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Decomposition of interaction indices: alternative interpretations of cardinal-probabilistic interaction indices*

Sébastien Courtin[†], Rodrigue Tido Takeng[‡], Frédéric Chantreuil[§]

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Abstract

In cooperative game theory, the concept of interaction index is an extension of the concept of value, considering interaction among players. In this paper we focus on cardinal-probabilistic interaction indices which are generalizations of the class of semivalues. We provide two types of decompositions. With the first one, a cardinal-probabilistic interaction index for a given coalition equals the difference between its external interaction index (or co-Möbius transform) and a weighted sum of the individual impact of the remaining players on the interaction index of the considered coalition. The second decomposition, based on the notion of the "decomposer", splits an interaction index into a direct part, the decomposer, which measures the interaction in the coalition considered, and an indirect part, which indicates how all remaining players individually affect the interaction of the coalition considered. We propose alternative characterization of the cardinal-probabilistic interaction indices. We then propose an illustration with a well-known example in Multicriteria Aid for Decisions.

Keywords: Game theory, Multicriteria Aid for Decisions, Cardinal-probabilistic interaction indices, External interaction index

JEL Codes: C71; D60; C44.

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1 Introduction

A large literature on cooperative games with transferable utility (TU-game), described as a situation within which players can obtain a pay-off by cooperating, is devoted to the definition and the characterization of a solution. A solution measures the player's importance in the game, *i.e.* the average added value that a player brings to all possible coalitions. The most well-known solutions are those of Shapley (Shapley [1953]) and Banzhaf (Banzhaf [1965]). As pointed out by Grabisch and Roubens [1999], a solution does not exhibit any information on the existing cooperation between players or their interactions. Consider two players i and j , and a value function v which assigns, to every subset of players in a game, a real number. We investigate the sign of the relation $v(ij) - (v(i) + v(j))$. If the sign is positive, it is natural to say that players i and j interact positively because considering the pair of players (ij) leads us to a situation within which the value of (ij) is more important. In a symmetric way, if the sign is negative, it seems reasonable to say that players i and j interact negatively because considering the pair of attributes (ij) leads us to a situation within which the value is lower. If the difference equals zero, one can assume that there is no interaction.

However, the presence of a third or fourth player should potentially modify the interaction between players i and j . Therefore, one needs to consider a more elaborate definition, one that takes into account what happens when i and j are in the presence of T , where T is a subset of players not containing players i and j . According to Kojadinovic [2005], Owen [1972] was the first to define a measure that evaluates the interaction between two players in the presence of T .

Grabisch [1997] suggested going even further by considering higher-order interactions, that is interactions between s players, with $s > 2$. He introduced the Shapley interaction index, which generalizes the Shapley solution. The notion of higher-order interactions has also been considered by Marichal and Roubens [1999] and Roubens [1996] who proposed respectively the Chaining interaction index and the Banzhaf interaction index. The latter is an extension of the Banzhaf solution, while the former is another extension of the Shapley solution. These three interaction indices belong to the class of cardinal-probabilistic interaction indices introduced by Fujimoto et al. [2006]. Cardinal-probabilistic interaction indices generalize the concept of cardinal-probabilistic values also known as semivalues (see Dubey et al. [1981]). In this paper, we present alternative interpretations of the cardinal-probabilistic

interaction indices. In order to do that, two disjoint decomposition approaches are put forward.

The first one splits a cardinal-probabilistic interaction index for a given coalition S into two parts: the first part is the external interaction index for this coalition S ; the second part is a weighted sum of the individual impact of the remaining players on the interaction index of the considered coalition. The external interaction index also called co-Möbius transform (Fujimoto et al. [2006]) is a higher-order extension of the marginal contribution of a player to the grand coalition. The second decomposition, based on Casajus and Huettner [2018], decomposes interaction indices into a direct part and an indirect part: the direct part measures the interaction of a given coalition in the game, while the indirect part indicates how the remaining players individually affect the interaction index of the coalition considered.

These two decompositions will permit us to present a new characterization of the class of cardinal-probabilistic interaction indices. The rest of the paper is a numerical illustration of the proposed methodology, using hypotheticalal data.

The paper is structured as follows: In Section 2, we present the first decomposition approach based on the external interaction index. Section 3 introduces the Casajus and Huettner decomposition approach and presents a new characterization of the well-known interaction indices. Section 4 is an illustration with Multicriteria Aid for Decisions (MCDA) and Section 5 concludes.

2 External interaction decomposition approach

2.1 Preliminaries

Let (N, v) be a game with transferable utility (TU game), where $N = \{1, 2, \dots, n\}$ is the set of players and let v be a characteristic function which assigns a real number $v(S)$ to every coalition $S \subseteq N$. We assume that $v(\emptyset) = 0$. The cardinality of sets S, T, \dots will be denoted by corresponding lower cases s, t, \dots . Following standard practice, we often refer to the game v instead of the game (N, v) . We denote by $TU(N)$ the set of cooperative game with transferable utility on N and denote by 2^N the set of non-empty coalitions of the players of N . We denote by $\mathcal{P}(N) = \{T \subseteq N\}$, the set of coalitions of the players of N .

In order to avoid heavy notation, we adopt the following conventions: we will omit

braces for singletons, e.g., by writing $v(i)$, $u \setminus i$ instead of $v(\{i\})$, $u \setminus \{i\}$. Similarly, for pairs, we will write ij instead of $\{i, j\}$.

A game $v \in TU(N)$ is monotonic if $v(S) \leq v(T)$ whenever $S \subset T$. A permutation on the set of players N is a bijective map π from N into itself. The pair $(N, \pi v)$ is also a TU game where $\pi v(S) := v(\pi^{-1}S)$ for all $S \subseteq N$. A player $i \in N$ is dummy in the game v if $v(S \cup i) - v(S) = v(i)$ for all $S \subseteq N \setminus i$.

For each game $v \in TU(N)$ and $S \subseteq N$, we denote by v^S , the game v restricted to S , given by $v^S(T) := v(T)$, $\forall T \subseteq S$. We call this game a *Restricted game*.

A *Reduced game*, $v_{\cup T}^S$, is the game $v \in TU(N)$ restricted to S in the presence of $T \subseteq N \setminus S$, i.e. $v_{\cup T}^S(A) := v(A \cup T) - v(T)$, $\forall A \subseteq S$.

Fujimoto et al. [2006] and Grabisch et al. [2000] propose an equivalent representation of a TU game. Any game $v \in TU(N)$ can be uniquely expressed in terms of its dividends $\{m(v, S)\}_{S \subseteq N}$ by

$$v(T) = \sum_{S \subseteq T} m(v, S), \quad \forall T \subseteq N.$$

In combinatorics, the set function $m(v, \cdot) : 2^N \rightarrow \mathbb{R}$ is called the Möbius transform of v and is given by

$$m(v, S) := \sum_{L \subseteq S} (-1)^{s-l} v(L), \quad \forall S \subseteq N$$

Fujimoto et al. [2006] and Grabisch et al. [2000] show also that $\sum_{T \supseteq S} m(v, T) = \sum_{L \subseteq S} (-1)^l v(N \setminus L)$, $\forall S \subseteq N$. The second part of this equation is called the co-Möbius transform of v applied to S , denoted by $m^*(v, S)$.

One can now introduce how we measure the simultaneous interaction among players. An interaction index of a game $v \in TU(N)$ is a function Ψ which assigns a vector of real values representing the interaction of each coalition $S \in 2^N$.

$$\begin{aligned} \Psi : TU(N) \times 2^N &\longrightarrow \mathbb{R} \\ (v, S) &\longmapsto \Psi(v, S) \end{aligned}$$

A positive (negative) sign can be interpreted as a positive (respectively negative) simultaneous interaction among all the players in S .

One need to introduce additional notations and definitions in order to present the

class of interaction index considered in that paper. Given a game $v \in TU(N)$, for any $S \in 2^N$ and for all $T \subseteq N \setminus S$, we define the discrete derivative with respect to the set S (see Fujimoto et al. [2006]), denoted $\Delta_S v(T)$, as follows:

$$\Delta_S v(T) := \sum_{L \subseteq S} (-1)^{s-l} v(L \cup T)$$

and with convention $\Delta_\emptyset v(T) := v(T)$. $\Delta_S v(T)$ ¹ can be interpreted as the simultaneous interaction amongs players in S in the presence of T . When $\Delta_S v(T) > 0$ (respectively < 0), it seems sensible to consider that there exists a positive (respectively negative) simultaneous interaction among the players in S in the presence of T . However, $\Delta_S v(T) = 0$ should obviously not be interpreted as an absence of interactions among players in S in the presence of T but as an absence of simultaneous interaction among the players in S in the presence of T .

If $S = i$, the first order derivative, $\Delta_i v(T) = v(T \cup i) - v(T)$, is the marginal contribution of a player i to a coalition T .

If $S = ij$, then the second order derivative (introduced by Owen [1972]),

$$\Delta_{ij} v(T) = v(T \cup ij) - v(T \cup i) - v(T \cup j) + v(T)$$

is the marginal contribution of two players i and j in the presence of a coalition T .

A probabilistic interaction index Ψ^P of a coalition $S \subseteq N$ in a game $v \in TU(N)$ is of the form

$$\Psi^P(v, S) = \sum_{T \subseteq N \setminus S} P_N^S(T) \Delta_S v(T)$$

where, for any $S \subseteq N$, the family of coefficients $\{P_N^S(T)\}_{T \subseteq N \setminus S}$ forms a probability distribution on $\mathcal{P}(N \setminus S)$. If $S \not\subseteq N$, we naturally set $\Psi^P(v, S) = 0$.

A cardinal-probabilistic interaction index is a probabilistic interaction index such that, additionally, for any $S \subseteq N$, the coefficients $P_N^S(T)$ ($T \subseteq N \setminus S$) depend only on the cardinalities of the coalitions S , T , and N , i.e., for any $s \in \{0, \dots, n\}$, there exists a family of nonnegative real numbers $\{P_n^s(t)\}_{t=0, \dots, n-s}$ fulfilling

$$\sum_{t=0}^{n-s} \binom{n-s}{t} P_n^s(t) = 1$$

¹Note that $\Delta_S v(\emptyset) = m(v, S)$ and $m^\star(v, S) = \Delta_S v(N \setminus S)$

such that, for any $S \subseteq N$ and any $T \subseteq N \setminus S$, we have $P_N^S(T) = P_n^s(t)$.

Well-known examples of cardinal-probabilistic interaction indices are the Shapley interaction index ($P_n^s(t) = \frac{t!(n-t-s)!}{(n-s+1)!}$, Grabisch [1997]), the Chaining interaction index ($P_n^s(t) = \frac{s(s+t-1)!(n-t-s)!}{n!}$, Marichal and Roubens [1999]) and the Banzhaf interaction index ($P_n^s(t) = \frac{1}{2^{n-s}}$, Roubens [1996]).

The co-Möbius transform of a game $v \in TU(N)$ is also a cardinal-probabilistic interaction indices with $P_n^s(t) = 1$ if $t = n - s$ and 0 otherwise. Fujimoto et al. [2006] and Grabisch et al. [2000] call this interaction index, the external interaction index (Ψ_{ext}). To resume, $\Psi_{ext}(v, S) := m^*(v, S) = \Delta_S v(N \setminus S) \quad \forall S \subseteq N$. If $S = i$, the external interaction index is simply the marginal contribution of a player to the grand coalition, *i.e.* $\Psi_{ext}(v, i) = v(N) - v(N \setminus i)$.

Let us now introduce some properties considered by Grabisch and Roubens [1999].

Definition 1 (Axioms).

- i) **Linearity:** An interaction index Ψ satisfies the Linearity property if for two games $v, w \in TU(N)$, $n \geq 1$, for any $S \in 2^N$, $\alpha \in \mathbb{R}$, we have, $\Psi(v + w, S) = \Psi(v, S) + \Psi(w, S)$ and $\Psi(\alpha v, S) = \alpha \Psi(v, S)$.*
- ii) **Dummy:** An interaction index Ψ satisfies the Dummy property if for all game $v \in TU(N)$, $n \geq 1$, for any dummy player $i \in N$ in the game v , then $\Psi(v, i) = v(i)$ and for any $S \subseteq N \setminus i$, $s \geq 1$, $\Psi(v, S \cup i) = 0$.*
- iii) **Symmetry:** An interaction index Ψ satisfies the Symmetry property if for all permutation π , for all TU game $v \in TU(N)$, $n \geq 1$, and for all $S \subseteq N$, $s \geq 1$, then $\Psi(\pi v, \pi(S)) = \Psi(v, S)$.*
- iv) **Recursivity:** An interaction index Ψ verifies the Recursivity property if for a game $v \in TU(N)$ ($n \geq 2$), for all $S \subseteq N$ ($s \geq 2$) and for all $j \in S$*

$$\Psi(v, S) = \Psi(v_{\cup_j}^{N \setminus j}, S \setminus j) - \Psi(v^{N \setminus j}, S \setminus j)$$

Following Fujimoto et al. [2006] and Grabisch and Roubens [1999], linearity axiom implies that interaction indices are linear combinations of the basic information related to the game: the worth of each coalition of players. The dummy axiom states that a dummy player has a value equal to its worth and that he does not interact

with any outsider coalition. The symmetry axiom indicates that the names of the players play no role in determining the outcome of an interaction index. According to the recursivity axiom, the interaction at a given level is linked to the difference in interactions defined at the previous level. For example, for two players i and j , the axiom says that their interaction should depend on the interaction when one of the players is removed (measured by the interaction in the corresponding reduced and restricted game).

Note that Shapley and Banzhaf interaction indices satisfy the recursivity axiom (Grabisch and Roubens [1999]), but not the Chaining interaction index. These three interaction indices satisfy the linearity, the dummy and the symmetry axioms.

The following theorem was shown by Grabisch and Roubens [1999].

Theorem 1 (Grabisch and Roubens [1999]).

An interaction index Ψ satisfies the Linearity, Dummy and Symmetry, if and only if, for any game $v \in TU(N)$, $n \geq 1$, and $S \subseteq N$, $s \geq 1$, there exists a collection of real constants $\{P_n^s(t)\}_{t=0, \dots, n-s}$, such that

$$\Psi(v, S) = \sum_{T \subseteq N \setminus S} P_n^s(t) \Delta_S v(T),$$

and for any $S \not\subseteq N$, we have $\Psi(v, S) = 0$.

2.2 Decomposition

We introduce a new class of interaction indices and then provide an axiomatic characterization. Moreover we present an alternative interpretation of well-known interaction indices.

Definition 2. *The class of Additive Interaction Decomposable (AID) index*

An interaction index $\Psi : TU(N) \times 2^N \rightarrow \mathbb{R}$ is an AID index if for all game $v \in TU(N)$ and for any coalition $S \subseteq N$ there exists a set of real numbers $\{f_n^s(t)\}_{t=0, \dots, n-s-1}$ such that,

$$\Psi(v, S) = \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \Delta_{S \cup j} v(T)$$

This definition states that an interaction index Ψ is an AID index if it can be split into two parts. The first one is the external interaction index for a given coalition S ,

and the second one is a weighted sum of the simultaneous interactions between the players of the coalition S and a single player j in the presence of all other coalitions T . Naturally, the case where $f_n^s(t) = 0$ leads to the external interaction index.

Remark 1. *If the set of real numbers $\{f_n^s(t)\}_{t=0, \dots, n-s-1}$ fulfills the following two conditions:*

$$i) f_n^s(t) \geq 0, \text{ for all } t = 0, \dots, n-s-1$$

$$ii) \sum_{t=0}^{n-s-1} \binom{n-s-1}{t} 2 f_n^s(t) = 1$$

then, $\Psi(v, S)$ can be interpreted as the difference between the external interaction index for a given coalition S and the sum under the player $j \in N \setminus S$, of the mathematical expectation on $\mathcal{P}(N \setminus (S \cup j))$ of the discrete derivative $\frac{\Delta_{S \cup j} v(T)}{2}$ with respect to the probability distribution $\{2 \times {}^j f_N^S(T)\}_{T \subseteq N \setminus (S \cup j)}$ where ${}^j f_N^S(T) = f_n^s(t)$.

The next result provides an axiomatic characterization of AID interaction indices on the class of TU game.

Theorem 2. *An interaction index is AID, if and only if, it satisfies Linearity, Dummy and Symmetry.*

Note that, the notion of interaction index is a generalization of one-point solution concept for cooperative games with transferable utility.

Corollary 1. *A solution is AID, if and only if, it satisfies the following classical properties of solutions (axioms used by Dubey et al. [1981]): Linearity, Dummy and Symmetry.*

Every cardinal-probabilistic index satisfies Linearity, Dummy and Symmetry. From Theorem 2, we can deduce a new rewriting (or interpretation) of cardinal-probabilistic interaction indices.

Corollary 2. *Every cardinal-probabilistic interaction index Ψ^P applied to a game $v \in TU(N)$ is AID. Moreover, for any coalition $S \subseteq N$,*

$$\Psi^P(v, S) = \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \Delta_{S \cup j} v(T)$$

where $f_n^s(t)$ is given by

$$f_n^s(t) = \begin{cases} \frac{P_n^s(0)}{n-s} & \text{if } t = 0 \\ \frac{t \times f_n^s(t-1) + P_n^s(t)}{n-t-s} & \text{otherwise} \end{cases}$$

and the set $\{P_n^s(t)\}_{t=0, \dots, n-s}$ are nonnegative constants associated to the cardinal-probabilistic interaction index.

From Corollary 2, we obtain a new rewriting (or interpretation) of the well-known interaction indices. The Shapley, the Chaining and the Banzhaf interaction indices are AID.

Corollary 3. *The real numbers of the Shapley interaction index as an AID index are given by*

$$f_n^s(t) = \frac{(t+1)!(n-t-s-1)!}{(n-s+1)!} \quad \text{for all } t = 0, \dots, n-s-1;$$

The real numbers of the Chaining interaction index as an AID index are given by

$$f_n^s(t) = \frac{(t+s)!(n-t-s-1)!}{n!} \quad \text{for all } t = 0, \dots, n-s-1$$

and the real numbers of the Banzhaf interaction index as an AID index are given by

$$f_n^s(t) = \begin{cases} \frac{1}{(n-s)2^{n-s}} & \text{if } t = 0 \\ \frac{t \times f_n^s(t-1) \times 2^{n-s} + 1}{(n-t-s)2^{n-s}} & \text{otherwise} \end{cases}$$

Note that, when $s = 1$ the cardinal-probabilistic interaction index becomes the semivalue. Every semivalues are AID.

Remark 2 (Chantreuil et al. [2019]). *The Shapley solution (Shapley [1953]) is AID.*

The following two propositions give another interpretation of the Shapley and the Banzhaf interaction indices.

Proposition 1. *For a given game $v \in TU(N)$, $n \geq 1$ and $S \subseteq N$, $s \geq 1$,*

$$\text{If } f_n^s(t) = \frac{(t+1)!(n-t-s-1)!}{(n-s+1)!} \text{ then } \sum_{t=0}^{n-s-1} \binom{n-s-1}{t} 2f_n^s(t) = 1.$$

The Shapley interaction index of the coalition S in the game v can be interpreted as the difference between the external interaction index for a given coalition S and the

sum under the player $j \in N \setminus S$, of the mathematical expectation on $\mathcal{P}(N \setminus (S \cup j))$ of the discrete derivative $\frac{\Delta_{S \cup j} v(T)}{2}$ with respect to the probability distribution $\{2 \times {}^j f_N^S(T)\}_{T \subseteq N \setminus (S \cup j)}$ where ${}^j f_N^S(T) = f_n^s(t)$.

Proposition 2. For a given game $v \in TU(N)$, $n \geq 1$ and $S \subseteq N$, $s \geq 1$,

$$\text{If } f_n^s(t) = \begin{cases} \frac{1}{(n-s)2^{n-s}} & \text{if } t = 0 \\ \frac{t \times f_n^s(t-1) \times 2^{n-s} + 1}{(n-t-s)2^{n-s}} & \text{otherwise} \end{cases} \quad \text{then } \sum_{t=0}^{n-s-1} \binom{n-s-1}{t} 2 f_n^s(t) = 1.$$

The Banzhaf interaction index new interpretation is similar to those of the Shapley interaction index.

The next Proposition gives the necessary and sufficient condition for which an AID interaction index is a cardinal-probabilistic interaction index.

Proposition 3. Let Ψ be an AID interaction index with the associated family of real numbers $\{f_n^s(t)\}_{t=0, \dots, n-s-1}$. Ψ is a cardinal-probabilistic interaction index, if and only if, $f_n^s(n-s-1) \leq \frac{1}{n-s}$, $f_n^s(0) \geq 0$ and $f_n^s(t) \geq \frac{t}{n-t-s} f_n^s(t-1)$ with $0 < t < n-s-1$.

Note that the class of AID index is not a subclass of the class of probabilistic interaction index and vice versa.

3 Casajus and Huettner decomposition approach

In this section, we are going to decompose an interaction index by using another interaction index called the decomposer.

3.1 Decomposability

Given a game $v \in TU(N)$, the multi-linear extension (Owen [1972]) of v is a multi-linear polynomial game $\bar{v} : [0, 1]^n \rightarrow \mathbb{R}$ given by

$$\bar{v}(x_1, \dots, x_n) := \sum_{\substack{S \subseteq N \\ S \neq \emptyset}} v(S) \prod_{i \in S} x_i \prod_{i \in N \setminus S} (1 - x_i)$$

Given an interaction index Ψ on the game $v \in TU(N)$, an auxiliary game is a game v^Ψ where $v^\Psi(S) = \sum_{i \in S} \Psi(v^S, i)$ for all $S \subseteq N$.

We can now introduce the different notions of decomposability under investigation in that section.

Definition 3 (CH-decomposability (Casajus and Huettner [2018])).

An interaction index Ψ is CH-decomposable if there exists an interaction index φ such that, for all game $v \in TU(N)$ and $i \in N$,

$$\Psi(v, i) = \varphi(v, i) + \sum_{j \in N \setminus i} [\varphi(v, j) - \varphi(v^{N \setminus i}, j)]$$

φ is then called the CH-decomposer of Ψ .

The expression $\varphi(v, i)$ reflects player i 's direct contribution subsumed under the interaction index φ , while the expression $\sum_{j \in N \setminus i} [\varphi(v, j) - \varphi(v^{N \setminus i}, j)]$ reflects player i 's indirect contributions.

In order to extend this decomposability notion to higher-order interactions, one need to introduce a rather similar notion of decomposability, the Individual-decomposability (I-decomposability).

Definition 4 (I-decomposability).

An interaction index Ψ is I-decomposable if there exists an interaction index φ such that, for all game $v \in TU(N)$ and $i \in N$,

$$\Psi(v, i) = \varphi(v, i) + \sum_{j \in N \setminus i} [\varphi(v, j) - \varphi(v^{N \setminus j}, i)]$$

φ is then called the I-decomposer of Ψ .

An interaction index is I-decomposable if it can be split into a direct part and an indirect part, where the direct part $\varphi(v, i)$ is the I-decomposer and the indirect part $\sum_{j \in N \setminus i} [\varphi(v, j) - \varphi(v^{N \setminus j}, i)]$ indicates how much the other players contribute to the payoff of a given player according to the I-decomposer. If the I-decomposer of Ψ exists, then it provides a kind of foundation for Ψ . The difference between the two previous notions of decomposability concerns the second part of the equations. However, the CH-decomposability is stronger than the I-decomposability as shown by the following proposition.

Proposition 4. *If an interaction index is CH-decomposable then it is I-decomposable. Moreover, the I-decomposer is unique and coincides with the unique CH-decomposer that is itself CH-decomposable.*

Considering higher-order interactions, we can extend the I-decomposability to Coalition-decomposability (C-decomposability) as follows.

Definition 5 (C-decomposability).

An interaction index Ψ is C-decomposable if there exists an interaction index φ such that, for all game $v \in TU(N)$, $S \in 2^N$,

$$\Psi(v, S) = \varphi(v, S) + \sum_{j \in N \setminus S} [\varphi(v, S) - \varphi(v^{N \setminus j}, S)]$$

φ is called the C-decomposer of Ψ .

For a given coalition S , the simultaneous interaction of the players of coalition S is split into two parts. The direct part is the simultaneous interaction of the players of coalition S measured by the C-decomposer. The indirect part indicates to what extent the other players contribute individually to the interactions of coalition S measured by the C-decomposer.

The next results present some features of the C-decomposability and of the C-decomposer.

Proposition 5. *If an interaction index Ψ is C-decomposable, then the C-decomposer of Ψ is unique.*

Proposition 5 clarifies the uniqueness of the C-decomposer. The next theorem shows that the recursivity property links the CH-decomposability and the C-decomposability.

Theorem 3. *Let an interaction index Ψ satisfying the Recursive axiom. If the interaction index Ψ is CH-decomposable, then it is also C-decomposable, and the C-decomposer is unique.*

One need to introduce a result of Casajus and Huettner [2018] before to give the explicit formulae for the C-decomposer of the C-decomposable interaction indices.

Proposition 6. (Casajus and Huettner [2018]) *If an interaction index Ψ is CH-decomposable, then its unique CH-decomposer φ is given by: For all game $v \in TU(N)$ and for all $i \in N$,*

$$\varphi(v, i) = \sum_{\substack{T \subset N \\ i \in T}} \frac{m(v^\Psi, T)}{t^2}$$

The dividends $m(v^\Psi, T)$ is the Möbius transform of V^Ψ applied to T .

The next result gives us the explicit formulae for the C-decomposer.

Proposition 7. *Let Ψ be an interaction index satisfying the Recursive axiom. If Ψ is C-decomposable and CH-decomposable, then its unique C-decomposer φ which is itself C-decomposable is given by: For all game $v \in TU(N)$, for any coalition $S \in 2^N$ and any player $i \in S$,*

$$\varphi(v, S) = \begin{cases} \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i \\ t=\beta}} \varphi(v_{\cup((S \setminus i) \setminus T)}^{N \setminus (S \setminus i)}, i) & \text{if } s \geq 2 \\ \varphi(v, i) & \text{if } s = 1 \end{cases}$$

φ is given by Proposition 6 when the coalition contains only one player.

3.2 Cardinal-probabilistic interaction index

We now study the interaction indices taken into consideration in that paper under the prism of the second decomposability approach.

Firstly, we show that Shapley and Banzhaf interaction indices are C-decomposable and that the corresponding C-decomposer is unique; and the Chaining interaction index is not C-decomposable.

Proposition 8. *The Shapley and the Banzhaf interaction indices are C-decomposables and the C-decomposer of each of these indices is unique.*

Proposition 9. *The Chaining interaction index is not C-decomposable.*

Secondly, we show that a cardinal-probabilistic interaction index is a unique C-decomposer of a given AID index.

Proposition 10. *Let Ψ^P be a cardinal-probabilistic interaction index . For all game $v \in TU(N)$, Ψ^P is the unique C-decomposer of the AID index χ given as follows: for any coalition $S \in 2^N$,*

$$\chi(v, S) = \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} [f_n^S(t) - P_{t+1}^S(n)] \Delta_{S \cup j} v(T)$$

Where the coefficients $\{f_n^s(t)\}_{t=0,\dots,n-s-1}$ are given by Corollary 2 and the coefficients $\{P_{t+1}^s(n)\}_{t=0,\dots,n-s-1}$ are the nonnegative real numbers of the cardinal-probabilistic interaction index Ψ^P .

The main implications of this result are the following.

Remark 3. (Implications)

- i) The Shapley interaction index is the unique C-decomposer of the external interaction index.²
- ii) The Banzhaf interaction index is the unique C-decomposer of the interaction index χ_B defined as follow: For all game $v \in TU(N)$ and for any coalition $S \in 2^N$,

$$\chi_B(v, S) = \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} [f_n^s(t) - \frac{1}{2^{n-s}}] \Delta_{S \cup j} v(T)$$

where $f_n^s(t)$ is given in Corollary 3.

- iii) The Chaining interaction index is the unique C-decomposer of the interaction index χ_C defined as follow: For all game $v \in TU(N)$ and for any coalition $S \in 2^N$,

$$\chi_C(v, S) = \Psi_{ext}(v, S) + \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} \frac{(s-1)(t+s)!(n-t-s-1)!}{n!} \Delta_{S \cup j} v(T)$$

4 An illustration with MCDA

Under the viewpoint of cooperative game theory and multicriteria decision making, everybody agrees that interaction phenomena do exist in real situations. The Cardinal-probabilistic interaction index has always been central in the modelling and analysis of preferences. The decompositions studied in this paper allow us to propose other interpretations of the importance of criteria.

4.1 Notations and basic notions

In the context of MCDA, the players are criteria and the set N is the set of n criteria. Let $X = X_1 \times X_2 \times \dots \times X_n$ be the set of alternatives, with $n \geq 2$. An alternative $x =$

²When $S = i$, we obtain Theorem 3 of Casajus and Huettner [2018], page 39.

(x_1, \dots, x_n) is identified to an element of X . \succeq is a preference relation on X which is complete (i.e., $x \succeq y$ or $y \succeq x$, for all $x, y \in X$) and transitive (i.e., $x \succeq y$ and $y \succeq z \implies x \succeq z$, for all $x, y, z \in X$).

For $T \subseteq N$, the notation $(x_T, y_{-T}) \in X$ is the compound alternative taking value x_i if $i \in T$ and value y_i otherwise.

Definition 6.

i) $T \subseteq N$ is preferentially independent of its complement $N \setminus T$ if for every $x, y, z, z' \in X$

$$(x_T, z_{-T}) \succeq (y_T, z_{-T}) \iff (x_T, z'_{-T}) \succeq (y_T, z'_{-T})$$

ii) The criteria X_1, \dots, X_n are (mutually) preferentially independent if every $T \subseteq N$ is preferentially independent of its complement.

Following Grabisch and Labreuche [2010] when the criteria are not mutually preferentially independent, there is interaction among the criteria, while there is no interaction if mutual preference independence holds.

$$\text{interaction} \iff \text{not}(\text{mutual preferential independence}).$$

The various criteria are recoded numerically using, for each $i \in N$, a function u_i from X_i into \mathbb{R} . We denote by U the overall utility function. For all $x \in X$, $U(x) = (u_1(x_1), \dots, u_n(x_n))$.

In the context of MCDA, we consider a monotonic TU game (N, μ) . μ is a capacity or fuzzy measure if $\mu(N) = 1$. A 2-additive capacity (Mayag et al. [2011]) is a capacity μ such that the Möbius transform of μ satisfies the following conditions:

- for all $S \subseteq N$ such that $s > 2$, $m(\mu, S) = 0$,
- there exist $T \subseteq N$ such that $t = 2$ and $m(\mu, T) \neq 0$

We recall that $\Delta_{ij}\mu(\emptyset) = m(\mu, ij) = \mu(ij) - \mu(i) - \mu(j)$ is the interaction between the two criteria i and j . The sign of $\Delta_{ij}\mu(\emptyset)$ is not always stable (see Mayag and Bouyssou [2020]). If $\Delta_{ij}\mu(\emptyset) > 0$ then criteria i and j are complementary. If $\Delta_{ij}\mu(\emptyset) < 0$ then criteria i and j are redundant. $\Delta_{ij}\mu(\emptyset)$ can also be zero, then criteria i and j are independent.

Following Mayag and Bouyssou [2020], for an alternative $x = (x_1, \dots, x_n) \in X$, the Choquet integral w.r.t. a 2-additive capacity μ , called for short the 2-additive Choquet integral, is given by:

$$C_\mu(u_1(x_1), \dots, u_n(x_n)) := \sum_{i \in N} u_i(x_i) \left[\mu(i) + \sum_{j \in N \setminus i} \Delta_{ij} \mu(\emptyset) \right] - \frac{1}{2} \sum_{\{i,j\} \subseteq N} \Delta_{ij} \mu(\emptyset) |u_i(x_i) - u_j(x_j)|$$

Definition 7 (Mayag and Bouyssou [2020]). *Let $i, j \in N$ be two distinct criteria. We say that:*

- i) there exists a possible positive (resp. null, negative) interaction between i and j if there exists a 2-additive capacity μ such that $\Delta_{ij} \mu(\emptyset) > 0$ (resp. $\Delta_{ij} \mu(\emptyset) = 0$, $\Delta_{ij} \mu(\emptyset) < 0$);*
- ii) there exists a necessary positive (resp. null, negative) interaction between i and j if $\Delta_{ij} \mu(\emptyset) > 0$ (resp. $\Delta_{ij} \mu(k) = 0$, $\Delta_{ij} \mu(\emptyset) < 0$) for all 2-additive capacity μ .*

4.2 Numerical illustration

We consider the following classical example in MCDA .

Example 4.1 (A classic example of Grabisch and Labreuche [2010]). *An assessment of students with the help of three criteria: Mathematics (M), Physics (P) and Languages kills (L) is given (the set of criteria is $N = \{M, P, L\}$). Assuming an evaluation scale from 0 to 20, consider four students a, b, c, d with the following marks (i.e. $X = \{a, b, c, d\}$):*

	Mathematics(M)	Physics(P)	Language skills(L)
<i>Student a</i>	8	18	12
<i>Student b</i>	8	12	18
<i>Student c</i>	16	18	12
<i>Student d</i>	16	12	18

To select the best students, the director expresses the following preferences

- *For a bad students in Mathematics, Physics is more important than language skills:*

$$\implies a > b \tag{1}$$

- For a good students in Mathematics, Language skills is more important than Physics:

$$\implies d > c \quad (2)$$

Let $T = \{P, L\}$ be a subset of N . We have $a = (8, 18, 12) = (a_{-T}, a_T)$, $c = (16, 18, 12) = (c_{-T}, c_T)$, $b = (8, 12, 18) = (b_{-T}, b_T)$ and $d = (16, 12, 18) = (d_{-T}, d_T)$.

If $T = \{P, L\}$ is preferentially independent of its complement $N \setminus T = \{M\}$, then (1) $\implies c > d$, which contradicts (2). So the preference informations $a > b$ and $d > c$ are not representable by a model verifying mutual preferential independence. Therefore there exists an interaction between criteria.

Modeling: M and P have a possible negative interaction; P and L (and similarly for M and L) have a possible positive interaction.

Subjects (S)	M	P	L	M,P	M,L	P,L	M,P,L
$\mu(S)$	0.3	0.3	0.2	0.4	0.7	0.7	1

Alternative x	a	b	c	d
$C_\mu(x)$	12.6	12	14.2	15.2

$\{i, j\}$	$\{M, P\}$	$\{M, L\}$	$\{P, L\}$
$\Delta_{ij}\mu(\emptyset)$	-0.2	0.2	0.2

These preferences are representable by the previous capacity ($d > c > a > b$). We are now going to focus on the study of possible or necessary interactions between the criteria:

i) What sort of link are we able to exhibit between the **Importance, Higher-order interaction terms** and **co-Möbius transform** ?

Answer: If the interaction index Ψ is AID then the link between these three notions can be given by the following relation:

$$\Psi(\mu, S) = \Psi_{ext}(\mu, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^S(t) \Delta_{S \cup j} \mu(T)$$

where $\Psi(\mu, S)$ measures the importance of criteria belonging to S . In another words, $\Psi(\mu, S)$ measures the degree of interaction between the criteria belonging to S . $\Psi_{ext}(\mu, S)$ is the co-Möbius transform of μ applied to S and $\sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^S(t) \Delta_{S \cup j} \mu(T)$ is the higher-order interaction of criteria belonging to S with the other criteria.

Remark 4. If $n \geq 3$ and $s \geq n - 1$ then $\sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \Delta_{S \cup j} \mu(T) = 0$ (since μ is 2-additive) and hence $\Psi(\mu, S) = \Psi_{ext}(\mu, S)$.

If we measure the importance with the Shapley interaction index (Ψ_{Sh}) (or the Banzhaf interaction index (Ψ_{Bz}) or the Chaining interaction index (Ψ_{Ch})) then we have the following table,

Subjects (S)	M	P	L	M,P	M,L	P,L	M,P,L
$\Psi_{Sh}(\mu, S) = \Psi_{Bz}(\mu, S) = \Psi_{Ch}(\mu, S)$	0.3	0.3	0.4	-0.2	0.2	0.2	0
$\Psi_{ext}(\mu, S)$	0.7	0.7	0.8	-0.2	0.2	0.2	0
$\sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \Delta_{S \cup j} \mu(T)$	0.4	0.4	0.4	0	0	0	0

ii) What sort of link are we able to exhibit between the **Importance, Decomposer** and **Contribution** ?

Answer: If the interaction index Ψ is C-decomposable then the degree of interaction between the criteria belonging to S can be given by the following relation:

$$\Psi(\mu, S) = \varphi(\mu, S) + \sum_{j \in N \setminus S} [\varphi(\mu, S) - \varphi(\mu^{N \setminus j}, S)]$$

$\varphi(\mu, S)$ is the simultaneous interaction of criteria belonging to S measured by φ (the decomposer). $\sum_{j \in N \setminus S} [\varphi(\mu, S) - \varphi(\mu^{N \setminus j}, S)]$ indicates to what extent the other criteria

contribute individually to the interactions of criteria belonging to S measured by φ . If we suppose that Ψ is the Shapley interaction index (or the Banzhaf interaction index (Ψ_{Bz}) or the Chaining interaction index (Ψ_{Ch})) then φ is the C-decomposer and we have the following table.

Subjects (S)	M	P	L	M,P	M,L	P,L	M,P,L
$\Psi_{Sh}(\mu, S) = \Psi_{Bz}(\mu, S) = \Psi_{Ch}(\mu, S)$	0.3	0.3	0.4	-0.2	0.2	0.2	0
$\varphi(\mu, S)$	0.3	0.3	0.3	-0.15	0.2	0.15	0
$\sum_{j \in N \setminus S} [\varphi(\mu, S) - \varphi(\mu^{N \setminus j}, S)]$	0	0	0.1	-0.05	0	0.05	0

5 Conclusion

This paper was devoted to the study of the cardinal-probabilistic interaction indices, especially the Shapley, the Chaining and the Banzhaf interaction indices. Owing to alternative decompositions of the cardinal-probabilistic interaction indices, we provide a numerical illustration of the proposed methodology, by using hypothetical data.

There are several ways in which the interaction index could be explored in further research. The most interesting direction is the extension of the interaction indices to games with a priori relationships between players. In a game with a coalition structure, it is supposed that players organize themselves into disjoint coalitions which are defined a priori. The most well-known solutions are the Owen-Shapley (Owen, 1977) and the Owen-Banzhaf solutions (Owen [1981]). A full characterization of the interaction indices in this context would be worthwhile pursuing.

Appendix A: Proof of Section 2 results

Proof of Theorem 2.

An AID interaction index satisfies Linearity, Dummy and Symmetry.

To proof this Theorem, it suffices for us to use Theorem 1 and show that: An interaction index Ψ is AID, if and only if, for any game $v \in TU(N)$, $n \geq 1$, and $S \subseteq N$, $s \geq 1$, there exists a collection of real constants $\{P_n^s(t)\}_{t=0, \dots, n-s}$, such that

$$\Psi(v, S) = \sum_{T \subseteq N \setminus S} P_n^s(t) \Delta_S v(T),$$

and for any $S \not\subseteq N$, we have $\Psi(v, S) = 0$.

Let Ψ be an interaction index.

\implies) For any game $v \in TU(N)$, $n \geq 1$, if Ψ is AID, then, for any $S \subseteq N$, $s \geq 1$, there exists a collection of real constants $\{f_n^s(t)\}_{t=0, \dots, n-s-1}$ and such that,

$$\Psi(v, S) = \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \Delta_{S \cup j} v(T)$$

We know that,

$$\Delta_S v(T) = \sum_{L \subseteq S} (-1)^{s-l} v(L \cup T)$$

For any player $i \in S$, $\Delta_S v(T)$ can be expressed recursively as follows:

$$\Delta_S v(T) = \Delta_{S \setminus i} v(T \cup i) - \Delta_{S \setminus i} v(T)$$

For any player $i \in S$, we have

$$\begin{aligned} \Psi(v, S) &= \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \Delta_{S \cup j} v(T) \\ &= \Delta_S v(N \setminus S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \times [\Delta_S v(T \cup j) - \Delta_S v(T)] \\ &= \Delta_S v(N \setminus S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \times [\Delta_{S \setminus i} v(T \cup ij) - \Delta_{S \setminus i} v(T \cup j) \\ &\quad - \Delta_{S \setminus i} v(T \cup i) + \Delta_{S \setminus i} v(T)] \\ &= \Delta_S v(N \setminus S) - a + c + b - d \end{aligned}$$

where

$$\begin{aligned} a &= \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \Delta_{S \setminus i} v(T \cup ij) = \sum_{j \in N \setminus S} \sum_{\substack{T \subseteq N \setminus S \\ j \in T}} f_n^s(t-1) \Delta_{S \setminus i} v(T \cup i) \\ &= \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} \sum_{j \in T} f_n^s(t-1) \Delta_{S \setminus i} v(T \cup i) = \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} t f_n^s(t-1) \Delta_{S \setminus i} v(T \cup i) \end{aligned}$$

$$\begin{aligned} b &= \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \Delta_{S \setminus i} v(T \cup i) = \sum_{j \in N \setminus S} \sum_{\substack{T \subseteq N \setminus S \\ j \notin T}} f_n^s(t) \Delta_{S \setminus i} v(T \cup i) \\ &= \sum_{T \subseteq N \setminus S} \sum_{j \notin T \cup S} f_n^s(t) \Delta_{S \setminus i} v(T \cup i) = \sum_{T \subseteq N \setminus S} (n-s-t) f_n^s(t) \Delta_{S \setminus i} v(T \cup i) \end{aligned}$$

$$\begin{aligned} c &= \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \Delta_{S \setminus i} v(T \cup j) = \sum_{j \in N \setminus S} \sum_{\substack{T \subseteq N \setminus S \\ j \in T}} f_n^s(t-1) \Delta_{S \setminus i} v(T) \\ &= \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} \sum_{j \in T} f_n^s(t-1) \Delta_{S \setminus i} v(T) = \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} t f_n^s(t-1) \Delta_{S \setminus i} v(T) \end{aligned}$$

and

$$\begin{aligned}
d &= \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \Delta_{S \setminus i} v(T) = \sum_{j \in N \setminus S} \sum_{\substack{T \subseteq N \setminus S \\ j \notin T}} f_n^s(t) \Delta_{S \setminus i} v(T) \\
&= \sum_{T \subseteq N \setminus S} \sum_{j \notin T \cup S} f_n^s(t) \Delta_{S \setminus i} v(T) = \sum_{T \subseteq N \setminus S} (n-s-t) f_n^s(t) \Delta_{S \setminus i} v(T)
\end{aligned}$$

Using the expressions of a , b , c and d , $\Psi(v, S)$ can be written as

$$\begin{aligned}
\Psi(v, S) &= \Delta_S v(N \setminus S) - \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} t f_n^s(t-1) \Delta_{S \setminus i} v(T \cup i) + \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} t f_n^s(t-1) \Delta_{S \setminus i} v(T) \\
&\quad + \sum_{T \subseteq N \setminus S} (n-s-t) f_n^s(t) \Delta_{S \setminus i} v(T \cup i) - \sum_{T \subseteq N \setminus S} (n-s-t) f_n^s(t) \Delta_{S \setminus i} v(T) \\
&= \Delta_S v(N \setminus S) - \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} t f_n^s(t-1) [\Delta_{S \setminus i} v(T \cup i) - \Delta_{S \setminus i} v(T)] \\
&\quad + \sum_{T \subseteq N \setminus S} (n-s-t) f_n^s(t) [\Delta_{S \setminus i} v(T \cup i) - \Delta_{S \setminus i} v(T)] \\
&= \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} [(n-s-t) f_n^s(t) - t f_n^s(t-1)] \Delta_S v(T) + (n-s) f_n^s(0) \Delta_S v(\emptyset) \\
&\quad + [1 - (n-s) f_n^s(n-s-1)] \Delta_S v(N \setminus S) \\
&= \sum_{T \subseteq N \setminus S} [(n-s-t) f_n^s(t) - t f_n^s(t-1)] q_n^s(t) \Delta_S v(T) + \sum_{T \subseteq N \setminus S} (n-s) f_n^s(0) r_n^s(t) \Delta_S v(T) \\
&\quad + \sum_{T \subseteq N \setminus S} [1 - (n-s) f_n^s(n-s-1)] k_n^s(t) \Delta_S v(T) \\
&= \sum_{T \subseteq N \setminus S} P_n^s(t) \Delta_S v(T)
\end{aligned}$$

where

$$q_n^s(t) = \begin{cases} 1 & \text{if } 0 < t < n-s \\ 0 & \text{otherwise} \end{cases} ; k_n^s(t) = \begin{cases} 1 & \text{if } t = n-s \\ 0 & \text{otherwise} \end{cases} ; r_n^s(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$P_n^s(t) = [(n-s-t) f_n^s(t) - t f_n^s(t-1)] q_n^s(t) + (n-s) f_n^s(0) r_n^s(t) + [1 - (n-s) f_n^s(n-s-1)] k_n^s(t)$$

Finally, for all $S \not\subseteq N$, for all $T \subseteq N \setminus S$, we have $\Delta_S v(T) = 0$ (Fujimoto et al. [2006] page 76). We can deduce that $\Psi(v, S) = 0$.

\Leftarrow) We suppose that, for any game $v \in TU(N)$, $n \geq 1$, and $S \subseteq N$, $s \geq 1$, there exists a collection of real constants $\{P_n^s(t)\}_{t=0, \dots, n-s}$, such that,

$$\Psi(v, S) = \sum_{T \subseteq N \setminus S} P_n^s(t) \Delta_S v(T)$$

We set,

$$f_n^s(t) = \begin{cases} \frac{P_n^s(0)}{n-s} & \text{if } t = 0 \\ \frac{1 - P_n^s(n-s)}{n-s} & \text{if } t = n-s-1 \\ \frac{t \times f_n^s(t-1) + P_n^s(t)}{n-t-s} & \text{otherwise} \end{cases}$$

$P_n^s(t)$ can be expressed as follows:

$$P_n^s(t) = [(n-s-t)f_n^s(t) - t f_n^s(t-1)]q_n^s(t) + (n-s)f_n^s(0)r_n^s(t) + [1 - (n-s)f_n^s(n-s-1)]k_n^s(t)$$

Finally, using the reverse of the first part of this proof, we can deduce that Ψ is an AID interaction index.

$f_n^s(t)$ can simply give as follows

$$f_n^s(t) = \begin{cases} \frac{P_n^s(0)}{n-s} & \text{if } t = 0 \\ \frac{t \times f_n^s(t-1) + P_n^s(t)}{n-t-s} & \text{otherwise} \end{cases}$$

since by linearity, everything is determined on dirac games ($e_K(S) = 1$ if $S = K$ and 0 otherwise).

$$\Psi(e_N, S) = P_n^s(n-s) = 1 - (n-s)f_n^s(n-s-1)$$

and then,

$$f_n^s(n-s-1) = \frac{1 - P_n^s(n-s)}{n-s}$$

We conclude that, an interaction index is AID, if and only if, it satisfies Linearity, Dummy and Symmetry.

Proof of Proposition 1.

The Shapley interaction index is AID and the associated coefficient is

$$f_n^s(t) = \frac{(t+1)!(n-t-s-1)!}{(n-s+1)!}, \text{ with } 0 \leq t \leq n-s-1.$$

Moreover,

$$\begin{aligned}
\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} 2f_n^s(t) &= 2 \sum_{t=0}^{n-s-1} \binom{n-s-1}{t} \frac{(t+1)!(n-t-s-1)!}{(n-s+1)!} \\
&= \frac{2}{(n-s+1)(n-s)} \sum_{t=0}^{n-s-1} (t+1) \\
&= \frac{2}{(n-s+1)(n-s)} \left[n-s + \frac{(n-s)(n-s-1)}{2} \right] \\
&= 1
\end{aligned}$$

Proof of Proposition 2.

The Banzhaf interaction index is AID and the associated coefficient is

$$f_n^s(t) = \begin{cases} \frac{1}{(n-s)2^{n-s}} & \text{if } t = 0 \\ \frac{t \times f_n^s(t-1) \times 2^{n-s} + 1}{(n-t-s)2^{n-s}} & \text{otherwise} \end{cases}$$

Moreover,

$$\begin{aligned}
\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} 2f_n^s(t) &= \frac{1}{(n-s)2^{n-s-1}} + 2 \sum_{t=1}^{n-s-1} \binom{n-s-1}{t} \frac{t \times f_n^s(t-1) \times 2^{n-s} + 1}{(n-t-s)2^{n-s}} \\
&= \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-1} \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} + 2 \sum_{t=1}^{n-s-1} \binom{n-s-1}{t-1} f_n^s(t-1) \\
&= \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-1} \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} + 2 \sum_{t=0}^{n-s-2} \binom{n-s-1}{t} f_n^s(t) \\
&= \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-1} \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} + \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-2} \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} \\
&\quad + 2 \sum_{t=0}^{n-s-3} \binom{n-s-1}{t} f_n^s(t) \\
&= \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-1} \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} + \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-2} \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} \\
&\quad + \dots + \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^1 \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} + \frac{1}{(n-s)2^{n-s-1}} \\
&= \binom{n-s-1}{0} \frac{1}{2^{n-s-1}} + \binom{n-s-1}{1} \frac{1}{2^{n-s-1}} + \binom{n-s-1}{2} \frac{1}{2^{n-s-1}} + \dots + \binom{n-s-1}{n-s-1} \frac{1}{2^{n-s-1}} \\
&= \frac{1}{2^{n-s-1}} \sum_{t=0}^{n-s-1} \binom{n-s-1}{t} \\
&= 1
\end{aligned}$$

Proof of Proposition 3.

This result come from the proof of Theorem 2.

Appendix B: Proof of Section 3 results

Proof of Proposition 4.

Given a game $v \in TU(N)$ with $n \geq 2$, i and j two players in N . It is easier to show by induction on n that, if an interaction index is I-decomposable, then its I-decomposer φ' is unique. Proposition 2 (page 39) of Casajus and Huettner [2018] states that if an interaction index is CH-decomposable, then there exists a unique CH-decomposer φ that is itself CH-decomposable. Furthermore, Theorem 4 (page 40) of Casajus and

Huettner [2018] shows that

$$\left[\varphi(v, i) - \varphi(v^{N \setminus j}, i) \right] = \left[\varphi(v, j) - \varphi(v^{N \setminus i}, j) \right]$$

Therefore, $\varphi = \varphi'$. Then, the unique CH-decomposable CH-decomposer of any CH-decomposable interaction index is a I-decomposer of this interaction index and hence coincides with the unique I-decomposer.

CH-decomposability \implies I-decomposability.

Proof of Proposition 5.

Let φ and φ' be two C-decomposers of Ψ . Given a game $v \in TU(N)$ and for all $S \in 2^N$ a coalition, we will show by induction on the cardinality of n that $\varphi(v, S) = \varphi'(v, S)$.

Induction basis: Given a game $v \in TU(N)$ and for all $S \in 2^N$, if $n = 1$ then $\Psi(v, S) = \varphi(v, S) = \varphi'(v, S)$.

Induction hypothesis: Given a game $v \in TU(N)$ and for all $S \in 2^N$ with $2 \leq n \leq k$ ($k \in \mathbb{N}$), we suppose that $\varphi(v, S) = \varphi'(v, S)$

Induction step: Given a game $v \in TU(N)$ with $n = k + 1$ and for all $S \in 2^N$,

$$\begin{aligned} \Psi(v, S) &= \varphi(v, S) + \sum_{j \in N \setminus S} \left[\varphi(v, S) - \varphi(v^{N \setminus j}, S) \right] \\ &= \varphi'(v, S) + \sum_{j \in N \setminus S} \left[\varphi'(v, S) - \varphi'(v^{N \setminus j}, S) \right] \end{aligned}$$

$$(n - s + 1)\varphi(v, S) - \sum_{j \in N \setminus S} \varphi(v^{N \setminus j}, S) = (n - s + 1)\varphi'(v, S) - \sum_{j \in N \setminus S} \varphi'(v^{N \setminus j}, S)$$

$$(n - s + 1)\varphi(v, S) = (n - s + 1)\varphi'(v, S)$$

$$\varphi(v, S) = \varphi'(v, S)$$

Lemma 1. *An interaction index Ψ satisfies the Recursive axiom if and only if Ψ is defined as follows.*

For every game $v \in TU(N)$ with $n \geq 2$, $S \subseteq N$ a coalition of N with $2 \leq s \leq n$ and any player $i \in S$, we have

$$\Psi(v, S) = \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i \\ t=\beta}} \Psi(v_{\cup((S \setminus i) \setminus T)}^{N \setminus (S \setminus i)}, i)$$

Proof of Lemma 1.

\Leftarrow) We will show that the interaction index Ψ satisfies the Recursive axiom.

For all game $v \in TU(N)$ with $n \geq 2$, $S \subseteq N$ a coalition of N with $2 \leq s \leq n$ and any player $i \in S$. W.l.o.g. we suppose that $S := \{i_1, i_2, \dots, i_s\}$ and $i = i_1$. We have,

$$\Psi(v_{\cup i_s}^{N \setminus i_s}, S \setminus i_s) = \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1)$$

and

$$\Psi(v^{N \setminus i_s}, S \setminus i_s) = \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus (T \cup i_s))}^{N \setminus (S \setminus i_1)}, i_1)$$

Furthermore,

$$\begin{aligned} \Psi(v, S) &= \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) \\ &= \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i_1 \\ t=\beta; i_s \notin T}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) + \sum_{\beta=1}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i_1 \\ t=\beta; i_s \in T}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) \\ &= \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) + \sum_{\beta=1}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta-1}} \Psi(v_{\cup((S \setminus i_1) \setminus (T \cup i_s))}^{N \setminus (S \setminus i_1)}, i_1) \\ &= \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) + \sum_{\beta=0}^{s-2} (-1)^{\beta+1} \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus (T \cup i_s))}^{N \setminus (S \setminus i_1)}, i_1) \\ &= \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) - \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus (T \cup i_s))}^{N \setminus (S \setminus i_1)}, i_1) \\ &= \Psi(v_{\cup i_s}^{N \setminus i_s}, S \setminus i_s) - \Psi(v^{N \setminus i_s}, S \setminus i_s) \end{aligned}$$

Ψ satisfies the Recursive axiom.

\implies) We suppose that an interaction index Ψ satisfies the Recursive axiom. Let us show by induction on the cardinality of the coalition that Ψ is defined as follows: For all game $v \in TU(N)$ with $n \geq 2$ and $S \subseteq N$ a coalition of N ($2 \leq s \leq n$). W.l.o.g. we suppose that $\{i_1, i_2, \dots, i_s\}$.

$$\Psi(v, S) = \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1)$$

Induction basis: For all game $v \in TU(N)$ and S any coalition of N containing two players.

Let i_1 and i_2 be these two players. Since Ψ satisfies the Recursive axiom, we have

$$\begin{aligned} \Psi(v, S) &= \Psi(v_{\cup i_2}^{N \setminus i_2}, i_1) - \Psi(v^{N \setminus i_2}, i_1) \\ &= \sum_{\beta=0}^{2-1} (-1)^\beta \sum_{\substack{T \subseteq i_2 \\ t=\beta}} \Psi(v_{\cup(i_2 \setminus T)}^{N \setminus i_2}, i_1) \end{aligned}$$

Induction hypothesis: For all game $v \in TU(N)$ with $n \geq 2$, let $k \in \mathbb{N}$ any integer such that $k \leq n$ and let $S = \{i_1, i_2, \dots, i_s\}$ any coalition of N with $2 < s \leq k$. We suppose that

$$\Psi(v, S) = \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1)$$

Induction step: For all game $v \in TU(N)$ with $n \geq 2$, $n > k$; $S := \{i_1, i_2, \dots, i_s\}$ any coalition of N with $s = k + 1$. Let us show that,

$$\Psi(v, S) = \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1)$$

$$\begin{aligned}
\Psi(v, S) &= \Psi(v_{\cup i_s}^{N \setminus i_s}, S \setminus i_s) - \Psi(v^{N \setminus i_s}, S \setminus i_s) \\
&= \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) - \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus (T \cup i_s))}^{N \setminus (S \setminus i_1)}, i_1) \\
&= \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i_1 \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1)
\end{aligned}$$

Proof of Theorem 3.

Let Ψ be an interaction index CH-decomposable. From Proposition 4, the I-decomposer is unique and coincides with the unique CH-decomposable CH-decomposer. Using Proposition 6, the I-decomposer is the interaction index φ given by: For all game $v \in TU(N)$ and for all $j \in N$,

$$\varphi(v, j) = \sum_{\substack{T \subseteq N \\ j \in T}} \frac{m(v^\Psi, T)}{t^2} \quad (3)$$

Let us show by induction that the unique C-decomposer of the interaction index Ψ is the interaction index Ψ_d defined as follows:

For all game $v \in TU(N)$, for all coalition $S \in 2^N$ and for any player $i \in S$,

$$\Psi_d(v, S) = \begin{cases} \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i \\ t=\beta}} \varphi(v_{\cup((S \setminus i) \setminus T)}^{N \setminus (S \setminus i)}, i) & \text{if } s \geq 2 \\ \varphi(v, i) & \text{if } s = 1 \end{cases}$$

From Lemma 1, Ψ_d satisfies the Recursive axiom when the coalition contains more than one player.

Induction basis: For all game $v \in TU(N)$, $S \in 2^N$, if $s = 1$ (that is $S = i$) then Ψ is I-decomposable from the hypothesis and the I-decomposer is the interaction index φ .

$$\Psi_d(v, i) = \varphi(v, i)$$

Induction hypothesis: For all game $v \in TU(N)$, let $k \in \mathbb{N}$ any integer such that $k \leq n$

and let S be any coalition of N with $2 \leq s \leq k$. We suppose that

$$\Psi(v, S) = \Psi_d(v, S) + \sum_{j \in N \setminus S} [\Psi_d(v, S) - \Psi_d(v^{N \setminus j}, S)]$$

Induction step: For all game $v \in TU(N)$, $n > k$ and $S \in 2^N$ such that $s = k + 1$. Let us show that

$$\Psi(v, S) = \Psi_d(v, S) + \sum_{j \in N \setminus S} [\Psi_d(v, S) - \Psi_d(v^{N \setminus j}, S)]$$

Let $i \in S$. Since Ψ satisfies the Recursive axiom, we have

$$\begin{aligned} \Psi(v, S) &= \Psi(v_{\cup i}^{N \setminus i}, S \setminus i) - \Psi(v^{N \setminus i}, S \setminus i) \\ &= \Psi_d(v_{\cup i}^{N \setminus i}, S \setminus i) + \sum_{j \in N \setminus S} [\Psi_d(v_{\cup i}^{N \setminus i}, S \setminus i) - \Psi_d(v_{\cup i}^{N \setminus ij}, S \setminus i)] \\ &\quad - \Psi_d(v^{N \setminus i}, S \setminus i) - \sum_{j \in N \setminus S} [\Psi_d(v^{N \setminus i}, S \setminus i) - \Psi_d(v^{N \setminus ij}, S \setminus i)] \\ &= \Psi_d(v_{\cup i}^{N \setminus i}, S \setminus i) - \Psi_d(v^{N \setminus i}, S \setminus i) + \sum_{j \in N \setminus S} [(\Psi_d(v_{\cup i}^{N \setminus i}, S \setminus i) \\ &\quad - \Psi_d(v^{N \setminus i}, S \setminus i)) - (\Psi_d(v_{\cup i}^{N \setminus ij}, S \setminus i) - \Psi_d(v^{N \setminus ij}, S \setminus i))] \\ &= \Psi_d(v, S) + \sum_{j \in N \setminus S} [\Psi_d(v, S) - \Psi_d(v^{N \setminus j}, S)] \end{aligned}$$

Since Ψ_d satisfies the Recursive axiom.

Ψ_d is a C-decomposer of the interaction index Ψ .

Ψ is C-decomposable and from Proposition 5, the C-decomposer is unique.

Proof of Proposition 7.

Let Ψ be an interaction index that satisfies the recursivity axiom. We suppose that Ψ is C-decomposable and CH-decomposable. Let φ be the C-decomposer. Given a game $v \in TU(N)$, for all $S \in 2^N$ and $\alpha \in \mathbb{N}$

If $S = \{i\}$ then from Proposition 4,

$$\Psi(v, i) = \varphi(v, i) + \sum_{l \in N \setminus S} [\varphi(v, i) - \varphi(v^{N \setminus l}, i)]$$

If $s \geq 2$ then, W.l.o.g, we suppose that $S = \{i_1, i_2, \dots, i_{s-1}, i_s\}$ and we have,

$\Psi(v, S)$

$$\begin{aligned}
&= \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i_1 \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) \\
&= \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i_1 \\ t=\beta}} \left[\varphi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) + \sum_{l \in N \setminus S} \left[\varphi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) - \varphi(v_{\cup((S \setminus i_1) \cup l)}^{N \setminus (S \setminus i_1)}) \right] \right] \\
&= \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i_1 \\ t=\beta}} \varphi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) + \sum_{l \in N \setminus S} \left[\sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i_1 \\ t=\beta}} \varphi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) \right. \\
&\quad \left. - \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i_1 \\ t=\beta}} \varphi(v_{\cup((S \setminus i_1) \cup l)}^{N \setminus (S \setminus i_1)}) \right] \\
&= \varphi(v, S) + \sum_{l \in N \setminus S} \left[\varphi(v, S) - \varphi(v^{N \setminus l}, S) \right]
\end{aligned}$$

Proof of Proposition 9.

Let $v \in TU(N)$ ($n \geq 2$) be a game on N and $S \in 2^N$ with $s \geq 2$. Let $j \in S$,

$\Psi_{Ch}(v, S)$

$$\begin{aligned}
&= \sum_{T \subseteq N \setminus S} \frac{s(s+t-1)!(n-t-s)!}{n!} \Delta_S v(T) \\
&= \sum_{T \subseteq N \setminus S} \frac{s(s+t-1)!(n-t-s)!}{n!} \Delta_{S \setminus j} v(T \cup j) - \sum_{T \subseteq N \setminus S} \frac{s(s+t-1)!(n-t-s)!}{n!} \Delta_{S \setminus j} v(T) \\
&= \sum_{\substack{T \subseteq N \setminus (S \setminus j) \\ j \in T}} \frac{s(s+t-2)!(n-t-s+1)!}{n!} \Delta_{S \setminus j} v(T) - \sum_{T \subseteq N \setminus S} \frac{s(s+t-1)!(n-t-s)!}{n!} \Delta_{S \setminus j} v(T) \\
&= \sum_{T \subseteq N \setminus (S \setminus j)} \frac{s(s+t-2)!(n-t-s+1)!}{n!} \Delta_{S \setminus j} v(T) - \sum_{\substack{T \subseteq N \setminus (S \setminus j) \\ j \in S}} \frac{s(s+t-2)!(n-t-s+1)!}{n!} \Delta_{S \setminus j} v(T) \\
&\quad - \sum_{T \subseteq N \setminus S} \frac{s(s+t-1)!(n-t-s)!}{n!} \Delta_{S \setminus j} v(T) \\
&= \sum_{T \subseteq N \setminus (S \setminus j)} \frac{s(s+t-2)!(n-t-s+1)!}{n!} \Delta_{S \setminus j} v(T) \\
&\quad - \sum_{T \subseteq N \setminus S} \frac{s(s+t-2)!(n-t-s)!(n-t-s+1+s+t-1)}{n!} \Delta_{S \setminus j} v(T) \\
&= s \sum_{T \subseteq N \setminus (S \setminus j)} \frac{(s+t-2)!(n-t-s+1)!}{n!} \Delta_{S \setminus j} v(T) - s \sum_{T \subseteq N \setminus S} \frac{(s+t-2)!(n-t-s)!}{(n-1)!} \Delta_{S \setminus j} v(T) \\
&= \frac{s}{s-1} \left[\Psi_{Ch}(v, S \setminus j) - \Psi_{Ch}(v^{N \setminus j}, S \setminus j) \right]
\end{aligned}$$

For any $i \in S$, we repeatedly apply this process and we obtain,

$$\Psi_{Ch}(v, S) = \left(\prod_{l=0}^{s-2} \frac{s-l}{s-l-1} \right) \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i \\ t=s-\beta-1}} \Psi_{Ch}(v^{N \setminus ((S \setminus i) \setminus T)}, i)$$

Finally, by induction on the cardinality of S , we obtain

$$\Psi_{Ch}(v, S) = s \Psi_{dc}(v, S) + \sum_{j \in N \setminus S} \left[\Psi_{dc}(v, S) - \Psi_{dc}(v^{N \setminus j}, S) \right]$$

where

$$\Psi_{dc}(v, S) = \begin{cases} \left(\prod_{l=0}^{s-2} \frac{s-l}{s-l-1} \right) \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i \\ t=s-\beta-1}} \varphi(v^{N \setminus ((S \setminus i) \setminus T)}, i) & \text{if } s \geq 2 \\ \varphi(v, i) & \text{if } s = 1 \end{cases}$$

φ is the unique I-decomposer of the Shapley solution and it is given by Equation (3). The Chaining interaction index is not C-decomposable.

Proof of Proposition 10.

Let Ψ^P be a cardinal-probabilistic interaction index applied to a game $v \in TU(N)$. From Corollary 2, for any coalition $S \subseteq N$,

$$\Psi^P(v, S) = \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \Delta_{S \cup j} v(T)$$

where $f_n^s(t)$ is given by

$$f_n^s(t) = \begin{cases} \frac{P_n^s(0)}{n-s} & \text{if } t = 0 \\ \frac{t \times f_n^s(t-1) + P_n^s(t)}{n-t-s} & \text{otherwise} \end{cases}$$

and the set $\{P_n^s(t)\}_{t=0, \dots, n-s}$ are nonnegative constants associated to the cardinal-probabilistic interaction index. By Fujimoto et al. [2006], the coefficients $P_n^s(t)$ obey the recurrence relation:

$$P_n^s(t) + P_n^s(t+1) = P_{n-1}^s(t)$$

$$\begin{aligned} & \Psi^P(V, S) + \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} [\Psi^P(v, S) - \Psi^P(v^{N \setminus j}, S)] \\ &= \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \Delta_{S \cup j} v(T) + \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} P_n^s(t+1) \Delta_{S \cup j} v(T) \\ &= \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} [f_n^s(t) - P_n^s(t+1)] \Delta_{S \cup j} v(T) \\ &= \chi(v, S) \end{aligned}$$

From Proposition 5, the C-decomposer is unique.

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