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ON EVOLUTION EQUATIONS FOR LIE GROUPOIDS

JEAN-MARIE LESCURE AND STÉPHANE VASSOUT

Abstract. Using the calculus of Fourier integral operators on Lie groupoids developped in [18], we study the fundamental solution of the evolution equation $(\frac{\partial}{\partial t}+iP)u=0$ where P is a self adjoint elliptic order one G-pseudodifferential operator on the Lie groupoid G. Along the way, we continue the study of distributions on Lie groupoids done in [17] by adding the reduced C^*-algebra of G in the picture and we investigate the local nature of the regularizing operators of [32].

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1. Introduction

The main motivation of this paper is the construction of an approximate solution to the problem

$$\begin{cases}(\dfrac{\partial}{\partial t}+iP)u=f\\ u(0)=g\end{cases}\tag{1}$$

in the framework of a Lie groupoid $G\rightrightarrows M$. This means that P here is a suitable order 1 pseudodifferential G-operator, that f,g live in suitable spaces of distributions and that the approximate solution will be seeked among Fourier integral G-operators. The present article can be considered as a continuation of [17], where properties of distributions on Lie groupoids, and convolution of them, are studied in a certain generality, and of [18], where Hörmander's notion and calculus of Fourier integral operators on manifolds [11, 12] are exported to the framework of Lie groupoids. We will frequently refer to the results of these papers, and one of their cornerstones, namely the symplectic groupoid structure of T^*G [6]: $s_\Gamma, r_\Gamma:\Gamma=T^*G\rightrightarrows A^*G$, will be of great importance here again.

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The Cauchy problem (1) has been and can be of course investigated in many situations under many different assumptions. We refer more precisely to [12, Theorem 29.1.1] to illustrate the kind of results that we want to achieve on Lie groupoids. This can be summarized by the following problem:

($\mathcal{P}$) *Under an ellipticity assumption on P, the fundamental solution of* (1) *should have, up to suitable regularizing error terms, an explicit approximation by Fourier integral G-operators that describes in a simple and geometric way how the singularities of the initial data g propagate at time t under the action of the principal symbol of P.*

To set the problem on firm foundations, we first study in Section 3 existence and unicity conditions for (1) in the general framework of C^*-algebras and Hilbertian modules, and we require there that P is an unbounded self-adjoint regular operator on a C^*-algebra A [2, 3, 33, 13, 32, 30]. Then the fundamental solution of (1) denoted by $E(t) = e^{-itP}$ is obtained by continuous functional calculus, which yields the existence of solutions, while easy computations identical to those for Hilbert spaces show the uniqueness. We get in particular:

Theorem 1. *Let A be a C^*-algebra, let H be a Hilbertian A-module and P be a selfadjoint regular operator on A. Let $H^\infty = \cap_k \operatorname{dom} P^k$. Then for any $f \in C^\infty(\mathbb{R}, H^\infty)$ and $g \in H^\infty$, the Cauchy problem* (1) *has a unique solution in $C^\infty(\mathbb{R}, H^\infty)$, given by*

$$u(t) = e^{-itP} g + \int_0^t e^{i(s-t)P} f(s) ds. \tag{2}$$

This preliminary result allows us to speak about the fundamental solution of (1) in the case of a Lie groupoid G with compact units space M and of a first order elliptic symmetric and compactly supported, polyhomogeneous pseudodifferential G-operator P. Indeed, we then know by [32] that (the closure of) P is selfadjoint and regular on, for instance, the reduced C^*-algebra of G, denoted by $C^*_r(G)$. In particular, the theorem above applies and the task to find a nice approximation to $E(t)$ among Fourier integral G-operators is meaningful. Note that, because of (2), the error term will automatically belong to the space $\mathcal{H}^\infty = H^\infty \cap (H^\infty)^*$. Our answer to the problem ($\mathcal{P}$) is the main result of the paper:

Theorem 2. *There exists a C^∞ family $\Lambda_t \subset T^*G$ of G-relations and a C^∞ family of compactly supported Fourier integral G-operators $U(t) \in I^{(n^{(1)}-n^{(0)})/4}(G, \Lambda_t; \Omega^{1/2})$ such that :*

$$(\frac{\partial}{\partial t} + iP)U(t) \in C_c^\infty(G, \Omega^{1/2}), \tag{3}$$

and for any t, we have: $E(t) - U(t) \in \mathcal{H}^\infty$.

Let us now explain in some details the ingredients and the intermediate results, some of them being interesting on their owns, required in the proof of the main theorem.

First of all, Theorem 2 immediately rises the preliminary question of the regularity of elements in $\mathcal{H}^\infty$. Strictly speaking, elements of $\mathcal{H}^\infty$ live in a noncommutative C^*-algebra so dealing with their local properties makes a priori no sense. We manage on the one hand to prove that elements of the reduced C^*-algebra $C^*_r(G)$ of a Lie groupoid G are distributions on G, in a canonical way, and on the other hand, to precise the regularity of elements in $\mathcal{H}^\infty$. These intermediate tasks are the subject of Sections 4 and 5 and the details can be summarized as follows.

The space of distributions we deal with, denoted by $\mathcal{D}'(G)$, is the one of distributions on G with values in the density bundle $\Omega^{1/2} := \Omega^{1/2}(r^*AG) \otimes \Omega^{1/2}(s^*AG)$ and thus the space of test functions we use, denoted by $\mathcal{D}(G)$, is the one of compactly supported C^∞ sections of the density bundle $\Omega_0^{1/2} := \Omega^{-1/2} \otimes \Omega_G^1$. Thus $\mathcal{D}'(G) = (\mathcal{D}(G))'$ and the choice of $\Omega^{1/2}$ is relevant because $C_c^\infty(G) := C_c^\infty(G, \Omega^{1/2}) \subset \mathcal{D}'(G)$ is in a *canonical* way an involutive algebra, whose a certain completion is precisely the algebra $C_r^*(G)$. Also, we have proved in [17] that the product $\star$ in $C_c^\infty(G)$ (called the convolution product for obvious reasons) widely generalizes, by continuity, to distributions in $\mathcal{D}'(G)$. For instance the space $\mathcal{E}'_{r,s}(G)$ of compactly supported distributions on G whose pushforwards by the source and range maps are C^∞ on M (transversal distributions) forms a unital involutive algebra for the convolution product. Another justification for these choices of densities comes from the present work, indeed we prove that transversal distributions also act by convolution on $\mathcal{D}(G)$ in a nice way, and that weak factorizations in the sense of [10] are available:

Theorem 3. *Let $\langle \cdot, \cdot \rangle$ denote the pairing $\mathcal{D}'(G) \times \mathcal{D}(G) \to \mathbb{C}$ and ι the inversion of the groupoid.*

(1) $\forall (u, \omega) \in \mathcal{D}'_{r,s}(G) \times \mathcal{D}(G), \quad \langle u, \omega \rangle = \langle \iota^* u, \iota^* \omega \rangle = \langle \delta_M, \iota^* u \star \omega \rangle$.

(2) The space $\mathcal{D}(G)$ is a bimodule over $\mathcal{E}'_{r,s}(G)$ and:

$$\forall u, v \in \mathcal{E}'_{r,s}(G), \forall \omega \in \mathcal{D}(G), \quad \langle u \star v, \omega \rangle = \langle v, \iota^* u \star \omega \rangle = \langle u, \omega \star \iota^* v \rangle$$

(3) Let $\omega \in \mathcal{D}(G)$. For any neighborhood V of M into G , on can write ω as a finite sum of elements $\xi \star \chi$ where $\xi \in C_c^\infty(G)$, $\mathrm{supp}(\xi) \subset V$ and $\chi \in \mathcal{D}(G)$, $\mathrm{supp}(\chi) \subset \mathrm{supp}(\omega)$.

This material allows us in Section 5 to answer to the question about the local nature of elements in $\mathcal{H}^\infty$, and along the way, that of elements in $C_r^*(G)$:

Theorem 4.

(1) There is a continuous embedding $C_r^*(G) \hookrightarrow \mathcal{D}'(G)$. This embedding extends the pairing $\langle \cdot, \cdot \rangle$ between $C_c^\infty(G)$ and $\mathcal{D}(G)$.

(2) The inclusions $C_{\mathrm{orb}}^{\infty,0}(G) \cap C_c(G) \subset \mathcal{H}^\infty \subset C_{\mathrm{orb}}^{\infty,0}(G) \cap C_r^*(G)$ hold true.
Here $C_{\mathrm{orb}}^{\infty,0}$ refers to the space of continuous functions on G that are C^∞ on the subgroupoids $G_{\mathcal{O}} = s^{-1}(\mathcal{O})$, as well as all their derivatives along the fibers of s and r, for every orbits $\mathcal{O} = r(s^{-1}(\{x\}))$ in M.

Next, we explain how the principal symbol p of P gives rise to the family of Lagrangian submanifolds Λ_t, $t \in \mathbb{R}$, that will describe the propagation of singularities as expected in Problem $(\mathcal{P})$. By definition, $P \in I^{1+(n^{(1)}-n^{(0)})/4}(G, M, \Omega^{1/2})$ is a polyhomogeneous conormal distribution, thus posseses a homogeneous principal symbol $p_0^0 \in C^\infty(A^*G \setminus 0)$. If one considers the family $(P_x)_{x \in M}$ of ordinary pseudodifferential operators in the fibers of s and collects their principal symbols into a homogeneous function $p^0 \in C^\infty(T_s^*G \setminus 0)$, $T_s^*G = (\ker ds)^*$, that will be called the *principal G-symbol of P*. After lifting p^0 to a function on $T^*G \setminus \ker r_\Gamma$, one gets the following identity:

$$\forall (\gamma, \xi) \in T^*G \setminus \ker r_\Gamma, \quad p(\gamma, \xi) = p_0(r_\Gamma(\gamma, \xi)).$$

Here r_Γ is the range map of the symplectic groupoid $\Gamma = T^*G$. The computations also give a local expression for the sub-principal G-symbol of P, that is, for the collection of the sub-principal symbols of the operators P_x. Now it turns out that the Hamiltonian flow χ of the principal G-symbol p is complete and right invariant, and we get the required Lagrangian

submanifolds that will describe the evolution of singularities:

$$\forall t \in \mathbb{R}, \qquad \Lambda_t = \chi_t(A^*G \setminus 0).$$

This already produces a C^∞ family of homogeneous Lagrangian submanifolds of $T^*G \setminus 0$ that satisfies the group relation, with respect to the product in T^*G:

$$\Lambda_t.\Lambda_s = \Lambda_{t+s}.$$

Moreover $\Lambda_t \subset \dot{T}^*G := T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma)$, that is, in the vocabulary of [18], every Λ_t is a G-relation, while the global object coming with the family $(\Lambda_t)_t$:

$$\Lambda = \{(t, \tau, \gamma, \xi) \ ; \ \tau + p(\gamma, \xi) = 0, \ (\gamma, \xi) \in \Lambda_t\} \subset T^*\mathbb{R} \times T^*G$$

is a family $\mathbb{R} \times G$-relation. As in [18], this construction highlights the important role of the symplectic groupoid structure of T^*G in analysis.

There is a last result, of technical nature, that intervenes in the proof of Theorem 2. Indeed, assuming that the Lagrangian submanifolds Λ_t provide the good candidate for Theorem 2, we are led to search a first order parametrix U_0 for $\partial_t + iP$ among the Fourier integral G-operators associated with $(\Lambda_t)_t$. This amounts to solve the transport equation for principal symbols:

$$\frac{\partial}{\partial t}\sigma_{\mathrm{pr}}(U_0) + i\sigma_{\mathrm{pr}}(PU_0) = 0$$

and thus it requires to express the principal symbol of the convolution product PU_0 of the lagrangian distributions P and U_0. Since by construction and on purpose, the principal symbol p_0 vanishes on $r_\Gamma\Lambda$, we need to look for the next term in the asymptotic expansion of the total symbol of PU_0. This is what is achieved, modulo some technical details, by using the following result:

Theorem 5. *Let $Q \in \Psi_c^m(G)$, with principal G-symbol q, sub-principal symbol q^{1s}, and let C be a G-relation such that q vanishes on C. Let $A \in I^{m'}(G, C; \Omega^{1/2})$ and a be a principal symbol of A.*
Then

$$QA \in I^{m+m'-1}(G, C; \Omega^{1/2}) \text{ and } \sigma_{\mathrm{pr}}(QA) = -i\mathcal{L}_q a + q^{1s} a.$$

Here $\mathcal{L}_q$ is the Lie derivative along the Hamiltonian vector field H_q of q.

Many interesting situations produce non compactly supported operators P: for instance, if Δ is a Laplacian on G then $\sqrt{\Delta} = P + S$ with P as above and $S \in \mathcal{H}^\infty$ [32]. The main theorem trivially extends to such non compactly supported operators: one just needs to replace $C_c^\infty(G)$ by $\mathcal{H}^\infty$ in (3). We describe at the end of the paper several situations where Theorem 2 applies:

(1) The usual pseudodifferential calculus on a compact manifold without boundary X. We use the pair groupoid $G = X \times X \rightrightarrows X$. Since X itself is an orbit, we have $C^{\infty,0}_{\mathrm{orb}}(G, E) = C^\infty(X \times X, E)$ and we just recover the classical result on manifolds.
(2) The longitudinal calculus on foliations [4]. We use the holonomy groupoid. We recover a construction in [15] by a quite different approach.
(3) Right invariant calculus on a Lie group G. We use G as a groupoid with units space $\{e\}$.
(4) The b-calculus on manifolds with corners [23]. We use the b-groupoid [25].
(5) The calculus on manifolds with fibred boundary or with iterated fibred corners [20, 8]. We use the groupoid of [8].

As far as we know, the results obtained for cases (3), (4), (5) above are new.

The next section contains the basic definitions and notation necessary for the sequel and can be considered as an extension of the introduction for the unfamiliar reader.

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2. Notation and reminders

Densities on manifolds. If E is a real vector space of dimension n and $\alpha \in \mathbb{R}$, we denote by $\Omega^\alpha(E)$ the vector space of maps $\omega : \Lambda^n E \setminus 0 \to \mathbb{C}$, called α-densities, such that $\omega(tV) = |t|^\alpha \omega(V)$ for any $t \neq 0$ and $V \in \Lambda^n E \setminus 0$. For any C^∞ real vector bundle $E \to X$, the vector bundle $\Omega^\alpha(E) = \cup_x \Omega^\alpha(E_x) \to X$ is a C^∞ line bundle, with transition functions given by $|\det(g_{ij})|^\alpha$ if (g_{ij}) is a set of transition functions for E. Sections of $\Omega^\alpha(E)$ are called α-densities on E and sections of $\Omega^\alpha_X := \Omega^\alpha(TX)$ are called α-densities on X. Densities bundles are always trivialisable, but not canonically in general: one can construct an everywhere positive section using local trivializations.

A fundamental point is that compactly supported one densities on X can be integrated over X. More precisely, there is a unique linear form $\int_X : C^\infty_c(X, \Omega^1_X) \longrightarrow \mathbb{R}$ such that if $f = \mathsf{f}(x)|dx|$ is compactly supported in a local chart U with local coordinates $x = (x_1, \dots, x_n)$, then

$$\int_X f = \int_{\mathbb{R}^n} \mathsf{f}(x) dx.$$

Above, $|dx|$ is the one density defined by $|dx| = |dx_1 \wedge \cdots \wedge dx_n|$. Diffeomorphisms $\phi : X \to Y$ provide isomorphisms $\phi^* : \Omega^\alpha_Y \to \Omega^\alpha_X$ given by $\phi^*\omega(V) = \omega(\phi_* V)$. By construction, the integral of one densities is invariant under the action of diffeomorphisms. Densities are usually handled with the following canonical isomorphisms:

- $\Omega^\alpha(E) \otimes \Omega^\beta(E) \simeq \Omega^{\alpha+\beta}(E)$
- $\Omega^\alpha(E \oplus F) \simeq \Omega^\alpha(E) \otimes \Omega^\alpha(F)$,
- $\Omega^\alpha(E^*) \simeq \Omega^{-\alpha}(E)$
- if $0 \to F \to E \to G \to 0$ is exact, then $\Omega^\alpha(E) \simeq \Omega^\alpha(F) \otimes \Omega^\alpha(G)$.

Lie groupoids. A Lie groupoid $G \rightrightarrows M$ is a pair of manifolds (G, M) of respective dimensions generally denoted by $n = n^{(1)} + n^{(0)}$ and $n^{(0)}$, together with the following data and required properties. The data are:

(a) two surjective submersions $r, s : G \to M$, called range and source,
(b) a C^∞ section $\upsilon : M \to G$ of both r and s, assimilated to an inclusion,
(c) a C^∞ map $\iota : G \to G$ called inversion, noted: $\gamma^{-1} := \iota(\gamma)$,
(d) a C^∞ map $G^{(2)} = \{(\gamma_1, \gamma_2) \in G^2 \ ; \ s(\gamma_1) = r(\gamma_2)\} \to G$ called multiplication: $\gamma_1\gamma_2 := m(\gamma_1, \gamma_2)$.

The required properties are those giving a sense to the following intuition: a groupoid is the algebraic structure obtained from a group G after spreading out its unit into a whole subset M, that is

(i) $r(\gamma_1\gamma_2) = r(\gamma_1)$, $s(\gamma_1\gamma_2) = s(\gamma_2)$ whenever it makes sense,
(ii) $\upsilon(r(\gamma))\gamma = \gamma$, $\gamma\upsilon(s(\gamma)) = \gamma$ for all γ,

(iii) $r(\gamma^{-1}) = s(\gamma)$, $s(\gamma^{-1}) = r(\gamma)$ for all γ,
(iv) $\gamma\gamma^{-1} = \upsilon(r(\gamma))$, $\gamma^{-1}\gamma = \upsilon(s(\gamma))$ for all γ,
(v) $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$ whenever it makes sense.

It follows that υ is an embedding (often omitted in the notation), that ι is an involutive diffeomorphism and m a surjective submersion. We note G_x, the s-fiber at $x \in M$, G^x its r-fiber, and we set $T_sG = \ker ds$, $T_rG = \ker dr$. We note L_γ, R_γ the left and right multiplication by γ. The Lie algebroid AG of $G \rightrightarrows M$ is by definition here the vector bundle $\ker ds|_M \to M$. The differential map $dr : AG \to TM$ is denoted by $\mathfrak{a}$ and called the anchor map. To any C^∞ section $X \in \Gamma(AG)$ corresponds a right invariant vector field $\widetilde{X} \in \Gamma(TG)$, defined by $\widetilde{X}_\gamma := dR_\gamma(X_{r(\gamma)})$, and conversely. The right invariance means $\widetilde{X}_{\gamma\eta} = dR_\eta(\widetilde{X}_\gamma)$. This allows to define a Lie algebra structure on $\Gamma(AG)$ that satisfies

$$\forall X, Y \in \Gamma(AG),\ \forall f \in C^\infty(M), \quad [X, fY] = f[X, Y] + (\mathfrak{a}(X)f)Y.$$

We refer to [24, 19] for a detailed account on Lie groupoids and Lie algebroids.

We will use several α-densities bundles over G, often for $\alpha = \pm 1/2, \pm 1$:

- The bundles $\Omega^\alpha(\ker d\pi)$ of densities along the fibers of $\pi = s, r$. They are conveniently replaced for computations by the respective isomorphic bundles $\Omega^\alpha_s = \Omega^\alpha(r^*AG)$ and $\Omega^\alpha_r = \Omega^\alpha(s^*AG)$. The isomorphisms are induced by:

$$\mathcal{R} : r^*AG \longrightarrow T_sG, \quad (\gamma, X) \longmapsto \big(\gamma, (dR_\gamma)_{r(\gamma)}(X)\big), \tag{4}$$

$$\mathcal{S} : s^*AG \longrightarrow T_rG, \quad (\gamma, X) \longmapsto \big(\gamma, (dL_\gamma \circ \iota)_{s(\gamma)}(X)\big). \tag{5}$$

- The "symmetrisation" of the preceeding ones: $\Omega^\alpha = \Omega^\alpha_s \otimes \Omega^\alpha_r$, which is suitable for convolution on G.
- The bundle $\Omega^{1/2}_0 = \Omega^{-1/2} \otimes \Omega^1_G$ necessary for the pairing:

$$f \in C^\infty(G, \Omega^{1/2}),\ \omega \in C^\infty_c(G, \Omega^{1/2}_0), \quad \langle f, \omega\rangle = \int_G fg.$$

Actually, there is a natural isomorphism $\Omega^{1/2}_0 \simeq \Omega^{1/2}(r^*TM) \otimes \Omega^{1/2}(s^*TM)$.

The cotangent groupoid The cotangent space T^*G has a non trivial groupoid structure: $\Gamma = T^*G \rightrightarrows A^*G$, with structure maps $r_\Gamma, s_\Gamma, m_\Gamma, \iota_\Gamma$ defined as follows:

- $r_\Gamma(\gamma, \xi) = \big(r(\gamma), {}^t dR_\gamma(\xi|_{T_\gamma G_{s(\gamma)}})\big)$ and $s_\Gamma(\gamma, \xi) = \big(s(\gamma), -{}^t d(L_\gamma \circ \iota)(\xi|_{T_\gamma G^{r(\gamma)}})\big)$,
- $(\gamma_1, \xi_1)(\gamma_2, \xi_2) = (\gamma_1\gamma_2, \xi)$ with $\xi(dm(t_1, t_2)) = \xi_1(t_1) + \xi_2(t_2)$,
- $(\gamma, \xi)^{-1} = (\gamma^{-1}, -{}^t d\iota(\xi))$.

This is a *symplectic* groupoid, which means that the graph of m_Γ is a Lagrangian submanifold of $(T^*G)^3$ provided with the symplectic form $\omega \oplus \omega \oplus -\omega$, with ω the canonical symplectic form of T^*G. We refer to [6, 19] for a detailed account on symplectic groupoids and on the related notion of VB-groupoids, as well as to [17, 18] for the interest of this symplectic structure regarding the theory of distributions on groupoids. We will denote

$$T^*_\bullet G = T^*G \setminus \ker r_\Gamma \quad \text{and} \quad \dot{T}^*G = T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma).$$

We will consider in this paper homogeneous lagrangian submanifods of $T^*G \setminus 0$ that avoids the kernel of r_Γ and s_Γ. We call them G-relations, in reference to the term *canonical relations* often employed for (product) manifolds. Under mild assumptions, G-relations compose well in the groupoid T^*G [18]. G-relations Λ such that $s_\Gamma|_\Lambda$ and $r_\Gamma|_\Lambda$ are diffeomorphisms onto

their ranges are called invertible. We will sometimes use densities along the s_Γ and r_Γ-fibers of the cotangent groupoid T^*G. Both are naturally isomorphic and:

$$\Omega^\alpha_{s_\Gamma} \simeq \Omega^\alpha_{r_\Gamma} \simeq \hat{\Omega}^\alpha \otimes \hat{\Omega}^{-\alpha}_G$$

where $\hat{E}$ denotes the pull back to T^*G of the bundle $E \to G$. Also, we note that $\Omega^1_{s_\Gamma}|_{A^*G} = \Omega^1(AT^*G) = (\mathcal{D}^{\mathrm{tr}}_{AG})^{-1}$ where $\mathcal{D}^{\mathrm{tr}}_{AG}$ is the transverse density bundle of AG [7].

The convolution algebra Throughout this paper we make the convention:

$$C^\infty(G) := C^\infty(G, \Omega^{1/2}),$$

that is, we omit the ubiquitous density bundle $\Omega^{1/2}$ in the notation. We apply the same convention for other sections of $\Omega^{1/2}$ with various regularity and support conditions. When the sections of a different bundle are considered, this bundle will be always mentionned.

The *convolution algebra structure* on $C^\infty_c(G)$ refers to the product $\star$ canonically defined from any of the following three intuitive formulas:

$$f \star g(\gamma) = \int_{\gamma_2 \in G_{s(\gamma)}} f(\gamma\gamma_2^{-1})g(\gamma_2) = \int_{\gamma_1 \in G^{r(\gamma)}} f(\gamma_1)g(\gamma_1^{-1}\gamma) = \int_{(\gamma_1,\gamma_2)\in m^{-1}(\gamma)} f(\gamma_1)g(\gamma_2) \tag{6}$$

This is justified as follows. Write $f = \mathsf{f}(\mu_s\mu_r)^{1/2}$, $g = \mathsf{g}(\mu_s\mu_r)^{1/2}$ with $\mathsf{f}, \mathsf{g} \in C^\infty_c(G, \mathbb{C})$, $\mu_s = r^*\mu \in C^\infty(G, \Omega^1_s)$, $\mu_r = s^*\mu \in C^\infty(G, \Omega^1_r)$ for some positive $\mu \in C^\infty(M, \Omega^1(AG))$. Then, whenever $\gamma_1\gamma_2 = \gamma$:

$$f(\gamma_1)g(\gamma_2) = \mathsf{f}(\gamma_1)\mathsf{g}(\gamma_2)\mu_r^{1/2}(\gamma_1)\mu_s^{1/2}(\gamma_2)(\mu_s\mu_r)^{1/2}(\gamma) \text{ and } \mu_r(\gamma_1) = \mu_s(\gamma_2).$$

We now may set rigorously:

$$f \star g(\gamma) = \Big(\int_{\gamma_2 \in G_{s(\gamma)}} \mathsf{f}(\gamma\gamma_2^{-1})\mathsf{g}(\gamma_2)\mathcal{R}_*\mu_s(\gamma_2)\Big)(\mu_s\mu_r)^{1/2}(\gamma). \tag{7}$$

This gives consistance to the first formula in (6). The second and third formulas are obtained from the first one using the diffeormorphisms $L_\gamma \circ \iota : G_{s(\gamma)} \to G^{r(\gamma)}$ and $(L_\gamma \circ \iota, \mathrm{Id}) : G_{s(\gamma)} \to m^{-1}(\gamma)$. Equivalently, one can directly define them as we did for the first one using the suitable structural isomorphisms to create the appropriate one densities on $G^{r(\gamma)}$ and $m^{-1}(\gamma)$. With the notation above, this concretely means:

$$f \star g(\gamma) = \Big(\int_{\gamma_1 \in G^{r(\gamma)}} \mathsf{f}(\gamma_1)\mathsf{g}(\gamma_1^{-1}\gamma)\mathcal{S}_*\mu_r(\gamma_1)\Big)(\mu_s\mu_r)^{1/2}(\gamma) \tag{8}$$

$$= \Big(\int_{(\gamma_1,\gamma_2)\in m^{-1}(\gamma)} \mathsf{f}(\gamma_1)\mathsf{g}(\gamma_2)\mathcal{M}_*\mu_s(\gamma_2)\Big)(\mu_s\mu_r)^{1/2}(\gamma). \tag{9}$$

with, for the last line:

$$\mathcal{M} : r^*AG|_{G_{s(\gamma)}} \longrightarrow Tm^{-1}(\gamma), \quad (\gamma_2, X) \longmapsto \big(\gamma\gamma_2^{-1}, \gamma_2, d(L_{\gamma\gamma_2^{-1}} \circ \iota)(X), dR_{\gamma_2}(X)\big). \tag{10}$$

By $C^{\infty,0}_\pi(G)$, we denote the space of elements in $C_c(G)$ that belong to $C(U_{(0)}, C^\infty(U_{(1)}))$ over any local trivializations $\kappa : U \xrightarrow{\simeq} U_{(0)} \times U_{(1)}$ of π (here $\pi = \mathrm{pr}_1 \circ \kappa$). The topology is modeled on that of $C(U_{(0)}, C^\infty(U_{(1)}))$ and is Fréchet. We write $C^{\infty,0}_{\pi,c}$ for $C^{\infty,0}_\pi \cap C_c$, and equip it with the corresponding LF-topology.

The reduced C^*-algebra of a groupoid. The space $C^\infty_c(G, \Omega^{1/2}_s)$ comes with a natural prehilbertian $C(M)$-module structure:

$$f, g \in C^\infty_c(G, \Omega^{1/2}_s), \quad \langle f \mid g\rangle_s(x) = \int_{G_x} \overline{f(\gamma)}g(\gamma). \tag{11}$$

Its completion as a hilbertian $C(M)$-module is denoted by $L^2_s(G)$. The homomorphism $\lambda : C^\infty_c(G) \longrightarrow \mathcal{L}(L^2_s(G))$ given by:

$$\forall f \in C^\infty_c(G),\ g \in C^\infty_c(G, \Omega^{1/2}_s), \quad \lambda(f)(g)(\gamma) = f \star g(\gamma) = \int_{G_{s(\gamma)}} f(\gamma\alpha^{-1})g(\alpha)$$

is well defined, injective, and the reduced C^*-algebra of G, denoted by $C^*_r(G)$, is the completion of $C^\infty_c(G)$ with respect to the C^*-norm $\|f\| = \|\lambda(f)\|_{\mathrm{op}}$. The extended homomorphism

$$\lambda : C^*_r(G) \longrightarrow \mathcal{L}(L^2_s(G))$$

is called the left regular representation. Starting from $C^\infty_c(G, \Omega^{1/2}_r)$, we get a Hilbert $C(M)$-module $L^2_r(G)$ and the right regular representation $\rho : C^*_r(G) \longrightarrow \mathcal{L}(L^2_r(G))$. The adjunction map $* : L^2_s(G) \longrightarrow L^2_r(G)$ provides a unitary anti-homomorphism. The unfamiliar reader may consult [29, 5, 14] for groupoids C^*-algebras and [33, 13, 30] for Hilbertian modules.

Distributions. We consider in this article various spaces of distributions on G, always valued in $\Omega^{1/2}$, which thus is safely omitted. We set:

$$\mathcal{D}'(G) := \mathcal{D}'(G, \Omega^{1/2}).$$

This is the topological dual of the space:

$$\mathcal{D}(G) := C^\infty_c(G, \Omega^{1/2}_0),$$

where $\Omega^{1/2}_0 := \Omega^{-1/2} \otimes \Omega^1_G$. The elements of $\mathcal{D}(G)$ will be called test functions, with a slight abuse of vocabulary. We denote by $\mathcal{E}'(G)$ the subspace of $\mathcal{D}'(G)$ consisting of compactly supported distributions. We set:

$$\mathcal{D}'_\pi(G) = \{u \in \mathcal{D}'(G)\ ;\ \forall f \in C^\infty_c(G, \Omega^{1/2}_s),\ \pi_!(uf) \in C^\infty(M, \Omega^{1/2}(AG))\} \tag{12}$$

where $\pi_!$ denotes the pushforward of distributions and $\pi = r, s$. Elements of $\mathcal{D}'_\pi(G)$ are called C^∞-transversal distributions with respect to π [1, 17, 31]. The convolution product $\star$ extends by continuity to transversal distibutions, providing $\mathcal{E}'_\pi(G)$ with the structure of a unital algebra and $\mathcal{E}'_{r,s}(G) = \mathcal{E}'_s(G) \cap \mathcal{E}'_r(G)$ with the structure of an involutive unital algebra. The unit is $\delta_M(f) = \int_M f$ and the involution is $u^* = \overline{\iota^* u}$. Elements of $\mathcal{D}'_\pi(G)$ can be restricted fiberwise, giving C^∞ families over M of distributions in the fibers, the space of whose families being denoted by $C^\infty_\pi(M, \mathcal{D}'(G))$, or viewed as $C^\infty(M)$-linear continuous operators, the space of whose operators being denoted by $\mathcal{L}_{C^\infty(M)}(C^\infty_c(G), C^\infty(M, \Omega^{1/2}(AG)))$, and there are canonical isomorphisms:

$$\mathcal{D}'_\pi(G) \simeq C^\infty_\pi(M, \mathcal{D}'(G)) \simeq \mathcal{L}_{C^\infty(M)}(C^\infty_c(G), C^\infty(M, \Omega^{1/2}(AG))).$$

We will also consider continuously transversal distributions with respect to $\pi = r, s$:

$$\mathcal{D}'_{\pi,0}(G) = \{u \in \mathcal{D}'(G)\ ;\ \forall f \in C^\infty_c(G, \Omega^{1/2}_s),\ \pi_!(uf) \in C(M, \Omega^{1/2}(AG))\}. \tag{13}$$

By rephrazing the arguments in [17], one gets:

$$\mathcal{D}'_{\pi,0}(G) \simeq C_\pi(M, \mathcal{D}'(G)) \simeq \mathcal{L}_{C(M)}(C^{\infty,0}_{\pi,c}(G), C(M, \Omega^{1/2}(AG))).$$

G-operators: they are the continuous linear maps $C^\infty_c(G) \to C^\infty(G)$ given by right invariant families of (linear continuous) operators acting in the s-fibers. More precisely, P is a G-operator if there exists a family $P_x : C^\infty_c(G_x, \Omega^{1/2}_{G_x}) \longrightarrow C^\infty(G_x, \Omega^{1/2}_{G_x})$, $x \in M$, such that for all $x \in M$, $\gamma \in G$, $f \in C^\infty_c(G)$:

$$P(f)|_{G_x} = P_x(f|_{G_x}) \text{ and } P_{r(\gamma)} \circ R_\gamma = R_\gamma \circ P_{s(\gamma)},\ \forall \gamma \in G. \tag{14}$$

This is equivalent to requiring that P maps $C_c^\infty(G) \to C^\infty(G)$ continuously and that:

$$P(f) \star g = P(f \star g) \quad \text{for any } f, g \in C_c^\infty(G).$$

A G-operator P as an adjoint if there exists a G-operator Q such that $(Pf)^* \star g = f^* \star (Qg)$ for any f, g. We denote by Op_G (resp. Op_G^*, $\mathrm{Op}_{G,c}^*$) the space of (resp. adjointable, compactly supported and adjointable) G-operators.

It is proved in [17] that the map

$$\mathcal{D}_r'(G) \to \mathrm{Op}_G, \quad u \mapsto u \star \cdot$$

is an isomorphism, with inverse $P \mapsto k_P$, given by $k_P(\gamma) = p_{r(\gamma)}(r(\gamma), \gamma^{-1})$ where p_x denotes the Schwartz kernel of P_x. The same map induces an isomorphism:

$$\mathrm{Op}_{G,c}^* \simeq \mathcal{E}_{r,s}'(G).$$

Pseudodifferential G-operators and regularizing operators. Among the class of G-operators one finds the well known subclass of pseudodifferential G-operators (G-PDO) [4, 26, 28, 32], that is, of right invariant families of pseudodifferential operators in the s-fibers: they coincide with left convolution by distributions in:

$$\Psi_G^* = I^{*+(n^{(1)}-n^{(0)})/4}(G, M; \Omega^{1/2}). \tag{15}$$

where here I refers to the space of conormal distributions. One has a principal symbol map:

$$\sigma_0 : \Psi_G^m \longrightarrow S^{[m]}(A^*G)$$

with kernel Ψ_G^{m-1}. Here $S^{[m]} = S^m/S^{m-1}$. It is well known that $(\Psi_{G,c}^*, \star)$ is an involutive unital algebra and σ_0 an algebra homomorphism. When $P \in \Psi_{G,c}^1$ is elliptic and symmetric, then its closure, as an unbounded operator on $C_r^*(G)$ with domain $C_c^\infty(G)$, is selfadjoint and *regular* [32, 2, 3, 30]. There is a canonical scale H^t, $t \in \mathbb{R}$, of Hilbert $C_r^*(G)$-modules, that we call intrinsic Sobolev modules, which do not depend, up to isomorphism of Hilbertian structures, on the symmetric elliptic operator $P \in \Psi_{G,c}^1$ used to define them:

$$H^t = \overline{C_c^\infty(G)}^{\langle\cdot|\cdot\rangle_t}, \quad \langle f \mid g\rangle_t = \langle (1+P^2)^t \star f \mid g\rangle \in C_r^*(G),$$

where $\langle a|b\rangle = a^*b$. Then any $Q \in \Psi_{G,c}^m$ gives a bounded homomorphism $Q \in \mathcal{L}(H^t, H^{t-m})$ and for any $m > 0$, the inclusion $H^t \hookrightarrow H^{t+m}$ is a compact homomorphism of Hilbert modules. All of this material is developped in [32]. Although we call the spaces H^t *Sobolev* modules, we may think of them as modules of abstract pseudodifferential operators of order $< -t$. Indeed, H^t is also the completion of $\Psi_{G,c}^{<-t}$ for the norm $\|Q\|_t = \|(1+P^2)^{t/2}Q\|_{C_r^*(G)}$. Scales of Hilbert modules closer to the usual notion of Sobolev regularity of order t for functions or distributions will be obtained using the left regular representation of H^t.

The algebra $\Psi_{G,c}^*$ is too small for practical purposes. For instance, the inverse of an elliptic element in $\Psi_{G,c}^*$ which is invertible as an operator between Sobolev modules, has no reason to be compactly supported. This phenomenom propagates to operators obtained by holomorphic functional calculus and we will eventually face it also when building an approximation of $E(t) = e^{itP}$ by Fourier integral G-operators. A suitable enlargement of $\Psi_{G,c}^*$ is provided by:

$$\Psi_G^* := \Psi_{G,c}^* + \mathcal{H}^\infty, \quad \text{where } \mathcal{H}^\infty = H^\infty \cap (H^\infty)^* \tag{16}$$

and $H^\infty = \cap_t H^t \subset C_r^*(G)$. Actually, $\mathcal{H}^\infty$ coincides with the ideal of regularizing operators introduced in [32].

Fourier integral G-operators. Another remarkable subclass of G-operators is given by that of lagrangian distributions on G with respect to arbitrary G-relations. We call them Fourier integral G-operators (G-FIO) and we set for a given G-relation Λ:

$$\Phi^*_G(\Lambda) = I^{*+(n^{(1)}-n^{(0)})/4}(G, \Lambda; \Omega^{1/2}). \tag{17}$$

where here I refers to the space of Lagrangian distributions. The convolution product gives a map

$$\Phi^*_{G,c}(\Lambda_1) \times \Phi^*_G(\Lambda_2) \to \Phi^*_G(\Lambda_1\Lambda_2) \tag{18}$$

as soon as $\Lambda_1 \times \Lambda_2$ has a clean intersection with $T^*G^{(2)}$. This proves in particular that $\Phi^*_G(\Lambda)$ is a bimodule over $\Psi^*_{G,c}$ and that $F^{-1}PF \in \Psi^*_{G,c}$ if $P \in \Psi^*_{G,c}$ and $F \in \Phi^*_{G,c}(\Lambda)$ is invertible. Also, when Λ is invertible, one gets $\Phi^0_{G,c}(\Lambda) \subset \mathcal{M}(C^*_r(G))$ and $\Phi^{<0}_{G,c}(\Lambda) \subset C^*_r(G)$. In general, if $A \in \Phi^*_G(\Lambda)$, the corresponding family $(A_x)_{x\in M}$ consists of operators A_x given by locally finite sums of oscillatory integrals and when Λ is transversal to $T^*_L G$, for any $L = s^{-1}(\mathcal{O})$ and $\mathcal{O} \in M/G$, (this is for instance the case if Λ is invertible), then each A_x is a genuine Fourier integral operator on the manifold G_x. All the statements here about G-FIOs are proved in [18].

3. The one parameter group e^{-itP}, $t \in \mathbb{R}$

Before analyzing evolution equations on groupoids, we study the functional analytic aspects of them in a reasonably general and simple framework. So, let us consider the Cauchy problem:

$$\begin{cases} (\frac{\partial}{\partial t} + iP)u = f \\ u(0) = g \end{cases} \tag{19}$$

in the following situation: P is a regular self-adjoint operator on H [2, 3, 30, 32] where H is a Hilbert module over some C^*-algebra A. It turns out that under natural assumptions on f and g, this problem has a unique solution given in term of the operator e^{-itP}. This operator is first defined in term of the unbounded continuous functional calculus for regular operators [30, Paragraph 14.3.3]. We recall that any nondegenerate representation

$$\pi : C_0(\mathbb{R}) \longrightarrow \mathscr{L}(H)$$

extends into a map $\widetilde{\pi}$ from $C(\mathbb{R})$ (viewed as regular operators on $C_0(\mathbb{R})$) to the set of regular operators on H. The map $\widetilde{\pi}$ is defined through the identification $C_0(\mathbb{R}) \otimes_\pi H \simeq H$ and the formula:

$$\widetilde{\pi}(f) = f \otimes_\pi \mathrm{Id}\,.$$

Moreover, there exists a unique such representation π such that $\widetilde{\pi}(\mathrm{Id}_\mathbb{R}) = P$ and we fix this particular one from now on. Introducing $f_t \in C(\mathbb{R})$, $f_t(\lambda) = e^{-it\lambda}$, we set:

$$e^{-itP} = \widetilde{\pi}(f_t).$$

Actually, the restriction of $\widetilde{\pi}$ to $C_b(\mathbb{R})$ is a strictly continuous homomorphism [30, Proposition 5.19] :

$$\overline{\pi} = \widetilde{\pi}|_{C_b(\mathbb{R})} : C_b(\mathbb{R}) \longrightarrow \mathcal{L}(H).$$

Here, strict continuity refers to the topologies of $C_b(\mathbb{R})$ and $\mathscr{L}(H)$ as multiplier algebras of $C_0(\mathbb{R})$ and $\mathcal{K}(H)$ respectively. The map $\mathbb{R} \ni t \mapsto f_t \in C_b(\mathbb{R})$ being strictly continuous, the map $\mathbb{R} \ni t \to e^{-itP} \in \mathcal{L}(H)$ is thus strictly continuous too. Specializing the semi-norms

giving the strict topology to rank one operators, this means that $t \longmapsto e^{-itP}x \in C(\mathbb{R}, H)$. The following properties are valid:

$$e^{-i(t+s)P} = e^{-itP}e^{-isP} \tag{20}$$

and defining $(Ef)(t) = e^{-itP}f(t)$, $t \in \mathbb{R}$, for $f \in C_b(\mathbb{R}, H)$ we also get:

$$E \in \mathcal{L}(C_b(\mathbb{R}, H)). \tag{21}$$

To further analyse e^{-itP}, we introduce the sequence of Hilbert A-modules associated to P:

$$\forall s \in \mathbb{R}_+,\ H^s = \mathrm{dom}(1+P^2)^{s/2} \text{ and } H^0 = H$$

Note that $H^k = \mathrm{dom}\, P^k$ for $k \in \mathbb{N}$. The Hilbertian structure of H^s is given by:

$$\langle u, v\rangle_s = \langle (1+P^2)^s u, v\rangle.$$

For negative order s, we define H^s to be the completion of $\mathrm{dom}\, P$ with respect to the pre-hilbertian structure given by the scalar product above. We refer to this family of Hilbert A-modules as the *intrinsic scale of Sobolev modules of* P. It was introduced in [32] in the framework of groupoid C^*-algebras.

We recall that $\widetilde{\pi}(f_t(\lambda)\lambda^k) = \widetilde{\pi}(f_t(\lambda))\widetilde{\pi}(\lambda^k)$ and that $\widetilde{\pi}(\lambda^k) = P^k$ for any $k \geq 0$, therefore:

$$e^{-itP}(H^k) = H^k \text{ and } e^{-itP}P^k = P^k e^{-itP}.$$

In particular, we get $e^{itP} \in \mathcal{L}(H^k)$ and $t \longmapsto e^{-itP}x \in C(\mathbb{R}, H^k)$ for any $x \in H^k$. Since $\frac{1}{t}(e^{-it\lambda} - 1) \overset{t\to 0}{\longrightarrow} i\lambda$ uniformly on compact subsets of $\mathbb{R}$, we get using [9, Appendix] that $\frac{1}{t}(e^{itP} - 1)$ converges to iP strongly, that is,

$$\|\frac{1}{t}(e^{itP}x - x) - iPx\|_H \overset{t\to 0}{\longrightarrow} 0, \qquad \text{for all } x \in H^1.$$

Therefore

$$\forall x \in H^1, \quad (t \longmapsto e^{-itP}x) \in C^1(\mathbb{R}, H) \cap C^0(\mathbb{R}, H^1) \tag{22}$$

and

$$\forall x \in H^1,\ \forall t \in \mathbb{R}, \quad \frac{d}{dt}e^{-itP}x = -iPe^{-itP}x. \tag{23}$$

Repeating the previous arguments gives for any natural number k:

$$\forall x \in H^k, \quad (t \longmapsto e^{-itP}x) \in \bigcap_{0\leq j\leq k} C^j(\mathbb{R}, H^{k-j}). \tag{24}$$

This eventually implies:

$$\forall x \in H^\infty, \quad (t \longmapsto e^{-itP}x) \in \bigcap_k C^\infty(\mathbb{R}, H^k) =: C^\infty(\mathbb{R}, H^\infty), \tag{25}$$

where $H^\infty = \cap_k H^k$ has Frechet space structure given by the seminorms $\|\cdot\|_{H^k}$, $k \geq 0$. We can now state the result:

Theorem 1. *Let k be a positive integer. For any $f \in C^{k-1}(\mathbb{R}, H^k)$ and $g \in H^k$, the Cauchy problem:*

$$\begin{cases} (\frac{\partial}{\partial t} + iP)u = f \\ u(0) = g \end{cases} \tag{26}$$

has a unique solution in $\bigcap_{0\leq j\leq k} C^j(\mathbb{R}, H^{k-j})$*, given by*

$$u(t) = e^{-itP}g + \int_0^t e^{i(s-t)P} f(s)ds. \tag{27}$$

Proof. That $e^{-itP}g$ is in the required space and satisfies the equation when $f = 0$ is done before the statement of the theorem. Straightforward arguments prove that the second term in the expression of $u(t)$ in (27) is in the required space too, and it is then obvious that u solves (19). For unicity, consider the case $f = g = 0$ and let u be a solution. Pairing the equation with u on both sides gives the relations

$$-i\langle \frac{\partial}{\partial t}u, u\rangle + \langle Pu, u\rangle = 0$$
$$i\langle u, \frac{\partial}{\partial t}u\rangle + \langle u, Pu\rangle = 0.$$

Since P is selfadjoint, substracting both relations gives

$$\frac{\partial}{\partial t}\langle u, u\rangle = \langle \frac{\partial}{\partial t}u, u\rangle + \langle u, \frac{\partial}{\partial t}u\rangle = 0.$$

Therefore, $\|u(t)\|_H^2 = \|\langle u(t), u(t)\rangle\|_A = \|\langle u(0), u(0)\rangle\|_A = 0$ for any t. □

Keeping the previous setting, let B be a C^*-algebra, L be a Hilbert B-module and $\lambda : A \longrightarrow \mathcal{L}(L)$ be a representation. Then $H_\lambda = H \otimes_\lambda L$ is a Hilbert B-module and $P_\lambda = P \otimes_\lambda \mathrm{Id}$ is a selfadjoint regular operator acting on it. Then, Proposition 1 applies to P_λ and we get the following corollary, using [30, 14.3.2].

Corollary 2.

$$\textit{for } k > 0, \quad \mathrm{dom}\, P_\lambda^k = H_\lambda^k \tag{28}$$

and we have the equality:

$$e^{itP_\lambda} = e^{itP} \otimes_\lambda \mathrm{Id}\,. \tag{29}$$

4. Distributions, test functions and weak factorizations for a Lie groupoid

From now on, and in the remaining parts of this article, we fix a Lie groupoid G of dimension $n = n^{(1)} + n^{(0)}$ with compact basis $G^{(0)} = M$ of dimension $n^{(0)}$. We recall that:

$$\Omega^{1/2} := \Omega_s^{1/2} \otimes \Omega_r^{1/2} = \Omega^{1/2}(r^*AG) \otimes \Omega^{1/2}(s^*AG) \simeq \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dr).$$

and that the bundle $\Omega_0^{1/2}$ used in the space of test functions $\mathcal{D}(G) = C_c^\infty(G, \Omega_0^{1/2})$ satisfies:

$$\Omega_0^{1/2} := \Omega^{-1/2} \otimes \Omega_G^1 \simeq \Omega_s^{1/2} \otimes \Omega_r^{-1/2} \otimes s^*\Omega_M \simeq \Omega_r^{1/2} \otimes \Omega_s^{-1/2} \otimes r^*\Omega_M \simeq r^*\Omega_M^{1/2} \otimes s^*\Omega_M^{1/2}. \tag{30}$$

All the isomorphisms above are easily checked using the isomorphisms

$$\Omega_G^\alpha \simeq \Omega_s^\alpha \otimes s^*\Omega_M^\alpha \simeq \Omega_r^\alpha \otimes r^*\Omega_M^\alpha$$

that result from the exact sequences:

$$0 \longrightarrow \ker d\sigma \longrightarrow TG \xrightarrow{d\sigma} \sigma^*TM \longrightarrow 0, \qquad \sigma = s, r$$

as well as straight properties of the calculus of densities. To finish with this description, we mention that $\Omega_0^{1/2}$ is related, but distinct, to the transverse density bundle $\mathcal{D}_{AG}^{\mathrm{tr}}$ of [7]. The latter is G-invariant and serves to produce geometric transverse measures useful for the geometry of groupoids and stacks, while our choice of "transverse" bundle is required for the

pairing with densities in $\Omega^{1/2}$, but only equivariant with respect to $\mathbb{R}$-actions provided by invariant vectors fields.

Moreover, besides its pairing with distributions, the space $\mathcal{D}(G)$ appears to be a bimodule over $C_c^\infty(G)$, with left and right multiplication given by the canonically defined integrals:

$$f \star \xi(\gamma) = \int_{G_{s(\gamma)}} f(\gamma\alpha^{-1})g(\alpha) \text{ and } \xi \star f(\gamma) = \int_{G_{s(\gamma)}} \xi(\gamma\alpha^{-1})f(\alpha). \tag{31}$$

Finally, we recall that the embedding $C_c^\infty(G) \subset \mathcal{D}'(G)$ is given by:

$$\forall u \in C_c^\infty(G), \omega \in \mathcal{D}(G), \quad \langle u, \omega\rangle = \int_G u(\gamma)\omega(\gamma)d\gamma.$$

The inversion map $\iota : G \to G$ acts on sections of $\Omega^{1/2}$ and $\Omega_0^{1/2}$ in the natural way. This gives involutive isomorphisms:

$$\iota^* : \mathcal{D}(G) \longrightarrow \mathcal{D}(G) \text{ and } \iota^* : C_c^\infty(G) \longrightarrow C_c^\infty(G). \tag{32}$$

The second one extends to an involutive isomorphism $\iota^* : \mathcal{D}'(G) \longrightarrow \mathcal{D}'(G)$.

Proposition 3.

(1) For any $(u, \omega) \in \mathcal{D}'_{r,s}(G) \times \mathcal{D}(G)$, we have

$$\langle u, \omega\rangle = \langle \delta_M, \iota^* u \star \omega\rangle = \langle \delta_M, \omega \star \iota^* u\rangle = \langle \iota^* u, \iota^* \omega\rangle \text{ (trace property).} \tag{33}$$

The trace property $\langle u, \omega\rangle = \langle \iota^* u, \iota^* \omega\rangle$ is still valid with $u \in \mathcal{D}'(G)$.

(2) The map ι^* is an anti-isomorphism of the algebra $\mathcal{E}'_{r,s}(G)$:

$$\forall u, v \in \mathcal{E}'_{r,s}(G), \quad \iota^*(u \star v) = \iota^* v \star \iota^* u, \tag{34}$$

(3) The space $\mathcal{D}(G)$ is a bimodule over $\mathcal{E}'_{r,s}(G)$ and ι^* is a bimodule antisomorphism:

$$\forall u, v \in \mathcal{E}'_{r,s}(G), \forall \omega \in \mathcal{D}(G), \quad \iota^*(u \star \omega \star v) = \iota^* v \star \iota^* \omega \star \iota^* u,$$

(4) For any $u, v \in \mathcal{E}'_{r,s}(G)$ and $\omega \in \mathcal{D}(G)$, we have:

$$\langle u \star v, \omega\rangle = \langle v, \iota^* u \star \omega\rangle = \langle u, \omega \star \iota^* v\rangle = \langle \delta_M, \iota^* u \star \omega \star \iota^* v\rangle$$

Proof. That $\mathcal{D}(G)$ is a bimodule over $\mathcal{E}'_{r,s}(G)$ follows directly from [1, 17]. If u is C^∞, the quantity $\langle u, \omega\rangle$ is the integral of the one density on G defined by the product $u\omega$, whose integral is then invariant by action of diffeomorphisms. In particular, $\langle u, \omega\rangle = \int_G u(\gamma^{-1})\omega(\gamma^{-1})$. On the other hand, one is allowed to write

$$\langle u, \omega\rangle = \int_M \Big(\int_{G_x} \iota^* u(\gamma^{-1})\omega(\gamma)\Big) dx = \int_M \iota^* u \star \omega(x) dx.$$

Both identities together give (1) when u is C^∞, and the general case follows by density and continuity. The identities given in (2) and (3) are then checked easily. □

Let $X \in \Gamma(AG)$. Since $AG \subset TG$, the vector field X provides at any $x \in M$ a local derivation $X_x : \mathcal{D}(G) \to \Omega^1(T_x M)$ and $x \mapsto X_x\omega$ is C^∞ for any $\omega \in \mathcal{D}(G)$. Therefore $X \in \Gamma(AG)$ provides a distribution

$$\tau_X \in \mathrm{Diff}(G) = \{u \in \Psi_G^* \ ; \ \mathrm{supp}(u) \subset M\} \subset \Psi_c^*(G),$$

via the formula:

$$\forall \omega \in C^\infty(G), \quad \langle \tau_X, \omega\rangle = \int_M X\omega.$$

We recall that the algebra isomorphism

$$\begin{aligned} \mathrm{op} :(\mathcal{E}'_{r,s}(G),\star) &\longrightarrow (\mathrm{Op}^{*,c}_G,\circ) \\ u &\longmapsto u\star\cdot \end{aligned} \tag{35}$$

maps $\Psi^*_{G,c}$ (resp. Diff^*_G) to the algebra of uniformly supported and equivariant C^∞ family of pseudodifferential (resp. equivariant C^∞ family of differential) operators on the fibers of s [28, 25, 17].

Note that the action of τ_X as a differential G-operator is given, up to inversion, by the right invariant vector field $\widetilde{X}$ associated with X:

$$\forall u\in C^\infty_c(G),\quad \iota^*\tau_X\star u=\widetilde{X}u.$$

Let φ be the flow of the vector field $\widetilde{X}$. By compacity of M, there exists $\varepsilon>0$ and a neighborhood U of M into G such that φ is defined on $]-\varepsilon,\varepsilon[\times U$. Since $\widetilde{X}_\gamma=dR_\gamma(X_{r(\gamma)})$ for any γ we get the relation $\varphi(t,\gamma\eta)=\varphi(t,\gamma)\eta$ whenever both terms are well defined. Therefore the flow φ is well defined on $]-\varepsilon,\varepsilon[\times G$, and then on $\mathbb{R}\times G$ using the one parameter group property. This proves that the flow of $\widetilde{X}$ is complete and commutes with right multiplication in G:

$$\forall t\in\mathbb{R},\forall\gamma,\eta\in G^{(2)},\quad \varphi(t,\gamma)\in G_{s(\gamma)} \text{ and } \varphi(t,\gamma\eta)=\varphi(t,\gamma)\eta. \tag{36}$$

In other words, X provides an action of $\mathbb{R}$ on the *manifold* G, which is equivariant with respect to right multiplication. Also, the map $\psi:=r\circ\varphi:\mathbb{R}\times M\longrightarrow M$ is the flow of the vector field $\mathfrak{a}(X)\in\Gamma(TM)$ where $\mathfrak{a}=dr|_{TM}$ is the anchor map of G [19] and the map:

$$\varphi:\mathbb{R}\ltimes_\psi M\longrightarrow G,\quad (t,x)\longmapsto\varphi(t,x)$$

is a (C^∞) homomorphism of groupoids over M. We recall that a groupoid homorphism $h:G_1\to G_2$ over (the identity map of) $X=G_1^{(0)}=G_2^{(0)}$ is a map satisfying $h(\alpha\beta)=h(\alpha)h(\beta)$ whenever it makes sense and $r\circ h=r$, $s\circ h=s$.

We record the following simple fact:

Proposition 4. *Let G,H be two Lie groupoids with same units space M.*

(1) Let $h:G\longrightarrow H$ a C^∞ be a homomorphism over M. Then the pushforward map $h_!$ gives rise to a (unital, involutive) algebra homomorphism:

$$h_!:\mathcal{E}'_{r,s}(G)\longrightarrow\mathcal{E}'_{r,s}(H). \tag{37}$$

(2) Let $h_1,h_2:G\longrightarrow H$ be two C^∞ homomorphisms over M and set $h_{12}:=m\circ(h_1\otimes h_2):G^{(2)}\longrightarrow H$. Then for any $u,v\in\mathcal{E}'_{r,s}(G)$, we have

$$h_{1!}u\star h_{2!}v=h_{12!}(u\otimes v|_{G^{(2)}}). \tag{38}$$

Proof. First of all, $h_!:\mathcal{E}'(G)\longrightarrow\mathcal{E}'(H)$ is well defined. Indeed, if $\omega\in\mathcal{D}(H)$ and $u\in C^\infty_c(G)$, then $\omega(h(\gamma))\in\Omega^{1/2}_{M,r(h(\gamma))}\otimes\Omega^{1/2}_{M,s(h(\gamma))}=\Omega^{1/2}_{M,r(\gamma)}\otimes\Omega^{1/2}_{M,s(\gamma)}$ and therefore:

$$\langle h_!u,\omega\rangle:=\langle u,\omega\circ h\rangle=\int_G u(\gamma)\omega(h(\gamma))$$

is canonically defined. The algebraic remaining assertions come from the identities: $m\circ(h\otimes h)=h\circ m$ on $G^{(2)}$, $h\iota=\iota h$ on G, from the functoriality of pushforwards: $f_!g_!=(fg)_!$, and from the definition of the convolution product of distributions: $u\star v=m_!(u\otimes v|_{G^{(2)}})$. $\square$

The goal now is to export to Lie groupoids (with compact unit spaces) a classic result by Dixmier and Malliavin about Lie groups [10]. This will be the main technical tool used to embed reduced C^*-algebras into distributions.

Theorem 5. *Let V be an open neighborhood of M into G and $\omega \in \mathcal{D}(G)$. Then ω is a finite sum of elements:*

$$\xi \star \chi \tag{39}$$

where $\xi \in C_c^\infty(G)$, $\mathrm{supp}(\xi) \subset V$ and $\chi \in \mathcal{D}(G)$, $\mathrm{supp}(\chi) \subset \mathrm{supp}(\omega)$. The result is still valid with the factors flipped in the convolution above.

We adapt the proof of [10, Theorem 3.1] to groupoids. Firstly, [10, Lemma 2.5 and Remark 2.6] gives rise to:

Lemma 6. *Let $X \in \Gamma(AG)$ and φ, ψ be the associated actions of $\mathbb{R}$ on G and M. Let $\varepsilon > 0$. For any test function $\omega \in \mathcal{D}(G)$, there exists $a_1, b_1 \in C_c^\infty(]-\varepsilon,\varepsilon[) \subset \mathcal{E}'_{r,s}(\mathbb{R} \ltimes_\psi M)$ and $\omega_1 \in \mathcal{D}(G)$ with $\mathrm{supp}(\omega_1) \subset \mathrm{supp}(\omega)$ such that*

$$\omega = \varphi_! a_1 \star \omega_1 + \varphi_! b_1 \star \omega. \tag{40}$$

Proof of the Lemma. First of all, we pick up a sequence (p_j) of semi-norms characterizing the topology of $\mathcal{D}(\mathrm{supp}(\omega))$, and set $\beta_k = k^{-2}\inf\{(p_j(D_X^{2i}\omega)+1)^{-1}\ ;\ i,j \le k\}$. Then the series $\sum(-1)^k\alpha_k D_X^{2k}\omega$ converges in $\mathcal{D}(G)$ for any sequence $0 \le \alpha_k \le \beta_k$. Next, we choose by [10, Lemma 2.5 and Remark 2.6], two functions $a_1, b_1 \in C_c^\infty(]-\varepsilon,\varepsilon[)$ and a sequence $0 \le \alpha_k \le \beta_k$ such that

$$\delta = a_1 \star \sum_{k=0}^{\infty}(-1)^k\alpha_k\delta^{(2k)} + b_1 = \sum_{k=0}^{\infty}(-1)^k\alpha_k a_1^{(2k)} + b_1 \text{ in } \mathcal{E}'(\mathbb{R}) \subset \mathcal{E}'_{r,s}(\mathbb{R} \ltimes_\psi M) \tag{41}$$

Now (1) of Proposition 4 gives the identity (40), with $\omega_1 = \sum_{k=0}^{\infty}(-1)^k\alpha_k D_X^{2k}\omega$. □

Proof of the theorem. Let $X_1, \dots, X_\ell$ be a family generating the $C^\infty(M)$-module $\Gamma(AG)$, and

$$\varphi_i : G_i := \mathbb{R} \ltimes_{\psi_i} M \to G$$

be the associated homomorphisms. Applying the lemma to ω with $\varphi = \varphi_1$, we get

$$\omega = \lambda_1 \star \omega_1 + \mu_1 \star \omega \text{ in } \mathcal{E}'_{r,s}(G), \tag{42}$$

with λ_1, μ_1 in $\varphi_{1!}(C_c^\infty(]-\varepsilon,\varepsilon[))$. Applying the lemma to ω and ω_1 with φ_2 we get, with intuitive notation:

$$\omega_1 = \lambda_{2,1} \star \omega_{2,1} + \mu_{2,1} \star \omega_1 \quad ; \quad \omega = \lambda_2 \star \omega_2 + \mu_2 \star \omega \text{ in } \mathcal{E}'_{r,s}(G). \tag{43}$$

Inserting (43) into (42), we get:

$$\omega = \lambda_1 \star \lambda_{2,1} \star \omega_{2,1} + \lambda_1 \star \mu_{2,1} \star \omega_1 + \mu_1 \star \mu_2 \star \omega_2 + \mu_1 \star \mu_2 \star \omega \text{ in } \mathcal{E}'_{r,s}(G), \tag{44}$$

where all the $\lambda_{j\bullet}, \mu_{j\bullet}$ are in the range of $C_c^\infty(]-\varepsilon,\varepsilon[)$ by $\varphi_{j!}$, $j = 1, 2$, and all $\omega_\bullet$ are test functions with support in $\mathrm{supp}(\omega)$.

Repeating the argument with $\varphi_{3!}, \dots, \varphi_{\ell!}$ we get that ω is equal to a sum of 2^ℓ distributions of the form:

$$\xi_1 \star \xi_2 \star \cdots \star \xi_\ell \star \chi \tag{45}$$

where $\xi_j = \varphi_{j!}(k_j) \in \mathcal{D}'(G)$ for some $k_j \in C_c^\infty(]-\varepsilon,\varepsilon[)$ and $\chi \in \mathcal{D}(G)$ with $\mathrm{supp}(\chi) \subset \mathrm{supp}(\omega)$. Setting as in Proposition (4):

$$\varphi = \varphi_{1\cdots\ell} : \mathbb{R}^\ell \times M \simeq G_1 \underset{M}{\times} \cdots \underset{M}{\times} G_\ell \longrightarrow G \tag{46}$$

and after an obvious induction, we get

$$\xi_1 \star \xi_2 \star \cdots \star \xi_\ell = \varphi_! k \quad \text{with} \quad k = k_1 \otimes \cdots \otimes k_\ell \in C_c^\infty(]-\varepsilon,\varepsilon[^\ell). \tag{47}$$

Since $\partial_{t_j}\varphi(t,x)|_{t=0} = X_j(x)$ and $\varphi(0,x) = x$ for any $1 \leq j \leq \ell$ and $x \in M$, we get that φ is a submersion on $]-\varepsilon,\varepsilon[^\ell \times M$ if $\varepsilon > 0$ is small enough. Since the push forward of a C^∞ distribution by a submersion is C^∞, we get that $\varphi_! k$:

$$\forall \eta \in \mathcal{D}(G), \quad \langle \varphi_! k, \omega \rangle = \int_{\mathbb{R}^\ell \times M} k(t)\eta(\varphi(t,x))dtdx, \tag{48}$$

is C^∞ and supported in $\varphi(]-\varepsilon,\varepsilon[^\ell \times M)$. Taking $\varepsilon > 0$ small enough ensures that this last set is contained in V. □

5. Embedding $C_r^*(G)$ into $\mathcal{D}'(G)$ and regularizing operators

From now on and in the remaining parts of this article, we fix a compactly supported, first order elliptic pseudodifferential G-operator $P \in \Psi^1_{G,c}$ and we denote by $C_r^*(G)$ the reduced C^*-algebra of G.

Theorem 7. *There is a continuous embedding:*

$$C_r^*(G) \hookrightarrow \mathcal{D}'(G) \tag{49}$$

that extends the pairing:

$$\forall u \in C_c^\infty(G),\ \forall \omega \in \mathcal{D}(G), \quad \langle u\ ,\ \omega \rangle = \int_M \iota^* u \star \omega \tag{50}$$

Proof. Let $v, \xi \in C_c^\infty(G)$ and $\chi \in \mathcal{D}(G)$. We have:

$$\begin{aligned} \langle v, \xi \star \chi \rangle &= \langle \iota^*\xi \star v, \chi \rangle \\ &= \langle \iota^* v \star \xi, \iota^*\chi \rangle \text{ (trace property)} \\ &= \int_G \iota^* v \star \xi(\alpha)\iota^*\chi(\alpha). \end{aligned} \tag{51}$$

Let μ and μ_0 be positive sections of, respectively, the degree 1 densities bundles of AG and TM. We define $\mu_r \in C^\infty(G, \Omega_r^1)$ and $\mu_{s,0} \in C^\infty(G, s^*\Omega_M^1)$ by

$$\mu_r(\gamma) = \mu(s(\gamma)) \quad \text{and} \quad \mu_{s,0}(\gamma) = \mu_0(s(\gamma)). \tag{52}$$

We observe:

$$(\iota^* v \star \xi).\mu_r^{-1/2} = \iota^* v \star \xi' \in C_c^\infty(G, \Omega_s^{1/2}) \text{ with } \xi' = \xi\mu_r^{-1/2} \in C_c^\infty(G, \Omega_s^{1/2})$$

and

$$\chi' = \chi.\mu_r^{1/2}.\mu_{s,0}^{-1} \in C_c^\infty(G, \Omega_s^{1/2}).$$

This allows us to write

$$\langle v, \xi \star \chi \rangle = \int_G \iota^* v \star \xi(\alpha)\iota^*\chi(\alpha) = \int_M \Big(\int_{G_x} \iota^* v \star \xi'(\alpha)\iota^*\chi'(\alpha) \Big) d\mu_0 \tag{53}$$

and to use the Cauchy Schwarz inequalities for the Hilbert spaces $(L^2(G_x,\Omega^{1/2}_{G_x}),\|\cdot\|_x)$ in the following computations:

$$\begin{aligned}|\langle v,\xi\star\chi\rangle| &\leq \int_M d\mu_0.\sup_{x\in M}|\int_{G_x}\iota^*v\star\xi'(\alpha)\iota^*\chi'(\alpha)|\\ &\leq c_M\sup_{x\in M}\|\iota^*v\star\xi'\|_x\|\iota^*\chi'\|_x\\ &\leq c_M\|\iota^*v\|_{C^*_r(G)}\|\xi'\|_{L^2_s(G)}\|\iota^*\chi'\|_{L^2_s(G)}\\ &\leq c\|v\|_{C^*_r(G)} \qquad (54)\end{aligned}$$

Now let $\omega\in\mathcal{D}(G)$ and pick up a weak factorisation $\omega=\sum_j\xi_j\star\chi_j$. Let $u\in C^*_r(G)$ and choose a sequence (u_k) with $u_k\in C^\infty_c(G)$ and $u_k\to u$ in $C^*_r(G)$. Using the previous estimates, we see that the sequence $\langle u_k,\omega\rangle\in\mathbb{C}$ satisfies the Cauchy criterium and thus converges. Setting $u(\omega)=\lim_{k\to+\infty}\langle u_k,\omega\rangle$ with get that $u\in\mathcal{D}'(G)$ and that $u_k\to u$ in $\mathcal{D}'(G)$. □

We now give some complements to the properties of the regularizing operators:

$$\Psi^{-\infty}_G:=\{R\in\mathcal{L}(C^*_r(G))\ ;\ R\in\mathcal{L}(H^s,H^t)\text{ for all }s,t\in\mathbb{N}\} \qquad (55)$$

$$=\{R\in\mathcal{L}(C^*_r(G))\ ;\ P_1RP_2\in\mathcal{L}(C^*_r(G))\text{ for all }P_j\in\Psi^{s_j}_{G,c},\ s_j\in\mathbb{N},\ j=1,2\}. \qquad (56)$$

introduced exaclty in this form in [32] and in an equivalent form in [16]. In both references, this ideal of the C^*-closure of $\Psi_{G,c}$ is proved to be stable under holomorphic functional calculus. Here H^s denotes the scale of intrinsic Sobolev $C^*_r(G)$-modules.

Proposition 8. *Operators in $\Psi^{-\infty}_G$ are exaclty convolution operators by elements of $\mathcal{H}^\infty$. In other words, as subsets of the multipliers algebra $\mathcal{M}(C^*_r(G))$, these sets coincide:*

$$\Psi^{-\infty}_G=\mathcal{H}^\infty\subset\mathcal{M}(C^*_r(G)).$$

Proof. We know that $\Psi^{-\infty}_G\subset\mathcal{K}(C^*_r(G))=C^*_r(G)$. Let $T\in\Psi^{-\infty}_G$. For any $k\in\mathbb{N}$, we have:

$$(1+P^2)^kT=S_k\in C^*_r(G)\text{ and }T(1+P^2)^k=S'_k\in C^*_r(G)$$

Then $T=(1+P^2)^{-k}S_k=S'_k(1+P^2)^{-k}\in H^{2k}\cap(H^{2k})^*$ for any k, which proves the first inclusion. The second one is obvious. □

All the previous statements hold true for the maximal C^*-algebra of G but we stay in the framework of the reduced C^*-algebra, because the embedding $C^*_r(G)\hookrightarrow\mathcal{D}'(G)$ and the regular representation allow us to precise in what extent elements of $\Psi^{-\infty}_G=\mathcal{H}^\infty$ are *regularizing*. For that purpose, we let Ψ^*_G act not on the scale of intrinsic Sobolev modules H^s, but on their representation via the left regular representation. These $C(M)$-modules are concretely given as follows, for $k\in\mathbb{Z}$:

$$H^k_s=\overline{C^\infty_c(G,\Omega^{1/2}_s)}^{\langle\cdot|\cdot\rangle_{k,s}},\quad\langle\omega\mid\eta\rangle_{k,s}=\langle(1+P^2)^k\star\omega\mid\eta\rangle_s\in C(M). \qquad (57)$$

Lemma 9. *We have:*

$$H^\infty_s:=\bigcap_{k\in\mathbb{Z}}H^k_s\subset C^{\infty,0}_s(G,\Omega^{1/2}_s)\ \textit{and}\ H^{-\infty}_s:=\bigcup_{k\in\mathbb{Z}}H^k_s\supset\mathcal{E}'_{s,0}(G,\Omega^{1/2}_s). \qquad (58)$$

Proof of the lemma. Let $\omega\in H^\infty_s$. Since pointwise multiplication operators by compactly supported C^∞ functions are in $\mathscr{L}(H^k_s)$ for any k, we can assume that ω is compactly supported in the domain U of a local trivialization $\kappa:U\to\mathbb{R}^{n^{(1)}}\times\mathbb{R}^{n^{(0)}}$, $\kappa(x)=(x',x'')$ of the submersion s. By assumption, we have

$$\forall k\geq 0,\quad\Delta^k_G\omega\in C^0_c(\mathbb{R}^{n^{(0)}},L^2(\mathbb{R}^{n^{(1)}})). \qquad (59)$$

Here $\Delta_G = d^*d \in \mathrm{Diff}^2(G)$ is the Laplacian associated with a given euclidean structure on AG. The ellipticity of each term of the C^∞ family $(\Delta_{G,x''})_{x''\in\mathbb{R}^{n^{(0)}}}$ and the compactness of $\mathrm{supp}(\omega)$ imply using usual Garding inequality that $\omega \in C^0(\mathbb{R}^{n^{(0)}}, H^{2k}(\mathbb{R}^{n^{(1)}}))$ for any k, where H^* denotes here the usual Sobolev spaces of euclidean spaces. We then conclude that $\omega \in C^0(\mathbb{R}^{n_0}, C^\infty(\mathbb{R}^{n^{(1)}})) = C^{\infty,0}_{\mathrm{pr}_2}(\mathbb{R}^{n^{(1)}} \times \mathbb{R}^{n^{(0)}})$. This proves $H^\infty_s \subset C^{\infty,0}_s(G, \Omega_s^{1/2})$.
Let $u \in \mathcal{E}'_{s,0}(G, \Omega_s^{1/2})$. The result [12, Theorem 4.4.7] extends immediately to continuous family of distributions so there exists $k \in \mathbb{N}$ and finite collections: $u_I \in C_c(G, \Omega_s^{1/2})$, $D_I \in \mathrm{Diff}^k_G$ such that

$$u = \sum_I D_I u_I. \tag{60}$$

Since $C_c(G, \Omega_s^{1/2}) \subset L^2_s(G)$ and $(1+P^2)^{-k/2} D_I \in \mathcal{L}(L^2_s(G))$, we then conclude that $u \in H^{-k}_s$. □

We recall [24, 19] that for any $x \in M$, the orbit $\mathcal{O} = r(s^{-1}(\{x\})) \subset M$ is an immersed submanifold, the map $r : G_x \longrightarrow \mathcal{O}$ is a submersion (actually a G^x_x principal bundle) and that $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$ is an immersed subgroupoid. We set:

$$C^{\infty,0}_{\mathrm{orb}}(G,E) = \{u \in C(G,E)\ ;\ \forall x \in M, \forall D \in \mathrm{Diff}(G), Du \in C^\infty(G_{\mathcal{O}}, E),\ \mathcal{O} = r(s^{-1}(\{x\})\}. \tag{61}$$

Theorem 10. *The following inclusions hold true:*

$$C^{\infty,0}_{\mathrm{orb}}(G) \cap C_c(G) \subset \mathcal{H}^\infty \subset C^{\infty,0}_{\mathrm{orb}}(G) \cap C^*_r(G). \tag{62}$$

In particular, since $\mathcal{H}^\infty$ is an ideal in $C^*_r(G)$:

Corollary 11. *Any $h \in \mathcal{H}^\infty$ provides continuous operators :*

$$h \star \cdot : C^*_r(G) \longrightarrow C^{\infty,0}_{\mathrm{orb}}(G) \cap C^*_r(G) \ \textit{and}\ \cdot \star h : C^*_r(G) \longrightarrow C^{\infty,0}_{\mathrm{orb}}(G) \cap C^*_r(G).$$

Proof of the theorem. Let $u \in \mathcal{H}^\infty$. By [32] and the left regular representation, u maps $H^{-k}_s \to H^k_s$ continuously for any $k \in \mathbb{N}$. Therefore, the previous lemma implies that u maps $\mathcal{E}'_{s,0}(G, \Omega_s^{1/2}) \to C^{\infty,0}_s(G, \Omega_s^{1/2})$ continuously. In particular for every $x \in M$, the distribution $\kappa_x(\gamma_1, \gamma_2) = u(\gamma_1\gamma_2^{-1}) \in \mathcal{D}'(G_x \times G_x, \Omega_x^{1/2})$ extends to a continuous map:

$$\kappa_x : \mathcal{E}'(G_x, \Omega^{1/2}_{G_x}) \to C^\infty(G_x, \Omega^{1/2}_{G_x}) \tag{63}$$

which implies that $\kappa_x \in C^\infty(G_x \times G_x, \Omega_x^{1/2})$ for fixed x. Next, consider $x \in M$, $\mathcal{O} = r(s^{-1}(\{x\}))$ the orbit of x in M and fix $(\gamma_1, \gamma_2) \in G_x \times G_x \subset G \underset{s}{\times} G$. We denote by $\pi : G \underset{s}{\times} G \longrightarrow M$ the obvious submersion. Since $r : G_x \longrightarrow \mathcal{O}$ is a submersion, there exists a C^∞ local section $\eta : U \in y \mapsto \eta_y \in G_x$ of r such that $\eta_x = x$, defined on some open neighborhood U of x into O. Then $V = \pi_{\mathcal{O}}^{-1}(U) = \{(\eta_1, \eta_2)\ ;\ s(\eta_1) = s(\eta_2) \in U\}$ is an open neighborhood of (γ_1, γ_2) into $G_{\mathcal{O}} \underset{s}{\times} G_{\mathcal{O}}$ and we have:

$$\forall (\eta_1, \eta_2) \in V, \quad \kappa_y(\eta_1, \eta_2) = \kappa_x(R_{\eta_y}\eta_1, R_{\eta_y}\eta_2) \tag{64}$$

which proves that κ is C^∞ on $G_{\mathcal{O}} \underset{s}{\times} G_{\mathcal{O}}$, and thus that u is C^∞ on $G_{\mathcal{O}}$. It is clear that $C^{\infty,0}_{\mathrm{orb}}(G) \cap C_c(G)$ is contained in $C^*_r(G)$ and is invariant under the left and right convolution by P. The inclusion $C^{\infty,0}_{\mathrm{orb}}(G) \cap C_c(G) \subset \mathcal{H}^\infty$ follows. □

Summarizing the content above, we have proved that regularizing operators are actually convolution operators by distributions on G lying in the class $\mathcal{H}^\infty$, the latter class being included in the class of functions that are continuous on G and infinitely differentiable over any orbit, and thus in particular along the fibers of s and r. Closely related results were obtained in [16] under the assumption of bounded geometry for G. In the following sections, we are going to prove that $E(t) = e^{itP}$ is a family $\mathbb{R} \times G$-FIO [18], modulo such regularizing operators.

6. Principal and Subprincipal symbols of G-PDOs

As a conormal distribution, any element of $\Psi_G^m = I^{m+(n^{(1)}-n^{(0)})/4}(G, M; \Omega^{1/2})$ has a principal symbol [12, Theorem 18.2.11] in:

$$S^{[m+(n^{(1)}-n^{(0)})/4+n/4]}(A^*G, \Omega^{1/2}_{A^*G} \otimes \hat{\Omega}^{1/2} \otimes \hat{\Omega}_G^{-1/2}). \tag{65}$$

The density bundle above is canonically trivial:

$$\Omega^{1/2}_{A^*G} \otimes \hat{\Omega}^{1/2} \otimes \hat{\Omega}_G^{-1/2} = \Omega^{1/2}(TM \oplus A^*G) \oplus \Omega^1(AG) \otimes \Omega^{-1/2}(TM \oplus AG) \simeq M \times \mathbb{C}, \tag{66}$$

and since half densities on A^*G contribute with a value of $n^{(1)}/2$ to the degree of symbols, the simplification above lowers the degree by the same value. In conclusion the principal symbol map is a well defined map:

$$\sigma_0 : \Psi_G^m \longrightarrow S^{[m]}(A^*G). \tag{67}$$

Alternatively, given $P \in \Psi_G^m$, one may consider the family $\widetilde{P} = (P_x)_{x\in M}$, $P_x \in \Psi^m(G_x, \Omega^{1/2}_{G_x})$ associated with P by the isomorphism (35) and collect the family of principal symbols $\sigma(P_x) \in S^{[m]}(T^*G_x)$ into the element $\sigma(P) \in S^{[m]}(T_s^*G)$, where $T_s^*G = (\ker ds)^*$, defined by:

$$\sigma(P)(\gamma, \xi) = \sigma(P_{s(\gamma)})(\gamma, \xi).$$

In this point of view, the principal symbol is a map :

$$\sigma : \Psi_G^m \longrightarrow S^{[m]}(T_s^*G). \tag{68}$$

Both notions are related by:

Proposition 12. *With the notation above, the following identity holds true:*

$$\sigma = \sigma_0 \circ r_\Gamma. \tag{69}$$

Remark 13. *Strictly speaking, the target map r_Γ is defined on T^*G. It is by construction the composition of the natural restriction map $T^*G \to T_s^*G$ with the natural map $T_s^*G \to A^*G$. It is understood in the Proposition above that r_Γ means the latter.*

Proof. Let $P \in \Psi_G^m$. Without loss of generality, we can assume that P is supported in a local chart U around some point of M and satisfying:

- the local coordinates trivialize the source map, that is $\gamma = (x'', x')$ with $s(\gamma) = x''$ on U,
- the domain U is invariant for the inversion map : $U^{-1} = U$.

We then pick up a positive one density μ on AG such that:

$$\forall x \in U \cap G^{(0)}, \quad \mu(x) = |dx'|,$$

and define $\mu_s \in C^\infty(G, \Omega_s^{1/2})$, $\mu_r \in C^\infty(G, \Omega_r^{1/2})$ by:

$$\mu_s(\gamma) = \mu(r(\gamma)) \text{ and } \mu_r(\gamma) = \mu(s(\gamma)).$$

We can set on U:

$$P(\gamma) = \mathsf{P}(\gamma).\mu_s^{1/2}\mu_r^{1/2} \tag{70}$$

where P is a scalar oscillatory integral conveniently given in the following form:

$$\mathsf{P}(\gamma) = \int e^{-i\gamma^{-1}.\xi'} p_0(r(\gamma), \xi') d\xi'. \tag{71}$$

Let us describe the various ingredients of this formula. First, $p_0 \in S^m(A^*G)$ is a (classical) symbol, and the integral (in the distribution sense) is performed with respect to $(0, \xi') \in A^*_{r(\gamma)}G \subset T^*_{r(\gamma)}G$. Secondly, it is understood that $\gamma^{-1} = \iota(\gamma)$ stands for the n-tuple of coordinates of the inverse of γ in G, and then $\gamma^{-1}.\xi'$ stands for its scalar product with $(0, \xi')$ in $\mathbb{R}^n$. We could use the inverse of an exponential map to give an invariant meaning to $\gamma^{-1}.\xi$ with $\xi \in A^*_{r(\gamma)}G$, but since we already work in local coordinates, this is pointless. Finally, we read from (71) that:

$$\sigma_0(P) = p_0 \mod S^{m-1}(A^*G), \tag{72}$$

and since the symbols used here are classical, we may identify $\sigma_0(P)$ with the leading homogeneous part p_0^0 of p_0. Now let $u \in C_c^\infty(G)$ with support in a local chart V of G, and set:

$$u(\gamma) = \mathsf{u}(\gamma).\mu_s^{1/2}\mu_r^{1/2} \tag{73}$$

with $\mathsf{u} \in C_c^\infty(V, \mathbb{C})$. To express Pu in local coordinates in terms of P and u, we need to recall the necessary identifications of densities allowing the convolution product:

$$P(u)(\gamma) = \int_{\alpha \in G_{s(\gamma)}} P(\gamma\alpha^{-1})u(\alpha). \tag{74}$$

For that purpose, note that for any γ, α with same source point:

$$\mu_s(\gamma\alpha^{-1}) = \mu(r(\gamma)) = \mu_s(\gamma),\ \mu_r(\gamma\alpha^{-1}) = \mu(r(\alpha)) = \mu_s(\alpha),\ \mu_r(\alpha) = \mu_r(\gamma).$$

Hence:

$$P(\gamma\alpha^{-1})u(\alpha) = \mathsf{P}(\gamma\alpha^{-1})\mathsf{u}(\alpha)\mu_s(\alpha)\mu_s^{1/2}(\gamma)\mu_r^{1/2}(\gamma).$$

It remains to express $\mu_s(\alpha)$ in term of a one density on $G_x \cap V$. We also assume that the coordinates fixed on V trivializes the source map s:

$$V \ni \alpha = (\alpha'', \alpha') \text{ with } s(\alpha) = \alpha''$$

In the coordinates fixed on U and V, we get using $(dR_\alpha)_{r(\alpha)} : (r^*AG)_\alpha \xrightarrow{\simeq} (\ker ds)_\alpha$:

$$\mu_s(\alpha) = |dx'| = |(dR_\alpha)_{r(\alpha)}|^{-1}|d\alpha'|.$$

It follows that, setting $\widetilde{P}(u) = \mathsf{v}\mu_s^{1/2}\mu_r^{1/2}$ on W:

$$\mathsf{v}(\gamma) = \int P(\gamma\alpha^{-1})\mathsf{u}(\alpha)|(dR_\alpha)_{r(\alpha)}|^{-1}d\alpha' \tag{75}$$

$$= \int e^{-i\alpha\gamma^{-1}.\xi'} p_0(r(\gamma), \xi')|(dR_\alpha)_{r(\alpha)}|^{-1}\mathsf{u}(\alpha)d\alpha' d\xi'. \tag{76}$$

Actually, the action of the induced family of operators $P_x \in \Psi^m(G_x, \Omega^{1/2}_{G_x})$ on half-densities $f \in C_c^\infty(G_x, \Omega^{1/2}_{G_x})$ is given by the same formula:

(77)

$$\text{if } f = \mathsf{f}\mu_s^{1/2},\ \text{then } P_x(f) = \mathsf{v}\mu_s^{1/2} \text{ with } \mathsf{v}(\gamma) = \int e^{-i\alpha\gamma^{-1}.\xi} p_0(r(\gamma), \xi)\mathsf{f}(\alpha)|(dR_\alpha)_{r(\alpha)}|^{-1}d\alpha' d\xi.$$

Let us set

$$\varphi(\alpha) = |(dR_\alpha)_{r(\alpha)}|^{-1}. \tag{78}$$

Since $\alpha\gamma^{-1}$ vanishes at $\alpha = \gamma \in G_{s(\gamma)}$, there exists a linear map

$$\psi(\alpha,\gamma) : \mathbb{R}^{n^{(1)}} \longrightarrow \mathbb{R}^{n^{(1)}} \tag{79}$$

which is C^∞ in (α,γ), bijective for α in a neighborhood of γ and satisfies:

$$\alpha\gamma^{-1} = \psi(\alpha,\gamma)(\alpha-\gamma). \tag{80}$$

By construction we have:

$$\psi(\gamma,\gamma) = (dR_{\gamma^{-1}})_\gamma = (dR_\gamma)^{-1}_{r(\gamma)}. \tag{81}$$

Now we work on (77) to find the amplitude and symbol of P_x in local coordinates on V:

$$\begin{aligned}
\mathsf{v}(\gamma) &= \int e^{-i\langle \alpha-\gamma, {}^t\psi(\alpha,\gamma)\xi'\rangle} p_0(r(\gamma),\xi')\mathsf{u}(\alpha)\varphi(\alpha)d\alpha' d\xi' \\
&= \int e^{i\langle \gamma-\alpha,\xi'\rangle}\big(p_0(r(\gamma), {}^t\psi(\alpha,\gamma)^{-1}\xi')\varphi(\alpha)|{}^t\psi(\alpha,\gamma)|^{-1}\ \mathsf{u}(\alpha)d\alpha' d\xi' \\
&= \int e^{i\langle \gamma-\alpha,\xi'\rangle}\widetilde{p}(\gamma,\alpha,\xi')\mathsf{u}(\alpha)d\alpha' d\xi' \\
&= \int e^{i\langle \gamma-\alpha,\xi'\rangle}p(\gamma,\xi')\mathsf{u}(\alpha)d\alpha' d\xi' \qquad (82)
\end{aligned}$$

where we have set

$$\widetilde{p}(\gamma,\alpha,\xi') = p_0(r(\gamma), {}^t\psi(\alpha,\gamma)^{-1}\xi')\varphi(\alpha)|{}^t\psi(\alpha,\gamma)|^{-1} \tag{83}$$

and

$$p(\gamma,\xi') = e^{i\langle D_{\alpha'}, D_{\xi'}\rangle}\widetilde{p}(\gamma,\alpha,\xi')|_{\alpha=\gamma} \tag{84}$$

which gives the asymptotic expansion:

$$p(\gamma,\xi') \sim \sum \frac{1}{k!}\langle iD_{\alpha'}, D_{\xi'}\rangle^k \widetilde{p}(\gamma,\alpha,\xi')\,|_{\alpha=\gamma}. \tag{85}$$

Since $(r(\gamma), {}^t\psi(\gamma,\gamma)^{-1}\xi') = (r(\gamma), {}^t(dR_\gamma)_{r(\gamma)}\xi') = r_\Gamma(\gamma,\xi')$, the expression of the principal symbol of P_x over V is the first term in the sum (85):

$$\sigma(P_x)(\gamma,\xi') = p_0(r_\Gamma(\gamma,\xi')) \mod S^{m-1}(T^*_sG), \tag{86}$$

or equivalently using homogeneous expansions: $\sigma(P) = p^0 = p^0_0 \circ r_\Gamma$. □

Remark 14. *We will often consider C^∞ functions on T^*_sG as C^∞ functions on T^*G, thanks to the convention $a(\gamma,\xi) = a(\gamma,\xi|_{T_{s(\gamma)}G})$.*

We now turn our attention to the sub-principal symbols. It is not obvious to us how to define the sub-principal symbol for general conormal distributions, but in the case of $\Psi^*_G = I(G,M,\Omega^{1/2})$, we may again consider the family of usual sub-principal symbols of the operators $P_x \in \Psi^m(G_x,\Omega^{1/2}_{G_x})$ and set:

$$(\gamma,\xi) \in T^*_sG, \quad \sigma^{1s}(P)(\gamma,\xi) := \sigma^{1s}(P_{s(\gamma)})(\gamma,\xi) \in S^{[m-1]}(T^*_sG). \tag{87}$$

This gives a well defined map:

$$\sigma^{1s} : \Psi^m_G \longrightarrow S^{[m-1]}(T^*_sG), \tag{88}$$

When P is given by (71), we recall that the sub-principal symbol above is given in terms of the homogeneous expansion of total symbol p, expressed in the last proof (see formula (83), (84) and (85)), by:

$$p^{1s}(\gamma,\xi') := \sigma^{1s}(P)(\gamma,\xi') = p^1(\gamma,\xi') - \frac{i}{2}\langle D_{\gamma'}, D_{\xi'}\rangle p^0(\gamma,\xi'), \tag{89}$$

We now consider $p^0 = \sigma(P)$ as a C^∞ homogeneous function on $T^*_\bullet G = T^*G \setminus \ker r_\Gamma$ (see Remark 14) and we denote $H_{p^0} \in \Gamma(TT^*_\bullet G)$ the hamiltonian vector field of p^0. We recall that the latter is defined by $dp^0(\cdot) = \omega_G(H_{p^0}, \cdot)$, and in local coordinates (γ,ξ) we get:

$$H_{p^0} = \sum_{j=1}^n \frac{\partial p^0}{\partial \xi_j}\frac{\partial}{\partial \gamma_j} - \frac{\partial p^0}{\partial \gamma_j}\frac{\partial}{\partial \xi_j}.$$

Now we shall compute the principal symbol of a product PA where P is a G-PDO and A a G-FIO in the situation later encountered in the construction of the parametrix of e^{itP}. To that purpose, we recall that the principal symbol of G-FIO is a homomorphism [12]:

$$I^m(G,\Lambda;\Omega^{1/2}) \longrightarrow S^{[m+n/4]}(\Lambda, M_\Lambda \otimes \Omega^{1/2}_\Lambda \otimes \hat{\Omega}^{1/2} \otimes \hat{\Omega}^{-1/2}_G) \tag{90}$$

where M_Λ is the Maslov bundle and $\hat{E}$ denotes the pull back of the vector bundle $E \to G$ over Λ. By [18], we know that there is canonical isomorphism:

$$\hat{\Omega}^{1/2} \otimes \hat{\Omega}^{-1/2}_G \simeq \Omega^{1/2}_{r_\Gamma} = s_\Gamma^*\Omega^{1/2}(AT^*G) \tag{91}$$

This isomorphism uses the product and inversion map of G but their contributions cancel and thus, elements in $\hat{\Omega}^{1/2} \otimes \hat{\Omega}^{-1/2}_G$ do define, without any other data, pull back of half densities on the vector bundle $AT^*G \longrightarrow A^*G$. We thus may consider the principal symbol of Fourier integral G-operators as a homomorphism:

$$I^m(G,\Lambda;\Omega^{1/2}) \longrightarrow S^{[m+n/4]}(\Lambda, M_\Lambda \otimes \Omega^{1/2}_\Lambda \otimes \Omega^{1/2}_{r_\Gamma}) \tag{92}$$

We recall that for a manifold X and a vector field V on X with flow ϕ_t, the Lie derivative of a α-density a is the α-density given by, in local coordinates $a = \mathsf{a}|dx|^\alpha$:

$$\mathcal{L}_V(\mathsf{a}|dx|^\alpha) = \frac{d}{dt}\phi_t^*\mathsf{a}|dx|^\alpha|_{t=0} = \big(V\cdot\mathsf{a} + \alpha\,\mathrm{div}(V)\mathsf{a}\big)|dx|^\alpha. \tag{93}$$

This is the same for sections $a \in C^\infty(\Lambda, M_\Lambda \otimes \Omega^{1/2}_\Lambda)$ and vector fields $V \in \Gamma(T\Lambda)$. Indeed, the transition functions of M_Λ are locally constant, so the bundle M_Λ can be factorized out of (93).

On the other hand, we are mainly interested in Hamiltonian vector fields $V = H_f$ that are also right invariant, which happens if and only if $f = f_0 \circ r_\Gamma$ [6], and such that $f|_\Lambda = 0$, which implies that V is tangent to Λ. For such vector fields, we can extend the Lie derivative above to a map $\mathcal{L}_f = \mathcal{L}_{H_f}$ acting on sections of the line bundle appearing in the symbols space in (92). To do that, consider $\nu_{r_\Gamma} = \nu \circ s_\Gamma \in C^\infty(T^*G, \Omega^{1/2}_{r_\Gamma})$ with ν a positive density on AT^*G. Since by assumption $s_{s_\Gamma} \circ \phi_t = s_\Gamma$, we get:

$$\mathcal{L}_f(\nu_{r_\Gamma}) := \frac{d}{dt}\phi_t^*(\nu_{r_\Gamma})|_{t=0} = \frac{d}{dt}\nu(s_\Gamma\circ\phi_t)|_{t=0} = \frac{d}{dt}\nu^{1/2}(s_\Gamma)|_{t=0} = 0.$$

Combining the usual action of $V|_\Lambda$ recalled in (93) with the above trivial one, we obtain that $V = H_f$ acts on $C^\infty(\Lambda, M_\Lambda \otimes \Omega^{1/2}_\Lambda \otimes \Omega^{1/2}_{r_\Gamma})$ by the formula:

$$\mathcal{L}_f(\mathsf{a}\mu^{1/2}_\Lambda \nu^{1/2}_{r_\Gamma}) = \big(H_f\cdot\mathsf{a} + \frac{1}{2}\,\mathrm{div}(H_f)\mathsf{a}\big)\mu^{1/2}_\Lambda\nu^{1/2}_{r_\Gamma}. \tag{94}$$

In the important particular case where the G-relation Λ is a bissection, that is, when s_Γ, r_Γ are diffeomorphisms from Λ to open subsets of A^*G, then

$$T\Lambda \oplus \ker ds_\Gamma|_\Lambda = T_\Lambda T^*G. \tag{95}$$

Since $v = H_{f_0 \circ r_\Gamma}$ is tangent to both Λ and to the s_Γ-fibers, we conclude that v vanishes on Λ, which implies that $\mathcal{L}_f = 0$ in (94).

Theorem 15. *Let Λ be a G-relation and $Q \in \Psi^m_{G,c}$ with principal and sub-principal G-symbols q^0 and q^{1s}. Assume that q^0 vanishes on Λ. Let $A \in I^{m'}(G, \Lambda; \Omega^{1/2})$ and let $a \in S^{m'+n/4}(\Lambda, M_\Lambda \otimes \Omega^{1/2}_\Lambda \otimes \hat{\Omega}^{1/2} \otimes \hat{\Omega}^{-1/2}_G)$ be a principal symbol of A.*
Then

$$QA \in I^{m+m'-1}(G, \Lambda; \Omega^{1/2}) \tag{96}$$

and PA has a principal symbol represented by

$$-i\mathcal{L}_{q^0} a + q^{1s} a. \tag{97}$$

We could consider the distribution QA as the family of operators $Q_x \circ A_x$ and apply [12, Theorem 25.2.4]. However, we are going to consider QA as a single lagrangian distribution on G given by the convolution in G of two distributions, and then make the minor necessary adaptations of the proof of [12, Theorem 25.2.4]. This yields more conceptual and self-contained explanations for the assertion to be proved.

Proof. We keep the assumptions and notation introduced for Q in the proof of Proposition 12. Using a partition of unity and [18], we can assume that A is supported in the domain V of local coordinates trivialising s such that there exists a conic open set C in $\mathbb{R}^n$ and a homogeneous C^∞ function h such that:

$$\Lambda \cap T^*V = \{(h'(\xi), \xi)\ ;\ \xi \in C\}. \tag{98}$$

The existence of such coordinates follows from [12, Lemma 25.2.5 and Theorem 21.2.16]. We can write in these local coordinates above:

$$A = \mathsf{A}.\mu_s^{1/2}\mu_r^{1/2} \text{ with } \mathsf{A}(\gamma) = \int e^{i(<\gamma,\eta> - h(\eta))}\mathsf{a}(\eta)d\eta, \tag{99}$$

where $\mathsf{a} \in S^{m'-n/4}(\mathbb{R}^n)$ has support in a conic neighborhood of C. Then on V:

$$QA = \mathsf{B}\mu_s^{1/2}\mu_r^{1/2} \text{ with } \mathsf{B}(\gamma) = \int e^{i(-<\alpha\gamma^{-1},\xi> + <\alpha,\eta> - h(\eta))} q_0(r(\gamma), \xi)\mathsf{a}(\eta)\varphi(\alpha)d\alpha d\xi d\eta. \tag{100}$$

Remember that, according to the decomposition $\eta = (\eta'', \eta')$ provided by the local trivialisation of s, the symbol $q(\gamma, \eta')$ of $Q_{s(\gamma)}$ (see (82) and above) is given by

$$q(\gamma, \eta') = e^{-i\langle\gamma,\eta'\rangle} Q_{s(\gamma)}(e^{i\langle\alpha,\eta'\rangle}) = \int e^{i(-<\alpha\gamma^{-1},\xi> + <\alpha-\gamma,\eta'>)} q_0(r(\gamma), \xi)\varphi(\alpha)d\alpha d\xi. \tag{101}$$

Since $\gamma, \alpha \in G_{s(\gamma)}$, the same identity is licit for q considered as a function of (γ, η), but does not define anymore a symbol in general (it satisfies symbolic estimates of order m in η' but is independent of η''). However, the assumption on the wave front set of A implies that the symbol a is or order $-\infty$ in some open cone around $(\eta'', 0)$. Indeed, $(\eta'', 0) \in \ker(s_\Gamma)$ and by assumption Λ is a G relation, hence $\mathrm{WF}\, A \cap \ker(s_\Gamma) = \emptyset$. Therefore, the product $\mathsf{b}(\gamma, \eta) = q(\gamma, \eta)\mathsf{a}(\eta)$ is a symbol of order $m + m' - n/4$. We get from (100) and (101):

$$\mathsf{B}(\gamma) = \int e^{i(<\gamma,\eta> - h(\eta))} q(\gamma, \eta)\mathsf{a}(\eta)d\eta, \tag{102}$$

It is a lagrangian distribution of order $m+m'$ a priori, but since the leading part of b, which is represented $\mathsf{b}^0(\gamma,\eta)=q^0(\gamma,\eta)\mathsf{a}(\eta)$, vanishes on Λ, it is actually of order $m+m'-1$ and we need to work out more the expression (102) to get its principal symbol. To this end, we set:

$$q(\gamma,\xi)=q^0(\gamma,\xi)+e(\gamma,\xi), \tag{103}$$

and using the assumption: $q^0(\gamma,\xi)=0$ whenever $\gamma=h'(\xi)$; we make the factorisation:

$$q^0(\gamma,\xi)=\sum_j q_j(\gamma,\xi)(\gamma_j-\frac{\partial h}{\partial\xi_j}). \tag{104}$$

Now, after an integration by parts in (102), we get:

$$\mathsf{B}(\gamma)=\int e^{i(<\gamma,\xi>-h(\xi))}(e\mathsf{a}-\sum_j D_{\xi_j}(q_j\mathsf{a}))d\xi. \tag{105}$$

It follows that QA has principal symbol represented by

$$(e\mathsf{a}-\sum_j D_{\xi_j}(q_j\mathsf{a}))(h'(\xi),\xi)|d\xi|^{1/2}(\mu_r\mu_s|d\gamma|^{-1})^{1/2} \tag{106}$$

while A has principal symbol represented by

$$\mathsf{a}(\xi)|d\xi|^{1/2}(\mu_r\mu_s|d\gamma|^{-1})^{1/2}. \tag{107}$$

Since H_{q^0} is tangent to Λ we have on Λ parametrized by ξ:

$$H_{q^0}(\xi)=-\sum_j\frac{\partial q^0}{\partial\gamma_j}(h'(\xi),\xi)\frac{\partial}{\partial\xi_j}=-\sum_j q_j(h'(\xi),\xi)\frac{\partial}{\partial\xi_j}. \tag{108}$$

Then, as a vector field on Λ, the divergence of $H_{q^0}\in\Gamma(T\Lambda)$ is given by:

$$\mathrm{div}(H_{q^0})=-\sum_j\frac{\partial}{\partial\xi_j}\Big[q_j(h'(\xi),\xi)\Big]=-\sum_{j,k}\frac{\partial^2 h}{\partial\xi_k\partial\xi_j}\frac{\partial q_j}{\partial\gamma_k}-\sum_j\frac{\partial q_j}{\partial\xi_j}. \tag{109}$$

On the other hand we have:

$$\begin{aligned}\frac{\partial^2 q^0}{\partial\xi_k\partial\gamma_k}(\gamma,\xi)&=\frac{\partial}{\partial\xi_k}\sum_j(\frac{\partial}{\partial\gamma_k}q_j).(\gamma_j-h'_j(\xi))+q_k\\&=\sum_j\frac{\partial^2 q_j}{\partial\gamma_k\partial\xi_k}.(\gamma_j-h'_j(\xi))-\sum_j\frac{\partial q_j}{\partial\gamma_k}\frac{\partial^2 h}{\partial\xi_k\partial\xi_j}+\frac{\partial q_k}{\partial\xi_k},\end{aligned}$$

which gives after evaluation at $\gamma=h'(\xi)$:

$$\sum_k\frac{\partial^2 q^0}{\partial\xi_k\partial\gamma_k}(h'(\xi),\xi)=\sum_k\frac{\partial q_k}{\partial\xi_k}-\sum_{j,k}\frac{\partial q_j}{\partial\gamma_k}\frac{\partial^2 h}{\partial\xi_k\partial\xi_j}=2\sum_k\frac{\partial q_k}{\partial\xi_k}+\mathrm{div}(H_{q^0}). \tag{110}$$

Setting

$$\mathsf{L}_{q^0}(a):=\mathcal{L}_{q^0}(a)(\mu_r\mu_s/|d\gamma|)^{-1/2}=-\sum_j q_j\frac{\partial\mathsf{a}}{\partial\xi_j}+\frac{1}{2}\mathrm{div}(H_{q^0})\mathsf{a},$$

we get, still for $\gamma = h'(\xi)$:

$$\begin{aligned}(e\mathsf{a} - \sum_j D_{\xi_j}(q_j\mathsf{a})) &= \Big(e + i\sum_j \frac{\partial q_j}{\partial \xi_j}\Big)\mathsf{a} + i\sum_j q_j \frac{\partial \mathsf{a}}{\partial \xi_j} \\ &= \Big(e + i\sum_j \frac{\partial q_j}{\partial \xi_j} + \frac{i}{2}\mathrm{div}(H_{q^0})\Big)\mathsf{a} - i\mathsf{L}_{q^0}(a) \\ &= \Big(e + \frac{i}{2}\sum_k \frac{\partial^2 q^0}{\partial \xi_k \partial \gamma_k}\Big)\mathsf{a} - i\mathsf{L}_{q^0}(a) \\ &= -i\mathsf{L}_{q^0}(a) + \left(e + \frac{i}{2}\sum_k \frac{\partial^2 q^0}{\partial \xi'_k \partial \gamma'_k}\right)\mathsf{a}. \qquad (111)\end{aligned}$$

In the last line, we have decomposed $\xi = (\xi', \xi'')$ where ξ' is cotangent to the s-fibers and used the fact that q^0 does not depend on the ξ'' variables. This proves, looking at formula (89), that $-i\mathcal{L}_{q^0}(a) + q^{1s}.a$ is a principal symbol of QA. □

7. The Hamiltonian flow of the principal symbol and the associated G-relations

We set:

$$\Lambda_0 = A^*G \setminus 0, \quad T^*_{\bullet}G = T^*G \setminus \ker r_\Gamma \quad \text{and} \quad \dot{T}^*G = T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma).$$

The operator P being elliptic, we have $\mathrm{WF}\, P = \Lambda_0$. Recall that $p_0^0 \in C^\infty(\Lambda_0)$ denotes the homogeneous representative of $\sigma_0(P)$ and that $p^0 = p_0^0 \circ r_\Gamma$ is then the homogeneous representative of $\sigma(P) \in C^\infty(T^*_sG \setminus 0)$.

For every $a \in \mathbb{R}$ and $(\gamma, \xi) \in T^*G$, we let $a.(\gamma, \xi) = \rho_a(\gamma, \xi) = (\gamma, a\xi)$.

Proposition 16. *The flow χ of H_{p^0} satisfies the following properties:*

(1) It is complete.

(2) It commutes with dilations in T^*G:

$$\forall a \in \mathbb{R}_+^*,\ t \in \mathbb{R},\ (\gamma, \xi) \in T^*_{\bullet}G, \quad a.\chi(t, \gamma, \xi) = \chi(t, \gamma, a\xi).$$

(3) It provides at each time t a section of s_Γ and commutes with right multiplication:

$$\forall t \in \mathbb{R},\ (\delta_1, \delta_2) \in \Gamma^{(2)}, \quad s_\Gamma(\chi(t, \delta_1)) = s_\Gamma(\delta_1) \text{ and } \chi(t, \delta_1\delta_2) = \chi(t, \delta_1)\delta_2.$$

In particular, the integral curves of H_{p^0} go along the fibers of s_Γ.

Proof. This is essentially contained in [6]. More precisely:

(2) The homogeneity of p^0 implies: $(\rho_a)_*(H_{p^0}) = H_{p^0}$ and therefore $\rho_a \circ \chi_t \circ \rho_a^{-1} = \chi_t$, which gives the result.

(3) By definition, we have $\omega(H_{p^0}, X) = dp^0(X) = dp_0^0(dr_\Gamma(X))$ which yields $H_{p^0} \in (d\ker r_\Gamma)^\omega = d\ker s_\Gamma$. Using the last part of the proof of [6, Lemma, p.22], we get that H_{p^0} is a right invariant vector field, which proves that the flow goes along the s-fibers and is right invariant.

(1) Now, by compacity of $M = G^{(0)}$ and the homogeneity of H_{p^0} in the fibers of Λ_0, we get the existence of $\epsilon > 0$ such that $\chi :]-\epsilon, \epsilon[\times\Lambda_0 \to T^*_{\bullet}G$ is well defined. By right invariance, we extend χ_p onto $]-\epsilon, \epsilon[\times(T^*_{\bullet}G)$ and using the group property of flows we can choose for any t an integer N such that $|t|/N < \epsilon$ and set

$$\chi(t, \alpha) = \chi_{t/N} \circ \ldots \circ \chi_{t/N}(\alpha)$$

which proves completeness of the flow.

□

Remark 17.

(1) The compacity of $M = G^{(0)}$ is only needed here for the completeness of H_{p^0}.
(2) By construction, χ_t is a diffeomorphism of $T^*_\bullet G$ and since $s_\Gamma \circ \chi_t = s_\Gamma$, we have:

$$\chi_t(\dot{T}^*G) = \dot{T}^*G. \tag{112}$$

We now set:

$$\forall t \in \mathbb{R}, \quad \Lambda_t = \chi_t(\Lambda_0) \subset \dot{T}^*G. \tag{113}$$

Proposition 18. *For any real number t, the set Λ_t is an invertible G-relation and:*

$$\forall t_1, t_2 \in \mathbb{R}, \quad \Lambda_{t_1}.\Lambda_{t_2} = \Lambda_{t_1+t_2}. \tag{114}$$

Proof. Since χ is the Hamilton flow of a homogeneous function, each χ_t is a homogeneous symplectomorphism. Since Λ_0 is a homogeneous Lagrangian, its image Λ_t by χ_t is then a Lagrangian homogeneous submanifold, contained by construction in $\dot{T}^*G$. Thus, Λ_t is a G-relation. Since $s_\Gamma \circ \chi_t|_{\Lambda_0} = \mathrm{Id}_{\Lambda_0}$, we get that $s_\Gamma|_{\Lambda_t}$ is a diffeomorphism. The same conclusion is true for $r_\Gamma|_{\Lambda_t}$ because the vector field H_{p^0} is right invariant and therefore $r_\Gamma \circ \chi|_{\mathbb{R}\times\Lambda_0}$ is the (complete) flow of the vector field $(r_\Gamma)_*(H_{p^0}) \in C^\infty(\Lambda_0, T\Lambda_0)$ defined by: $(r_\Gamma)_*(H_{p^0})(\delta) = (dr_\Gamma)_\delta(H_{p^0}(\delta)), \delta \in \Lambda_0$. That $s_\Gamma|_{\Lambda_t}$ and $r_\Gamma|_{\Lambda_t}$ are diffeomorphisms mean precisely that Λ_t is an invertible G-relation [18] (or a lagrangian bissection, if one accepts as bissections submanifolds of Γ onto which r_Γ and s_Γ are diffeomorphisms onto open subsets of $A^*G = (T^*G)^{(0)}$).

Let us proceed to the proof of the one parameter group relation. Let $\delta_j = \chi_{t_j}(u_j) \in \Lambda_{t_j}$, $j = 1, 2$, be two composable elements. Then $u_1 = s_\Gamma(\delta_1) = r_\Gamma(\delta_2)$ and by commutation of χ with right multiplication:

$$\delta_1.\delta_2 = \chi_{t_1}(r_\Gamma(\delta_2))\delta_2 = \chi_{t_1}(\delta_2) = \chi_{t_1}(\chi_{t_2}(u_2)) = \chi_{t_1+t_2}(u_2) \in \Lambda_{t_1+t_2}.$$

The converse inclusion is then obvious. □

8. Global aspects of the family $(\Lambda_t)_t$

In the vocabulary of [18], the family $(\Lambda_t)_{t\in\mathbb{R}}$ admits a gluing into a single Lagrangian submanifold $\Lambda \subset T^*(\mathbb{R} \times G)$. The expression of Λ is actually straightforward and we shall study it in relation with the both groupoid structures on $\mathbb{R} \times G$.

Proposition 19. *Let $i_t : G \to \mathbb{R} \times G$ be the inclusion $\gamma \mapsto (t, \gamma)$. The set*

$$\Lambda = \left\{(t, -p^0(\chi_t(x,\xi)), \chi_t(x,\xi)) \in T^*(\mathbb{R} \times G) \ ; \ t \in \mathbb{R}, (x,\xi) \in A^*G \setminus 0\right\} \tag{115}$$

is a conic Lagrangian submanifold of $T^*(\mathbb{R} \times G)$ satisfying:

$$\forall t \in \mathbb{R}, \quad i_t^*\Lambda = \Lambda_t.$$

Proof. the map $F(t, \delta) = (t, \chi_t(\delta))$ being a diffeomorphism of $\mathbb{R} \times (T^*_\bullet G)$, the set $F(\mathbb{R} \times \Lambda_0)$ is a submanifold of $\mathbb{R} \times T^*G$ and therefore Λ, as a graph, is a submanifold of $T^*(\mathbb{R} \times G)$, obviously homogeneous and $\phi(t, x, \xi) = (t, -p^0(\chi_t(x,\xi)), \chi_t(x,\xi))$ is a parametrization.

We check that Λ is lagrangian, which is equivalent by homogeneity of Λ to the vanishing on it of the canonical one form $\alpha_{\mathbb{R}\times G} = \tau dt + \alpha_G$, that is to the vanishing of the one form on $\mathbb{R}\times\Lambda_0$ defined by $\phi^*\alpha_{\mathbb{R}\times G}$. We have:

$$\begin{aligned}\phi^*\alpha_{\mathbb{R}\times G} &= -p^0(\chi)dt + \chi^*\alpha_G \\ &= -p^0(\chi)dt + \alpha_G(\frac{\partial\chi}{\partial t})dt + (\chi_t)^*\alpha_G\end{aligned}$$

Since χ_t is a homogeneous symplectomorphism of $T^*_\bullet G$, the one form $(\chi_t)^*\alpha_G$ vanishes on Λ_0. On the other hand, by homogeneity of p^0 and Euler formula:

$$\alpha_G(\frac{\partial\chi}{\partial t}) = \alpha_G(H_{p^0}(\chi_t)) = \sum_j \xi_j \frac{\partial p^0}{\partial \xi_j}(\chi) = p^0(\chi), \tag{116}$$

which proves the required assertion. □

There are two natural structures of groupoid on $\mathbb{R}\times G$, with different unit space:

$$\widetilde{G} = \mathbb{R}\times G \rightrightarrows \mathbb{R}\times G^{(0)} \text{ and } G_+ = \mathbb{R}\times G \rightrightarrows G^{(0)}$$

The first is the (constant) family of groupoids $G_t = G$ parametrized by the space $\mathbb{R}$, and the second one is the cartesian product of G with the additive group $\mathbb{R}$. The corresponding symplectic groupoid structures on $T^*(\mathbb{R}\times G)$ will be denoted by:

$$\widetilde{\Gamma} = T^*(\mathbb{R}\times G) \rightrightarrows \mathrm{pr}_2^* A^*G \quad\text{and}\quad \Gamma_+ = T^*(\mathbb{R}\times G) \rightrightarrows \mathbb{R}\times A^*G \tag{117}$$

where $\mathrm{pr}_2 : \mathbb{R}\times M \to M$ is the second projection and $\mathbb{R}\times A^*G$ denotes the bundle over M with fibers $\mathbb{R}\times A^*_xG$.

We will say that a subset $A\subset\mathbb{R}\times X$ is $\mathbb{R}$-proper if $\mathrm{pr}_1 : A\to\mathbb{R}$ is proper, that is

$$A\cap[a,b]\times X \text{ is compact for any } a,b\in\mathbb{R}.$$

We will call support of $A\subset E$ the set $\mathrm{supp}(A) = \pi(A)\subset X$ for any bundle map $\pi : E\to X$.

Proposition 20. *The submanifold Λ of $T^*(\mathbb{R}\times G)$ satisfies the following:*

(1) It is contained in $(T^*\mathbb{R}\setminus 0)\times\dot{T}^*G$ and closed in $T^*\mathbb{R}\times(T^*_\bullet G)$.
(2) It is both an invertible $\widetilde{G}$-relation and a family G_+-relation.
(3) The support of Λ is $\mathbb{R}$-proper.

Proof. (1) We first check that Λ is closed in $T^*\mathbb{R}\times T^*_\bullet G$. The map

$$\phi : \mathbb{R}\times T^*_\bullet G \longrightarrow \mathbb{R}\times T^*_\bullet G, \quad (t,\lambda)\longmapsto(t,\chi(t,\lambda))$$

is a diffeomorphism and $\mathbb{R}\times\Lambda_0 = \mathbb{R}\times T^*_\bullet G\cap\mathbb{R}\times A^*G$ is closed in $\mathbb{R}\times T^*_\bullet G$ since $\mathbb{R}\times A^*G$ is closed in $T^*(\mathbb{R}\times G)$. Thus $\phi(\mathbb{R}\times\Lambda_0)$ is closed in $\mathbb{R}\times T^*_\bullet G$. It follows that

$$\Lambda = \{(t,-p^0(\lambda),\lambda)\in T^*\mathbb{R}\times T^*_\bullet G\ ;\ (t,\lambda)\in\phi(\mathbb{R}\times\Lambda_0)\}$$

is closed in $T^*\mathbb{R}\times T^*_\bullet G$.

(2) By remark 17, the inclusion $\Lambda\subset T^*\mathbb{R}\times\dot{T}^*G$ holds true and by ellipticity of P, the function $p^0 = p^0\circ r_\Gamma$ does not vanish on $\dot{T}^*G$, hence

$$\Lambda\subset(T^*\mathbb{R}\setminus 0)\times\dot{T}^*G.$$

(3) Since $r_{\widetilde{\Gamma}}(t,\tau,\lambda) = (t, r_\Gamma(\lambda))$ and $s_{\widetilde{\Gamma}}(t,\tau,\lambda) = (t, s_\Gamma(\lambda))$, we immediately deduce the invertibility of Λ from the invertibility of the G-relations Λ_t for all t.

(4) Since $r_{\Gamma_+}(t,\tau,\lambda) = (\tau, r_\Gamma(\lambda))$, $s_{\Gamma_+}(t,\tau,\lambda) = (\tau, s_\Gamma(\lambda))$ and $\Lambda \subset (T^*\mathbb{R}\setminus 0)\times \dot{T}^*G$ we get $\Lambda \cap \ker r_{\Gamma_+} = \emptyset$ and the same for s_{Γ_+} so Λ is a G_+-relation. Moreover, denoting by π, π_0, π_2 the natural projection maps:

$$\pi : T^*(\mathbb{R}\times G) \to \mathbb{R}\times G, \quad \pi_0 : A^*G \to M, \quad \pi_2 : T^*(\mathbb{R}\times G) \to T^*G,$$

since $(t,\tau,\lambda) \in \Lambda \mapsto s_\Gamma(\lambda) \in A^*G$ is a submersion, the composition

$$\pi_0 \circ s_\Gamma \circ \pi_2 = s_{G_+} \circ \pi|_\Lambda : \Lambda \longrightarrow A^*G_+ \longrightarrow M, (t,\tau,\gamma,\xi) \longmapsto s(\gamma)$$

is a submersion. This proves that Λ is a G_+-family by [18, Remark 15 and below]

(5) This is a straightforward consequence of the compacity of $M = G^{(0)}$, of the homogeneity of χ, and of standard continuity arguments.

□

9. Approximation of e^{-itP} by G-FIOs

The manifold $\mathbb{R}\times G$ will be provided by the pull back of the half density bundle used for G, and it will still be denoted by $\Omega^{1/2}$.

Let Λ be the $\widetilde{G}$-relation defined by P as in (115). Since Λ is a family $\widetilde{G}$-relation, any $U \in I^m(\mathbb{R}\times G, \Lambda; \Omega^{1/2})$ is a Fourier integral $\widetilde{G}$-operator (see [18] for the details), also given as a distribution on $\widetilde{G}$ by the C^∞ family $U(t) \in I^{m+1/4}(G, \Lambda_t; \Omega^{1/2})$ of G-FIOs defined by $U(t) = i_t^*(U)$. Here $i_t : G \to \mathbb{R}\times G$ is the inclusion $i_t(\gamma) = (t,\gamma)$. The converse is true: any such family gives a single distribution in $I^m(\mathbb{R}\times G, \Lambda; \Omega^{1/2})$.

Theorem 21. *There exists a Fourier integral $\widetilde{G}$-operator $U \in I^{-\frac{1}{4}+(n^{(1)}-n^{(0)})/4}(\mathbb{R}\times G, \Lambda; \Omega^{1/2})$ with $\mathbb{R}$-proper support such that:*

$$(\frac{\partial}{\partial t} + iP)U \in C^\infty(\mathbb{R}\times G, \Omega^{1/2}). \tag{118}$$

Moreover, if $E = (e^{-itP})_{t\in\mathbb{R}}$ denotes the one parameter group defined in Section 3, we have:

$$E - U \in C^\infty(\mathbb{R}, \mathcal{H}^\infty). \tag{119}$$

Remark 22.

(1) It follows that $(E(t))_{t\in\mathbb{R}}$ is a C^∞ family of distributions, equivalently $E \in \mathcal{D}'_{\mathrm{pr}_1}(\mathbb{R}\times G, \Omega^{1/2})$.

(2) Recall that $\mathcal{H}^\infty \subset C^{\infty,0}_{\mathrm{orb}}(G)$ by Section 5, in particular the error term (119) is C^∞ on $\mathbb{R}\times G_{\mathcal{O}}$ for any orbit $\mathcal{O} \subset M$.

(3) Theorem 21 also gives information about the operators e^{itP_x} on the (usually non compact, complete, with bounded geometry) manifolds G_x, $x \in M$. In the latter situation, we refer to [21] for related results.

Proof of the theorem. Let $U \in I^m(\mathbb{R}\times G, \Lambda; \Omega^{1/2})$. We first check that:

$$\frac{\partial}{\partial t}U \in I^{m+1}(\mathbb{R}\times G, \Lambda; \Omega^{1/2}) \text{ and } PU \in I^{m+1}(\mathbb{R}\times G, \Lambda; \Omega^{1/2}). \tag{120}$$

The distribution PU is given by convolution product in $\widetilde{G}$ of the $\widetilde{G}$-PDO P with the $\widetilde{G}$-FIO U. Therefore, the composition theorem of [18] applies and proves $PU \in I^{m+1}(\mathbb{R}\times G, \Lambda, \Omega^{1/2})$. Note that PU is also a convolution of distributions in G_+:

$$PU = (\delta_{t-s} \otimes P) *_{G_+} U, \tag{121}$$

but this time it is not a composition of G_+-FIO because $\delta_{t-s} \otimes P$ fails to be in general a G_+-PDO. The other assertion in (120) can be checked either by directly differentiating with

respect to t the family $(U(t))_t$ expressed in local coordinates with oscillatory integrals, or by composing the differential G_+-operator $\frac{\partial}{\partial t}$ with the G_+-FIO U.

The next task is to prove that the sum $(\frac{\partial}{\partial t}+iP)U$ is actually of order m and has principal symbol given by:

$$\mathcal{L}_{\tau+p^0}(u)+i\sigma^{1s}(P)u. \tag{122}$$

Since $\frac{\partial}{\partial t}+iP$ is neither a $\widetilde{G}$ nor G_+ pseudodifferential operator, we can not directly apply Proposition 15 to extract the principal symbol of (120). We propose two ways to overcome this difficulty, both containing useful technics.

First approach. Both distributions $\frac{\partial}{\partial t}U$ and PU are $\widetilde{G}$-FIO. Working as before in suitable local coordinates (t,γ,ξ), and using for instance [18, Theorems 5 and 6], there exists a C^∞ function $h(t,\xi)$, homogeneous of order 1 in ξ, and a symbol $u(t,\xi)$, such that:

$$(t,\tau,\gamma,\xi)\in\Lambda \iff \tau=-h'_t(t,\xi),\ \gamma=h'_\xi(t,\xi), \tag{123}$$

$$U(t,\gamma)=\int e^{i(\langle\gamma,\xi\rangle-h(t,\xi))}u(t,\xi)d\xi. \tag{124}$$

It immediately follows that

$$(\frac{\partial}{\partial t}+iP)U(t,\gamma)=\int e^{i(\langle\gamma,\xi\rangle-h(t,\xi))}\frac{\partial u}{\partial t}(t,\xi)d\xi+\int e^{i(\langle\gamma,\xi\rangle-h(t,\xi))}i(p(\gamma,\xi)-h'_t(t,\xi))u(t,\xi)d\xi. \tag{125}$$

The right hand side is again a sum of Lagrangian distributions. The principal symbol of the first term in the right hand side of (125) is just the restriction to Λ of:

$$\frac{\partial u}{\partial t}=\mathcal{L}_\tau u \tag{126}$$

In the second term, although $p-h'_t$ does not satisfy symbol estimates in ξ, the product $i(p(\gamma,\xi)-h'_t(t,\xi))u(t,\xi)$ does and its leading part, which is represented by $i(p^0-h'_t)u$, vanishes on Λ_t for any t. We then reproduce the computations starting with (102), just replacing $h(\xi)$ by $h(t,\xi)$, $q(\gamma,\xi)$ by $p(\gamma,\xi)-h'_t(t,\xi)$ and $a(\xi)$ by $u(t,\xi)$, without omitting an extra factor i. The reminder e is unchanged $e=(p-h'_t)-(p^0-h'_t)=p-p^0$. The vector field $H_{p^0-h'_t}$ being tangent to Λ_t for any t, we get, since h'_t is independent of γ:

$$H_{p^0-h'_t}=-\frac{\partial}{\partial\gamma_j}(p^0-h'_t)\frac{\partial}{\partial\xi_j}=-\frac{\partial p^0}{\partial\gamma_j}\frac{\partial}{\partial\xi_j}=H_{p^0}.$$

Now we can read the expression for the required principal symbol in (111):

$$\mathcal{L}_{p^0}(u)+\left(ie-\frac{1}{2}\sum_k\frac{\partial^2(p^0-h'_t)}{\partial\xi_k\partial\gamma_k}\right)u. \tag{127}$$

Again, since h'_t is independent of γ and p^0 independent of ξ'', the last expression simplifies to:

$$\mathcal{L}_{p^0}(u)+\left(ie-\frac{1}{2}\sum_k\frac{\partial^2 p^0}{\partial\xi'_k\partial\gamma'_k}\right)u=\mathcal{L}_{p^0}(u)+i\sigma^{1s}(P)u. \tag{128}$$

Summing up (126) and (128), we conclude that the principal symbol of $(\frac{\partial}{\partial t}+iP)U$ is (122).

Second approach. We wish to use Proposition 15 in the framework of the groupoid G_+. However, we need to have the convolution of a pseudodifferential G_+-operator with a G_+-FIO. The problem is that the distribution $(\frac{\partial}{\partial t}+i\delta_{t-s}\otimes P)$ is not a G_+-pseudodifferential

operator, unless P is differential. The trick (similar to the one used in the proof of [12, Theorem 25.2.4]), consists in finding a suitable microlocal approximation $\delta_{t-s}\otimes P = P_1 + P_2$ of $\delta_{t-s}\otimes P$ by a G_+-PDO P_1 such that $P_2U \in C^\infty(\mathbb{R}\times G)$. For that purpose, observe that we can deduce from (115) that there exists constants $c_1, c_2 > 0$ such for any $(\tau, x, \xi) \in r_{\Gamma_+}(\Lambda)$, we have

$$c_1 \leq \frac{|\tau|}{|\xi|} \leq c_2. \tag{129}$$

Indeed, we know that $(\tau, x, \xi) = r_{\Gamma_+}(t, -p(\lambda), \lambda) = (-p(\lambda), r_\Gamma(\lambda))$ for some $t \in \mathbb{R}$ and $\lambda \in \Lambda_t$. Denoting $\lambda = (\gamma, \eta) \in T^*G \setminus 0$ and $r_\Gamma(\gamma, \eta) = (x, \xi) \in A^*G \setminus 0$, we then get by homegenity of p_0,

$$\frac{|\tau|}{|\xi|} = \frac{|p_0 \circ r_\Gamma(\gamma, \eta)|}{|\xi|} = \frac{|p_0(x, \xi)|}{|\xi|} = |p_0(x, \frac{\xi}{|\xi|})|$$

and the result follows by continuity of p_0 and compacity of $M = G^{(0)}$ (which implies that $S^*G = \{(x, \xi) \in A^*G \setminus 0, |\xi| = 1\}$ is compact.) We will use

Lemma 23. *The distribution $\delta_{t-s}\otimes P$ on G_+ can be written $\delta_{t-s}\otimes P = P_1 + P_2$ with P_1 a G_+-pseudodifferential operator and P_2 a distribution on G_+ such that* $\mathrm{WF}\, P_2 \subset \{(t, \tau, \lambda) \in T^*G_+ \setminus 0, s_\Gamma(\lambda) = (x, \xi)$ *with* $\frac{|\xi|}{|\tau|} < \varepsilon\}$. *In particular the total symbol of P_1 and P coincide in a neighborhood of $r_{\Gamma_+}(\Lambda)$ and one has that $P_2U \in C^\infty(G_+)$ if $U \in I^m(\mathbb{R}\times G, \Lambda, \Omega^{1/2})$.*

Proof. Consider a map χ on $A^*(G_+) = \mathbb{R}_\tau \times A^*G$ such that χ is homogeneous of degree 0 in the cotangent variables outside a compact set, and such that for a chosen ε , one has

(1) $\chi(x, \xi, \tau) = 0$ unless $1 < \varepsilon|\tau|$ and $|\xi| < \varepsilon|\tau|$;
(2) $\chi(x, \xi, \tau) = 1$ if $2 < \varepsilon|\tau|$ and $2|\xi| < \varepsilon|\tau|$.

If $P(x, \xi)$ is a total symbol for P, then one can write

$$P(x, \xi) = p_1(x, \xi, \tau) + p_2(x, \xi, \tau) = (1 - \chi(x, \xi, \tau))P(x, \xi) + \chi(x, \xi, \tau)P(x, \xi).$$

It is clear that $p_1(x, \xi, \tau) \in S^1(A^*G_+)$, so that the corresponding operator

$$P_1(t, \gamma) = \int e^{i<\kappa(\gamma),\xi>+i<t,\tau>} p_1(r(\gamma), \xi, \tau) d\xi d\tau\, \mu_s^{1/2}(\gamma)\mu_r^{1/2}(\gamma) \in \Psi^1_{G_+}.$$

Moreover, in the neighbourhood of $r_{\Gamma_+}(\Lambda)$, one has that $\chi(x, \xi, \tau) = 0$, because of (129) and hence the symbol of P_1 is the symbol of P.
Now the wave front of the distribution :

$$P_2(t, \gamma) = \int e^{i<\kappa(\gamma),\xi>+i<t,\tau>} \chi(r(\gamma), \xi, \tau) P(r(\gamma), \xi) d\xi d\tau\, \mu_s^{1/2}(\gamma)\mu_r^{1/2}(\gamma)$$

is such that if $(t, \tau, \lambda) \in T^*G_+ \setminus 0$ and $s_\Gamma(\lambda) = (x, \xi) \in A^*G \setminus 0$, $r_\Gamma(\lambda) = (y, \eta) \in A^*G \setminus 0$, then $(t, \tau, \lambda) \in \mathrm{WF}\, P_2 \implies \max(\frac{|\xi|}{|\tau|}, \frac{|\eta|}{|\tau|}) \leq \varepsilon$. This implies in particular that $\mathrm{WF}\, P_2.\Lambda = \emptyset$. □

To conclude this second approach, note that the principal symbol of $(\frac{\partial}{\partial t} + iP_1)$ is equal to $\tau + p_0$ in a neighboorhood of $r_{\Gamma_+}(\Lambda)$ and vanishes on $r_{\Gamma_+}(\Lambda)$, because $(\tau + p_0)\circ r_{\Gamma_+} = \tau + p_0 \circ r_\Gamma$ vanishes on Λ. Thus we may apply Proposition 15 with G_+ as underlying groupoid to the operators $(\frac{\partial}{\partial t} + iP_1)$ and U, which allows to recover the formula (122) for the principal symbol of their product by remarking that the subprincipal symbol of P_1 is also equal to the subprincipal of P in a neighboorhood of $r_{\Gamma_+}(\Lambda)$.

The rest of the proof is essentially identical to the proof of [12, Theorem 29.1.1]. Indeed the (transport) equation

$$\begin{cases} (\frac{\partial}{\partial t} + \mathcal{L}_{p^0} + i\sigma^{1s}(P))u^0 = 0 \\ u^0(0,.) = 1 \end{cases} \tag{130}$$

has a unique solution $u^0 \in C^\infty(\Lambda)$, and u^0 homogeneous of degree 0 with respect to the $\mathbb{R}_+$ action on each Λ_t. Let us fix a $\mathbb{R}$-proper set $\mathcal{V} \subset \mathbb{R} \times G$ such that $\mathrm{supp}(\Lambda) \subset \overset{\circ}{\mathcal{V}}$. choose $U^0 \in I^{(n^{(1)}-n^{(0)}-1)/4}(\mathbb{R} \times G, \Lambda, \Omega^{1/2})$ with principal symbol u^0 and support in $\mathcal{V}$. Note that $U^0(0) \in \Psi^0_{G,c}$ because $\Lambda_0 = A^*G \setminus 0$. It follows that:

$$I - U^0(0) \in \Psi^{-1}_{G,c} \text{ and } (\frac{\partial}{\partial t} + iP)U^0 = F^1 \in I^{-1+(n^{(1)}-n^{(0)}-1)/4}(\mathbb{R} \times G, \Lambda, \Omega^{1/2}). \tag{131}$$

Next one chooses $U^1 \in I^{-1+(n^{(1)}-n^{(0)}-1)/4}(\mathbb{R} \times G, \Lambda, \Omega^{1/2})$ with support in $\mathcal{V}$ and principal symbol u^1 solving the transport equation

$$\begin{cases} (\frac{\partial}{\partial t} + \mathcal{L}_{p^0} + i\sigma^{1s}(P))u^1 = -f^1 \\ u^1(0,.) = \sigma(I - U^0(0)) \end{cases} \tag{132}$$

and so on. We construct in this way a sequence $U^j \in I^{-j+(n^{(1)}-n^{(0)}-1)/4}(\mathbb{R} \times G, \Lambda, \Omega^{1/2})$. Finally we choose $U \in I^{(n^{(1)}-n^{(0)}-1)/4}(\mathbb{R} \times G, \Lambda, \Omega^{1/2})$ with support in $\mathcal{V}$ such that:

$$U \sim \sum U^j.$$

By construction, we get

$$R := (\frac{\partial}{\partial t} + iP)U \in C^\infty(\mathbb{R} \times G) \quad \text{and} \quad S := \mathrm{Id} - U(0) \in C^\infty_c(G). \tag{133}$$

Modifying U into $U + \varphi S$ with $\varphi \in C^\infty_c(\mathbb{R})$ and $\varphi(0) = 1$, we can directly assume that $U(0) = \mathrm{Id}$. Also, the support of R is contained in

$$\mathcal{V}' = \mathcal{V} \cup (\mathbb{R} \times \mathrm{supp}(P)) \cdot_{\widetilde{G}} \mathcal{V} = \mathcal{V} \cup \{(t, \gamma) \ ; \ \gamma \in \mathrm{supp}(P) \cdot \mathcal{V}_t\}.$$

The set $\mathcal{V}'$ is again $\mathbb{R}$-proper. This implies:

$$R \in C^\infty(\mathbb{R}, C^\infty_c(G)) \subset C^\infty(\mathbb{R}, \mathcal{H}^\infty)$$

We obtain, using (27) and following verbatim the proof of [12, Theorem 29.1.1]

$$U(t) - e^{-itP} = \tilde{R}(t) := i \int_0^t e^{i(t-s)P} R(s) ds \tag{134}$$

Using the results of Section 3, we get $\tilde{R} \in C^\infty(\mathbb{R}, \mathcal{H}^\infty)$, which ends the proof. □

The previous theorem is only stated for compactly supported operators, but it admits the following slight generalization:

Corollary 24. *Let $T = P_c + S \in \Psi^1_G$, with $P_c \in \Psi^1_{G,c}$ satisfying the assumption of Theorem 21 and $S \in \mathcal{H}^\infty$. There exists $U \in I^{-\frac{1}{4}+(n^{(1)}-n^{(0)})/4}(\mathbb{R} \times G, \Lambda; \Omega^{1/2})$ with $\mathbb{R}$-proper support such that*

$$(\frac{\partial}{\partial t} + iT)U \in C^\infty(\mathbb{R}, \mathcal{H}^\infty). \tag{135}$$

Proof. Apply Theorem 21 to P_c and let $U \in I^{-\frac{1}{4}+(n^{(1)}-n^{(0)})/4}(\mathbb{R}\times G,\Lambda;\Omega^{1/2})$ be the corresponding parametrix. Then

$$(\frac{\partial}{\partial t}+iT)U = iSU + R, \qquad R\in C^\infty(\mathbb{R},\mathcal{H}^\infty). \tag{136}$$

Using the continuity theorems for G-FIO [18], one gets that for any t, U_t acts continuously on the scale of Sobolev modules, which immediately implies that $SU \in C^\infty(\mathbb{R},\mathcal{H}^\infty)$. □

As examples of situations into which Theorem 21 and Corollary 24 apply, we mention:

(1) The pair groupoid $G = X\times X \rightrightarrows X$ of a compact manifold without boundary X. Since X itself is an orbit, we have $C^{\infty,0}_{\text{orb}}(G) = C^\infty(X\times X,\Omega^{1/2})$ and we just recover the classical result (see [12, Theorem 29.1.1] for instance), after the obvious identification between G-operators and continuous linear operators $C^\infty(X,\Omega^{1/2}_X)\to C^\infty(X,\Omega^{1/2}_X)$.

(2) The holonomy groupoid G of a compact foliated manifold X. We recover the construction of the leafwise geometrical optic approximation of e^{itP} given in [15]. The latter is worked out for small time and by solving eikonal equations to find the required phases in local coordinates as well as by solving transport equations. Our construction here can be viewed as a complement, available for arbitrary time and regarding the evolution of singularities as well as the kind of Fourier integral operators involved in the problem.

(3) $G \rightrightarrows \{e\}$ a Lie group. Here again, there is only one orbit so $C^{\infty,0}_{\text{orb}}(G) = C^\infty(G)$. The result applies for instance to the square root $\sqrt{\Delta}$ of any elliptic laplacian $\Delta = -\sum X_j^2 \in \text{Diff}^2_G$, viewed as right invariant operators on G. That $\sqrt{\Delta} = P_c + S \in \Psi^1_G$ with $\sigma^0_G(P_c) = \sqrt{\sum \xi_j^2}$ follows from [32] and we get here the existence of a C^∞ family U_t of right invariant FIO on G [27, 18] such that $(\frac{\partial}{\partial t}+i\sqrt{\Delta})U_t \in C^\infty(G)\cap C^*_r(G)$ for any t.

(4) The groupoid $G_b \rightrightarrows X$ of the b-calculus of a manifold with embedded corners X [25]. We recall that G_b is the open submanifold with corners of the b-stretched product of R. Melrose X^2_b in which all the lateral faces are removed. Identifying G_b-operators with pseudodifferential operators in the b-calculus, and their restrictions at boundary hypersurfaces with indicial operators, we get for any elliptic symmetric $P\in\Psi^1_b(X)$ in the small calculus the existence of a C^∞ family U_t of b-FIO on X [22, 18] such that

$$(\frac{\partial}{\partial t}+iP)U_t = R_t \in C^\infty((X\setminus\partial X)^2)\cap\mathcal{L}(L^2_b(X)). \tag{137}$$

and for any boundary hypersurfaces H (with normal bundle trivialized with a boundary defining function):

$$I_H(R_t) = (\frac{\partial}{\partial t}+iI_H(P))I_H(U_t)\in C^\infty(H^2\times\mathbb{R})\cap\mathcal{L}(L^2_b(H\times\mathbb{R})).$$

The error term R_t is C^0 on G_b and there is no reason neither to expect that it is C^∞ on G_b, nor that it extends continuously to X^2_b.

(5) This discussion is similar to the previous one for the groupoid $G_\pi \rightrightarrows X$ [8] and its associated pseudodifferential calculus, where X is a manifold with iterated fibred corners. In both cases, the regularity result that we reach for the error term R is likely not optimal. This will be investigated, among other applications to singular spaces, in future works.

As far as we know, examples (3–5) above are new.

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