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Sign sequences of log-atomic numbers

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Log-atomic numbers are surreal numbers whose iterated logarithms are monomials, i.e. additively irreducible numbers. Presenting surreal numbers as sign sequences, we give the sign sequence formula for log-atomic numbers. In doing so, we relate log-atomic numbers to fixed-points of certain surreal functions.

INTRODUCTION

The class \mathbf{No} of *surreal numbers* of J. H. Conway [7] is an inductively defined ordered field with additional structure. Conway uses the abstract notion of Dedekind inspired cut as a fundamental constructor to define numbers by well-founded induction.

Indeed, any number $x \in \mathbf{No}$ is obtained from sets L, R of previously defined numbers, as the “simplest” number with $L < x$ and $x < R$. This relation is denoted $x = \{L|R\}$. Conversely, any sets of surreal numbers L, R with $L < R$ give rise to a unique simplest number $\{L|R\}$. Thus the definition of \mathbf{No} comes with several features: an inductively defined order $<$, an order-saturation property, a corresponding ordinal rank called the birthday $\beta(x)$ which represents the minimal ordinal number of inductive steps required to yield x . For instance $0 := \{\emptyset|\emptyset\}$ has birthday 0 whereas $1 := \{\{0\}|\emptyset\}$ has birthday 1 and $1/2 := \{\{0\}|\{1\}\}$ has birthday 2.

The versatility of this construction allowed Conway and several authors after him to define a rich structure on \mathbf{No} . In particular, he defined ring operations that are compatible with the ordering and turn $(\mathbf{No}, +, \times, \leq)$ into an ordered field extension of the reals, as well as an extension of the ordered semi-ring $(\mathbf{On}, \oplus, \otimes, \in)$ of ordinal numbers under Hessenberg sum and product.

Conway also discovered that \mathbf{No} enjoys a natural structure of field of Hahn series as per [10]. Every surreal number can be expressed as a possibly transfinite sum of additively irreducible numbers called *monomials*. Moreover, monomials can be parametrized by a morphism $x \mapsto \omega^x$, called the ω -map, for which Conway gave an equation using the cut presentation of numbers. Whereas the latter presentation is useful to produce inductive definitions of functions, by induction on the birthday for instance, the presentation of numbers as Hahn series is useful to compute certain operations on \mathbf{No} . In particular, it can be used to describe H. Gonshor's exponential function [9], and A. Berarducci and V. Mantova's surreal derivation [5].

The sign sequence presentation of surreal numbers, invented and studied by H. Gonshor in [9], is a way to give precise description of surreal numbers. In this picture, numbers are sequences of signs $+1, -1$ indexed by ordinals, or equivalently, nodes in the binary tree $\{-1, +1\}^{<\mathbf{On}}$. The birthdate measure then coincides with the domain, called *length*, of the sign sequences. A natural relation of *simplicity*, which corresponds to the inclusion of sign sequences in one another and is denoted \sqsubseteq , emerges as a more precise measure of the complexity of numbers.

Sign sequences are by no means an optimal tool to describe surreal numbers in the context of the ordered field, exponential ordered field, Hahn series field, or differential field structures. For instance given numbers x, y with known sign sequences, computing the sign sequence of $x + y$, or even formulating the properties of that of xy in general are open problems. However, sign sequences behave relatively well with respect to operations that preserve simplicity under certain conditions. This includes the ω -map, the transfinite summation which identifies numbers with Hahn series, and the exponentiation of monomials. In this context, sign sequences can be useful tools to understand the behavior of length under the operations, as well as deriving general properties of certain classes of numbers with respect to the relations \leq and \sqsubseteq .

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Describing the sign sequences of elements in a class \mathbf{Y} often requires a reference with respect to which they can be given. One way to give such a reference is to find a parametrization of \mathbf{Y} , i.e. a bijection $F: \mathbf{X} \rightarrow \mathbf{Y}$ for a certain subclass $\mathbf{X} \subseteq \mathbf{No}$, and then describe the sign sequence of $f(x) = y$ in terms of that of x for each $x \in \mathbf{X}$. This presumes that the behavior of function F on sign sequences may be understood, suggesting that F should be compatible in a sense with the relations \leq and \sqsubseteq . There enters the notion of surreal substructure of [3]. Surreal substructures are subclasses \mathbf{S} of \mathbf{No} that are isomorphic to $(\mathbf{No}, \leq, \sqsubseteq)$ under the restrictions of \leq and \sqsubseteq to $\mathbf{S} \times \mathbf{S}$. The isomorphism $\Xi_{\mathbf{S}}: \mathbf{No} \rightarrow \mathbf{S}$ then being unique, one may rely on it to relate the sign sequence of $\Xi_{\mathbf{S}} z \in \mathbf{S}$ with that of z .

Surreal substructures naturally appear when defining certain operations on \mathbf{No} , see for instance [3, Sections 6 and 7]. The study of surreal substructure yields tools to express and compute sign sequences. In particular, expressing a structure $\Xi_{\mathbf{S}}$ using classes of fixed points for given parametrizations yields a method to compute the sign sequence formula of $\Xi_{\mathbf{S}}$. Relying extensively on [3], we will develop the relevant notions in Sections 1 and 2.

A surreal substructure of particular interest is the class \mathbf{La} of *log-atomic numbers*. Those are numbers a such that the n -fold iteration of $\log^{\circ n} a$ of the logarithm at a yields a monomial for each $n \in \mathbb{N}$. This structure plays a crucial role in the definition of derivations on \mathbf{No} that are compatible with the exponential and the structure of field of series of \mathbf{No} . It is also used in the investigation of the properties of expansions of numbers as transseries. This class was characterized by Berarducci and Mantova [5, Section 5] who defined such a derivation and proved a fundamental structure property for \mathbf{No} . Finally \mathbf{La} plays a role in the definition of the first surreal hyperexponential function E_{ω} [4]. Our goal in this article is to compute the sign sequence formula for $\Xi_{\mathbf{La}}$. This will in particular give the sign sequence for $E_{\omega}(a)$ for each surreal number $a > \mathbb{R}$ whose length is strictly below the first ε -number $\varepsilon_0 \in \mathbf{On}$.

In this task, we are continuing work of S. Kuhlmann and M. Matusinski in [13]. Indeed, they considered a surreal substructure \mathbf{K} properly contained in \mathbf{La} and determined its sign sequence formula. The relation between \mathbf{K} and \mathbf{La} was subject to the conjecture

$$\mathbf{La} \stackrel{?}{=} \log^{\circ \mathbb{Z}} \mathbf{K} \quad [13, \text{Conjecture 5.2}],$$

which turned out to be false [5, Proposition 5.24]. The correct relation between \mathbf{La} and \mathbf{K} was later found by Mantova and Matusinski [14]. We will rely on their sign sequence formula and a presentation of \mathbf{La} using classes of fixed points to derive our formula.

1 Numbers and sign sequences

1.1 Numbers as sign sequences

As Gonshor, we define numbers as sign sequences.

DEFINITION 1. *A surreal number is a map $x: \ell(x) \rightarrow \{-1, 1\}; \alpha \mapsto x[\alpha]$, where $\ell(x) \in \mathbf{On}$ is an ordinal number. We call $\ell(x)$ the **length** of x and the map $x: \ell(x) \rightarrow \{-1, 1\}$ the **sign sequence** of x . We write \mathbf{No} for the class of surreal numbers.*

Given a surreal number $x \in \mathbf{No}$, we extend its sign sequence with $x[\alpha] = 0$ for all $\alpha \geq \ell(x)$. Given $x \in \mathbf{No}$ and $\alpha \in \mathbf{On}$, we also introduce the *restriction* $y = x \upharpoonright \alpha \in \mathbf{No}$ to α as being the initial segment of x of length α , i.e. $y[\beta] = x[\beta]$ for $\beta < \alpha$ and $y[\beta] = 0$ for $\beta \geq \alpha$.

The ordering \leq on \mathbf{No} is lexicographical: given distinct elements $x, y \in \mathbf{No}$, there exists a smallest ordinal α with $x[\alpha] \neq y[\alpha]$ and we set $x < y$ if and only if $x[\alpha] < y[\alpha]$.

For $x, y \in \mathbf{No}$, we say that x is *simpler* than y , and write $x \sqsubseteq y$, if $x = y \upharpoonright \ell(x)$. The partially ordered class $(\mathbf{No}, \sqsubseteq)$ is well-founded. We write $x \sqsubset y$ if $x \sqsubseteq y$ and $x \neq y$. For $x \in \mathbf{No}$, we write

$$x_{\sqsubset} := \{a \in \mathbf{No} : a \sqsubset x\}$$

for the set of numbers that are strictly simpler than x . The set $(x_{\sqsubseteq}, \sqsubseteq)$ is well-ordered with order type $\text{ot}(x_{\sqsubseteq}, \sqsubseteq) = \ell(x)$. Moreover, x_{\sqsubseteq} is the union of the sets

$$x_L := \{y \in \mathbf{No} : y \sqsubset x, y < x\} \quad \text{and} \quad x_R := \{y \in \mathbf{No} : y \sqsubset x, y > x\}.$$

Every linearly ordered subset X of $(\mathbf{No}, \sqsubseteq)$ has a *supremum* $s = \sup_{\sqsubseteq} X$ in \mathbf{No} . Indeed, we have $\ell(s) = \sup \ell(Z)$, and $s[\alpha] = x[\alpha]$ for all $x \in X$ with $\alpha < \ell(x)$. Numbers x that are equal to $\sup_{\sqsubseteq} x_{\sqsubseteq}$ are called *limit numbers*; other numbers are called *successor numbers*. Limit numbers are exactly the numbers whose length is a limit ordinal.

Notice that \mathbf{No} is a proper class. For instance, the linearly ordered class (\mathbf{On}, \leq) is embedded into the partial order $(\mathbf{No}, \sqsubseteq)$ through the map $\alpha \mapsto (1)_{\beta < \alpha} : \mathbf{On} \rightarrow \mathbf{No}$. We will thus identify ordinals numbers as surreal numbers.

1.2 Monomials

Any non-archimedean ordered field K admits a non-trivial valuation called the natural valuation [1, Section 3.5]. The natural valuation of $x \in K^>$ is related to its *archimedean class*, i.e. the class

$$\mathcal{H}[x] := \{y \in K^> : \exists q \in \mathbb{Q}^>, q^{-1}x < y < qx\}.$$

In NBG, the corresponding valued ordered field can be embedded [11] into a field of generalized Hahn series, as defined by Hahn [10]. The existence of such embeddings, called Kaplansky embeddings, usually requires the axiom of choice.

Having defined the structure of ordered field, Conway noticed that for \mathbf{No} , the instances where choice is required could be circumvented by the use of the inductive nature of \mathbf{No} . Indeed, in \mathbf{No} , each archimedean class $\mathcal{H}[x]$ has a unique \sqsubseteq -minimal element denoted \flat_x , and the class $\mathbf{Mo} := \{\flat_x : x \in \mathbf{No}^>\}$ is a subgroup of $(\mathbf{No}^>, \times)$ which can be used to carry the definition of the Kaplansky embedding.

DEFINITION 2. A **monomial** is a number m which is simplest in the class $\mathcal{H}[m]$, i.e. a number of the form $m = \flat_x$ for a certain $x \in \mathbf{No}^>$.

Moreover, Conway defined a parametrization $z \mapsto \hat{\omega}^z$ of \mathbf{Mo} . This is a strictly increasing morphism $(\mathbf{No}, +, \leq) \rightarrow (\mathbf{Mo}, \times, \leq)$ which also preserves simplicity:

$$\forall x, y \in \mathbf{No}, x \sqsubseteq y \iff \hat{\omega}^x \sqsubseteq \hat{\omega}^y.$$

The operation $\alpha \mapsto \hat{\omega}^\alpha : \mathbf{On} \rightarrow \mathbf{No}$ coincides with the ordinal exponentiation with basis ω .

1.3 Arithmetic on sign sequences

For ordinals α, β , we will denote their ordinal sum, product, and exponentiation by $\alpha \dot{+} \beta$, $\alpha \dot{\times} \beta$ and $\hat{\alpha}^\beta$. Here, we introduce the operations $\dot{+}$ and $\dot{\times}$ of [3, Section 3.2] on \mathbf{No} . Those are natural extensions of ordinal arithmetic to \mathbf{No} , which we will use in order to describe sign sequences.

For numbers x, y , we write $x \dot{+} y$ for the number whose sign sequence is the concatenation of that of y at the end of that of x . So $x \dot{+} y$ is the number of length $\ell(x \dot{+} y) = \ell(x) \dot{+} \ell(y)$, which satisfies

$$\begin{aligned} (x \dot{+} y)[\alpha] &= x[\alpha] & (\alpha < \ell(x)) \\ (x \dot{+} y)[\ell(x) \dot{+} \beta] &= y[\beta] & (\beta < \ell(y)) \end{aligned}$$

The operation $\dot{+}$ clearly extends ordinal sum. We have $x \dot{+} 0 = 0 \dot{+} x = x$ for all $x \in \mathbf{No}$.

We write $x \dot{\times} y$ for the number of length $\ell(x) \dot{\times} \ell(y)$ whose sign sequence is defined by

$$(x \dot{\times} y)[\ell(x) \dot{\times} \alpha \dot{+} \beta] = y[\alpha] x[\beta] \quad (\alpha < \ell(y), \beta < \ell(x))$$

The operation $\dot{\times}$ extends ordinal product. Special cases can be described informally: given $x \in \mathbf{No}$ and $\alpha \in \mathbf{On}$, the number $x \dot{\times} \alpha$ is the concatenation of x with itself to the right “ α -many times”, whereas $\alpha \dot{\times} x$ is the number obtained from x by replacing each sign by itself α -many times. We have $x \dot{\times} 1 = x$ and $x \dot{\times} (-1) = -x$ for all $x \in \mathbf{No}$.

The operations also enjoy the properties which extend that of their ordinal counterparts as is illustrated hereafter. We refer to [3, Section 3.2] for more details.

LEMMA 3. [3, Lemma 3.1] *For $x, y, z \in \mathbf{No}$, we have*

- a) $x \dot{+} (y \dot{+} z) = (x \dot{+} y) \dot{+} z$, $x \dot{+} 0 = 0 \dot{+} x = x$, and $x \dot{+} y = \sup_{\square} (x \dot{+} y_{\square})$ if y is a limit.
- b) $x \dot{\times} (y \dot{\times} z) = (x \dot{\times} y) \dot{\times} z$, $x \dot{\times} (y \dot{+} z) = (x \dot{\times} y) \dot{+} (x \dot{\times} z)$, and $x \dot{\times} y = \sup_{\square} (x \dot{\times} y_{\square})$ if y is a limit.

1.4 Sign sequences formulas

If α is an ordinal and f is a function $\alpha \rightarrow \mathbf{No}$, then (α, f) uniquely determines a surreal number $[\alpha, f]$ defined inductively by the rules

- $[0, f] = 0$,
- $[\gamma \dot{+} 1, f \upharpoonright (\gamma \dot{+} 1)] = [\gamma, f \upharpoonright \gamma] \dot{+} f(\gamma)$ for $\beta < \alpha$,
- $[\gamma, f \upharpoonright \gamma] = \sup_{\square} \{[\mu, f \upharpoonright \mu] : \mu < \gamma\}$ for limit ordinals $\gamma \leq \alpha$.

We say that (α, f) is a *sign sequence formula* for $[\alpha, f]$, which we identify with the informal expression

$$\sum_{\gamma < \alpha} f(\gamma) = f(0) \dot{+} f(1) \dot{+} \dots \dot{+} f(\beta) \dot{+} \dots$$

where γ ranges in α . For instance, the ordered pair $(\omega, ((-1)^n)_{n < \omega})$ is a sign sequence formula of $1/2 \dot{\times} \omega = 1 \dot{+} (-1) \dot{+} 1 \dot{+} (-1) \dot{+} \dots = 2/3$. In general, we look for such formulas where f alternates between $\mathbf{On}^>$ and $-\mathbf{On}^>$, that is, where f ranges in $\mathbf{On}^> \cup -\mathbf{On}^>$ and where, for each ordinal β with $\beta \dot{+} 1 < \alpha$, we have $f(\beta) f(\beta \dot{+} 1) < 0$. It is easy to see that every surreal number admits a unique such alternating sign sequence formula. We refer to this formula as *the sign sequence formula* of said number.

If $F: \mathbf{No} \rightarrow \mathbf{No}$ is a function, we may look for a function Ψ whose value at each number x is a sign sequence formula of $F(x)$. We then consider Ψ as a *sign sequence formula* for F .

As an example, we now state Gonshor's results regarding the sign sequences of monomials. For $z \in \mathbf{No}$, we write τ_z for the order type of (z_L, \square) , that is the order type $z^{-1}[\{1\}]$ when z is seen as a sign sequence, or equivalently the ordinal number of signs 1 in the sign sequence of z . We have $\tau_z = 0$ if and only if $z \in -\mathbf{On}$. Gonshor found the following sign sequence formula for monomials:

PROPOSITION 4. [9, Theorem 5.11] *Let z be a number. The sign sequence formula for $\dot{\omega}^z$ is*

$$\dot{\omega}^z = 1 \dot{+} \sum_{\beta < \ell(x)} (z[\beta] \dot{\omega}^{\tau_z(\beta+1)+1}).$$

For instance, we have

$$\begin{aligned} \sqrt{\dot{\omega}} &= \dot{\omega}^{1/2} = \dot{\omega} \dot{+} (-\dot{\omega}^2) \quad \text{and} \\ \log \dot{\omega} &= \dot{\omega}^{\dot{\omega}-1} = \dot{\omega} \dot{+} (-\dot{\omega}^3). \end{aligned}$$

Let us also specify a consequence that we will use often in what follows.

LEMMA 5. *Let $x, y \in \mathbf{No}$ be such that the maximal ordinal α with $\alpha \square y$ is a limit. We have $\dot{\omega}^{x \dot{+} y} = \dot{\omega}^x \dot{+} \dot{\omega}^{\tau_x \dot{+} y}$.*

PROOF. Let α be maximal with $\alpha \sqsubseteq y$ and write $y = \alpha \dot{+} z$. Since y is strictly positive, we have $\alpha > 0$, whence $\alpha \geq \omega$. Let

$$m := (z[0] \dot{\omega}^{\tau_x + \alpha + z + 1}) \dot{+} (z[1] \dot{\omega}^{\tau_x + \alpha + z + 1}) \dot{+} \dots \dot{+} (z[\gamma] \dot{\omega}^{\tau_x + \alpha + (z^{(\gamma+1)}) + 1}) \dot{+} \dots$$

where γ ranges in $\ell(z)$. Proposition 4 yields $\dot{\omega}^{\tau_x + y} = \omega^{\tau_x + \alpha + z} = \omega^{\tau_x + \alpha} \dot{+} m$. On the other hand, we have $\omega^{x + y} = \omega^x \dot{+} n \dot{+} m$ where n is defined as the transfinite concatenation

$$n = (y[0] \dot{\omega}^{\tau_x + 1 + 1}) \dot{+} \dots \dot{+} (y[\beta] \dot{\omega}^{\tau_x + \beta + 1}) \dot{+} \dots,$$

where β ranges in α . For $\beta < \alpha$, we have $y[\beta] = 1$ and $\tau_x + \beta = \tau_x \dot{+} \beta$, so

$$\begin{aligned} n &= \dot{\omega}^{\tau_x + 1 + 1} \dot{+} \dots \dot{+} \dot{\omega}^{\tau_x + \beta + 1} \dot{+} \dots \\ &= \dot{\omega}^{\tau_x + \alpha}. \end{aligned}$$

It follows that $\dot{\omega}^{x + y} = \dot{\omega}^x \dot{+} \dot{\omega}^{\tau_x + y}$. □

2 Surreal substructures

2.1 Surreal substructures

DEFINITION 6. A **surreal substructure** is a subclass \mathbf{S} of \mathbf{No} such that $(\mathbf{No}, \leq, \sqsubseteq)$ and $(\mathbf{S}, \leq, \sqsubseteq)$ are isomorphic. The isomorphism $(\mathbf{No}, \leq, \sqsubseteq) \rightarrow (\mathbf{S}, \sqsubseteq, \leq)$ is unique, denoted $\Xi_{\mathbf{S}}$ and called the **parametrization** of \mathbf{S} .

Example 7. Here are examples of common surreal substructures and corresponding parametrizations.

- For $x \in \mathbf{No}$, the class $x \dot{+} \mathbf{No} = \mathbf{No}^{\dot{+}x} := \{y \in \mathbf{No} : x \sqsubseteq y\}$ has parametrization $z \mapsto x \dot{+} z$. We have

$$\begin{aligned} \mathbf{No}^{\dot{+}1} &= \mathbf{No}^> := \{y \in \mathbf{No} : y > 0\}, \\ \mathbf{No}^{\dot{+}\omega} &= \mathbf{No}^{>, >} := \{y \in \mathbf{No} : y > \mathbb{R}\}, \\ \mathbf{No}^{\dot{+}\omega^{-1}} &= \mathbf{No}^{>, <} := \{y \in \mathbf{No} : 0 < y < \mathbb{R}^>\}. \end{aligned}$$

- For $x \in \mathbf{No}^>$, the class $x \dot{\times} \mathbf{No} := \{x \dot{\times} z : z \in \mathbf{No}\}$ has parametrization $z \mapsto x \dot{\times} z$. We have

$$\begin{aligned} 1 \dot{\times} \mathbf{No} &= \mathbf{No} \\ \omega \dot{\times} \mathbf{No} &= \mathbf{No}_{>} := \{y \in \mathbf{No} : \text{supp } y > 1\}. \end{aligned}$$

- The class \mathbf{Mo} of monomials has parametrization $z \mapsto \dot{\omega}^z$.

2.2 Convexity

Let \mathbf{S} be a surreal substructure and let \mathbf{C} be a *convex* subclass of \mathbf{S} . This means that for $x, y, z \in \mathbf{S}$ with $x \leq y \leq z$, we have $x, y \in \mathbf{C} \implies y \in \mathbf{C}$. Then we have a simple criterion to decide whether \mathbf{C} is a surreal substructure:

PROPOSITION 8. [3, Proposition 4.29] *A convex subclass \mathbf{C} of \mathbf{S} is a surreal substructure if and only if every subset of \mathbf{C} has strict upper and lower bounds in \mathbf{C} .*

In particular, non-empty open intervals $(a, b) \cap \mathbf{S}$, where $a, b \in \mathbf{S} \cup \{-\infty, +\infty\}$, are surreal substructures. The following proposition gives a sign sequence formula for $\Xi_{(a,b)}$ in certain particular cases.

LEMMA 9. *Let α be a non-zero ordinal. Let a, b and z be numbers.*

- We have $\Xi_{(a, a + \alpha)} z = a \dot{+} 1 \dot{+} z$ if $z \not\prec \alpha_L$,*

and $\Xi_{(a, a+\alpha)}(\alpha \dot{+} z) = a \dot{+} \alpha \dot{+} (-1) \dot{+} z$.

- b) We have $\Xi_{(b \dot{+} (-\alpha), b)} z = b \dot{+} (-1) \dot{+} z$ if $z \not\prec -(\alpha_L)$,
and $\Xi_{(b \dot{+} (-\alpha), b)}((-\alpha) \dot{+} z) = b \dot{+} (-\alpha) \dot{+} 1 \dot{+} z$.

PROOF. First notice that we have $\alpha \sqsubseteq z$ if and only if $z > (\alpha_L)$ and $(-\alpha) \sqsubseteq z$ if and only if $z < -(\alpha_L)$ so the given descriptions cover all cases. We will only derive the formulas for the interval $(a, a + \alpha)$, the other ones being symmetric. The number $a + 1$ is the root of $(a, a + \alpha)$ so for $z \in \mathbf{No}$, we have $\Xi_{(a, a+\alpha)} z = a + 1 + u_z$ for a certain number u_z . The class $\mathbf{No}^{\not\prec \alpha_L}$ is convex in \mathbf{No} and for $z \in \mathbf{No}^{\not\prec \alpha_L}$, we have $a + 1 + z \in (a, a + \alpha)$. The function $z \mapsto a + 1 + z$ defined on \mathbf{No} defines a surreal isomorphism, so we have $\Xi_{(a, a+\alpha)} z = a + 1 + z$ if $z \in \mathbf{No}^{\not\prec \alpha_L}$. On the other hand, we have

$$\begin{aligned} \mathbf{No} \setminus \mathbf{No}^{\not\prec \alpha_L} &= \mathbf{No}^{\supseteq \alpha}, \quad \text{and} \\ (a, a + \alpha) \setminus (a + 1 + \mathbf{No}^{\not\prec \alpha_L}) &= a + \alpha + (-1) \dot{+} \mathbf{No}. \end{aligned}$$

It follows that $\Xi_{(a, a+\alpha)} \upharpoonright \mathbf{No}^{\supseteq \alpha} = \Xi_{\mathbf{No}^{\supseteq a+\alpha+(-1)}}$, which yields the other part of the description. \square

2.3 Imbrication

Let \mathbf{U}, \mathbf{V} be two surreal substructures. Then there is a unique (\leq, \sqsubseteq) -isomorphism

$$\Xi_{\mathbf{V}}^{\mathbf{U}} := \Xi_{\mathbf{V}} \Xi_{\mathbf{U}}^{-1}: \mathbf{U} \rightarrow \mathbf{V}$$

that we call the *surreal isomorphism* between \mathbf{S} and \mathbf{T} . The composition $\Xi_{\mathbf{U}} \circ \Xi_{\mathbf{V}}$ is also an embedding, so its image $\mathbf{U} \prec \mathbf{V} := \Xi_{\mathbf{U}} \Xi_{\mathbf{V}} \mathbf{No}$ is again a surreal substructure that we call the *imbrication* of \mathbf{V} into \mathbf{U} . Given $n \in \mathbb{N}$, we write $\mathbf{U}^{\prec n}$ for the n -fold imbrication of \mathbf{U} into itself. We have $\Xi_{\mathbf{U}^{\prec n}} = \Xi_{\mathbf{U}}^n$.

Example 10. For $x, y \in \mathbf{No}$, it follows from Lemma 3 that we have

$$\begin{aligned} (x \dot{+} \mathbf{No}) \prec (y \dot{+} \mathbf{No}) &= (x \dot{+} y) \dot{+} \mathbf{No}, \\ (x \dot{\times} \mathbf{No}) \prec (y \dot{\times} \mathbf{No}) &= (x \dot{\times} y) \dot{\times} \mathbf{No}, \quad \text{and} \\ (x \dot{\times} \mathbf{No}) \prec (y \dot{+} \mathbf{No}) &= (x \dot{\times} y) \dot{+} (x \dot{\times} \mathbf{No}). \end{aligned}$$

Less elementary results include the relation $\mathbf{Mo} \prec \mathbf{No}^{\supset} = \mathbf{No}^{\supset} \prec \mathbf{Mo}$ of [3, Proposition 7.3].

The relation of imbrication is related to the inclusion of surreal substructures in the following way:

PROPOSITION 11. [3, Proposition 4.11] *Given two surreal substructures \mathbf{U}, \mathbf{V} , we have $\mathbf{U} \subseteq \mathbf{V}$ if and only if there is a surreal substructure \mathbf{W} with $\mathbf{U} = \mathbf{V} \prec \mathbf{W}$.*

2.4 Fixed points

DEFINITION 12. *Let \mathbf{S} be a surreal substructure. We say that a number z is **S-fixed** if $\Xi_{\mathbf{S}} z = z$. We write $\mathbf{S}^{\prec \omega}$ for the class of **S-fixed** numbers.*

If $\mathbf{U}, \mathbf{V}, \mathbf{W}$ are surreal substructures with $\mathbf{U} = \mathbf{V} \prec \mathbf{W}$, then the class $\mathbf{W}^{\prec \omega}$ is that of numbers z with $\Xi_{\mathbf{U}} z = \Xi_{\mathbf{V}} z$.

PROPOSITION 13. [3, Proposition 5.2] *If \mathbf{S} is a surreal substructure, then $\mathbf{S}^{\prec \omega} = \bigcap_{n \in \mathbb{N}} \mathbf{S}^{\prec n}$.*

Notice that \mathbf{S} may not itself be a surreal substructure in general. We thus consider the following notion of closure:

DEFINITION 14. We say that \mathbf{S} is **closed** if we have $\Xi_{\mathbf{S}} \sup_{\sqsubseteq} Z = \sup_{\sqsubseteq} \Xi_{\mathbf{S}} Z$ for all non-empty set $Z \subset \mathbf{No}$ such that (Z, \sqsubseteq) is linearly ordered.

LEMMA 15. [3, Lemma 5.12] If \mathbf{U} and \mathbf{V} are closed surreal substructures, then so is $\mathbf{U} \prec \mathbf{V}$.

PROPOSITION 16. [3, Corollary 5.14] If \mathbf{S} is closed, then $\mathbf{S}^{<\omega}$ is a closed surreal substructure.

In general, when $\mathbf{S}^{<\omega}$ is a surreal substructure, we write $\Xi_{\mathbf{S}^{<\omega}} =: \Xi_{\mathbf{S}}^{\omega}$.

Example 17. Here are some examples of structures of fixed points (see [3, Example 5.3]).

- For $x \in \mathbf{No}$, the structure $\mathbf{No}^{\exists x}$ is closed, with $(\mathbf{No}^{\exists x})^{<\omega} = \mathbf{No}^{\exists x \times \omega}$.
- For $x \in \mathbf{No}^>$, the structure $x \times \mathbf{No}$ is closed, with $(x \times \mathbf{No})^{<\omega} = x^{\infty} \times \mathbf{No}$ where $x^{\infty} := \sup_{\sqsubseteq} \{x, x \times x, x \times x \times x\}$.
- The structure \mathbf{Mo} is closed, and the parametrization $\Xi_{\mathbf{Mo}}^{\omega}$ of $(\mathbf{Mo})^{<\omega}$ is usually denoted $z \mapsto \varepsilon_z$, since it extends the ordinal function $\zeta \mapsto \varepsilon_{\zeta}$ which parametrizes the fixed points of $\zeta \mapsto \omega^{\zeta}$.
- The structure $1 + \mathbf{Mo}^{<}$ (where $\mathbf{Mo}^{<} = \mathbf{Mo} \cap \mathbf{No}^{<}$) is closed. Moreover, its structure of fixed points $(1 + \mathbf{Mo}^{<})^{<\omega}$ coincides with the class of fixed points of the function $z \mapsto 1 + \dot{\omega}^{z-3/2}$, although this last one is not a surreal isomorphism. Informal expansions

$$x = 1 + \dot{\omega}^{-1/2 + \dot{\omega}^{-1/2 + \dot{\omega}^{\cdot}}},$$

of numbers in this class can be replaced by the formulation $\exists z \in \mathbf{No}, x = \Xi_{1 + \mathbf{Mo}^{<}}^{\omega} z$. The sign sequence of x can be computed in terms of that of z using Proposition 19 below.

2.5 Sign sequence formulas for closed structures

Let \mathbf{S} be a closed surreal substructure and let $\Psi_{\mathbf{S}} = (\alpha_z, f_z)_{z \in \mathbf{No}}$ denote its alternating sign sequence formula. This formula is “continuous” or “closed” in the sense that for any limit number z , we have $\alpha_z = \sup_{\sqsubseteq} \alpha_{z_{\sqsubset}}$ and $f_z = \bigcup f_{z_{\sqsubset}}$. If for each surreal number u and for $\sigma \in \{-1, 1\}$ we write $\phi_{u+\sigma}$ for the non-zero surreal number defined by $(\Xi_{\mathbf{S}} u) \dot{+} \phi_{u+\sigma}^{\mathbf{S}} = \Xi_{\mathbf{S}}(u \dot{+} \sigma)$, we then have

$$\forall z \in \mathbf{No}, \Xi_{\mathbf{S}} z = \Xi_{\mathbf{S}} 0 \dot{+} \phi_{z1}^{\mathbf{S}} \dot{+} \phi_{z12}^{\mathbf{S}} \dot{+} \cdots \dot{+} \phi_{z1(\beta+1)}^{\mathbf{S}} \dot{+} \cdots$$

where β ranges in $\ell(z)$. Thus it is enough to compute the numbers $\phi_{u+\sigma}$ to determine sign sequences of element of \mathbf{S} . In the case in Gonshor's formulas for the ω -map and the ε -map and Kuhlmann-Matusinski's formula for the κ -map, those numbers are ordinals or opposites of ordinals. If \mathbf{S} is not closed, then for every limit number a for which there will be an additional term $\iota(a) \in \mathbf{No}$ such that $\delta(a)$ is not always zero, and that we have, for all $z \in \mathbf{No}$,

$$\Xi_{\mathbf{S}} z = \Xi_{\mathbf{S}} 0 \dot{+} \phi_{z1}^{\mathbf{S}} \dot{+} \phi_{z12}^{\mathbf{S}} \dot{+} \cdots \dot{+} \delta(z1\omega) \dot{+} \phi_{z1(\omega+1)}^{\mathbf{S}} \dot{+} \cdots \dot{+} \delta(z1(\omega \times \eta)) \dot{+} \cdots \dot{+} \phi_{z1(\beta+1)}^{\mathbf{S}} \dot{+} \cdots,$$

where $\omega \times \eta \leq \beta < \ell(z)$. Such terms $\delta(a)$ must then be computed independently. Thus it may be a good first step towards computing a sign sequence formula for \mathbf{S} to check whether it is closed or not, and where closure defects occur.

Example 18. The parameters $\phi_{z+\sigma}$ for the ω -map are

$$\begin{aligned} \phi_{z+(-1)}^{\mathbf{Mo}} &= \dot{\omega}^{\tau_z+1} \quad \text{and} \\ \phi_{z+1}^{\mathbf{Mo}} &= \dot{\omega}^{\tau_z+2} \quad \text{for all } z \in \mathbf{No}. \end{aligned}$$

By [9, Chapter 9], the sign sequence formula for ε is given by the parameters:

$$\begin{aligned}\phi_{z+(-1)}^{\mathbf{Mo}^{<\omega}} &= -\varepsilon_z^\omega \quad \text{and} \\ \phi_{z+1}^{\mathbf{Mo}^{<\omega}} &= \varepsilon_{\tau_{z+1}} \quad \text{for all } z \in \mathbf{No}.\end{aligned}$$

Assume again that \mathbf{S} is a closed surreal substructure and let $z \in \mathbf{No}$. Our goal in this paragraph is to compute $\Xi_{\mathbf{S}}^\omega z$ provided we know $\Xi_{\mathbf{S}}^\omega$ on z_{\sqsubset} . The identity $\mathbf{S}^{<\omega} = \bigcap_{n \in \mathbb{N}} \Xi_{\mathbf{S}}^n \mathbf{No}$ suggests that $\Xi_{\mathbf{S}}^\omega$ may be a form of limit of $\Xi_{\mathbf{S}}^n$ as n tends to infinity. Indeed, we claim that one can always find $a_z \in \mathbf{No}$ such that $\Xi_{\mathbf{S}}^{\mathbb{N}} a_z$ is a \sqsubseteq -chain with supremum $\Xi_{\mathbf{S}}^\omega z$. More precisely, we have:

PROPOSITION 19. *Let \mathbf{S} is a closed surreal substructure and let $z \in \mathbf{No}$. We have*

$$\begin{aligned}\Xi_{\mathbf{S}}^\omega z &= \sup_{\sqsubseteq} \Xi_{\mathbf{S}}^\omega z_{\sqsubset} \quad \text{if } z \text{ is a limit, and} \\ \Xi_{\mathbf{S}}^\omega z &= \sup_{\sqsubseteq} \Xi_{\mathbf{S}}^{\mathbb{N}} (\Xi_{\mathbf{S}}^\omega u \dot{+} \sigma) \quad \text{of } z = u \dot{+} \sigma \text{ for certain } u \in \mathbf{No}, \sigma \in \{-1, 1\}.\end{aligned}$$

PROOF. Let $Z = \Xi_{\mathbf{S}}^\omega z_{\sqsubset}$, and set $s_z = \sup_{\sqsubseteq} Z$. Since \mathbf{S} is closed, so is $\mathbf{S}^{<\omega}$, which implies that s_z is \mathbf{S} -fixed.

Assume that z is a limit number. Notice that s_z is the simplest element of $\mathbf{S}^{<\omega}$ with $\Xi_{\mathbf{S}}^\omega z_L < s_z < \Xi_{\mathbf{S}}^\omega z_R$, whereas z is the simplest number with $z_L < z < z_R$. We deduce that $s_z = \Xi_{\mathbf{S}}^\omega z$.

Otherwise, there is a number u and a sign $\sigma \in \{-1, 1\}$ with $z = u \dot{+} \sigma$. Let

$$a_z := (\Xi_{\mathbf{S}}^\omega u) \dot{+} \sigma.$$

Since the sign of $a_z - (\Xi_{\mathbf{S}}^\omega u)$ is σ , the sign of $\Xi_{\mathbf{S}} a_z - \Xi_{\mathbf{S}} (\Xi_{\mathbf{S}}^\omega u) = \Xi_{\mathbf{S}} a_z - (\Xi_{\mathbf{S}}^\omega u)$ is σ as well, so $a_z \sqsubseteq \Xi_{\mathbf{S}} a_z$. It follows by induction that $\Xi_{\mathbf{S}}^{\mathbb{N}} a_z$ is a \sqsubseteq -chain whose supremum y lies in each $\Xi_{\mathbf{S}}^n \mathbf{S}$ for $n \in \mathbb{N}$ by closure of those structures, so $y \in \mathbf{S}^{<\omega}$. Since the sign of $\Xi_{\mathbf{S}}^\omega z - \Xi_{\mathbf{S}}^\omega u$ is σ , we have $a_z \sqsubseteq \Xi_{\mathbf{S}}^\omega z$, whence $y \sqsubseteq \Xi_{\mathbf{S}}^\omega z$. Now $\Xi_{\mathbf{S}}^\omega z$ is the simplest \mathbf{S} -fixed number such that the sign of $\Xi_{\mathbf{S}}^\omega z - \Xi_{\mathbf{S}}^\omega u$ is σ , whence $y = \Xi_{\mathbf{S}}^\omega z$. \square

3 Exponentiation

In this section, we define the exponential function of [9, Chapter 10] as well as the surreal substructures \mathbf{La} and \mathbf{K} .

3.1 Surreal exponentiation

Inductive equations Recall that by Conway's construction, or in Gonshor's setting by [9, Theorem 2.1], given sets L, R of numbers with $L < R$, there is a unique \sqsubseteq -minimal, or simplest, number $\{L|R\}$ with

$$L < \{L|R\} < R.$$

Let \mathbf{S} be a surreal substructure. Carrying the previous property through the parametrization of \mathbf{S} , we obtain, given a surreal substructure \mathbf{S} and subsets $L, R \subset \mathbf{S}$ with $L < R$, a unique \sqsubseteq -minimal element $\{L|R\}_{\mathbf{S}}$ of \mathbf{S} with

$$L < \{L|R\}_{\mathbf{S}} < R.$$

In order to write certain equations, we will write, for $x \in \mathbf{S}$, $x_L^{\mathbf{S}} = \mathbf{S} \cap x_L$ and $x_R^{\mathbf{S}} := x_R \cap \mathbf{S}$. Notice that we have $x = \{x_L^{\mathbf{S}} | x_R^{\mathbf{S}}\}_{\mathbf{S}}$ by definition. Thus if \mathbf{T} is another surreal substructure, the surreal isomorphism $\Xi_{\mathbf{T}}^{\mathbf{S}}$ sends x onto

$$\Xi_{\mathbf{T}}^{\mathbf{S}} x = \{\Xi_{\mathbf{T}}^{\mathbf{S}} x_L^{\mathbf{S}} | \Xi_{\mathbf{T}}^{\mathbf{S}} x_R^{\mathbf{S}}\}_{\mathbf{T}}. \quad (1)$$

Surreal exponentiation Gonshor uses an inductive equation to define the exponential function. Given $n \in \mathbb{N}$ and $x \in \mathbf{No}$, set

$$[x]_n = \sum_{k \leq n} \frac{x^k}{k!}.$$

Let $x \in \mathbf{No}$ be such that \exp is defined on x_{\square} . Notice that for instance, for $x' \in x_L$ and $n \in \mathbb{N}$, we should have

$$\exp(x) = \exp(x') \exp(x - x') > \exp(x') [x - x']_n,$$

whereas similar inequalities hold for $x'' \in x_R$. This led Gonshor to define

$$\exp x = \left\{ 0, [x - x_L]_{\mathbb{N}} \exp x_L, [x - x_R]_{2\mathbb{N}+1} \exp x_R \mid \frac{\exp x_R}{[x - x_R]_{2\mathbb{N}+1}}, \frac{\exp x_L}{[x_L - x]_{\mathbb{N}}} \right\}.$$

Gonshor proved that this equation is warranted and that it does define a strictly increasing bijective morphism $(\mathbf{No}, +) \rightarrow (\mathbf{No}^{\succ}, \times)$. We write \log for the reciprocal of \exp .

The functions h and g The function \exp interacts with the ω -map in the following way:

$$\exp(\mathbf{Mo}^{\succ}) = \mathbf{Mo} \prec \mathbf{Mo} \quad \text{by [9, Chapter 10].}$$

More precisely, for every strictly positive number x and every number y , we have

$$\begin{aligned} \exp(\dot{\omega}^x) &= \dot{\omega}^{\dot{\omega}^{g(x)}}, \\ \log(\dot{\omega}^y) &= \dot{\omega}^{h(y)}, \end{aligned}$$

where the strictly increasing and bijective function $g: \mathbf{No}^{\succ} \rightarrow \mathbf{No}$ and its reciprocal h have the following equations in \mathbf{No}^{\succ} and \mathbf{No} [9, Theorems 10.11 and 10.12]:

$$\begin{aligned} \forall z = \{L|R\} \in \mathbf{No}^{\succ}, g(z) &= \{\Xi_{\mathbf{Mo}}^{-1} \flat_z, g(L \cap \mathbf{No}^{\succ}) | g(R)\}, \\ \forall z = \{L|R\} \in \mathbf{No}, h(z) &= \{h(L) | h(R), \mathbb{R}^{\succ} \dot{\omega}^z\}_{\mathbf{No}^{\succ}}. \end{aligned}$$

The function g was entirely studied by Gonshor who gave formal results such as the characterization of its fixed points and as well as a somewhat informal description of the sign sequence of $g(z)$ for any strictly positive number z given that of z . We will recover part of his results in a different approach in Section 4.

3.2 Log-atomic numbers

The class \mathbf{La} of log-atomic numbers is defined as the class of numbers $m \in \mathbf{Mo}^{\succ}$ with $\log^n m \in \mathbf{Mo}^{\succ}$ for all $n \in \mathbb{N}$. In other words, we have the equality

$$\mathbf{La} := \bigcap_{n \in \mathbb{N}} \exp^n \mathbf{Mo}^{\succ}.$$

The class \mathbf{La} was first described by Berarducci and Mantova [5] in order to define a derivation on \mathbf{No} . We next describe the parametrization of \mathbf{La} .

Consider, for $r \in \mathbb{R}^{\succ}$ and $n \in \mathbb{N}$ the function

$$f_{n,r} := x \mapsto \exp^n(r \log^n z): \mathbf{No}^{\succ, \succ} \rightarrow \mathbf{No}^{\succ, \succ}.$$

Recall that the class of monomials is that of numbers m which are simplest in each class

$$\mathcal{M}[m] = \{x \in \mathbf{No}^{\succ} : \exists r \in \mathbb{R}^{\succ}, f_{0,r^{-1}}(m) < x < f_{0,r}(m)\}.$$

Similarly, Berarducci and Mantova proved that log-atomic numbers λ are exactly the simplest numbers of each class

$$\mathcal{E}[\lambda] := \{x \in \mathbf{No}^{\succ, \succ} : \exists n \in \mathbb{N}, \exists r \in \mathbb{R}^{\succ}, f_{n, r^{-1}(m)} < x < f_{n, r(m)}\}.$$

Moreover, they derived [5, Definition 5.12 and Corollary 5.17] the following equation for the parametrization $\lambda := \Xi_{\mathbf{La}}$ of \mathbf{La} :

$$\begin{aligned} \forall z \in \mathbf{No}, \lambda_z &= \{\mathbb{R}, f_{\mathbb{N}, \mathbb{R}^{\succ}}(\lambda_{z_L}) | f_{\mathbb{N}, \mathbb{R}^{\succ}}(\lambda_{z_R})\} \\ &= \{\mathbb{R}, \exp^n(r \log^n \lambda_{z'}) | \exp^n(r \log^n \lambda_{z''})\}, \end{aligned} \quad (2)$$

where r, n, z', z'' respectively range in $\mathbb{R}^{\succ}, \mathbb{N}, z_L$ and z_R .

Moreover, the following formula [2, Proposition 2.5] is known for λ :

$$\forall z \in \mathbf{No}, \lambda_{z+1} = \exp \lambda_z. \quad (3)$$

3.3 Surreal hyperexponentiation

In [4], we defined a strictly increasing bijection $E_\omega: \mathbf{No}^{\succ, \succ} \rightarrow \mathbf{No}^{\succ, \succ}$ which satisfies the equation

$$\forall x \in \mathbf{No}^{\succ, \succ}, E_\omega(x+1) = \exp(E_\omega(x)). \quad (4)$$

Moreover, this function is surreal-analytic at every point in the sense of [6, Definition 7.8] and satisfies $E_\omega(x) > E_{\mathbb{N}}(x)$ for all $x \in \mathbf{No}^{\succ, \succ}$. This function can be seen as a surreal counterpart to Knieser's transexponential function [12].

In order to define E_ω , we relied on a surreal substructure \mathbf{Tr} of so-called truncated numbers. They can be characterized as numbers $\varphi \in \mathbf{No}^{\succ, \succ}$ with $\text{supp } \varphi > \frac{1}{L_{\mathbb{N}} E_\omega(\varphi)}$. The function E_ω is defined on the class of truncated series by the equation:

$$\forall \varphi \in \mathbf{Tr}, E_\omega(\varphi) = \{E_{\mathbb{N}}(\varphi), f_{\mathbb{N}, \mathbb{R}^{\succ}}(E_\omega(\varphi_L^{\mathbf{Tr}})) | f_{\mathbb{N}, \mathbb{R}^{\succ}}(E_\omega(\varphi_R^{\mathbf{Tr}}))\} \in \mathbf{La}$$

This equation realizes a strictly increasing bijection $\mathbf{Tr} \rightarrow \mathbf{La}$.

Now consider the set $\mathbf{No}(\varepsilon_0)$ of surreal numbers x with length $\ell(x) < \varepsilon_0$. By [8, Corollary 5.5], the structure $(\mathbf{No}(\varepsilon_0), +, \times, \exp)$ is an elementary extension of the real exponential field. Moreover, for $x \in \mathbf{No}(\varepsilon_0)$, there is $n \in \mathbb{N}$ with $x < E_n(\omega)$. It follows that each element of $\mathbf{No}(\varepsilon_0)^{\succ, \succ} := \mathbf{No}(\varepsilon_0) \cap \mathbf{No}^{\succ, \succ}$ is truncated, and that for $\varphi \in \mathbf{No}(\varepsilon_0) \cap \mathbf{No}^{\succ, \succ}$, the sets $E_{\mathbb{N}}(\varphi)$ and $\lambda_{\mathbb{N}} = \{\omega, \exp(\omega), \exp(\exp(\omega)), \dots\}$ are mutually cofinal with respect to one another. Thus on $\mathbf{No}(\varepsilon_0)^{\succ, \succ}$, the equation for E_ω becomes

$$\forall \varphi \in \mathbf{No}(\varepsilon_0)^{\succ, \succ}, E_\omega(\varphi) = \{\lambda_{\mathbb{N}}, f_{\mathbb{N}, \mathbb{R}^{\succ}}(E_\omega(\varphi_L^{\mathbf{Tr}})) | f_{\mathbb{N}, \mathbb{R}^{\succ}}(E_\omega(\varphi_R^{\mathbf{Tr}}))\}.$$

We claim that $E_\omega(\varphi) = \lambda_\varphi$ for all $\varphi \in \mathbf{No}(\varepsilon_0)$. We prove this by induction on $(\mathbf{No}(\varepsilon_0)^{\succ, \succ}, \sqsubseteq)$. Let $\varphi \in \mathbf{No}(\varepsilon_0)^{\succ, \succ}$ such that the result holds on φ_{\sqsubseteq} . Noticing that $\mathbf{No}(\varepsilon_0)$ is \sqsubseteq -initial in \mathbf{No} , we get

$$\begin{aligned} E_\omega(\varphi) &= \{\lambda_{\mathbb{N}}, f_{\mathbb{N}, \mathbb{R}^{\succ}}(E_\omega(\varphi_L^{\mathbf{Tr}})) | f_{\mathbb{N}, \mathbb{R}^{\succ}}(E_\omega(\varphi_R^{\mathbf{Tr}}))\} \\ &= \{\lambda_{\mathbb{N}}, f_{\mathbb{N}, \mathbb{R}^{\succ}}(\lambda_{\varphi_L^{\mathbf{Tr}}}) | f_{\mathbb{N}, \mathbb{R}^{\succ}}(\lambda_{\varphi_R^{\mathbf{Tr}}})\} \quad \text{by the inductive hypothesis.} \\ &= \{\lambda_{\mathbb{N}}, f_{\mathbb{N}, \mathbb{R}^{\succ}}(\lambda_{\varphi_L^{\mathbf{No}^{\succ, \succ}}}) | f_{\mathbb{N}, \mathbb{R}^{\succ}}(\lambda_{\varphi_R^{\mathbf{No}^{\succ, \succ}}})\} \quad \text{because } \mathbf{No}(\varepsilon_0)^{\succ, \succ} \subseteq \mathbf{Tr} \\ &= \{\mathbb{R}, f_{\mathbb{N}, \mathbb{R}^{\succ}}(\lambda_{\varphi_L}) | f_{\mathbb{N}, \mathbb{R}^{\succ}}(\lambda_{\varphi_R})\} \quad \text{because } \varphi_L = \varphi_L^{\succ, \succ} \cup \mathbb{N} \text{ and } \varphi_R = \varphi_R^{\mathbf{No}^{\succ, \succ}} \\ &= \lambda_\varphi \quad \text{by (2).} \end{aligned}$$

So E_ω and λ coincide on $\mathbf{No}(\varepsilon_0)^{\succ, \succ}$.

3.4 Kappa numbers

The structure \mathbf{K} of κ -numbers was introduced first and studied in detail by S. Kuhlmann and M. Matusinski in [13]. It was designed as an intermediate subclass between fundamental monomials and log-atomic numbers. The relation between \mathbf{K} and \mathbf{La} is given by the imbrication $\mathbf{K} = \mathbf{La} \prec \mathbf{No}_{\succ}$ of [14, p 21].

Similarly to monomials and log-atomic numbers, \mathbf{K} numbers can be characterized as simplest elements in each convex class

$$\mathcal{E}^*[x] := \{y \in \mathbf{No}^{\succ, \succ} : \exists n \in \mathbb{N}, \log^n x \leq y \leq \exp^n x\}, \text{ for } x \in \mathbf{No}^{\succ, \succ}.$$

The parametrization $\kappa := \Xi_{\mathbf{K}}: \mathbf{No} \rightarrow \mathbf{K}$ of \mathbf{K} is given by the equation

$$\forall z \in \mathbf{No}, \kappa_z = \{\mathbb{R}, \exp^{\mathbb{N}} \kappa_{z_L} | \log^{\mathbb{N}} \kappa_{z_R}\}.$$

Moreover, by [3, Corollary 13 and Theorem 6.16], the structure \mathbf{K} is closed.

In order to compute the sign sequence of κ -numbers, Kuhlmann and Matusinski rely on an intermediate surreal substructure denoted \mathbf{I} . This surreal substructure is defined by the imbrication relation: $\mathbf{K} = \mathbf{Mo}^{\prec 2} \prec \mathbf{I}$. Indeed, since $\mathbf{K} \subseteq \mathbf{La} \subseteq \exp(\mathbf{Mo}^{\succ}) = \mathbf{Mo}^{\prec 2}$, the structure \mathbf{I} exists and is unique.

For $z \in \mathbf{No}$ we let $\flat z$ denote the number with $z = 1 \dot{+} \flat z$ if $z > 0$ and $\flat z = -\infty$ if $z \leq 0$. We also extend the functions ε and τ to $-\infty$ with $\tau_{-\infty} := -\infty$ and $\varepsilon_{-\infty} := 0$. For $z \in \mathbf{No}$, we define $\phi_{z \dot{+} 1}^{\mathbf{I}} := \varepsilon_{\tau_{\flat z}}$ and $\phi_{z \dot{+} (-1)}^{\mathbf{I}} := -\omega$. The parametrization $\iota := \Xi_{\mathbf{I}}$ of \mathbf{I} admits the following sign sequence formula

$$\forall z \in \mathbf{No}, \iota_z = \sum_{\beta < \ell(z)} \phi_{z \dot{+} 1(\beta+1)}^{\mathbf{I}} \quad [13, \text{Lemma 4.2}].$$

4 Log-atomic numbers and fixed points

Our goal is to compute the sign sequence of $\Xi_{\mathbf{La}} z = \lambda_z$ in terms of that of z , for all numbers z . Recall that $\mathbf{La} \subseteq \mathbf{Mo}^{\prec 2}$, so there is a surreal substructure \mathbf{R} with $\mathbf{La} = \mathbf{Mo}^{\prec 2} \prec \mathbf{R}$. We write $\rho_z := \Xi_{\mathbf{R}} z$ for all $z \in \mathbf{No}$. We have $\mathbf{K} = \mathbf{Mo}^{\prec 2} \prec \mathbf{I} = \mathbf{La} \prec \mathbf{No}_{\succ}$, whence $\mathbf{I} = \mathbf{R} \prec \mathbf{No}_{\succ}$. The computation in [13] of sign sequences of numbers in $\mathbf{La} \prec (\mathbf{No}_{\succ} + \mathbb{Z}) = \exp^{\mathbb{Z}}(\mathbf{La} \prec \mathbf{No}_{\succ}) = \log^{\mathbb{Z}} \mathbf{K}$ can thus be used to derive part of the result.

For $z = \omega \dot{\times} a + n \in \mathbf{No}_{\succ} + \mathbb{Z}$ where $a \in \mathbf{No}$ and $n \in \mathbb{Z}$, we have $\lambda_z = \exp^n \lambda_{\omega \dot{\times} a} = \exp^n \kappa_a$. By [13, Theorem 4.3], we have

$$\lambda_z = \dot{\omega}^{\dot{\omega}^{a \dot{+} n}} \quad \text{if } n \leq 0. \quad (5)$$

$$\lambda_z = \dot{\omega}^{\dot{\omega}^{a \dot{+} \Xi_{\mathbf{Mo}}^n(\varepsilon_{\tau_{\flat a} + 1})}} \quad \text{if } n > 0. \quad (6)$$

Our goal is to extend this description to the sign sequences of numbers ρ_z for all $z \in \mathbf{No}$, relying on the known values $\rho_{\omega \dot{\times} a + n}$ for all $a \in \mathbf{No}$ and $n \in \mathbb{Z}$. More precisely, we will compute ρ on all intervals

$$\mathbf{I}_{a,n} := (\omega \dot{\times} a - (n+1), \omega \dot{\times} a - n) \quad \text{where } a \in \mathbf{No} \text{ and } n \in \mathbb{Z}.$$

Since \mathbf{No} can be realized as the reunion

$$\mathbf{No} = \left(\bigsqcup_{\substack{a \in \mathbf{No} \\ n \in \mathbb{Z}}} \mathbf{I}_{a,n} \right) \sqcup \{\omega \dot{\times} a + n : a \in \mathbf{No}, n \in \mathbb{N}\},$$

this will cover all cases. The sign sequence of λ_z can then be computed using Proposition 4 twice. In order to compute the sign sequence for \mathbf{R} , we first describe the action of h and g on sign sequences in certain cases.

4.1 Computing h

Let $a \in \mathbf{No}$ be fixed and write $\theta_a := \omega \dot{\times} a$.

LEMMA 20. *We have $\mathbf{I} \subseteq \mathbf{No}_{\succ}$, whence $\mathbf{R} \prec \mathbf{No}_{\succ} \subseteq \mathbf{No}_{\succ}$.*

PROOF. The first result is a consequence of the identity $\mathbf{No}_> = \omega \dot{\times} \mathbf{No}$ and [13, Lemma 4.2]. The second immediately follows. \square

Recall that we have $\kappa_a = \dot{\omega}^{\dot{\omega}^a}$ so $\rho_{\theta_a} = \iota_a$. By [13, Lemma 4.2], we have $\tau_a = \varepsilon_b \tau_a$. We set

$$\begin{aligned} \zeta_a &:= \tau_a \in \mathbf{On} \quad \text{and} \\ \delta_a &:= \dot{\omega}^{\zeta_a + 1} = \dot{\omega}^{\zeta_a} \dot{\times} \omega \in \mathbf{On}, \end{aligned}$$

so $\delta_a = \omega$ if $\tau_a = 0$ and $\delta_a = \zeta_a \dot{\times} \omega$ if $\tau_a > 0$. We will treat the cases $\tau_a = 0$ and $\tau_a > 0$ in a uniform way.

LEMMA 21. For $n \in \mathbb{N}$, we have $\rho_{\theta_a - n} = \rho_{\theta_a} - n = \rho_{\theta_a} \dot{+} (-n)$

PROOF. We have $\rho_{\theta_a - n} = \log^n \rho_{\theta_a} = \log^n \kappa_a = \dot{\omega}^{\dot{\omega}^{\iota_a + (-n)}}$ by [13, Theorem 4.3(1)]. By Lemma 20, we have $\iota_a \in \mathbf{No}_>$ so $\iota_a \dot{+} (-n) = \iota_a - n$, so $\dot{\omega}^{\dot{\omega}^{\rho_{\theta_a - n}}} = \rho_{\theta_a - n} = \dot{\omega}^{\dot{\omega}^{\iota_a - n}}$. Since $\rho_{\theta_a} = \dot{\omega}^{\dot{\omega}^{\iota_a}}$, we have $\rho_{\theta_a - n} = \rho_{\theta_a} - n$. It follows that $\rho_{\theta_a} - n = \rho_{\theta_a} \dot{+} (-n)$. \square

We now fix $n \in \mathbb{N}$ and we set $\mathbf{J}_{a,n} := h(\mathbf{I}_{a,n})$. By the previous lemma, we can write the interval $\mathbf{I}_{a,n}$ as the surreal substructure

$$\begin{aligned} \mathbf{I}_{a,n} &= (\rho_{\theta_a - n - 1}, \rho_{\theta_a - n}) \\ &= (\rho_{\theta_a} - n - 1, \rho_{\theta_a} - n) \\ &= \mathbf{No}^{\supseteq \rho_{\theta_a} - n - 1/2}. \end{aligned}$$

Thus $\Xi_{\mathbf{I}_{a,n}}$ is straightforward to compute in terms of sign sequences.

We have $h(\rho_{\theta_a - n}) = \dot{\omega}^{\rho_{\theta_a} - n - 1}$ and $h(\rho_{\theta_a - n - 1}) = \dot{\omega}^{\rho_{\theta_a} - n - 2} = \dot{\omega}^{\rho_{\theta_a} - n - 1} \dot{+} (-\delta_a)$. Thus the interval $\mathbf{J}_{a,n}$ is subject to the computation given in Lemma 9(b). That is, we have:

LEMMA 22. For $z \in \mathbf{No}$, we have

- $\Xi_{\mathbf{J}_{a,n}} z = \dot{\omega}^{\rho_{\theta_a} - n - 1} \dot{+} (-1) \dot{+} z$ if $z \not\prec -(\delta_a)_L$.
- $\Xi_{\mathbf{J}_{a,n}} ((-\delta_a) \dot{+} z) = \dot{\omega}^{\rho_{\theta_a} - n - 2} \dot{+} 1 \dot{+} z$.

Moreover, for $x \in \mathbf{I}_{a,n}$, we have $\dot{\omega}^{\rho_{\theta_a} - n - 1} - 1 \in x_L$ and $\dot{\omega}^{\rho_{\theta_a} - n - 1} \in x_R$. This implies that $\dot{\omega}^{\rho_{\theta_a} - n - 2} = h(\rho_{\theta_a - n - 1}) \in h(x_L)$ and that $\dot{\omega}^{\rho_{\theta_a} - n - 1} = h(\rho_{\theta_a - n}) \in h(x_R)$. Since with $\mathbb{R}^> \dot{\omega}^x > \dot{\omega}^{\rho_{\theta_a} - n - 2}$, we deduce the following equation for h on $\mathbf{I}_{a,n}$:

$$\begin{aligned} \forall x \in \mathbf{I}_{a,n}, h(x) &= \{ \dot{\omega}^{\rho_{\theta_a} - n - 2}, h(x_L^{\mathbf{I}_{a,n}}) | h(x_R^{\mathbf{I}_{a,n}}), \dot{\omega}^{\rho_{\theta_a} - n - 1} \} \\ &= \{ h(x_L^{\mathbf{I}_{a,n}}) | h(x_R^{\mathbf{I}_{a,n}}) \}_{\mathbf{J}_{a,n}}. \end{aligned}$$

We see that $h \upharpoonright \mathbf{I}_\alpha$ is the surreal isomorphism $\mathbf{I}_\alpha \rightarrow \mathbf{J}_\alpha$, so $h \upharpoonright \mathbf{I}_\alpha = \Xi_{\mathbf{J}_\alpha} \circ \Xi_{\mathbf{I}_\alpha}^{-1}$.

PROPOSITION 23. For $z \in \mathbf{No}$, we have

- $h(\rho_{\theta_a - n - 1} \dot{+} 1 \dot{+} z) = \dot{\omega}^{\rho_{\theta_a} - n - 1} \dot{+} (-1) \dot{+} z$ if $z \not\prec -(\delta_a)_L$.
- $h(\rho_{\theta_a - n - 1} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} z) = \dot{\omega}^{\rho_{\theta_a} - n - 2} \dot{+} 1 \dot{+} z$.

PROOF. This is a direct consequence of the previous lemma and the identity $h \upharpoonright \mathbf{I}_{a,n} = \Xi_{\mathbf{J}_{a,n}}^{\mathbf{I}_{a,n}}$. \square

LEMMA 24. For $z \in \mathbf{I}_{a,n}$, if the number $h(z)$ is a monomial, then there is $u \in \mathbf{No}$ with $\delta_a \sqsubseteq u$ and $z = \rho_{\theta_a - n - 1} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} u$.

PROOF. Assume $z = \rho_{\theta_a - n - 1} \dot{+} 1 \dot{+} u$ where $u \not\prec -(\delta_a)_L$. Since $h(z) = \dot{\omega}^{\rho_{\theta_a - n - 1}} \dot{+} (-1) \dot{+} u$ is a monomial with $\tau_{\rho_{\theta_a - n - 1}} = \zeta_a$, by Proposition 4, the sign sequence of u must start with $(-\delta_a)$, which contradicts the hypothesis on u . We deduce that there is a number u with $z = \rho_{\theta_a - n - 1} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} u$. Since $h(z) = \dot{\omega}^{\rho_{\theta_a - n - 2}} \dot{+} 1 \dot{+} u$ is a monomial and by the same argument, the sign sequence of u must start with at least δ_a many signs 1, that is, we must have $\delta_a \sqsubseteq u$. \square

4.2 Computing g

For $n \in \mathbb{N}$, we set

$$\mathbf{K}_{a,n} := (\dot{\omega}^{\rho_{\theta_a + n - 1}}, \dot{\omega}^{\rho_{\theta_a + n}})$$

and

$$\mathbf{H}_{a,n} := g(\mathbf{K}_{a,n}) = (\rho_{\theta_a + n}, \rho_{\theta_a + n + 1}).$$

Recall that if $n > 0$, we have $\rho_{\theta_a + n} = \rho_{\theta_a} \dot{+} \Xi_{\mathbf{Mo}}^n(\zeta_a + 1)$, so $\Xi_{\mathbf{K}_{a,n}}$ is subject to the computations of Lemma 9(a). We have $\rho_{\theta_a - 1} = \rho_{\theta_a} \dot{+} (-1)$, so $\Xi_{\mathbf{K}_{a,0}}$ is subject to the computations of Lemma 9(b), whence

LEMMA 25. For $z \in \mathbf{No}$, and $n > 0$, we have

- a) $\Xi_{\mathbf{K}_{a,n}} z = \dot{\omega}^{\rho_{\theta_a + n - 1}} \dot{+} 1 \dot{+} z$ if $z \not\prec (\Xi_{\mathbf{Mo}}^{n+1}(\zeta_a + 1))_L$.
- b) $\Xi_{\mathbf{K}_{a,n}}(\Xi_{\mathbf{Mo}}^{n+1}(\zeta_a + 1) \dot{+} z) = \dot{\omega}^{\rho_{\theta_a + n}} \dot{+} (-1) \dot{+} z$ if $z > 0$.
- c) $\Xi_{\mathbf{K}_{a,0}} z = \dot{\omega}^{\rho_{\theta_a}} \dot{+} (-1) \dot{+} z$ if $z \not\prec -(\delta_a)_L$.
- d) $\Xi_{\mathbf{K}_{a,n}}((-\delta_a) \dot{+} z) = \dot{\omega}^{\rho_{\theta_a - 1}} \dot{+} 1 \dot{+} z$ if $z < 0$.

LEMMA 26. For $z \in \mathbf{No}$ and $n \in \mathbb{N}$ we have

- a) $\Xi_{\mathbf{H}_{a,n}} z = \rho_{\theta_a + n - 1} \dot{+} 1 \dot{+} z$ if $z \not\prec (\Xi_{\mathbf{Mo}}^n(\zeta_a + 1))_L$.
- b) $\Xi_{\mathbf{H}_{a,n}}(\dot{\omega}^{\rho_{\theta_a + n - 1}} \dot{+} \Xi_{\mathbf{Mo}}^n(\zeta_a + 1) \dot{+} z) = \rho_{\theta_a + n} \dot{+} (-1) \dot{+} z$ if $z > 0$.

PROOF. Again, this is a direct consequence of Lemma 9(b). \square

LEMMA 27. We have $g \upharpoonright \mathbf{K}_{a,n} = \Xi_{\mathbf{H}_{a,n}}^{\mathbf{K}_{a,n}}$.

PROOF. Let $z \in (\dot{\omega}^{\rho_{\theta_a + n - 1}}, \dot{\omega}^{\rho_{\theta_a + n}})$. We have $\delta_z < \dot{\omega}^{\rho_{\theta_a + n}}$ so $\Xi_{\mathbf{Mo}}^{-1} \delta_z < \rho_{\theta_a + n} = g(\dot{\omega}^{\rho_{\theta_a + n - 1}})$. We now compute

$$\begin{aligned} g(z) &= g(\{z_L^>, \dot{\omega}^{\rho_{\theta_a + n - 1}} | z_R^>, \dot{\omega}^{\rho_{\theta_a + n}}\}_>) \\ &= \{\Xi_{\mathbf{Mo}}^{-1} \delta_z, g(\dot{\omega}^{\rho_{\theta_a + n - 1}}), g(z_L^{\mathbf{K}_{a,n}}) | g(z_R^{\mathbf{K}_{a,n}}), g(\dot{\omega}^{\rho_{\theta_a + n}})\} \\ &= \{\Xi_{\mathbf{Mo}}^{-1} \delta_z, g(\dot{\omega}^{\rho_{\theta_a + n - 1}}), g(z_L^{\mathbf{K}_{a,n}}) | g(z_R^{\mathbf{K}_{a,n}}), g(\dot{\omega}^{\rho_{\theta_a + n}})\}. \\ &= \{g(\dot{\omega}^{\rho_{\theta_a + n - 1}}), g(z_L^{\mathbf{K}_{a,n}}) | g(z_R^{\mathbf{K}_{a,n}}), \rho_{\theta_a} \dot{+} \zeta_a \dot{\omega}\} \\ &= \{g(z_L^{\mathbf{K}_{a,n}}) | g(z_R^{\mathbf{K}_{a,n}})\}_{\mathbf{H}_{a,n}}. \end{aligned}$$

It follows that $g \upharpoonright \mathbf{K}_{a,n} = \Xi_{\mathbf{H}_{a,n}}^{\mathbf{K}_{a,n}}$. \square

PROPOSITION 28. For $z \in \mathbf{No}$, and $n > 0$, we have

- a) $g(\dot{\omega}^{\rho_{\theta_a + n - 1}} \dot{+} 1 \dot{+} z) = \rho_{\theta_a + n} \dot{+} 1 \dot{+} z$ if $z \not\prec (\Xi_{\mathbf{Mo}}^{n+1}(\zeta_a + 1))_L$.
- b) $g(\dot{\omega}^{\rho_{\theta_a + n - 1}} \dot{+} \Xi_{\mathbf{Mo}}^{n+1}(\zeta_a + 1) \dot{+} z) = \rho_{\theta_a + n + 1} \dot{+} (-1) \dot{+} z$ if $z < 0$.
- c) $g(\dot{\omega}^{\rho_{\theta_a}} \dot{+} (-1) \dot{+} z) = \rho_{\theta_a} \dot{+} 1 \dot{+} z$ if $z \not\prec -(\delta_a)_L$.
- d) $g(\dot{\omega}^{\rho_{\theta_a}} \dot{+} (-\delta_a) \dot{+} z) = \rho_{\theta_a} \dot{+} z$ if $z > 0$.

PROOF. This follows from the three previous lemmas. \square

4.3 The characterization theorem

Piecing the descriptions of Propositions 23 and 28 together, one obtains a full description of the sign sequence of $h(z)$ for $z \in \mathbf{No}$. We will only require part of those descriptions to reach our goal of describing sign sequences in \mathbf{R} . For $n \in \mathbb{Z}$, we set $\mathbf{R}_{a,n} = \Xi_{\mathbf{R}} \mathbf{I}_{a,n}$. We will characterize $\mathbf{R}_{a,n}$ using fixed points of the surreal substructure

$$\mathbf{V}_a := \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a) \dot{+} \mathbf{Mo}.$$

Notice that this structure depends only on τ_a . Both \mathbf{Mo} and $\mathbf{No}^{\exists \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a)}$ are closed so \mathbf{V}_a is closed by Lemma 15, so the class $\mathbf{V}_a^{\prec \omega}$ is also closed.

THEOREM 29. For $n \in \mathbb{N}$, we have

a)

$$\mathbf{R}_{a,n} = \rho_{\theta_{a-(n+1)}} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} \mathbf{V}_a^{\prec \omega}.$$

b) For $u = \rho_{\theta_{a-(n+1)}} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v$ where $v \in \mathbf{V}_a^{\prec \omega}$ and $k \in \mathbb{N}$, we have

$$\log^k \dot{\omega} \dot{\omega}^u = \dot{\omega}^{\dot{\omega}^{\rho_{\theta_{a-(n+k+1)}} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v}}.$$

c)

$$\mathbf{R}_{a,-(n+1)} = \rho_{\theta_{a+n}} \dot{+} \mathbf{V}_a^{\prec \omega}.$$

d) For $u = \rho_{\theta_{a-(n+1)}} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v$ where $v \in \mathbf{V}_a^{\prec \omega}$ and $k \in \mathbb{N}^>$, we have

$$\exp^k \dot{\omega} \dot{\omega}^u = \dot{\omega}^{\dot{\omega}^{\rho_{\theta_{a+k-(n+1)}} \dot{+} v}}.$$

PROOF. We first prove by induction on k that we have

$$\forall n \in \mathbb{N}, \mathbf{R}_{a,n} \subseteq \rho_{\theta_{a-(n+1)}} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} \mathbf{V}_a^{\prec k}.$$

First assume that $k = 0$. For $n \in \mathbb{N}$ and $u \in \mathbf{R}_{a,n}$, the number $h(u)$ is a monomial, which implies by Lemma 24 that u has the form $\rho_{\theta_{a-(n+1)}} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} z$ with $\delta_a \sqsubseteq z$. This implies in particular that $u \in \rho_{\theta_{a-(n+1)}} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} \mathbf{No}$.

Let $k \in \mathbb{N}$ such that the result holds at k and consider $n \in \mathbb{N}$ and $u_0 \in \mathbf{R}_{a,n}$. There is a $v_k \in \mathbf{V}_a^{\prec k}$ with $u_0 = \rho_{\theta_{a-(n+1)}} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v_k$. Since $u_0 \in \mathbf{R}_{a,n}$, we have $h(u_0) \in \dot{\omega}^{\mathbf{R}_{a,n+1}}$. Now $h(u_0) = \dot{\omega}^{\rho_{\theta_{a-(n+2)}} \dot{+} 1 \dot{+} v_k} = \dot{\omega}^{\rho_{\theta_{a-(n+2)}} \dot{+} v_k}$ by Proposition 23(b). Let $u_1 \in \mathbf{R}_{a,n+1}$ with $h(u_0) = \dot{\omega}^{u_1}$. We have $h(u_1) \in \mathbf{Mo}$, so by Lemma 24, there is a number v_{k+1} with $u_1 = \rho_{\theta_{a-(n+2)}} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v_{k+1}$ and $\delta_a \sqsubseteq v_{k+1}$. Thus $h(u_0) = \dot{\omega}^{\rho_{\theta_{a-(n+2)}} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v_{k+1}}$. By Proposition 4 and Lemma 5, we have

$$h(u_0) = \dot{\omega}^{\rho_{\theta_{a-(n+2)}} \dot{+} \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a) \dot{+} \dot{\omega}^{\zeta_a \dot{+} 1 \dot{+} v_{k+1}}}$$

We identify

$$\begin{aligned} v_k &= \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a) \dot{+} \dot{\omega}^{\zeta_a \dot{+} 1 \dot{+} v_{k+1}} \\ &= \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a) \dot{+} \dot{\omega}^{v_{k+1}} \quad \text{since } \delta_a = \zeta_a \dot{+} 1 \dot{+} \delta_a \sqsubseteq v_{k+1} \\ &= \Xi_{\mathbf{V}_a} v_{k+1}. \end{aligned}$$

The inductive hypothesis applied at $(u_1, n+1)$ yields $v_{k+1} \in \mathbf{V}_a^{\prec k}$, so $v_k \in \mathbf{V}_a^{\prec (k+1)}$. We thus have

$$\mathbf{R}_{a,n} \subseteq \bigcap_{k \in \mathbb{N}} \rho_{\theta_{a-(n+1)}} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} \mathbf{V}_a^{\prec k} = \rho_{\theta_{a-(n+1)}} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} \mathbf{V}_a^{\prec \omega}.$$

We next prove *b*). Since it implies that $\log^{\mathbb{N}} \dot{\omega}^{\mathbf{R}_{a,n}^{\omega}} \subseteq \mathbf{Mo}$, this will yield *a*). We prove *b*) by induction on k . Note that the result is immediate for $k = 0$. Let $k \in \mathbb{N}$ be such that the formula holds at k , let $v \in \mathbf{V}_a^{<\omega}$ and consider $u = \rho_{\theta_a - (n+1)} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v$. Our inductive hypothesis is

$$\log^k \dot{\omega}^u = \dot{\omega}^{\rho_{\theta_a - (n+k+1)} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v}.$$

Thus

$$\begin{aligned} \log^{k+1} \dot{\omega}^u &= \log \dot{\omega}^{\rho_{\theta_a - (n+k+1)} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v} \\ &= \dot{\omega}^{h(\rho_{\theta_a - (n+k+1)} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v)} \\ &= \dot{\omega}^{\rho_{\theta_a - (n+k+2)} \dot{+} 1 \dot{+} v} \quad \text{by Proposition 23(b)} \\ &= \dot{\omega}^{\rho_{\theta_a - (n+k+2)} \dot{+} v} \\ &= \dot{\omega}^{\rho_{\theta_a - (n+k+2)} \dot{+} \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a) \dot{+} \dot{\omega}^v} \quad \text{since } v \in \mathbf{V}_a^{<\omega} \text{ and } \mathbf{V}_a^{<\omega} = \mathbf{Fix} \mathbf{V}_a \\ &= \dot{\omega}^{\rho_{\theta_a - (n+k+2)} \dot{+} \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a) \dot{+} \dot{\omega}^{\zeta_a \dot{+} 1 \dot{+} v}} \quad \text{since } \delta_a = \zeta_a \dot{+} 1 \dot{+} \delta_a \sqsubseteq v_{k+1} \\ &= \dot{\omega}^{\rho_{\theta_a - (n+k+2)} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} \dot{\omega}^{\zeta_a \dot{+} 1 \dot{+} v}} \quad \text{by Proposition 4} \\ &= \dot{\omega}^{\rho_{\theta_a - (n+k+2)} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v} \quad \text{by Lemma 5.} \end{aligned}$$

This are the desired results. The formula follows by induction.

We next prove *d*). Let $u = \rho_{\theta_a - 1} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v$ where $v \in \mathbf{V}_a^{<\omega}$, so that u is a generic element of $\mathbf{R}_{a,0}$. By the same computations as above, we have

$$\begin{aligned} g(\dot{\omega}^u) &= g(\dot{\omega}^{\rho_{\theta_a - 1} \dot{+} 1 \dot{+} (-\delta_a) \dot{+} v}) \\ &= g(\dot{\omega}^{\rho_{\theta_a - 1} \dot{+} \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a) \dot{+} \dot{\omega}^{\zeta_a \dot{+} 1 \dot{+} v}}) \quad \text{by Proposition 4 and Lemma 5} \\ &= g(\dot{\omega}^{\rho_{\theta_a} \dot{+} (-1) \dot{+} \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a) \dot{+} \dot{\omega}^v}) \quad \text{since } \delta_a = \zeta_a \dot{+} 1 \dot{+} \delta_a \sqsubseteq v \\ &= g(\dot{\omega}^{\rho_{\theta_a} \dot{+} (-1) \dot{+} v}) \quad \text{since } v \in \mathbf{V}_a^{<\omega} \\ &= g(\dot{\omega}^{\rho_{\theta_a} \dot{+} (-\delta_a) \dot{+} v}) \quad \text{by Proposition 4} \\ &= \rho_{\theta_a} \dot{+} 1 \dot{+} v \quad \text{by Proposition 28(d)} \\ &= \rho_{\theta_a} \dot{+} v. \end{aligned}$$

This proves the formula for $n = 0$. Now let $n \in \mathbb{N}$ be such that the formula holds at n . Let $u = \rho_{\theta_a + n} \dot{+} v$ with $v \in \mathbf{V}_a^{<\omega}$ be a generic element of $\mathbf{R}_{a, -(n+1)}$. We have $v < \delta_a$ so $v \not\prec (\exists_{\mathbf{Mo}}^{n+1}(\delta_a + 1))_L$, thus Proposition 28(a) yields

$$\begin{aligned} g(\dot{\omega}^u) &= g(\dot{\omega}^{\rho_{\theta_a + n} \dot{+} v}) \\ &= g(\dot{\omega}^{\rho_{\theta_a + n} \dot{+} 1 \dot{+} v}) \\ &= \rho_{\theta_a + n + 1} \dot{+} v \end{aligned}$$

By induction, the formula for g is true for all $n \in \mathbb{N}$. □

4.4 Application to the closure of \mathbf{La}

COROLLARY 30. *For $n \in \mathbb{Z}$, the structure $\mathbf{R}_{a,n}$ is closed.*

PROOF. Since both $\mathbf{No}^{\supseteq \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a)}$ and \mathbf{Mo} are closed and by Lemma 15, the surreal substructure \mathbf{V}_a is closed. Therefore $\mathbf{V}_a^{<\omega}$ is closed. By the previous theorem, the structure $\mathbf{R}_{a,n}$ can be expressed as $\mathbf{No}^{\supseteq b} \prec \mathbf{V}_a^{<\omega}$ for a certain number b depending on a and n . So by Lemma 15, it is closed. □

PROPOSITION 31. *The structure \mathbf{La} is closed.*

PROOF. Let C be a non-empty chain in \mathbf{La} . We want to prove that $\sup_{\sqsubseteq} C$ is log-atomic. We may assume that C has no \sqsubseteq -maximum.

If the set K of κ -numbers κ_z with $\exists c \in C, \kappa_z \sqsubseteq c$ is \sqsubseteq -cofinal in C , then $\sup_{\sqsubseteq} C = \sup_{\sqsubseteq} K \in \mathbf{K}$ by closure of \mathbf{K} , so $\sup_{\sqsubseteq} C \in \mathbf{La}$.

Assume that K is not \sqsubseteq -cofinal in C . Let $c_0 \in C$ be \sqsubseteq -minimal with $K \sqsubset c_0$. Let $C_0 = \{c \in C: c_0 \sqsubseteq c\}$ and notice that $\sup_{\sqsubseteq} C = \sup_{\sqsubseteq} C_0$. Let $a \in \mathbf{No}$ with $\kappa_a = \sup_{\sqsubseteq} K \in \mathbf{K}$. For $c \in C_0$, the number κ_a is \sqsubseteq -maximal in \mathbf{K} with $\kappa_a \sqsubseteq c$. By [3, Corollary 7.13 and Theorem 6.16], we have $c \in \mathcal{E}^*[\kappa_a]$ for all $c \in C_0$, whence $C_0 \subseteq \mathcal{E}^*[\kappa_a] = \mathcal{E}^*[\lambda_{\theta_a}]$. The set $\log^{\mathbb{Z}} \lambda_{\theta_a}$ is cofinal and cointial in $\mathcal{E}^*[\lambda_{\theta_a}]$ for the order \leq . So for each element $c \in C_0$ there is a unique integer $n_c \in \mathbb{Z}$ with $c \in \mathbf{La} \prec \mathbf{I}_{a, n_c}$. The equations (5) and (6) imply that if c, c' are elements of C_0 with $c \sqsubset c'$, then we must have $n_c \sqsubseteq n_{c'}$. It follows that $v := \sup_{\sqsubseteq} n_{C_0}$ exists in $\mathbb{Z} \cup \{-\omega, \omega\}$. If $v = \sigma \omega$ where $\sigma \in \{-1, 1\}$, then $\sup_{\sqsubseteq} C = \lambda_{\theta_{a+\sigma}} \in \mathbf{La}$. If $v \in \mathbb{Z}$, then for $c_1 \in C_0$ with $n_{c_1} = v$ and $C_1 = \{c \in C: c_1 \sqsubseteq c\}$, we have $\sup_{\sqsubseteq} C = \sup_{\sqsubseteq} C_1$ and $C_1 \subseteq \mathbf{La} \prec \mathbf{I}_{a, v} = \mathbf{Mo}^{\prec 2} \prec \mathbf{R}_{a, v}$. The structure $\mathbf{Mo}^{\prec 2}$ is closed, so by Corollary 30 and Lemma 15, the structure $\mathbf{Mo}^{\prec 2} \prec \mathbf{R}_{a, v}$ is closed. So $\sup_{\sqsubseteq} C \in \mathbf{Mo}^{\prec 2} \prec \mathbf{R}_{a, v}$ is log-atomic. \square

5 The sign sequence formula

We apply the results of the previous sections to give the sign sequence formula for \mathbf{R} .

PROPOSITION 32. Let $V(a) = \sup_{\sqsubseteq} \Xi_{\mathbf{V}_a}^{\mathbb{N}} 0$. We have $V(a) = \Xi_{\mathbf{V}_a}^{\omega} 0$, which is the transfinite concatenation

$$V(a) = \dot{\omega}^{\zeta_a+1} \dot{+} (-\dot{\omega}^{\zeta_a+2+\zeta_a+1}) \dot{+} \sum_{n \in \mathbb{N}} \dot{\omega}^{\Xi_{\mathbf{Mo}}^n \delta_a} \dot{+} \dot{\omega}^{(\sum_{k=0}^n \Xi_{\mathbf{Mo}}^k \delta_a) + 1 + \zeta_a + 2 + \zeta_a + 1}.$$

As a consequence, we have $\tau_{V(a)} = \varepsilon_b \tau_{a+1} \in \mathbf{Mo}^{\prec \omega}$.

PROOF. The identity $V(a) = \sup_{\sqsubseteq} \Xi_{\mathbf{V}_a}^{\mathbb{N}} 0$ is a direct consequence of the closure of \mathbf{V}_a . The computation of this supremum is left to the reader. \square

PROPOSITION 33. For $z \in \mathbf{No}$ define $\phi_{z+1}^a = \varepsilon_b \tau_{a+1+z+1}$ and $\phi_{z+(-1)}^a = -\varepsilon_b^{\omega} \tau_{a+1+z}$. Let Φ^a be the function defined on \mathbf{No} by $\Phi^a(z) = \sum_{\beta < \ell(z)} \phi_{z+1}^a(\beta+1)$. We have

$$\forall z \in \mathbf{No}, \Xi_{\mathbf{V}_a}^{\omega} z = V(a) \dot{+} \Phi^a(z).$$

PROOF. Since \mathbf{V}_a is closed, we may rely on Proposition 19. We need only prove that for $z \in \mathbf{No}$ and $\sigma \in \{-1, 1\}$, we have $\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} \phi_{z+\sigma}^a = \sup_{\sqsubseteq} \Xi_{\mathbf{V}_a}^{\mathbb{N}} (\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} \sigma)$. We prove this by induction on $(\mathbf{No}, \sqsubseteq)$ along with the claim that $\forall z \in \mathbf{No}, \tau_{\Xi_{\mathbf{V}_a}^{\omega} z} = \varepsilon_b \tau_{a+1+z}$.

Note that the functions $\tau, \varepsilon_b, z \mapsto a+1+z$ and $\Xi_{\mathbf{V}_a}^{\omega}$ preserve non-empty suprema. Moreover, the identity $\tau_{\Xi_{\mathbf{V}_a}^{\omega} z} = \varepsilon_b \tau_{a+1+z}$ is valid for $z=0$. So by the previous lemma, we need only prove the claim at successors cases. Let $z \in \mathbf{No}$, set $\beta_0 = 0$, and define

$$\beta_{n+1} := \dot{\omega}^{\varepsilon_b \tau_{a+1+z} + 1 + \beta_n} \quad \text{for all } n \in \mathbb{N}.$$

We have $\sup_{\sqsubseteq} \beta_{\mathbb{N}} = \varepsilon_b \tau_{a+1+z+1} = \phi_{z+1}^a$. Let $n \in \mathbb{N}$ with $\Xi_{\mathbf{V}_a}^n (\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} 1) = \Xi_{\mathbf{V}_a}^{\omega} z \dot{+} 1 \dot{+} \beta_n$ (this is trivially the case for $n=0$). We have

$$\begin{aligned} \Xi_{\mathbf{V}_a}^{n+1} (\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} 1) &= \Xi_{\mathbf{V}_a} (\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} 1 \dot{+} \beta_n) \\ &= \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a) \dot{+} \dot{\omega}^{\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} 1 \dot{+} \beta_n} \\ &= \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a) \dot{+} \dot{\omega}^{\Xi_{\mathbf{V}_a}^{\omega} z} \dot{+} \dot{\omega}^{\tau_{\Xi_{\mathbf{V}_a}^{\omega} z} + 1 \dot{+} \beta_n} \quad \text{by Lemma 5} \\ &= \Xi_{\mathbf{V}_a} \Xi_{\mathbf{V}_a}^{\omega} z \dot{+} \dot{\omega}^{\varepsilon_b \tau_{a+1+z} + 1 \dot{+} \beta_n} \\ &= \Xi_{\mathbf{V}_a}^{\omega} z \dot{+} \beta_{n+1} \\ &= \Xi_{\mathbf{V}_a}^{\omega} z \dot{+} 1 \dot{+} \beta_{n+1}. \end{aligned}$$

It follows that $\sup_{\square} \Xi_{\mathbf{V}_a}^{\mathbb{N}}(\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} 1) = \Xi_{\mathbf{V}_a}^{\omega} z \dot{+} \sup_{\square} \beta_{\mathbb{N}} = \Xi_{\mathbf{V}_a}^{\omega} z \dot{+} \phi_{z+1}^a$. The second claim is valid since we have $\tau_{\Xi_{\mathbf{V}_a}^{\omega}(z+1)} = \tau_{\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} \phi_{z+1}^a} = \varepsilon_b \tau_{a+1+z} \dot{+} \varepsilon_b \tau_{a+1+z+1} = \varepsilon_b \tau_{a+1+z+1}$. Let $\gamma_0 = 1$, and for $n \in \mathbb{N}$ define $\gamma_{n+1} = \varepsilon_b \tau_{a+1+z} \dot{\times} \omega \dot{\times} \gamma_n$. We have $\sup_{\square} (-\gamma_{\mathbb{N}}) = -\varepsilon_b^{\omega} \tau_{a+1+z} = \phi_{z+1}^a(-1)$. Let $n \in \mathbb{N}$ with $\Xi_{\mathbf{V}_a}^n(\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} 1) = \Xi_{\mathbf{V}_a}^{\omega} z \dot{+} (-\gamma_n)$ (this is trivially the case for $n = 0$). We have

$$\begin{aligned} \Xi_{\mathbf{V}_a}^{n+1}(\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} 1) &= \Xi_{\mathbf{V}_a}(\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} (-\gamma_n)) \\ &= \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a) \dot{+} \dot{\omega}^{\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} (-\gamma_n)} \\ &= \delta_a \dot{+} (-\delta_a \dot{\times} \omega \dot{\times} \delta_a) \dot{+} \dot{\omega}^{\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} (-\omega^{\tau_{\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} 1}} \dot{\times} \gamma_n)} \quad \text{Proposition 4} \\ &= \Xi_{\mathbf{V}_a} \Xi_{\mathbf{V}_a}^{\omega}(z \dot{+} (-\omega^{\varepsilon_b \tau_{a+1+z}} \dot{\times} \omega \dot{\times} \gamma_n)) \\ &= \Xi_{\mathbf{V}_a}^{\omega} z \dot{+} (-\varepsilon_b \tau_{a+1+z} \dot{\times} \omega \dot{\times} \gamma_n) \\ &= \Xi_{\mathbf{V}_a}^{\omega} z \dot{+} (-\gamma_{n+1}). \end{aligned}$$

It follows that $\sup_{\square} \Xi_{\mathbf{V}_a}^{\mathbb{N}}(\Xi_{\mathbf{V}_a}^{\omega} z \dot{+} 1) = \Xi_{\mathbf{V}_a}^{\omega} z \dot{+} \phi_{z+1}^a(-1)$. The second claim is valid since we have $\tau_{\Xi_{\mathbf{V}_a}^{\omega}(z+(-1))} = \tau_{\Xi_{\mathbf{V}_a}^{\omega} z} = \varepsilon_b \tau_{a+1+z} = \varepsilon_b \tau_{a+1+z+(-1)}$. This concludes the proof. \square

Our work on sign sequences is summed up in the following table where we distinguish the cases $\tau_a > 0$ and $\tau_a = 0$.

	If $a \in -\mathbf{On}$	If $a \notin -\mathbf{On}$
$\theta_a = \omega \dot{\times} a$	$\theta_a \in -\mathbf{On}$	$\theta_a \notin -\mathbf{On}$
$\tau_a = \text{ot}(a_L, \square)$	$-a$	$\tau_a > 0$
$\zeta_a = \varepsilon_b \tau_a$	0	$\zeta_a \in \varepsilon \mathbf{On}$
$\delta_a = \dot{\omega}^{\zeta_a+1}$	ω	$\zeta_a \times \omega$
\mathbf{V}_a	$\dot{\omega}^{\omega^{-1}} \dot{+} \mathbf{Mo}$	$\dot{\omega}^{\zeta_a+1 \dot{+} (-\delta_a)} \dot{+} \mathbf{Mo}$
$V(a) = \Xi_{\mathbf{V}_a}^{\omega} 0$	$\dot{\omega}^{\omega^{-1}} \dot{+} \sum_{p \in \mathbb{N}} \dot{\omega}^{\Xi_{\mathbf{Mo}}^p \omega} \dot{+} (-\dot{\omega}^{\sum_{k=0}^p \Xi_{\mathbf{Mo}}^k \omega}) \dot{+} 4$	$\dot{\omega}^{\zeta_a+1 \dot{+} (-\delta_a)} \dot{+} \sum_{p \in \mathbb{N}} \dot{\omega}^{\Xi_{\mathbf{Mo}}^p \delta_a} \dot{+} (-\dot{\omega}^{\sum_{k=0}^p \Xi_{\mathbf{Mo}}^k \delta_a \dot{\times} \delta_a^2})$
ϕ_{z+1}^a	$\varepsilon \tau_z \dot{+} 1$	$\varepsilon_b \tau_{a+1+z+1}$
$\phi_{z+1}^a(-1)$	$-\dot{\varepsilon}^{\omega} \tau_z$	$-\dot{\varepsilon}^{\omega} \tau_{a+1+z}$
$\Phi^a(z)$	$\sum_{\beta < \ell(z)} \phi_{z1}^0(\beta+1)$	$\sum_{\beta < \ell(z)} \phi_{z1}^a(\beta+1)$
$\phi_{z+1}^{\mathbf{I}}$	$\varepsilon_b \tau_z$	
$\phi_{z+1}^{\mathbf{I}}(-1)$	$-\omega$	
$\rho \theta_a$	θ_a	$\sum_{\gamma < \ell(a)} \phi_{a1}^{\mathbf{I}}(\gamma+1)$
$\rho \theta_{a-n}$	$\rho \theta_a \dot{+} (-n)$	
$\rho \theta_{a+n+1}$	$\rho \theta_a \dot{+} \Xi_{\mathbf{Mo}}^{n+1} 1$	$\rho \theta_a \dot{+} \Xi_{\mathbf{Mo}}^{n+1}(\zeta_a + 1)$
$\mathbf{R}_{a,n}$	$\rho \theta_{a-n-1} \dot{+} \mathbf{V}_0^{\omega}$	$\rho \theta_{a-n-1} \dot{+} \mathbf{V}_0^{\omega}$
$\mathbf{R}_{a,-(n+1)}$	$\rho \theta_{a+n} \dot{+} \mathbf{V}_0^{\omega}$	$\rho \theta_{a+n} \dot{+} \mathbf{V}_0^{\omega}$
$\rho(\theta_{a-n-1/2}) \dot{+} z$	$\rho \theta_{a-n-1} \dot{+} V(0) \dot{+} \Phi^0(z)$	$\rho \theta_{a-n-1} \dot{+} V(a) \dot{+} \Phi^a(z)$
$\rho(\theta_{a+n+1/2}) \dot{+} z$	$\rho \theta_{a+n} \dot{+} V(0) \dot{+} \Phi^0(z)$	$\rho \theta_{a+n} \dot{+} V(a) \dot{+} \Phi^a(z)$

Sign sequence formulas for \mathbf{R} where $a \in \mathbf{No}$ and $z \in \mathbf{No}$ and $n \in \mathbb{N}$.

In order to obtain the formula for λ_z , one need only use the relation $\lambda_z = \dot{\omega}^{\dot{\omega}^{\rho z}}$ and apply Proposition 4 twice.

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GLOSSARY

No class of surreal numbers 1

On	class of ordinal numbers	1
\sqsubseteq	simplicity relation	1
$x[\alpha]$	α -th sign of x	2
$\ell(x)$	length of x	2
$x \sqsubseteq y$	x is simpler than y	2
$x \sqsubset$	set of strictly simpler numbers	2
$\sup_{\sqsubseteq} X$	supremum of the chain (X, \sqsubseteq) in $(\mathbf{No}, \sqsubseteq)$	3
δ_x	unique monomial in the archimedean class of x	3
Mo	class of monomials	3
$z \mapsto \dot{\omega}^z$	ω -map, i.e. parametrization of Mo	3
$\alpha \dot{+} \beta$	ordinal sum	3
$\alpha \dot{\times} \beta$	ordinal product	3
α^β	ordinal exponentiation	3
$x \dot{+} y$	sum concatenation	3
$x \dot{\times} y$	product concatenation	3
$\Xi_{\mathbf{U}}^{\mathbf{V}}$	(\leq, \sqsubseteq) -isomorphism $\mathbf{U} \rightarrow \mathbf{V}$	6
$\mathbf{U} \prec \mathbf{V}$	imbrication $\Xi_{\mathbf{U}} \Xi_{\mathbf{V}} \mathbf{No}$ of \mathbf{V} into \mathbf{U}	6
$\mathbf{S}^{<\omega}$	class of fixed points of $\Xi_{\mathbf{S}}$	6
ε	parametrization of $\mathbf{Mo}^{<\omega}$	7
exp	Gonshor's exponential function	9
La	class of log-atomic numbers	9
λ	parametrization of La	10
K	class of κ -numbers	10
κ	parametrization of K	11
I	structure with $\mathbf{K} = \mathbf{Mo}^{<2} \prec \mathbf{I}$	11
ι	parametrization of I	11
R	structure with $\mathbf{La} = \mathbf{Mo}^{<2} \prec \mathbf{R}$	11
ρ	parametrization of R	11

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