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Lagrangian discretization of variational mean field games

Clément SARRAZIN

Abstract

In this article, we introduce a method to approximate solutions of some variational mean field game problems with congestion, by finite sets of player trajectories. These trajectories are obtained by solving a minimization problem, similar to the initial variational problem. In this discretized problem, congestion is penalized by a Moreau envelop with the 2-Wasserstein distance. Study of this envelop as well as efficient computation of its values and variations is done using semi-discrete optimal transport.

We show convergence of the discrete sets of trajectories toward a solution of the mean field game, as well as conditions on the discretization in order to get this convergence. We also show strong convergence (in particular almost everywhere) in some cases for the underlying densities resulting from the Moreau projection of solution to the discrete problem.

1 Introduction

Mean field games were introduced by Lasry and Lions in [12, 13] and, independently by Caines, Huang and Malhamé in [10]. In these games, an infinite population of indistinguishable players evolve in a domain while minimizing an energy that depends on the trajectory of the whole population. In a simple case, players, which are represented by curves on the domain Ω , $x \in \mathcal{C}^0([0; T], \Omega)$, are trying to minimize an accumulated energy

$$\int_0^T \left[\frac{\|x'(t)\|^2}{2} + g(\mu(t, x(t))) + V(x(t)) \right] dt + \Phi(x(T)).$$

Here, g is an increasing function on \mathbb{R} , penalizing $\mu(t, x(t))$, the density of the population around player x at time t , V and Φ are scalar functions on \mathbb{R}^d . Players are then pushed towards areas where V is low (and, at the end of the trajectory, Φ), while also trying to avoid overcrowded areas, where $g(\mu(t, x))$ is too large. At equilibrium, each player is following an optimal path for his/her own energy, the evolution of the population's density μ is linked to this of the value function:

$$\phi : (t_0, x_0) \mapsto \inf \left\{ \int_{t_0}^T \left[\frac{\|x'(t)\|^2}{2} + g(\mu(t, x(t))) + V(x(t)) \right] dt + \Phi(x(T)) \mid x(t_0) = x_0 \right\}$$

via a mean-field game system (with no-flux conditions for the second equation):

$$\begin{cases} -\partial_t \phi + \frac{\|\nabla \phi\|^2}{2} = g(\mu) + V \\ \partial_t \mu - \nabla \cdot (\mu \nabla \phi) = 0 \\ \mu(0, \cdot) = \mu^0, \phi(T, \cdot) = \Phi \end{cases}$$

If we set $f' = g$, the first (Hamilton-Jacobi) equation in this system is interpreted as the belonging of $-\partial_t \phi + \frac{\|\nabla \phi\|^2}{2} - V$ to the subgradient of the (convex) function f . Using some tools

from convex analysis, one can show that μ , seen as a curve in $\mathcal{C}^0([0; T], \mathcal{P}(\Omega))$ is a minimizer for the variational mean field game problem:

$$\inf_{\substack{\partial_t \mu + \nabla \cdot (\mu v) = 0 \\ \mu(0) = \mu_0}} \int_0^T \int_{\Omega} \left[\frac{\|v(t, x)\|^2}{2} \mu(t, x) + f(\mu(t, x)) + V(x) \mu(t, x) \right] dx dt + \int_{\Omega} \Phi(x) \mu(T, x) dx$$

This is a convex minimization problem, up to the change of variables $\mu = \mu$, $w = \mu v$. Conversely, any minimizer for this problem induces weak solutions to the mean field system (for a more precise statement of this correspondance we refer the reader to the course notes on mean field games by Cardaliaguet [5], and by Santambrogio [19] as well as the survey by Santambrogio, Carlier and Benamou, [2]).

In this article, we will consider the more general case of (μ, v) minimizing the following global energy:

$$J(\mu, v) = \int_0^T \left[\int_{\Omega} L(v(t, x)) d\mu(t, x) dx + F(\mu(t, \cdot)) \right] dt + G(\mu)$$

under the constraints $\partial_t \mu + \nabla \cdot (\mu v) = 0$ (with no flux boundary conditions) and the initial distribution $\mu(0, \cdot) = \mu^0 \in \mathcal{P}(\Omega)$.

The first term in J , $\int_0^T \int_{\Omega} L(v(t, x)) d\mu(t, x) dx$ measures the cost of displacement associated with the evolution of μ following the velocity v and will be referred to as the "kinetic" part of the energy. We make the assumption that $L : \mathbb{R}^d \rightarrow \mathbb{R}$ is a convex continuous function on \mathbb{R}^d which behaves like $\|\cdot\|^r$ for some $r > 1$. More precisely, there exists $C > 0$,

$$\forall p \in \mathbb{R}^d, \frac{1}{rC} \|p\|^r - C \leq L(p) \leq \frac{C}{r} \|p\|^r + C \quad (1)$$

The second term, involving F , modelizes the impact of congestion. It penalizes high densities, dissuading players from concentrating during their movement. F is now a convex lower semi-continuous function (for the topology associated with the narrow topology) over $\mathcal{M}_+(\Omega)$ (the space of positive finite measures on Ω). In the previous variational mean field games, it was defined by:

$$F : \mu \in \mathcal{M}(\Omega) \begin{cases} \int_{\Omega} f(\mu(x)) dx & \text{if } 0 \leq \mu \ll dx \\ +\infty & \text{otherwise} \end{cases}.$$

To handle its possibly bad behaviour at singular measures we use the particular Moreau-Yosida regularization defined in Section 6. The simplest expression, for which we compute approximate optimal trajectories in Section 8, is

$$F : \mu \in \mathcal{M}(\Omega) \mapsto \begin{cases} 0 & \text{if } 0 \leq \mu \leq dx \\ +\infty & \text{otherwise} \end{cases}$$

which corresponds to a hard congestion constraint, forbidding the presence of more than one player at a given position.

Finally, G is a continuous (again, for the topology associated with the narrow topology) convex function over $\mathcal{C}^0([0; T], \mathcal{P}(\mathbb{R}^d))$, bounded from below. By driving the players toward a goal (the trajectories at which it is the lowest) this term will often favorize congestion. In our numerical simulations, it will be given by a potential, at intermediate times (V in the introductory example above) and/or at final time (Φ in this example).

Discretization of mean field games: In Parts 1 to 3 of this paper, we study a lagrangian approximation of the solutions to the mean field game described above. In practice, this means reformulating it as a minimization over $\mathcal{P}(\mathcal{C}^0([0;T], \mathbb{R}^d))$, then looking for minimizers of a similar problem, over discrete probability measures this time. For such measures, the term $\int_0^T F(\mu(t))dt$ could be ill-defined and congestion must be penalized in a different manner. The main goal of this article is to show that replacing the functional F with a regularized version, in the form of a Moreau envelope in the Wasserstein space:

$$F_\epsilon(\mu) := \inf_{\rho \in \mathcal{P}(\Omega)} \frac{W_2^2(\rho, \mu)}{2\epsilon} + F(\rho)$$

circumvents this issue and allows us to build our approximate trajectories as solutions of the discretized problem:

$$\inf \left\{ \int_0^T \left[\int_{\Omega} L(v(t, x)) d\mu(t, x) + F_\epsilon(\mu(t)) \right] dt + G(\mu) \mid \mu \in \mathcal{C}^0([0;T], \mathcal{P}_N(\mathbb{R}^d)), \mu(0) = \mu_N^0 \right\} \quad (2)$$

Here, minimization is done on the space of continuous curves valued into the space of discrete uniform probability measures on \mathbb{R}^d , $\mathcal{P}_N(\mathbb{R}^d)$, for which there exists a vector field v such that $\partial_t \mu + \nabla \cdot (\mu v) = 0$. Note that this problem is non-convex, due to the term F_ϵ , however, after time discretization of the time integrals above, it is a finite dimensional minimization problem, and we can look for its local minimizers, using standard gradient descent algorithms.

A general weak convergence of the discrete minimizers of (2) is shown in Sections 4 and 5, as N goes to infinity and ϵ to 0. Such a convergence is very reminiscent of Γ -convergence and guarantees that we recover the continuous solutions, as the number of players goes to infinity. In particular cases of functional F , one can extract from the Moreau envelope a density that approximates the optimal density at all time, in a stronger L^p sense, as shown in Section 7.

Such a discretization is applied, in [14], to the similar problem of Wasserstein gradient descent, on the same kind of congestion penalizations F . However, in this particular case, the same convergence results as those we claim above can only be proven under strong assumptions that are deemed "unnatural" by the authors.

Moreau envelope in the Wasserstein space: In Part 6, we further study these Moreau envelopes in the 2-Wasserstein space. Indeed, their restrictions to the space of uniform discrete measures on N points enjoy some good regularity properties (when seen as functions on $(\mathbb{R}^d)^N$, the positions of the Diracs). We give explicit expressions for their derivatives, which are then used in Section 8 to compute good candidates for the minimizers of these discrete problems (which are non-convex).

Values of the Moreau envelope can be numerically computed using a Newton algorithm (as the corresponding problem can be cast as a concave problem), whereas the minimizing discrete trajectories, are approximated using an implementation of the L-BFGS algorithm on the finite dimensional problem, after time discretization. The images from section 8 can be obtained using the code available on Github¹.

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¹https://github.com/CSarrazin-prog/Congested_MFG.git

Leclerc for his work on the `pysdot`² library for numerical computations of integrals over Laguerre cells, as well as Jean-Marie Mirebeau for his work on approximating the solutions of Eikhonal equations³. We make ample use of these libraries in Section 8 when we compute our approximated optimal trajectories.

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2 Preliminaries

To guarantee existence of minimizers for our our minimization problems, we make the assumption that the various functions we use are *lower semi-continuous (l.s.c)*, often for the narrow convergence described below. That is to say, if x_n converges to x (for the corresponding topology), then $H(x) \leq \liminf_{n \rightarrow \infty} H(x_n)$ for the l.s.c function H . A standard way to show that such functions admit minimizers according to a given set of closed constraints is to take a minimizing sequence (of admissible points) and extract from it a converging subsequence. Then the limit point is a minimizer for the problem.

For a Polish space X , the space $\mathcal{M}(X)$ of finite measures over X can be endowed with the topology of *narrow convergence*, which is defined by duality with the space of continuous bounded functions on X . It is, in particular, in this sense that we express the continuity in time of solutions to the continuity equation mentioned in Theorem 1.

Given $\mu \in \mathcal{P}(X)$ and $T : X \rightarrow Y$ another Polish space, one defines the *push-forward* (or *image measure of μ by T*), which we write $T\#\mu$, as verifying: for any $\phi \in \mathcal{C}_b^0(Y)$,

$$\int_Y \phi(y) d(T\#\mu)(y) = \int_X \phi(T(x)) d\mu(x)$$

We use in particular this notation when dealing with a probability measure on $\mathcal{C}^0([0; T], \mathbb{R}^d)$ to access the corresponding measure at time t , $e_t\#Q$. Here, the map used is the evaluation at time $t \in [0; T]$, $e_t : \mathcal{C}^0([0; T], \mathbb{R}^d) \in \Gamma \mapsto \gamma(t)$.

In the specific case where the Polish space X is compact, the narrow topology described above is metrizable, using the celebrated Wasserstein distance which derives from the notion of optimal transport between measures (see [20], [22] and [21]):

Given $\mu, \rho \in \mathcal{P}(X)$ and a cost function $c : X^2 \mapsto \mathbb{R} \cup +\infty$ which is lower semi-continuous and bounded from below, the *optimal transport cost from μ to ρ according to c* is:

$$I_c(\mu, \rho) := \inf_{\pi \in \Pi(\mu, \rho)} \int_{X^2} c(x, y) d\pi(x, y)$$

where the *transport plans* π are coupling probabilities, with marginals μ and ρ , i.e. the infimum is taken over $\Pi(\mu, \rho) := \{\pi \in \mathcal{P}(X^2) \mid p_x\#\pi = \mu, p_y\#\pi = \rho\}$ (p_x and p_y being the projectors on the first and second coordinates for X^2). With these hypotheses on X and c , this infimum is always attained by an optimal transport plan π .

²<https://github.com/sd-ot/pysdot>

³<https://github.com/Mirebeau/HamiltonFastMarching.git>, implementation based on [17]

This convex minimization problem over $\mathcal{P}(X^2)$ admits a dual concave formulation:

$$I_c(\rho, \mu) = \sup \left\{ \int_X \phi d\mu + \int_Y \psi d\rho \mid \phi \in \mathcal{C}_b^0(X), \psi \in \mathcal{C}_b^0(Y) : \forall (x, y) \in X \times Y, \phi(x) + \psi(y) \leq c(x, y) \right\}$$

Taking, for each $\phi \in \mathcal{C}_b^0(X)$ the optimal $\psi = \phi^c := \inf_{x \in X} c(x, \cdot) - \phi(x)$, we have:

$$I_c(\rho, \mu) = \sup \left\{ \int_X \phi d\mu + \int_Y \phi^c d\rho \mid \phi \in \mathcal{C}_b^0(X) \right\}$$

When they exist, optimal ϕ and $\psi = \phi^c$ are called *Kantorovich potentials* for the transport from μ to ρ , and symmetrically, $\phi = \psi^c$. Functions that are the c -transform of functions in \mathcal{C}_b^0 ($\psi = \phi^c$) are said to be c -concave so that the optimisation in $\phi \in \mathcal{C}_b^0(X)$ above can in fact be done on c -concave functions only.

In this paper, we focus on optimal transport where one of these measures, let us say μ with our previous notations, is supported on a finite set, $\mu = \sum_{i=1}^N \mu_i \delta_{y_i}$, $y_1, \dots, y_N \in (\mathbb{R}^d)^N$, whereas ρ admits a density with respect to the Lebesgue measure. In this case, a theorem attributed to Brenier, states that an optimal transport plan from ρ (on Ω) to μ , is induced by a transport map $T: \pi = (T, id) \# \rho$. Ω is in return partitioned into *Laguerre cells*, $\text{Lag}_i = T^{-1}(y_i)$, $i = 1, \dots, N$. The optimal transport map T can be immediately obtained, given these Laguerre cells. On the other hand, these cells can be computed using the Kantorovich potentials, (ϕ, ϕ^c) , for this optimal transport. Indeed, setting (for readability purposes) $\phi_i = \phi(y_i)$, one has the characterization

$$\text{Lag}_i = \text{Lag}_i(y, \phi) := \{x \in \Omega \mid c(x, y_i) - \phi_i \leq c(x, y_j) - \phi_j \text{ for all } j\}.$$

Solving optimal transport amounts to solving the (non-linear) system in $\phi \in \mathbb{R}^d$:

$$\rho(\text{Lag}_i(y, \phi)) = \mu_i, \text{ for every } i$$

which simply states the mass conservation during transport. This can be done using a quasi Newton algorithm on the weights ϕ , as described in [11].

The Wasserstein distance is obtained from the optimal cost with c being a power of the distance d on X . Taking the appropriate root of this transport cost naturally gives a distance between probability measures (albeit only on a subset of $\mathcal{P}(X)$):

Take the cost $c_p : x, y \in X^2 \mapsto d(x, y)^p$, $p > 1$. For $\mu, \rho \in \mathcal{P}(X)$, we will write the p -Wasserstein distance between μ and ρ ,

$$W_p(\mu, \rho) := (I_{c_p}(\mu, \rho))^{1/p}$$

It defines a distance on the set of probabilities over X , with finite p -th order moment. Furthermore, in the case where X is compact, W_p metrizes the narrow topology on $\mathcal{P}(X)$.

Notations: In this article, Ω will always denote a smooth compact domain in \mathbb{R}^d , and will be the domain to which our particles are restrained in the continuous setting. However, we will also consider trajectories going outside the domain and therefore, our variational optimization problems will be written with the space $\Gamma := \mathcal{C}^0([0; T], \mathbb{R}^d)$ as the space of possible trajectories.

For a convex proper l.s.c function f on a locally convex space E in duality with a space E^* , we define the subdifferential of f at $x \in E$ via

$$\partial f(x) := \{p \in E^* \mid \forall y \in E, f(y) \geq f(x) + p \cdot (y - x)\}$$

and the Legendre transform of f at $p \in E^*$:

$$f^*(p) = \sup_{x \in E} p \cdot x - f(x)$$

Finally, to simplify some lengthy computations, we will write \lesssim for inequalities up to a multiplicative positive constant which only depends on the domain Ω .

3 The lagrangian setting and the continuous case

A curve in $\mathcal{C}^0([0; T], \mathcal{P}(\Omega))$ solution of our variational mean field game described in Part 1 can be seen as a probability measure in $\mathcal{P}(\Gamma)$ thanks to the following theorem (see Theorems 4 and 5 of [15]):

Theorem 1. *Let $\rho \in \mathcal{C}^0([0; T]; \mathcal{P}(\Omega))$, $r > 1$, be solution (in the sense of distributions) of the continuity equation $\partial_t \rho + \nabla \cdot (\rho v) = 0$, with a $\mathbb{L}^r(d\rho_t dt)$ velocity vector field v . Then there exists a probability $Q \in \mathcal{P}(\Gamma)$ such that:*

1. Q -almost every $\gamma \in \Gamma$ is in $\mathbb{W}^{1,r}([0; T]; \Omega)$ and satisfies $\gamma'(t) = v(\gamma(t))$ for \mathcal{L}^1 -almost every $t \in [0; T]$.
2. $\rho(t) = e_t \# Q$ for every $t \in [0; T]$.

Conversely, any $Q \in \mathcal{P}(\Gamma)$ which satisfies $\int \|\gamma'\|_{\mathbb{L}^r}^r dQ(\gamma) < +\infty$ induces an absolutely continuous curve in $\mathcal{C}^0([0; T]; \mathcal{P}(\Omega))$ solution to the continuity equation, via $\rho(t) = e_t \# Q$.

For the rest of this article, we fix an initial distribution of players, $\mu^0 \in \mathcal{P}(\Omega)$, that admits a density with respect to the Lebesgue measure on Ω . The variational mean-field game we consider can be rewritten as a minimization problem over $\mathcal{P}(\Gamma)$, using the representation of Theorem 1:

$$(\mathcal{M}_{\mu^0}) : \inf_{\substack{Q \in \mathcal{P}(\Gamma) \\ e_0 \# Q = \mu^0}} J(Q) := \int_{\Gamma} L(\gamma) dQ(\gamma) + \int_0^T F(e_t \# Q) dt + G(Q). \quad (3)$$

Remark. We have made here the abuse of notations: for $\gamma \in \Gamma$,

$$L(\gamma) := \int_0^T L(\gamma(t)) dt$$

We also now take for G , a continuous function, on $\mathcal{P}(\Gamma)$, (for the narrow topology).

Let us briefly recall why this problem admits minimizers, using to the direct method in calculus of variations:

Proposition 2. *The functional J is l.s.c on $\mathcal{P}(\Gamma)$, and (\mathcal{M}_{μ^0}) has minimizers.*

Proof. We first prove that J is l.s.c by treating separately its three terms. Theorem 4.5 of [9] and the bounds in (1) on L directly imply the lower-semicontinuity of the kinetic energy on $\mathcal{P}(\Gamma)$, $Q \mapsto \int_{\Gamma} L(\gamma) dQ(\gamma)$.

We prove the lower semi-continuity of $Q \mapsto \int_{\Gamma} F(e_t \# Q) dt$ directly. Let (Q_n) be a sequence converging to Q_{∞} in $\mathcal{P}(\Gamma)$. By continuity of e_t on Γ , for every t , $e_t \# Q_n$ narrowly converges to

$e_t \# Q_\infty$ as n goes to infinity. Thus, by lower semi-continuity of F and Fatou Lemma, we get as desired

$$\int_0^1 F(e_t \# Q_\infty) dt \leq \int_0^1 \liminf_{N \rightarrow \infty} F(e_t \# Q_N) dt \leq \liminf_{N \rightarrow \infty} \int_0^1 F(e_t \# Q_N) dt$$

Finally, G is continuous (and thus l.s.c) by assumption.

To show existence, we first note that for any upper bound $C > 0$, the set

$$K_C := \{\gamma \in \Gamma \mid L(\gamma) \leq C, \gamma(0) \in \Omega\}$$

is compact, which follows from Arzela-Ascoli's theorem and from the fact that any curve in K_C is Hölder-continuous. If we take a minimizing sequence $(Q_n)_n$ for (\mathcal{M}_{μ^0}) we notice that it is tight since for any n and C ,

$$Q_n(\Gamma \setminus K_C) < \frac{J(Q_n)}{C}.$$

Using Prokhorov's theorem, we can extract from it a sequence converging to a $Q_\infty \in \mathcal{P}(\Gamma)$ for the narrow topology. In particular, $e_0 \# Q = \mu^0$. But then, by lower-semi-continuity, Q_∞ is a minimizer for our problem since $J(Q_\infty) \leq \liminf_n J(Q_n) = \inf_{\substack{Q \in \mathcal{P}(\Gamma) \\ e_0 \# Q = \mu^0}} J(Q)$ \square

For now on, Q_{\min} will denote a (any) minimizer of (\mathcal{M}_{μ^0}) .

4 Space discretization in $\mathcal{P}(\Gamma)$

Following the lagrangian point of view of Theorem 1, we wish to approximate solutions of the previous continuous problem, (\mathcal{M}_{μ^0}) by probabilities $Q_N \in \mathcal{P}_N(\Gamma)$, with a fixed initial distribution $e_0 \# Q_N = \mu_N^0 \in \mathcal{P}_N(\Omega)$, where $\mathcal{P}_N(X) := \left\{ \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \mid x_1, \dots, x_N \in X \right\}$. For such distribution of trajectories, the value of J defined above can be $+\infty$, even for distribution very close to a minimizer.

We avoid this problem by replacing the congestion term by a regularized version of it, (which behaves well for discrete probability distributions): in a Hilbert space H , the Moreau envelope of a convex function g , with parameter ϵ , is given by the inf convolution $g_\epsilon(x) = \inf_{y \in H} \frac{\|x-y\|_H^2}{2\epsilon} + g(y)$. It has the advantages of being finite, and even differentiable, for any $x \in H$, upon some mild assumptions on g . Notice also that, $g_\epsilon(x)$ has limit $g(x)$ as ϵ goes to 0, whereas the limit is $\inf g$ as ϵ goes to $+\infty$. As we are on the space $\mathcal{P}(\Omega)$, a natural replacement for the squared norm is the 2-Wasserstein distance squared defined in the Preliminaries section. For $\epsilon > 0$, we set

$$F_\epsilon : \mu \in \mathcal{P}(\mathbb{R}^d) \mapsto \min_{\rho \in \mathcal{P}(\Omega)} \frac{W_2^2(\mu, \rho)}{2\epsilon} + F(\rho).$$

We call F_ϵ the Moreau envelop of F with parameter ϵ (by analogy with the Hilbert case). Note that $F(\rho) < +\infty$ implies $\text{spt}(\rho) \subset \Omega$, and therefore, as ϵ goes to 0, points outside Ω highly penalize the value of F_ϵ .

The corresponding regularization of our energy J , which we describe now, is inspired by [16] where a similar treatment is applied to a variational formulation for the incompressible Euler equations. The other terms in J are well defined for discrete probabilities also, and therefore, we keep them unchanged in the following discretized problem:

$$\begin{aligned} (\mathcal{M}_{N, \mu_N^0, \epsilon}) : \inf \{ J_\epsilon(Q) \mid Q \in \mathcal{P}_N(\Gamma), e_0 \# Q = \mu_N^0 \} \\ \text{where } J_\epsilon(Q) := \int_\Gamma L dQ + \int_0^1 F_\epsilon(e_t \# Q) dt + G(Q) \end{aligned}$$

We will come back to F_ϵ more extensively in Section 6. For now, we will only use its lower semi-continuity on $\mathcal{P}(\mathbb{R}^d)$ and the limits as $\epsilon \rightarrow 0/ +\infty$, mentioned in proposition 8. Immediately, by similar arguments as in Section 3, we have existence of minimizers for this discrete problem:

Proposition 3. *For every $N \in \mathbb{N}^*$, $\epsilon > 0$, J_ϵ is l.s.c for the narrow convergence and for every $\mu_N^0 \in \mathcal{P}_N(\Omega)$, the infimum in $(\mathcal{M}_{N,\mu_N^0,\epsilon})$ is attained.*

Similarly to the Hilbert case, one would expect minimizers for $(\mathcal{M}_{N,\mu_N^0,\epsilon})$ to converge to a minimizer of (\mathcal{M}_{μ^0}) as $N \rightarrow \infty$ and $\epsilon \rightarrow 0$. This is the case, but only provided ϵ does not vanish too quickly. Proposition 4, very much in the spirit of Γ -convergence states such a convergence. Note however that the result stated in (upper bound) is weaker than the usual Γ -limsup one.

The proof of this proposition uses a quantization argument on a solution Q_{\min} of (\mathcal{M}_{μ^0}) . From standard sobolev inclusions, we can find $\frac{1}{2} < s \leq 1$ such that

$$W^{1,r}([0;T], \mathbb{R}^d) \hookrightarrow \mathbb{H}^s([0;T], \mathbb{R}^d) \hookrightarrow \mathcal{C}^0([0;T], \mathbb{R}^d)$$

and these injections are compact (recall that r is the exponent in the definition of the lagrangian L). From now on, we will denote $\mathbb{H}^s([0;T], \mathbb{R}^d)$ by \mathbb{H}^s and the 2-Wasserstein distance on $\mathcal{P}(\mathbb{H}^s)$ by $W_{\mathbb{H}^s}$, as no confusion can arise. In particular, Q_{\min} is supported on \mathbb{H}^s and we will take our quantization measures supported in this same space.

The reason behind this choice is the following: to prove the (Upper bound) part of proposition 4, we need to approximate Q_{\min} using discrete probabilities which have lower kinetic energy. Although the approximation can be done by quantization measures according to most Wasserstein distances, taking one associated with a Hilbert norm, the \mathbb{H}^s norm here, gives quantization measures supported on suitable barycenters, in some sense. L being a convex function, this gives us measures with a lower kinetic energy.

Lemma 1. *For every $N \in \mathbb{N}^*$, there exists \tilde{Q}_N minimizing $W_{\mathbb{H}^s}^2(\cdot, Q_{\min})$ on $\mathcal{P}_N(\Gamma)$. Moreover, if we set $\tau_N := W_{\mathbb{H}^s}^2(\tilde{Q}_N, Q_{\min})$, then $\tau_N \rightarrow 0$ as N goes to infinity. In particular, \tilde{Q}_N narrowly converges towards Q_{\min} in $\mathcal{P}(\Gamma)$.*

Proof. $W_{\mathbb{H}^s}^2(\cdot, Q_{\min})$ is l.s.c for the narrow convergence on $\mathcal{P}_N(\Gamma)$, from the lower semi-continuity of the \mathbb{H}^s norm with respect to the uniform norm on Γ . Take a minimizing sequence $(Q_n)_n$ for our problem. We can choose Q_n to have lower kinetic energy than Q_{\min} :

To see this, fix $n \in \mathbb{N}$, and set $Q_n = \frac{1}{N} \sum_{i=1}^N \delta_{\tilde{\gamma}^i}$ and $P = \frac{1}{N} \sum_{i=1}^N \delta_{\tilde{\gamma}^i} \times Q_{\min}^i$ an optimal transport plan from Q_n to Q_{\min} . For each i , set $\eta^i = \int_{\Gamma} \gamma dQ_{\min}^i(\gamma)$. For every i , η^i is a minimizer of the convex functional $\int_{\Gamma} \|\cdot - \gamma\|_{\mathbb{H}^s}^2 dQ_{\min}^i(\gamma)$ over \mathbb{H}^s (since its gradient is given by an \mathbb{H}^s scalar product and vanishes at η_i). Therefore,

$$\begin{aligned} W_{\mathbb{H}^s}^2 \left(\frac{1}{N} \sum_{i=1}^N \delta_{\eta^i}, Q_{\min} \right) &\leq \frac{1}{N} \sum_{i=1}^N \int_{\Gamma} \|\eta^i - \gamma\|_{\mathbb{H}^s}^2 dQ_{\min}^i(\gamma) \\ &\leq \frac{1}{N} \sum_{i=1}^N \int_{\Gamma} \|\tilde{\gamma}^i - \gamma\|_{\mathbb{H}^s}^2 dQ_{\min}^i(\gamma) = W_{\mathbb{H}^s}^2(Q_n, Q_{\min}) \end{aligned}$$

and we can assume that Q_n is supported on the barycenters η_i .

Then we have $\int_{\Gamma} LdQ_n \leq \int_{\Gamma} LdQ_{\min}$ for every n (by convexity of L). Similarly to proposition 2, we can conclude that $(Q_n)_n$ is tight and, up to a subsequence, it narrowly converges towards a

minimizer \tilde{Q}_N of $W_{\mathbb{H}^s}^2(\cdot, Q_{\min})$ over $\mathcal{P}_N(\Gamma)$, which verifies $\int_{\Gamma} Ld\tilde{Q}_N \leq \int_{\Gamma} LdQ_{\min}$.

Now, to show that τ_N vanishes at infinity, it is sufficient to show that there exists $(Q_N)_N$, such that, for every N , $Q_N \in \mathcal{P}_N(\mathbb{H}^s)$, Q_N narrowly converges towards Q_{\min} in $\mathcal{P}_N(\mathbb{H}^s)$, and $\int_{\Gamma} \|\gamma\|_{\mathbb{H}^s}^2 dQ_N(\gamma)$ converges towards $\int_{\Gamma} \|\gamma\|_{\mathbb{H}^s}^2 dQ_{\min}(\gamma)$, as N goes to infinity. This can be done, for instance, as in Theorem 2.13 of [3], by sampling trajectories in the support of Q_{\min} and using a law of large numbers. Finally, since $s > \frac{1}{2}$, \mathbb{H}^s is continuously injected in Γ , and we have the narrow convergence in $\mathcal{P}(\Gamma)$ (for the uniform norm, this time). \square

Proposition 4. *Let $(\epsilon_N)_N$ be a positive sequence vanishing at infinity and assume that μ_N^0 narrowly converges towards μ^0 in $\mathcal{P}(\mathbb{R}^d)$.*

- **(Lower bound)** *Let $(Q_N)_N$ narrowly converge to Q_{∞} in $\mathcal{P}(\Gamma)$. Then, we have*

$$J(Q_{\infty}) \leq \liminf_{N \rightarrow \infty} J_{\epsilon_N}(Q_N).$$

- **(Upper bound)** *Assume that $\tau_N = o_{N \rightarrow \infty}(\epsilon_N)$ and $W_2^2(\mu_N^0, \mu^0) = o_{N \rightarrow \infty}(\epsilon_N)$. Then for any sequence $(Q_N)_N$ where Q_N is a minimizer, respectively for $(\mathcal{M}_{N, \mu_N^0, \epsilon_N})$,*

$$\limsup_{N \rightarrow \infty} J_{\epsilon_N}(Q_N) \leq J(Q_{\min}).$$

Proof of proposition 4 (lower bound). Take Q_N and Q_{∞} as in the proposition. For every $t \in [0; T]$ and every N , define ρ_N^t as a minimizer in the problem defining $F_{\epsilon_N}(e_t \# Q_N)$.

One can assume that $J_{\epsilon_N}(Q_N)$ is bounded from above. Therefore, there exists $C > 0$ such that $\int_0^T W_2^2(e_t \# Q_N, \rho_N^t) dt \leq C\epsilon_N$ for every N , since F and G are also bounded from below. Up to extracting a subsequence, we can assume that for almost all $t \in [0; T]$, ρ_N^t narrowly converges, as N goes to infinity, towards $e_t \# Q_{\infty}$. Using Fatou lemma, we get

$$\int_0^T F(e_t \# Q_{\infty}) dt \leq \int_0^T \liminf_{N \rightarrow \infty} F_{\epsilon_N}(e_t \# Q_N) dt \leq \liminf_{N \rightarrow \infty} \int_0^T F_{\epsilon_N}(e_t \# Q_N) dt$$

The lagrangian part (as well as G of course) is l.s.c for the narrow convergence, and we can write:

$$\begin{aligned} \int_{\Gamma} L(\gamma) dQ_{\infty}(\gamma) + \int_0^T F(e_t \# Q_{\infty}) dt + G(Q_{\infty}) &\leq \liminf_{N \rightarrow \infty} \int_{\Gamma} L(\gamma) dQ_N(\gamma) \\ &\quad + \liminf_{N \rightarrow \infty} \int_0^T F_{\epsilon_N}(e_t \# Q_N) dt + \liminf_{N \rightarrow \infty} G(Q_N) \\ &\leq \liminf_{N \rightarrow \infty} J_{\epsilon_N}(Q_N) \end{aligned}$$

which is our claim. \square

Lemma 2. *Let $Q, Q' \in \mathcal{P}(\mathbb{H}^s)$. Then, there exists $C > 0$, such that for every $t \in [0; T]$,*

$$W_2(e_t \# Q, e_t \# Q') \leq CW_{\mathbb{H}^s, 2}(Q, Q')$$

Proof. Since $s > \frac{1}{2}$, Sobolev injections give the existence of $C > 0$ such that $\|\cdot\|_\infty \leq C\|\cdot\|_{\mathbb{H}^s}$ on \mathbb{H}^s . Take P , an optimal transport plan from Q to Q' with the cost given by $\|\cdot\|_{\mathbb{H}^s}^2$, and $t \in [0; T]$. Then, $(e_t, e_t) \# P$ is a transport plan from $e_t \# Q$ to $e_t \# Q'$, and we can write:

$$\begin{aligned} W_2^2(e_t \# Q, e_t \# Q') &\leq \int_{\Gamma \times \Gamma} \|\gamma^1(t) - \gamma^2(t)\|^2 dP(\gamma^1, \gamma^2) \\ &\leq C^2 \int_{\Gamma \times \Gamma} \|\gamma^1 - \gamma^2\|_{\mathbb{H}^s, 2}^2 dP(\gamma^1, \gamma^2). \end{aligned} \quad \square$$

Proof of prop 4 (upper bound): Set $N \in \mathbb{N}^*$ and \tilde{Q}_N and τ_N as in Lemma 1. As is, \tilde{Q}_N is not necessarily admissible since it may not satisfy $e_0 \# \tilde{Q}_N = \mu_N^0$. However, since they are discrete measures with the same amount of Diracs and the same masses, we can simply translate the curves in $\text{spt}(\tilde{Q}_N)$ in order for it to be admissible for $(\mathcal{M}_{N, \mu_N^0, \epsilon_N})$, using vectors that are constant in time. This new measure, which we denote by $\tilde{Q}_{\mu_N^0, N}$ is admissible for $(\mathcal{M}_{N, \mu_N^0, \epsilon_N})$, has the same kinetic energy as \tilde{Q}_N and satisfies $W_{\mathbb{H}^s}^2(\tilde{Q}_{\mu_N^0, N}, \tilde{Q}_N) = W_2^2(\mu_N^0, e_0 \# \tilde{Q}_N)$.

Now, if Q_N is a minimizer for $(\mathcal{M}_{N, \mu_N^0, \epsilon_N})$, then

$$\begin{aligned} J_{\epsilon_N}(Q_N) &\leq J_{\epsilon_N}(\tilde{Q}_{\mu_N^0, N}) \leq \int_{\Gamma} \mathcal{L}(\gamma) d\tilde{Q}_N(\gamma) + G(\tilde{Q}_{\mu_N^0, N}) \\ &\quad + \int_0^T \frac{W_2^2(e_t \# \tilde{Q}_{\mu_N^0, N}, e_t \# Q_{\min})}{2\epsilon_N} + F(e_t \# Q_{\min}) dt \\ &\leq \int_{\Gamma} \mathcal{L}(\gamma) dQ_{\min}(\gamma) + G(\tilde{Q}_{\mu_N^0, N}) + T \frac{W_2^2(e_0 \# \tilde{Q}_N, \mu_N^0)}{\epsilon_N} \\ &\quad + \int_0^T \frac{W_2^2(e_t \# \tilde{Q}_N, e_t \# Q_{\min})}{\epsilon_N} + F(e_t \# Q_{\min}) dt \end{aligned} \quad (4)$$

Besides, $\int_0^1 W_2^2(e_t \# \tilde{Q}_N, e_t \# Q_{\min}) dt \lesssim W_{\mathbb{H}^s}^2(\tilde{Q}_N, Q_{\min}) = \tau_N$ following lemma 2 and by convexity of the transport cost,

$$W_2^2(e_0 \# \tilde{Q}_N, \mu_N^0) \leq 2 \left(W_2^2(e_0 \# \tilde{Q}_N, \mu^0) + W_2^2(\mu^0, \mu_N^0) \right) \lesssim (\tau_N + W_2^2(\mu^0, \mu_N^0)).$$

Now, $\tilde{Q}_{\mu_N^0, N}$ narrowly converges to Q_{\min} in $\mathcal{P}(\Gamma)$, by construction. If we take $(\epsilon_N)_N$ and μ_N^0 such that τ_N and $W_2^2(\mu^0, \mu_N^0)$ are negligible compared to ϵ_N as N goes to infinity, then, taking the limsup in inequalities 4, we get $\limsup_N J_{\epsilon_N}(Q_N) \leq J(Q_{\min})$, as we wanted. \square

Corollary 4.1. *With the same notations and assumptions on $(\epsilon_N)_N$ and $(\mu_N^0)_N$ as in the (upper bound) property, up to a subsequence, Q_N narrowly converges towards a minimizer of J . In particular, if (\mathcal{M}_{μ^0}) has a unique minimizer Q_{\min} , then any such sequence $(Q_N)_N$ narrowly converges toward Q_{\min} .*

Proof. By the (Upper bound) property, up to a subsequence, $J_{\epsilon_N}(Q_N)$ converges towards $l \leq \min J$. Then as before, $\int_{\Gamma} LdQ_N$ is bounded in N and $(\mu_N^0)_N$ is tight, therefore, $(Q_N)_N$ is tight, in $\mathcal{P}(\Gamma)$. Extract from it a subsequence converging towards $Q_\infty \in \mathcal{P}(\Gamma)$. Then $e_0 \# Q_\infty = \mu^0$, and by the (Lower bound) property, $J(Q_\infty) \leq l \leq \min J$ hence, Q_∞ is a minimizer of (\mathcal{M}_{μ^0}) . \square

The proper sequence ϵ_N of parameters (or rather their precise behaviour as $N \rightarrow \infty$), remains beyond our reach even in the simpler convex case of Part 8. A hint however, on how to bound the sequence τ_N is given by the following correspondence between optimal quantization and optimal covering of a set:

Proposition 5. *In X a metric space, take $Q \in \mathcal{P}(X)$ supported on $\Sigma \subset X$. Define the optimal quantization radius $\tau_N = \min W_{d_X, 2}^2(Q, \tilde{Q})$ as in lemma 1, and the optimal covering radius by $r_N = \inf_{\substack{P \subset X \\ |P| \leq N}} d_H(\Sigma, P)$ (here, d_H is the Hausdorff distance between subsets of X).*

$$\text{Then, assuming } r_N = O_{N \rightarrow \infty} \left(N^{-\frac{1}{D}} \right) \text{ one has: } \tau_N = \begin{cases} O_{N \rightarrow \infty} (N^{-1}) & \text{if } D < 2 \\ O_{N \rightarrow \infty} (N^{-1} \ln N) & \text{if } D = 2 \\ O_{N \rightarrow \infty} \left(N^{-\frac{2}{D}} \right) & \text{if } D > 2 \end{cases}$$

The constant D in this proposition is often referred to as the box-dimension or Minkowsky dimension of the set Σ .

Remark. A first point to make is that proposition 5 with $X = \mathbb{R}^d$ guarantees that we can choose μ_N^0 in such a way that $W_2^2(\mu_N^0, \mu^0) = O_{N \rightarrow \infty}(N^{-2/d})$. This is an information to take into account when choosing ϵ_N (although, it is likely to be redundant with the one given by the growth of τ_N).

To bound the covering radius r_N of the support of Q_{\min} , one can recall that this measure gives us a solution (ρ, v) to the continuity equation, according to proposition 1. In return, Q_{\min} -almost every curve is solution almost everywhere on $[0; T]$ of the EDO $\gamma' = v(\gamma)$. Provided we have some uniform Lipschitz-continuity of v , ρ is then given by the pushforward of μ^0 along the general solution of $x' = v(t, x)$ (see for instance [1], chapter 8.1 for more precise assumptions on v , and a demonstration). It is then immediate that $\text{spt}(Q_{\min})$ is of box-dimension $D = d$ (the dimension of Ω) in \mathbb{H}^s and, in that case, we can take any ϵ_N dominating $N^{-2/d}$ (or $\ln(N)/N$ in dimension $d=2$).

5 The fully discrete problem

We now use a uniform time discretization $0, \delta, \dots, M\delta = T$ to compute a fully discretized version of the energy J . Rather than writing heavy formulas for the new global energy, we change the subset upon which the minimization is done which allows for an energy almost identical to J_ϵ . The lagrangian and potential parts are then the same as in J and J_ϵ , whereas the congestion term is approximated by a Riemann sum. This is mainly done to simplify computations, and any time-discretization of curves in Γ which allows $\mathbb{W}^{1,r}$ bounds of the sort of (5) and (6) should also work here.

We perform our optimization on the space $\Gamma_\delta^{\text{lin}}$, of functions in Γ that are affine on each interval $[i\delta, (i+1)\delta]$ with $i = 0..M-1$. Our fully discrete problem is then:

$$\begin{aligned} (\mathcal{M}_{N, \mu_N^0, \delta, \epsilon}) : \inf & \left\{ J_{\delta, \epsilon}(Q) \mid Q \in \mathcal{P}_N(\Gamma_\delta^{\text{lin}}) \text{ s.t. } e_0 \# Q = \mu_N^0 \right\} \\ \text{with } J_{\delta, \epsilon}(Q) = & \int_\Gamma \mathcal{L} dQ + \delta \sum_{i=1}^{M-1} F_\epsilon(e_{i\delta} \# Q) + G(Q) \end{aligned}$$

Similarly to $(\mathcal{M}_{N, \mu_N^0, \delta, \epsilon})$ and (\mathcal{M}_{μ^0}) , we have existence of minimizers for any value of the parameters:

Proposition 6. *For every $N \in \mathbb{N}^*$, $\delta, \epsilon > 0$, $J_{\delta, \epsilon}$ is l.s.c for the narrow convergence and for every $\mu_N^0 \in \mathcal{P}_N(\Omega)$, the infimum in $(\mathcal{M}_{N, \mu_N^0, \delta, \epsilon})$ is attained.*

What is more interesting is a similar convergence result to the one in proposition 4, with an additional constraint on the parameters of the time discretizations, δ_N :

Proposition 7. *Assume that $(\delta_N)_N, (\epsilon_N)_N$ converge to 0, and that μ_N^0 narrowly converges towards μ^0 in $\mathcal{P}(\mathbb{R}^d)$ as $N \rightarrow \infty$:*

- **(lower bound)** *Let $(Q_N)_N$ narrowly converge to Q_∞ in $\mathcal{P}(\Gamma)$. Then, we have*

$$J(Q_\infty) \leq \liminf_{N \rightarrow \infty} J_{\delta_N, \epsilon_N}(Q_N).$$

- **(upper bound)** *Under the same assumptions as in proposition 4, and also assuming that $(\delta_N)^{2/r'} = o(\epsilon_N)$ where $r' = \frac{r}{r-1}$ is the dual exponent for r (set in the Introduction). Then for every sequence $(Q_N)_N$, with Q_N a minimizer respectively for $(\mathcal{M}_{N, \mu_N^0, \delta_N, \epsilon_N})$, we have*

$$\limsup_{N \rightarrow \infty} J_{\delta_N, \epsilon_N}(Q_N) \leq J(Q_{\min})$$

Proof of Proposition 7 (lower bound). We can assume that $J_{\delta_N, \epsilon_N}(Q_N)$ is bounded from above uniformly in N . In particular, Q_N is supported in $\mathbb{W}^{1,r}$ for every N . Then, as before, $\int_\Gamma LdQ_\infty \leq \liminf_N \int_\Gamma LdQ_N$.

For every $\gamma \in \mathbb{W}^{1,r}$, and $i\delta_N \leq t \leq (i+1)\delta_N$,

$$\|\gamma(t) - \gamma(i\delta_N)\|^2 \leq \delta_N^{2/r'} \left(\int_0^1 \|\gamma'(u)\|^r du \right)^{2/r} \quad (5)$$

and, integrating this inequality along Q_N , we get $W_2^2(e_{i\delta_N} \# Q_N, e_t \# Q_N) \leq C\delta_N^{2/r'}$ for every t in $]i\delta_N; \leq (i+1)\delta_N]$, since $\int LdQ_N$ is bounded. In particular, for every t , $e_{\lfloor t/\delta_N \rfloor \delta_N} \# Q_N$ narrowly converges towards $e_t \# Q_\infty$. Then, by Fatou lemma,

$$\begin{aligned} \int_0^1 F(e_t \# Q_\infty) dt &\leq \liminf_{N \rightarrow \infty} \sum_{i=0}^{M_N-1} \int_{i\delta_N}^{(i+1)\delta_N} F_{\epsilon_N}(e_{i\delta_N} \# Q_N) \\ &\leq \liminf_{N \rightarrow \infty} \delta_N \sum_{i=1}^{M_N-1} F_{\epsilon_N}(e_{i\delta_N} \# Q_N) \end{aligned}$$

and that last term is exactly the congestion term in J_{δ_N, ϵ_N} . Finally, continuity of G gives us our (lower bound) inequality. \square

Proof of Proposition 7 (upper bound). We momentarily fix $N \in \mathbb{N}^*$. Take \tilde{Q}_N and $\tilde{Q}_{\mu_N^0, N}$ as in lemma 1 and the proof of proposition 4 and define the piecewise affine interpolation operator, $T_{\delta_N}^{\text{lin}} : \gamma \in \Gamma \mapsto \gamma_{\delta_N}^{\text{lin}}$ where for t in $[i\delta_N; (i+1)\delta_N]$, and $\gamma \in \Gamma$, $\gamma_{\delta_N}^{\text{lin}}(t) = \gamma(i\delta_N) + \frac{\gamma((i+1)\delta_N) - \gamma(i\delta_N)}{\delta_N}(t - i\delta_N)$. The measure $\tilde{Q}_{N, \delta_N}^{\text{lin}} = T_{\delta_N}^{\text{lin}} \# \tilde{Q}_{\mu_N^0, N}$ will take the role of competitor for the problem $(\mathcal{M}_{N, \mu_N^0, \delta_N, \epsilon_N})$ that $\tilde{Q}_{\mu_N^0, N}$ had for problem $(\mathcal{M}_{N, \mu_N^0, \epsilon_N})$.

Then, convexity of \mathcal{L} gives us for every N , the inequality

$$\int_\Gamma \mathcal{L}d\tilde{Q}_{N, \delta_N}^{\text{lin}} \leq \int_\Gamma \mathcal{L}d\tilde{Q}_N \leq \int_\Gamma \mathcal{L}dQ_{\min}.$$

For F_{ϵ_N} , we have, as previously,

$$\begin{aligned}
\delta_N \sum_{i=1}^N F_{\epsilon_N}(e_{i\delta_N} \# \tilde{Q}_{N,\delta_N}^{\text{lin}}) dt &\leq \sum_{i=1}^N \int_{i\delta_N}^{(i+1)\delta_N} \frac{W_2^2(e_{i\delta_N} \# \tilde{Q}_{\mu_N^0, N}, e_t \# Q_{\min})}{2\epsilon_N} + F(e_t \# Q_{\min}) dt \\
&\leq \sum_{i=1}^N \int_{i\delta_N}^{(i+1)\delta_N} \frac{W_2^2(e_{i\delta_N} \# \tilde{Q}_N, e_t \# \tilde{Q}_N)}{\epsilon_N} + \frac{W_2^2(e_t \# \tilde{Q}_N, e_t \# Q_{\min})}{\epsilon_N} dt \\
&\quad + T \frac{W_2^2(\mu_N^0, e_0 \# \tilde{Q}_N)}{\epsilon_N} + \int_0^T F(e_t \# Q_{\min}) dt \\
&\leq C \left[\frac{W_2^2(\mu^0, \mu_N^0)}{\epsilon_N} + \frac{\delta_N^{2/r'}}{\epsilon_N} + \frac{\tau_N}{\epsilon_N} \right] + \int_0^1 F(e_t \# Q_{\min}) dt
\end{aligned}$$

Finally, for $\gamma \in \mathbb{W}^{1,r}$ and $i\delta_N < t \leq (i+1)\delta_N$

$$\begin{aligned}
\left\| \gamma(t) - \gamma(i\delta_N) - \frac{\gamma((i+1)\delta_N) - \gamma(i\delta_N)}{\delta_N} (t - i\delta_N) \right\| &\leq \int_{i\delta_N}^t \|\gamma'(u)\| du \\
&\quad + \frac{t - i\delta_N}{\delta_N} \int_{i\delta_N}^{(i+1)\delta_N} \|\gamma'(u)\| du \quad (6) \\
&\leq 2\delta_N^{1/r'} \left(\int_0^1 \|\gamma'(u)\|^r du \right)^{1/r}
\end{aligned}$$

and, integrating along \tilde{Q}_N , $\lim_{N \rightarrow \infty} W_{\mathbb{L}^\infty}^1(\tilde{Q}_{N,\delta_N}^{\text{lin}}, \tilde{Q}_N) = 0$, therefore, by continuity of G on Γ , $\lim_{N \rightarrow \infty} G(\tilde{Q}_{N,\delta_N}^{\text{lin}}) = G(Q_{\min})$. Now, to conclude, we observe as before that,

$$J_{\delta_N, \epsilon_N}(Q_N) \leq J_{\delta_N, \epsilon_N}(\tilde{Q}_{N,\delta_N}^{\text{lin}}) \leq J(Q) + C \frac{\delta_N^{2/r'} + W_2^2(\mu^0, \mu_N^0) + \tau_N}{\epsilon_N} + G(\tilde{Q}_{N,\delta_N}^{\text{lin}}) - G(Q)$$

and, as soon as $(\delta_N)_N$ is taken such that $\delta_N^{2/r'} = o(\epsilon_N)$ along with the same growth for the other parameters as proposition 4, one can conclude $\limsup_{N \rightarrow \infty} J_{\delta_N, \epsilon_N}(Q_N) \leq J(Q)$. \square

As previously, minimizers of J_{δ_N, ϵ_N} narrowly converge to minimizers of J , under these assumptions on ϵ_N , δ_N and μ_N^0 .

6 The Moreau envelope in the Wasserstein space

To make expressions more concise, we will use from time to time the optimal transport cost associated to the cost $c(x, y) = \frac{\|x-y\|^2}{2\epsilon}$ (and more importantly the associated Kantorovich potentials), instead of the standard squared norm. Let us quickly recall the definition, for $\mu \in \mathcal{P}(\mathbb{R}^d)$:

$$F_\epsilon(\mu) := \inf_{\rho \in \mathcal{M}(\Omega)} \frac{W_2^2(\rho, \mu)}{2\epsilon} + F(\rho) = \inf_{\rho \in \mathcal{M}(\Omega)} I_c(\rho, \mu) + F(\rho).$$

The transport cost $I_c(\cdot, \rho)$ is $+\infty$ outside of $\mathcal{P}(\mathbb{R}^d)$ and F is $+\infty$ outside of $\text{Dom}(F)$ so that the infimum is, *in fine*, only taken on the intersection of those 2 sets.

Proposition 8. *For every $\epsilon > 0$, the infimum defining F_ϵ is attained and F_ϵ is l.s.c on $\mathcal{P}(\mathbb{R}^d)$. Therefore, $Q \in \mathcal{P}(\Gamma) \mapsto \int_0^1 F_\epsilon(e_t \# Q) dt$ is l.s.c on $\mathcal{P}(\Gamma)$.*

Furthermore, $\lim_{\epsilon \rightarrow 0} F_\epsilon(\mu) = F(\mu)$ whereas $\lim_{\epsilon \rightarrow +\infty} F_\epsilon(\mu) = \inf F$, assuming μ has finite second order moment.

Proof. The fact that the infimum is attained for every $\epsilon > 0$ is straightforward and can be shown by the direct method of calculus of variations. Furthermore, take a sequence μ_n narrowly converging to μ_∞ in $\mathcal{P}(\Omega)$, and for every n , a measure $\rho_n \in \text{Dom}(F) \cap \mathcal{P}(\Omega)$ optimal for the problem defining $F_\epsilon(\mu_n)$. We may assume that $F_\epsilon(\mu_n)$ has a finite limit l as n goes to infinity.

Using Prokhorov theorem, we can extract a subsequence from $(\rho_n)_n$, narrowly converging towards a $\rho_\infty \in \text{Dom}(F)$. We extract the corresponding subsequence from μ_n and rename these new sequences, ρ_n and μ_n . Then,

$$\begin{aligned} F_\epsilon(\mu_\infty) &\leq \frac{W_2^2(\rho_\infty, \mu_\infty)}{2\epsilon} + F(\rho_\infty) \\ &\leq \liminf_n \frac{W_2^2(\rho_n, \mu_n)}{2\epsilon} + \liminf_n F(\rho_n) \\ &\leq \liminf_n F_\epsilon(\mu_n) = l \end{aligned}$$

and this is the lower semi-continuity inequality. The function on $\mathcal{P}(\Gamma)$ is then l.s.c by continuity of e_t on Γ and Fatou lemma. \square

Being defined as the minimum of a convex function, F_ϵ can be rewritten as the supremum of a concave dual problem using Fenchel-Rockafellar duality (see [7]). We make here two hypotheses on our congestion penalizing function F to ensure that this dual problem has solutions. These may seem like demanding restrictions, but in the cases that interest us, they are very natural.

Proposition 9. *For any $\mu \in \mathcal{P}(\mathbb{R}^d)$,*

$$F_\epsilon(\mu) = \sup_{\phi \text{ } c\text{-concave}} \int_{\mathbb{R}^d} \phi d\mu - F^*(-\phi^c)$$

with the definition of the Legendre transform F^ of F and the c -transform ϕ^c of ϕ given in section 2. This supremum is attained in ϕ if and only if for any ρ optimal for the primal problem,*

- * (ϕ, ϕ^c) is a pair of c -concave Kantorovich potentials for the optimal transport from μ to ρ .
- * $(-\phi^c) \in \partial F(\rho)$ (or, equivalently, $\rho \in \partial F^*(-\phi^c)$).

Assume Furthermore that F has non-empty subgradient at two measures ρ^- and ρ^+ such that $\rho^-(\Omega) < 1$ and $\rho^+(\Omega) > 1$, and that μ is supported on a compact set. Then, this supremum is indeed attained

Proof. Take

$$G : \rho \in \mathcal{M}(\Omega) \mapsto \begin{cases} I_c(\rho, \mu) & \text{if } \rho \in \mathcal{P}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

For $\rho \in \mathcal{M}(\Omega)$, $G(\rho) = \sup_{\psi \in \mathcal{C}^0(\Omega)} \int_\Omega \psi d\rho + \int_{\mathbb{R}^d} \psi^c d\mu = \bar{G}^*(\rho)$, where this last functional $\bar{G} : \psi \in \mathcal{C}^0(\Omega) \mapsto -\int_\Omega \psi^c d\mu$ is a convex continuous function. Therefore, $G^* = \bar{G}$.

Now, we have G^* a convex continuous function, and F^* which is convex l.s.c and not $+\infty$ everywhere. Applying Fenchel-Rockafellar duality theorem to the following infimum problem:

$$\inf_{\phi \in \mathcal{C}^0(\Omega)} G^*(\phi) + F^*(-\phi),$$

we can write:

$$\begin{aligned} \inf_{\phi \in \mathcal{C}^0(\Omega)} G^*(\phi) + F^*(-\phi) &= \max_{\rho \in \mathcal{M}(\Omega)} -G(\rho) - F(\rho) \\ &= - \min_{\rho \in \mathcal{M}(\Omega)} G(\rho) + F(\rho) \\ &= - \min_{\rho \in \mathcal{P}(\Omega)} \frac{W_2^2(\rho, \mu)}{2\epsilon} + F(\rho) \end{aligned}$$

But, this inf problem also rewrites:

$$\begin{aligned} \inf_{\phi \in \mathcal{C}^0(\Omega)} G^*(\phi) + F^*(-\phi) &= - \sup_{\phi \in \mathcal{C}^0(\Omega)} \int_{\Omega} \phi^c(x) d\mu(x) - F^*(-\phi) \\ &= - \sup_{\phi \text{ c-concave}} \int_{\Omega} \phi^c(x) d\mu(x) - F^*(-\phi) \end{aligned}$$

where the last equality is a consequence of $F^*(-\phi) \geq F^*(-\phi^{cc})$ for all $\phi \in \mathcal{C}^0(\Omega)$, true since ∂F^* is composed of positive measures only, by hypothesis. We can therefore take our supremum only on the ϕ that are c-concave.

Finally, up to a change of variable $\phi \mapsto \phi^c$ we obtain the primal and dual problems that we claimed:

$$\min_{\rho \in \mathcal{P}(\Omega)} \frac{W_2^2(\rho, \mu)}{2\epsilon} + F(\rho) = \sup_{\phi \text{ c-concave}} \int_{\Omega} \phi(x) d\mu(x) - F^*(-\phi^c)$$

Optimality conditions for both problems are straightforward. Indeed, for every ϕ and ρ , both admissible for their respective problem, we have

$$\int_{\Omega} \phi d\mu + \int_{\Omega} \phi^c d\rho \leq \frac{W_2^2(\mu, \rho)}{2\epsilon}$$

with equality if and only if (ϕ, ϕ^c) are Kantorovich potentials for the transport from μ to ρ , and,

$$F^*(-\phi^c) + F(\rho) \geq - \int_{\Omega} \phi^c(x) d\rho(x)$$

with equality iff $\rho \in \partial F^*(-\phi^c)$. Summing up these inequalities, and canceling the opposite terms, we get exactly

$$F(\rho) + \frac{W_2^2(\rho; \mu)}{2\epsilon} \geq \int_{\Omega} \phi(x) d\mu(x) - F^*(-\phi^c)$$

with equality if and only if (ϕ, ϕ^c) are Kantorovich potentials for the transport from μ to ρ and, $\rho \in \partial F^*(-\phi^c)$.

Now, for the existence part of the proposition, let K be the compact support of μ . Following the standard method in calculus of variations, consider a maximizing sequence of c-concave functions for the dual problem, $(\phi_n)_n$. These functions all have the same lipschitz constant as c , on the compact set K . Let L be this common Lipschitz constant, and for $n \in \mathbb{N}$ $M_n = \max_K \phi_n$, so that

for any $y \in K$, $M_n - L.\text{diam}(K) \leq \phi_n(y) \leq M_n$. We also set $C = \max_{x \in K, y \in \Omega} c(x, y)$. This gives us the bounds, for any $y \in \Omega$ and $n \in \mathbb{N}$,

$$-M_n \leq \phi_n^c(y) \leq C - M_n - L.\text{diam}(K) = A - M_n,$$

the constant A depending only on c , Ω and μ .

Now, assume M_n diverges towards $+\infty$ as $n \rightarrow \infty$ (this is equivalent to $(\phi_n)_n$ not uniformly bounded from above on K). Since there exists ϕ^+ such that $\phi^+ \in \partial F(\rho^+)$ with ρ^+ of mass strictly more than 1, or equivalently, $\rho^+ \in \partial F^*(\phi^+)$, we can write:

$$\begin{aligned} \int_{\Omega} \phi_n(x) d\mu(x) - F^*(-\phi_n^c) &\leq M_n - F^*(\phi^+) - \int_{\Omega} (-\phi^c - \phi^+) d\rho^+ \\ &\leq M_n(1 - \rho^+(\Omega)) + A\rho^+(\Omega) - F^*(\phi^+) + \int_{\Omega} \phi^+ d\rho^+ \end{aligned}$$

and that last part diverges to $-\infty$ as $n \rightarrow \infty$, which is absurd since $(\phi_n)_n$ is a maximizing sequence. Similarly, if M_n diverges towards $-\infty$ as $N \rightarrow \infty$, the fact that F has a non-empty subgradient at a measure of mass strictly less than 1 gives us again that ϕ_n cannot be a maximizing sequence.

Therefore, $(\phi_n)_n$ is uniformly bounded and, using Arzela-Ascoli theorem, we can extract from it a subsequence that converges uniformly on K , as $n \rightarrow \infty$. By upper semi-continuity of the functional in the dual problem, this limit is a maximizer. \square

Remark. The hypotheses on F are very natural ones considering our congestion terms have the integral form 7 in our numerical simulations. However, they are not the sharpest ones to obtain existence as one can see in this simple example: On a domain with area 1, for F of the form $F(\rho) = \int_{\Omega} f(\rho(x)) dx$ and $f = \chi_{\{1\}}$ (only a density equal to 1 almost everywhere is allowed), one can check that the dual problem admits solutions since $F_{\epsilon} = \frac{W_2^2(\cdot, dx)}{2\epsilon}$, which are the classical Kantorovich potentials for the corresponding transport. However, F itself does not satisfy the assumptions in proposition 9, since it is only finite at the Lebesgue measure.

Allowing F to be very general can allow the use of congestions such as $F(\rho) = \int_{\Omega} \frac{\|\nabla \rho\|^2}{\rho}$, which can appear when viewing second-order mean field games with entropy penalization as first order mean field games (see for instance [8] or [6]). However, in this paper, we will concentrate on the cases where F has the following form:

$$F(\rho) = \begin{cases} \int_{\Omega} f(\rho(x)) dx & \text{if } \rho \ll dx \\ +\infty & \text{otherwise.} \end{cases} \quad (7)$$

For F to be l.s.c, convex, we have to take a function f l.s.c, convex, and superlinear (on \mathbb{R}). Finally, our hypothesis on the subgradients of F is equivalent to assuming that $\frac{1}{|\Omega|}$ is in the interior of $\text{Dom}(f)$ as the following proposition justifies:

Proposition 10. *Let F be defined as above, with f a convex, l.s.c, superlinear function. Let $\phi \in C^0(\Omega)$, and $\rho \in \mathcal{M}(\Omega)$, then $\phi \in \partial F(\rho)$ if and only if $\rho \ll dx$ and for a.e $x \in \Omega$, $\phi(x) \in \partial f(\rho(x))$.*

Proof. Let ρ, ϕ be as in the proposition. Then by definition of the Legendre transform, $\phi \in (\partial F)(\rho)$ if and only if $\rho \ll dx$ and $\int_{\Omega} f(\rho(x)) dx + \int_{\Omega} f^*(\phi(x)) dx = \int_{\Omega} \phi(x) \rho(x) dx$.

But then, for a.e. $x \in \Omega$, $f(\rho(x)) + f^*(\phi(x)) \geq \phi(x) \rho(x)$ and so this is an equality almost everywhere, which exactly says that $\phi(x) \in (\partial f)(\rho(x))$. The converse implication is straightforward. \square

In the cases of interest to us in Parts 4 and 5, μ is a discrete measure of \mathbb{R}^d of the form $e_t \# Q$ for some $Q \in \mathcal{P}_N(\Gamma)$. A similar system of optimality conditions as the one in semi-discrete optimal transport can be found for the dual formulation of F_ϵ , when it is computed at a discrete measure. In such cases, F_ϵ can be seen as a function of the locations of the Dirac masses in $(\mathbb{R}^d)^N$. We claim the following expression for its dual formulation, as a finite dimensional concave maximization problem:

Proposition 11. *Assume that $\frac{1}{|\Omega|} \in \text{int}(\text{Dom}(f))$.*

Then, for $y \in (\mathbb{R}^d)^N$, we have

$$F_\epsilon(y) := F_\epsilon \left(\frac{1}{N} \sum_{i=1}^N \delta_{y_i} \right) = \max_{\phi \in \mathbb{R}^N} \sum_{i=1}^N \left[\frac{\phi_i}{N} - \int_{\text{Lag}_i(y, \phi)} f^* \left(\phi_i - \frac{\|x - y_i\|^2}{2\epsilon} \right) dx \right]$$

A pair $(\rho, \phi) \in \mathcal{P}(\Omega) \times \mathbb{R}^N$ is optimal for the primal/dual problems defining $F_\epsilon(y)$ if and only if the following conditions hold: For every $i = 1, \dots, N$

- * $\int_{\text{Lag}_i(y, \phi)} (f^*)'(\phi_i - c(x, y_i)) dx = \frac{1}{N}$ (**area**)
- * $\rho(x) = (f^*)'(\phi_i - c(x, y_i))$, for a.e. $x \in \text{Lag}_i(y, \phi)$ (**density**)

In particular, the optimal ρ is unique..

Remark. Uniqueness of ρ depends very much on the regularity of ϕ^c on Ω . As such regularity cannot be demanded of ρ , ϕ could not be unique.

Proof. One clearly has for any $y \in (\mathbb{R}^d)^N$,

$$F_\epsilon(y) \leq \sup_{\phi \in \mathbb{R}^N} \sum_{i=1}^N \left[\frac{\phi_i}{N} - \int_{\text{Lag}_i(y, \phi)} f^*(\phi_i - c(x, y_i)) dx \right].$$

For the other inequality, similar arguments as for proposition 9 guarantee that this last supremum is a maximum, attained at an optimal $\phi \in \mathbb{R}^N$ if and only if it satisfies (**area**). This implies that every Laguerre cell $\text{Lag}_i(y, \phi)$ is not Lebesgue-negligible, and in particular, not empty. Taking $\psi = \phi^c := \min_i c(\cdot, y_i) - \phi_i$ it is immediate that $\psi \in \mathcal{C}_b^0(\Omega)$ and $\psi^c(y_i) = \phi_i$ for every i . But then, (**area**) and (**density**) exactly express that ψ^c is optimal for the dual formulation of $F_\epsilon \left(\frac{1}{N} \sum_{i=1}^N \delta_{y_i} \right)$.

Now taking ρ optimal for the primal problem, all that remains is rewriting the optimality condition $\rho(x) \in \partial f^*(-\phi^c(x))$ a.e. on Ω , into the equality almost everywhere, (**density**): On one hand, the level-sets of ϕ^c are Lebesgue negligible (because μ is discrete) and on the other one, f^* is convex, continuous on \mathbb{R} because f is superlinear and therefore, differentiable on \mathbb{R} except at most at a countable number of points. Therefore, $\partial f^*(-\phi^c(x))$ is almost everywhere a singleton, and we have for a.e. $x \in \Omega$, $\rho(x) = (f^*)'(-\phi^c(x))$ with the appropriate expression on each cell. Note also that, since this equality has to be true for every ρ and ϕ optimal, ρ has to be unique. \square

Let us conclude this section by computing the gradient for the maximized functional in the dual problem, that is $F_{\epsilon, \mu} : \phi \in \mathbb{R}^N \mapsto \sum_{i=1}^N \left[\frac{\phi_i}{N} - \int_{\text{Lag}_i(y, \phi)} f^* \left(\phi_i - \frac{\|x - y_i\|^2}{2\epsilon} \right) dx \right]$ with respect to the weights, as well as the gradient of F_ϵ with respect to the Diracs' positions:

Proposition 12. $F_{\epsilon,\mu}$ is \mathcal{C}^1 , concave, and for $i = 1..N$, and $\phi \in \mathbb{R}^N$, we have

$$\frac{\partial}{\partial \phi_i} F_{\epsilon,\mu}(\phi) = \frac{1}{N} - \int_{\text{Lag}_i(y,\phi)} (f^*)' \left(\phi_i - \frac{\|x - y_i\|^2}{2\epsilon_N} \right) dx$$

Proof. Take, $\phi, \psi \in \mathbb{R}^N$:

$$\begin{aligned} F_{\epsilon,\mu}(\phi) - F_{\epsilon,\mu}(\psi) &= \sum_{i=1}^N \left[\frac{\phi_i - \psi_i}{N} - \left(\int_{\text{Lag}_i(y,\phi)} f^* \left(\phi_i - \frac{\|x - y_i\|^2}{2\epsilon_N} \right) dx \right. \right. \\ &\quad \left. \left. - \int_{\text{Lag}_i(y,\psi)} f^* \left(\psi_i - \frac{\|x - y_i\|^2}{2\epsilon_N} \right) dx \right) \right] \\ &\leq \sum_{i=1}^N \left[\frac{\phi_i - \psi_i}{N} - \left(\int_{\text{Lag}_i(y,\psi)} f^* \left(\phi_i - \frac{\|x - y_i\|^2}{2\epsilon_N} \right) \right. \right. \\ &\quad \left. \left. - f^* \left(\psi_i - \frac{\|x - y_i\|^2}{2\epsilon_N} \right) dx \right) \right] \end{aligned}$$

$$\text{and } F_{\epsilon,\mu}(\phi) - F_{\epsilon,\mu}(\psi) \leq \sum_{i=1}^N \left[\frac{\phi_i - \psi_i}{N} - \int_{\text{Lag}_i(y,\psi)} (f^*)' \left(\psi_i - \frac{\|x - y_i\|^2}{2\epsilon_N} \right) (\phi_i - \psi_i) dx \right]$$

The first inequality comes from the definition of $\text{Lag}_i(y, \phi)$ and the increasing nature of f^* and the second one, from the convexity of f^* and the fact that $-\psi^c$ has Lebesgue-negligible level sets.

Ergo, $F_{\epsilon,\mu}$ is concave and $\left(\frac{1}{N} - \int_{\text{Lag}_i(y,\psi)} (f^*)' \left(\psi_i - \frac{\|x - y_i\|^2}{2\epsilon_N} \right) dx \right)_{i=1..N} \in \partial F_{\epsilon,\mu}(\psi)$ for every ψ (in this particular case, "∂" denotes the supergradient of the concave function $F_{\epsilon,\mu}$ defined similarly to the subgradient of a convex one). $(f^*)'$ is continuous except on an at most countable set of reals, and the level set of $x \mapsto \phi_i - \frac{\|x - y_i\|^2}{2\epsilon}$ are negligible, as are the boundaries of the Laguerre cells. By dominated convergence, we have a continuous choice of supergradients for $F_{\epsilon,\mu}$ and so, $F_{\epsilon,\mu}$ is \mathcal{C}^1 with the partial derivatives that we claimed. \square

Remark. $F_{\epsilon,\mu}$ being a concave function, this proposition is another way to show the optimality condition $\int_{\text{Lag}_i(y,\phi)} (f^*)' \left(\phi_i - \frac{\|x - y_i\|^2}{2\epsilon_N} \right) dx = \frac{1}{N}$ for every i , that we claimed earlier. Note also, although this is beyond the scopes of this paper, that this expression is differentiable one more time (in ϕ), using the co-area formula. The resulting Hessian is invertible when the Laguerre cells associated to the weights do not have zero mass. This encourages the use of Newton-type algorithm similarly to what is done in semi-discrete optimal transport to find the optimal weights.

These expressions are very reminiscent of those obtained by Bourne, Wirth and Schmitzer in [4] for unbalanced semi-discrete optimal transport. Here, our congestion penalization plays the role of the mass discrepancy penalization between our discrete measure μ and the Lebesgue measure on Ω , which does not have mass 1 under the assumptions of proposition 11. This suggests that one could rewrite F_ϵ as an unbalanced transport problem between these measures.

Now, in order to numerically solve $(\mathcal{M}_{N,\mu_N^0,\delta_N,\epsilon_N})$ in part 8, we regard J_{δ_N,ϵ_N} as a function of the positions of the Dirac masses at various time steps. Its gradient is then given by the following result.

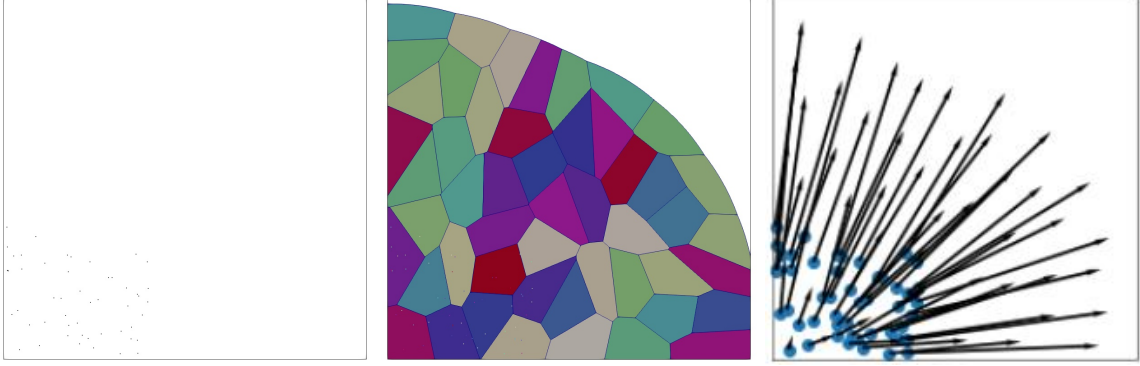


Figure 1: From left to right, (1) 50 points scattered in the bottom left corner of a 5×5 square, (2) the "charged" Laguerre cells obtained by intersecting the Laguerre cells with the support of the optimal ρ defining F_ϵ and (3) the vectors joining each point to the barycenter of its Laguerre cell. $\partial_{y_i} F_\epsilon$ is collinear, opposite, to the corresponding vector $b_{y,i} - y_i$, for $i=1\dots N$.

Proposition 13 (Gradient of F_ϵ). *For $y = (y_1, \dots, y_N) \in (\mathbb{R}^d)^N$, let ρ_y, ϕ_y be optimal for the primal problem defining $F_\epsilon(y)$, and for $i = 1..N$, $\rho_{y,i}$ such that $\sum \frac{1}{N} \delta_{y_i} \otimes \rho_{y,i}$ is the optimal transport plan between μ_y and ρ_y . Finally, we call for $i = 1, \dots, N$, $b_{y,i}$ the barycenter of the i th Laguerre cell, according to the probability $\rho_{y,i}$:*

$$b_{y,i} = \int_{\Omega} x d\rho_{y,i}(x) = N \int_{\text{Lag}_i(y, \phi_y)} x d\rho_y(x) \quad (8)$$

Then, F_ϵ is C^1 as a function of y , on $D_N = (\mathbb{R}^d)^N \setminus \{y | \exists i \neq j, y_i = y_j\}$ and for every $y \in D_N$,

$$\partial_{y_i} F_\epsilon(y) = \frac{y_i - b_{y,i}}{N\epsilon}$$

Proof. We show that $\Psi : y \mapsto F_\epsilon(y) - \frac{\sum_{i=1}^N \mu_i \|y_i\|^2}{2\epsilon}$ is concave on $(\mathbb{R}^d)^N$ with the vector $-\frac{1}{\epsilon}(\mu_i b_{y,i})_i$ in its supergradient at y . Then showing that this is a selection of subgradients continuous on D_N will, as before guarantee that Ψ and therefore F_ϵ is C^1 , on this set. The supergradient is still valid in configurations when $y_i = y_j$ occurs, but continuity, and more generally, the fact that the supergradient is a singleton, does not hold at these points.

First, take $x, y \in (\mathbb{R}^d)^N$,

$$\begin{aligned} \Psi(y) &\leq \sum_{i=1}^N \int_{\text{Lag}_i(x, \phi_x)} \frac{1}{2\epsilon} (\|z\|^2 - 2z \cdot y_i + \|y_i\|^2) \rho_x(z) + f(\rho_x(z)) dz - \frac{1}{2\epsilon} \sum_{i=1}^N \mu_i \|y_i\|^2 \\ &\leq \Psi(x) - \sum_{i=1}^N \frac{1}{\epsilon} \int_{\text{Lag}_i(x, \phi_x)} z d\rho_x(z) \cdot (y_i - x_i) \\ &\leq \Psi(x) - \sum_{i=1}^N \frac{b_{x,i}}{N\epsilon} \cdot (y_i - x_i) \end{aligned}$$

which exactly means that Ψ is concave and that $(-\frac{1}{N\epsilon} b_{x,i})_i \in \partial\Psi(x)$ for every $x \in (\mathbb{R}^d)^N$.

Let us now show that $y \in D_N \mapsto (b_{y,i})_i$ is continuous: First, in order to make some parts of the proof more concise, we will write for $y \in (\mathbb{R}^d)^N$, $\mu_y = \frac{1}{N} \sum_{i=1}^N \delta_{y_i}$ and for $\phi \in \mathbb{R}^N$, ϕ^c the continuous bounded function on Ω equal to $c(\cdot, y_i) - \phi_i$ on the Laguerre cell $\text{Lag}_i(y, \phi)$.

Take $(y^n)_n$ a sequence converging to y in D_N . Let for all n , ρ^n , ϕ^n be optimal for the primal and dual problems defining $F_\epsilon(y^n)$ respectively (ρ^n is uniquely defined for all n). The functions $(\phi^n)^c$ are c -concave Kantorovich potential in the transport from ρ^n to δ_{y^n} , and almost everywhere on Ω , we have $\rho^n(x) = (f^*)'(-(\phi^n)^c(x))$.

By similar arguments as for Proposition 9, up to a subsequence, $(\phi^n)_n$ converges towards a $\phi \in \mathbb{R}^N$ (and so, $(\phi^n)^c$ uniformly converges on Ω towards ϕ^c). By the optimality conditions of Proposition 11, $\rho^n = (f^*)'(-(\phi^n)^c)$ converges almost everywhere (up to a subsequence) towards $\rho = (f^*)'(-\phi^c)$. And since ρ^n is bounded in $\mathbb{L}^\infty(\Omega)$, the convergence is also a weak-* convergence in $\mathbb{L}^\infty(\Omega)$. In particular, ϕ^c is a c -concave Kantorovich potential for the optimal transport from ρ to μ_y , since μ_{y^n} narrowly converges towards μ_y and ρ^n towards ρ in their respective spaces. But then $\rho = \rho_y$ is the unique minimizer for the primal problem. In particular, $b_{y,i} = N \int_{\text{Lag}_i(y, \phi)} x \rho(x) dx$.

Finally, for all i and n ,

$$\begin{aligned} \|b_{y^n, i} - b_{y, i}\| &= N \left\| \int_{\text{Lag}_i(y^n, \phi^n)} x \rho^n(x) dx - \int_{\text{Lag}_i(y, \phi)} x \rho(x) dx \right\| \\ &\lesssim \left\| \int_{\text{Lag}_i(y^n, \phi^n)} x \rho^n(x) dx - \int_{\text{Lag}_i(y, \phi)} x \rho^n(x) dx \right\| \\ &\quad + \left\| \int_{\text{Lag}_i(y, \phi)} x \rho^n(x) dx - \int_{\text{Lag}_i(y, \phi)} x \rho(x) dx \right\| \end{aligned}$$

Now, let T^n (resp T) be the Brenier map for the optimal transport from ρ^n (resp ρ_y) to μ_{y^n} (resp μ_y), with cost c . For n large enough,

$$\begin{aligned} \left\| \int_{\text{Lag}_i(y^n, \phi^n)} x \rho^n(x) dx - \int_{\text{Lag}_i(y, \phi)} x \rho^n(x) dx \right\| \\ \lesssim \sum_{i \neq j} \rho^n \left((T^n = y_i^n \cap T = y_j) \cup (T^n = y_j^n \cap T = y_i) \right) \end{aligned}$$

But since y_j^n converges to y_j for every j , this last term is bounded by $\rho^n(\|T^n - T\| \geq \omega_i)$ with $\omega_i = \frac{1}{2} \inf_j \|y_i - y_j\| (> 0)$ and we can write

$$\|b_{y^n, i} - b_{y, i}\| \lesssim \rho^n(\|T^n - T\| \geq \omega_i) + o_{n \rightarrow \infty}(1)$$

In a similar fashion as [21], Corollary 5.21, we show that $\lim_{N \rightarrow \infty} \rho^n(\|T^n - T\| \geq \omega_i) = 0$ and we immediately have the continuity of $y \mapsto b_y$ on D_N , which concludes our proof:

Take $\eta > 0$. Lusin theorem allows us to find a (compact) set A such that $\rho_y(\Omega \setminus A) \leq \eta$ and T coincides with a continuous function \tilde{T} on A . Then also $\rho^n(\Omega \setminus A) \leq 2\eta$ for n large enough, by the weak-* convergence we established, and,

$$\begin{aligned} \rho^n(\|T^n - \tilde{T}\| < \omega_i) &\leq \rho^n(\|T^n - T\| < \omega_i) \cap A + \rho^n(\Omega \setminus A) \\ &\leq \rho^n(\|T^n - T\| < \omega_i) + 2\eta \end{aligned}$$

Usual stability theorems, with the fact that ρ^n and ρ are absolutely continuous with respect to Lebesgue state that the optimal transport plan π^n from ρ^n to μ_{y_n} narrowly converges to the optimal transport plan π from ρ_y to μ_y .

Consider then the (open) set $\tilde{A} = \left\{x, y \mid \left\|y - \tilde{T}(x)\right\| < \omega_i\right\}$. One has

$$\begin{aligned} 1 - \eta &\leq \rho(\left\|\tilde{T} - T\right\| < \omega_i) = \pi(\tilde{A}) \\ &\leq \liminf \pi^n(\tilde{A}) \\ &\leq \liminf \rho^n(\left\|T^n - \tilde{T}\right\| < \omega_i) \\ &\leq \liminf \rho^n(\|T^n - T\| < \omega_i) + 2\eta \end{aligned}$$

and, this is true for any $\eta > 0$, hence, $\lim \rho_n(\|T^n - T\| < \omega_i) = 1$ and we have the limit we wanted. \square

7 Strong convergence in strictly convex cases

Propositions 4 and 7 allow us to express a stronger convergence, for the Moreau projections

$$\rho_N : t \in [0; T] \mapsto \operatorname{argmin}_\rho \frac{W_2^2(\rho, e_t \# Q_N)}{2\epsilon_N} + \int_\Omega f(\rho(x)) dx$$

associated with a sequence $(Q_N)_N$ of minimizers for problem $(\mathcal{M}_{N, \mu_N^0, \epsilon_N})$, or its fully discrete version.

We will prove this convergence in the case discrete in space, continuous in time:

Let us assume that problem (\mathcal{M}_{μ^0}) has a unique solution, Q_{\min} and that for all $N \in \mathbb{N}$, Q_N is a minimizer for problem $(\mathcal{M}_{N, \mu_N^0, \epsilon_N})$. We take the appropriate values for the parameters ϵ_N and μ_N^0 such that (up to a subsequence), Q_N narrowly converges to Q_{\min} and $J_{\epsilon_N}(Q_N)$ converges to $J(Q_{\min})$. Now,

Lemma 3. *With these notations,*

$$\lim_{N \rightarrow \infty} \int_0^T \int_\Omega f(\rho_N(t)(x)) dx = \int_0^T \int_\Omega f(e_t \# Q_{\min}(x)) dx$$

Proof. This is a direct consequence of proposition 4. Indeed from lower, semi-continuity of F , we already have for almost any $t \in [0; T]$,

$$\int_{[0; T] \times \Omega} f(e_t \# Q_{\min}(x)) dx dt \leq \liminf_{N \rightarrow \infty} \int_{[0; T] \times \Omega} f(\rho_N(t)(x)) dx dt$$

But, on the other hand,

$$\begin{aligned} \int_{[0; T] \times \Omega} f(\rho_N(t)(x)) dx dt &\leq \int_{[0; T] \times \Gamma} Ld(Q_{\min} - Q_N) + \int_{[0; T] \times \Omega} f(e_t \# Q_{\min}(x)) dx dt \\ &\quad + G(Q_{\min}) - G(Q_N) + o_{N \rightarrow \infty}(1) \end{aligned}$$

and taking the limsup as $N \rightarrow \infty$, we obtain

$$\limsup_{N \rightarrow \infty} \int_{[0; T] \times \Omega} f(\rho_N(t)(x)) dx dt \leq \int_{[0; T] \times \Omega} f(e_t \# Q_{\min}(x)) dx dt$$

and our lemma. \square

Strong convergence immediately follows in two cases:

Proposition 14. *Under the assumptions on ϵ_N , μ_N^0 , Q_N and Q_{\min} listed above,*

- *If f is strongly convex, then ρ_N strongly converges in $\mathbb{L}^2([0; T] \times \Omega)$ (as a function of t and x), towards $\rho_{\min} : (t, x) \in [0; T] \times \Omega \mapsto e_t \# Q_{\min}(x)$.*
- *If $f : \rho \mapsto |\rho|^m$, $m \geq 2$ is a power, then a similar strong convergence is true, this time in $\mathbb{L}^m([0; T] \times \Omega)$.*

Proof. This is immediate from lemma 3.

The power case is the simplest one. Indeed, in this case, our lemma guarantees that $\|\rho_N\|_{\mathbb{L}^m}$ converges to $\|\rho_{\min}\|_{\mathbb{L}^m}$. Since ρ_N narrowly converges towards ρ_{\min} already, using a simple argument of approximation by continuous functions, we have the weak convergence in duality with $\mathbb{L}^{m'}$ where $m' = \frac{m}{m-1}$. But, from the convergence of the norms, we can conclude the strong convergence in \mathbb{L}^m using the Radon-Riesz property.

If f is strongly convex, denoting by r the minimum point of f on \mathbb{R} , there exists $C > 0$ such that for $N \in \mathbb{N}$:

$$\int_{[0; T] \times \Omega} f(\rho_N(t, x)) dx dt \geq T|\Omega|f(r) + \frac{C}{2} \int_{[0; T] \times \Omega} \|\rho_N(t, x) - r\|^2 dx dt.$$

and $(\rho_N)_N$ is bounded in \mathbb{L}^2 , therefore it weakly-* converges to ρ_{\min} in \mathbb{L}^2 . But one can also see F as a strongly convex function over $\mathbb{L}^2([0; T] \times \Omega)$ and there exists another constant $C > 0$ and a function $p \in \mathbb{L}^2([0; T] \times \Omega)$ such that for any $N \in \mathbb{N}$,

$$\begin{aligned} \int_{[0; T] \times \Omega} f(\rho_N(t, x)) dx dt &\geq \int_{[0; T] \times \Omega} f(\rho_{\min}(t, x)) dx dt + \int_{[0; T] \times \Omega} (\rho_N(t, x) - \rho_{\min}(t, x)) \cdot p(t, x) dx dt \\ &\quad + \frac{C}{2} \int_{[0; T] \times \Omega} \|\rho_N(t, x) - \rho_{\min}(t, x)\|^2 dx dt \end{aligned}$$

and taking the limit as $N \rightarrow \infty$, we obtain the strong convergence in \mathbb{L}^2 of ρ_N to ρ_{\min} . \square

Remark. We briefly note that, if there exists functions j and j^* on \mathbb{R} and a constant $C > 0$ such that for p and ρ in \mathbb{R} ,

$$f(\rho) + f^*(p) \geq p \cdot \rho + C|j(\rho) - j^*(p)|^2,$$

we get strong convergence of the functionals $j(\rho_N)$ towards $j(\rho_{\min})$ in \mathbb{L}^2 , provided some invertibility on j , which implies strict convexity for f . This is a common assumption to show regularity results on the solutions of a convex problem, using duality (see for instance [18]) and $j = j^* \equiv 0$ are always suitable for any convex function f . However, in the case of f strongly convex, $j = Id$ and $j^* = (f^*)'$ are suitable and we recover the first case of proposition 14. Similarly, for the case $f \equiv |\cdot|^m$, $j(\rho) = \rho \cdot |\rho|^{m/2-1}$ and $j^*(p) = p \cdot |p|^{m'/2-1}$ are suitable. Again, we recover the \mathbb{L}^m convergence claimed above.

8 Numerics

In the following numerical simulations, we restrained ourself to a simple lagrangian given by the squared norm, $L : x \mapsto \frac{1}{2} \|x\|^2$. The congestion term F and the potential energy G will be specific

to each case.

The values of F_ϵ at each time steps are computed using the pysdot module for python, which allows for fast computation of integrals over Laguerre cells. The functional J_{δ_N, ϵ_N} , seen as a function on the positions at each time is not a convex function and we used the scipy implementation of the L-BFGS algorithm (with the gradient of J_{δ_N, ϵ_N} computed using proposition 13) to find local minimizers.

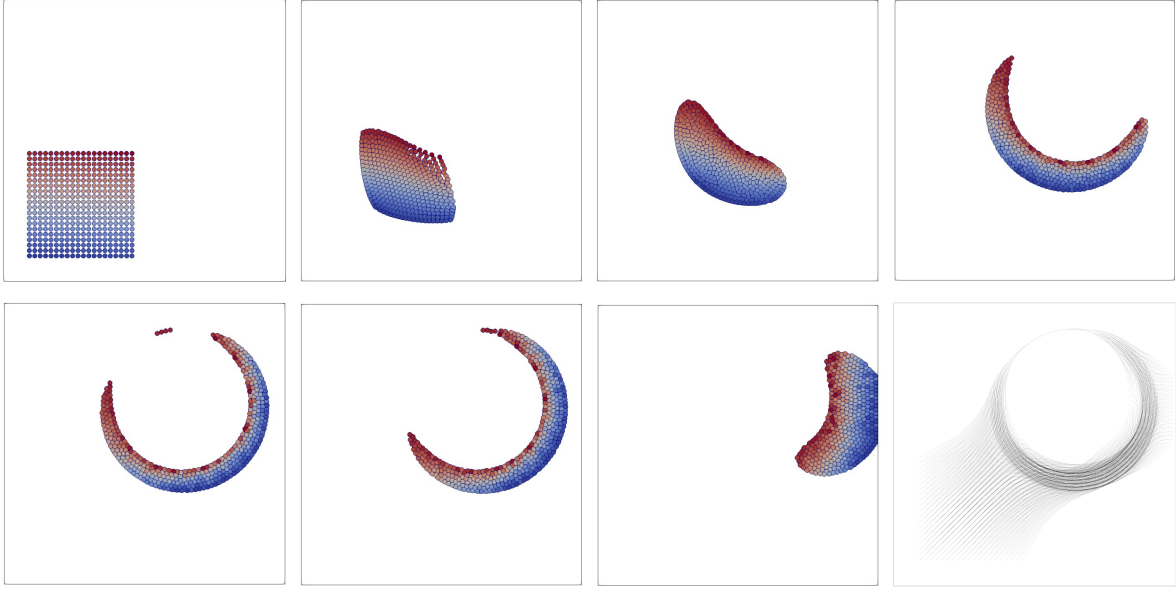


Figure 2: On the first 7 images (top left to bottom right), the evolution of the "charged" Laguerre cells (intersected with the support of $\rho_N(t)$) at several time steps for 400 particles, in the square. The final picture (bottom right) represents the full trajectories of the particles.

Evolution in a convex domain: For this case, we took the simple congestion penalization,

$$F : \rho \in \mathcal{M}(\Omega) \mapsto \int_{\Omega} \chi_{[0;1]}(\rho(x)) dx = \begin{cases} 0 & \text{if } 0 \leq \rho \leq dx \\ +\infty & \text{otherwise} \end{cases}$$

with $\chi_{[0;1]}$ being the convex indicator function of $[0; 1]$. Admissible population trajectories for the continuous problem cannot have a density higher than 1 at any time or position. The conclusions of part 6 apply in this case with $f = \chi_{[0;1]}$ and $f^* = \max(\cdot, 0)$ being the positive part function on \mathbb{R} . For $y \in (\mathbb{R}^d)^N$, the associated optimal density in $F_\epsilon(y)$ is given on $\text{Lag}_i(y, \phi)$ by:

$$\rho(x) = \left(\phi_i - \frac{\|x - y_i\|^2}{2\epsilon} \right)_+ = \begin{cases} 1 & \text{if } \|x - y_i\|^2 \leq 2\epsilon\phi_i \\ 0 & \text{otherwise} \end{cases}$$

and the charged Laguerre cells (intersected with the support of ρ) are the intersection of the actual Laguerre cells, with the respective balls $B(y_i, \sqrt{2\epsilon\phi_i})$. Although proposition 14 does not apply in this case, one can expect them to give a good idea of the support of the limit measure $e_t \# Q_{\min}$, and we have highlighted them on the pictures instead of the actual player's positions for this reason.

For this first experiment, we chose the convex domain $\Omega = [-1; 10]^2$ and a "potential energy"

$$G : Q \in \mathcal{P}(\Gamma) \mapsto \int_{\Gamma} \int_0^T V(\gamma(t)) dt + \Phi(\gamma(T)) dQ(\gamma)$$

with $V(x) = (\|x - (6, 6)\|^2 - 9)^2$ and $\Phi(x) = \|x - (11, 6)\|^2$. 400 players, each of mass 1/40 (for a total mass 10, in order to have visible charged laguerre cells) started aligned on a regular grid on the square $[0; 4] \times [0; 4]$. The potential term then drove them towards the circle $\mathcal{C}((6, 6), 3)$ during the trajectory and to the point $(10, 6)$ at the end. The trajectories on figure 2 were obtained for the values of the parameters $\epsilon = 0.01$, $\delta = 1/64$ and $T = 15$.

In this case, the hypothesis of a lipschitz velocity field for the continuous solution of our mean field game seems contradicted by the experiment, as all players do not pass around the same side of the circle \mathcal{C} . However this seems to be the only point of splitting for our optimal trajectories, which suggests that $spt(Q_{\min})$ should still be of dimension 2. In such a case, any ϵ_N dominating $\ln(N)/N$ should be suitable to obtain the convergence of Proposition 7.

Evolution in a non-convex domain: Here, we had to adapt our congestion term, as the Newton algorithm did not always converge for diracs too far away from Ω (which was the case for some particles not using the corridor). To make optimization easier we fix a small maximum density $m > 0$ for the area outside Ω but inside its convex envelope $\text{conv}(\Omega)$, and take the congestion penalization:

$$F : \rho \in \mathcal{M}(\text{conv}(\Omega)) \mapsto \int_{\text{conv}(\Omega)} f(x, \rho(x)) dx$$

$$\text{where } f : (x, \rho) \in \text{conv}(\Omega) \times \mathbb{R} \mapsto \begin{cases} 0 & \text{if } 0 \leq \rho \leq 1 \text{ and } x \in \Omega \\ 0 & \text{if } 0 \leq \rho \leq m \text{ and } x \in \text{conv}(\Omega) \setminus \Omega \\ +\infty & \text{otherwise} \end{cases}$$

Although this isn't quite the framework of Propositions 11 to 13, these can be easily adapted to this form of congestion. The support of the Moreau projection will still be an intersection of balls with the Laguerre cells, but the value of the optimal density ρ will not be 1 everywhere on this support. Instead, $\rho(x) = m$ a.e. on $\text{conv}(\Omega) \setminus \Omega$ giving us larger charged Laguerre cells for the points passing near the border (or outside) of Ω . For low values of the outside density m , only very few particles can fit outside the corridor and we should recover the strong penalization of the convex example.

In this case, our particles evolved on the domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ constituted of two rooms, $\Omega_1 = [0; 8]^2$ and $\Omega_3 = [11; 19] \times [0; 8]$ separated by a narrow corridor, $\Omega_2 = [8; 11] \times [3.5; 4.5]$.

The particles are driven by a potential energy featuring a fast marching distance on $\text{conv}(\Omega)$:

$$G : Q \in \mathcal{P}(\Gamma) \mapsto \int_{\Gamma} \Phi(\gamma(T)) dQ(\gamma)$$

with Φ being solution of the Eikonal equation on $\text{conv}(\Omega)$:

$$\begin{cases} \|\nabla \Phi(x)\| = 1 & \text{on } \Omega. \\ \|\nabla \Phi(x)\| = v & \text{outside.} \\ \Phi(18, 1) = \Phi(18, 7) = 0 \end{cases}$$

with v being a small value of the velocity, outside the corridor. Such a potential "guides", so to speak, the players to the closest point between $(18,1)$ and $(18,7)$, while encouraging them to move inside Ω . Notice that, unlike the one made for F , this prescription $\|\nabla V(x)\| = v$ outside Ω is dictated by the theory since our discrete trajectories could pass outside the corridor and we do not regularize G . However, even for our value $v = 0.1$, trajectories leaving Ω were, in the end, rejected by the optimization.

To obtain figure 3, we ran the optimization for 400 particles, each of mass $1/8$, starting on a regular grid over the first square Ω_1 . The trajectories on these images were obtained for $\epsilon = 0.1$, a time step of $\delta_N = 1/2^8$ and a maximum time $T = 600$. We chose to take a fairly strong congestion penalization outside the corridor, with $m = 10^{-3}$, putting a much weaker penalization on the speed via the Eikonal equation, with a value $v = 0.1$.

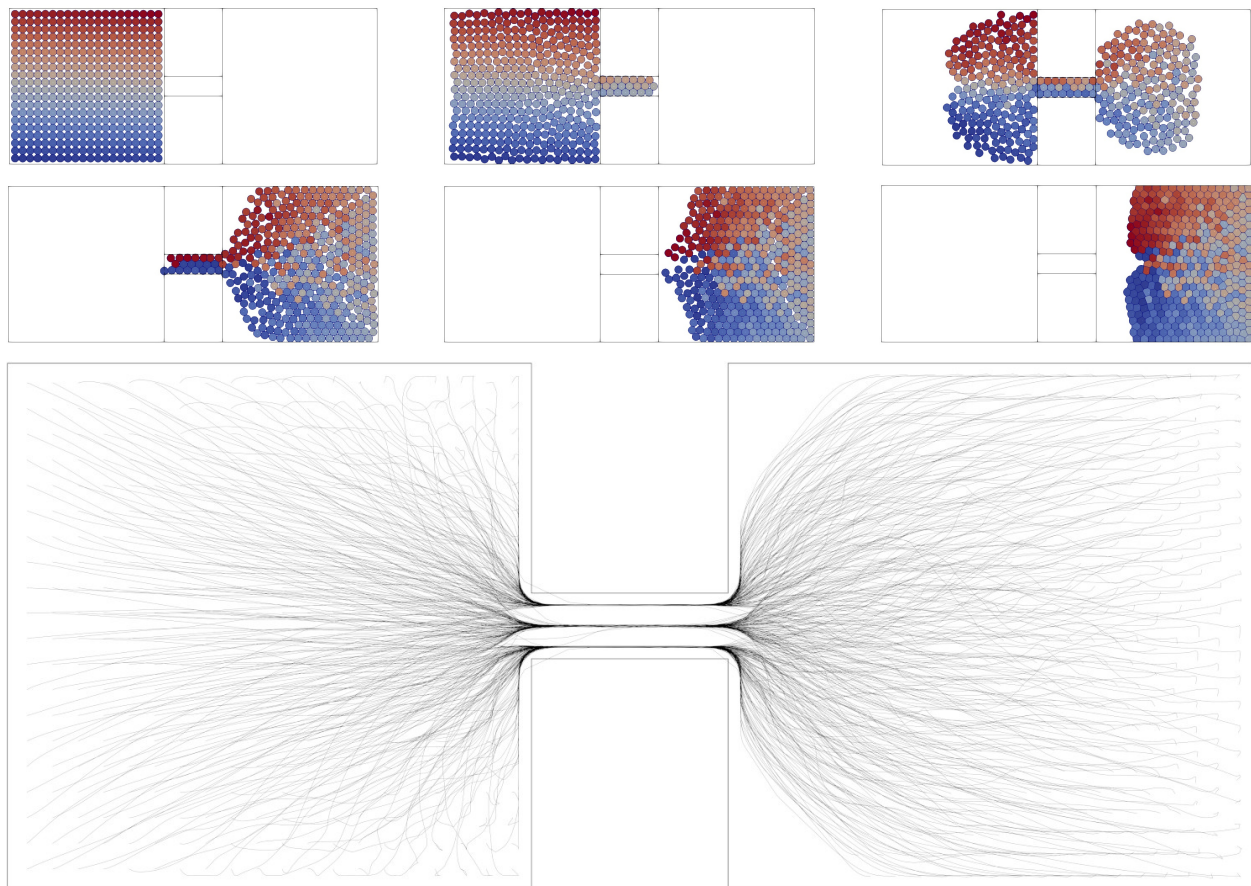


Figure 3: On the top six images are represented the positions and "charged" Laguerre cells of 400 particles moving in Ω . The bottom picture shows the trajectories of all the particles.

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