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Higher-rank discrete symmetries in the IBM. II Octahedral shapes: Dynamical symmetries

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Abstract

The symmetries of the *sdg*-IBM, the interacting boson model with *s*, *d* and *g* bosons, are studied as regards the occurrence of shapes with octahedral symmetry. It is shown that no *sdg*-IBM Hamiltonian with a dynamical symmetry displays in its classical limit an isolated minimum with octahedral shape. However, a degenerate minimum that includes a shape with octahedral symmetry can be obtained from a Hamiltonian that is transitional between two limits, $U_g(9) \otimes U_d(5)$ and $SO_{sg}(10) \otimes U_d(5)$, and the conditions for its existence are derived. An isolated minimum with octahedral shape, either an octahedron or a cube, may arise through a modification of two-body interactions between the *g* bosons. Comments on the observational consequences of this construction are made.

Key words: discrete octahedral symmetry, interacting boson model, *g* bosons
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1 Introduction

This paper is the second in the series initiated with Ref. [1], henceforth referred to as I. The overall purpose of this series is the study of nuclear shapes with a higher-rank discrete symmetry, in particular of the tetrahedral or octahedral type, in the framework of a variety of interacting boson models. For the general context of this study we refer the reader to the introduction in I, of which only the essential points and references are mentioned here.

The paper by Li and Dudek [2] pointed out the possibility of intrinsic nuclear shapes with a higher-rank discrete symmetry. Subsequent publications by the Strasbourg group and their collaborators [3–6] showed the possible occurrence

of tetrahedral, octahedral and icosahedral symmetries through combinations of deformations of specific multipolarity. In parallel with these theoretical developments, a search was initiated for experimental manifestations of such symmetries in nuclei [7–9]. In addition to these experimental searches cited in I, a more recent study of this type appeared in 2018 [10], formulating criteria for the identification of tetrahedral and/or octahedral symmetries in nuclei.

From the theory side most work related to higher-rank discrete symmetries has been carried out in the context of mean-field models. The aim of this series of papers is to address the question of the possible occurrence of higher-rank discrete symmetries from a different theoretical perspective. Our study is carried out in the context of algebraic collective models inspired by the interacting boson model (IBM) of Arima and Iachello [11–13], which proposes a description of quadrupole collective nuclear states in terms of s and d bosons with angular momentum $\ell = 0$ and $\ell = 2$. The application of this idea to collective states of different nature, notably of octupole and hexadecapole character, requires the introduction of other bosons [14], in particular f and g bosons with angular momentum $\ell = 3$ and $\ell = 4$. Since shapes with tetrahedral symmetry arise in lowest order through a particular octupole deformation and those with octahedral symmetry emanate from a combination of hexadecapole deformations, the study of such shapes in an algebraic context requires the introduction of f and g bosons, respectively.

We initiated this program in I with the study of octahedral shapes in the sdg -IBM. We considered the most general rotationally invariant Hamiltonian with up to two-body interactions between the bosons and derived the conditions for this Hamiltonian to have in its classical limit a minimum with an intrinsic shape with octahedral symmetry. Owing to the general nature of the analysis in I, only qualitative conclusions could be drawn with regard to the (non-)existence of such minima in the sdg -IBM. In the present paper we arrive at more concrete conclusions by considering a subset of all possible Hamiltonians of the sdg -IBM, namely those that have a dynamical symmetry and are analytically solvable. Although none of the symmetry Hamiltonians leads to a stable shape with octahedral symmetry, a subset can be used to propose a generalization with additional interactions between the g bosons that drive the system toward such a shape.

The paper is structured as follows. To avoid repeatedly referring to equations in I, we list in Section 2 formulas from I that are needed in this paper. In Section 3 we focus on two dynamical symmetries of the sdg -IBM of particular relevance in our quest for octahedral shapes and in Section 4 the classical limit is derived for the Hamiltonian that is transitional between these two limits. The central results of this paper are presented in Section 5, with a catastrophe analysis of the energy surface associated with the transitional symmetry Hamiltonian, and suitable generalizations thereof, to uncover the

existence of minima at stable shapes with octahedral symmetry. Finally, in Section 6 the conclusions of the second paper in this series are summarized.

2 The *sdg*-IBM and its classical limit

A boson-number-conserving, rotationally invariant Hamiltonian of the *sdg*-IBM with up to two-body interactions is of the form

$$\hat{H} = \epsilon_s \hat{n}_s + \epsilon_d \hat{n}_d + \epsilon_g \hat{n}_g + \sum_{\ell_1 \leq \ell_2, \ell'_1 \leq \ell'_2, L} \frac{(-)^L v_{\ell_1 \ell_2 \ell'_1 \ell'_2}^L}{\sqrt{(1 + \delta_{\ell_1 \ell_2})(1 + \delta_{\ell'_1 \ell'_2})}} [b_{\ell_1}^\dagger \times b_{\ell_2}^\dagger]^{(L)} \cdot [\tilde{b}_{\ell'_2} \times \tilde{b}_{\ell'_1}]^{(L)}, \quad (1)$$

where \hat{n}_ℓ is the number operator for the ℓ boson, ϵ_ℓ is the energy of the ℓ boson and $v_{\ell_1 \ell_2 \ell'_1 \ell'_2}^L$ is a boson–boson interaction matrix element.

A geometric understanding of this quantum-mechanical Hamiltonian is obtained by considering its expectation value in a coherent state. As discussed in I, a study of shapes with octahedral symmetry in *sdg*-IBM requires a coherent state of the form

$$|N; \beta_2, \beta_4, \gamma_2, \gamma_4, \delta_4\rangle = \sqrt{\frac{1}{N!(1 + \beta_2^2 + \beta_4^2)^N}} \Gamma(\beta_2, \beta_4, \gamma_2, \gamma_4, \delta_4)^N |o\rangle, \quad (2)$$

with

$$\begin{aligned} \Gamma(\beta_2, \beta_4, \gamma_2, \gamma_4, \delta_4) = & s^\dagger + \beta_2 \left[\cos \gamma_2 d_0^\dagger + \sqrt{\frac{1}{2}} \sin \gamma_2 (d_{-2}^\dagger + d_{+2}^\dagger) \right] \\ & + \beta_4 \left[\left(\sqrt{\frac{7}{12}} \cos \delta_4 + \sqrt{\frac{5}{12}} \sin \delta_4 \cos \gamma_4 \right) g_0^\dagger \right. \\ & \quad - \sqrt{\frac{1}{2}} \sin \delta_4 \sin \gamma_4 (g_{-2}^\dagger + g_{+2}^\dagger) \\ & \quad \left. + \left(\sqrt{\frac{5}{24}} \cos \delta_4 - \sqrt{\frac{7}{24}} \sin \delta_4 \cos \gamma_4 \right) (g_{-4}^\dagger + g_{+4}^\dagger) \right], \end{aligned} \quad (3)$$

in terms of the deformation parameters β_2 , β_4 , γ_2 , γ_4 and δ_4 , following the convention of Rohoziński and Sobiczewski [15]. A shape with octahedral symmetry is obtained for $\beta_2 = 0$, $\beta_4 \neq 0$ and $\delta_4 = 0$ (octahedron) or $\delta_4 = \pi$ (cube) with arbitrary γ_4 . Another solution with octahedral symmetry exists for $\gamma_4 = 0$ and $\delta_4 = \arccos(1/6)$, corresponding to a rotated octahedron, i.e. with the same intrinsic shape as for $\delta_4 = 0$. The parameterization introduced in Ref. [15] therefore does *not* define a unique intrinsic state.

The expectation value of the Hamiltonian (1) in the coherent state (2) (or the classical limit of \hat{H}) leads to the energy surface

$$\begin{aligned} \langle \hat{H} \rangle &\equiv E(\beta_2, \beta_4, \gamma_2, \gamma_4, \delta_4) \\ &= \frac{N(N-1)}{(1 + \beta_2^2 + \beta_4^2)^2} \sum_{kl} \beta_2^k \beta_4^l \left[c'_{kl} + \sum_{ij} c_{kl}^{ij} \cos(i\gamma_2 + j\gamma_4) \phi_{kl}^{ij}(\delta_4) \right], \end{aligned} \quad (4)$$

where $\phi_{kl}^{ij}(\delta_4)$ are trigonometric functions defined in I. The coefficients c'_{kl} are known in terms of the scaled single-boson energies $\epsilon'_\ell \equiv \epsilon_\ell/(N-1)$ and the interaction matrix elements $v_{\ell_1 \ell_2 \ell'_1 \ell'_2}^L$,

$$\begin{aligned} c'_{00} &= \frac{1}{2} v_{ssss}^0 + \epsilon'_s, \\ c'_{20} &= \sqrt{\frac{1}{5}} v_{ssdd}^0 + v_{sdsd}^2 + \epsilon'_s + \epsilon'_d, \\ c'_{02} &= \frac{1}{3} v_{ssgg}^0 + v_{sgsg}^4 + \epsilon'_s + \epsilon'_g, \\ c'_{40} &= \frac{1}{10} v_{dddd}^0 + \frac{1}{7} v_{dddd}^2 + \frac{9}{35} v_{dddd}^4 + \epsilon'_d, \\ c'_{22} &= \frac{1}{\sqrt{45}} v_{ddgg}^0 + \frac{7}{\sqrt{715}} v_{ddgg}^4 + \frac{1}{6} v_{dgdg}^2 + \frac{4}{11} v_{dgdg}^4 + \frac{1}{6} v_{dgdg}^5 + \frac{10}{33} v_{dgdg}^6 + \epsilon'_d + \epsilon'_g, \\ c'_{04} &= \frac{1}{18} v_{gggg}^0 + \frac{38}{693} v_{gggg}^2 + \frac{89}{1001} v_{gggg}^4 + \frac{62}{495} v_{gggg}^6 + \frac{1129}{6435} v_{gggg}^8 + \epsilon'_g. \end{aligned} \quad (5)$$

Only a single coefficient c_{kl}^{ij} is needed in the subsequent analysis, *viz.*

$$c_{04}^{00} = -\frac{2}{693} v_{gggg}^2 + \frac{4}{3003} v_{gggg}^4 + \frac{2}{495} v_{gggg}^6 - \frac{16}{6435} v_{gggg}^8, \quad (6)$$

introducing a δ_4 dependence in the energy surface (4) since

$$\phi_{04}^{00}(\delta_4) = 2 \cos 2\delta_4 + 17 \cos 4\delta_4. \quad (7)$$

It is assumed in the following that the boson Hamiltonian is Hermitian and therefore that $v_{\ell_1 \ell_2 \ell'_1 \ell'_2}^L = v_{\ell'_1 \ell'_2 \ell_1 \ell_2}^L$. With this assumption the expressions for the coefficients c_{kl} and c_{kl}^{ij} , given in Eqs. (20) and (21) of I, are valid with $v_{\ell_1 \ell_2 \ell'_1 \ell'_2}^L = v_{\ell_1 \ell_2 \ell'_1 \ell'_2}^L$. Note that for a general Hamiltonian one has $v_{\ell_1 \ell_2 \ell'_1 \ell'_2}^L = (v_{\ell_1 \ell_2 \ell'_1 \ell'_2}^L + v_{\ell'_1 \ell'_2 \ell_1 \ell_2}^L)/2$, which corrects by a factor 2 the expression given in I.

3 The $U_g(9)$ and $SO_{sg}(10)$ limits

Since the Hamiltonian of the *sdg*-IBM conserves the total number of bosons, it can be written in terms of the $(1+5+9)^2 = 225$ operators $b_{\ell m}^\dagger b_{\ell' m'}$. The 225 operators generate the Lie algebra $U(15)$ with a substructure that determines the dynamical symmetries of the *sdg*-IBM.

A comprehensive list of the dynamical symmetries of the *sdg*-IBM is given by De Meyer *et al.* [16], and their group-theoretical properties are extensively dis-

cussed by Kota *et al.* [17]. It is found that the model has seven major dynamical symmetries, four of strong coupling, SU(3), SU(6), SU(5) and SO(15), and three of weak coupling, $U_s(1) \otimes U_{dg}(14)$, $U_{sd}(6) \otimes U_g(9)$ and $U_{sg}(10) \otimes U_d(5)$. The question treated in this paper is whether any of the dynamical symmetries of the *sdg*-IBM corresponds to a shape with octahedral symmetry. In this section a choice of the limits that possibly have such property is made on the basis of intuitive arguments. Subsequently, in Sections 4 and 5, the conditions for the existence of a shape with octahedral symmetry are derived rigorously on the basis of the results obtained in I.

A minimum with octahedral shape requires mixing of *s* and *g* bosons, so as to induce hexadecapole deformation, and no or weak mixing of these with the *d* boson to ensure zero quadrupole deformation. These conditions rule out all limits where *s*, *d* and *g* bosons are strongly mixed on an equal footing, that is, they discard the SU(3), SU(6), SU(5) and SO(15) limits [18]. A strict decoupling of the *s* and *g* from the *d* bosons is obtained by the reduction

$$\begin{array}{ccc}
 U(15) \supset U_d(5) \otimes U_{sg}(10) & & \\
 \downarrow \quad \downarrow \quad \downarrow & , & (8) \\
 [N] \quad n_d \quad n_{sg} & &
 \end{array}$$

and, furthermore, zero quadrupole deformation follows from a U(5) classification for the *d* bosons,

$$\begin{array}{ccc}
 U_d(5) \supset SO_d(5) \supset SO_d(3) & & \\
 \downarrow \quad \downarrow \quad \downarrow & , & (9) \\
 n_d \quad v_d \quad \nu_d \quad L_d & &
 \end{array}$$

where underneath each algebra the associated quantum number is given. Specifically, N is the total number of bosons while n_ℓ is the number of ℓ bosons and $n_{\ell\ell'}$ is the number of ℓ plus ℓ' bosons. The seniority label associated with an ℓ boson is denoted as ν_ℓ and corresponds to the number of ℓ bosons not in pairs coupled to angular momentum zero. Additional (or missing) labels, not associated with any algebra, are indicated with ν_ℓ . Finally, the angular momentum generated by the ℓ bosons is denoted as L_ℓ .

The $U_{sg}(10)$ algebra in Eq. (8) allows two classifications of interest. The first is obtained by eliminating from the generators of $U_{sg}(10)$ those that involve

the s boson, leading to

$$\begin{array}{ccccccc}
 & & \text{U}_{sg}(10) \supset \text{U}_g(9) \supset \text{SO}_g(9) \supset \text{SO}_g(3) & & & & \\
 \text{(I)} & & \downarrow & \downarrow & \downarrow & \downarrow & \cdot \\
 & & n_{sg} & n_g & \nu_g & \nu_g & L_g
 \end{array} \tag{10}$$

In this limit, which for brevity shall be referred to as $\text{U}_g(9)$ or limit I, the separate boson numbers n_s and n_g are conserved. The resulting spectrum is vibrational-like with a spherical ground state and excited states that correspond to oscillations in the hexadecapole degree of freedom.

The second classification of $\text{U}_{sg}(10)$ is specified by the following chain of nested algebras:

$$\begin{array}{ccccccc}
 & & \text{U}_{sg}(10) \supset \text{SO}_{sg}(10) \supset \text{SO}_g(9) \supset \text{SO}_g(3) & & & & \\
 \text{(II)} & & \downarrow & \downarrow & \downarrow & \downarrow & , \\
 & & n_{sg} & \nu_{sg} & \nu_g & \nu_g & L_g
 \end{array} \tag{11}$$

which for brevity shall be referred to as $\text{SO}_{sg}(10)$ or limit II. The defining feature of the reduction (11) is the appearance of the algebra $\text{SO}_{sg}(10)$ and its label ν_{sg} , associated with the pairing of s and g bosons. As is shown in Section 5, the ground state in this limit acquires a permanent hexadecapole deformation and the limit is therefore of interest in our quest for octahedral shapes. On its own, however, limit II implies degenerate energies of the s and g boson, and as such it is not realistic. It is therefore necessary to study a combination of the two limits I and II.

In Sections 4 and 5 we investigate to what extent non-degenerate energies can be taken for the s and g boson that still lead to a hexadecapole-deformed minimum and whether that minimum can have octahedral symmetry. In the remainder of this section we list some of the properties of limits I and II that are necessary to carry out this analysis.

The classification of limits I and II can be summarized with the algebraic

lattice

$$\begin{array}{ccc}
\text{U}(15) \supset \text{U}_d(5) & \otimes & \text{U}_{sg}(10) \\
| & & \swarrow \searrow \\
\downarrow & & \text{U}_g(9) \quad \text{SO}_{sg}(10) \\
\text{SO}_d(5) & & \searrow \swarrow \\
| & & \text{SO}_g(9) \quad , \\
\downarrow & & \downarrow \\
\text{SO}_d(3) & & \text{SO}_g(3) \\
\searrow & & \swarrow \\
& & \text{SO}(3)
\end{array} \tag{12}$$

where $\text{SO}(3)$ is associated with the total angular momentum L , which results from the coupling of L_d and L_g . The generators of the different algebras in the lattice (12) are as follows:

$$\begin{aligned}
\text{U}_d(5): & \quad \{[d^\dagger \times \tilde{d}]_\mu^{(\lambda)}, \lambda = 0, \dots, 4\}, \\
\text{SO}_d(5): & \quad \{[d^\dagger \times \tilde{d}]_\mu^{(\lambda)}, \lambda = 1, 3\}, \\
\text{SO}_d(3): & \quad \{\hat{L}_{d,\mu} \equiv \sqrt{10}[d^\dagger \times \tilde{d}]_\mu^{(1)}\}, \\
\text{U}_{sg}(10): & \quad \{[s^\dagger \times \tilde{s}]_0^{(0)}, [s^\dagger \times \tilde{g}]_\mu^{(4)}, [g^\dagger \times \tilde{s}]_\mu^{(4)}, [g^\dagger \times \tilde{g}]_\mu^{(\lambda)}, \lambda = 0, \dots, 8\}, \\
\text{U}_g(9): & \quad \{[g^\dagger \times \tilde{g}]_\mu^{(\lambda)}, \lambda = 0, \dots, 8\}, \\
\text{SO}_{sg}(10): & \quad \{[s^\dagger \times \tilde{g} + g^\dagger \times \tilde{s}]_\mu^{(4)}, [g^\dagger \times \tilde{g}]_\mu^{(\lambda)}, \lambda = 1, 3, 5, 7\}, \\
\text{SO}_g(9): & \quad \{[g^\dagger \times \tilde{g}]_\mu^{(\lambda)}, \lambda = 1, 3, 5, 7\}, \\
\text{SO}_g(3): & \quad \{\hat{L}_{g,\mu} \equiv \sqrt{60}[g^\dagger \times \tilde{g}]_\mu^{(1)}\}, \\
\text{SO}(3): & \quad \{\hat{L}_\mu \equiv \hat{L}_{d,\mu} + \hat{L}_{g,\mu}\}.
\end{aligned} \tag{13}$$

The linear and quadratic Casimir operators of the algebras appearing in the lattice (12) can be expressed as follows in terms of the generators (13):

$$\begin{aligned}
\hat{C}_1[\text{U}(15)] &= \hat{N} = \hat{n}_s + \hat{n}_d + \hat{n}_g, \\
\hat{C}_2[\text{U}(15)] &= \hat{N}(\hat{N} + 14), \\
\hat{C}_1[\text{U}_d(5)] &= \hat{n}_d, \\
\hat{C}_2[\text{U}_d(5)] &= \hat{n}_d(\hat{n}_d + 4), \\
\hat{C}_2[\text{SO}_d(5)] &= 2 \sum_{\lambda \text{ odd}} [d^\dagger \times \tilde{d}]^{(\lambda)} \cdot [d^\dagger \times \tilde{d}]^{(\lambda)},
\end{aligned}$$

$$\begin{aligned}
\hat{C}_2[\text{SO}_d(3)] &= \hat{L}_d \cdot \hat{L}_d, \\
\hat{C}_1[\text{U}_{sg}(10)] &= \hat{n}_s + \hat{n}_g, \\
\hat{C}_2[\text{U}_{sg}(10)] &= (\hat{n}_s + \hat{n}_g)(\hat{n}_s + \hat{n}_g + 9), \\
\hat{C}_1[\text{U}_g(9)] &= \hat{n}_g, \\
\hat{C}_2[\text{U}_g(9)] &= \hat{n}_g(\hat{n}_g + 8), \\
\hat{C}_2[\text{SO}_{sg}(10)] &= [s^\dagger \times \tilde{g} + g^\dagger \times \tilde{s}]^{(4)} \cdot [s^\dagger \times \tilde{g} + g^\dagger \times \tilde{s}]^{(4)} + \hat{C}_2[\text{SO}_g(9)], \\
\hat{C}_2[\text{SO}_g(9)] &= 2 \sum_{\lambda \text{ odd}} [g^\dagger \times \tilde{g}]^{(\lambda)} \cdot [g^\dagger \times \tilde{g}]^{(\lambda)}, \\
\hat{C}_2[\text{SO}_g(3)] &= \hat{L}_g \cdot \hat{L}_g, \\
\hat{C}_2[\text{SO}(3)] &= \hat{L} \cdot \hat{L}.
\end{aligned} \tag{14}$$

The expressions for the quadratic Casimir operators of unitary algebras are not general but are valid in a symmetric irreducible representation. A rotationally invariant Hamiltonian with up to two-body interactions can be written in terms of the Casimir operators (14):

$$\begin{aligned}
\hat{H}_{\text{sym}} &= \epsilon_d \hat{n}_d + a_d \hat{C}_2[\text{U}_d(5)] + b_d \hat{C}_2[\text{SO}_d(5)] + c_d \hat{C}_2[\text{SO}_d(3)] \\
&\quad + \epsilon_s \hat{n}_s + \epsilon_g \hat{n}_g + a_{sg} \hat{C}_2[\text{U}_{sg}(10)] + a_g \hat{C}_2[\text{U}_g(9)] + b_{sg} \hat{P}_{sg}^\dagger \hat{P}_{sg} \\
&\quad + b_g \hat{C}_2[\text{SO}_g(9)] + c_g \hat{C}_2[\text{SO}_g(3)] + c \hat{C}_2[\text{SO}(3)],
\end{aligned} \tag{15}$$

where ϵ_ℓ , a_ℓ , $a_{\ell\ell'}$, b_ℓ , $b_{\ell\ell'}$ and c_ℓ are parameters. The quadratic Casimir operator of $\text{U}(15)$ is omitted for simplicity since it gives a constant contribution for a fixed boson number $N = n_s + n_d + n_g$. Furthermore, it is convenient to define, instead of the quadratic Casimir operator $\hat{C}_2[\text{SO}_{sg}(10)]$, the combination

$$\hat{P}_{sg}^\dagger \hat{P}_{sg} = \hat{C}_2[\text{U}_{sg}(10)] - \hat{C}_1[\text{U}_{sg}(10)] - \hat{C}_2[\text{SO}_{sg}(10)], \tag{16}$$

where $\hat{P}_{sg}^\dagger \equiv s^\dagger s^\dagger - g^\dagger \cdot g^\dagger$ is the pairing operator for s and g bosons. The symmetry Hamiltonian (15) is less general than Eq. (1) but it is the most general one that can be written in terms of invariant operators of the lattice (12) and as such it is intermediate between the limits I and II.

The $\text{U}_g(9)$ limit occurs for $b_{sg} = 0$, leading to the eigenvalues

$$\begin{aligned}
E_I &= \epsilon_d n_d + a_d n_d(n_d + 4) + b_d v_d(v_d + 3) + c_d L_d(L_d + 1) \\
&\quad + \epsilon_s n_s + \epsilon_g n_g + a_{sg} n_{sg}(n_{sg} + 9) + a_g n_g(n_g + 8) + \\
&\quad + b_g v_g(v_g + 7) + c_g L_g(L_g + 1) + c L(L + 1).
\end{aligned} \tag{17}$$

The $\text{SO}_{sg}(10)$ limit is attained for $\epsilon_s = \epsilon_g \equiv \epsilon_{sg}$ and $a_g = 0$, in which case the Hamiltonian's eigenstates have the eigenvalues

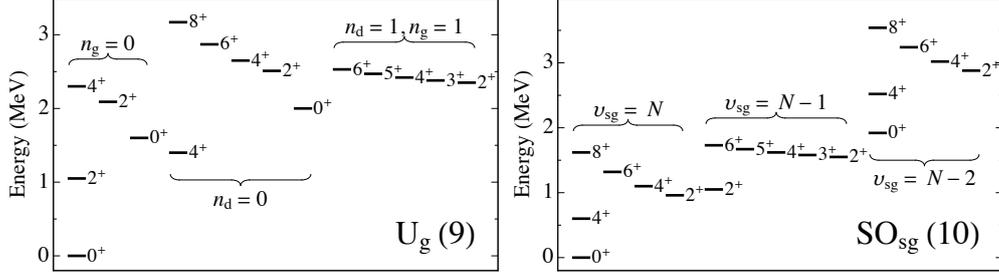


Fig. 1. Energy spectra in the $U_g(9)$ and $SO_{sg}(10)$ limits of the sdg -IBM for $N = 5$ bosons. For the $U_g(9)$ spectrum the non-zero parameters in the Hamiltonian (15) are $\epsilon_d - \epsilon_s = 800$, $\epsilon_g - \epsilon_s = 1000$, $b_d = 40$, $c_d = 10$, $b_g = 25$ and $c_g = c = 5$ keV. For the $SO_{sg}(10)$ spectrum the non-zero parameters are $\epsilon_d - \epsilon_s = 800$, $\epsilon_g - \epsilon_s = 0$, $b_d = 40$, $c_d = 10$, $b_{sg} = 60$, $b_g = 50$ and $c_g = c = 5$ keV.

$$\begin{aligned}
E_{II} = & \epsilon_d n_d + a_d n_d(n_d + 4) + b_d v_d(v_d + 3) + c_d L_d(L_d + 1) \\
& + \epsilon_{sg} n_{sg} + a_{sg} n_{sg}(n_{sg} + 9) + b_{sg}[n_{sg}(n_{sg} + 8) - v_{sg}(v_{sg} + 8)] \\
& + b_g v_g(v_g + 7) + c_g L_g(L_g + 1) + c L(L + 1).
\end{aligned} \tag{18}$$

The eigenspectra are then determined with the help of the necessary branching rules. The reduction $U(15) \supset U_d(5) \otimes U_{sg}(10)$ implies the relation $N = n_d + n_{sg}$ or the branching rule

$$[N] \mapsto (n_d, n_{sg}) = (0, N), (1, N - 1), \dots, (N, 0). \tag{19}$$

The branching rules for the classification (9) are known from the $U(5)$ limit of the sd -IBM [11] and those for the classifications (10) and (11) can be found in Ref. [17].

Typical energy spectra in the $U_g(9)$ and $SO_{sg}(10)$ limits are shown in Fig. 1. The $U_g(9)$ spectrum displays quadrupole- and hexadecapole-phonon multiplets characterized by a fixed number of d and g bosons. The multiplets are further structured by a seniority quantum number: for example, the $n_d = 2$ multiplet has $v_d = 2$ except for the 0^+ level, which has $v_d = 0$, and similarly for the $n_g = 2$ multiplet and the v_g seniority. Also combined quadrupole-hexadecapole multiplets occur in the spectrum. The $SO_{sg}(10)$ spectrum contains sets of levels with $v_{sg} = N, N - 2, \dots$ (for $n_d = 0$) with $v_{sg} = N - 1, N - 3, \dots$ (for $n_d = 1$) etc., and $v_{sg} = N$ levels are lowest in energy due the repulsive sg -pairing. Multiplets characterized by the seniority quantum number $v_g = 0, 1, \dots$ occur within each $SO_{sg}(10)$ multiplet. Note that in this limit, unless b_{sg} is small, the first-excited state has $J^\pi = 4^+$. This is a consequence of the unrealistic condition that the energies of the s and g bosons are degenerate, $\epsilon_s = \epsilon_g$.

4 Classical limit of the symmetry Hamiltonian

The classical limit of the symmetry Hamiltonian (15) can be obtained with the general procedure outlined in I. First, a conversion to the standard representation (1) is carried out. A quadratic Casimir operator $\hat{C}_2(G)$ is given generically as an expansion over the generators,

$$\hat{C}_2(G) = \sum_{\lambda r} a_{\lambda r} \left(\sum_{\ell_1 \ell_2} \alpha_{\ell_1 \ell_2}^{\lambda r} (b_{\ell_1}^\dagger \times \tilde{b}_{\ell_2})^{(\lambda)} \right) \cdot \left(\sum_{\ell'_1 \ell'_2} \alpha_{\ell'_1 \ell'_2}^{\lambda r} (b_{\ell'_1}^\dagger \times \tilde{b}_{\ell'_2})^{(\lambda)} \right), \quad (20)$$

with coefficients $\alpha_{\ell_1 \ell_2}^{\lambda r}$ and $a_{\lambda r}$ that are specific to each algebra G , as given in Eq. (14). The overall sum in Eq. (20) is over the multipolarity λ of the generators and over an additional index r to distinguish different generators with the same λ . With use of the expansion (20) one finds the following expression for the matrix element of $\hat{C}_2(G)$ between two-boson states [19]:

$$\begin{aligned} & \langle \ell_1 \ell_2; L | \hat{C}_2(G) | \ell'_1 \ell'_2; L \rangle \\ &= [f_{\ell_1}(G) + f_{\ell_2}(G)] \delta_{\ell_1 \ell_3} \delta_{\ell_2 \ell_4} + \frac{2(-)^{\ell_2 + \ell_3}}{\sqrt{(1 + \delta_{\ell_1 \ell_2})(1 + \delta_{\ell_3 \ell_4})}} \sum_{\lambda r} a_{\lambda r} (2\lambda + 1) \\ & \quad \times \left[\alpha_{\ell_1 \ell_4}^{\lambda r} \alpha_{\ell_2 \ell_3}^{\lambda r} \left\{ \begin{matrix} \ell_1 & \ell_4 & \lambda \\ \ell_3 & \ell_2 & L \end{matrix} \right\} + (-)^L \alpha_{\ell_1 \ell_3}^{\lambda r} \alpha_{\ell_2 \ell_4}^{\lambda r} \left\{ \begin{matrix} \ell_1 & \ell_3 & \lambda \\ \ell_4 & \ell_2 & L \end{matrix} \right\} \right], \end{aligned} \quad (21)$$

where the symbol between curly brackets is a Racah coefficient [20]. The quantity $f_\ell(G)$ is the expectation value of the operator $\hat{C}_2(G)$ between single-boson states,

$$f_\ell(G) \equiv \langle \ell | \hat{C}_2(G) | \ell \rangle = \sum_{\lambda r} \frac{2\lambda + 1}{2\ell + 1} a_{\lambda r} \sum_{\ell'} (-)^{\ell + \ell'} \alpha_{\ell \ell'}^{\lambda r} \alpha_{\ell' \ell}^{\lambda r}. \quad (22)$$

The Hamiltonian (15) can, with use of Eqs. (21) and (22), be converted into its standard representation on which the classical-limit expression (4) can be applied. The procedure results in the energy surface

$$E(\beta_2, \beta_4) = \frac{N(N-1)}{(1 + \beta_2^2 + \beta_4^2)^2} \sum_{kl} c'_{kl} \beta_2^k \beta_4^l, \quad (23)$$

where the non-zero coefficients c'_{kl} are

$$\begin{aligned} c'_{00} &= a_{sg} + b_{sg} + \Gamma_s, & c'_{20} &= \Gamma_s + \Gamma_d, & c'_{02} &= 2a_{sg} - 2b_{sg} + \Gamma_s + \Gamma_g, \\ c'_{40} &= a_d + \Gamma_d, & c'_{22} &= \Gamma_d + \Gamma_g, & c'_{04} &= a_{sg} + b_{sg} + a_g + \Gamma_g, \end{aligned} \quad (24)$$

in terms of the combinations

$$\begin{aligned}
\Gamma_s &\equiv \frac{1}{N-1}(\epsilon_s + 10a_{sg}), \\
\Gamma_d &\equiv \frac{1}{N-1}(\epsilon_d + 5a_d + 4b_d + 6c_d + 6c), \\
\Gamma_g &\equiv \frac{1}{N-1}(\epsilon_g + 10a_{sg} + 9a_g + 8b_g + 20c_g + 20c).
\end{aligned} \tag{25}$$

All coefficients c_{kl}^{ij} of Eq. (4) vanish identically in the classical limit of the symmetry Hamiltonian (15).

5 Octahedral shapes

From the outset it should be clear that energy surface (23) cannot have an isolated minimum with octahedral shape since it is independent of γ_2 , γ_4 and δ_4 . What can still happen, however, is the occurrence of a minimum with zero quadrupole and non-zero hexadecapole deformation ($\beta_2^* = 0$ and $\beta_4^* \neq 0$), which, given the instability in γ_4 and δ_4 , *includes* a shape with octahedral symmetry. Therefore, the goal of this section is to establish the conditions on the parameters in the symmetry Hamiltonian (15) such that its classical limit displays a minimum with $\beta_2^* = 0$ and $\beta_4^* \neq 0$, and, subsequently, to identify the interactions in the general Hamiltonian (1) that generate a dependence on δ_4 , enabling the formation of an isolated minimum with octahedral shape. This problem can be investigated with the procedure outlined in I.

According to the analysis of the previous section, the classical energy (23) is a two-variable function $E(\beta_2, \beta_4)$. The conditions for this energy surface to have an extremum are

$$\left. \frac{\partial E}{\partial \beta_2} \right|_{p^*} = \left. \frac{\partial E}{\partial \beta_4} \right|_{p^*} = 0, \tag{26}$$

where $p^* \equiv (\beta_2^*, \beta_4^*)$ is a short-hand notation for an arbitrary critical point. Furthermore, a critical point at an extremum with $\beta_2^* = 0$ and $\beta_4^* \neq 0$ shall be denoted as h^* . The condition (26) in β_2 is identically satisfied for $p^* = h^*$ and does not lead to any constraints on the coefficients c'_{kl} . The condition in β_4 leads to a cubic equation with the solutions

$$\beta_4^* = 0, \quad \beta_4^* = \pm \sqrt{\frac{2c'_{00} - c'_{02}}{2c'_{04} - c'_{02}}}. \tag{27}$$

Only the last solution corresponds to an extremum h^* and implies the following condition on the ratio of coefficients:

$$\frac{2c'_{00} - c'_{02}}{2c'_{04} - c'_{02}} > 0. \quad (28)$$

While the condition (28) is necessary and sufficient to have an *extremum* at $\beta_2^* = 0$ and $\beta_4^* \neq 0$, a *minimum* at these values implies further constraints. They are obtained by requiring that the eigenvalues of the stability matrix [*i.e.*, the partial derivatives of $E(\beta_2, \beta_4)$ of second order] are all positive. Since the off-diagonal element of the stability matrix vanishes for the energy surface (23), the existence of a minimum follows from the uncoupled conditions

$$\left. \frac{\partial^2 E}{\partial \beta_2^2} \right|_{h^*} > 0, \quad \left. \frac{\partial^2 E}{\partial \beta_4^2} \right|_{h^*} > 0, \quad (29)$$

or, in terms of the coefficients c'_{kl} in Eq. (23),

$$\begin{aligned} \frac{(2c'_{04} - c'_{02})[2c'_{00}(c'_{22} - 2c'_{04}) + c'_{02}(c'_{02} - c'_{20} - c'_{22}) + 2c'_{04}c'_{20}]}{(c'_{00} - c'_{02} + c'_{04})^2} > 0, \\ \frac{(2c'_{00} - c'_{02})(2c'_{04} - c'_{02})^3}{(c'_{00} - c'_{02} + c'_{04})^3} > 0. \end{aligned} \quad (30)$$

If we write Eq. (28) as $A/B > 0$, the second inequality in Eq. (30) becomes $AB^3/(A+B)^3 > 0$ and therefore both A and B should be positive, $A \equiv 2c'_{00} - c'_{02} > 0$ and $B \equiv 2c'_{04} - c'_{02} > 0$, leading to the constraint

$$-4b_{sg} - 2a_g < \Gamma_g - \Gamma_s < 4b_{sg}. \quad (31)$$

The first inequality in Eq. (30) can be reduced to

$$2b_{sg}(2\Gamma_d - \Gamma_s - \Gamma_g - 4a_{sg} - a_g) + a_g(\Gamma_d - \Gamma_s - 2a_{sg}) > 0. \quad (32)$$

The conditions (31) and (32) are necessary and sufficient for the energy surface $E(\beta_2, \beta_4)$ to have a minimum at zero quadrupole and non-zero hexadecapole deformation. To obtain an intuitive understanding of them, we note that $\Gamma_g - \Gamma_s$, for a reasonable choice of parameters, is positive. The upper part of the inequality (31) therefore expresses the need for b_{sg} to be positive and sufficiently large, corresponding to a repulsive sg -pairing interaction that puts the configuration with *maximal* sg seniority $v_{sg} = n_{sg}$ at lowest energy. For $b_{sg} > 0$ and $a_g > 0$, the lower part of the inequality (31) is automatically satisfied. The condition (32) is easier to appreciate if it is assumed that the coefficients in front of the quadratic Casimir operators of the unitary algebras

$U_d(5)$, $U_g(9)$ and $U_{sg}(10)$ vanish, $a_d = a_{sg} = a_g = 0$. This assumption is justified if anharmonicities are neglected in the various limits. Given that $b_{sg} > 0$, it then follows that

$$2\Gamma_d - \Gamma_s - \Gamma_g > 0. \quad (33)$$

In terms of the original parameters in the symmetry Hamiltonian (15) (assuming $a_d = a_{sg} = a_g = 0$) the conditions to have a minimum at $\beta_2^* = 0$ and $\beta_4^* \neq 0$ can be summarized as

$$\begin{aligned} -4(N-1)b_{sg} < \epsilon_g - \epsilon_s + 8b_g + 20(c_g + c) < 4(N-1)b_{sg}, \\ 2\epsilon_d - \epsilon_s - \epsilon_g + 8(b_d - b_g) + 4(3c_d - 5c_g - 2c) > 0. \end{aligned} \quad (34)$$

We now ask the question whether two-body interactions can be added to the symmetry Hamiltonian (15), which lift the (γ_4, δ_4) instability and create a minimum at $\delta_4 = 0$, $\delta_4 = \arccos(1/6)$ or $\delta_4 = \pi$. To achieve this goal, we recall the result from I that for a general Hamiltonian of the *sdg*-IBM an extremum with $\beta_2^* = 0$ and $\beta_4^* \neq 0$ occurs for

$$\beta_4^* = \pm \sqrt{\frac{2c'_{00} - c'_{02}}{2c'_{04} - c'_{02} + 38c'_{04}{}^{00}}}. \quad (35)$$

The term in $c'_{04}{}^{00}$ introduces a dependence in δ_4 that may lead to an isolated minimum with octahedral symmetry. Given the expression (6) for $c'_{04}{}^{00}$, this argument suggests adding g -boson interactions v_{gggg}^L to the Hamiltonian (15). The classical energy (23) then becomes a three-variable function $E(\beta_2, \beta_4, \delta_4)$, for which the above catastrophe analysis can be repeated. With these additional two-body interactions the extremum and stability conditions (26) and (29) become

$$\begin{aligned} 2c'_{00} - c'_{02} > 0, \quad 2c'_{04} - c'_{02} + 38c'_{04}{}^{00} > 0, \quad c'_{04}{}^{00} < 0, \\ 2c'_{00}(c'_{22} - 2c'_{04}) + c'_{02}(c'_{02} - c'_{20} - c'_{22}) + 2c'_{04}c'_{20} - 38c'_{04}{}^{00}(2c'_{00} - c'_{20}) > 0. \end{aligned} \quad (36)$$

For $a_d = a_{sg} = a_g = 0$, these conditions imply the inequalities

$$\begin{aligned} -2b_{sg} - \tilde{v} < \Gamma_g - \Gamma_s < 4b_{sg}, \\ \bar{v} \equiv 65v_{gggg}^2 - 30v_{gggg}^4 - 91v_{gggg}^6 + 56v_{gggg}^8 > 0, \\ 4b_{sg}(2\Gamma_d - \Gamma_s - \Gamma_g) + (\Gamma_d - \Gamma_s - 2b_{sg})(\tilde{v} - 2b_{sg}) > 0, \end{aligned} \quad (37)$$

where \tilde{v} is the following linear combination of g -boson interaction matrix ele-

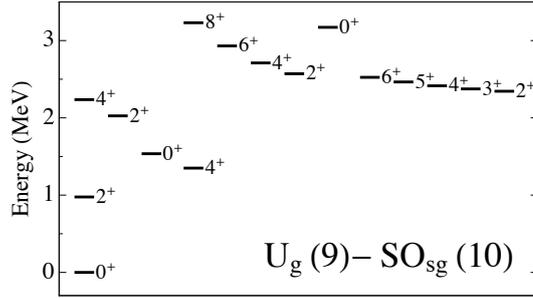


Fig. 2. Energy spectrum of a $U_g(9)$ - $SO_{sg}(10)$ transitional Hamiltonian of the sdg -IBM for $N = 5$ bosons. The non-zero parameters of the Hamiltonian (15) are $\epsilon_d - \epsilon_s = 1200$, $\epsilon_g - \epsilon_s = 1500$, $b_d = 40$, $c_d = 10$, $b_{sg} = 150$, $b_g = 25$ and $c_g = c = 5$ keV.

ments:

$$\tilde{v} = \frac{1}{9}v_{gggg}^0 + \frac{98}{429}v_{gggg}^4 + \frac{40}{99}v_{gggg}^6 + \frac{10}{39}v_{gggg}^8. \quad (38)$$

For the Hamiltonian (15) the linear combination \bar{v} vanishes identically and the second inequality in Eq. (37) is not fulfilled. This expresses the δ_4 independence of the symmetry Hamiltonian and the fact that its classical limit does not acquire an isolated minimum with octahedral shape. Furthermore, for the symmetry Hamiltonian one has $\tilde{v} = 2b_{sg}$ and the conditions (37) reduce to Eq. (34).

There are clearly many ways to find matrix elements v_{gggg}^L that satisfy all conditions (37) but one way is particularly simple. Note that the quadrupole matrix element v_{gggg}^2 does not appear in the combination \tilde{v} . By making this matrix element more repulsive, the second inequality in Eq. (37) is satisfied while the other two conditions are not modified with respect those in Eq. (34) valid for the symmetry Hamiltonian (15). A possible procedure to construct an sdg -IBM Hamiltonian whose classical energy displays a minimum with octahedral shape is therefore to add to a hexadecapole-deformed symmetry Hamiltonian (15) a repulsive v_{gggg}^2 interaction.

Let us illustrate this procedure with an example. The starting point is a $U_g(9)$ - $SO_{sg}(10)$ transitional Hamiltonian associated with the lattice (12), giving rise to the spectrum shown in Fig. 2. Note that the choice of the single-boson energies ϵ_ℓ for this figure is realistic in the sense that the g -boson energy is higher than that of the d boson. The sg -pairing strength b_{sg} , of which little is known either empirically or microscopically, is chosen such that a hexadecapole-deformed minimum occurs in the classical limit. Other parameters in the Hamiltonian (15) are of lesser importance and are chosen as to lift degeneracies in the spectrum. Note that with this choice of parameters the resulting spectrum, as shown in Fig. 2, is rather closer to the $U_g(9)$ and than to the $SO_{sg}(10)$ limit.

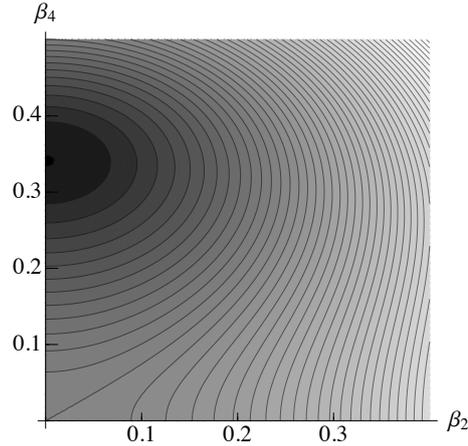


Fig. 3. The energy surface $E(\beta_2, \beta_4)$ obtained in the classical limit of a $U_g(9)$ - $SO_{sg}(10)$ transitional Hamiltonian of the sdg -IBM. Parameters of the Hamiltonian (15) are given in the caption of Fig. 2. Black corresponds to low energies and the lines indicate changes by 10 keV.

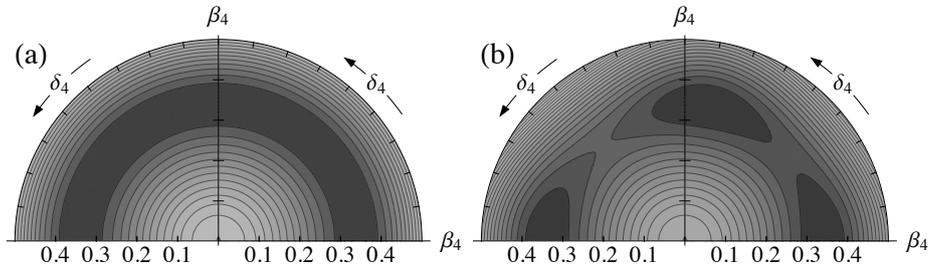


Fig. 4. Energy surfaces $E(\beta_4, \delta_4)$ obtained in the classical limit of two different Hamiltonians of the sdg -IBM for $N = 5$ bosons. The dependence on $\beta_4 > 0$ and $0 \leq \delta_4 \leq \pi$ is shown for $\beta_2^* = 0$. Black corresponds to low energies and the lines indicate changes by 10 keV. (a) The $U_g(9)$ - $SO_{sg}(10)$ transitional Hamiltonian is taken with the parameters given in the caption of Fig. 2. (b) The Hamiltonian of (a) is modified by taking a repulsive interaction $v_{gggg}^2 = 500$ keV.

The parameters quoted in the caption of Fig. 2 satisfy both conditions (34). As a result, the energy surface in the classical limit of the corresponding Hamiltonian has a minimum for $\beta_2^* = 0$ and $\beta_4^* \approx 0.34$, as shown in Fig. 3. According to the preceding discussion, the surface is independent of δ_4 , which is indeed confirmed by Fig. 4(a). If the v_{gggg}^2 matrix element is modified, a dependence in δ_4 is introduced, as illustrated in Fig. 4(b) for the value $v_{gggg}^2 = 500$ keV. It is seen that the energy surface displays *three* isolated minima that are exactly degenerate. The three minima all have an octahedral symmetry, corresponding to either an octahedron [$\delta_4^* = 0$ and $\delta_4^* = \arccos(1/6) \approx 84.4^\circ$] or a cube ($\delta_4^* = \pi$).

Although this analysis shows that isolated minima with octahedral symmetry can be obtained in the classical limit of an sdg -IBM Hamiltonian with reasonable parameters, it can be expected that such minima are rather shallow.

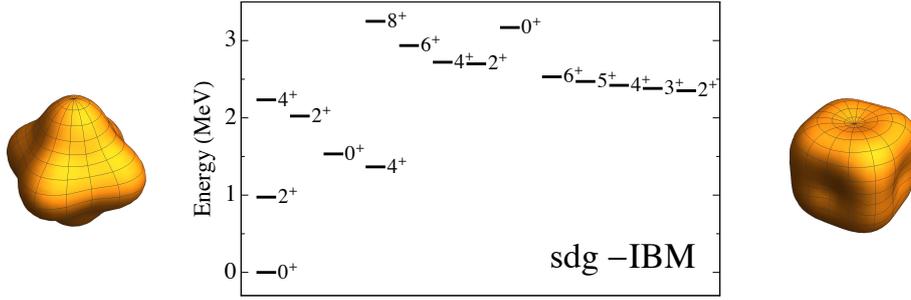


Fig. 5. Energy spectrum of a general Hamiltonian of the *sdg*-IBM for $N = 5$ bosons. The same Hamiltonian is taken as in Fig. 2 but one g -boson two-body matrix element is modified to $v_{gggg}^2 = 500$ keV. On the left- and right-hand sides are shown the shapes at the minima in the energy surface obtained in the classical limit of this Hamiltonian. They have octahedral symmetry and correspond to either an octahedron or a cube.

Even for the fairly large value of the interaction matrix element in the above example, $v_{gggg}^2 = 500$ keV, the three minima are separated by a barrier of ~ 20 keV, inducing only very weak observable effects. This point is illustrated with Fig. 5, which shows the spectrum of the $U_g(9)$ - $SO_{sg}(10)$ transitional Hamiltonian with the modified v_{gggg}^2 matrix element. Except for some minute changes the spectrum is essentially the same as that shown in Fig. 2.

One subtle point made clear by the current study is that it is not sufficient to carry out a catastrophe analysis of the generic surface (4) obtained in the classical limit of the most general *sdg*-IBM Hamiltonian (1) with up to two-body interactions between the bosons. The coefficients c'_{kl} and c_{kl}^{ij} cannot be treated as free parameters but their expressions in terms of the single-bosons energies and boson-boson interactions are an essential part of the analysis. To illustrate this point consider the energy surface shown in Fig. 4(b). The minima at $\delta_4^* = 0$ and $\delta_4^* \approx 84.4^\circ$ correspond to the same intrinsic shape (an octahedron, shown on the left-hand side of Fig. 5) and, as a consequence, the minima *must* be exactly degenerate. This behavior is generally valid. Therefore, whatever single-boson energies and boson-boson interactions one adopts in the Hamiltonian (1), the energy surface in its classical limit must satisfy the constraint that points on the surface with same intrinsic shape (e.g., $\delta_4^* = 0$ and $\delta_4^* \approx 84.4^\circ$) are at the same energy.

6 Conclusions

The main conclusion of this paper is that no *sdg*-IBM Hamiltonian with a dynamical symmetry that includes Casimir operators of up to second order displays in its classical limit an isolated minimum with octahedral shape. Nevertheless, a degenerate minimum that *includes* a shape with octahedral

symmetry can be obtained from a Hamiltonian transitional between two limits. In the limits in question the d boson is decoupled from s and g bosons. Furthermore, limit I, $U_g(9)$, has hexadecapole vibrational characteristics while in limit II, $SO_{sg}(10)$, s - and g -boson states are mixed through an sg -pairing interaction. A catastrophe analysis of the energy surface obtained in the classical limit of this transitional symmetry Hamiltonian indicates that a minimum with zero quadrupole and non-zero hexadecapole deformation can be obtained with reasonable parameters. However, this minimum is always δ_4 independent, meaning that it ranges from an octahedron to a cube and includes intermediate shapes without octahedral symmetry. Isolated minima with octahedral symmetry can be obtained by adding two-body interactions between the g bosons to the transitional symmetry Hamiltonian. The resulting energy surface displays in this case minima with octahedral symmetry, with the shape of either an octahedron or a cube, separated by a barrier with low energy even for fairly strong interactions between the g bosons. The conclusion of this analysis in the context of the sdg -IBM is therefore that it will be difficult to find experimental manifestations of octahedral symmetry in nuclei.

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