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DISPERSIVE ESTIMATES FOR THE SEMI-CLASSICAL SCHRÖDINGER EQUATION IN A STRICTLY CONVEX DOMAIN

OANA IVANOVICI

ABSTRACT. We consider a model case for a strictly convex domain $\Omega \subset \mathbb{R}^d$ of dimension $d \geq 2$ with smooth boundary $\partial\Omega \neq \emptyset$ and we describe dispersion for the semi-classical Schrödinger equation with Dirichlet boundary condition. More specifically, we obtain the optimal fixed time decay rate for the linear semi-classical flow : a loss of $(\frac{h}{t})^{1/4}$ occurs with respect to the boundary less case due to repeated swallowtail type singularities. The result is optimal and implies corresponding Strichartz estimates.

1. INTRODUCTION

Let us consider the Schrödinger equation on a manifold (Ω, g) , with a strictly convex boundary $\partial\Omega$ (a precise definition of strict convexity will be provided later):

$$(1) \quad i\partial_t v + \Delta_g v = 0, \quad v|_{t=0} = v_0, \quad v|_{\mathbb{R} \times \partial\Omega} = 0,$$

where Δ_g denotes the Laplace operator. The boundary condition will be assumed to be Dirichlet.

In all recent work on hyperbolic equations on manifolds, understanding the linear flow is a pre-requisite to studying nonlinear problems: when dealing with the Cauchy problem for nonlinear wave equation, one starts with perturbative techniques and faces the difficulty of controlling the size of solutions to the linear equation in term of the size of the initial data. For this (especially at low regularities), mixed norms of Strichartz type $(L_t^q L_x^r)$ are particularly useful. In the case of the Schrödinger flow $e^{it\Delta_g} v_0$ of (1), local Strichartz estimates (in their most general form) state that

$$(2) \quad \|e^{it\Delta_g} v_0\|_{L^q(0,T)L^r(\Omega)} \leq C_T \|v_0\|_{H^\sigma(\Omega)},$$

where $2 \leq q, r \leq \infty$ satisfy the Schrödinger admissibility condition, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, $(q, r, d) \neq (2, \infty, 2)$ and $\frac{2}{q} + \frac{d}{r} \geq \frac{d}{2} - \sigma$ (scale-invariant when equality; otherwise, *loss of derivatives* in the estimate (2) as it deviates from the optimal regularity predicted by scale invariance.) Such estimates are well known in the flat Euclidean space \mathbb{R}^d with $g = (\delta_{ij})$. In that case $\sigma = 0$ and $T = \infty$, so that (2) holds globally in time.

The canonical path leading to such Strichartz estimates is to obtain a stronger, fixed time, dispersion estimate, which is then combined with energy conservation, interpolation and a duality argument to obtain (2). The dispersive estimates for the classical Schrödinger

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flow read as:

$$(3) \quad \|e^{\pm it\Delta_{\mathbb{R}^d}}\|_{L^1(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)} \leq C(d)t^{-d/2}, \text{ for all } t \neq 0.$$

Indeed, (3) and the unitary of the propagator on $L^2(\mathbb{R}^d)$ are all that is required to obtain all known Strichartz estimates ; the endpoint cases are more delicate (see [21], [11], [35]).

On any boundary-less Riemann manifold (Ω, g) one may follow the same path, replacing the exact formula by a parametrix which may be constructed locally within a small ball, thanks to finite speed of propagation for waves or in semi-classical time for Schrödinger - where one deals with very short time, wavelength sized intervals (e.g. the size is the inverse of the frequency), allowing a reduction to almost finite speed of propagation. In semi-classical time, the dispersive estimates for the Schrödinger equation in the Euclidian space read as follows

$$(4) \quad \sup \left| \psi(hD_t)e^{\pm it\Delta_{\mathbb{R}^d}} \right| \lesssim \frac{1}{h^d} \min(1, (\frac{h}{t})^{\frac{d}{2}}) \text{ for all } 0 < |t| \lesssim 1.$$

When $\Omega = \mathbb{R}^d$, dispersive properties of (1) are well understood and play a crucial role in understanding the nonlinear problems. Studying the same hyperbolic or dispersive equations on manifolds (curved geometry, e.g. variable metric) started in part with Bourgain's work on KdV and Schrödinger on the torus, and then expanded in several different directions, all of them with low regularity requirements (e.g. Staffilani-Tataru [32], Burq-Gérard-Tzvetkov [9], [8] for Schrödinger, Smith [26], [27], Tataru [33], Bahouri-Chemin [4], [3], Klainerman-Rodnianski [24] and Smith-Tataru [31], [30] for wave equations).

Even though the boundary-less case has been well understood for some time, our understanding of dispersion on domains is far from complete. Indeed, one may think safely that while the subject has seen considerable progress in recent years, we only have a very partial knowledge of dispersive properties of solutions to the wave equation on domains with boundary, while for Schrödinger on a bounded domain we do not have any satisfactory estimates to deal with important applications (as an example, the cubic nonlinear Schrödinger equation on even a 3D ball is not well-understood in the natural energy space). For compact manifolds (even without boundary) one cannot expect estimates of the same form as for the Euclidian space: eventually a loss will occur, due to the volume being finite. There can be no-long time dispersion of wave packets, since they simply have nowhere to disperse. Long time estimates for the wave equation are unknown, while in the case of the Schrödinger equation, the infinite speed of propagation immediately produces unavoidable losses of derivatives in dispersive estimates. Informally, this may be related to the existence of eigenfunctions, but the complete understanding of the loss mechanism is still a delicate issue. During the last decade, we have studied dispersive properties of the wave equation on domains with boundary. Our first result [18], which deals with the model case of a strictly convex domain (the Friedlander domain, which can be seen as a simplified model for the disk) highlights a loss in dispersion for the solution to the linear wave equation which we informally relate to the presence of caustics generated in arbitrarily small time near the boundary. Such caustics appear when optical rays are no longer diverging from each other in the normal direction, where less dispersion occurs as compared to the \mathbb{R}^d case.

The Friedlander's model domain is the half-space, for $d \geq 2$, $\Omega_d = \{(x, y) | x > 0, y \in \mathbb{R}^{d-1}\}$ with the metric g_F inherited from the following Laplace operator

$$(5) \quad \Delta_F = \partial_x^2 + \sum_j \partial_{y_j}^2 + x \sum_{j,k} q_{j,k} \partial_{y_j} \partial_{y_k},$$

where $q_{j,k}$ are constants and $q(\theta) = \sum_{j,k} q_{j,k} \theta_j \theta_k$ is a positive definite quadratic form. Notice that the strict convexity of the domain defined by Ω_d with the metric inherited from Δ_F is equivalent to the ellipticity of $\sum_{j,k} q_{j,k} \partial_{y_j} \partial_{y_k}$. When $q_{j,k} = \delta^{j,k}$ (i.e. when $q(\theta) = |\theta|^2$) the domain (Ω_d, g_F) is a first order approximation of the unit disk in polar coordinates $(r, \theta) : \text{set } r = 1 - \frac{x}{2}, \theta = y$. Let $h \in (0, 1]$ and let $0 < a \leq 1$ be small (comparable to a positive power of h) : if $u_a(t, x, y) = \cos(t\sqrt{|\Delta_F|})(\delta_{x=a, y=0})$ denotes the linear wave flow on $(\Omega, g) = (\Omega_d, g_F)$ with data $\delta_{x=a, y=0}$ and satisfying the Dirichlet boundary condition, then

$$(6) \quad \|\psi(hD_t)u_a(t, \cdot)\|_{L^\infty} \leq C(d)h^{-d} \min \left\{ 1, (h/t)^{\frac{d-2}{2}} \left(\left(\frac{h}{t}\right)^{1/2} + \left(\frac{h}{t}\right)^{1/3} + a^{1/4} \left(\frac{h}{t}\right)^{1/4} \right) \right\}.$$

Moreover, there exists a sequence $(t_n)_n$ for which equality holds in the estimates (6), making them sharp. This optimal loss of $\frac{1}{4}$ in the $\frac{h}{t}$ exponent is unavoidable for small a and is due to swallowtail type singularities in the wave front set of u_a . This preliminary steps opened several directions, from obtaining the generic convex case to understanding more complicated boundary shapes.

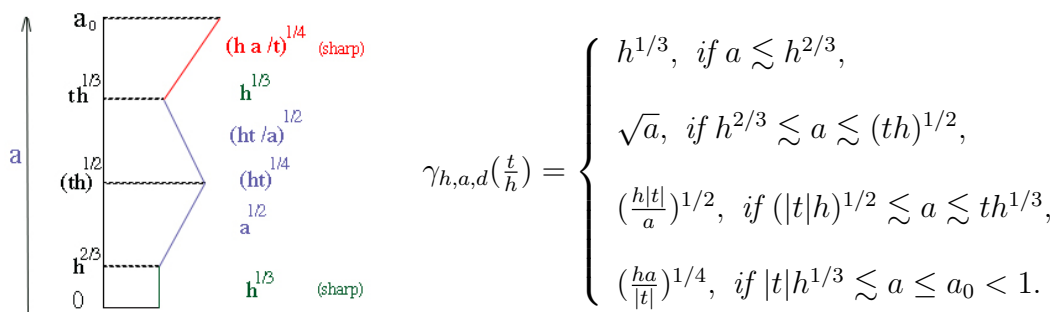
In the present work, we address the same set of issues for the Schrödinger equation, where parallel developments were expected, at least in the so called semiclassical setting (recall that is a shorthand for saying that one deals with time intervals of the size of the wavelength h , allowing a reduction to almost finite speed of propagation). It should be understood that obtaining sharp results for the *classical* Schrödinger equation - in the non-trapping case - is far from routine, the main issue being the infinite speed of propagation of the flow. In the case of a convex boundary, even the wavelength sized time behaviour is complicated due to the existence of gliding rays. Let $h \in (0, 1]$ and consider the semiclassical Schrödinger equation inside the Friedlander domain (Ω_d, g_F) , with Δ_F given in (5) and Dirichlet boundary condition

$$(7) \quad ih\partial_t v - h^2 \Delta_F v = 0, \quad v|_{t=0} = v_0, \quad v|_{\partial\Omega_d} = 0.$$

The main result of this paper is the following :

Theorem 1. *Let $\psi \in C_0^\infty([\frac{1}{2}, \frac{3}{2}])$, $0 \leq \psi \leq 1$. There exists $C(d) > 0$, $T_0 > 0$ and $a_0 < 1$ such that for all $a \in (0, a_0]$, $h \in (0, 1]$ and $|t| \in [-T_0, T_0]$, the solution $v(t, \cdot)$ to (7) with data $v_0(x, y) = \psi(hD_y)\delta_{x=a, y=0}$ satisfies the following estimates:*

$$\|\psi(hD_t)v(t, \cdot)\|_{L^\infty(\Omega_d)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{\frac{d-1}{2}} \min\{1, \gamma_{h,a,d}(\frac{t}{h})\}, \quad \text{where}$$



Moreover, for every $|t| \in (\sqrt{a}, T_0]$ and every $|t|h^{1/3} \ll a \lesssim 1$, these bounds are sharp as $\left(\frac{ha}{|t|}\right)^{1/4} \gg h^{1/3}$ and

$$(8) \quad \|\psi(hD_t)v(t, \cdot)\|_{L^\infty(\Omega_d)} \simeq \frac{1}{h^d} \left(\frac{h}{t}\right)^{\frac{(d-1)}{2}} \left(\frac{ha}{|t|}\right)^{1/4}.$$

Remark 1. Notice that when $a \lesssim |t|h^{1/3}$ we have $\gamma_{h,a,d}\left(\frac{t}{h}\right) \lesssim (h|t|)^{1/4}$, when $a \simeq |t|h^{1/3}$ we have $\gamma_{h,a,d}\left(\frac{t}{h}\right) \lesssim h^{1/3}$, while when $|t|h^{1/3} \ll a \leq a_0$, (8) yields a sharp loss of 1/4 powers of h/t compared to the Euclidian case (4).

We notice in particular two main differences with respect to the wave equation which imply important additional difficulties.

- When a is not too small, the Green function for the wave flow can be explicitly expressed as a sum of "time-almost-orthogonal" waves, which essentially "live" between a finite number of consecutive reflections; we are therefore reduced to obtaining good dispersion bounds for a *finite* sum of waves well localised in both time and tangential variable. Using a subordination formula yields a similar representation of the Schrödinger flow as a sum of wave packets; nonetheless, at a given time t , *all* the waves in this sum provide important contributions, which occurs because they travel with different speeds. To sum up all these contributions we need very refined bounds for each of them, similar to those obtained in [17] for waves.
- When a is very small, writing a parametrix as a sum over reflections doesn't help anymore since there are too many terms to deal with (the number of terms becomes unbounded even in the case of the wave equation, but the finite speed of propagation allows to handle the situation at least as long as $a \gg h^{2/3}$); using the spectral decomposition of the data in terms of eigenfunctions of the Laplace operator allows to obtain a parametrix as a sum over the zeros of the Airy function. When dealing to the wave equation, each term in this sum satisfies the usual dispersive estimate, hence we can sum up sufficiently many of them and still get sufficiently good bounds. On the other hand, when considering the semi-classical Schrödinger flow, even the very first modes - localised at distance $h^{2/3}$ from $\partial\Omega$ (that are known as gallery modes) yield a *sharp* loss of 1/6 in both dispersion and Strichartz estimates (in [13] we compare the two situations).

Theorem 2. *The Strichartz estimates for the semi-classical flow hold with at most $\frac{1}{4}$ loss : let (q, r) be a such that, for $d \geq 2$,*

$$\frac{1}{q} \leq \left(\frac{d}{2} - \frac{1}{4}\right)\left(\frac{1}{2} - \frac{1}{r}\right),$$

let $s = d/2 - 2/q - d/r$, then the solution to (7) with data v_0 , satisfies

$$\|\psi(hD_t)v\|_{L^q([0, T_0], L^r(\Omega_d))} \lesssim h^{-s} \|v_0\|_{L^2(\Omega_d)}.$$

The proof of Theorem 2 follows from Theorem 1 using the classical TT^* argument and the endpoint argument of Keel-Tao [21] for $q = 2$ when $d \geq 3$. The (scale-invariant) loss at the semi-classical level corresponds to $1/4$ derivative in space, as illustrated with $d = 2$, for which the (forbidden) endpoint $(2, \infty)$ with $s = 0$ is replaced by $(8/3, \infty)$ with $s = 1/4$.

In [12], we proved that there must be a loss of at least $\frac{1}{6}$ derivatives in the Strichartz estimates for the semi-classical Schrödinger flow, which is obtained when the data is a gallery mode. It is currently unknown whether or not this result is sharp, nor even if a loss in the semi-classical setting *should* provide losses in classical time in the case of a generic non-trapping domain where concave portions of the boundary could act like mirrors and refocus wave packets (yielding unavoidable losses in dispersion). In fact, understanding the Strichartz estimates in exterior domains seems to be a very delicate task: the obstructions from the bounded situation no longer apply, at least in the case of non-trapping obstacles. Thus, one may reasonable expect *all Strichartz estimates to hold*, but the full answer remains unknown. The conflict between this questioning and the failure of semi-classical Strichartz (and dispersion) near the boundary is only apparent: for non trapping domains, a wave packet would spend too short a time in a too narrow to be a contradiction by itself.

For the wave equation, Strichartz estimates with losses were obtained in [5] using short time parametrices constructions from [29]. As already noticed, the main advantage of [5] is also its main weakness: by considering only time intervals that allow for no more than one reflection of a given wave packet, one may handle any boundary but one does not see the full effect of dispersion in the tangential variables. New results in both positive and negative directions have recently been obtained, which improve on known results for strictly convex domains in all dimensions: in [17] it is shown that Strichartz estimates for the wave equation hold true on the domain $(\Omega_{d=2}, g_F)$ with at least $1/9$ loss. For $d = 2$, [5] obtained $\frac{1}{6}$ instead of $\frac{1}{9}$ (but for any boundary), while [18] provides $\frac{1}{4}$. The proof of [17] rely on improving the parametrix construction of [18] and the resulting bounds on the Green function : in particular, the degenerate stationary phase estimates in [18] may be refined to isolate precisely the space-time location of the worst case scenario of a swallowtail singularity. It turns out that, for the wave equation, such singularities only happen at an exceptional, discrete set of times. The proof of Theorem 1 relies on the refinements of the degenerate stationary phase estimates from [17] as well as on refined estimates on gallery modes from [12], all of which are of independent interest:

- the parametrix construction from [18] may be done for initial data $\delta_{(x=a, y=0)}$ with $a > h^{2/3-\varepsilon}$, for any $\varepsilon > 0$;

- the degenerate stationary phase estimates in [18] may be refined to isolate precisely the space-time location of the worst case scenario of a swallowtail singularity. If in the case of the wave equation it turns out that such singularities only happen (see (6)) at exceptional, discrete set of values of t , in the case of the semiclassical Schrödinger equation for a given time $t \in (h, 1]$ we can expect to get $1/4$ losses for all y in a whole interval around $2t$; this happens because wave packets propagate with different speeds.
- gallery modes provide $1/6$ losses in the usual dispersive and Strichartz estimates (as already proved in [12]) but with uniform constant with respect to the order of the mode : this allows to deal with the $a < h^{2/3-\varepsilon}$ region.

Remark 2. *Adapting the construction of a parametrrix for the wave flow from [14], one should be able to generalize the proof of Theorem 1 to a domain Ω whose boundary is everywhere strictly (geodesically) convex : what we mean by that is that for every point $(0, y_0) \in \partial\Omega$ there exists $(0, y_0, \xi_0, \eta_0) \in T^*\Omega$ where the boundary is micro-locally strictly convex, i.e. such that there exists a bicharacteristic passing through $(0, y_0, \xi_0, \eta_0)$ that intersects $\partial\Omega$ tangentially having exactly second order contact with the boundary and remaining in the complement of $\partial\bar{\Omega}$. Under this assumption, local coordinates can be chosen in order to work on the half space Ω_d with a Laplace operator of the form $\Delta = \partial_x^2 + R(x, y, \partial_y)$ such that*

$$\begin{aligned} \tau^2 &= R(0, y_0, \eta_0), \quad \{\xi^2 + R(x, y, \eta), x\}|_{\kappa_0} = 2\xi_0 = 0, \\ \{\{\xi^2 + R(x, y, \eta), x\}, \xi^2 + R(x, y, \eta)\}|_{\kappa_0} &= 2\partial_x R(0, y_0, \eta_0) > 0, \end{aligned}$$

where $\{.,.\}$ denotes the Poisson bracket. This is work in progress.

For general domains, there have been important developments especially by H. Smith and C. Sogge (see for example [6], [29] and references therein). All these positive results rely on a clever reduction from a boundary problem with smooth metric to a boundary-less problem with a Lipschitz metric across an interface, and then use the machinery originally developed for low regularity metrics [26, 33] and spectral cluster estimates [29]. Such constructions do away with multiply reflected rays by suitable microlocalizations: one ends up working on a possibly very small time interval, depending on the incidence of the wave packet under consideration, such that all corresponding rays are only reflected once. Summing these intervals induces (scale-invariant) losses, which get worse with dimension; while Strichartz estimates are obtained in a more direct way in [6] in the case of the wave equation, one can observe that the corresponding dispersion estimate would have at most $1/t$ decay for $d \geq 4$, as the argument is blind to the full dispersion which should occur in tangential directions. As a result, one gets sub-optimal estimates for $d \geq 3$ because the new metric is not smooth enough (only Lipschitz along an hyper-surface, which is meant to be the boundary). However, as we have shown in [18], caustics (points where light is singularly intense) appear right between the first and the second reflection of the wave front: hence, while very flexible with respect to the boundary shape, any microlocalization procedure that reduces to wave-packets crossing the boundary only once cannot provide

optimal results.

Depending on the trajectory of light rays, we distinguish two main situations :

The non-trapping case. In order to assure dispersive effects, certain authors introduced the non-trapping assumption for a manifold: (Ω, g) is non-trapping if any geodesic of the metric g reflecting on the boundary according to the laws of geometric optics goes to infinity. In the case of the wave equation, this condition is essentially equivalent to the local decay of the energy which is "transported" along optical rays, which therefore go to infinity. The Schrödinger equation on non-trapping domains induces a gain of spatial regularity for almost all t when compared to the initial data which, if ψ is a smooth, compactly supported function, reads as

$$(9) \quad \|\psi e^{it\Delta_g} v_0\|_{L_t^2 H^{1/2}(\Omega)} \leq C \|v_0\|_{L^2(\Omega)}.$$

The non-trapping condition is known to be essentially necessary and sufficient for (9) to hold. The local smoothing estimate guarantees that the wave packets only spend a bounded amount of time next to the obstacle. While for (9) the situation is fairly well understood, as far as Strichartz estimates are concerned much less is known and it is not really clear under what geometric assumptions a loss in (2) must occur. Despite important progress on the study of Strichartz estimates (and nonlinear Schrödinger equations) in exterior domains (see the papers [2], [8], [22], [23], [19], [25], [32], [20], [36], [1] and the references therein), when the domain is non-trapping, we only dispose of some partial results (a restricted set of non sharp, but scale invariant Strichartz estimates that are known to hold, together with square function type estimates) : results with loss of derivatives were obtained in [8] (see also [1] and [5]). For the Schrödinger equation, Strichartz estimates without loss of derivatives have been obtained by Blair, Smith and Sogge in [7], where the authors have shown that (2) holds true with $\sigma = \frac{1}{q}$ for a range of sub-critical indices (q, r) such that $3/q + 2/r \leq 1$, if $d = 2$; $1/q + 1/r \geq 1/2$, if $d \geq 3$. This is currently the best known result in the case of general non-trapping exterior domains. For the wave equation, Burq, Lebeau and Planchon [10] used the square function estimates from [29] to prove Strichartz without loss for solutions to the linear wave equation on Ω for the admissible pair $(d = 3, q = 5, r = 10)$, that allowed them to show that there is global existence for the H^1 - critical nonlinear wave equation. Both works [7] and [10] are based on [29] and the restriction of indices is naturally imposed by the local nature of the parametrix construction in [29] and is clearly not optimal for $d \geq 3$.

In the particular case of the exterior of a strictly convex obstacle the situation has been much better understood lately : let $\Theta \subset \mathbb{R}^d$ be an obstacle whose boundary is everywhere strictly geodesically convex and let $\Omega = \mathbb{R}^d \setminus \Theta$, then the Strichartz estimates are known to hold like in \mathbb{R}^d (see [13]). The first step in [13] consisted in proving sharp, scale-invariant Strichartz for the semi-classical Schrödinger equation (e.g. on a time interval of size the wavelength for the classical equation) on compact manifolds with strictly concave boundaries of dimension $d \geq 2$ (an example of such manifold is the so-called Sinai billiard,

e.g. a punctured torus). Following the approach of [9], one then sums up over such small, wavelength size intervals of time and obtain Strichartz estimates on a fixed time interval for the classical flow with a loss of $1/q$ derivative and the same admissible pairs as in flat space. We want to stress that in order to obtain Strichartz in semi-classical time in [13] we bypassed dispersion: in fact, we side-stepped this issue by taking advantage, as in [28], of the L^2 continuity of certain operators to reduce consideration to operators like those on a manifold without boundary: we wish to point out that this approach is very unlikely to work when one is interested in obtaining dispersion.

While many positive results on dispersive effects had been established lately in exterior domains, the question about whether or not dispersion estimates hold remained completely open, even for the exterior of a sphere, until very recently. In [15], we provided some interesting answers to this question: outside a ball in \mathbb{R}^3 we obtained a precise description of the behaviour of the flow and proved that dispersive estimates hold like in \mathbb{R}^3 , while in higher dimensions $d \geq 4$ we constructed explicit counterexamples to (3) for both wave equation and Schrödinger equation : in fact, a loss in dispersion may be informally related to a cluster point: such clusters occur because optical rays (sent along different directions) are no longer diverging from each other. Our intuition tells us that if there is a location where dispersion could fail, this could only happen at the Poisson-Arago spot (a bright point that appears at the center of a circular obstacle's shadow due to Fresnel diffraction).

Compact domains. For both compact manifolds and bounded domains, finiteness of the volume means that there can be no long-time dispersion of wave packets: because of this, all Strichartz estimates must be local in time. Further, due to the existence of conjugate points for the geodesic flow, high frequency waves can refocus and this can happen arbitrarily quickly. As a consequence, the corresponding Strichartz estimates lose derivatives relative to the Euclidean setting.

On a compact domain with smooth boundary, the same set of estimates as in the exterior case are still known to hold, but only at semi-classical time scales. In fact, the estimates in the non-trapping case are obtained by combining these semi-classical estimates via the local smoothing effect, following a strategy pioneered by [32]. Thus, the best available Strichartz estimates are due to [7] and they exhibit losses with respect to scaling.

In the remaining of the paper, $A \lesssim B$ means that there exists a constant C such that $A \leq CB$ and this constant may change from line to line but is independent of all parameters. It will be explicit when (very occasionally) needed. Similarly, $A \sim B$ means both $A \lesssim B$ and $B \lesssim A$.

2. THE SEMI-CLASSICAL SCHRÖDINGER PROPAGATOR: SPECTRAL ANALYSIS AND PARAMETRIX CONSTRUCTION

We recall a few notations, where Ai denotes the standard Airy function: define

$$(10) \quad A_{\pm}(z) = e^{\mp i\pi/3} Ai(e^{\mp i\pi/3} z) = \Psi(e^{\mp i\pi/3} z) e^{\mp \frac{2}{3} iz^3/2}, \quad \text{for } z \in \mathbb{C},$$

then $Ai(-z) = A_+(z) + A_-(z)$. We have $\Psi(z) \simeq z^{-1/4} \sum_{j=0}^{\infty} a_j z^{-3j/2}$, $a_0 = \frac{1}{4\pi^{3/2}}$.

Lemma 1. (see [16, Lemma 1]) *Define*

$$(11) \quad L(\omega) = \pi + i \log \frac{A_-(\omega)}{A_+(\omega)}, \quad \text{for } \omega \in \mathbb{R},$$

then L is real analytic and strictly increasing. We also have

$$(12) \quad L(0) = \pi/3, \quad \lim_{\omega \rightarrow -\infty} L(\omega) = 0, \quad L(\omega) = \frac{4}{3}\omega^{\frac{3}{2}} + \frac{\pi}{2} - B(\omega^{\frac{3}{2}}), \quad \text{for } \omega \geq 1,$$

with $B(u) \simeq \sum_{k=1}^{\infty} b_k u^{-k}$, $b_k \in \mathbb{R}$, $b_1 > 0$. Finally, $Ai(-\omega_k) = 0 \iff L(\omega_k) = 2\pi k$ and $L'(\omega_k) = 2\pi \int_0^{\infty} Ai^2(x - \omega_k) dx$ where here and thereafter, $\{-\omega_k\}_{k \geq 1}$ denote the zeros of the Airy function in decreasing order. Recall that $\omega_1 \simeq 2.33$.

2.1. Spectral analysis of the Friedlander model. Our domain is the half space $\Omega_d = \{(x, y) \in \mathbb{R}^d \mid x > 0, y \in \mathbb{R}^{d-1}\}$ and the Laplacian Δ_F is given in (5). As Δ_F has constant coefficients in y , taking the Fourier transform in the y variable, it transforms into $-\partial_x^2 + |\theta|^2 + xq(\theta)$. For $\theta \neq 0$, this operator is a positive self-adjoint operator on $L^2(\mathbb{R}_+)$, with compact resolvent.

Lemma 2. (see [16, Lemma 2]) *There exist orthonormal eigenfunctions $\{e_k(x, \theta)\}_{k \geq 0}$ of $-\partial_x^2 + |\theta|^2 + xq(\theta)$ with corresponding eigenvalues $\lambda_k(\theta) = |\theta|^2 + \omega_k q(\theta)^{2/3}$, which form a Hilbert basis of $L^2(\mathbb{R}_+)$. These eigenfunctions have an explicit form in terms of Airy functions :*

$$(13) \quad e_k(x, \theta) = \frac{\sqrt{2\pi} q(\theta)^{1/6}}{\sqrt{L'(\omega_k)}} Ai\left(xq(\theta)^{1/3} - \omega_k\right),$$

where $L'(\omega_k)$ (with L as in (11)) is such that $\|e_k(\cdot, \theta)\|_{L^2(\mathbb{R}_+)} = 1$.

In a classical way, for $x_0 > 0$, the Dirac distribution $\delta_{x=x_0}$ on \mathbb{R}_+ may be decomposed as $\delta_{x=x_0} = \sum_{k \geq 1} e_k(x, \theta) e_k(x_0, \theta)$. If, for some fixed t_0 , we consider a data at time $t = t_0$ such that $u(t_0, x, y) = \psi(hD_y) \delta_{x=x_0, y=y_0}$, where $h \in (0, 1]$ is a small parameter and $\psi \in C_0^\infty([\frac{1}{2}, \frac{3}{2}])$, we can write the (localized in θ) Green function associated to the semi-classical Schrödinger operator on Ω_d as

$$(14) \quad G_h((t, x, y), (t_0, x_0, y_0)) = \sum_{k \geq 1} \int_{\mathbb{R}^{d-1}} e^{i(t-t_0)\lambda_k(\theta)} e^{i\langle y-y_0, \theta \rangle} \psi(h|\theta|) e_k(x, \theta) e_k(x_0, \theta) d\theta.$$

In addition to the cut-off $\psi(h|\theta|)$, which localizes the Fourier variable dual to y , we may add a spectral cut-off $\psi_1(h\sqrt{\lambda_k(\theta)})$ under the θ integral, where ψ_1 is also such that $\psi_1 \in C_0^\infty([\frac{1}{2}, \frac{3}{2}])$. For that, we use

$$-\Delta_F \left(\psi(h|\theta|) e^{iy\theta} e_k(x, \theta) \right) = \lambda_k(\theta) \psi(h|\theta|) e^{iy\theta} e_k(x, \theta).$$

On the flow, this is nothing but $\psi_1(hD_t)$ and this smoothes out the Green function.

Remark 3. *As remarked in [18] where the wave flow was considered (see also [17]), after adding $\psi_1(h\sqrt{\lambda_k(\theta)})$, the significant part of the sum over k in (14) becomes then a finite sum over $k \lesssim 1/h$. Indeed, with the usual notations $\tau = \frac{h}{i} \partial_t = hD_t$, $\xi = \frac{h}{i} \partial_x = hD_x$, $\eta =$*

$\frac{h}{i}\nabla_y = hD_y$, the characteristic set of the semi-classical Schrödinger operator $ih\partial_t - h^2\Delta_F$ is given by

$$\tau = \xi^2 + |\eta|^2 + xq(\eta).$$

The hyperbolic (resp. elliptic) subset of the cotangent bundle of the boundary $x = 0$ is $\tau > |\eta|^2$ (resp. $\tau < |\eta|^2$) and the gliding subset is $\tau = |\eta|^2$. Using $\tau = hD_t = h\lambda_k(D_y)$, one obtains (at the symbolic level) that on the micro-support of any gallery mode associated to ω_k we have

$$(15) \quad h^{2/3}\omega_k q^{2/3}(\eta) = |\xi|^2 + xq(\eta).$$

We may assume that on the support of $\psi(\eta)\psi_1(h\sqrt{\lambda_k(\eta/h)})$ one has $h^{2/3}\omega_k \leq \varepsilon_0$ with ε_0 small. This is compatible with (15) since it is equivalent to $|\xi|^2 \lesssim \varepsilon_0$. Considering the asymptotic expansion of $\omega_k \sim k^{2/3}$ the condition $h^{2/3}\omega_k \leq \varepsilon_0$ yields $k \lesssim \varepsilon_0/h$.

Remark 4. As in [18], irrespective of the position of x_0 relative to h , the remaining part of the Green function (corresponding to larger values of k) will be essentially transverse and will see at most one reflexion for $t \in [0, T_0]$ with T_0 small (depending on the above choice of ε_0). Hence, this regime can be dealt with as in [5] to get the free space decay and we will ignore it in the upcoming analysis.

Reducing the sum to $k \leq \varepsilon_0/h$ is equivalent to adding a spectral cut-off $\phi_{\varepsilon_0}(x + h^2D_x^2/q(\theta))$ in the Green function, where $\phi_{\varepsilon_0} = \phi(\cdot/\varepsilon_0)$ for some smooth cut-off function $\phi \in C_0^\infty([-1, 1])$: using that the eigenfunctions of the operator $-\partial_x^2 + xq(\theta)$ are also $e_k(x, \theta)$ but associated to the eigenvalues $\lambda_k(\theta) - |\theta|^2 = \omega_k q^{2/3}(\theta)$, we can localize with respect to $x + h^2D_x^2/q(\theta)$: notice $(x + h^2D_x^2/q(\theta))e_k(x, \theta) = (\omega_k q^{2/3}(\theta)/q(\theta))e_k(x, \theta)$ and this new localization operator is exactly associated by symbolic calculus to the cut-off $\phi_{\varepsilon_0}(\omega_k/q(\theta)^{1/3})$. We therefore set, for $(t_0, x_0, y_0) = (0, a, 0)$,

$$(16) \quad G_{h, \varepsilon_0}(t, x, y, 0, a, 0) = \sum_{k \geq 1} \int_{\mathbb{R}^{d-1}} e^{it\lambda_k(\theta)} e^{i(y-b)\eta/h} \psi(h|\theta|) \psi_1(h\sqrt{\lambda_k(\theta)}) \\ \times \phi_{\varepsilon_0}(\omega_k/q(\theta)^{1/3}) e_k(x, \theta) e_k(a, \theta) d\theta.$$

In the following we introduce a new, small parameter γ satisfying $\max(a, h^{2/3}) \lesssim \gamma \leq \varepsilon_0$ and then split the (tangential part of the) Green function into a dyadic sum $G_{h, \gamma}$ corresponding to a dyadic partition of unity supported for $\omega_k/q(\theta)^{1/3} \simeq \gamma \simeq 2^j \max(a, h^{2/3}) \leq \varepsilon_0$. Let $\psi_2(\cdot/\gamma) := \phi_\gamma(\cdot) - \phi_{\gamma/2}(\cdot)$ and decompose ϕ_{ε_0} as follows

$$(17) \quad \phi_{\varepsilon_0}(\cdot) = \phi_{\max(a, h^{2/3})}(\cdot) + \sum_{1 \leq j < \log_2(\varepsilon_0/\max(a, h^{2/3}))} (\phi_{2^j \max(a, h^{2/3})} - \phi_{2^{j-1} \max(a, h^{2/3})})(\cdot) \\ = \phi_{\max(a, h^{2/3})}(\cdot) + \sum_{\gamma=2^j \max(a, h^{2/3}), 1 \leq j < \log_2(\varepsilon_0/\max(a, h^{2/3}))} \psi_2(\cdot/\gamma),$$

which allows to write $G_{h,\varepsilon_0} = \sum_{\max(a, h^{2/3}) \leq \gamma < 1} G_{h,\gamma}$ where (rescaling the θ variable for later convenience) $G_{h,\gamma}$ takes the form

$$(18) \quad G_{h,\gamma}(t, x, a, y) = \sum_{k \geq 1} \frac{1}{h^{d-1}} \int_{\mathbb{R}^{d-1}} e^{it\lambda_k(\eta/h)} e^{iy\eta/h} \psi(|\eta|) \psi_1(h\sqrt{\lambda_k(\eta/h)}) \\ \times \psi_2(h^{2/3}\omega_k/(q(\eta)^{1/3}\gamma)) e_k(x, \eta/h) e_k(a, \eta/h) d\eta.$$

Remark 5. Notice that, when $\gamma = \max(a, h^{2/3})$, according to the decomposition (17), we should have written $\phi_{\max(a, h^{2/3})}$ instead of $\psi_2(\cdot/\max(a, h^{2/3}))$ in (18) : it turns out that, for values $h^{2/3}\omega_k \lesssim \frac{1}{2} \max(a, h^{2/3})$, the corresponding Airy factors are exponentially decreasing and provide an irrelevant part; therefore writing $\phi_{\max(a, h^{2/3})}$ or $\psi_2(\cdot/\max(a, h^{2/3})) = \phi_{\max(a, h^{2/3})}(\cdot) - \phi_{\frac{1}{2}\max(a, h^{2/3})}(\cdot)$ yields the same contribution in $G_{h, \max(a, h^{2/3})}$ modulo $O(h^\infty)$. In fact, when $a < h^{2/3}$ is sufficiently small, there is no ω_k satisfying $h^{2/3}\omega_k/q^{1/3}(\eta) < h^{2/3}/2$ as $\omega_k \geq \omega_1 \simeq 2.33$ and $|\eta| \in [\frac{1}{2}, \frac{3}{2}]$; on the other hand, when $a \gtrsim h^{2/3}$ and $h^{2/3}\omega_k/q^{1/3}(\eta) \leq a/2$ then the Airy factor of $e_k(a, \eta/h)$ is exponentially decreasing (see [34, Section 2.1.4.3] for details). In order to keep similar notations, we use the same formula (18) for each $G_{h,\gamma}$.

Note that from an operator point of view, if $G_h(\cdot)$ is the true semi-classical Schrödinger propagator, we are in fact considering (with $D_x = \partial_x/i$, $D_y = \nabla_y/i$)

$$G_{h,\gamma} = \psi(hD_y) \psi_1(h\sqrt{-\Delta_F}) \psi_2((x + h^2 D_x^2/q(hD_y))/\gamma) G_h.$$

We are left to evaluate $\|G_{h,\varepsilon_0}(t, \cdot)\|_{L^\infty(\Omega_d)}$.

Remark 6. For $a \lesssim h^{2/3}$, dispersive estimates with unavoidable loss of 1/6 have been proved in [12] :

$$(19) \quad \|G_{h, h^{2/3}}(t, \cdot, a \lesssim h^{2/3}, \cdot)\|_{L^\infty} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} h^{1/3}.$$

The proof in [12] is given for $q(\eta) = |\eta|^2$ but can be easily generalised to any positive definite quadratic form q . This loss of 1/6 is optimal in the regime $a \lesssim h^{2/3}$: in [12, Theorem 1.8] we have suitably chosen a gaussian data whose associated semi-classical Schrödinger flow satisfied (19) with \lesssim replaced by \simeq . This regime (called the regime of "whispering gallery modes") has been well understood by now. It is worthwhile recalling that (see [12]), in the case of the wave flow, this regime provides no loss in the dispersive and Strichartz estimates.

Remark 7. For $a \gtrsim h^{2/3}$, the main contribution in the sum (16) comes from values of k and γ such that $\gamma \simeq a$ and $a/h^{2/3} \simeq \omega_k$, (when the maximum of $e_k(a, \eta/h)$ is reached). When a lies in a $h^{2/3}$ -neighborhood of the boundary, the sum defining $G_{h, \max(a, h^{2/3})}$ is essentially finite (as $\omega_k \simeq k^{2/3} \simeq a/h^{2/3}$). Each term (or wave packet) in this sum has degenerate critical points and provides a 1/6 loss at $x = a$; obtaining a sharp bound for each wave packet and then summing-up a finite number of terms still gives a sharp bound. When $a \gg h^{2/3}$, in the sum defining $G_{h,a}$ the number of wave packets that provide important

contributions can be very large, as $k \simeq a^{3/2}/h := \lambda$: summing $\lambda \gg 1$ terms provides additional losses. Moreover, in the sum over eigenmodes e_k we cannot expect to localize the essential supports of the wave packets : for a given (t, x, y) , each of the $k \simeq \lambda$ terms provide contributions corresponding to $1/6$ loss.

Remark 8. We will see later on that for $a \gg h^{2/3}$, the "tangential" part $G_{h,a}$ of (14), corresponding to $\gamma \simeq a$ and values $k \simeq \lambda$ provides a loss in the dispersion of \sqrt{a} (when $a \simeq h^{2/3}$ we find $h^{1/3}$ in (19)) : when $1 > a > h^{1/2}$ this is obviously (much) worst than announced in Theorem 1; it becomes clear that this regime has to be dealt with in a different manner. Below we will show how a Poisson summation-type formula allows to transform the sum over eigenmodes into a sum over "reflected waves" that will be used later on to handle the case of larger values of a .

We briefly recall a variant of the Poisson summation formula that will be crucial to analyse the spectral sum defining $G_{h,\gamma}$ (see [16, Lemma 3] for the proof).

Lemma 3. In $\mathcal{D}'(\mathbb{R}_\omega)$, one has $\sum_{N \in \mathbb{Z}} e^{-iNL(\omega)} = 2\pi \sum_{k \in \mathbb{N}^*} \frac{1}{L'(\omega_k)} \delta(\omega - \omega_k)$. In other words, for $\phi(\omega) \in C_0^\infty$,

$$(20) \quad \sum_{N \in \mathbb{Z}} \int e^{-iNL(\omega)} \phi(\omega) d\omega = 2\pi \sum_{k \in \mathbb{N}^*} \frac{1}{L'(\omega_k)} \phi(\omega_k).$$

Using (20) to $G_{h,\gamma}$, we transform the sum over k into a sum over $N \in \mathbb{Z}$, as follows

$$(21) \quad \hat{G}_{h,\gamma}(t, x, a, \eta/h) = \frac{1}{2\pi} \sum_{N \in \mathbb{Z}} \int_{\mathbb{R}} e^{-iNL(\omega)} (|\eta|/h)^{2/3} q^{1/3}(\eta/|\eta|) e^{\frac{i}{h}t|\eta|^2(1+h^{2/3}\omega q^{1/3}(\eta/|\eta|)/|\eta|^{2/3})} \\ \times \psi_1\left(|\eta| \sqrt{1 + h^{2/3}\omega q^{2/3}(\eta/|\eta|)/|\eta|^{2/3}}\right) \psi_2(h^{2/3}\omega/(q^{1/3}(\eta)\gamma)) \\ \times Ai(xq^{1/3}(\eta)/h^{2/3} - \omega) Ai(aq^{1/3}(\eta)/h^{2/3} - \omega) d\omega,$$

where $\hat{G}_{h,\gamma}$ is the Fourier transform in y . For $\max(a, h^{2/3}) \leq \gamma < 1$, we let $\lambda_\gamma = \frac{\gamma^{3/2}}{h}$; when $h^{2/3} \lesssim a$ and $\gamma \simeq a$ we write $\lambda := \frac{a^{3/2}}{h}$. Replacing the Airy factors by their integral formulas yields

$$(22) \quad Ai(xq^{1/3}(\eta)/h^{2/3} - \omega) = \frac{1}{2\pi} \int e^{i(\frac{\sigma^3}{3} + \sigma q^{1/3}(\eta)\lambda_\gamma^{2/3}(\frac{x}{\gamma} - \omega/(q^{1/3}(\eta)\lambda_\gamma^{2/3}))} d\sigma$$

$$(23) \quad = \frac{q^{1/6}(\eta)\lambda_\gamma^{1/3}}{2\pi} \int e^{iq^{1/2}(\eta)\lambda_\gamma(\frac{\sigma^3}{3} + \sigma(\frac{x}{\gamma} - \omega/(q^{1/3}(\eta)\lambda_\gamma^{2/3}))} d\sigma.$$

Rescaling $\omega = q^{1/3}(\eta)\lambda_\gamma^{2/3}\alpha = q^{1/3}(\eta)\gamma\alpha/h^{2/3}$ in (21) yields

$$(24) \quad \hat{G}_{h,\gamma}(t, x, a, \eta/h) = \frac{\lambda_\gamma^{4/3}}{(2\pi)^3 h^{2/3}} \sum_{N \in \mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{\frac{i}{h}\tilde{\Phi}_{N,a,\gamma}(\eta,\alpha,s,\sigma,t,x)} q(\eta) \\ \times \psi_1\left(|\eta| \sqrt{1 + \gamma\alpha q(\eta/|\eta|)}\right) \psi_2(\alpha) ds d\sigma d\alpha,$$

where $\tilde{\Phi}_{N,a,\gamma}(\eta, \alpha, s, \sigma, t, x)$ is given by

$$(25) \quad \tilde{\Phi}_{N,a,\gamma}(\eta, \alpha, s, \sigma, t, x) = t|\eta|^2(1 + \gamma\alpha q(\eta/|\eta|)) - NhL(q^{1/3}(\eta)\lambda_\gamma^{2/3}\alpha) \\ + \gamma^{3/2}q^{1/2}(\eta)\left(\frac{\sigma^3}{3} + \sigma\left(\frac{x}{\gamma} - \alpha\right) + \frac{s^3}{3} + s\left(\frac{a}{\gamma} - \alpha\right)\right).$$

Here $NhL(q^{1/3}(\eta)\lambda_\gamma^{2/3}\alpha) = \frac{4}{3}Nq^{1/2}(\eta)(\gamma\alpha)^{3/2} - NhB(q^{1/2}(\eta)\lambda_\gamma\alpha^{3/2})$ and we recall that $B(q^{1/2}(\eta)\lambda_\gamma\alpha^{3/2}) \simeq \sum_{k \geq 1} \frac{b_k}{(q^{1/2}(\eta)\lambda_\gamma\alpha^{3/2})^k}$, where on the support of $\psi_2(\alpha)$ we have $\alpha \simeq 1$. At this point, it is worth noticing that, as $|\eta| \in [1/2, 3/2]$, we may drop the ψ_1 localization in (24) by support considerations (slightly changing any cut-off support if necessary). Therefore we obtain

$$(26) \quad G_{h,\gamma}(t, x, a, y) = \frac{1}{(2\pi)^3} \frac{\gamma^2}{h^{d+1}} \sum_{N \in \mathbb{Z}} \int_{\mathbb{R} \times \mathbb{R}^{d-1}} \int_{\mathbb{R}^2} e^{\frac{i}{h}(\langle y, \eta \rangle + \tilde{\Phi}_{N,a,\gamma})} q(\eta) \psi(|\eta|) \\ \times \psi_2(\alpha) ds d\sigma d\alpha d\eta.$$

Remark 9. Notice that formulas (26) and (18) define exactly the same object and both will be necessary in order to prove the dispersive estimates. The sum over the eigenmodes e_k will be particularly useful for small values of $a \lesssim (ht)^{1/2}$, while for large values of the initial distance to the boundary the sum over N will provide the announced bounds. Note that, while both formulas coincide, there is a duality between the two on them: when a is small, there are less terms in the sum over k in (18), while when $a > (ht)^{1/2}$ there are less terms in the sum over the reflections N .

Remark 10. As noticed in [18], the symmetry of the Green function (or its suitable spectral truncations) with respect to x and a allows to restrict the computations of the L^∞ norm to the region $0 \leq x \leq a$. In other words, instead of evaluating $\|G_{h,\varepsilon_0}\|_{L^\infty(0 \leq x, y)}(t, \cdot)$ it will be enough to bound $\|G_{h,\varepsilon_0}(t, \cdot)\|_{L^\infty(0 \leq x \leq a, y)}$.

Remark 11. In order to generalize Theorem 1 to a convex domain as in Remark 2, our construction of gallery modes from [14] will turn out to be crucial. Notice that in the general situation even the regime $a \leq h$ has its own difficulties: even deciding how the initial data should be chosen in order the Dirichlet condition to be satisfied on the boundary becomes a non-trivial issue. Indeed, an initial data of the form $\chi_0(hD_x)\psi(hD_y)\delta_{(a,0)}$ can provide a non-trivial contribution on the boundary.

In the model case, Lemma 2 provides a Hilbert basis of $L^2(\mathbb{R}_+)$ in terms of eigenfunctions of the (model) operator $-\partial_x^2 + |\theta|^2 + xq(\theta)$. In the general case, the eigenfunctions of the Laplace operator in a compact Ω form a basis in $L^2(\Omega)$ and one can safely think that this basis can be used to expand a data of the form $\varkappa(-h^2\Delta)\psi(hD_y)\delta_{(a,0)}$, but it turns out that we don't have a complete understanding of the eigenfunctions of $\Delta = \partial_x^2 + R(x, y, \partial_y)$ from Remark 2 on Ω_d . Instead, we chose to use the spectral theory for the model Laplace operator (5) in order to expand $\varkappa(-h^2\Delta_M)\psi(hD_y)\delta_{(a,0)}$ in terms of the eigenfunctions e_k of $-\Delta_M$. Thus, constructing of a parametrix in the model case (in terms of both eigenmodes and sum

over reflections) and obtaining its best possible decay properties is particularly important in order to be able to further generalize Theorem 1.

3. DISPERSIVE ESTIMATES FOR THE SEMI-CLASSICAL SCHRÖDINGER FLOW

In this section we prove dispersive bounds for $G_{h,\varepsilon_0}(t, x, a, y)$ on Ω_d for fixed $|t| \in [h, T_0]$, where $T_0 > 0$ is a suitably chosen small constant. To do that, we estimate separately $\|G_{h,\gamma}(t, \cdot)\|_{0 \leq x \leq a, y \in \mathbb{R}^{d-1}}$ for every γ such that $\max(a, h^{2/3}) \lesssim \gamma \leq \varepsilon_0$. The reduction to the interval $[0, a]$ in the normal variable is due to the symmetry of the Green function with respect to x and a (see Remark 10).

Henceforth we assume $t > 0$. We deal separately with the following situations :

- (1) $\max(h^{2/3-\epsilon}, (ht)^{1/2}) \leq a \leq \varepsilon_0$ for some small $\epsilon > 0$: in this case, for all γ such that $\max(a, h^{2/3}) \lesssim \gamma \leq \varepsilon_0$ we have

$$\max(h^{2/3-\epsilon}, (ht)^{1/2}) \leq a \lesssim \gamma \leq \varepsilon_0.$$

This is, in some sense, the main situation, when only formula (26) is useful; the integrals with respect to σ, s have up to third order degenerate critical points and we need to perform a fine analysis of these integrals. In particular, the "tangential" case $\gamma \simeq a$ provides the worst decay estimates. When $8a \leq \gamma$, the integrals in (26) have degenerate critical points of order at most two. We call this regime "transverse" : summing up $\sum_{8a \leq \gamma} \|G_{h,\gamma}(t, \cdot)\|_{L^\infty}$ still provides a better contribution than $\|G_{h,a}(t, \cdot)\|_{L^\infty}$.

- (2) $a \lesssim \max(h^{2/3-\epsilon}, (ht)^{1/2})$ for some small $\epsilon > 0$: in this case we consider separately the following situations:
- (a) when $\max(h^{2/3-\epsilon}, (ht)^{1/2}) \leq \gamma \leq \varepsilon_0$, the situation corresponds to the "transverse" regime of the preceding case, and the same kind of estimates follow using (26).
- (b) when $\max(a, h^{2/3}) \lesssim \gamma \lesssim \max(h^{2/3-\epsilon}, (ht)^{1/2})$ we use the form (18) of $G_{h,\gamma}$ in terms of eigenmodes to evaluate its L^∞ norms.

3.1. Case $\max(h^{2/3-\epsilon}, (ht)^{1/2}) \leq a \leq \varepsilon_0$ for some small $\epsilon > 0$. In this section we use (26). Since in this case $\max(a, h^{2/3}) = a$, we consider γ such that $a \lesssim \gamma \leq \varepsilon_0$. Let $\lambda_\gamma := \gamma^{3/2}/h$, then we also have $\lambda_\gamma \geq h^{-3\epsilon/2}$.

Remark 12. *The approach followed in this part applies for all $h^{2/3-\epsilon} \lesssim a \leq \varepsilon_0$ (and not only for $\max(h^{2/3-\epsilon}, (ht)^{1/2}) \leq a \leq \varepsilon_0$) and provides sharp dispersive estimates for each $G_{h,\gamma}$ for all $h^{2/3-\epsilon} \lesssim a \leq \gamma \leq \varepsilon_0$; however, when summing up over $a \lesssim \gamma \leq (ht)^{1/2}$ we obtain bounds for G_{h,ε_0} that are worst than those announced in Theorem 1. This is why we consider here only values $\max(h^{2/3-\epsilon}, (ht)^{1/2}) \leq a \leq \varepsilon_0$.*

We first show that the sum defining $G_{h,\gamma}$ in (26) over N is essentially finite and we estimate of the maximum number of terms in this sum.

Proposition 1. *For a fixed $t \in (h, T_0]$ the sum (26) over N is essentially finite and $|N| \lesssim \frac{1}{\sqrt{\gamma}}$. In other words,*

$$\frac{1}{(2\pi)^3} \frac{\gamma^2}{h^{d+1}} \sum_{N \in \mathbb{Z}, |N| \sqrt{\gamma} \notin O(t)} \int_{\mathbb{R} \times \mathbb{R}^{d-1}} \int_{\mathbb{R}^2} e^{\frac{i}{h} \langle y, \eta \rangle + \tilde{\Phi}_{N, a, \gamma}} q(\eta) \psi(|\eta|) \psi_2(\alpha) ds d\sigma d\alpha d\eta = O(h^\infty).$$

Proof. The proof follows easily using non-stationary phase arguments for $N \geq M \frac{t}{\sqrt{\gamma}}$ for some M sufficiently large. In fact, the critical points with respect to σ, s satisfy

$$(27) \quad \sigma^2 = \alpha - \frac{x}{\gamma}, \quad s^2 = \alpha - \frac{a}{\gamma},$$

and since $x \geq 0$, it follows that $\tilde{\Phi}_{N, a, \gamma}$ may be stationary in σ, s only if $|(\sigma, s)| \leq \sqrt{\alpha}$. If $|(\sigma, s)| \geq (1 + |N|^\epsilon) \sqrt{\alpha}$ for some $\epsilon > 0$ we can perform repeated integrations by parts in σ, s to obtain $O(((1 + N^\epsilon) \lambda_\gamma)^{-n})$ for all $n \geq 1$. Let χ a smooth cutoff supported in $[-1, 1]$ and write $1 = \chi(\sigma/(N^\epsilon \sqrt{\alpha})) + (1 - \chi)(\sigma/(N^\epsilon \sqrt{\alpha}))$, then

$$(28) \quad \begin{aligned} \psi(|\eta|) \sum_{N \in \mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{\frac{i}{h} \tilde{\Phi}_{N, a, \gamma}} \psi_2(\alpha) \chi(s/(N^\epsilon \sqrt{\alpha})) (1 - \chi)(\sigma/(N^\epsilon \sqrt{\alpha})) ds d\sigma d\alpha \\ \lesssim \lambda_\gamma^{-1/3} \sup_{\alpha, |\eta| \in [1/2, 3/2]} \left| Ai\left((a - \gamma\alpha) q^{1/3}(\eta)/h^{2/3}\right) \right| \sum_{N \in \mathbb{Z}} \left((1 + N^\epsilon) \lambda_\gamma \right)^{-n} \\ = O(\lambda_\gamma^{-\infty}) = O(h^\infty), \end{aligned}$$

where in the last line we have used that in this regime $\lambda_\gamma \geq h^{-3\epsilon/2}$, $\epsilon > 0$. In the same way, we can sum-up on the support of $(1 - \chi)(s/(N^\epsilon \sqrt{\alpha}))$ and obtain a $O(h^\infty)$ contribution. Therefore, we can introduce the cut-offs $\chi(\sigma/(N^\epsilon \sqrt{\alpha})) \chi(s/(N^\epsilon \sqrt{\alpha}))$ in the integral defining $G_{h, \gamma}$ without changing its contribution modulo $O(h^\infty)$.

Using again (25), the critical point of $\tilde{\Phi}_{N, a, \gamma}$ with respect to α satisfies :

$$(29) \quad \frac{t}{\sqrt{\gamma}} q(\eta) - q^{1/2}(\eta)(s + \sigma) = 2N q^{1/2}(\eta) \sqrt{\alpha} \left(1 - \frac{3}{4} B'(\eta \lambda \alpha^{3/2})\right),$$

and as $|(\sigma, s)| \lesssim (1 + |N|^\epsilon) \sqrt{\alpha}$ on the support of $\chi(\sigma/(N^\epsilon \sqrt{\alpha})) \chi(s/(N^\epsilon \sqrt{\alpha}))$, it follows that $\tilde{\Phi}_{N, a, \gamma}$ may be stationary with respect to α only when $\frac{t}{\sqrt{\gamma}} \simeq 2N$. As $B'(\eta \lambda \alpha^{3/2}) = O(\lambda_\gamma^{-3}) = O(h^{9\epsilon/2})$, this contribution is very small. From (27) and (29), it follows that if

$$(30) \quad \frac{t}{\sqrt{\gamma}} \frac{|\eta|}{\sqrt{\alpha}} q^{1/2}(\eta/|\eta|) \notin [2(N-1), 2(N+1)],$$

then the phase is non-stationary in α . Recall that q is a positive definite quadratic form : let

$$(31) \quad m_0 := \inf_{\Theta \in \mathbb{S}^{d-2}} q^{1/2}(\Theta), \quad M_0 = \sup_{\Theta \in \mathbb{S}^{d-2}} q^{1/2}(\Theta).$$

As $|\eta|, \alpha \in [\frac{1}{2}, \frac{3}{2}]$ on the support of the symbol, it follows that, it $2(N-1) > \frac{t}{\sqrt{\gamma}} \times M_0 \frac{3/2}{\sqrt{1/2}}$ or if $2(N+1) < \frac{t}{\sqrt{\gamma}} \times m_0 \frac{1/2}{\sqrt{3/2}}$, then the phase is non-stationary in α as its first order derivative behaves like N . Repeated integrations by parts allow to sum up in N as above, and conclude. \square

Remark 13. Notice that we can introduce a even better localization with respect to σ and s : indeed, on the support of $(1-\chi)(\sigma/(2\sqrt{\alpha}))$ and $(1-\chi)(s/(2\sqrt{\alpha}))$ the phase is still non-stationary in σ or s , and repeated integrations by parts yield a $O(\lambda_\gamma^{-\infty})$ contribution there. According to Proposition 1, for $h \leq t \leq T_0 \leq 1$ the sum over N has finitely many terms, and therefore summing up finitely many $O(\lambda_\gamma^{-\infty})$ contributions still yields $O(\lambda_\gamma^{-\infty}) = O(h^\infty)$.

Remark 14. Notice that we can bring the factor $e^{iNB(q^{1/2}(\eta)\lambda_\gamma\alpha^{3/2})}$ into the symbol as it doesn't oscillate : indeed, as q is a positive definite quadratic form, $\alpha, |\eta| \in [\frac{1}{2}, \frac{3}{2}]$ on the support of ψ_2, ψ and $N \simeq \frac{t}{\sqrt{\gamma}}$, we obtain, using Lemma 1

$$NB(q^{1/2}(\eta)\lambda_\gamma\alpha^{3/2}) \simeq N \sum_{k \geq 1} \frac{b_k}{(q^{1/2}(\eta)\lambda_\gamma\alpha^{3/2})^k} \simeq \frac{Nb_1}{q^{1/2}(\eta)\lambda_\gamma} \simeq \frac{ht}{\gamma^2},$$

and since we consider here the case $(ht)^{1/2} \lesssim \gamma$, this term remains bounded. Notice that we consider here only values $N \lesssim \lambda_\gamma$ as $N \simeq \frac{t}{\sqrt{\gamma}} \lesssim \frac{\gamma^2/h}{\sqrt{\gamma}} = \lambda_\gamma$.

We set $\Phi_{N,a,\gamma} = \langle y, \eta \rangle + \tilde{\Phi}_{N,a,\gamma} - NhB(q^{1/2}(\eta)\lambda_\gamma\alpha^{3/2})$: from Remark 14 it follows that, in this regime, $\Phi_{N,a,\gamma}$ is the phase function of $G_{h,\gamma}$ defined by (26). We have

$$(32) \quad \Phi_{N,a,\gamma}(\eta, \alpha, s, \sigma, t, x, y) = \langle y, \eta \rangle + t|\eta|^2(1 + \gamma\alpha q(\eta/|\eta|)) \\ + \gamma^{3/2}q^{1/2}(\eta) \left(\frac{\sigma^3}{3} + \sigma \left(\frac{x}{\gamma} - \alpha \right) + \frac{s^3}{3} + s \left(\frac{a}{\gamma} - \alpha \right) - \frac{4}{3}N\alpha^{3/2} \right).$$

In the following we study, for each $|N| \lesssim \frac{1}{\sqrt{\gamma}}$, the integrals in the sum (26). Notice that when $N = 0$ we deal with the free semi-classical Schrödinger flow (that hasn't reached the boundary yet). As we have fixed a sign for $t > 0$, we consider $N \geq 1$.

Proposition 2. Let $N \geq 1$. The phase function $\Phi_{N,a,\gamma}$ can have at most one critical point (α_c, η_c) on the support $[\frac{1}{2}, \frac{3}{2}]$ of the symbol. When $\nabla_\eta \Phi_{N,a,\gamma} = 0$ and $\partial_\alpha \Phi_{N,a,\gamma} = 0$, the determinant of the matrix of second order derivatives with respect to η and α is $\simeq t^{d-1} \times \gamma^{3/2}N$. In other words the stationary phase applies in both $\alpha \in [1/2, 3/2]$ and $\eta \in \mathbb{R}^{d-1}$ and yields a factor $(\frac{h}{t})^{(d-1)/2} \times \frac{1}{\sqrt{\lambda_\gamma N}}$.

Proof. The derivatives of the phase $\Phi_{N,a,\gamma}$ with respect to α, η are

$$\partial_\alpha \Phi_{N,a,\gamma} = \gamma^{3/2}q^{1/2}(\eta) \left(\frac{t}{\sqrt{\gamma}}q^{1/2}(\eta) - (\sigma + s) - 2N\sqrt{\alpha} \right), \\ \nabla_\eta \Phi_{N,a,\gamma} = y + 2\eta t + \gamma^{3/2} \frac{\nabla q(\eta)}{2q^{1/2}(\eta)} \left(\frac{\sigma^3}{3} + \sigma \left(\frac{x}{\gamma} - \alpha \right) + \frac{s^3}{3} + s \left(\frac{a}{\gamma} - \alpha \right) - \frac{4}{3}N\alpha^{3/2} + 2\alpha \frac{t}{\sqrt{\gamma}}q^{1/2}(\eta) \right).$$

At $\partial_\alpha \Phi_{N,a,\gamma} = 0$ and $\nabla_\eta \Phi_{N,a,\gamma} = 0$ the equations satisfied by the critical points are

$$(33) \quad \sqrt{\alpha} = \frac{tq^{1/2}(\eta)}{2N\sqrt{\gamma}} - \frac{s + \sigma}{2N}$$

and also (replacing $2N\sqrt{\alpha}$ by $\frac{t}{\sqrt{\gamma}}q^{1/2}(\eta) - (s + \sigma)$ in the expression of $\nabla_\eta \Phi_{N,a,\gamma}$)

$$(34) \quad 2t\left(\eta + \frac{1}{2}\gamma\alpha\nabla q(\eta)\right) = -y - \gamma^{3/2}\frac{\nabla q(\eta)}{2q^{1/2}(\eta)}\left[\frac{\sigma^3}{3} + \sigma\frac{x}{\gamma} + \frac{s^3}{s} + s\frac{a}{\gamma} - \frac{(s + \sigma)\alpha}{3}\right].$$

It follows from (30) (and the support condition on η and α) that a critical point $\alpha_c \in [\frac{1}{2}, \frac{3}{2}]$ does exist only if

$$(35) \quad (1 - 1/N)\frac{\sqrt{1/2}}{3M_0/2} \leq \frac{t}{2N\sqrt{\gamma}} \leq (1 + 1/N)\frac{\sqrt{3/2}}{m_0/2}.$$

- For $N \geq 2$, fix M sufficiently large such that $[(1 - 1/2)\frac{\sqrt{1/2}}{3M_0/2}, (1 + 1/2)\frac{\sqrt{3/2}}{m_0/2}] \subset [1/M, M]$, then (33) may have a solution on the support of ψ_2 only when $\frac{t}{2N\sqrt{\gamma}} \in [1/M, M]$ (notice that if $q(\eta) = |\eta|^2$, we can take $M = 3\sqrt{2}$).
- For $N = 1$, we obtain the upper bound $\frac{t}{2\sqrt{\gamma}} \leq \frac{4}{m_0}\sqrt{3/2}$ but also, using (27), the following lower bounds : either $s + \sigma \geq -\frac{3}{2}\sqrt{\alpha}$, in which case $\frac{t}{2\sqrt{\gamma}} \geq \frac{\sqrt{\alpha}}{4|\eta|M_0}$, or $(s + \sigma) \leq -\frac{3}{2}\sqrt{\alpha}$ in which case both s and σ must take non positive values and in this case

$$q^{1/3}(\eta)\frac{t}{2\sqrt{\gamma}} \geq \sqrt{\alpha} + \frac{s + \sigma}{2} \geq \frac{a/\gamma}{2(\sqrt{\alpha} - s)} + \frac{x/\gamma}{2(\sqrt{\alpha} - \sigma)} \geq \frac{a/\gamma}{4\sqrt{\alpha}}.$$

This last lower bound on t states precisely that for $t \leq \frac{a/\sqrt{\gamma}}{2\sqrt{3/2}M_0^{2/3}}$ the flow doesn't reach the boundary (hence there are no reflections).

Let $N \geq 1$ hence $t \geq \frac{a/\sqrt{\gamma}}{2\sqrt{3/2}M_0^{2/3}}$ (since otherwise the phase is non-stationary). As $\alpha \in [\frac{1}{2}, \frac{3}{2}]$ and $\gamma \leq \varepsilon_0$ it follows that (34) may have a critical point η_c only when $|y|/2t \in [\frac{1}{2} + O(\varepsilon_0), \frac{3}{2} + O(\varepsilon_0)]$.

Using $\partial_{\eta_j} q(\eta) = 2q_{j,j}\eta_j + \sum_{k \neq j} q_{j,k}\eta_k$, $q_{j,k} = q_{k,j}$ the second order derivatives become

$$(36) \quad \partial_{\alpha,\alpha}^2 \Phi_{N,a,\gamma} = -\gamma^{3/2} q^{1/2}(\eta) \frac{N}{\sqrt{\alpha}},$$

$$(37) \quad \partial_{\eta_j} \partial_{\alpha} \Phi_{N,a,\gamma} = \frac{\partial_{\eta_j} q(\eta)}{2q(\eta)} \partial_{\alpha} \Phi_{N,a,\gamma} + \gamma^{3/2} \frac{t}{2\sqrt{\gamma}} \partial_{\eta_j} q(\eta),$$

$$(38) \quad \partial_{\eta_j, \eta_j}^2 \Phi_{N,a,\gamma} = 2t \left(1 + \gamma \alpha \frac{(\partial_{\eta_j} q(\eta))^2}{4q(\eta)} \right)$$

$$(39) \quad + \frac{\gamma^{3/2}}{q^{1/2}(\eta)} \left(q_{j,j} - \frac{(\partial_{\eta_j} q(\eta))^2}{4q(\eta)} \right) \left(\frac{\sigma^3}{3} + \sigma \left(\frac{x}{\gamma} - \alpha \right) + \frac{s^3}{3} + s \left(\frac{a}{\gamma} - \alpha \right) - \frac{4}{3} N \alpha^{3/2} + 2\alpha \frac{t}{\sqrt{\gamma}} q^{1/2}(\eta) \right),$$

$$(40) \quad \partial_{\eta_j, \eta_k}^2 \Phi_{N,a,\gamma} = 2t\gamma\alpha \frac{\partial_{\eta_j} q(\eta)}{2q^{1/2}(\eta)} \frac{\partial_{\eta_k} q(\eta)}{2q^{1/2}(\eta)}$$

$$(41) \quad + \frac{\gamma^{3/2}}{q^{1/2}(\eta)} \left(q_{j,k} - \frac{\partial_{\eta_j} q(\eta) \partial_{\eta_k} q(\eta)}{4q(\eta)} \right) \left(\frac{\sigma^3}{3} + \sigma \left(\frac{x}{\gamma} - \alpha \right) + \frac{s^3}{3} + s \left(\frac{a}{\gamma} - \alpha \right) - \frac{4}{3} N \alpha^{3/2} + 2\alpha \frac{t}{\sqrt{\gamma}} q^{1/2}(\eta) \right).$$

At the stationary points we easily see that $\nabla_{\eta,\eta}^2 \Phi_{N,a,\gamma} \simeq 2t(1 + O(\gamma)) \mathbb{I}_{d-1} d + O(\gamma^{3/2})$ where \mathbb{I}_{d-1} denotes the identity matrix in dimension $d-1$; as $\varepsilon_0 < 1$ is small we deduce $\nabla_{\eta,\eta}^2 \Phi_{N,a,\gamma} \simeq 2t \mathbb{I}_{d-1}$. Hence, the stationary phase with respect to η yields a factor $\left(\frac{h}{t}\right)^{\frac{d-1}{2}}$, while the stationary phase in α yields a factor $\frac{1}{\sqrt{\lambda_\gamma N}}$ for $N \geq 1$. \square

Lemma 4. *Let $N \geq 1$ and $a \lesssim \gamma \leq \varepsilon_0$. The critical point η_c of $\Phi_{N,a,\gamma}$ is a function of $s + \sigma$, $(\sigma - s)^2$, $(\sigma - s) \frac{(x-a)}{\gamma}$, $\frac{y}{2t}$ and $\frac{t}{2N\sqrt{\gamma}}$. There exists smooth, uniformly bounded vector valued functions $\Theta, \tilde{\Theta}$ depending on the small parameter γ satisfying*

$$\Theta\left(\frac{y}{2t}, \frac{t}{2N\sqrt{\gamma}}, \gamma\right) = -\frac{1}{2} \left(\frac{t}{2N\sqrt{\gamma}}\right)^2 (q \nabla q) \left(-\frac{y}{2t}\right) + \gamma \tilde{\Theta}\left(\frac{y}{2t}, \frac{t}{2N\sqrt{\gamma}}, \gamma\right),$$

such that the value at $\sigma = s = 0$ of the critical point, denoted $\eta_c^0 := \eta_c|_{\sigma=s=0}$, satisfies

$$\eta_c^0 = -\frac{y}{2t} + \gamma \Theta\left(\frac{y}{2t}, \frac{t}{2N\sqrt{\gamma}}, \gamma\right).$$

Moreover, the vector valued functions defined as follows $\Theta_1 := \frac{t}{\gamma^{3/2}} \partial_\sigma \eta_c$ and $\Theta_2 := \frac{t}{\gamma^{3/2}} \partial_s \eta_c$ are smooth and uniformly bounded.

Proof. We start with the second statement. Let first $N \geq 2$ and define M as follows

$$(42) \quad M := 4 \max \left\{ \frac{\sqrt{3/2}}{m_0 - \varepsilon_0}, \frac{M_0 + \varepsilon_0}{\sqrt{1/2}} \right\}, \text{ with } m_0, M_0 \text{ introduced in (31),}$$

and where we can assume, without loss of generality, that $0 < \varepsilon_0 < m_0/2$. Then M is large enough so that $\left[(1 - 1/2)\frac{\sqrt{1/2}}{3M_0/2}, (1 + 1/2)\frac{\sqrt{3/2}}{m_0/2}\right] \subset [1/M, M]$ and for $\frac{t}{2N\sqrt{\gamma}} \in [1/M, M]$ and $\frac{|y|}{2t} \in [\frac{1}{4}, 2]$, the critical points α_c and η_c of $\Phi_{N,a,\gamma}$ solve the system (33) and (34). Let $\eta_c^0 := \eta_c|_{\sigma=s=0}$ denote the value of η_c at $\sigma = s = 0$, then, using (34) we find that η_c^0 solves the following equation :

$$\eta_c^0 + \frac{1}{2}\gamma\left(\frac{t}{2N\sqrt{\gamma}}\right)^2 q(\eta_c^0)\nabla q(\eta_c^0) = -\frac{y}{2t}.$$

For $\frac{t}{2N\sqrt{\gamma}} \in [1/M, M]$, writing $\eta_c^0 = -\frac{y}{2t} + \gamma\Theta\left(\frac{y}{2t}, \frac{t}{2N\sqrt{\gamma}}, \gamma\right)$, yields the following equation for $\Theta\left(\frac{y}{2t}, \frac{t}{2N\sqrt{\gamma}}, \gamma\right)$

$$(43) \quad \Theta + \frac{1}{2}\left(\frac{t}{2N\sqrt{\gamma}}\right)^2 (q\nabla q)\left(-\frac{y}{2t} + \gamma\Theta\right) = 0,$$

which further reads as follows, with $\Theta = (\Theta^{(1)}, \dots, \Theta^{(d-1)})$ and for all $1 \leq l \leq d-1$

$$\Theta^{(l)} + \left(\frac{t}{2N\sqrt{\gamma}}\right)^2 \sum_{j,k,p} q_{j,k} q_{p,l} \left(-\frac{y_j}{2t} + \gamma\Theta^{(j)}\right) \left(-\frac{y_k}{2t} + \gamma\Theta^{(k)}\right) \left(-\frac{y_p}{2t} + \gamma\Theta^{(p)}\right) = 0.$$

As $\gamma \leq \varepsilon_0 < 1$, the last equation has an unique solution that is a smooth function of $\left(\frac{y}{2t}, \frac{t}{2N\sqrt{\gamma}}, \gamma\right)$ of the form $\Theta^{(l)} = \left(\frac{t}{2N\sqrt{\gamma}}\right)^2 \left(\sum_{j,k,p} q_{j,k} q_{p,l} \left(\frac{y_j}{2t}\right) \left(\frac{y_k}{2t}\right) \left(\frac{y_p}{2t}\right)\right) + \gamma\tilde{\Theta}^{(l)}$, where $\tilde{\Theta} = (\tilde{\Theta}^{(1)}, \dots, \tilde{\Theta}^{(d-1)})$ is a smooth function of $\left(\frac{y}{2t}, \frac{t}{2N\sqrt{\gamma}}, \gamma\right)$. When $N = 1$, t can take (very) small values but doesn't vanish there where the phase $\Phi_{1,a,\gamma}$ may be stationary and therefore (43) still holds and $\frac{|y|}{2t} \in [\frac{1}{4}, 2]$, hence we obtain Θ in the same way.

We now show that for all $N \geq 1$, η_c is a function of $s + \sigma$, $(\sigma - s)^2$, $(\sigma - s)\frac{(x-a)}{\gamma}$, $\frac{y}{2t}$ and $\frac{t}{2N\sqrt{\gamma}}$. This will be useful in the last part of the paper, and in particular in the proof of Proposition 5. Inserting (33) in (34) yields

$$(44) \quad \eta_c + \frac{\gamma}{2}\left(\frac{t}{2N\sqrt{\gamma}}q^{1/2}(\eta_c) - \frac{\sigma + s}{2N}\right)^2 \nabla q(\eta_c) \\ = -\frac{y}{2t} - \frac{\gamma^{3/2}}{2t} \frac{\nabla q(\eta_c)}{2q^{1/2}(\eta_c)} \left[\frac{\sigma^3}{3} + \sigma\frac{x}{\gamma} + \frac{s^3}{3} + s\frac{a}{\gamma} - \frac{(s + \sigma)}{3}\left(\frac{t}{2N\sqrt{\gamma}}q^{1/2}(\eta_c) - \frac{\sigma + s}{2N}\right)^2\right].$$

It follows that η_c is a function of $(s + \sigma)$ and $\frac{\sigma^3}{3} + \sigma\frac{x}{\gamma} + \frac{s^3}{3} + s\frac{a}{\gamma}$ and writing the last term under the form $\frac{(s+\sigma)^3}{3} - 4(s + \sigma)\left((s + \sigma)^2 - (s - \sigma)^2\right) + (s + \sigma)\frac{(x+a)}{2\gamma} + (\sigma - s)\frac{(x-a)}{2\gamma}$ allows to conclude. Taking now the derivative with respect to σ in (44) yields

$$(45) \quad \partial_\sigma \eta_c \left(\mathbb{I}_{d-1} + O(\gamma) + O(\gamma^{3/2}/t)\right) \\ = \frac{\gamma}{2N} \nabla q(\eta_c) + \frac{\gamma^{3/2}}{2t} \frac{\nabla q(\eta_c)}{2q^{1/2}(\eta_c)} \left[\sigma^2 + \frac{x}{\gamma} + \frac{\sqrt{\alpha_c}}{3}\left(\frac{s + \sigma}{N} - \sqrt{\alpha_c}\right)\right],$$

where the second and third terms in brackets in the first line of (45) are smooth, bounded functions of η_c , $\frac{t}{2N\sqrt{\gamma}}$, $(s+\sigma)$ and $\frac{\sigma^3}{3} + \sigma\frac{x}{\gamma} + \frac{s^3}{3} + s\frac{a}{\gamma}$ with coefficients γ and $\gamma^{3/2}/t$, respectively. Let first $N \geq 2$, then using $\frac{t}{2N\sqrt{\gamma}} \in [1/M, M]$ we find $\gamma^{3/2}/t \simeq \gamma/N$ and therefore $\partial_\sigma \eta_c = O(\gamma^{3/2}/t)$. In the same way we obtain $\partial_s \eta_c = O(\gamma^{3/2}/t)$. Let now $N = 1$, then $\gamma^{3/2}/t \gtrsim \gamma$ there where the phase may be stationary, and therefore we still find $\partial_\sigma \eta_c = O(\gamma^{3/2}/t)$ and $\partial_s \eta_c = O(\gamma^{3/2}/t)$. It follows that $\Theta_1 := \frac{t}{\gamma^{3/2}} \partial_\sigma \eta_c$ (and $\Theta_2 := \frac{t}{\gamma^{3/2}} \partial_s \eta_c$, respectively) is a smooth and uniformly bounded vector valued function depending on $\sigma + s, \sigma^2 + \frac{x}{\gamma}, \sigma^3/3 + \sigma\frac{x}{\gamma} + s^3/3 + s\frac{a}{\gamma}$ and $(\frac{t}{2N\sqrt{\gamma}}, \frac{y}{2t}, \gamma)$ (and, respectively, on $\sigma + s, s^2 + \frac{a}{\gamma}, \sigma^3/3 + \sigma\frac{x}{\gamma} + s^3/3 + s\frac{a}{\gamma}$ and $(\frac{t}{2N\sqrt{\gamma}}, \frac{y}{2t}, \gamma)$). In the following we write $\Theta_j = \Theta_j\left(\sigma, s, \frac{t}{2N\sqrt{\gamma}}, \frac{x}{\gamma}, \frac{a}{\gamma}, \frac{y}{2t}, \gamma\right)$ for $j \in \{1, 2\}$. \square

The next lemma gives the form of the critical point α_c .

Lemma 5. *For all $N \geq 1$, the critical point α_c has the following form*

$$(46) \quad \sqrt{\alpha_c} = \frac{t}{2N\sqrt{\gamma}} q^{1/2} \left(\eta_c^0 \right) - \frac{\sigma}{2N} (1 - \gamma \mathcal{E}_1) - \frac{s}{2N} (1 - \gamma \mathcal{E}_2),$$

where \mathcal{E}_j are smooth, uniformly bounded functions, defined as follows

$$(47) \quad \mathcal{E}_1 := \left\langle \int_0^1 \Theta_1 \left(o\sigma, os, \frac{t}{2N\sqrt{\gamma}}, \frac{x}{\gamma}, \frac{a}{\gamma}, \frac{y}{2t}, \gamma \right) do, \int_0^1 \frac{\nabla q}{2q^{1/2}} (o\eta_c^0 + (1-o)\eta_c) do \right\rangle,$$

$$(48) \quad \mathcal{E}_2 := \left\langle \int_0^1 \Theta_2 \left(o\sigma, os, \frac{t}{2N\sqrt{\gamma}}, \frac{x}{\gamma}, \frac{a}{\gamma}, \frac{y}{2t}, \gamma \right) do, \int_0^1 \frac{\nabla q}{2q^{1/2}} (o\eta_c^0 + (1-o)\eta_c) do \right\rangle.$$

Proof. Equation (33) can be written as

$$\sqrt{\alpha_c} = \frac{t}{2N\sqrt{\gamma}} q^{1/2} \left(\eta_c^0 \right) - \frac{(\sigma + s)}{2N} + \frac{t}{2N\sqrt{\gamma}} (q^{1/2}(\eta_c) - q^{1/2}(\eta_c^0)).$$

As $\eta_c - \eta_c^0 = \frac{\gamma^{3/2}}{t} \langle (\sigma, s), \int_0^1 (\Theta_1, \Theta_2) \left(o\sigma, os, \frac{t}{2N\sqrt{\gamma}}, \frac{x}{\gamma}, \frac{a}{\gamma}, \frac{y}{2t}, \gamma \right) do \rangle$ and

$$(49) \quad q^{1/2}(\eta_c) - q^{1/2}(\eta_c^0) = (\eta_c - \eta_c^0) \int_0^1 \left(\frac{\nabla q}{2q^{1/2}} \right) (o\eta_c^0 + (1-o)\eta_c) do,$$

defining \mathcal{E}_j as in (47) and (48) yields (46). \square

Corollary 1. *Applying the stationary phase in both α and η yields*

$$G_{h,\gamma}(t, x, y) = \frac{C}{h^d} \left(\frac{h}{t} \right)^{(d-1)/2} \tilde{\psi} \left(\frac{|y|}{2t} \right) \sum_{\frac{t}{\sqrt{\gamma}} \simeq N \lesssim \frac{1}{\sqrt{\gamma}}} V_{N,h,\gamma}(t, x, y) + O(h^\infty),$$

where $C \neq 0$ is a constant (independent of h, a, γ), $\tilde{\psi} \in C_0^\infty([\frac{1}{4}, 2])$ with $\tilde{\psi} = 1$ on the support of ψ and where we set

$$V_{N,h,\gamma}(t, x, y) = \frac{\gamma^2}{h} \frac{1}{\sqrt{\lambda_\gamma N}} \int e^{\frac{i}{h} \phi_{N,a,\gamma}(\sigma,s,t,x,y)} \mathcal{K}(\sigma, s, t, x, y; h, \gamma, 1/N) d\sigma ds,$$

with phase $\phi_{N,a,\gamma}(\sigma, s, t, x, y) = \Phi_{N,a,\gamma}(\eta_c, \alpha_c, \sigma, s, t, x, y)$ and symbols $\varkappa(\cdot; h, \gamma, 1/N)$ obtained from $q(\eta)\psi(|\eta|)\psi_2(\alpha)e^{iNB(q^{1/2}(\eta)\lambda_\gamma\alpha^{3/2})}$ after the stationary phase in η, α .

Remark 15. The new symbol $\varkappa(\cdot; h, \gamma, 1/N)$ has main contribution $q(\eta_c)\psi(|\eta_c|)\psi_2(\alpha_c)e^{iNB(q^{1/2}(\eta)\lambda_\gamma\alpha^{3/2})}$. Its dependence on the parameters $h, a, \gamma, 1/N$ is harmless as $\varkappa(\cdot, h, \gamma, 1/N)$ reads as an asymptotic expansion with small parameter $(\lambda_\gamma N)^{-1} = h/(N\gamma^{3/2})$ in α and small parameter (h/t) in η , and the terms in the expansions are smooth functions of α_c, η_c .

Remark 16. Notice that, due to Remark 13, we can introduce the cut-offs $\chi(\sigma/(2\sqrt{\alpha_c}))$ and $\chi(s/(2\sqrt{\alpha_c}))$ supported for $|(\sigma, s)| \leq 2\sqrt{\alpha_c}$ in the integral defining $V_{N,h,\gamma}$ without changing its contribution modulo $O(h^\infty)$.

We are left with integrals with respect to the variables s, σ and we need to estimate $\|V_{N,h,\gamma}(t, \cdot)\|_{L^\infty}$: in order to address this issue in a meaningful way, we first need to compute the higher order derivatives of the critical value $\Phi_{N,a,\gamma}(\eta_c, \alpha_c, s, \sigma, t, y, x)$. The critical points in σ, s satisfy the following system

$$(50) \quad \partial_\sigma \left(\Phi_{N,a,\gamma}(\eta_c, \alpha_c, s, \sigma, \cdot) \right) = \gamma^{3/2} q^{1/2}(\eta_c) \left(\sigma^2 + \frac{x}{\gamma} - \alpha_c \right),$$

$$(51) \quad \partial_s \left(\Phi_{N,a,\gamma}(\eta_c, \alpha_c, s, \sigma, \cdot) \right) = \gamma^{3/2} q^{1/2}(\eta_c) \left(s^2 + \frac{a}{\gamma} - \alpha_c \right).$$

The higher order derivatives of $\phi_{N,a,\gamma}(\sigma, s, \cdot) := \Phi_{N,a,\gamma}(\eta_c, \alpha_c, \sigma, s, \cdot)$ involve derivatives of the critical points α_c, η_c with respect to σ, s :

$$(52) \quad \partial_{\sigma,\sigma}^2 \left(\Phi_{N,a,\gamma}(\eta_c, \alpha_c, s, \sigma, \cdot) \right) = \partial_\sigma \eta_c \frac{\nabla q(\eta)}{2q(\eta)} \Big|_{\eta=\eta_c} \partial_\sigma \phi_{N,a,\gamma} + \gamma^{3/2} q^{1/2}(\eta_c) (2\sigma - 2\sqrt{\alpha_c} \partial_\sigma \sqrt{\alpha_c}),$$

$$(53) \quad \partial_{s,s}^2 \left(\Phi_{N,a,\gamma}(\eta_c, \alpha_c, s, \sigma, \cdot) \right) = \partial_s \eta_c \frac{\nabla q(\eta)}{2q(\eta)} \Big|_{\eta=\eta_c} \partial_s \phi_{N,a,\gamma} + \gamma^{3/2} q^{1/2}(\eta_c) (2s - 2\sqrt{\alpha_c} \partial_s \sqrt{\alpha_c}),$$

$$(54) \quad \partial_{\sigma,s}^2 \left(\Phi_{N,a,\gamma}(\eta_c, \alpha_c, s, \sigma, \cdot) \right) = \partial_\sigma \eta_c \frac{\nabla q(\eta)}{2q(\eta)} \Big|_{\eta=\eta_c} \partial_s \phi_{N,a,\gamma} - \gamma^{3/2} q^{1/2}(\eta_c) (2\sqrt{\alpha_c} \partial_\sigma \sqrt{\alpha_c}),$$

and therefore, when $\partial_\sigma \phi_{N,a,\gamma} = \partial_s \phi_{N,a,\gamma} = 0$, we have

$$\begin{aligned} \partial_{\sigma,\sigma}^2 \phi_{N,a,\gamma}(\eta_c, \alpha_c, s, \sigma, \cdot) \Big|_{\partial_\sigma \phi_{N,a,\gamma} = \partial_s \phi_{N,a,\gamma} = 0} &= 2\gamma^{3/2} q^{1/2}(\eta_c) (\sigma - \sqrt{\alpha_c} \partial_\sigma \sqrt{\alpha_c}), \\ \partial_{s,s}^2 \phi_{N,a,\gamma}(\eta_c, \alpha_c, s, \sigma, \cdot) \Big|_{\partial_\sigma \phi_{N,a,\gamma} = \partial_s \phi_{N,a,\gamma} = 0} &= 2\gamma^{3/2} q^{1/2}(\eta_c) (s - \sqrt{\alpha_c} \partial_s \sqrt{\alpha_c}), \\ \partial_{\sigma,s}^2 \phi_{N,a,\gamma}(\eta_c, \alpha_c, s, \sigma, \cdot) \Big|_{\partial_\sigma \phi_{N,a,\gamma} = \partial_s \phi_{N,a,\gamma} = 0} &= -2\gamma^{3/2} q^{1/2}(\eta_c) \sqrt{\alpha_c} \partial_\sigma \sqrt{\alpha_c}. \end{aligned}$$

Remark 17. Notice that, at the critical points we indeed have $\partial_\sigma \sqrt{\alpha_c} = \partial_s \sqrt{\alpha_c}$: in fact, the derivatives of α_c depend on η_c that solves (34); from (34), the derivatives of η_c with respect to σ (and s , respectively) depend upon $(s + \sigma)$, $\sigma^2 + \frac{x}{\gamma}$ and $\sigma^3/3 + \sigma \frac{x}{\gamma} + s^3/3 + s \frac{a}{\gamma}$ (and upon $(s + \sigma)$, $s^2 + \frac{a}{\gamma}$ and $\sigma^3/3 + \sigma \frac{x}{\gamma} + s^3/3 + s \frac{a}{\gamma}$); as at the critical points σ, s we have $\sigma^2 + \frac{x}{\gamma} = s^2 + \frac{a}{\gamma} = \alpha_c$, we find $\partial_\sigma \eta_c = \partial_s \eta_c$.

3.1.1. *Tangent waves* $a \in [\frac{1}{8}\gamma, 8\gamma]$. To deal with the situation $\gamma \simeq a$ it will be crucial to obtain very precise dispersive bounds for $V_{N,h,a}$: such bounds are given in Propositions 3, 4 and 5 below (similar to those obtained for the wave flow in [17]). In this section we show how they can be applied to prove Theorem 1 for $a \geq (ht)^{1/2}$.

To eliminate any confusion, we write in this case $G_{h,a}$ instead of $G_{h,\gamma \simeq a}$, $\lambda = a^{3/2}/h$ instead of $\lambda_{\gamma \simeq a}$ and we recall from Corollary 1 that

$$(55) \quad G_{h,a}(t, x, y) = \frac{C}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \tilde{\psi}\left(\frac{|y|}{2t}\right) \sum_{\frac{t}{\sqrt{a}} \simeq N \lesssim \frac{1}{\sqrt{a}}} V_{N,h,a}(t, x, y) + O(h^\infty),$$

where $C \neq 0$ is constant (independent of h, a), $\tilde{\psi} \in C_0^\infty([1/4, 2])$ with $\tilde{\psi} = 1$ on the support of ψ and

$$(56) \quad V_{N,h,a}(t, x, y) = \frac{a^2}{h} \frac{1}{\sqrt{\lambda N}} \int e^{i\frac{1}{h}\phi_{N,a}(\sigma, s, t, x, y)} \varkappa(\sigma, s, t, x, y, h, a, 1/N) d\sigma ds,$$

where $\phi_{N,a}(\sigma, s, t, x, y) = \Phi_{N,a,a}(\eta_c, \alpha_c, \sigma, s, t, x, y)$ and where the symbol \varkappa is obtained from $q(\eta)\psi(|\eta|)\psi_2(\alpha)e^{iNB(q^{1/2}(\eta)\lambda\alpha^{3/2})}$ after the stationary phase in η, α .

Remark 18. Recall from Remark 14 that we consider here only values $N \lesssim \lambda$. After a careful examination of the second order derivatives of $\phi_{N,a}$, it turns out that one needs to deal separately with the cases $N < \lambda^{1/3}$ and $\lambda^{1/3} \lesssim N$. Fix t and set $T = \frac{t}{\sqrt{a}}$: if $\lambda^{1/3} \lesssim T \simeq N$, then $\phi_{N,a}$ behaves like the phase of a product of two Airy functions and can be bounded using mainly their asymptotic behaviour.

When $N \simeq T \lesssim \lambda^{1/3}$, the situation is more delicate as $\phi_{N,a}$ may have degenerate critical points up to order 3. We will show that for any t such that $T := \frac{t}{\sqrt{a}} \ll \lambda^{1/3}$ and for any $N \simeq T$ there exists a locus of points $\mathcal{Y}_N(T) := \{Y \in \mathbb{R}^{d-1} | K_a(\frac{Y}{4N}, \frac{T}{4N}) = 1\}$, where K_a is the smooth function defined in (57) such that, for all $Y \in \mathcal{Y}_N(T)$ we have $\|G_{h,a}(t, \cdot)\|_{L^\infty(\Omega)} = |G_{h,a}(t, a, a, y)|_{y \in \sqrt{a}\mathcal{Y}_N(t/\sqrt{a})} \simeq \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} a^{1/4} \left(\frac{h}{t}\right)^{1/4}$ and for all $(ht)^{1/2} \lesssim a \lesssim \varepsilon_0$, which will prove that the announced estimates are optimal.

Remark 19. When dealing with the wave flow inside the same domain, a parametrix is also obtained as a sum of reflected waves: due to the finite speed of propagation, the main contribution in this sum at a given moment t comes from the waves that are located between the $(N-1)$ th and $(N+1)$ th reflections, where $N = [\frac{t}{\sqrt{a}}]$. For each $N \ll \lambda^{1/3}$ there exists a unique time t_N when the worst bound is reached and this happens at $x = a$ and for a unique y_N ; in the case of the Schrödinger equation, for any $t/\sqrt{a} \ll \lambda^{1/3}$ and any $N \simeq t/\sqrt{a}$ there is a locus of points $\mathcal{Y}_N(t/\sqrt{a})$ such that, for $x = a$, $|V_{N,h,a}(t, a, y)|_{y \in \sqrt{a}\mathcal{Y}_N(t/\sqrt{a})} \simeq \|G_{h,a}(t, \cdot)\|_{L^\infty}$, where $\mathcal{Y}_N(t/\sqrt{a}) \cap \mathcal{Y}_{N'}(t/\sqrt{a}) = \emptyset$ for all $N \neq N'$.

We denote α_c^0 the value of the critical point α_c obtained in (46) at $\sigma = s = 0$. Recall from Lemma 4 (with γ replaced by a), that $\eta_c^0 = -\frac{y}{2t} + a\Theta(\frac{y}{2t}, \frac{t}{2N\sqrt{a}}, a)$ is a smooth function of $(\frac{y}{2t}, \frac{t}{2N\sqrt{a}}, a)$, hence so is $\sqrt{\alpha_c^0} = \frac{t}{2N\sqrt{a}} q^{1/2}(\eta_c^0)$. Let $T = t/\sqrt{a}$, $Y = y/\sqrt{a}$ and define

$K_a(\frac{Y}{4N}, \frac{T}{2N}) = \sqrt{\alpha_c^0(\frac{Y}{4N}, \frac{2N}{T}, \frac{T}{2N}, a)}$. We then have

$$(57) \quad K_a\left(\frac{Y}{4N}, \frac{T}{2N}\right) = \frac{|Y|}{4N} q^{1/2} \left(-\frac{Y}{|Y|} + a \frac{T}{2N} \frac{4N}{|Y|} \Theta\left(\frac{Y}{4N}, \frac{2N}{T}, \frac{T}{2N}, a\right) \right),$$

with K_a smooth in all variables. The following estimates, proven in Section 4, are crucial in the proof of Theorem 1 :

Proposition 3. *For $\lambda^{1/3} \lesssim T \simeq N$, we have*

$$|V_{N,h,a}(t, x, y)| \lesssim \frac{h^{1/3}}{\left((N/\lambda^{1/3})^{1/2} + \lambda^{1/6} \sqrt{4N} |K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1|^{1/2}\right)}.$$

Proposition 4. *For $1 \leq N < \lambda^{1/3}$ and $|K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1| \gtrsim 1/N^2$, we have*

$$(58) \quad |V_{N,h,a}(t, x, y)| \lesssim \frac{h^{1/3}}{\left(1 + 2N |K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1|^{1/2}\right)}.$$

Proposition 5. *For $1 \leq N < \lambda^{1/3}$ and $|K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1| \leq \frac{1}{4N^2}$, we have*

$$(59) \quad |V_{N,h,a}(t, x, y)| \lesssim \frac{h^{1/3}}{\left((N/\lambda^{1/3})^{1/4} + N^{1/3} |K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1|^{1/6}\right)}.$$

We postpone the proofs of Propositions 3, 4 and 5 to Section 4 and we achieve the proof of Theorem 1 in the case $(ht)^{1/2} \lesssim a \simeq \gamma \leq \varepsilon_0 < 1$. Let therefore $\sqrt{a} \lesssim t \lesssim 1$ be fixed and let $N_t \geq 1$ be the unique positive integer such that $T = \frac{t}{\sqrt{a}} \leq N_t < \frac{t}{\sqrt{a}} + 1 = T + 1$, hence $N_t = [T]$, where $[T]$ denotes the integer part of T . If N_t is bounded then the number of $V_{N,h,a}$ with $N \simeq N_t$ in the sum (55) is also bounded and we can easily conclude adding the (worst) bound (5) a finite number of times. Assume $N_t \geq 2$ is large enough. We introduce the following notation: for $k \in \mathbb{Z}$ let

$$I_{N_t, k} := [4(N_t + k) - 2, 4(N_t + k) + 2].$$

As $\alpha_c, \eta_c \in [\frac{1}{2}, \frac{3}{2}]$ and $\sqrt{\alpha_c} = \frac{T}{2N} q^{1/2}(\eta_c) - \frac{(\sigma+s)}{2N}$ with $|(\sigma, s)| \leq 2\sqrt{\alpha_c}$ on the support of χ (see Remark 16), we deduce (using (35)) that, for M defined in (42), we have $2N \in [\frac{T}{M}, MT] \subset [\frac{N_t}{M}, M(N_t + 1)]$. Using (55), we then bound $G_{h,a}(t, \cdot)$ as follows

$$\|G_{h,a}(t, \cdot)\|_{L^\infty(0 \leq x \leq a, y)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \sup_{x \leq a, y} \sum_{N_t/M \leq 2N \leq M(N_t+1)} |V_{N,h,a}(t, x, y)|.$$

It will follow from the proofs of Propositions 3, 4 and 5 that the worst dispersive bounds for $V_{N,h,a}$ always occur at $x = a$ (when $\phi_{N,a}$ may have a critical point of order 3). Therefore, we will seek for bounds for $G_{h,a}$ especially at $x = a$.

For a fixed y on the support of $\tilde{\psi}(\frac{|y|}{2t})$ we let $Y = \frac{y}{\sqrt{a}}$, then $\frac{1}{4} \leq \frac{|Y|}{2T} \leq 2$, and therefore $|Y| \in [T/2, 4T] \subset [N_t/2, 4(N_t + 1)]$. We want to study the set of points where K_a may get close to 1 : using (57) and the fact that $q^{1/2}$ is homogeneous of order 1, this happens

when $q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a))$ is sufficiently close to $4N$. As $2 < N_t \leq T \leq 1/\sqrt{a}$, $|Y|/T \in [1/2, 4]$, Θ is bounded and $0 < a \leq \varepsilon_0$ is small,

$$\begin{aligned} q^{1/2}\left(-Y + 2aT\Theta\left(\frac{Y}{2T}, \frac{T}{2N}, a\right)\right) &\in |Y|[m_0 + O(a), M_0 + O(a)] \\ &\subset [N_t(m_0 - \varepsilon_0)/2, 4(N_t + 1)(M_0 + \varepsilon_0)], \end{aligned}$$

where m_0 and M_0 are defined in (31). Setting

$$k_1 = -N_t(1 - (m_0 - \varepsilon_0)/8), \quad k_2 = (N_t + 1)(M_0 + \varepsilon_0 - 1) - 1,$$

we have $N_t + k \simeq N_t$ and

$$[N_t(m_0 - \varepsilon_0)/2, 4(N_t + 1)(M_0 + \varepsilon_0)] \subset \cup_{k_1 \leq k \leq k_2} I_{N_t, k}.$$

Let $\tilde{I}_{N_t, k} := (4(N_t + k) - 1, 4(N_t + k) + 1) \subset I_{N_t, k}$. We now write

$$\begin{aligned} (60) \quad &\sup_{x, y} \sum_{N_t/M \leq 2N \leq M(N_t+1)} |V_{N, h, a}(t, x, y)| \\ &= \sup_{k_1 \leq k \leq k_2} \sup_{q^{1/2}\left(-Y + 2aT\Theta\left(\frac{Y}{2T}, \frac{T}{2N}, a\right)\right) \in I_{N_t, k}} \sum_{N_t/M \leq 2N \leq M(N_t+1)} |V_{N, h, a}(t, a, y)| \\ &\geq \sup_{k_1 \leq k \leq k_2} \sup_{q^{1/2}\left(-Y + 2aT\Theta\left(\frac{Y}{2T}, \frac{T}{2N}, a\right)\right) \in \tilde{I}_{N_t, k}} \sum_{N_t/M \leq 2N \leq M(N_t+1)} |V_{N, h, a}(t, a, y)|. \end{aligned}$$

In the next three propositions we prove dispersive bounds for $G_{h, a}(t, \cdot)$: we start with the case $\lambda^{1/3} \ll N_t (\lesssim \lambda)$, when only Proposition 3 applies, then we deal with the case $\lambda^{1/3} \gg N_t$ when only Propositions 4 and 5 may be used and eventually we deal with the case $N_t \simeq \lambda^{1/3}$ when we can encounter all the three situations.

Proposition 6. *There exists a constant $C > 0$ independent on h, a such that, if $N_t := \lfloor \frac{t}{\sqrt{a}} \rfloor \gg \lambda^{1/3}$, then*

$$\|G_{h, a}(t, \cdot)\|_{L^\infty(\Omega_d)} \leq \frac{C}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \left(\left(\frac{ht}{a}\right)^{1/2} + h^{1/3}\right).$$

Remark 20. *Notice that in this regime we always have $h^{1/3} \ll \left(\frac{ht}{a}\right)^{1/2}$ as this is equivalent to $\frac{t}{\sqrt{a}} \gg \frac{\sqrt{a}}{h^{1/3}} = \lambda^{1/3}$, which is the case we consider here. As $(ht)^{1/2} \lesssim a$, we also have $\left(\frac{ht}{a}\right)^{1/2} \lesssim (ht)^{1/4}$, and for $t \leq 1$ this yields a dispersive bound which is better than announced in Theorem 1.*

Proof. If $\lambda^{1/3} \ll N_t$, then $N_t + k \gg \lambda^{1/3}$ for all $k \in [k_1, k_2]$ and we estimate the L^∞ norms of $G_{h, a}(t, \cdot)$ using the first two lines of (60) and Proposition 3: if $k_y \in [k_1, k_2]$ is such that $q^{1/2}(-Y) \in I_{N_t, k_y}$, then, using again that a is small, it follows that $4NK_a(\frac{Y}{4N}, \frac{T}{2N}) =$

$q^{1/2}\left(-Y + 2aT\Theta\left(\frac{Y}{2T}, \frac{T}{2N}, a\right)\right) \in \cup_{|k'-k_y|\leq 1} I_{N_t, k'}$ and therefore the second line in (60) can be (uniformly) bounded as follows

$$(61) \quad \sup_{k_1 \leq k \leq k_2} \sup_{4NK_a\left(\frac{Y}{4N}, \frac{T}{2N}\right) \in I_{N_t, k}} \sum_{2N \in [N_t/M, M(N_t+1)]} |V_{N, h, a}(t, a, y)| \\ \leq \sup_{|k'-k_y|\leq 1} \sup_{4NK_a\left(\frac{Y}{4N}, \frac{T}{2N}\right) \in I_{N_t, k'}} \sum_{2N \in [N_t/M, M(N_t+1)]} |V_{N, h, a}(t, a, y)| \\ \leq \sup_{4NK_a\left(\frac{Y}{4N}, \frac{T}{2N}\right) \in \cup_{|k'-k_y|\leq 1} I_{N_t, k'}} \sum_{2N \in [N_t/M, M(N_t+1)]} \frac{h^{1/3}}{\left((N/\lambda^{1/3})^{1/2} + \lambda^{1/6} \left|4NK_a\left(\frac{Y}{4N}, \frac{T}{2N}\right) - 4N\right|^{1/2}\right)}.$$

As $4NK_a\left(\frac{Y}{4N}, \frac{T}{2N}\right) \in \cup_{|k'-k_y|\leq 1} I_{N_t, k'}$, we find, for $N = N_t + k_y + j$ and $|j| \geq 2$, that

$$\left|4NK_a\left(\frac{Y}{4N}, \frac{T}{2N}\right) - 4N\right| \geq |j| - 1,$$

and therefore the last line in (61) can be further bounded by

$$(62) \quad \frac{h^{1/3}(N_t + k_y)^{1/2}}{(N_t + k_y)} \left(3\lambda^{1/6} + \sum_{|N - (N_t + k_y)| = |j| \geq 2} \frac{1}{(1 + j/(N_t + k_y))\lambda^{-1/6} + \lambda^{1/6}(|j| - 1)/(N_t + k_y)^{1/2}}\right).$$

As $\lambda^{1/3} \ll \frac{t}{\sqrt{a}} \simeq N_t \simeq N_t + k_y$, we bound the first term in the last sum as follows

$$\frac{h^{1/3}\lambda^{1/6}}{(N_t + k_y)^{1/2}} \simeq h^{1/3} \left(\frac{\lambda^{1/3}}{(t/\sqrt{a})}\right)^{1/2} \leq h^{1/3}.$$

The sum over $N > N_t + k_y + 1$ reads as

$$(63) \quad \frac{h^{1/3}(N_t + k_y)^{1/2}}{\lambda^{1/6}(N_t + k_y)} \sum_{N=N_t+k_y+1+j, j \geq 1} \frac{1}{(1 + (j+1)/(N_t + k_y))\lambda^{-1/3} + |j/(N_t + k_y)|^{1/2}} \\ \leq h^{1/3} \frac{(N_t + k_y)^{1/2}}{\lambda^{1/6}} \int_0^1 \frac{dx}{\sqrt{x} + \lambda^{-1/3}(1 + (N_t + k_y)^{-1} + x)^{1/2}},$$

and the last integral is bounded by $\sqrt{x}\Big|_0^1 = \frac{1}{2}$. The sum over $N < N_t + k_y$ reads as

$$(64) \quad \frac{h^{1/3}(N_t + k_y)^{1/2}}{\lambda^{1/6}(N_t + k_y)} \sum_{N=N_t+k_y-1-j \geq N_t/(2M), j \geq 1} \frac{1}{(1 - (j+1)/(N_t + k_y))\lambda^{-1/3} + |j/(N_t + k_y)|^{1/2}} \\ = h^{1/3} \frac{(N_t + k_y)^{1/2}}{\lambda^{1/6}} \int_0^{1 - (N_t + k_y)^{-1}(1 + N_t/(2M))} \frac{dx}{\sqrt{x} + \lambda^{-1/3}(1 - (N_t + k_y)^{-1} - x)^{1/2}},$$

where the last integral is taken on $[0, 1 - (N_t + k_y)^{-1}(1 + N_t/(2M))]$ as in the previous sum the following restriction $1 - (N_t + k_y)^{-1}(1 + N_t/(2M)) \geq j/(N_t + k_y)$ holds. As $k_y \geq k_1$,

we have $N_t + k_y \geq N_t(1 + (m_0 - \varepsilon_0)/8)$ and therefore, using (42),

$$\frac{N_t}{2M(N_t + k_y)} \leq \frac{4}{M(m_0 - \varepsilon_0)} \leq \frac{1}{\sqrt{3/2}}.$$

As the integral is bounded by $\frac{1}{2}$, the contribution coming from the sum over $|N - (N_t + k_y)| \geq 2$ in (62) is $h^{1/3}(N_t + k_y)^{1/2}/\lambda^{1/6}$ and as $N_t + k_y \leq (N_t + 1)(M_0 + \varepsilon_0 - 1)$ where M_0 is fixed, depending only on q , and $N_t \leq \frac{t}{\sqrt{a}}$, we obtain

$$(65) \quad \sup_{4NK_a(\frac{Y}{4N}, \frac{T}{2N}) \in \cup_{|k' - k_y| \leq 1} I_{N_t, k'} \quad 2N \in [N_t/M, M(N_t + 1)]} |V_{N, h, a}(t, a, y)| \leq \sqrt{M_0} h^{1/3} \left(\frac{t}{\sqrt{a}}\right)^{1/2} \lambda^{-1/6},$$

and $h^{1/3}(\frac{t}{\sqrt{a}})^{1/2} \lambda^{-1/6} = \left(\frac{ht}{a}\right)^{1/2}$. We can conclude as at fixed y there are at most three values of k that satisfy $q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a)) \in I_{N_y, k}$, that the bounds in (65) are independent of k , and that the intervals $I_{N_t, k}$ are disjoint. \square

Before dealing with the case $N_t \lesssim \lambda^{1/3}$, we need to introduce one more notation. As a is small, for a fixed $Y = \frac{y}{\sqrt{a}}$ there exists at most one value of k such that $q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a)) \in \tilde{I}_{N_t, k}$. If y is such that $q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a)) \in \tilde{I}_{N_t, k}$ for some $k_1 \leq k \leq k_2$, then k is unique and we denote it $k_y^\#$ (recall that t is fixed). If $2(N_t + k_y^\#) \in [N_t/M, M(N_t + 1)]$, we can either have $\lambda^{1/3} \lesssim N_t + k_y^\#$, or $N_t + k_y^\# < \lambda^{1/3}$ (notice that this last situation always occur if $N_t \ll \lambda^{1/3}$ as $k_y^\# \leq k_2 < 2M_0 N_y$ and M_0 is fixed, depending only on q).

Remark 21. When $N_t + k_y^\# < \lambda^{1/3}$, Proposition 5 may apply only for $N = N_t + k_y^\#$, as for $k_y^\# \neq k \in [k_1, k_2]$ and $n = N_t + k$ we necessarily have

$$(66) \quad \left| q^{1/2} \left(-Y + 2aT\Theta\left(\frac{Y}{2T}, \frac{T}{2N}, a\right) \right) - 4n \right| \geq 4|n - (N_t + k_y^\#)| \\ - \left| q^{1/2} \left(-Y + 2aT\Theta\left(\frac{Y}{2T}, \frac{T}{2N}, a\right) \right) - 4(N_t + k_y^\#) \right| \geq 3 \gg \frac{1}{n}.$$

Proposition 7. There exists a constant $C > 0$ independent on h, a such that, if $N_t := \lfloor \frac{t}{\sqrt{a}} \rfloor \ll \lambda^{1/3}$, then

$$(67) \quad \|G_{h, a}(t, \cdot)\|_{L^\infty(\Omega_a)} \simeq \frac{C}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \left(\frac{ha}{t}\right)^{1/4}.$$

Proof. If y is such that $q^{1/2}(-Y) \in I_{N_t, k_y}$ for some $k_y \in [k_1, k_2]$, using that $a \leq \varepsilon_0$ is small enough, it follows that

$$\left| q^{1/2} \left(-Y + 2aT\Theta\left(\frac{Y}{2T}, \frac{T}{2N}, a\right) \right) - 4n \right| \geq 4|n - (N_t + k_y)| - \left| q^{1/2} \left(-Y + 2aT\Theta\left(\frac{Y}{2T}, \frac{T}{2N}, a\right) \right) - 4(N_t + k_y) \right|$$

for all $n \neq N_t + k_y$; the second term in the right hand side is smaller than 2, while the first one is at least 4; therefore the assumption of Proposition 5 cannot hold for $n \neq N_t + k_y$.

For all such n we then use Proposition 4 that yields

$$\begin{aligned}
 (68) \quad & \sup_{q^{1/2}(-Y) \in I_{N_t, k_y}} \sum_{2n \in [N_t/M, M(N_t+1)], n \neq N_t + k_y} |V_{n, h, a}(t, a, y)| \\
 & \lesssim h^{1/3} \sum_{2n \in [N_t/M, M(N_t+1)], n \neq N_t + k_y} \frac{1}{(1 + |n(q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a)) - 4n)|^{1/2})} \\
 & \lesssim h^{1/3} \sum_{n=N_t+k_y+j, 1 \leq |j| \lesssim N_t} \frac{1}{(1 + (N_t + k_y + j)^{1/2}|j|^{1/2})} \\
 & \leq h^{1/3} \left(\int_0^{1-(N_t+k_y)^{-1}(1+N_t/(2M))} \frac{dx}{x^{1/2}(1-x)^{1/2} + (N_t + k_y)^{-1}} + \int_0^1 \frac{dx}{x^{1/2}(1+x)^{1/2} + (N_t + k_y)^{-1}} \right),
 \end{aligned}$$

where the last two integrals are uniform bounds for the sum over $N < N_t + k_y$ and $N > N_t + k_y$, respectively; when $N > N_t + k_y$, the integral over $[0, 1]$ is bounded by a uniform constant ; when $N < N_t + k_y$ we write $x = \sin^2 \theta$, $\theta \in [0, \pi/2)$ and therefore $1 - x = \cos^2 \theta$, $dx = 2 \sin \theta \cos \theta$, and therefore the first integral is also bounded by at most π . This bound is also uniform with respect to $k_y \in [k_1, k_2]$.

We are left with $N = N_t + k_y$. If $q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a)) \notin \tilde{I}_{N_t, k_y}$, then we use again Proposition 4. If, on the contrary, $q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a)) \in \tilde{I}_{N_t, k_y}$, then $k_y^\# = k_y \in [k_1, k_2]$ and we may be able to apply Proposition 5 with $N = N_t + k_y^\#$ if moreover the following holds: $|q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a)) - 4N| \lesssim \frac{1}{N}$; if this is not the case we apply again Proposition 4 for $N = N_t + k_y^\#$. We then have

$$\begin{aligned}
 (69) \quad & \sup_{q^{1/2}(-Y) \in I_{N_t, k_y}} |V_{N=N_t+k_y, h, a}(t, a, y)| \\
 & \lesssim \frac{h^{1/3}}{(N/\lambda^{1/3})^{1/4}} + \frac{h^{1/3}}{(1 + |N(q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a)) - 4N)|^{1/2})} \\
 & \simeq \left(\frac{ha}{t}\right)^{1/4} + h^{1/3}.
 \end{aligned}$$

As for $N_t \simeq \frac{t}{\sqrt{a}} \ll \frac{\sqrt{a}}{h^{1/3}} = \lambda^{1/3}$ we have $h^{1/3} \ll \left(\frac{ha}{t}\right)^{1/4}$ it follows that, at fixed t , the supremum of the sum over $V_{N, h, a}(t, x, y)$ is reached for $x = a$ and y such that $q^{1/2}(-Y + 2aT\Theta(\frac{Y}{2T}, \frac{T}{2N}, a)) = 4N$ where $N = N_t + k_y$. As the contribution from (68) the sum over $n \neq N_t + k_y$ is $h^{1/3}$, we obtain an upper bound for $G_{h, a}(t, \cdot)$. Using the last line of (60) and the fact that $h^{1/3} \ll \left(\frac{ha}{t}\right)^{1/4}$ yields a lower bound for $G_{h, a}(t, \cdot)$ of the same form and therefore (70) holds true. \square

Proposition 8. *There exists a constant $C > 0$ independent on h, a such that, if $N_t := \lfloor \frac{t}{\sqrt{a}} \rfloor \simeq \lambda^{1/3}$, then*

$$(70) \quad \|G_{h,a}(t, \cdot)\|_{L^\infty(\Omega_a)} \leq \frac{C}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \left(\left(\frac{ha}{t}\right)^{1/4} + \left(\frac{ht}{a}\right)^{1/2} + h^{1/3} \right).$$

Remark 22. *When $N_t \simeq \lambda^{1/3}$ we find $a \simeq h^{1/3}t$ and all the terms in brackets in the right hand side of (70) behave like $h^{1/3}$, hence $\|G_{h,a}(t, \cdot)\|_{L^\infty(\Omega_a)} \leq \frac{C}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} h^{1/3}$.*

Proof. If $N_t \simeq \lambda^{1/3}$ and $k \simeq N_t$, we split according to whether y is such that $N_t + k_y < \lambda^{1/3}$ or $N_t + k_y \geq \lambda^{1/3}$ and proceed as in the previous cases using Propositions 3, 4 and 5. As, for such N_t , we have $(\frac{ha}{t})^{1/4} \simeq h^{1/3} \simeq (\frac{th}{a})^{1/2}$, we cannot deduce that the supremum equals $(\frac{ha}{t})^{1/4}$ but obtain the uniform bound $h^{1/3}$ for $\gamma_{h,a,d}(\frac{t}{h})$. \square

3.1.2. *Transverse waves.* Let $\gamma > 8a$, $a \geq x \geq 0$.

Proposition 9. *Let $\frac{t}{2\sqrt{\gamma}} \geq \frac{8}{m_0}$, then*

$$(71) \quad \|G_{h,\gamma}(t, \cdot)\|_{L^\infty(0 \leq x \leq a, y)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \sqrt{\frac{th}{\gamma}}.$$

Moreover,

$$(72) \quad \sum_{\gamma=2^{2j}(ht)^{1/2}} \|G_{h,\gamma}(t, \cdot)\|_{L^\infty(0 \leq x \leq a, y)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} (ht)^{1/4}.$$

Proof. As $\frac{t}{2\sqrt{\gamma}} \geq \frac{8}{m_0}$ it follows, using (35), that only $V_{N,h,\gamma}(t, \cdot)$ with $N \geq 2$ may provide significant contributions in $G_{h,\gamma}(t, \cdot)$. Let therefore $N \geq 2$. We will show that the usual stationary phase applies in σ, s : the critical points satisfy $\sigma^2 + \frac{x}{\gamma} = \alpha_c$, $s^2 + \frac{a}{\gamma} = \alpha_c$ and since $\frac{x}{\gamma} \leq \frac{a}{\gamma} \leq \frac{1}{8}$ and $\alpha_c \in [\frac{1}{2}, \frac{3}{2}]$ on the support of ψ_2 , we find $|\sigma_c s_c| \geq |\alpha_c - \frac{a}{\gamma}| \geq \alpha_c - \frac{1}{8}$ and $|\sigma + s| \leq 2\sqrt{\alpha_c}$. Moreover, $\partial_\sigma^2 \phi_{N,a,\gamma} = 2(\sigma - \sqrt{\alpha_c} \partial_\sigma \sqrt{\alpha_c})$, $\partial_s^2 \phi_{N,a,\gamma} = 2(s - \sqrt{\alpha_c} \partial_s \sqrt{\alpha_c})$, $\partial_{\sigma,s}^2 \phi_{N,a,\gamma} = -2\sqrt{\alpha_c} \partial_s \sqrt{\alpha_c}$. At the critical points we do have $\partial_s \sqrt{\alpha_c} = \partial_\sigma \sqrt{\alpha_c}$ (see Remark 17). Using Lemma 5 yields

$$\partial_\sigma \sqrt{\alpha} = -\frac{1}{2N} (1 - \gamma(\mathcal{E}_1 + \sigma \partial_\sigma \mathcal{E}_1)), \quad \partial_s \sqrt{\alpha} = -\frac{1}{2N} (1 - \gamma(\mathcal{E}_2 + \sigma \partial_\sigma \mathcal{E}_2)),$$

where \mathcal{E}_j are smooth, bounded, defined in (47), (48). Putting all this together allows to obtain a lower bound for the determinant of the matrix of second order derivatives at the

stationary points as follows

(73)

$$\begin{aligned} \det \text{Hess}_{\sigma,s} \phi_{N,a,\gamma} |_{\nabla_{\sigma,s} \phi_{N,a,\gamma}=0} &= 4\gamma^3 \left| (\sigma - \sqrt{\alpha_c} \partial_\sigma \sqrt{\alpha_c})(s - \sqrt{\alpha_c} \partial_s \sqrt{\alpha_c}) - \alpha_c \partial_\sigma \sqrt{\alpha_c} \partial_s \sqrt{\alpha_c} \right| \\ &\geq 4\gamma^3 \left(|\sigma s| - |\sigma + s| \frac{\sqrt{\alpha_c}}{2N} (1 + O(\gamma)) \right) \Big|_{s=s_c, \sigma=\sigma_c} \geq 4\gamma^3 \left(\alpha_c - \frac{1}{8} - \frac{\alpha_c}{N} (1 + O(\gamma)) \right) \\ &\geq 4\gamma^3 \left| \frac{\alpha_c}{3} - \frac{1}{8} \right| \geq \frac{1}{6} \gamma^3 \quad \forall N \geq 2, \end{aligned}$$

where in the last line we have used that $\gamma \leq \varepsilon_0$ and that ε_0 can be taken sufficiently small such that $\frac{2}{3}\alpha_c \geq \frac{\alpha_c}{2}(1 + O(\varepsilon_0))$ for $\alpha_c \in [\frac{1}{2}, \frac{2}{3}]$; in the last inequality we have used that $\alpha_c \geq \frac{1}{2}$. The stationary phase applies in both σ and s for all $N \geq 2$ and provides a factor λ_γ^{-1} : the sum over transverse waves $G_{h,\gamma}$ with $\gamma > 8a$ can be bounded as follows

$$\begin{aligned} (74) \quad \|G_{h,\gamma}(t, \cdot)\|_{L^\infty(0 \leq x \leq a,y)} &\lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \sum_{N \simeq \frac{t}{\sqrt{\gamma}}} \frac{\gamma^2}{h} \times \frac{1}{\sqrt{N\lambda_\gamma}} \times \frac{1}{\lambda_\gamma} \\ &\simeq \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \frac{\gamma^2}{h} \times \frac{1}{\lambda_\gamma^{3/2}} \left(\frac{t}{\sqrt{\gamma}}\right)^{1/2} \simeq \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \sqrt{\frac{th}{\gamma}}. \end{aligned}$$

Taking the sum over dyadic $(ht)^{1/2} \leq \gamma \leq \varepsilon_0$ yields (72). \square

Proposition 10. *Let $\frac{a}{\gamma} \lesssim \frac{t}{2\sqrt{\gamma}} \leq \frac{8}{m_0}$, then*

$$\|G_{h,\gamma}(t, \cdot)\|_{L^\infty(0 \leq x \leq a,y)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \left(h^{1/3} + \frac{h^{1/2}}{\gamma^{1/4}}\right).$$

Moreover,

$$\sum_{\gamma=2^{2j}(ht)^{1/2}} \|G_{h,\gamma}(t, \cdot)\|_{L^\infty(0 \leq x \leq a,y)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} (h^{1/3} \log(h) + h^{1/4} \left(\frac{h}{t}\right)^{1/8}).$$

Proof. For bounded values of $t/\sqrt{\gamma}$, only a bounded number of $V_{N,h,\gamma}$ may provide non-trivial contributions in $G_{h,\gamma}(t, \cdot)$; in particular, we can have $N = 1$. Since for $N = 2$ we can use the preceding bounds (and sum only a small, uniformly bounded number of them), we focus on $N = 1$. Notice that if $|s + \sigma| < 5\sqrt{\alpha_c}/4$, then

$$\det \text{Hess}_{\sigma,s} \phi_{1,a,\gamma} |_{\nabla_{\sigma,s} \phi_{1,a,\gamma}=0} \geq 4\gamma^3 \left(\alpha_c - \frac{1}{8} - \frac{5}{8} \alpha_c \right) \geq \frac{1}{2} \gamma^3 (3\alpha_c - 1) \geq \gamma^3/4$$

and we can act as in the proof of (74). We are left with the case $|s + \sigma| \geq 5\sqrt{\alpha_c}/4$: making the change of variables $\xi_1 = s + \sigma$ and $\xi_2 = s - \sigma$ yields a new phase for which the usual stationary method applies in ξ_2 whenever $\xi_1 > c$ for any small constant $c > 0$; as we consider here $|\xi_1| = |s + \sigma| \geq 5\sqrt{\alpha_c}/4$, where now $\alpha_c = \alpha_c((\xi_1 - \xi_2)/2, (\xi_1 + \xi_2)/2)$, this is indeed the case. We state that the critical value may be degenerate in ξ_1 of order at most two : this follows in the same way as in Lemma 7 from the last section. As such, the

contribution on the set $|\xi_1| \geq c$ of $V_{1,h,\gamma}(t, \cdot)$ is $\lambda_\gamma^{-1/2} \times \lambda_\gamma^{-1/3}$, where the first factor comes from the non-degenerate stationary phase in ξ_2 and the second one is coming applying Van der Corput in ξ_1 . This yields

$$(75) \quad \|V_{1,h,\gamma}(t, \cdot)\|_{L^\infty(0 \leq x \leq a,y)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \frac{\gamma^2}{h} \times \frac{1}{\lambda_\gamma^{1/2+5/6}} = \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} h^{1/3}.$$

Taking the sum over $\gamma > h^{2/3-\epsilon}$, yields

$$\sum_{\gamma=2^{2j}(ht)^{1/2}} \|V_{1,h,\gamma}(t, \cdot)\|_{L^\infty(0 \leq x \leq a,y)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} h^{1/3} \log(h).$$

For $2 \leq N$ bounded we obtain as before

$$\|V_{N,h,\gamma}(t, \cdot)\|_{L^\infty(0 \leq x \leq a,y)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \frac{\gamma^2}{h} \times \frac{1}{\lambda_\gamma^{3/2}} = \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \frac{h^{1/2}}{\gamma^{1/4}},$$

and summing up in γ yields $\sum_{\gamma=2^{2j}(ht)^{1/2}} \|V_{N,h,\gamma}(t, \cdot)\|_{L^\infty(0 \leq x \leq a,y)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} h^{1/4} \left(\frac{h}{t}\right)^{1/8}$. \square

3.2. Case $a \lesssim \max(h^{2/3-\epsilon}, (ht)^{1/2})$ for some small $\epsilon > 0$.

3.2.1. *The sum over $8 \max(h^{2/3-\epsilon}, (ht)^{1/2}) \leq \gamma \leq \varepsilon_0$.* This part is easy to deal with as we can apply the estimates obtained in the previous section. As we necessarily have $8a \leq \gamma$ and as in this regime we can use the parametrix (26), we get from (74)

$$(76) \quad \left\| \sum_{8a \lesssim 8 \max(h^{2/3-\epsilon}, (ht)^{1/2}) \leq \gamma \leq \varepsilon_0} G_{h,\gamma}(t, \cdot) \right\|_{L^\infty(0 \leq x \leq a,y)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} (ht)^{1/4}.$$

3.2.2. *The sum over $\max(a, h^{2/3}) \lesssim \gamma \lesssim \max(h^{2/3-\epsilon}, (ht)^{1/2})$.* This part will be entirely dealt with using the formula (18) and next Lemma.

Lemma 6. (see [18]) *There exists C_0 such that for $L \geq 1$ the following holds true*

$$(77) \quad \sup_{b \in \mathbb{R}} \left(\sum_{1 \leq k \leq L} \omega_k^{-1/2} Ai^2(b - \omega_k) \right) \leq C_0 L^{1/3}.$$

Moreover, we have

$$(78) \quad \sup_{b \in \mathbb{R}_+} \left(\sum_{1 \leq k \leq L} \omega_k^{-1/2} h^{2/3} Ai^2(b - \omega_k) \right) \leq C_0 h^{2/3} L.$$

Although the proof of (77) has been given in [18], we recall it here since it is useful to understand how the same arguments can be used in order to obtain (83) below.

Proof. From $|Ai(z)| \leq C(1 + |z|)^{-1/4}$, we get

$$J(b) := \sum_{1 \leq k \leq L} \omega_k^{-1/2} Ai^2(b - \omega_k) \lesssim \sum_{1 \leq k \leq L} \omega_k^{-1/2} \frac{1}{1 + |b - \omega_k|^{1/2}}.$$

Using $\omega_k \simeq k^{2/3}$, we get easily with C independent of L and D large enough

$$\sup_{b \leq 0} J(b) \leq CL^{1/3}, \quad \sup_{b \geq DL^{2/3}} J(b) \leq CL^{1/3}.$$

Thus we may assume $b = L^{2/3}b'$ with $b' \in [0, D]$. As $\omega_k = k^{2/3}g(k)$ with g being an elliptic symbol of degree 0, we are left to prove that

$$I(b') = L^{-1/3} \sum_{1 \leq k \leq L} (k/L)^{-1/3} \frac{1}{1 + L^{1/3}|b' - (k/L)^{2/3}|^{1/2}}$$

satisfies $\sup_{b' \in \mathbb{R}} I(b') \leq C_0 L^{1/3}$. We split $[0, 1]$ into a finite union of intervals on which the function $\frac{t^{-1/3}}{1 + L^{1/3}|b' - t^{2/3}|^{1/2}}$ is monotone : as each term in the sum is bounded by 1, we get

$$I(b') \lesssim Cte + L^{2/3} \int_0^1 \frac{t^{-1/3}}{1 + L^{1/3}|b' - t^{2/3}|^{1/2}} dt \leq Cte + L^{1/3} \int_0^1 \frac{3}{2|b' - s|^{1/2}} ds,$$

which achieves the proof of (77). Inequality (78) follows from similar arguments together with $|Ai'(z)| \leq C(1 + |z|)^{1/4}$. \square

Write, for $\gamma_{\max} := \max(h^{2/3-\epsilon}, (ht)^{1/2})$, $\gamma_{\min} := \max(a, h^{2/3})$,

$$\begin{aligned} (79) \quad & \sum_{\gamma_{\min} \leq \gamma \leq \gamma_{\max}} G_{h,\gamma}(t, x, a, y) = \sum_{\gamma_{\min} \leq \gamma \leq \gamma_{\max}} \frac{h^{1/3}}{h^d} \int e^{\frac{i}{h}\langle y, \eta \rangle} \psi(|\eta|) \\ & \times \sum_{\omega_k \leq \epsilon_0/h^{2/3}} e^{\frac{i}{h}t(|\eta|^2 + \omega_k h^{2/3} q^{2/3}(\eta))} \frac{q^{1/3}(\eta)}{L'(\omega_k)} \psi_2(h^{2/3} \omega_k / (q^{1/3}(\eta) \gamma)) \\ & \times Ai(xq^{1/3}(\eta)/h^{2/3} - \omega_k) Ai(aq^{1/3}(\eta)/h^{2/3} - \omega_k) d\eta \\ = & \sum_{k \simeq \lambda_\gamma, \gamma_{\min} \leq \gamma \leq \gamma_{\max}} \frac{h^{1/3}}{h^d} \int e^{\frac{i}{h}\langle y, \eta \rangle} \psi(|\eta|) e^{\frac{i}{h}t(|\eta|^2 + \omega_k h^{2/3} q^{2/3}(\eta))} \frac{q^{1/3}(\eta)}{L'(\omega_k)} \psi_2(h^{2/3} \omega_k / (q^{1/3}(\eta) \gamma)) \\ & \times Ai(xq^{1/3}(\eta)/h^{2/3} - \omega_k) Ai(aq^{1/3}(\eta)/h^{2/3} - \omega_k) d\eta + O(h^\infty), \end{aligned}$$

where we have used the fact that ψ_2 and ψ are supported on $[\frac{1}{2}, \frac{3}{2}]$ to deduce $k \simeq \omega_k^{3/2} \simeq \lambda_\gamma q^{1/2}(\eta) \simeq \lambda_\gamma$ on the support of $\psi_2(h^{2/3} \omega_k / (q^{1/3}(\eta) \gamma)) \psi(|\eta|)$; the term $O(h^\infty)$ comes from the (finite) sum over $1 \leq k \ll \lambda_\gamma$ and $\lambda_\gamma \ll k \lesssim 1/h$.

We are left with the sum over k in the last two lines of (79). Notice that if $t \leq h^{1/3-2\epsilon}$ then $(ht)^{1/2} \leq h^{2/3-\epsilon}$, which yields $\gamma_{\max} = h^{2/3-\epsilon}$ and therefore for such t we only need to consider values $a \leq h^{2/3-\epsilon}$ (otherwise $\gamma_{\min} = a > h^{2/3-\epsilon} > \gamma_{\max}$ and there are no terms to consider in the sum (79)). Recall that $\lambda_\gamma := \gamma^{3/2}/h$: for $t \leq h^{1/3-2\epsilon}$ and $\gamma \leq \gamma_{\max} = h^{2/3-\epsilon}$ it follows that $\lambda_\gamma \lesssim h^{-3/2\epsilon}$ for $\epsilon > 0$ small and we cannot perform stationary phase arguments

with parameter λ_γ ; formula (26) becomes therefore useless. The integral formulas of the two Airy factors in (18) have phase functions with parameter λ_γ (these two Airy functions correspond to the integrals in σ, s in (26)), but the advantage of (18) compared to (26) is that there is no integration with respect to α (instead we have to sum over $k \lesssim \lambda_\gamma$, hence over an essentially bounded number of k ; recall that the Airy-Poisson formula (20) transformed ω_k into $\omega = q^{1/3}(\eta)\lambda_\gamma^{2/3}\alpha$). As we cannot take advantage of stationary phase arguments in the integral formulas of the Airy functions, we instead deal with the two Airy factors in (18) as if they were part of the symbol (using the fact that they don't oscillate "too much", as we will see below). In order to do that we have to check that the stationary phase in η does apply, and it turns out that this is indeed the case if t is not "too" small. On the other hand, when $t \geq h^{1/3-2\epsilon}$, then $h^{2/3-\epsilon} \leq (ht)^{1/2} = \gamma_{\max}$ and for $\gamma \leq (ht)^{1/2}$ we can apply again the stationary phase with respect to η with the two Airy factors as part of the symbol (using that t is definitely not "too small"). We consider separately the situations $t \geq h^{1/3-2\epsilon}$ and $t \leq h^{1/3-2\epsilon}$, although the arguments in the corresponding proofs are similar.

3.2.3. *Let $t \geq h^{1/3-2\epsilon}$, in which case $(ht)^{1/2} \geq h^{2/3-\epsilon}$.* We will be able to bring the Airy functions into the symbol and apply the stationary phase in $\eta \in \mathbb{R}^{d-1}$. The sum over k is taken over $1 \leq k \lesssim (ht)^{3/4}/h$ and on the support of ψ_2 we have $k^{2/3} \simeq \omega_k \simeq \lambda_\gamma^{2/3}$ with $\gamma \leq \gamma_{\max} := (ht)^{1/2}$.

Proposition 11. *For $t \geq h^{1/3-2\epsilon}$, the following dispersive estimates hold*

$$\begin{aligned} \left\| \sum_{\max(a, h^{2/3}) \leq \gamma \leq (ht)^{1/2}} G_{h,\gamma}(t, \cdot) \right\|_{L^\infty(0 \leq x \leq a, y)} &\lesssim \frac{h^{1/3}}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \left(\frac{(ht)^{3/4}}{h}\right)^{1/3} \\ &\simeq \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} (ht)^{1/4}. \end{aligned}$$

Proof. Let $z = y/t$ and let $\frac{t}{h}$ be the large parameter in the integrals in the fourth line of (79) whose phase function are, for each $\omega_k \simeq \lambda_\gamma^{2/3}$, of the form

$$\langle z, \eta \rangle + |\eta|^2 + \omega_k h^{2/3} q^{2/3}(\eta).$$

For each $\omega_k \lesssim \lambda_{\max}/h^{2/3} = (ht)^{1/2}/h^{2/3}$, the corresponding critical point η_c satisfies $z + 2\eta_c + O(\omega_k h^{2/3}) = 0$ and using that $\omega_k h^{2/3} \leq \epsilon_0$, we obtain that the matrix of second order derivatives behaves like $2\mathbb{I}_{d-1} + O(\epsilon_0)$. In order to apply the stationary phase with symbol

$$q^{1/3}(\eta)\psi(|\eta|)\psi_2(\omega_k/(q^{1/3}(\eta)\lambda_\gamma^{2/3}))\text{Ai}\left(q^{1/3}(\eta)\lambda_\gamma^{2/3}\frac{x}{\gamma} - \omega_k\right)\text{Ai}\left(q^{1/3}(\eta)\lambda_\gamma^{2/3}\frac{a}{\gamma} - \omega_k\right)$$

we need to check that there exists some $\nu > 0$ such that for all $j \geq 1$ and for all multi-indices α with $|\alpha| = j$ one has

$$(80) \quad \left| \partial_\eta^\alpha \left(\text{Ai}\left(q^{1/3}(\eta)\lambda_\gamma^{2/3}\frac{x}{\gamma} - \omega_k \right) \right) \right| \leq C_j \left(\frac{t}{h}\right)^{j(1/2-\nu)}, \quad \forall |\alpha| = j.$$

In particular, this allows to deduce that, for η on the support of ψ we have

$$(81) \quad \partial_{\eta_i, \eta_j}^2 \left(q^{1/3}(\eta) \psi_2(\omega_k / (q^{1/3}(\eta) \lambda_\gamma^{2/3})) \text{Ai} \left(q^{1/3}(\eta) \lambda_\gamma^{2/3} \frac{x}{\gamma} - \omega_k \right) \text{Ai} \left(q^{1/3}(\eta) \lambda_\gamma^{2/3} \frac{a}{\gamma} - \omega_k \right) \right) \lesssim \left(\frac{t}{h} \right)^{1-2\nu},$$

and assures that the stationary phase can be applied with the Airy factors as part of the symbol. As one has, for all $l \geq 0$,

$$\sup_{b \geq 0} \left| b^l \text{Ai}^{(l)}(b - \omega_k) \right| \leq C_l \omega_k^{3l/2},$$

it is sufficient to check that for $t \geq h^{1/3-\epsilon}$ and $k \leq (ht)^{3/4}/h$ the following holds

$$(82) \quad \omega_k^{3/2} \lesssim \left(\frac{t}{h} \right)^{1/2-\nu}.$$

Using that $\omega_k \simeq k^{2/3} \lesssim \lambda_{\gamma_{\max}}^{2/3} \simeq ((ht)^{3/4}/h)^{2/3}$ for $k \leq (ht)^{3/4}/h$, (82) holds if we show that

$$t^{1/2} \left(\frac{t}{h} \right)^{1/4} = \frac{(ht)^{3/4}}{h} \lesssim \left(\frac{t}{h} \right)^{1/2-\nu}$$

which is obviously true as it reduces to $t \lesssim \left(\frac{t}{h} \right)^{1/2-2\nu}$ for some $\nu > 0$ (recall that we consider here only values $t \lesssim 1$). The sum of the main contributions of the symbols obtained after applying the stationary phase equals in η equals

$$(83) \quad \left| \sum_{k \lesssim (ht)^{3/4}/h} \omega_k^{-1/2} \text{Ai} \left(x \frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k \right) \text{Ai} \left(a \frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k \right) \right| \\ \leq \left| \sum_{k \lesssim (ht)^{3/4}/h} \omega_k^{-1/2} \text{Ai}^2 \left(x \frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k \right) \right|^{1/2} \\ \times \left| \sum_{k \lesssim (ht)^{3/4}/h} \omega_k^{-1/2} \text{Ai}^2 \left(a \frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k \right) \right|^{1/2} \lesssim (\lambda_{\gamma_{\max}})^{1/3},$$

where we have applied the Cauchy-Schwarz inequality followed by (77) from Lemma 6 with $L \simeq \lambda_{\gamma_{\max}} = (ht)^{3/4}/h$. Notice that we can indeed use (77) as the critical points $\eta_c(\frac{y}{2t}, \omega_k h^{2/3})$ satisfy $\eta_c(\frac{y}{t}, \omega_k h^{2/3}) = -\frac{y}{2t} + O(\omega_k h^{2/3})$ with $|\frac{y}{t}| \in [\frac{1}{2} + O(\epsilon_0), \frac{3}{2} + O(\epsilon_0)]$ and $O(\omega_k h^{2/3}) = O(\epsilon_0)$ for all ω_k on the support of ψ_2 and using the asymptotic behaviour of the Airy function $|\text{Ai}(z)| \leq \frac{C}{(1+|z|)^{1/4}}$, the factors $\text{Ai}^2 \left(x \frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k \right)$ with $x \leq a$

or $Ai^2\left(a\frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k\right)$ can be bounded as follows

$$(84) \quad \left| Ai^2\left(a\frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k\right) \right| \simeq \left| Ai^2\left(q^{1/3}\left(-\frac{y}{2t} + O(\omega_k h^{2/3})\right)\frac{a}{h^{2/3}} - \omega_k\right) \right|$$

$$\lesssim \frac{1}{(1 + |\omega_k - q^{1/3}(-\frac{y}{2t})(1 + O(\omega_k h^{2/3}))\frac{a}{h^{2/3}}|)^{1/2}}$$

$$\simeq \frac{1}{(1 + |\omega_k(1 + O(a)) - q^{1/3}(-\frac{y}{2t})\frac{a}{h^{2/3}}|)^{1/2}},$$

and therefore the proof of (77) of Lemma 6 does indeed apply with $b = q^{1/3}(-\frac{y}{2t})\frac{a}{h^{2/3}}$. However, this is not enough to conclude : we also need to show that the lower order terms of the symbol obtained after the stationary phase do sum up and provide smaller contributions : the second main contributions of these symbols are obtained by taking two derivatives with respect to η and are of the form (81) (with the factor $\frac{1}{L'(\omega_k)}$) : although the condition (82) assures that these contributions are small enough for each $k \leq (ht)^{3/4}/h$, in order to prove Theorem 1 in this regime we also need to estimate the following sums for $x \leq a$ (where we recall that $\lambda = a^{3/2}/h$): the first sum involves one derivative on each Airy factor

$$(85) \quad \frac{h}{t} \left| \sum_{k \lesssim \lambda_{\gamma_{\max}}} \omega_k^{-1/2} \frac{xa}{h^{4/3}} Ai'\left(x\frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k\right) Ai'\left(a\frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k\right) \right|$$

$$\lesssim \left(\frac{h}{t} \frac{\lambda^{4/3}}{h^{2/3}}\right) \times \left| \sum_{k \lesssim \lambda_{\gamma_{\max}}} \frac{h^{2/3}}{\omega_k^{1/2}} Ai'\left(x\frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k\right) Ai'\left(a\frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k\right) \right|$$

$$\lesssim \left(\frac{h}{t} \frac{\lambda^{4/3}}{h^{2/3}}\right) \times h^{2/3} \lambda_{\gamma_{\max}} = \lambda_{\gamma_{\max}}^{1/3} \times a \times \frac{a}{(ht)^{1/2}} \leq \lambda_{\gamma_{\max}}^{1/3} \times a,$$

where we have first used (78) from Lemma 6 with $L \simeq \lambda_{\gamma_{\max}}$ together with Cauchy-Schwarz, and then the fact that we consider only values $a \leq (ht)^{1/2}$; the second sum involves two derivatives on the same Airy factor

$$(86) \quad \frac{h}{t} \left| \sum_{k \leq \lambda_{\gamma_{\max}}} \omega_k^{-1/2} \lambda^{4/3} \left| \left(a\frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k \right) \right| Ai\left(x\frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k\right) \right.$$

$$\left. \times Ai\left(a\frac{q^{1/3}(\eta_c(z, \omega_k h^{2/3}))}{h^{2/3}} - \omega_k\right) \right| \lesssim (\text{LHS})(83) \times \left(\frac{h}{t}\right) \lambda^{4/3} \lambda_{\gamma_{\max}}^{2/3} \lesssim \lambda_{\gamma_{\max}}^{1/3} \times a,$$

where in the first inequality in the second line we have bounded each ω_k by $\lambda_{\gamma_{\max}}^{2/3}$ and then used (83), while in the last inequality we used the fact that $\frac{h}{t} \lambda^{4/3} \lambda_{\gamma_{\max}}^{2/3} \leq a \times \frac{a}{(ht)^{1/2}} \leq a$ when $a \lesssim (ht)^{1/2}$.

Using the equation satisfied by the Airy function, it follows that each contribution obtained by taking $2n$ derivatives of the symbol will be a product of either two Airy functions

as in (86) or a product of two derivatives of Airy functions as in (85) and we can apply Lemma 6 at each step to obtain a factor a^n and conclude. \square

Remark 23. Notice that applying the stationary phase with respect to η provides a dispersive bound of the form

$$\frac{h^{1/3}}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \left(\frac{\gamma_{\max}^{3/2}}{h}\right)^{1/3} = \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} \sqrt{\gamma_{\max}},$$

where the sum over k is taken for $k \lesssim \gamma_{\max}^{3/2}/h$. It is clear that for $\gamma \gg h^{1/2}$ this method cannot provide estimates sufficiently good for our purpose. Although $\sqrt{\gamma_{\max}}$ may not be sharp, it coincides with the one obtained for $t \simeq 1$ in the regime $(ht)^{1/2} \lesssim a < h^{1/3}$. In this situation, and in particular when $a \simeq h^{1/2}$ and $t \simeq 1$, there are too many wave packets producing important contributions that overlap. Still, the sum of all these contribution gives a loss less important than the one obtained for $a > h^{1/3}$, when the oscillatory integrals describing the reflected wave packets may have degenerate critical points of order exactly 3.

3.2.4. Let $t \leq h^{1/3-2\epsilon}$ for some small $\epsilon > 0$. Then $\max(h^{2/3-\epsilon}, (ht)^{1/2}) = h^{2/3-\epsilon}$ and, as we have already noticed, we only need to consider values of γ such that $\max(h^{2/3}, a) \lesssim \gamma \lesssim h^{2/3-\epsilon}$, as the sum over $\gamma > h^{2/3-\epsilon} > (ht)^{1/2}$ can be handled as in (76). In this case $\lambda_{\gamma_{\max}} = (h^{2/3-\epsilon})^{3/2}/h = h^{-3\epsilon/2}$.

Proposition 12. Let $0 < \epsilon < \frac{2}{9(d+1)}$. For all $h^{1/3+\epsilon} \lesssim t \leq h^{1/3-2\epsilon}$ we have

$$\left\| \sum_{\max(a, h^{2/3}) \lesssim \gamma \lesssim h^{2/3-\epsilon}} G_{h,\gamma}(t, \cdot) \right\|_{L^\infty(0 \leq x \leq a, y)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-1)/2} h^{1/3-\epsilon/2}.$$

For all $0 < t \lesssim h^{1/3+\epsilon}$ we have

$$\left\| \sum_{\max(a, h^{2/3}) \lesssim \gamma \lesssim h^{2/3-\epsilon}} G_{h,\gamma}(t, \cdot) \right\|_{L^\infty(0 \leq x \leq a, y)} \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{d/2}.$$

Proof. Notice first that

$$\left(\frac{h}{t}\right)^{1/2} \leq h^{1/3-\epsilon/2} \quad \text{if and only if } t \geq h^{1/3+\epsilon},$$

therefore when $h^{1/3+\epsilon} \lesssim t \leq h^{1/3-2\epsilon}$ a loss appears when compared to the flat case. Consider $0 < \epsilon < \frac{2}{9(d+1)}$ and set

$$t(h, \epsilon) := h^{1-3\epsilon-2\epsilon/d}.$$

Notice that the condition $0 < \epsilon < \frac{2}{9(d+1)}$ implies $t(h, \epsilon) \ll h^{1/3+\epsilon}$ for all $d \geq 1$. Let $t(h, \epsilon) \lesssim t \lesssim h^{1/3-2\epsilon}$ when the same proof as in the previous case applies. In fact, to use the stationary phase with the Airy factors in the symbol we need the condition (82) to be satisfied for all $k \lesssim \lambda_{\gamma_{\max}}$, which translates into

$$(87) \quad h^{-3\epsilon/2} \lesssim \left(\frac{t}{h}\right)^{1/2-\nu} \quad \text{for some } \nu > 0.$$

Let $\nu = \frac{2}{2+3d}$, then (87) holds since it is equivalent to $t \gtrsim h^{1-\frac{3\epsilon}{1-\nu}} = t(h, \epsilon)$.

- If $h^{1/3+\epsilon} \leq t \leq h^{1/3-2\epsilon}$, we obtain a loss as $h^{1/3-\epsilon/2} \geq \left(\frac{h}{t}\right)^{1/2}$.
- If $t(h, \epsilon) \lesssim t \leq h^{1/3+\epsilon}$ we bound $h^{1/3-\epsilon/2}$ by $\left(\frac{h}{t}\right)^{1/2}$.

Let now $t \lesssim t(h, \epsilon)$. We set $L := 8h^{-3\epsilon/2}$, then the sum over k in (79) is taken for $k \leq L$. Applying the Cauchy-Schwarz inequality in (79) and using (77) yields

$$\begin{aligned}
(88) \quad & \left\| \sum_{\max(a, h^{2/3}) \lesssim \gamma \lesssim h^{2/3-\epsilon}} G_{h, \gamma}(t, \cdot) \right\|_{L^\infty(0 \leq x \leq a, y)} \\
& \lesssim \frac{h^{1/3}}{h^d} \sum_{k \leq L} \omega_k^{-1/2} Ai(xq^{1/3}(\eta)/h^{2/3} - \omega_k) Ai(aq^{1/3}(\eta)/h^{2/3} - \omega_k) \\
& \lesssim \frac{h^{1/3}}{h^d} \left(\sum_{1 \leq k \leq L} \omega_k^{-1/2} Ai^2(xq^{1/3}(\eta)/h^{2/3} - \omega_k) \right)^{1/2} \left(\sum_{1 \leq k \leq L} \omega_k^{-1/2} Ai^2(aq^{1/3}(\eta)/h^{2/3} - \omega_k) \right)^{1/2} \\
& \lesssim \frac{h^{1/3}}{h^d} L^{1/3}.
\end{aligned}$$

For $t \lesssim t(h, \epsilon)$ we have $\frac{1}{t(h, \epsilon)} \lesssim \frac{1}{t}$ and as $\left(\frac{h}{t(h, \epsilon)}\right)^{d/2} h^{-3(d+1)\epsilon/2} = h^{-\epsilon/2}$ we find

$$h^{1/3} L^{1/3} = 2h^{1/3-\epsilon/2} = 2\left(\frac{h}{t(h, \epsilon)}\right)^{d/2} h^{1/3-3(d+1)\epsilon/2} \lesssim \left(\frac{h}{t}\right)^{d/2} h^{1/3-3(d+1)\epsilon/2} \leq \left(\frac{h}{t}\right)^{d/2},$$

as the condition $\epsilon < \frac{2}{9(d+1)}$ implies $1/3 - 3(d+1)\epsilon/2 > 0$. \square

4. PROOF OF PROPOSITIONS 3, 4 AND 5

Here we need to analyse in details the structure of higher order derivatives of the phase functions $\phi_{N, a}$. Let T be fixed, $N \in [\frac{T}{M}, MT]$ with $M > 8$ large enough and let $Y = \frac{y}{\sqrt{a}}$ with $\frac{Y}{2T} \in [\frac{1}{4}, 2]$. We prove Propositions 3, 4 and 5 for $V_{N, h, a}$ defined in (56), whose integral formula is recalled here

$$V_{N, h, a}(t, x, y) = \frac{a^2}{h} \frac{1}{\sqrt{\lambda N}} \int e^{i\hbar \phi_{N, a}} \varkappa(\sigma, s, t, x, y, h, a, 1/N) d\sigma ds,$$

where, from Remark 16, we can assume (without changing the contribution of $V_{N, h, a}$ modulo $O(h^\infty)$) that the symbol \varkappa is supported for $|(\sigma, s)| \leq 2\sqrt{a_c}$. In order to integrate with respect to the remaining variables σ, s we need precise information on the higher order derivatives of the phase functions.

Proof of Proposition 3. We start with the case where $\lambda^{1/3} \lesssim N$ and we follow closely the proof of [17, Prop.7]. We will prove the following :

$$(89) \quad \left| \int_{\mathbb{R}^2} e^{i\hbar \phi_{N, a}} \varkappa(\sigma, s, t, x, y, h, a, 1/N) ds d\sigma \right| \lesssim \frac{\lambda^{-2/3}}{1 + \lambda^{1/3} |K_a^2(\frac{Y}{4N}, \frac{T}{2N}) - 1|^{1/2}}.$$

Let $X = \frac{x}{a}$. We rescale variables with $\sigma = \lambda^{-1/3}p$ and $s = \lambda^{-1/3}q$ and define

$$(90) \quad A = \lambda^{2/3} \left(K_a^2 \left(\frac{Y}{4N}, \frac{T}{2N} \right) - X \right) \quad \text{and} \quad B = \lambda^{2/3} \left(K_a^2 \left(\frac{Y}{4N}, \frac{T}{2N} \right) - 1 \right),$$

and we are reduced to proving that the following holds uniformly in (A, B) :

$$(91) \quad \left| \int_{\mathbb{R}^2} e^{iG_{N,a,\lambda}(p,q,t,x,y)} \chi(\lambda^{-1/3}p, \lambda^{-1/3}q, t, x, y, h, a, 1/N) dpdq \right| \lesssim \frac{1}{1 + |B|^{1/2}},$$

where the rescaled phase is

$$(92) \quad G_{N,a,\lambda}(p, q, t, x, y) := \frac{1}{h} \left(\phi_{N,a}(\lambda^{-1/3}p, \lambda^{-1/3}q, t, x, y) - \phi_{N,a}(0, 0, t, x, y) \right).$$

Replacing γ by a in the expression of the first order derivatives of $\phi_{N,a,\gamma}$ in (50) and (51) yields

$$\begin{aligned} \partial_p G_{N,a,\lambda} &= \frac{1}{h} \frac{\partial \sigma}{\partial p} \partial_\sigma (\phi_{N,a})|_{(\sigma,s)=(\lambda^{-1/3}p, \lambda^{-1/3}q)} = \lambda^{2/3} q^{1/2} (\eta_c) (\lambda^{-2/3} p^2 + X - \alpha_c) \\ &= q^{1/2} (\eta_c) (p^2 - \lambda^{2/3} (\alpha_c - X)), \\ \partial_q G_{N,a,\lambda} &= \frac{1}{h} \frac{\partial s}{\partial q} \partial_s (\phi_{N,a})|_{(\sigma,s)=(\lambda^{-1/3}p, \lambda^{-1/3}q)} = \lambda^{2/3} q^{1/2} (\eta_c) (\lambda^{-2/3} q^2 + 1 - \alpha_c) \\ &= q^{1/2} (\eta_c) (q^2 - \lambda^{2/3} (\alpha_c - 1)). \end{aligned}$$

From (46) it follows that, in the new variables, α_c has the following form

$$(93) \quad \alpha_c|_{(\lambda^{-1/3}p, \lambda^{-1/3}q)} = \left(K_a \left(\frac{Y}{4N}, \frac{T}{2N} \right) - \lambda^{-1/3} \frac{p}{2N} (1 - a\mathcal{E}_1) - \lambda^{-1/3} \frac{q}{2N} (1 - a\mathcal{E}_2) \right)^2,$$

where f_j are smooth functions of $(\sigma, s) = \lambda^{-1/3}(p, q)$ and of $\frac{T}{2N}$, X and $\frac{Y}{4N}$. With these notations and with $K_a = K_a(\frac{Y}{4N}, \frac{T}{2N})$, we re-write the first order derivatives of $G_{N,a,\lambda}$ as follows

$$\begin{aligned} \partial_p G_{N,a,\lambda} &= q^{1/2} (\eta_c) \left(p^2 - A + \frac{\lambda^{1/3}}{N} K_a (p(1 - a\mathcal{E}_1) + q(1 - a\mathcal{E}_2)) - \frac{1}{4N^2} (p(1 - a\mathcal{E}_1) + q(1 - a\mathcal{E}_2))^2 \right), \\ \partial_q G_{N,a,\lambda} &= q^{1/2} (\eta_c) \left(q^2 - B + \frac{\lambda^{1/3}}{N} K_a (p(1 - a\mathcal{E}_1) + q(1 - a\mathcal{E}_2)) - \frac{1}{4N^2} (p(1 - a\mathcal{E}_1) + q(1 - a\mathcal{E}_2))^2 \right), \end{aligned}$$

Since $\lambda^{1/3} \leq N$, we notice that if A, B are bounded, then (91) obviously holds for $|(p, q)|$ bounded and by integration by parts if $|(p, q)|$ is large. So we can assume that $|(A, B)| \geq r_0$ with $r_0 \gg 1$. Set $(A, B) = r(\cos(\theta), \sin(\theta))$ and rescale again $(p, q) = r^{1/2}(\tilde{p}, \tilde{q})$: we aim at

$$(94) \quad \left| \int_{\mathbb{R}^2} e^{ir^{3/2}\tilde{G}_{N,a,\gamma}} \chi(\lambda^{-1/3}r^{1/2}\tilde{p}, \lambda^{-1/3}r^{1/2}\tilde{q}, t, x, y, h, a, 1/N) d\tilde{p}d\tilde{q} \right| \lesssim \frac{1}{r^{5/4}},$$

where $r_0 < r \lesssim \lambda^{2/3}$ is our large parameter, and the phase $\tilde{G}_{N,a,\lambda}$ is given by $\tilde{G}_{N,a,\lambda}(\tilde{p}, \tilde{q}, t, x, y) = r^{-3/2} G_{N,a,\lambda}(r^{1/2}p, r^{1/2}q, t, x, y)$. Let us compute, using the formulas of the first order derivatives of $G_{N,a,\lambda}$

(95)

$$\begin{aligned} \frac{\partial_{\tilde{p}} \tilde{G}_{N,a,\lambda}}{q^{1/2}(\eta_c)} &= \tilde{p}^2 - \cos \theta + \frac{\lambda^{1/3} K_a}{N r^{1/2}} (\tilde{p}(1 - a\mathcal{E}_1) + \tilde{q}(1 - a\mathcal{E}_2)) - \frac{(\tilde{p}(1 - a\mathcal{E}_1) + \tilde{q}(1 - a\mathcal{E}_2))^2}{4N^2}, \\ \frac{\partial_{\tilde{q}} \tilde{G}_{N,a,\lambda}}{q^{1/2}(\eta_c)} &= \tilde{q}^2 - \sin \theta + \frac{\lambda^{1/3} K_a}{N r^{1/2}} (\tilde{p}(1 - a\mathcal{E}_1) + \tilde{q}(1 - a\mathcal{E}_2)) - \frac{(\tilde{p}(1 - a\mathcal{E}_1) + \tilde{q}(1 - a\mathcal{E}_2))^2}{4N^2}, \end{aligned}$$

where, abusing notations, \mathcal{E}_j is now $\mathcal{E}_j(r^{1/2}\lambda^{-1/3}(\tilde{q}, \tilde{p}), \frac{T}{2N}, \frac{Y}{4N})$. Since on the support of the symbol $\varkappa(\lambda^{-1/3}r^{1/2}\tilde{p}, \lambda^{-1/3}r^{1/2}\tilde{q}, t, x, y, h, a, 1/N)$ we must have $|(\tilde{p}, \tilde{q})| \lesssim \lambda^{1/3}r^{-1/2} \lesssim \lambda^{1/3}r_0^{-1/2}$, it follows that, for $\lambda^{1/3} \lesssim N$, the last term in both derivatives is $O(r_0^{-1})$, while the next to last term is $r_0^{-1/2}O(\tilde{p}, \tilde{q})$. Indeed, using that $\mathcal{E}_{1,2}$ are uniformly bounded, K_a is bounded for $\frac{Y}{2T}$ on the support of $\tilde{\psi}$ and $r \geq r_0$ we obtain $|\frac{\lambda^{1/3}}{N} K_a \frac{(\tilde{p}(1-a\mathcal{E}_1) + \tilde{q}(1-a\mathcal{E}_2))}{r^{1/2}}| \lesssim r_0^{-1/2}|\tilde{p} + \tilde{q}|$. Therefore, when $|(\tilde{p}, \tilde{q})| > \tilde{C}$ with \tilde{C} sufficiently large, the corresponding part of the integral is $O(r^{-\infty})$ by integration by parts (since $(\sin(\theta), \cos(\theta))$ stay bounded). So we are left with restricting our integral to a compact region in (\tilde{p}, \tilde{q}) .

We remark that, from the restriction $X \leq 1$, we have $A \geq B$ (and $A = B$ if and only if $X = 1$), that is to say, $\cos \theta \geq \sin \theta$ and therefore $\theta \in (-\frac{3\pi}{4}, \frac{\pi}{4})$. We proceed depending upon the size of $B = r \sin \theta$. If $\sin \theta < -C/r^{1/2}$ for some $C > 0$ sufficiently large then $\partial_{\tilde{q}} \tilde{G}_{N,a,\lambda} > c/(2r^{1/2})$ for some $C > c > 0$ and the phase is non stationary. Indeed, in this case we have

$$\frac{\partial_{\tilde{q}} \tilde{G}_{N,a,\lambda}}{q^{1/2}(\eta_c)} \geq \tilde{q}^2 + \frac{C}{2r^{1/2}} + \frac{\lambda^{1/3} K_a}{N r^{1/2}} (\tilde{p}(1 - a\mathcal{E}_1) + \tilde{q}(1 - a\mathcal{E}_2)) - \frac{(\tilde{p}(1 - a\mathcal{E}_1) + \tilde{q}(1 - a\mathcal{E}_2))^2}{4N^2}$$

and using that \tilde{p}, \tilde{q} are bounded, that on the support of \varkappa we have $|r^{1/2}(\tilde{p}, \tilde{q})| \lesssim \lambda^{1/3}$ and that $\frac{1}{N} \lesssim \frac{1}{\lambda^{1/3}} \ll 1$, we then have, for some C large enough

$$\frac{\lambda^{1/3}}{N} (\tilde{p} + \tilde{q}) \left[\frac{K_a}{r^{1/2}} - \frac{(\tilde{p}(1 - a\mathcal{E}_1) + \tilde{q}(1 - a\mathcal{E}_2))}{4N\lambda^{1/3}} \right] \lesssim \frac{C}{4r^{1/2}}.$$

We recall that on the support of $\psi_2(\alpha)$ we had $\alpha \in [\frac{1}{2}, \frac{3}{2}]$ and the critical point α satisfied (33) (with γ replaced by a in this case) hence $K_a = K_a(\frac{Y}{4N}, \frac{T}{2N})$ introduced in (57) stays close to 1 as main contribution of the critical point α_c . It follows that $\partial_{\tilde{q}} \tilde{G}_{N,a,\lambda} > C/(2r^{1/2})$ and integrations by parts yield a bound $O(r^{-n})$ for all $n \geq 1$. Next, let $\sin \theta > -C/r^{1/2}$ and assume $A > 0$ (since otherwise the non-stationary phase applies), which in turn implies $A > r_0/2$. Indeed, since $\cos \theta \geq \sin \theta > -C/r^{1/2}$ it follows that $\theta \in (-\frac{C}{\sqrt{r_0}}, \frac{\pi}{4})$ and therefore in this regime $\cos \theta \geq \frac{\sqrt{2}}{2}$.

Consider first the case $|\sin \theta| < C/r^{1/2}$, with $C > 0$ like before. The non degenerate stationary phase always applies in \tilde{p} , at two (almost) opposite values of \tilde{p} , such that $|\tilde{p}_{\pm}| \simeq$

$|\pm \sqrt{\cos \theta}| \geq 1/4$, and the integral in (94) can be written as follows

$$(96) \quad r \int_{\mathbb{R}^2} e^{ir^{3/2}\tilde{G}_{N,a,\lambda}} \varkappa(\lambda^{-1/3}r^{1/2}\tilde{p}, \lambda^{-1/3}r^{1/2}\tilde{q}, t, x, y, h, a, 1/N) d\tilde{p}d\tilde{q} \\ = \frac{r}{r^{3/4}} \left(\int_{\mathbb{R}} e^{ir^{3/2}\tilde{G}_{N,a,\lambda}^+} \varkappa^+(\tilde{q}, t, x, y, h, a, 1/N) d\tilde{q} + \int_{\mathbb{R}} e^{ir^{3/2}\tilde{G}_{N,a,\lambda}^-} \varkappa^-(\tilde{q}, h, a, 1/N) d\tilde{q} \right).$$

Indeed, the phase is stationary in \tilde{p} when

$$\tilde{p}^2 = \cos \theta - \frac{\lambda^{1/3}K_a}{Nr^{1/2}}(\tilde{p}(1 - a\mathcal{E}_1) + \tilde{q}(1 - a\mathcal{E}_2)) + \frac{(\tilde{p}(1 - a\mathcal{E}_1) + \tilde{q}(1 - a\mathcal{E}_2))^2}{4N^2},$$

and since $\cos \theta \geq \frac{\sqrt{2}}{2}$ and $\frac{1}{r} \leq \frac{1}{r_0} \ll 1$, it follows that there are exactly two distinct solutions to $\partial_{\tilde{p}}\tilde{G}_{N,a,\lambda} = 0$, that we denote $\tilde{p}_{\pm} \simeq \pm\sqrt{\cos \theta} + O(r^{-1/2})$. Using (95), the second derivative of the phase with respect to \tilde{p} at the critical points becomes

$$\partial_{\tilde{p},\tilde{p}}^2\tilde{G}_{N,a,\lambda}|_{\tilde{p}_{\pm}} = q^{1/2}(\eta_c) \left(2\tilde{p} + \frac{\lambda^{1/3}K_a}{Nr^{1/2}}(1 + O(a)) \right) + O(N^{-2})|_{\tilde{p}_{\pm}},$$

where we have used the fact that \tilde{p}, \tilde{q} are bounded and that $\partial_{\tilde{p}}\mathcal{E}_j = O(\frac{r^{1/2}\lambda^{-1/3}}{N})$ to deduce that all the terms except the first one are small. Since $\lambda^{1/3} \lesssim N$ and $r^{-1/2} \ll 1$, and since K_a is bounded, close to 1, it follows that for $\tilde{p} \in \{\tilde{p}_{\pm}\}$ the second order derivative equals $\partial_{\tilde{p},\tilde{p}}^2\tilde{G}_{N,a,\lambda}|_{\tilde{p}_{\pm}} \simeq 2\tilde{p}_{\pm} + O(r^{-1/2})$, and since $|\tilde{p}_{\pm}| \geq \frac{1}{4} - O(r^{-1/2})$, the stationary phase applies. The critical values at \tilde{p}_{\pm} , denoted $\tilde{G}_{N,a,\lambda}^{\pm}$, satisfy

$$(97) \quad \partial_{\tilde{q}}\tilde{G}_{N,a,\lambda}^{\pm}(\tilde{q}, \cdot) := \partial_{\tilde{q}}\tilde{G}_{N,a,\lambda}(\tilde{q}, \tilde{p}_{\pm}, \cdot) = q^{1/2}(\eta_c) \left(\tilde{q}^2 - \sin \theta \right. \\ \left. + \frac{\lambda^{1/3}K_a(\tilde{p}(1 - a\mathcal{E}_1) + \tilde{q}(1 - a\mathcal{E}_2))}{Nr^{1/2}} - \frac{(\tilde{p}(1 - a\mathcal{E}_1) + \tilde{q}(1 - a\mathcal{E}_2))^2}{4N^2} \Big|_{\tilde{p}=\tilde{p}_{\pm}} \right).$$

Since $|\sin \theta| < C/r^{1/2}$, the phases $\tilde{G}_{N,a,\lambda}^{\pm}$ may be stationary but degenerate; taking two derivatives in (97), one easily checks that $|\partial_{\tilde{q}}^3\tilde{G}_{N,a,\lambda}^{\pm}| \geq q^{1/2}(\eta_c)(2 - O(r_0^{-1/2}))$. Hence we get, by Van der Corput Lemma

$$(98) \quad \left| \int_{\mathbb{R}} e^{ir^{3/2}\tilde{G}_{N,a,\lambda}^{\pm}} \varkappa^{\pm}(\tilde{q}, t, x, y, h, a, 1/N) d\tilde{q} \right| \lesssim (r^{3/2})^{-1/3}.$$

Using (96) and (98) eventually gives

$$(99) \quad \left| r \int_{\mathbb{R}^2} e^{ir^{3/2}\tilde{G}_{N,a,\lambda}} \varkappa(\lambda^{-1/3}r^{1/2}\tilde{p}, \lambda^{-1/3}r^{1/2}\tilde{q}, t, x, y, h, a, 1/N) d\tilde{p}d\tilde{q} \right| \lesssim r^{-1/4}.$$

Notice moreover that $|B| = |r \sin \theta| \leq Cr^{1/2}$, hence from $r^2 = A^2 + B^2$, we have $A \sim r$ and since r is large $r^{-1/4} \lesssim 1/(1 + |B|^{1/2})$, which means that (91) holds true and, replacing B by $\lambda^{2/3}(K_a^2 - 1)$, it yields (89). Recall that A, B are given in (90); using that $a^2 = (h\lambda)^{4/3}$,

we obtain from (89)

$$(100) \quad |V_{N,h,a}(t, x, y)| \leq \frac{a^2}{h} \frac{1}{\sqrt{\lambda N}} \frac{\lambda^{-2/3}}{(1 + \lambda^{1/3}|K_a^2 - 1|^{1/2})} \\ = \frac{2h^{1/3}}{2\sqrt{N/\lambda^{1/3}} + \lambda^{1/6}\sqrt{K_a + 1}|4NK_a - 4N|^{1/2}}.$$

In the last case $\sin \theta > C/r^{1/2}$, which means $A \geq B \geq Cr^{1/2}$, we directly perform stationary phase in (\tilde{p}, \tilde{q}) : the determinant of the Hessian matrix is at least $C\sqrt{\cos \theta}\sqrt{\sin \theta}$ and we get a bound for the integral in the left hand side term in (94)

$$|(\text{LHS})(94)| \lesssim \frac{1}{(\sqrt{\cos \theta}\sqrt{\sin \theta})^{1/2}r^{3/2}} \lesssim \frac{1}{r} \frac{1}{(r\sqrt{\cos \theta}\sqrt{\sin \theta})^{1/2}} \lesssim \frac{1}{r} \frac{1}{(AB)^{1/4}}$$

so in this case the estimates is slightly better than (89), as we have

$$(101) \quad \left| \int_{\mathbb{R}^2} e^{\frac{i}{h}\phi_{N,a}} \varkappa(s, \sigma, t, x, y, h, a, 1/N) ds d\sigma \right| \lesssim \frac{1}{\lambda^{2/3}|AB|^{1/4}} \leq \frac{1}{\lambda^{2/3}|B|^{1/2}}.$$

This eventually yields

$$(102) \quad |V_{N,h,a}(t, x, y)| \lesssim \frac{(h\lambda)^{4/3} \lambda^{-1/2}}{h} \frac{1}{N^{1/2} \lambda^{2/3}|B|^{1/2}} \simeq h^{1/3} \frac{\lambda^{1/6}}{N^{1/2} \lambda^{1/3}|K_a^2 - 1|^{1/2}}.$$

The proof of Proposition 3 is achieved.

Proof of Propositions 4 and 5. The only differences between the proof of Proposition 4 and that of [17, Prop.5] occur from the existence of the additional critical point η_c , which is not considered in the case of the wave equation. The proof of Proposition 5 follow the same path as the one of [17, Prop.6], but in the last part the analysis is slightly more delicate as one has to carefully deal with the contributions coming from the higher order derivatives of η_c . Let $1 \leq N < \lambda^{1/3}$: we aim at proving

$$(103) \quad \left| \int_{\mathbb{R}^2} e^{\frac{i}{h}\phi_{N,a}} \varkappa(\sigma, s, t, x, y, h, a, 1/N) ds d\sigma \right| \lesssim N^{1/4} \lambda^{-3/4}.$$

Since N is bounded by $\lambda^{1/3}$, we cannot ignore anymore the last two terms in the first order derivatives of $\phi_{N,a}$, as we did in the previous case when they were small enough to make a meaningful contribution. Set $\Lambda = \lambda/N^3 = \frac{1}{N^3} \times a^{3/2}/h$ to be the new parameter. We rescale again variables with $\sigma = p'/N$ and $s = q'/N$ and set now

$$\Lambda G_{N,a}(p', q', t, x, y) = \frac{1}{h} \left(\phi_{N,a}(\sigma, s, t, x, y) - \phi_{N,a}(0, 0, t, x, y) \right).$$

On the support of \varkappa we then have $|(p', q')| \lesssim N$. We are reduced to proving

$$(104) \quad \left| \int_{\mathbb{R}^2} e^{i\Lambda G_{N,a}} \varkappa(p'/N, q'/N, t, x, y, h, a, 1/N) dp' dq' \right| \lesssim \Lambda^{-3/4}.$$

The first order derivatives of $G_{N,a}$ with respect to (p', q') are given by

$$(105) \quad \nabla_{(p',q')} G_{N,a} = \frac{N^3}{h} \left(\frac{\partial \sigma}{\partial p'} \partial_\sigma \phi_{N,a}, \frac{\partial s}{\partial q'} \partial_s \phi_{N,a} \right) \Big|_{(p'/N, q'/N)} \\ = q^{1/2}(\eta_c) \left(p'^2 + N^2(X - \alpha_c), q'^2 + N^2(1 - \alpha_c) \right),$$

where, using (46), $\alpha_c(\sigma, s, \cdot) \Big|_{(\sigma=p'/N, s=q'/N)}$ is given by

$$(106) \quad \alpha_c \Big|_{(p'/N, q'/N)} = \left(K_a - \frac{p'}{2N^2}(1 - af_1) - \frac{q'}{2N^2}(1 - af_2) \right)^2.$$

Recall that $K_a = K_a(\frac{Y}{4N}, \frac{T}{2N})$, given in (57), equals $\sqrt{\alpha_c^0}$ and stays therefore close to 1 on the support of the symbol. We define

$$(107) \quad A' = (K_a^2 - X)N^2 \quad \text{and} \quad B' = (K_a^2 - 1)N^2.$$

With these notations, the first order derivatives of $G_{N,a,\lambda}$ read as follows

$$\partial_{p'} G_{N,a} = q^{1/2}(\eta_c) \left(p'^2 - A' + K_a(p'(1 - a\mathcal{E}_1) + q'(1 - a\mathcal{E}_2)) - \frac{1}{4N^2}(p'(1 - a\mathcal{E}_1) + q'(1 - a\mathcal{E}_2))^2 \right),$$

$$\partial_{q'} G_{N,a} = q^{1/2}(\eta_c) \left(q'^2 - B' + K_a(p'(1 - a\mathcal{E}_1) + q'(1 - a\mathcal{E}_2)) - \frac{1}{4N^2}(p'(1 - a\mathcal{E}_1) + q'(1 - a\mathcal{E}_2))^2 \right).$$

Unlike the previous case, we can no longer ignore the two last terms (since even the terms with $\frac{1}{N^2}$ factors may provide important contributions !).

We start with $|(A', B')| \geq r_0$ for some large, fixed r_0 , in which case we can follow the same approach as in the previous case. Set again $A' = r \cos \theta$ and $B' = r \sin \theta$. If $|(p', q')| < r_0/2$, then the corresponding integral is non stationary and we get decay by integration by parts. We change variables $(p', q') = r^{1/2}(\tilde{p}', \tilde{q}')$ with $r_0 \leq r \lesssim N^2$ and aim at proving the following

$$(108) \quad \left| r \int_{\mathbb{R}^2} e^{ir^{3/2}\Lambda \tilde{G}_{N,a}} \varkappa(r^{1/2}\tilde{p}'/N, r^{1/2}\tilde{q}'/N, t, x, y, h, a, 1/N) d\tilde{p}' d\tilde{q}' \right| \lesssim r^{-1/4} \Lambda^{-5/6},$$

where \varkappa is now compactly supported in an annulus. The new phase is $\tilde{G}_{N,a}(\tilde{p}', \tilde{q}', t, x, y) = r^{-3/2} G_{N,a}(r^{1/2}\tilde{p}', r^{1/2}\tilde{q}', t, x, y)$. We compute

$$\frac{\partial_{\tilde{p}'} \tilde{G}_{N,a}}{q^{1/2}(\eta_c)} = \tilde{p}'^2 - \cos \theta + \frac{K_a}{r^{1/2}}(\tilde{p}'(1 - a\mathcal{E}_1) + \tilde{q}'(1 - a\mathcal{E}_2)) - \frac{(\tilde{p}'(1 - a\mathcal{E}_1) + \tilde{q}'(1 - a\mathcal{E}_2))^2}{4N^2}, \\ \frac{\partial_{\tilde{q}'} \tilde{G}_{N,a}}{q^{1/2}(\eta_c)} = \tilde{q}'^2 - \sin \theta + \frac{K_a}{r^{1/2}}(\tilde{p}'(1 - a\mathcal{E}_1) + \tilde{q}'(1 - a\mathcal{E}_2)) - \frac{(\tilde{p}'(1 - a\mathcal{E}_1) + \tilde{q}'(1 - a\mathcal{E}_2))^2}{4N^2}.$$

We intend to follow the same approach, as long as possible, as in the case $\lambda^{1/3} \lesssim N$. Since $X \leq 1$ we have $A' \geq B'$ which implies $\cos \theta \geq \sin \theta$. If $|(\tilde{p}', \tilde{q}')| \geq \tilde{C}$ for some large $\tilde{C} \geq 1$, then the critical points $(\tilde{p}'_c, \tilde{q}'_c)$ satisfy $\tilde{p}'_c^2 \geq \tilde{q}'_c^2$ and if \tilde{C} is sufficiently large the non-stationary phase applies (in fact it is enough to pick any $\tilde{C} > 4$). Therefore we are reduced, like in the previous case, to the situation where $|(\tilde{p}', \tilde{q}')|$ stay bounded. We

deal again with three situations, depending upon $B' = r \sin \theta$: if $\sin \theta < -\frac{C}{\sqrt{r}}$ for some sufficiently large constant $C > 0$, then

$$\frac{\partial_{\tilde{q}'} \tilde{G}_{N,a}}{q^{1/2}(\eta_c)} \geq \tilde{q}'^2 + \frac{C}{r^{1/2}} + \frac{K_a}{r^{1/2}}(\tilde{p}'(1 - a\mathcal{E}_1) + \tilde{q}'(1 - a\mathcal{E}_2)) - \frac{(\tilde{p}'(1 - a\mathcal{E}_1) + \tilde{q}'(1 - a\mathcal{E}_2))^2}{4N^2},$$

and as $|(\tilde{p}', \tilde{q}')|$ is bounded, $f_{1,2}$ are bounded, N is sufficiently large in this case (indeed, recall that $r_0 \leq r \lesssim N^2$ so that $\frac{1}{\sqrt{r}} \geq \frac{1}{N}$), it follows that the non-stationary applies since the sum of the last three terms in the previous inequality is greater than $C/(2r^{1/2})$ if C is large enough. If $|\sin \theta| \leq \frac{C}{\sqrt{r}}$ then, again, $\theta \in (-\frac{C}{\sqrt{r_0}}, \frac{\pi}{4})$ and $\cos \theta \geq \frac{\sqrt{2}}{2}$. We have $|B'| = |r \sin \theta| \leq C\sqrt{r}$; if $|B'| < C$, then $1 + |B'| \lesssim r^{1/2}$, while $|A'| \simeq r$. As in the previous case the stationary phase applies in \tilde{p}' with non-degenerate critical points \tilde{p}'_{\pm} and yields a factor $(r^{3/2}\Lambda)^{-1/2}$; the critical value of the phase function at these critical points, that we denote $\tilde{G}_{N,a}^{\pm}$, always satisfies $|\partial_{\tilde{q}'}^3 \tilde{G}_{N,a}^{\pm}| \geq q^{1/2}(\eta_c)(2 - O(\frac{1}{r_0^{1/2}}))$ and the integral in \tilde{q}' is bounded by $(r^{3/2}\Lambda)^{-1/3}$. We therefore obtain (108) which yields, using that $|B'| = |N^2(K_a^2 - 1)| \leq r^{1/2}$,

$$\begin{aligned} |V_{N,a,h}(t, x, y)| &= \frac{h^{1/3}\lambda^{4/3}}{\sqrt{\lambda NN^2}} \left| r \int_{\mathbb{R}^2} e^{ir^{3/2}\Lambda \tilde{G}_{N,a}} \varkappa(r^{1/2}\tilde{p}'/N, r^{1/2}\tilde{q}'/N, t, x, y, h, a, 1/N) d\tilde{p}' d\tilde{q}' \right| \\ &\lesssim \frac{h^{1/3}\lambda^{5/6}}{N^{5/2}} r^{-1/4} \left(\frac{\lambda}{N^3} \right)^{-5/6} \lesssim \frac{h^{1/3}}{(1 + |B'|^{1/2})} \simeq \frac{h^{1/3}}{(1 + N|K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1|^{1/2})}. \end{aligned}$$

If $\sin \theta > \frac{C}{\sqrt{r}}$, then $B' = r \sin \theta > C\sqrt{r}$ and therefore $N^2|K_a^2 - 1| > Cr^{1/2}$. We directly perform the stationary phase with large parameter $r^{3/2}\Lambda$ as the determinant of the Hessian matrix at the critical point is at least $C\sqrt{\cos \theta \sin \theta}$, and obtain a bound for the left hand side term in (108) of the form

$$\frac{cr}{(\sqrt{\sin \theta} \sqrt{\cos \theta})^{1/2} r^{3/2} \Lambda} = \frac{1}{\Lambda} \frac{1}{(A'B')^{1/4}} \leq \frac{1}{\Lambda} \frac{1}{B'^{1/2}}.$$

We just proved that for $N < \lambda^{1/3}$ and when $N^2|K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1|$ is not too small, the following holds

$$(109) \quad |V_{N,h,a}(t, x, y)| \lesssim \frac{h^{1/3}}{\lambda^{1/6} \sqrt{N} |K_a(\frac{Y}{4N}, \frac{T}{2N}) - 1|^{1/2}}.$$

We now move to the most delicate case $|(A', B')| \leq r_0$. For $|(p', q')|$ large, the phase is non stationary and integrations by parts provide $O(\Lambda^{-\infty})$ decay. So we may replace \varkappa by a cut-off, that we still call \varkappa , that is compactly supported in $|(p', q')| < R$. We proceed as in [17, proof of Prop. 6], by identifying one variable where the usual stationary phase may be performed and then evaluating the remaining $1D$ oscillating integral using Van der Corput lemma with different decay rates depending on the lower bounds on the derivatives of order at most 4.

Using (105), we compute the derivatives of $G_{N,a}$ as follows

$$(110) \quad \partial_{p'} G_{N,a} = q^{1/2}(\eta_c)(p'^2 + N^2(X - \alpha_c)), \quad \partial_{q'} G_{N,a} = q^{1/2}(\eta_c)(q'^2 + N^2(1 - \alpha_c)).$$

The second order derivatives of $G_{N,a}$ follow from (52), (53) and (54)

$$(111) \quad \partial_{p'p'}^2 G_{N,a} = q^{1/2}(\eta_c)(2p' - N^2 \partial_{p'} \alpha_c) + \frac{\partial_{p'} \eta_c \nabla q(\eta_c)}{2q^{1/2}(\eta_c)}(p'^2 + N^2(X - \alpha_c)),$$

$$(112) \quad \partial_{q'q'}^2 G_{N,a} = q^{1/2}(\eta_c)(2q' - N^2 \partial_{q'} \alpha_c) + \frac{\partial_{q'} \eta_c \nabla q(\eta_c)}{2q^{1/2}(\eta_c)}(q'^2 + N^2(1 - \alpha_c)),$$

$$(113) \quad \begin{aligned} \partial_{q'p'}^2 G_{N,a} &= q^{1/2}(\eta_c)(-N^2 \partial_{q'} \alpha_c) + \frac{\partial_{q'} \eta_c \nabla q(\eta_c)}{2q^{1/2}(\eta_c)}(p'^2 + N^2(X - \alpha_c)) \\ &= \partial_{p'q'}^2 G_{N,a} = q^{1/2}(\eta_c)(-N^2 \partial_{p'} \alpha_c) + \frac{\partial_{p'} \eta_c \nabla q(\eta_c)}{2q^{1/2}(\eta_c)}(q'^2 + N^2(1 - \alpha_c)). \end{aligned}$$

At the critical points where $\partial_{p'} G_{N,a} = \partial_{q'} G_{N,a} = 0$, the determinant of the Hessian matrix of $G_{N,a}$ is given by

$$\det \text{Hess}_{(p',q')} G_{N,a} |_{\nabla_{(p',q')} G_{N,a} = 0} = q(\eta_c) \left(4p'q' - N^2(p' + q') \partial_{p'} \alpha_c \right).$$

When $\nabla_{(p',q')} G_{N,a} = 0$ and $|\det \text{Hess}_{(p',q')} G_{N,a}| > c > 0$ for some small $c > 0$ we can apply the usual stationary phase in both variables p', q' . We expect the worst contributions to occur in a neighborhood of the critical points where $|\det \text{Hess}_{(p',q')} G_{N,a}| \leq c$ for some c sufficiently small. We turn variables with $\xi_1 = (p' + q')/2$ and $\xi_2 = (p' - q')/2$. Then $p' = \xi_1 + \xi_2$ and $q' = \xi_1 - \xi_2$, and we also let

$$\mu := A' + B' = N^2(2K_a^2 - 1 - X), \quad \nu := A' - B' = N^2(1 - X).$$

We will show that the most degenerate situation corresponds to $\nu = \mu = 0$ and $\xi_1 = 0, \xi_2 = 0$. Let $g_{N,a}(\xi_1, \xi_2) = G_{N,a}(\xi_1 + \xi_2, \xi_1 - \xi_2)$.

Case $c \lesssim |\xi_1|$. We first prove that, for ξ_1 outside a small neighbourhood of 0, the usual stationary phase applies in ξ_2 and the critical value $g_{N,a}(\xi_1, \xi_{2,c})$ may have degenerate critical points of order at most 2. The phase $g_{N,a}$ is stationary in ξ_2 when $\partial_{p'} G_{N,a} = \partial_{q'} G_{N,a}$ and from Remark 17 it follows that in this situation $\partial_{p'} \eta_c = \partial_{q'} \eta_c$ and $\partial_{p'} \alpha_c = \partial_{q'} \alpha_c$. The second derivative of $g_{N,a}$ with respect to ξ_2 is given by

$$\partial_{\xi_2, \xi_2}^2 g_{N,a}(\xi_1, \xi_2) = \left(\partial_{p'p'}^2 G_{N,a} - 2\partial_{p'q'}^2 G_{N,a} + \partial_{q'q'}^2 G_{N,a} \right) (p', q') |_{\xi_1, \xi_2}.$$

Using the explicit form of the second order derivatives of $G_{N,a}$ given above we obtain, at $p' = \xi_1 + \xi_2, q' = \xi_1 - \xi_2$ such that $p'^2 + N^2(X - \alpha_c) = q'^2 + N^2(1 - \alpha_c)$ and for which $\partial_{p'} \eta_c = \partial_{q'} \eta_c$, the following form for $\partial_{\xi_2, \xi_2}^2 g_{N,a}(\xi_1, \xi_2)$:

$$\partial_{\xi_2, \xi_2}^2 g_{N,a}(\xi_1, \xi_2) |_{\partial_{\xi_2} g_{N,a} = 0} = 2q^{1/2}(\eta_c)(p' + q') = 4q^{1/2}(\eta_c)\xi_1.$$

Since $q(\eta_c) = |\eta_c|q(\eta_c/|\eta_c|) \in [\frac{1}{2}m_0^2, \frac{3}{2}M_0^2]$ with m_0, M_0 defined in (31), it follows that the stationary phase applies in ξ_2 when $|\xi_1| \gtrsim c$. We denote $\xi_{2,c}$ the critical point that satisfies

$$\partial_{\xi_2} g_{N,a}(\xi_1, \xi_2) = \left(\partial_{p'} G_{N,a} - \partial_{q'} G_{N,a} \right) (p', q') |_{p'=\xi_1+\xi_2, q'=\xi_1-\xi_2} = 0,$$

and obtain

$$(\xi_1 + \xi_{2,c})^2 + N^2(X - \alpha_c) = (\xi_1 - \xi_{2,c})^2 + N^2(1 - \alpha_c),$$

which, in turn, yields $4\xi_1\xi_{2,c} = N^2(1 - X) = \nu$ and therefore $\xi_{2,c} = \frac{\nu}{4\xi_1}$. In the following we compute the higher order derivatives of the critical value of the phase $g_{N,a}(\xi_1, \xi_{2,c})$ with respect to ξ_1 and prove the following lemma :

Lemma 7. *For $|N| \geq 1$, the phase $g_{N,a}(\xi_1, \xi_{2,c})$ may have critical points degenerate of order at most 2.*

Proof. Recall that at the critical point $\xi_{2,c}$, Remark (17) implies $\partial_{p'}\eta_c = \partial_{q'}\eta_c$ and $\partial_{p'}\alpha_c = \partial_{q'}\alpha_c$. This in turn implies that the functions $\Theta_{1,2}$ in Lemma 4 coincide there, hence the functions $\mathcal{E}_{1,2}$ defined in (47),(48) coincide also at $\xi_{2,c}$. Notice that we kept the same notation $\mathcal{E}_{1,2}$ but we refer to now as functions of $(p'/N, q'/N) = (\xi_1 + \xi_2)/N, (\xi_1 - \xi_2)/N$. As $\mathcal{E}_1|_{p'^2+N^2X=q'^2+N^2} = \mathcal{E}_2|_{p'^2+N^2X=q'^2+N^2}$, setting $\mathcal{E} := \mathcal{E}_1|_{p'^2+N^2X=q'^2+N^2} = \mathcal{E}_2|_{p'^2+N^2X=q'^2+N^2}$ in the expression (46) yields

$$\sqrt{\alpha_c}|_{\partial_{\xi_2}g_{N,a}=0} = K_a - \frac{\xi_1}{N^2}(1 - a\mathcal{E})$$

and therefore

$$\begin{aligned} \partial_{\xi_1}(g_{N,a}(\xi_1, \xi_{2,c})) &= \partial_{\xi_1}g_{N,a}(\xi_1, \xi_{2,c}) + \frac{\partial\xi_{2,c}}{\partial\xi_1}\partial_{\xi_2}g_{N,a}(\xi_1, \xi_2)|_{\xi_2=\xi_{2,c}} \\ &= \partial_{\xi_1}g_{N,a}(\xi_1, \xi_{2,c}) \\ &= \left(\partial_{p'}G_{N,a} + \partial_{q'}G_{N,a}\right)(p', q')|_{\xi_1, \xi_{2,c}} \\ &= q^{1/2}(\eta_c) \left((\xi_1 + \xi_{2,c})^2 + N^2(X - \alpha_c) + (\xi_1 - \xi_{2,c})^2 + N^2(1 - \alpha_c) \right) \\ &= q^{1/2}(\eta_c) \left(2\xi_1^2 + 2\xi_{2,c}^2 - \mu - 2N^2 \left(\left(K_a - \frac{\xi_1}{N^2}(1 - a\mathcal{E}) \right)^2 - K_a^2 \right) \right) \\ &= q^{1/2}(\eta_c) \left(2\xi_1^2 \left(1 - \frac{1}{N^2}(1 - a\mathcal{E}) \right) + 2\xi_{2,c}^2 - \mu + 4K_a\xi_1(1 - a\mathcal{E}) \right) \\ (114) \quad &= q^{1/2}(\eta_c) \left(2\xi_1^2 \left(1 - \frac{1}{N^2}(1 - a\mathcal{E}) \right) + 2\frac{\nu^2}{16\xi_1^2} - \mu + 4K_a\xi_1(1 - a\mathcal{E}) \right). \end{aligned}$$

Taking the derivatives of (114) with respect to ξ_1 yields

$$\begin{aligned} \partial_{\xi_1, \xi_1}^2(g_{N,a}(\xi_1, \xi_{2,c})) &= q^{1/2}(\eta_c) \left[4\xi_1 \left(1 - \frac{1}{N^2}(1 - a(\mathcal{E} + \frac{1}{2}\xi_1\partial_{\xi_1}\mathcal{E})) \right) - \frac{\nu^2}{8\xi_1^3} \right. \\ &\quad \left. + 4K_a \left(1 - a(\mathcal{E} + \xi_1\partial_{\xi_1}\mathcal{E}) \right) \right] \\ &+ \left(\partial_{p'}(q^{1/2}(\eta_c)) + \partial_{q'}(q^{1/2}(\eta_c)) + \frac{\partial\xi_{2,c}}{\partial\xi_1}(\partial_{p'}(q^{1/2}(\eta_c)) - \partial_{q'}(q^{1/2}(\eta_c))) \right) \frac{\partial_{\xi_1}g_{N,a}(\xi_1, \xi_{2,c})}{q^{1/2}(\eta_c)}, \end{aligned}$$

where the last line vanishes when $\partial_{\xi_1} g_{N,a}(\xi_1, \xi_{2,c}) = 0$. In the same way we find

$$\partial_{\xi_1, \xi_1, \xi_1}^3 (g_{N,a}(\xi_1, \xi_{2,c}))|_{\partial_{\xi_1} (g_{N,a}(\xi_1, \xi_{2,c})) = \partial_{\xi_1, \xi_1}^2 (g_{N,a}(\xi_1, \xi_{2,c})) = 0} = q^{1/2}(\eta_c) \left(4 \left(1 - \frac{1}{N^2} \right) + \frac{3\nu^2}{8\xi_1^4} + O(a) \right).$$

Let first $|N| \geq 2$, then we immediately see that the third order derivative takes positive values and stays bounded from below by a fixed constant, $\partial_{\xi_1, \xi_1, \xi_1}^3 (g_{N,a}(\xi_1, \xi_{2,c})) \geq 2$, and therefore the critical points may be degenerate (when $\partial_{\xi_1, \xi_1}^2 (g_{N,a}(\xi_1, \xi_{2,c})) = 0$) of order at most 2. Let now $|N| = 1$ when the coefficient of $2\xi_1^2$ in (114) is $O(a)$. Assume that for $c \lesssim |\xi_1|$ the first two derivative vanish, then $\frac{\nu^2}{8\xi_1^3} = 4K_a + O(a)$ and therefore the third derivative cannot vanish since its main contribution is $\frac{3\nu^2}{8\xi_1^4}$. \square

Case $|\xi_1| \lesssim c$ for some small $0 < c < 1/2$. Let now $\xi_1 = p' + q'$ belong to a small neighbourhood of 0, $|\xi_1| \leq c < 1/2$: we first show that the stationary phase with non-degenerate critical point applies this time in ξ_1 . We compute (in the same way as we did for $|\xi_1| \gtrsim c$)

$$(115) \quad \begin{aligned} \partial_{\xi_1} g_{N,a}(\xi_1, \xi_2) &= \left(\partial_{p'} G_{N,a} + \partial_{q'} G_{N,a} \right) (p', q')|_{\xi_1, \xi_2} \\ &= q^{1/2}(\eta_c) \left((\xi_1 + \xi_2)^2 + N^2(X - \alpha_c) + (\xi_1 - \xi_2)^2 + N^2(1 - \alpha_c) \right). \end{aligned}$$

Using (46), we write again, with $K_a = K_a(\frac{Y}{4N}, \frac{T}{2N}) = \frac{T}{2N} q^{1/2}(\eta_c^0)$,

$$\sqrt{\alpha_c} = K_a - \frac{(\sigma + s)}{2N} + \frac{T}{2N} (q^{1/2}(\eta_c) - q^{1/2}(\eta_c^0)),$$

where in the new variables $\sigma + s = 2\xi_1/N$. Using (49), we have $(q^{1/2}(\eta_c) - q^{1/2}(\eta_c^0)) = \frac{a}{NT} O(\xi_1, \xi_2)$ and since $|\xi_1| \leq c < \frac{1}{2}$ is small, $a \leq \varepsilon_0$ and $\alpha_c \in [\frac{1}{2}, \frac{3}{2}]$ it follows from $K_a = \sqrt{\alpha_c} + O(c/N^2)$ that we must have $K_a \in [1/4, 2]$ for all $N \geq 1$. The derivative of $g_{N,a}(\xi_1, \xi_2)$ becomes

$$(116) \quad \begin{aligned} \partial_{\xi_1} g_{N,a}(\xi_1, \xi_2) &= q^{1/2}(\eta_c) \left\{ 2\xi_1^2 + 2\xi_2^2 - \mu - 2N^2 \left[\left(K_a - \frac{\xi_1}{N^2} + \frac{a}{N^2} O(\xi_1, \xi_2) \right)^2 - K_a^2 \right] \right\} \\ &= q^{1/2}(\eta_c) \left(2\xi_1^2 \left(1 - \frac{1}{N^2} \right) + 2\xi_2^2 - \mu + 4K_a \xi_1 + aO(\xi_1, \xi_2) \right). \end{aligned}$$

At the critical point, the second derivative with respect to ξ_1 is given by

$$(117) \quad \partial_{\xi_1, \xi_1}^2 g_{N,a}(\xi_1, \xi_2)|_{\partial_{\xi_1} g_{N,a}(\xi_1, \xi_2) = 0} = q^{1/2}(\eta_c) \left(4\xi_1 \left(1 - \frac{1}{N^2} \right) + 4K_a + O(a) \right),$$

and as $K_a \in [\frac{1}{4}, 2]$ it follows that the second derivative behaves like $4q^{1/2}(\eta_c)K_a$. The stationary phase applies for any $|N| \geq 1$ and provides a factor $\Lambda^{-1/2}$. We are left with the integral with respect to ξ_2 and we will show that it may be degenerate up to order 3. In

order to obtain precise bounds for it we need the exact form of the critical point ξ_1 as a function of ξ_2 . Let $\xi_{1,c}$ denote the solution to $\partial_{\xi_1} g_{N,a}(\xi_1, \xi_2) = 0$, then $\xi_{1,c}$ satisfies

$$(118) \quad 2\xi_{1,c}^2 + 2\xi_2^2 = \mu + 2N^2 \left[K_a^2 - \left(K_a^2 - \frac{\xi_1}{N^2} + \frac{T}{2N} (q^{1/2}(\eta_c) - q^{1/2}(\eta_c^0)) \right)^2 \Big|_{\xi_1, \xi_2} \right],$$

where, using (49), $\frac{T}{2N} (q^{1/2}(\eta_c) - q^{1/2}(\eta_c^0)) = O(\frac{a}{N^2})$. Notice that requiring $|\xi_{1,c}| \leq c$ for some small c implies $|\mu/2 - \xi_2^2| \lesssim c$. More precisely, if $|\mu/2 - \xi_2^2| \geq 4c$, the equation (118) has no real solution $\xi_{1,c}$ such that $|\xi_{1,c}| \leq c$.

Lemma 8. *For all $|N| \geq 1$ and $|\mu/2 - \xi_2^2| \leq 4c$, the equation (118) has one real solution of the following form*

$$(119) \quad \xi_{1,c} = (\mu/2 - \xi_2^2)\Xi_0 + a \left((\mu/2 - \xi_2^2)\Xi_1 + \xi_2^2\Xi_2 + \xi_2 \frac{\nu}{N^2}\Xi_3 \right),$$

where $K_{a=0} = \frac{|Y|}{4N} q^{1/2}(-Y/|Y|)$ and $\Xi_0 = \Xi_0(\mu/2 - \xi_2^2, K_{a=0}, 1/N^2)$ is given by

$$(120) \quad \Xi_0(\mu/2 - \xi_2^2, K_{a=0}, 1/N^2) = \frac{1}{K_{a=0} + \sqrt{K_{a=0}^2 + (\mu/2 - \xi_2^2)(1 - 1/N^2)}}$$

and where $\Xi_{1,2,3}$ are a smooth functions of $(\xi_2, \mu/2 - \xi_2^2, \nu/N^2, K_a, 1/N, a)$ such that $|\partial_{\xi_2}^k \Xi_j| \leq C_k$, for all $k \geq 0$, where C_k are positive constants.

Proof. Notice that for $a = 0$, the equation (118) has an unique, explicit solution $\xi_{1,c}|_{a=0}$ that reads as follows

$$\xi_{1,c}|_{a=0} = \frac{\mu/2 - \xi_2^2}{K_{a=0} + \sqrt{K_{a=0}^2 + (\mu/2 - \xi_2^2)(1 - 1/N^2)}},$$

that we rename $(\mu/2 - \xi_2^2)\Xi_0$ where $\Xi_0 = \Xi_0(\mu/2 - \xi_2^2, K_{a=0}, 1/N^2)$ is defined in (120). Let now $a \neq 0$. Using Lemma 4 with $s + \sigma = (p' + q')/N = 2\xi_1/N$, $\sigma - s = (p' - q')/N = 2\xi_2/N$, $(a - x)/a = \nu/N^2$, it follows that the critical point η_c is a function of ξ_1/N , ξ_2^2/N^2 and $\xi_2\nu/N^3$. Write $\xi_{1,c}$ under the form $\xi_{1,c} = (\mu/2 - \xi_2^2)\Xi_0 + a\Xi$ for some function Ξ to be determined; introducing this in (118) allows to obtain Ξ as a sum of smooth functions with factors $\mu/2 - \xi_2^2$, ξ_2^2 and $\xi_2\nu/N^2$ as follows

$$\Xi = (\mu/2 - \xi_2^2)\Xi_1 + \xi_2^2\Xi_2 + \xi_2 \frac{\nu}{N^2}\Xi_3,$$

where Ξ_j are smooth functions of $\mu/2 - \xi_2^2, \xi_2^2/N^2, \xi_2\nu/N^3$. \square

Let $\tilde{g}_{N,a}(\xi_2) := g_{N,a}(\xi_{1,c}, \xi_2)$: the first derivative of $\tilde{g}_{N,a}$ with respect to ξ_2 vanishes when $(\partial_{p'} G_{N,a} - \partial_{q'} G_{N,a})(p', q')|_{(\xi_{1,c}, \xi_2)} = 0$ which is equivalent to $4\xi_{1,c}\xi_2 = \nu$. We compute the second derivative of $\tilde{g}_{N,a}$ using $\partial_{\xi_2} \tilde{g}_{N,a} = \nu - 4\xi_{1,c}\xi_2$, with $\xi_{1,c}$ given in (119) as follows

$$\partial_{\xi_2}^2 \tilde{g}_{N,a} = -4(\xi_2 \partial_{\xi_2} \xi_{1,c} + \xi_{1,c}).$$

It follows that the critical points ξ_2 are degenerate when

$$(121) \quad (\mu/2 - \xi_2^2)\Xi_0 + a\left((\mu/2 - \xi_2^2)\Xi_1 + \xi_2^2\Xi_2 + \xi_2\frac{\nu}{N^2}\Xi_3\right) \\ = 2\xi_2^2\Xi_0\left(1 - (\mu/2 - \xi_2^2)\tilde{\Xi}_0(\mu/2 - \xi_2^2, K_a, 1/N^2)\right) \\ + a\left(2\xi_2^2(\Xi_1 - \Xi_2 - \frac{1}{2}\xi_2\partial_{\xi_2}\Xi_2 - \frac{\nu}{N^2}\partial_{\xi_2}\Xi_3) - \xi_2(\mu/2 - \xi_2^2)\partial_{\xi_2}\Xi_1 - \xi_2\frac{\nu}{N^2}\Xi_3\right),$$

where the term in the second line of (121) equals $\xi_2\partial_{\xi_2}\Xi_0$. We have thus set

$$\tilde{\Xi}_0(\mu/2 - \xi_2^2, K_a, 1/N^2) := \frac{(1 - 1/N^2)\Xi_0(\mu/2 - \xi_2^2, K_a, 1/N^2)}{2\sqrt{K_a^2 + (\mu/2 - \xi_2^2)(1 - 1/N^2)}}.$$

Consider $a = 0$ in (121) for the moment, then the critical points are degenerate if

$$\mu/2 - \xi_2^2 = 2\xi_2^2\left(1 - (\mu/2 - \xi_2^2)\tilde{\Xi}_0(\mu/2 - \xi_2^2, K_0, 1/N^2)\right).$$

Recall that $K_a \in [1/4, 2]$ and that $|\mu/2 - \xi_2^2| \leq 4c$ with c small enough. The last equation can be further written as follows

$$(\mu/2 - \xi_2^2)\left(2 + \frac{1}{1 - (\mu/2 - \xi_2^2)\tilde{\Xi}_0}\right) = \mu$$

which may have solutions only if μ is also small enough, $|\mu| \leq 10c$. Let $z = \mu/2 - \xi_2^2$; for $|z| \leq 4c$ and $|\mu| \leq 10c$ with c small enough, we may now seek the solution to $z(2 + 1/(1 - z\tilde{\Xi}_0(z, K_0, 1/N^2))) = \mu$ under the form $z = \mu Z_0(\mu, K_0, 1/N^2)$ and obtain an explicit form for $Z_0(\mu, K_0, 1/N^2)$ which satisfies $Z_0(0, K_0, 1/N^2) = \frac{1}{3}$. It follows that the solutions to (121) when $a = 0$ are functions of $\sqrt{\mu}$ which coincide at $\mu = 0$ when they both vanish. They have the form

$$\xi_{2,\pm}|_{a=0} = \pm\frac{\sqrt{\mu}}{\sqrt{6}}\left(1 + \mu\zeta(\mu, K_0, 1/N^2)\right), \text{ for some smooth function } \zeta.$$

Let now $a \neq 0$, then the solutions ξ_2 to (121) are functions of $\sqrt{\mu}, \nu/N^2, a$ that coincide at $\mu = \nu = 0$ when they are both equal to 0. In fact, since Ξ_1 is a function of $\mu/2 - \xi_2^2, \xi_2^2, \xi_2\nu/N^2$, it follows that $\xi_2\partial_{\xi_2}\Xi_1$ is also a function of $\mu/2 - \xi_2^2, \xi_2^2, \xi_2\nu/N^2$ and we can write $\mu/2 - \xi_2^2$ as a function of $\mu/2, \xi_2^2$ and $\xi_2\nu/N^2$,

$$(122) \quad \mu/2 - \xi_2^2 = 2\xi_2^2(1 - (\mu/2 - \xi_2^2)\tilde{\Xi}_0(\mu/2 - \xi_2^2, K_a, 1/N^2)) \\ + a\left(\xi_2^2F_1(\xi_2^2, \xi_2\nu/N^2, \mu) + \xi_2\frac{\nu}{N^2}F_2(\xi_2^2, \xi_2\nu/N^2, \mu)\right),$$

for some smooth functions $F_{1,2}$. Notice that, as $|\mu/2 - \xi_2^2| \leq 4c$ and a is small, (122) may have real solutions ξ_2 only for $|\xi_2^2| \leq 4c$. For such small ξ_2 , equation (122) has at most two distinct solutions (that coincide at $\mu = \nu = 0$) of the form

$$(123) \quad \xi_{2,\pm} = \pm\frac{\sqrt{\mu}}{\sqrt{6}}(1 + \mu\zeta(\mu, K_a, 1/N^2)) + a < (\sqrt{\mu}, \frac{\nu}{N^2}), (\zeta_{1,\pm}, \zeta_{2,\pm}) > (\sqrt{\mu}, \frac{\nu}{N^2}, K_a, a),$$

for some smooth functions $\zeta, \zeta_{j,\pm}$. We compute the third derivative of $\tilde{g}_{N,a}$ at the points $\xi_{2,\pm}$ defined in (123) where its second derivative vanishes. Using (121) yields

$$\begin{aligned}
(124) \quad \partial_{\xi_2, \xi_2, \xi_2}^3 \tilde{g}_{N,a}(\xi_{1,c}, \xi_2)|_{\xi_2=\xi_{2,\pm}} &= -4(2\partial_{\xi_2} \xi_{1,c} + \xi_2 \partial_{\xi_2, \xi_2}^2 \xi_{1,c})|_{\xi_{2,\pm}} \\
&= -8 \frac{\xi_{1,c}}{\xi_2} |_{\xi_2=\xi_{2,\pm}} - 4\xi_{2,\pm} \partial_{\xi_2, \xi_2}^2 \xi_{1,c} \\
&= 16\xi_2 \Xi_0 \left(1 - (\mu/2 - \xi_2^2) \tilde{\Xi}_0(\mu/2 - \xi_2^2, K_a, 1/N^2) \right) \\
&+ 8a \left(2\xi_2(\Xi_1 - \Xi_2 - \frac{1}{2}\xi_2 \partial_{\xi_2} \Xi_2 - \frac{\nu}{N^2} \partial_{\xi_2} \Xi_3) - (\mu/2 - \xi_2^2) \partial_{\xi_2} \Xi_1 - \frac{\nu}{N^2} \Xi_3 \right) |_{\xi_{2,\pm}} \\
&+ 8\xi_2 \Xi_0 (1 + O(\mu/2 - \xi_2^2) + O(a)) |_{\xi_{2,\pm}},
\end{aligned}$$

where the last line in (124) equals $-4\xi_{2,\pm} \partial_{\xi_2, \xi_2}^2 \xi_{1,c}$: we don't expand this formula since it comes with the small factor $\xi_{2,\pm}$, which is sufficiently small for what we need. The third and fourth lines of (124) represent the formula of $-8\partial_{\xi_2} \xi_{1,c}$ already obtained in (121) (where $\partial_{\xi_2} \xi_{1,c}$ comes with a factor ξ_2). Since the third derivative of $\tilde{g}_{N,a}$ is evaluated at $\xi_{2,\pm}$ we can replace (122) in (124) and obtain

$$\partial_{\xi_2, \xi_2, \xi_2}^3 \tilde{g}_{N,a}(\xi_{1,c}, \xi_2)|_{\xi_2=\xi_{2,\pm}} = \frac{12\xi_{2,\pm}}{K_a} (1 + O(\xi_{2,\pm}^2) + O(a)) + O(a\nu/N^2).$$

It follows that at $\mu = \nu = 0$ the order of degeneracy is higher as $\xi_{2,\pm}|_{\mu=\nu=0} = 0$ and $\partial_{\xi_2, \xi_2, \xi_3}^3 \tilde{g}_{N,a}|_{\xi_{2,\pm}, \mu=\nu=0} = 0$. We now write

$$\tilde{g}_{N,a}(\xi_2) = \tilde{g}_{N,a}(\xi_{2,\pm}) + (\xi_2 - \xi_{2,\pm}) \partial_{\xi_2} \tilde{g}_{N,a}(\xi_{2,\pm}) + \frac{(\xi_2 - \xi_{2,\pm})^3}{6} \partial_{\xi_2, \xi_2, \xi_2}^3 \tilde{g}_{N,a}(\xi_{2,\pm}) + O((\xi_2 - \xi_{2,\pm})^4),$$

where the fourth derivative doesn't cancel at $\xi_{2,\pm}$ since it stays close to $12/K_a \in [6, 48]$ as $K_a \in [1/4, 2]$. Since we are to have $\partial_{\xi_2} \tilde{g}_{N,a}(\xi_{2,\pm}) = 0$, we also deduce $\nu = 4\xi_{1,c}|_{\xi_{2,\pm}} \xi_{2,\pm}$, which gives

$$\begin{aligned}
(125) \quad \nu &= 4 \left(\pm \frac{\sqrt{\mu}}{\sqrt{6}} (1 + \mu\zeta(\mu)) + a(\sqrt{\mu}\zeta_{1,\pm} + \frac{\nu}{N^2} \zeta_{2,\pm}) \right) \\
&\times \left((\mu/2 - \xi_{2,\pm}^2) \Xi_0 + a((\mu/2 - \xi_{2,\pm}^2) \Xi_1 + \xi_{2,\pm}^2 \Xi_2 + \xi_{2,\pm} \frac{\nu}{N^2} \Xi_3) \right)
\end{aligned}$$

and replacing (123) in (119) yields ν of the form $\nu = \pm \frac{\sqrt{2}\mu^{3/2}}{3\sqrt{3}K_a} (1 + O(a))$, which is at leading order the equation of a cusp. At the degenerate critical points $\xi_{2,\pm}$ where $\nu = \pm \frac{\sqrt{2}\mu^{3/2}}{3\sqrt{3}K_a} (1 + O(a))$, the phase integral behaves like

$$I = \int_{\xi_2} \rho(\xi_2) e^{\mp i\Lambda \frac{\sqrt{2}\sqrt{\mu}}{K_a\sqrt{3}} (\xi_2 - \xi_{2,\pm})^3} d\xi_2,$$

and we may conclude in a small neighborhood of the set $\{\xi_2^2 + |\mu| + |\nu|^{2/3} \lesssim c\}$ (as outside this set, the non-stationary phase applies) by using Van der Corput lemma on the remaining oscillatory integral in ξ_2 with phase $\tilde{g}_{N,a}(\xi_2)$. In fact, on this set, $\partial_{\xi_2}^4 \tilde{g}_{N,a}$ is bounded from below, which yields an upper bound $\Lambda^{-1/4}$, uniformly in all parameters. When $\mu \neq 0$, the

third order derivative of the phase is bounded from below by $\frac{|\xi_2|}{K_a}$: either $|\mu/6 - \xi_2^2| \leq \mu/12$ and then $|\partial_{\xi_2}^3 \tilde{g}_{N,a}|$ is bounded from below by $|\mu|^{1/2}/(12K_a)$ or $|\mu/6 - \xi_2^2| \leq |\mu|/12$ in which case $|\partial_{\xi_2}^2 \tilde{g}_{N,a}|$ is bounded from below by $|\mu|/(12K_a)$. Hence, using that $K_a \in [1/4, 2]$, we find $|\partial_{\xi_2}^3 \tilde{g}_{N,a}| + |\partial_{\xi_2}^2 \tilde{g}_{N,a}| \gtrsim \sqrt{|\mu|}$ (recall that here μ is small so $\sqrt{|\mu|} \geq |\mu|$) which yields an upper bound $\sqrt{|\mu|}\Lambda^{-1/3}$. Eventually we obtain

$$(126) \quad |I| \lesssim \inf \left\{ \frac{1}{\Lambda^{1/4}}, \frac{1}{|\mu|^{1/6}\Lambda^{1/3}} \right\}.$$

From $\mu = A' + B'$ and $\nu = A' - B' \simeq \pm\mu^{3/2} \ll \mu$ for $\mu < 1$, we deduce that $A' \sim B'$ and therefore $\mu \sim 2B'$, which is our desired bound, as the non degenerate stationary phase in ξ_1 provided a factor $\Lambda^{-1/2}$.

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