Non-contact rheology of finite-size air-water interfaces – Supplementary Material –

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GENERAL MODEL

The modulus $|Z^*|$ of the amplitude Z^* of the vertical displacement Z of the sphere is assumed to be much smaller than the average sphere-bubble distance D, so that we can neglect the contribution of Z in the non-linear ($\propto h^3$) factor within the Reynolds equation (see Eq. (4) in the main text). The complex version of the latter becomes:

$$12\eta r i\omega\left[Z^*+u^*(r)\right]=\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(D+\frac{r^2}{2R_{\mathrm{eff}}}\right)^3\frac{\mathrm{d}p^*(r)}{\mathrm{d}r}\right]\,, \tag{S1}$$

with identical notations as in the main text. The excess pressure and deformation fields are set to zero above a given cut-off radius b, *i.e.*:

$$p^*(r)=0\,,\ u^*(r)=0\,,\qquad r>b\,. \tag{S2}$$

We then introduce the discrete Hankel transform. The amplitudes of the excess pressure and deformation fields are expanded in terms of Fourier-Bessel series [1]. For $r\in[0,b]$, one thus has:

$$p^*(r)=\sum_{k=1}^{\infty}p_k J_0(\phi_k r)\,,\qquad u^*(r)=\sum_{k=1}^{\infty}u_k J_0(\phi_k r)\,, \tag{S3}$$

where J_0 is the Bessel function of the first kind with index 0, and $b\,\phi_k$ the kth root of J_0. The series coefficients are related to the amplitude of the deformation field via the inversion relation:

$$u_k=\frac{2}{b^2J_1^2(\phi_k b)}\int_0^b\mathrm{d}r\,r\,u^*(r)J_0(\phi_k r)\,, \tag{S4}$$

where J_1 is the Bessel function of the first kind with index 1. We further introduce the dimensionless variables:

$$x=\frac{r}{\sqrt{2R_{\mathrm{eff}}D}}\,,\qquad U^*=u^*/Z^*\,,\qquad P^*=\frac{p^*}{\frac{\eta R_{\mathrm{eff}}Z^*\omega}{D^2}}\,,\qquad B=\frac{b}{\sqrt{2R_{\mathrm{eff}}D}}\,,\qquad \psi_k=\phi_k\sqrt{2R_{\mathrm{eff}}D}\,, \tag{S5}$$

so that Eq. (S1) and the linearized Young-Laplace equation (Eq. (5) in the main text) respectively become:

$$24ix\left(1+U^*\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left[x\left(1+x^2\right)^3\frac{\mathrm{d}P^*}{\mathrm{d}x}\right]\,, \tag{S6}$$

and:

$$\frac{1}{8}\left(\frac{D_{\mathrm{c}}}{D}\right)P^*=\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}U^*}{\mathrm{d}x}\right)\,, \tag{S7}$$

where $D_{\mathrm{c}}=16R_{\mathrm{eff}}^2\eta\omega/\gamma$ is a typical viscocapillary distance that emerges from the rescaling. By integrating Eq. (S6) with respect to x, and invoking the Fourier-Bessel decomposition, we find:

$$-x(1+x^2)^3\sum_{k=1}^{\infty}\psi_k P_k J_1(\psi_k x)=12ix^2+24i\sum_{k=1}^{\infty}U_k\frac{J_1(\psi_k x)}{\psi_k}x\,, \tag{S8}$$

where $U_k = u_k/Z^*$ and $P_k = p_k D^2/(\eta R_{\rm eff} Z^* \omega)$. Using Eq. (S7), one can show that:

$$\psi_k^2\, U_k = -\frac{1}{8}\left(\frac{D_{\rm c}}{D}\right) P_k \ . \tag{S9}$$

Injecting the last relation in Eq. (S8), one gets:

$$-\sum_{l=1}^{\infty} \psi_l P_l J_1(\psi_l x) = \frac{12ix}{(1+x^2)^3} + \frac{3i}{(1+x^2)^3}\left(\frac{D_{\rm c}}{D}\right) \sum_{l=1}^{\infty} P_l \frac{J_1(\psi_l x)}{\psi_l^3} \ . \tag{S10}$$

Then, we perform the Fourier-Bessel inversion by applying to Eq. (S10) the operator $\frac{2}{B^2 J_1^2(\psi_k B)} \int_0^B {\rm d}y\, y J_1(\psi_k y)(*)$, where $(*)$ stands for a function of y, and we find:

$$P_k = A_k + \sum_{l=1}^{\infty} M_{k,l} P_l \ , \quad \forall k \in \mathbb{N}^* \ , \tag{S11}$$

where A_k and $M_{k,l}$ are defined as:

$$A_k = -12i \frac{2}{B^2 J_1^2(\psi_k B)} \int_0^B {\rm d}y\, \frac{y^2}{(1+y^2)^3} \frac{J_1(\psi_k y)}{\psi_k} \ , \tag{S12}$$

and:

$$M_{k,l} = -\,3i\left(\frac{D_{\rm c}}{D}\right) \frac{2}{B^2 J_1^2(\psi_k B)} \int_0^B {\rm d}y\, \frac{y}{(1+y^2)^3} \frac{J_1(\psi_k y)}{\psi_k} \frac{J_1(\psi_l y)}{\psi_l^3} \ . \tag{S13}$$

Finally, the amplitude of the hydrodynamic force is $F_{\rm h}^* = 2\pi \int_0^b {\rm d}r\, r\, p^*(r)$. The mechanical impedance $G^* = -F^*/Z^*$ can thus be expressed as [2]:

$$\begin{aligned} G^* &= -\frac{4\pi\eta\omega R_{\rm eff}^2}{D} \int_0^B x\, P^*(x)\, {\rm d}x \\ &= -\frac{4\pi\eta\omega R_{\rm eff}^2}{D} \sum_{k=1}^{\infty} P_k \frac{B J_1(\psi_k B)}{\psi_k} \ . \end{aligned} \tag{S14}$$

The mechanical impedance can be estimated numerically from this last equation, restricting the summation to *e.g.* $k \leq 1000$. We checked the robustness of the results against a change of that arbitrary upper bound.

LARGE-DISTANCE ASYMPTOTIC MODEL

We assume that the ratio $D_{\rm c}/D$ is a small parameter, and we expend the amplitude of the dimensionless excess pressure field as:

$$P^*(x) \simeq P_0^*(x) + \frac{D_{\rm c}}{D} P_1^*(x) \ . \tag{S15}$$

The leading-order term P_0^* is given by Eq. (S6) in the undeformed-bubble limit, and reads:

$$P_0^*(x) = -\frac{3i}{(1+x^2)^2} \ . \tag{S16}$$

Similarly, the leading-order amplitude U_1^* of the dimensionless deformation field can be found by inserting P_0^* in Eq. (S7), and by invoking the boundary condition $U_1^*(x=B) = 0$. We find:

$$U_1^*(x) = -\frac{3i}{32} \frac{D_{\rm c}}{D} \left[\log(1+B^2) - \log(1+x^2) \right] . \tag{S17}$$

Then, introducing U_1^* in Eq. (S6) allows us to express the next-order correction in pressure:

$$P_1^*(x) = \frac{3}{32}\left[-3\frac{\log(1+B^2)}{(1+x^2)^2} - \pi^2 + \frac{3}{(1+x^2)^2} + \frac{3}{(1+x^2)} + 2\frac{\log(1+x^2)}{1+x^2} - \log^2(1+x^2) - 2\mathrm{Li}_2(-x^2)\right], \tag{S18}$$

where Li_2 denotes the dilogarithm function [3]. After some algebra, we find that:

$$\mathcal{G}' = \mathcal{R}[\mathcal{G}^*] \simeq \frac{3}{32}\left(\frac{D_\mathrm{c}}{D}\right)^2\left[-1 + \log(1+B^2)\right], \tag{S19}$$

which is the dimensionless equivalent of Eq. (6) in the main text.

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