n)}[J, \omega, \theta, Z, |\Psi|^2]}{1 - \left( \int \frac{\kappa}{N} \omega\left(J, \theta - \frac{|Z-Z'|}{c}, Z'\right) \left| \Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right) \right|^2 T(Z, Z') dZ' \right) G'[J, \omega, \theta, Z, |\Psi|^2]} \\ & \simeq \frac{(\frac{\kappa}{N} T(Z, Z_1))^n \hat{G}^{(n)}[J, \omega, \theta, Z, |\Psi|^2]}{1 - \left( \int \frac{\kappa}{N} \omega\left(J, \theta - \frac{|Z-Z'|}{c}, Z'\right) \left| \Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right) \right|^2 T(Z, Z') dZ' \right) G'[J, \omega, \theta, Z, |\Psi|^2]} \end{aligned}$$

As a consequence, using (275) and (146):

$$\begin{aligned} \frac{\delta^n \omega_1^{-1}(J(\theta), \theta, Z, \mathcal{G}_0)}{\delta^n \mathcal{G}_0(0, Z)} & \simeq (\omega(J, \theta, Z) \bar{\Xi}_1(\bar{J}, |Z_i - Z|))^n \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}) \\ & \simeq \frac{\delta^n \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0)}{\delta^n \mathcal{G}_0(0, Z)} \hat{\Xi}_{1,n}(Z_i, \{Z_j\}_{j \neq i}, \theta_i^{(i)}) \end{aligned}$$

as needed.

The term  $\omega_2^{-1}(J(\theta), \theta, Z, \Psi)$  is defined by:

$$\begin{aligned} & \omega_2^{-1}(J(\theta), \theta, Z, |\Psi|^2) \\ & = F^2 (\Psi^\dagger \mathcal{G}_0^{-1} \Psi) \left( \frac{(\alpha(x) - Fx \exp(-x)) \left( \bar{\Xi}_{1,\infty}(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}) \right)}{(x + \exp(-x)) (\langle 1 \rangle - x + (y^2 - x^2))^4} \right) \\ & \quad - F^2 (\Psi^\dagger \mathcal{G}_0^{-1} \Psi) \left( \frac{(\beta(x) - F \exp(-x)) y \left( \bar{\Xi}_{1,2}(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}) \right)}{(x + \exp(-x)) (\langle 1 \rangle - x + (y^2 - x^2))^4} \right) \\ & \simeq -F^2 \left( \Psi_0^\dagger U''(X_0) \delta \Psi(\theta, Z) - \nabla_\theta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) \Psi_0^\dagger \Psi_0 \right) \\ & \quad \times \left( \frac{(\alpha(x) - Fx \exp(-x)) \left( \bar{\Xi}_{1,\infty}(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}) \right) - (\beta(x) - F \exp(-x)) y \left( \bar{\Xi}_{1,2}(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}) \right)}{(x + \exp(-x)) (\langle 1 \rangle - x + (y^2 - x^2))^4} \right) \\ & = -F^2 \left( \sqrt{X_0} U''(X_0) \delta \Psi(\theta, Z) - \nabla_\theta \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0) X_0 \right) \\ & \quad \times \left( \frac{(\alpha(x) - Fx \exp(-x)) \left( \bar{\Xi}_{1,\infty}(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}) \right) - (\beta(x) - F \exp(-x)) y \left( \bar{\Xi}_{1,2}(Z_i, \theta_{i'}^{(i)}, \{Z_j\}_{j \neq i}) \right)}{(x + \exp(-x)) (\langle 1 \rangle - x + (y^2 - x^2))^4} \right) \end{aligned}$$

## Appendix 6. Dynamic equations for connectivity functions

We adapt the description of ([47]) to our context. The transfer function  $T$  from  $i$  to  $j$  satisfies the following equation:

$$\begin{aligned} & \nabla_{\theta^{(i)}(n_i)} T \left( \left( Z_i, \theta^{(i)}(n_i), \omega_i(n_i) \right), \left( Z_j, \theta^{(j)}(n_j), \omega_j(n_j) \right) \right) \\ &= -\frac{1}{\tau} T \left( \left( Z_i, \theta^{(i)}(n_i), \omega_i(n_i) \right), \left( Z_j, \theta^{(j)}(n_j), \omega_j(n_j) \right) \right) \\ & \quad + \lambda \left( \hat{T} \left( \left( Z_i, \theta^{(i)}(n_i), \omega_i(n_i) \right), \left( Z_j, \theta^{(j)}(n_j), \omega_j(n_j) \right) \right) \right) \delta \left( \theta^{(i)}(n_i) - \theta^{(j)}(n_j) - \frac{|Z_i - Z_j|}{c} \right) \end{aligned} \quad (276)$$

where  $\hat{T}$  measures the variation of  $T$  due to the signals sent from  $j$  to  $i$  and the signals emitted by  $i$ . It satisfies the following equation:

$$\begin{aligned} & \nabla_{\theta^{(i)}(n_i)} \hat{T} \left( \left( Z_i, \theta^{(i)}(n_i), \omega_i(n_i) \right), \left( Z_j, \theta^{(j)}(n_j), \omega_j(n_j) \right) \right) \\ &= \rho \delta \left( \theta^{(i)}(n_i) - \theta^{(j)}(n_j) - \frac{|Z_i - Z_j|}{c} \right) \\ & \quad \times \left\{ \left( h(Z, Z_1) - \hat{T} \left( \left( Z_i, \theta^{(i)}(n_i), \omega_i(n_i) \right), \left( Z_j, \theta^{(j)}(n_j), \omega_j(n_j) \right) \right) \right) C \left( \theta^{(i)}(n-1) \right) h_C \left( \omega_i(n_i) \right) \right. \\ & \quad \left. - D \left( \theta^{(i)}(n-1) \right) \hat{T} \left( \left( Z_i, \theta^{(i)}(n_i), \omega_i(n_i) \right), \left( Z_j, \theta^{(j)}(n_j), \omega_j(n_j) \right) \right) h_D \left( \omega_j(n_j) \right) \right\} \end{aligned} \quad (277)$$

where  $h_C$  and  $h_D$  are increasing functions. We depart slightly from ([47]) by the introduction of the function  $h(Z, Z_1)$  (they chose  $h(Z, Z_1) = 1$ ), to implement some loss due to the distance. We may chose for example:

$$h(Z, Z_1) = \exp \left( -\frac{|Z_i - Z_j|}{\nu c} \right)$$

where  $\nu$  is a parameter. Equation (277) involves two dynamic factors  $C \left( \theta^{(i)}(n-1) \right)$  and  $D \left( \theta^{(i)}(n-1) \right)$ . The factor  $C \left( \theta^{(i)}(n-1) \right)$  describes the accumulation of input spikes. It is solution of the differential equation:

$$\nabla_{\theta^{(i)}(n-1)} C \left( \theta^{(i)}(n-1) \right) = -\frac{C \left( \theta^{(i)}(n-1) \right)}{\tau_C} + \alpha_C \left( 1 - C \left( \theta^{(i)}(n-1) \right) \right) \omega_j \left( Z_j, \theta^{(i)}(n-1) - \frac{|Z_i - Z_j|}{c} \right) \quad (278)$$

In the continuous approximation, the solution of (278) is:

$$\begin{aligned} C \left( \theta^{(i)}(n-1) \right) &= \int \exp \left( - \left( \frac{\left( \theta^{(i)}(n-1) - \theta^{(i)'} \right)}{\tau_C} + \alpha_C \int_{\theta^{(i)'}}^{\theta^{(i)}(n-1)} \omega_j \left( Z_j, \theta' - \frac{|Z_i - Z_j|}{c} \right) d\theta' \right) \right) \\ & \quad \times \omega_j \left( Z_j, \theta^{(i)'} - \frac{|Z_i - Z_j|}{c} \right) d\theta^{(i)'} \end{aligned}$$

If a static equilibrium  $\omega_0(Z_j)$  exists, expanding around this equilibrium leads to approximate the integral:

$$\int_{\theta_i'}^{\theta_i} \omega_j \left( Z_j, \theta' - \frac{|Z_i - Z_j|}{c} \right) d\theta'$$

by the quantity:

$$\omega_0(Z_j) \left( \theta^{(i)}(n-1) - \theta^{(i)'} \right)$$

so that:

$$C \left( \theta^{(i)}(n-1) \right) = \int \exp \left( - \left( \frac{1}{\tau_C} + \alpha_C \omega_0(Z_j) \right) \left( \theta^{(i)}(n-1) - \theta^{(i)'} \right) \right) \left( C_0 + \omega_j \left( Z_j, \theta^{(i)'} - \frac{|Z_i - Z_j|}{c} \right) \right) d\theta_i' \quad (279)$$

The term  $D(\theta^{(i)}(n-1))$  is proportional to the accumulation of output spikes and is solution of:

$$\nabla_{\theta^{(i)}(n-1)} D(\theta^{(i)}(n-1)) = -\frac{D(\theta^{(i)}(n-1))}{\tau_D} + \alpha_D \left(1 - D(\theta^{(i)}(n-1))\right) \omega_i(Z_i) \quad (280)$$

In the continuous approximation, the solution of (280) is:

$$D(\theta^{(i)}(n-1)) = \int \exp\left(-\left(\frac{1}{\tau_D} + \alpha_D \omega_0(Z_i)\right) (\theta^{(i)}(n-1) - \theta^{(i)'})\right) (D_0 + \omega_i(Z_i, \theta^{(i)'})) d\theta^{(i)'} \quad (281)$$

As a consequence, the dynamics for transfer functions is a set of two equations:

$$\begin{aligned} & \nabla_{\theta^{(i)}(n_i)} T\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right), \left(Z_j, \theta^{(j)}(n_j), \omega_j(n_j)\right)\right) \\ &= -\frac{1}{\tau} T\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right), \left(Z_j, \theta^{(j)}(n_j), \omega_j(n_j)\right)\right) \\ &+ \lambda \left(\hat{T}\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right), \left(Z_j, \theta^{(j)}(n_j), \omega_j(n_j)\right)\right)\right) \delta\left(\theta^{(i)}(n_i) - \theta^{(j)}(n_j) - \frac{|Z_i - Z_j|}{c}\right) \end{aligned} \quad (282)$$

$$\begin{aligned} & \nabla_{\theta^{(i)}(n_i)} \hat{T}\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right), \left(Z_j, \theta^{(j)}(n_j), \omega_j(n_j)\right)\right) \\ &= \rho \delta\left(\theta^{(i)}(n_i) - \theta^{(j)}(n_j) - \frac{|Z_i - Z_j|}{c}\right) \\ &\times \left\{ \left(h(Z, Z_1) - \hat{T}\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right), \left(Z_j, \theta^{(j)}(n_j), \omega_j(n_j)\right)\right)\right) C(\theta^{(i)}(n-1)) h_C(\omega_i(n_i)) \right. \\ &\left. - D(\theta^{(i)}(n-1)) \hat{T}\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right), \left(Z_j, \theta^{(j)}(n_j), \omega_j(n_j)\right)\right) h_D(\omega_j(n_j)) \right\} \end{aligned} \quad (283)$$

with  $C(\theta^{(i)}(n-1))$  and  $D(\theta^{(i)}(n-1))$  given by (279) and (281).

The field translation of (282) and (283) is obtained by including the following potential terms in the action for the field:

$$\begin{aligned} & \int \left( \nabla_{\theta} T((Z, \theta, \omega), (Z_1, \theta_1, \omega_1)) + \frac{T((Z, \theta, \omega), (Z_1, \theta_1, \omega_1))}{\tau} \right. \\ & \left. - \lambda \left(\hat{T}((Z, \theta, \omega), (Z_1, \theta_1, \omega_1))\right) \delta\left(\theta - \theta_1 - \frac{|Z - Z_1|}{c}\right) \right) \\ & \times |\Psi(\theta, Z, \omega)|^2 |\Psi(\theta_1, Z_1, \omega_1)|^2 \end{aligned} \quad (284)$$

corresponding to (282) and:

$$\begin{aligned} & \int \left( \nabla_{\theta} \hat{T}((Z, \theta, \omega), (Z_1, \theta_1, \omega_1)) - \rho \delta\left(\theta^{(i)}(n_i) - \theta^{(j)}(n_j) - \frac{|Z - Z_1|}{c}\right) \right. \\ & \times \left\{ \left(h(Z, Z_1) - \hat{T}((Z, \theta, \omega), (Z_1, \theta_1, \omega_1))\right) C(\theta, Z, Z_1) h_C(\omega) - D(\theta, Z) \hat{T}((Z, \theta, \omega), (Z_1, \theta_1, \omega_1)) h_D(\omega_1) \right\} \\ & \left. \times |\Psi(\theta, Z, \omega)|^2 |\Psi(\theta_1, Z_1, \omega_1)|^2 \right) \end{aligned} \quad (285)$$

for (283), with  $C(\theta, Z, Z_1)$  and  $D(\theta, Z)$  are defined as:

$$\begin{aligned} C(\theta, Z, Z_1) &= \int^{\theta} \exp\left(-\left(\frac{1}{\tau_C} + \alpha_C \omega_0(Z_1)\right) (\theta - \theta')\right) \left(C_0 + \omega\left(Z_1, \theta' - \frac{|Z - Z_1|}{c}\right)\right) d\theta' \\ D(\theta, Z) &= \int^{\theta} \exp\left(-\left(\frac{1}{\tau_D} + \alpha_D \omega_0(Z)\right) (\theta - \theta')\right) (D_0 + \omega(Z, \theta')) d\theta' \end{aligned}$$



For

$$\begin{aligned}\tau_C(Z_1) &= \frac{1}{\tau_C} + \alpha_C \omega_0(Z_1) < 1 \\ \tau_D(Z) &= \frac{1}{\tau_D} + \alpha_D \omega_0(Z) < 1\end{aligned}$$

and if the transfer function adapts slowly with respect to  $\omega(Z, \theta)$ , we can simplify the expressions for  $C(\theta, Z, Z_1)$  and  $D(\theta, Z)$ :

$$\begin{aligned}C(\theta, Z, Z_1) &\simeq C(Z_1) = \frac{C_0 + \omega_0(Z_1)}{\tau_C(Z_1)} \\ D(\theta, Z) &\simeq D(Z) = \frac{D_0 + \omega_0(Z)}{\tau_D(Z)}\end{aligned}$$

After projection on the dependent frequency states the transfer functions become functions  $T((Z, \theta), (Z_1, \theta_1))$  and  $\hat{T}((Z, \theta), (Z_1, \theta_1))$  respectively. Moreover, we can simplify the action by finding the configurations for  $T((Z, \theta), (Z_1, \theta_1))$  and  $\hat{T}((Z, \theta), (Z_1, \theta_1))$  that minimize the potential terms (284), (285). It corresponds to set:

$$\nabla_{\theta} T((Z, \theta, \omega), (Z_1, \theta_1, \omega_1)) + \frac{T((Z, \theta, \omega), (Z_1, \theta_1, \omega_1))}{\tau} - \lambda \left( \hat{T}((Z, \theta, \omega), (Z_1, \theta_1, \omega_1)) \right) \delta \left( \theta - \theta_1 - \frac{|Z - Z_1|}{c} \right) = 0 \quad (286)$$

and:

$$\begin{aligned}0 &= \left( \nabla_{\theta} \hat{T}((Z, \theta, \omega), (Z_1, \theta_1, \omega_1)) - \rho \delta \left( \theta^{(i)}(n_i) - \theta^{(j)}(n_j) - \frac{|Z - Z_1|}{c} \right) \right. \\ &\quad \left. \times \left\{ \left( h(Z, Z_1) - \hat{T}((Z, \theta, \omega), (Z_1, \theta_1, \omega_1)) \right) C(\theta, Z, Z_1) h_C(\omega) - D(\theta, Z) \hat{T}((Z, \theta, \omega), (Z_1, \theta_1, \omega_1)) h_D(\omega_1) \right\} \right.\end{aligned} \quad (287)$$

We look for solutions of the form:

$$\begin{aligned}T \left( Z, \theta, Z_1, \theta - \frac{|Z - Z_1|}{c} \right) &\equiv T(Z, \theta, Z_1) \\ \hat{T} \left( Z, \theta, Z_1, \theta - \frac{|Z - Z_1|}{c} \right) &\equiv \hat{T}(Z, \theta, Z_1)\end{aligned}$$

so that  $T(Z, \theta, Z_1)$  and  $\hat{T}(Z, \theta, Z_1)$  satisfy:

$$\nabla_{\theta} T(Z, \theta, Z_1) + \left( \frac{T(Z, \theta, Z_1)}{\tau} - \lambda \hat{T}(Z, \theta, Z_1) \right) = 0 \quad (288)$$

$$\nabla_{\theta} \hat{T}(Z, \theta, Z_1) = \rho \left( \left( h(Z, Z_1) - \hat{T}(Z, \theta, Z_1) \right) C(Z_1) h_C(\omega(Z, \theta)) - \hat{T}(Z, \theta, Z_1) D(Z) h_D \left( \omega \left( Z_1, \theta - \frac{|Z - Z_1|}{c} \right) \right) \right) \quad (289)$$

Using (288), we replace  $\hat{T}(Z, \theta, Z_1)$  in (289):

$$\hat{T}(Z, \theta, Z_1) = \frac{\nabla_{\theta} T(Z, \theta, Z_1)}{\lambda} + \frac{T(Z, \theta, Z_1)}{\lambda \tau}$$

and we arrive to the differential equation satisfied by  $T(Z, \theta, Z_1)$ :

$$\frac{\nabla_{\theta}^2 T(Z, \theta, Z_1)}{\lambda} + U_1(\omega) \nabla_{\theta} T(Z, \theta, Z_1) + U_2(\omega) T(Z, \theta, Z_1) = \rho C(Z_1) h(Z, Z_1) h_C(\omega(Z, \theta)) \quad (290)$$

where:

$$\begin{aligned}
U_1(\omega) &= \left( \frac{1}{\lambda\tau} + \frac{\rho}{\lambda} \left( C(Z_1) h_C(\omega(Z, \theta)) + D(Z) h_D \left( \omega \left( Z_1, \theta - \frac{|Z - Z_1|}{c} \right) \right) \right) \right) \\
U_2(\omega) &= \frac{\rho}{\lambda\tau} \left( C(Z_1) h_C(\omega(Z, \theta)) + D(Z) h_D \left( \omega \left( Z_1, \theta - \frac{|Z - Z_1|}{c} \right) \right) \right)
\end{aligned}$$

If we consider that the transfer function varies slowly compared to the oscillations of the thread, we can approximate (290) by a quite static equation:

$$U_2(\omega) T(Z, \theta, Z_1) = \rho C(Z_1) h(Z, Z_1) h_C(\omega(Z, \theta))$$

whose solution is:

$$\begin{aligned}
T(Z, \theta, Z_1) &= \frac{\lambda\tau C(Z_1) h(Z, Z_1) h_C(\omega(Z, \theta))}{C(Z_1) h_C(\omega(Z, \theta)) + D(Z) h_D \left( \omega \left( Z_1, \theta - \frac{|Z - Z_1|}{c} \right) \right)} \quad (291) \\
&\simeq \frac{\lambda\tau h(Z, Z_1)}{1 + \frac{D(Z)}{C(Z_1)} \frac{h_D \left( \omega \left( Z_1, \theta - \frac{|Z - Z_1|}{c} \right) \right)}{h_C(\omega(Z, \theta))}}
\end{aligned}$$

Thus  $T(Z, \theta, Z_1)$  is a decreasing function of  $\omega \left( Z_1, \theta - \frac{|Z - Z_1|}{c} \right)$  and an increasing function of  $\omega(Z, \theta)$ , as hypothesized in the text. The fully static solution associated to (291) is:

$$T_0(Z, Z_1) = \frac{\lambda\tau h(Z, Z_1)}{1 + \frac{D(Z)}{C(Z_1)} \frac{h_D(\omega_0(Z_1))}{h_C(\omega_0(Z))}}$$

We conclude this section by giving the linearized version of (290) around the static solution  $(\omega_0(Z), T_0(Z, Z_1))$ . It is:

$$\begin{aligned}
0 &= \frac{\nabla_{\theta}^2 \hat{T}(Z, \theta, Z_1)}{\lambda} + U_1(\omega_0) \nabla_{\theta} \hat{T}(Z, \theta, Z_1) + U_2(\omega_0) \hat{T}(Z, \theta, Z_1) \quad (292) \\
&\quad - \rho C(Z_1) \left( 1 - \frac{\hat{T}_0(Z, Z_1)}{\lambda\tau} \right) h'_C(\omega_0(Z)) \Omega(Z, \theta) \\
&\quad + \frac{\rho \hat{T}_0(Z, Z_1)}{\lambda\tau} \left( D(Z) h'_D(\omega_0(Z_1)) \Omega \left( Z_1, \theta - \frac{|Z - Z_1|}{c} \right) \right)
\end{aligned}$$

where:

$$\begin{aligned}
U_1(\omega_0) &= \frac{1}{\lambda\tau} + \frac{\rho}{\lambda} (C(Z_1) h_C(\omega_0(Z)) + D(Z) h_D(\omega_0(Z_1))) \\
U_2(\omega_0) &= \frac{\rho}{\lambda\tau} (C(Z_1) h_C(\omega_0(Z)) + D(Z) h_D(\omega_0(Z_1))) \\
T_0(Z, Z_1) &= \frac{\lambda\tau h(Z, Z_1)}{1 + \frac{D(Z)}{C(Z_1)} \frac{h_D(\omega_0(Z_1))}{h_C(\omega_0(Z))}} \\
\hat{T}_0(Z, Z_1) &= \frac{T_0(Z, Z_1)}{h(Z, Z_1)} \\
\hat{T}(Z, \theta, Z_1) &= \frac{T(Z, \theta, Z_1) - T_0(Z, Z_1)}{h(Z, Z_1)} \\
\Omega(Z, \theta) &= \omega(Z, \theta) - \omega_0(Z)
\end{aligned}$$

for  $\omega_0(Z) \equiv \omega_0$ , this reduces to:

$$\begin{aligned}
&\frac{\nabla_{\theta}^2 \hat{T}(Z, \theta, Z_1)}{\lambda} + U_1(\omega_0) \nabla_{\theta} \hat{T}(Z, \theta, Z_1) + U_2(\omega) \hat{T}(Z, \theta, Z_1) \quad (293) \\
&= - \frac{\rho \hat{T}_0(Z, Z_1) D(Z) h'_D(\omega_0) \Omega \left( Z_1, \theta - \frac{|Z - Z_1|}{c} \right)}{\lambda\tau} + \rho C(Z_1) \left( 1 - \frac{\hat{T}_0(Z, Z_1)}{\lambda\tau} \right) h'_C(\omega_0) \Omega(Z, \theta)
\end{aligned}$$

where:

$$\begin{aligned}
U_1(\omega) &= \frac{1}{\lambda\tau} + \frac{\rho}{\lambda} (Ch_C(\omega_0) + Dh_D(\omega_0)) \\
U_2(\omega) &= \frac{\rho}{\lambda\tau} (Ch_C(\omega_0) + Dh_D(\omega_0)) \\
T_0(Z, Z_1) &= \frac{\lambda\tau h(Z, Z_1)}{1 + \frac{D}{C} \frac{h_D(\omega_0)}{h_C(\omega_0)}} \\
C &= \frac{C_0 + \omega_0}{\tau_C} \\
D &= \frac{D_0 + \omega_0}{\tau_D}
\end{aligned}$$

which can also be written, up to the second order in derivatives:

$$\begin{aligned}
& \frac{\nabla_\theta^2 \hat{T}(Z, \theta, Z_1)}{\lambda} + U_1(\omega_0) \nabla_\theta \hat{T}(Z, \theta, Z_1) + U_2(\omega) \hat{T}(Z, \theta, Z_1) \\
&= \left( \rho C(Z_1) h'_C(\omega_0) - \frac{\rho \hat{T}_0(Z, Z_1) (D(Z) h'_D(\omega_0) + C(Z_1) h'_C(\omega_0))}{\lambda\tau} \right) \Omega(Z, \theta) \\
&+ \frac{\rho \hat{T}_0(Z, Z_1) D(Z) h'_D(\omega_0) \left( \frac{|Z-Z_1|}{c} \nabla_\theta \Omega(Z, \theta) - \frac{(Z-Z_1)^2}{2c^2} \nabla_\theta^2 \Omega(Z, \theta) - \frac{(Z-Z_1)^2}{2} \nabla_Z^2 \Omega(Z, \theta) \right)}{\lambda\tau}
\end{aligned} \tag{294}$$

Then, to separate the dependences in time and position, we define:

$$\begin{aligned}
\hat{T}(Z, \theta) &= \int h(Z, Z_1) \frac{\hat{T}(Z, \theta, Z_1)}{\sqrt{\frac{\pi}{8} \left( \frac{1}{X_r} \right)^2 + \frac{\pi}{2} \alpha}} \\
\bar{C}(Z) &= \frac{1}{\sqrt{\frac{\pi}{8} \left( \frac{1}{X_r} \right)^2 + \frac{\pi}{2} \alpha}} \int h(Z, Z_1) C(Z_1) \\
\bar{C}_0(Z) &= \frac{1}{\sqrt{\frac{\pi}{8} \left( \frac{1}{X_r} \right)^2 + \frac{\pi}{2} \alpha}} \int h(Z, Z_1) C(Z_1) \hat{T}_0(Z, Z_1) \\
\hat{T}_0(Z) &= \frac{1}{\sqrt{\frac{\pi}{8} \left( \frac{1}{X_r} \right)^2 + \frac{\pi}{2} \alpha}} \int h(Z, Z_1) \hat{T}_0(Z, Z_1)
\end{aligned}$$

and  $\hat{T}(Z, \theta)$  satisfies:

$$\begin{aligned}
& \frac{\nabla_\theta^2 \hat{T}(Z, \theta)}{\lambda} + U_1(\omega_0) \nabla_\theta \hat{T}(Z, \theta) + U_2(\omega) \hat{T}(Z, \theta) \\
&= \left( \rho \bar{C}(Z) h'_C(\omega_0) - \frac{\rho \left( D(Z) \hat{T}_0(Z) h'_D(\omega_0) + \bar{C}_0(Z) h'_C(\omega_0) \right)}{\lambda\tau} \right) \Omega(Z, \theta) \\
&+ \frac{\rho D(Z) h'_D(\omega_0) \left( \Gamma_1 \nabla_\theta \Omega(Z, \theta) - (\Gamma_1 \nabla_\theta^2 \Omega(Z, \theta) + c^2 \Gamma_2 \nabla_Z^2 \Omega(Z, \theta)) \right)}{\lambda\tau}
\end{aligned}$$

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