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## Applications

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Marc Briane, Loïc Hervé. Asymptotics of ODE's flow on the torus through a singleton condition and a perturbation result. Applications. 2020. hal-02949388v1

**HAL Id: hal-02949388**

**<https://hal.science/hal-02949388v1>**

Preprint submitted on 25 Sep 2020 (v1), last revised 29 Sep 2021 (v2)

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# Asymptotics of ODE's flow on the torus through a singleton condition and a perturbation result. Applications

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Friday 25<sup>th</sup> September, 2020

## Abstract

This paper deals with the long time asymptotics  $X(t, x)/t$  of the flow  $X$  solution to the autonomous vector-valued ODE:  $X'(t, x) = b(X(t, x))$  for  $t \in \mathbb{R}$ , with  $X(0, x) = x$  a point of the torus  $Y_d := \mathbb{R}^d/\mathbb{Z}^d$ . We assume that the vector field  $b$  reads as the product  $\rho \Phi$ , where  $\rho : Y_d \rightarrow [0, \infty)$  is a non negative regular function and  $\Phi : Y_d \rightarrow \mathbb{R}^d$  is a non vanishing regular vector field. In this work, the singleton condition means that the rotation set  $\mathbf{C}_b$  composed of the average values of  $b$  with respect to the invariant probability measures for the flow  $X$  is a singleton  $\{\zeta\}$ , or equivalently, that  $\lim_{t \rightarrow \infty} X(t, x)/t = \zeta$  for any  $x \in Y_d$ . This combined with Liouville's theorem regarded as a divergence-curl lemma, first allows us to obtain the asymptotics of the flow  $X$  when  $b$  is a current field. Then, we prove a general perturbation result assuming that  $\rho$  is the uniform limit in  $Y_d$  of a positive sequence  $(\rho_n)_{n \in \mathbb{N}}$  satisfying for any  $n \in \mathbb{N}$ ,  $\rho \leq \rho_n$  and  $\mathbf{C}_{\rho_n \Phi}$  is a singleton  $\{\zeta_n\}$ . It turns out that the limit set  $\mathbf{C}_b$  either remains a singleton, or enlarges to the closed line set  $[0, \lim_n \zeta_n]$  of  $\mathbb{R}^d$ . We provide various corollaries of this perturbation result involving or not the classical ergodic condition, according to the positivity or not of some harmonic means of  $\rho$ . These results are illustrated by different examples which show that the perturbation result is limited to the scalar perturbation of  $\rho$ , and which highlight the alternative satisfied by the rotation set  $\mathbf{C}_b$ . Finally, we prove that the singleton condition allows us to homogenize in any dimension the linear transport equation induced by the oscillating velocity  $b(x/\varepsilon)$  beyond any ergodic condition satisfied by the flow  $X$ .

**Keywords:** ODE's flow, asymptotics, perturbation, homogenization, conductivity equation, transport equation, invariant measure, rotation set, ergodic

**Mathematics Subject Classification:** 34E10, 35B27, 37C10



# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Some variants of classical ergodicity results</b>	<b>6</b>
2.1	The singleton result . . . . .	6
2.2	A divergence-curl result . . . . .	8
2.3	A first application . . . . .	10
<b>3</b>	<b>Some new results involving the singleton condition</b>	<b>11</b>
3.1	A perturbation result . . . . .	11
3.2	Applications to the asymptotics of the ODE's flow . . . . .	14
<b>4</b>	<b>Examples</b>	<b>22</b>
<b>5</b>	<b>Homogenization of linear transport equations</b>	<b>28</b>
<b>A</b>	<b>Derivation of invariant probability measures</b>	<b>29</b>

## 1 Introduction

In this paper we study the large time asymptotics of the solution  $X(\cdot, x)$  for  $x \in \mathbb{R}^d$ , to the ODE

$$\begin{cases} \frac{\partial X}{\partial t}(t, x) = b(X(t, x)), & t \in \mathbb{R} \\ X(0, x) = x, \end{cases} \quad (1.1)$$

where  $b$  is a  $C^1$ -regular vector field defined in the torus  $Y_d := \mathbb{R}^d \setminus \mathbb{Z}^d$  (denoted by  $b \in C_\#^1(Y_d)^d$ ) according to the  $\mathbb{Z}^d$ -periodicity (1.13). The solution  $X(\cdot, x)$  is well-defined for any  $x \in Y_d$  by virtue of (1.17). More precisely, we focus on the existence of the limit of  $X(t, x)/t$  as  $t \rightarrow \infty$  for  $x \in Y_d$ . This question naturally arises in ergodic theory, since it involves the *dynamic flow*  $X$  induced by ODE (1.1), and the Borel measures  $\mu$  on the torus  $Y_d$  which are *invariant for the flow*  $X$ , *i.e.*

$$\forall t \in \mathbb{R}, \forall \psi \in C_\#^0(Y_d), \quad \int_{Y_d} \psi(X(t, y)) d\mu(y) = \int_{Y_d} \psi(y) d\mu(y). \quad (1.2)$$

A strengthened variant of the famous Birkhoff ergodic theorem [13, Theorem 2, Section 1.8] claims that if the flow is *uniquely ergodic*, *i.e.* there exists a unique probability measure  $\mu$  on  $Y_d$  which is invariant for the flow, then any function  $f \in C_\#^0(Y_d)$  satisfies

$$\forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \left[ \frac{1}{t} \int_0^t f(X(s, x)) ds \right] = \int_{Y_d} f(y) d\mu(y), \quad (1.3)$$

and the converse actually holds true. In the particular case where  $f = b$ , limit (1.3) yields

$$\forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \int_{Y_d} b(y) d\mu(y). \quad (1.4)$$

The unique ergodicity condition is a rather restrictive condition on the flow (1.1). Alternatively, define the set

$$\mathcal{I}_b := \{ \mu \in \mathcal{M}_p(Y_d) : \mu \text{ invariant for the flow } X \}, \quad (1.5)$$



where  $\mathcal{M}_p(Y_d)$  is the set of probability measures on  $Y_d$ , and the subset of  $\mathcal{J}_b$

$$\mathcal{E}_b := \{\mu \in \mathcal{J}_b : \mu \text{ ergodic for the flow } X\}. \quad (1.6)$$

It is known that the ergodic measures for the flow  $X$  are the extremal points of the convex set  $\mathcal{J}_b$  so that

$$\mathcal{J}_b = \text{conv}(\mathcal{E}_b). \quad (1.7)$$

Also define for any vector field  $b \in C^1_{\sharp}(Y_2)^d$  the two following non empty subsets of  $\mathbb{R}^d$ :

- The set of all the limit points of the sequences  $(X(n, x)/n)_{n \geq 1}$  for  $x \in Y_d$  (denoted by  $\rho_p(b)$  in [23])

$$\mathbf{A}_b := \bigcup_{x \in Y_d} \left[ \bigcap_{n \geq 1} \overline{\left\{ \frac{X(k, x)}{k} : k \geq n \right\}} \right]. \quad (1.8)$$

- The so-called Herman [18] rotation set

$$\mathbf{C}_b := \left\{ \int_{Y_2} b(y) d\mu(y) : \mu \in \mathcal{J}_b \right\} = \text{conv} \left\{ \int_{Y_2} b(y) d\mu(y) : \mu \in \mathcal{E}_b \right\}, \quad (1.9)$$

which is compact and convex.

An implicit consequence of [23, Theorem 2.4, Remark 2.5, Corollary 2.6] shows that

$$\mathbf{A}_b \subset \mathbf{C}_b = \text{conv}(\mathbf{A}_b) \quad \text{and} \quad \#\mathbf{A}_b = 1 \Leftrightarrow \#\mathbf{C}_b = 1. \quad (1.10)$$

Note that by definition (1.8) the equivalence of (1.10) can be written for any  $\zeta \in \mathbb{R}^d$ ,

$$\mathbf{C}_b = \{\zeta\} \quad \Leftrightarrow \quad \forall x \in Y_d, \quad \lim_{n \rightarrow \infty} \frac{X(n, x)}{n} = \zeta \quad \Leftrightarrow \quad \forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \zeta. \quad (1.11)$$

In the sequel the “singleton condition” means that Herman’s rotation set  $\mathbf{C}_b$  is a singleton  $\{\zeta\}$ . Proposition 2.1 below provides an alternative proof of (1.11). Then, the “singleton approach” consists in establishing sufficient conditions on the vector field  $b$  to ensure the singleton condition. The aim of this paper is to exploit this approach either to get the asymptotics of the flow  $X$  under suitable vector fields  $b$ , or in the less favorable cases to determine Herman’s rotation set  $\mathbf{C}_b$  as a closed line segment of  $\mathbb{R}^d$ .

First of all, revisiting Liouville’s theorem for invariant probability measures (see, *e.g.*, [13, Theorem 1, Section 2.2]) as a divergence-curl lemma (see Proposition 2.2) we obtain (see Proposition 2.4) a rather surprising nul asymptotics of the flow  $X$  associated with any current field of type  $b = A \nabla v$ , where  $A$  is a positive definite matrix-valued conductivity in  $C^1_{\sharp}(Y_d)^{d \times d}$  and  $v$  is a scalar potential in  $C^2_{\sharp}(Y_d)$ .

Actually, except the one-dimensional case where  $b$  is parallel to a fixed direction so that the flow can be computed explicitly (see Example 4.1), there are very few examples of vector field  $b$  for which the asymptotics of the flow (1.1) is completely known in dimension  $d \geq 2$ . There are at least two cases:

- If  $b$  is a non vanishing regular field in dimension two, Peirone [24, Theorem 3.1] has proved that the asymptotics of the flow  $X(\cdot, x)$  does exist at each point of  $x \in Y_2$ . Moreover, the singleton condition is satisfied under the extra ergodic condition that for any  $x \in Y_2$ , the flow  $X(\cdot, x)$  is not periodic in  $Y_2$  according to (1.18).



- Under the global rectification condition  $\nabla \Psi b = \zeta$  in  $Y_d$ , where  $\Psi$  is a  $C^2$ -diffeomorphism on the torus  $Y_d$  and  $\zeta$  is a non zero constant vector in  $\mathbb{R}^d$ , the set  $\mathbf{C}_b$  is the singleton  $\{\zeta\}$  in any dimension (see [8, Corollary 4.1] and Remark 2.1).

In the two former situations the vector field  $b$  does not vanish. In order to extend these two results among others to a vanishing vector field  $b$ , a natural question is to know if the singleton condition is stable under a uniform non vanishing perturbation  $b_n$  of  $b$  for  $n \in \mathbb{N}$ . We provide a partial answer to this question with the main result Theorem 3.1 of the paper. Restricting ourselves to a perturbation  $b_n = \rho_n \Phi$ , where  $(\rho_n)_{n \in \mathbb{N}}$  is a sequence of positive functions in  $C_{\sharp}^1(Y_d)$  converging uniformly in  $Y_d$  to some regular function  $\rho \leq \rho_n$  and where  $\Phi$  is a fixed vector field, we prove that if  $\mathbf{C}_{b_n} = \{\zeta_n\}$  for any  $n \in \mathbb{N}$ , then the sequence  $(\zeta_n)_{n \in \mathbb{N}}$  converges to some  $\zeta \in \mathbb{R}^d$ . Moreover, we get that the limit set  $\mathbf{C}_b$  is the singleton  $\{\zeta\}$  if  $\rho$  is positive, and that  $\mathbf{C}_b$  is the closed line segment  $[0, \zeta]$  if  $\rho$  is only non negative. In Theorem 3.1 it is essential that the vector field  $\Phi$  in  $b_n = \rho_n \Phi$ , is independent  $n$ , otherwise the perturbation result does not hold in general (see Remark 3.2 and Example 4.1). Moreover, the two-dimensional Example 4.2 shows that the sequence of singletons  $(\mathbf{C}_{b_n})_{n \in \mathbb{N}}$  may be actually enlarged to the limit closed line segment  $\mathbf{C}_b = [0, \zeta]$  with  $\zeta \neq 0$ . From the perturbation result we first deduce (see Corollary 3.1) that for a fixed vector field  $\Phi \in C_{\sharp}^1(Y_d)^d$ , the set of the positive functions  $\rho \in C_{\sharp}^1(Y_d)$  with  $\# \mathbf{C}_{\rho \Phi} = 1$  is closed for the uniform convergence in  $C_{\sharp}^1(Y_d)^d$ . This result does not extend to the larger set composed of the non negative functions  $\rho$  as shown in Remark 3.3. Then, we prove various corollaries of Theorem 3.1 in terms of the asymptotics of the ODE's flow (1.1):

- We show (see Corollary 3.2) that the asymptotics of the flow  $X(\cdot, x)$  associated with  $b = \rho \Phi$  does exist at any point  $x$  such that the orbit  $X(\mathbb{R}, x)$  is far enough from the set  $\{\rho = 0\}$ .
- When the flow associated with the vector field  $\Phi$  admits an invariant probability measure with a positive density  $\sigma \in C_{\sharp}^1(Y_d)$  with respect to Lebesgue's measure, we prove (see Corollary 3.3) the following alternative:
  - $\mathbf{C}_{\rho \Phi} = \{0\}$  if the harmonic mean of  $\rho/\sigma$  is equal to 0,
  - $\mathbf{C}_{\rho \Phi}$  is some closed line set  $[0, \zeta]$  of  $\mathbb{R}^d$  if the harmonic mean of  $\rho/\sigma$  is positive (see the two-dimensional Example 4.2).
- As a by-product of Corollary 3.3 we determine (see Corollary 3.4) the set  $\mathbf{C}_{\rho \Phi}$  in dimension two when  $\Phi$  is parallel to an orthogonal gradient satisfying an ergodic condition, extending Peirone's result [24, Theorem 3.1] (see Remark 3.5) to the case where the vector field  $b$  does vanish. Corollary 3.4 is illustrated in Example 4.4 by the case where  $b$  is a two-dimensional electric field, while dimension three is shown to be quite different. Similarly, we extend (see Corollary 3.5) the non ergodic case of [8, Corollary 4.1] to a vanishing vector field  $b$  in any dimension (see Example 4.3).

It turns out that the singleton approach cannot be regarded exclusively as an ergodic approach. It may contain an ergodic condition as in Corollary 3.4. But it may be also independent of any ergodic condition as in Corollary 3.5. This does make this approach an original alternative to the classical ergodic approach.

Finally, we apply the asymptotics of the flow (1.1) to the homogenization of the linear transport equation with an oscillating velocity

$$\frac{\partial u_{\varepsilon}}{\partial t}(t, x) + b\left(\frac{x}{\varepsilon}\right) \cdot \nabla_x u_{\varepsilon}(t, x) = 0 \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^d, \quad (1.12)$$



where the vector field  $b$  belongs to  $C^1_\#(Y_d)^d$ . Tartar [28] and Amirat *et al* [2, 3, 4] have shown that in general the homogenization of equation (1.12) leads to a nonlocal limit problem. Here, we focus on the cases where the homogenized equation remains a linear transport equation with some average velocity  $\langle b \rangle$  of the vector field  $b$ . In this perspective, the following works may be quoted: First, assuming that  $b$  is divergence free and the flow associated with  $b$  is ergodic, Brenier [6] has proved the convergence of the solution  $u_\varepsilon$  in any dimension. This result has been extended by Golse [16, Theorem 8] (see also [17]) for a more general velocity  $b(x, x/\varepsilon)$  with  $\operatorname{div}_y b(x, \cdot) = 0$ , assuming the ergodicity of the flows associated with the vector fields  $b(x, \cdot)$ . Hou and Xin [20] have performed the homogenization of (1.12) in dimension two with an oscillating initial condition  $u_\varepsilon(0, x) = u^0(x, x/\varepsilon)$ , assuming that  $b$  is a non vanishing divergence free vector field in  $\mathbb{R}^2$  and that the flow (1.1) is ergodic. To this end, they have used a two-scale convergence approach combined with Kolmogorov's theorem [21] involving some rotation number. Tassa [26] has extended the two-dimensional homogenization result of [20], assuming that the flow  $X$  associated with  $b$  has an invariant probability measure with a positive regular density with respect to Lebesgue's measure. More generally, Peirone [24] has proved the convergence of the solution  $u_\varepsilon$  to equation (1.12) in dimension two, under the sole assumption that  $b$  does not vanish in  $Y_2$ . More recently, we have proved in [8, Corollary 4.4] (see also [9] for an extension to the non periodic case) the homogenization of (1.12), replacing the classical ergodic condition by the rectification condition  $\nabla \Psi b = \zeta$  for some  $C^2$ -diffeomorphism  $\Psi$  on  $Y_d$  and a non zero constant vector  $\zeta$  in  $\mathbb{R}^d$ .

In the present case, extending the previous results in the case where the initial condition  $u_\varepsilon(0, \cdot)$  does not oscillate, we prove a new result (see Theorem 5.1) on the homogenization of the linear transport equation (1.12) in any dimension, only assuming the singleton condition. The results of Section 2 and Section 3 provide various situations where the singleton condition holds true.

The paper is organized as follows. In Section 2 we revisit the singleton condition and the Liouville theorem, from which we deduce the asymptotics of the flow (1.1) when  $b$  is a current field. In Section 3 we establish the perturbation Theorem 3.1 which is the main result of the paper, and we derive various corollaries on the set  $C_b$  and on the asymptotics of the flow  $X$ . Section 4 presents four examples which illustrate the results of Section 2 and Section 3. Section 5 is devoted to the homogenization of the transport equation (1.12) in connection with the asymptotics of the flow.

## Notation

- $|\cdot|$  denotes the euclidian norm in  $\mathbb{R}^N$ .
- $Y_d$  for  $d \geq 1$ , denotes the  $d$ -dimensional torus  $\mathbb{R}^d/\mathbb{Z}^d$ , which is identified to the cube  $[0, 1)^d$  in  $\mathbb{R}^d$ .
- $C^k_c(\mathbb{R}^d)$  for  $k \in \mathbb{N} \cup \{\infty\}$ , denotes the space of the real-valued functions in  $C^k(\mathbb{R}^d)$  with compact support.
- $C^k_\#(Y_d)$  for  $k \in \mathbb{N} \cup \{\infty\}$ , denotes the space of the real-valued functions  $f \in C^k(\mathbb{R}^d)$  which are  $\mathbb{Z}^d$ -periodic, *i.e.*

$$\forall \kappa \in \mathbb{Z}^d, \forall x \in \mathbb{R}^d, \quad f(x + \kappa) = f(x). \quad (1.13)$$

- $L^p_\#(Y_d)$  for  $p \geq 1$ , denotes the space of the real-valued functions in  $L^p_{\text{loc}}(\mathbb{R}^d)$  which are  $\mathbb{Z}^d$ -periodic.



- $\mathcal{M}(\mathbb{R}^d)$ , resp.  $\mathcal{M}(Y_d)$ , denotes the space of the Radon measures on  $\mathbb{R}^d$ , resp.  $Y_d$ , and  $\mathcal{M}_p(Y_d)$  denotes the space of the probability measures on  $Y_d$ .
- The notation  $\mathcal{J}_b$  in (1.5) will be used throughout the paper.
- $\mathcal{D}'(\mathbb{R}^d)$  denotes the space of the distributions on  $\mathbb{R}^d$ .
- Let  $a$  be a non negative function in  $L^1_\#(Y_d)$ , the arithmetic mean  $\bar{a}$  and the harmonic mean  $\underline{a}$  of  $a$  are defined by

$$\bar{a} := \int_{Y_d} a(y) dy \quad \text{and} \quad \underline{a} := \left( \int_{Y_d} \frac{dy}{a(y)} \right)^{-1}.$$

## Definitions and recalls

Let  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a vector-valued function in  $C^1_\#(Y_d)^d$ . Consider the dynamical system

$$\begin{cases} \frac{\partial X}{\partial t}(t, x) = b(X(t, x)), & t \in \mathbb{R} \\ X(0, x) = x \in \mathbb{R}^d. \end{cases} \quad (1.14)$$

The solution  $X(\cdot, x)$  to (1.14) which is known to be unique (see, *e.g.*, [19, Section 17.4]) induces the dynamic flow  $X$  defined by

$$\begin{aligned} X : \mathbb{R} \times \mathbb{R}^d &\rightarrow \mathbb{R}^d \\ (t, x) &\mapsto X(t, x), \end{aligned} \quad (1.15)$$

which satisfies the semi-group property

$$\forall s, t \in \mathbb{R}, \forall x \in \mathbb{R}^d, \quad X(s + t, x) = X(s, X(t, x)). \quad (1.16)$$

The flow  $X$  is actually well defined in the torus  $Y_d$ , since

$$\forall t \in \mathbb{R}, \forall x \in \mathbb{R}^d, \forall \kappa \in \mathbb{Z}^d, \quad X(t, x + \kappa) = X(t, x) + \kappa. \quad (1.17)$$

Property (1.17) follows immediately from the uniqueness of the solution  $X(\cdot, x)$  to (1.14) combined with the  $\mathbb{Z}^d$ -periodicity of  $b$ .

For any  $x \in Y_d$ , the solution  $X(\cdot, x)$  to (1.14) is said to be *periodic in the torus  $Y_d$*  if there exist  $T > 0$  and  $\kappa \in \mathbb{Z}^d$  such that

$$\forall t \in \mathbb{R}, \quad X(t + T, x) = X(t, x) + \kappa. \quad (1.18)$$

If  $\kappa = 0$  the solution is said to be periodic in  $\mathbb{R}^d$ .

## 2 Some variants of classical ergodicity results

### 2.1 The singleton result

Equivalences (1.11) have been obtained in [23] as a consequence of the so-called ergodic decomposition theorem. Here, we provide a simpler and more direct proof of (1.11), which is based on Proposition A.1 (in the Appendix) only involving the weak-\* compactness of  $\mathcal{M}_p(Y_d)$ .



**Proposition 2.1** *Let  $b \in C^1_\#(Y_d)^d$ . Then, the following equivalence holds for any  $\zeta \in \mathbb{R}^d$ ,*

$$\mathbf{C}_b = \{\zeta\} \quad \Leftrightarrow \quad \forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \zeta. \quad (2.1)$$

*Proof.* First, assume that  $\mathbf{C}_b = \{\zeta\}$ . Let  $x \in \mathbb{R}^d$ , let  $(t_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$  be such that  $\lim_n t_n = \infty$ , and define the sequence  $(u_n)_{n \in \mathbb{N}}$  by (A.2) with the function  $g := b \cdot \xi$  for  $\xi \in \mathbb{R}^d$  and  $x_n := x$ . Let  $a$  be a limit point of the sequence  $(u_n)_{n \in \mathbb{N}}$ . By Proposition A.1 there exists an invariant probability measure  $\mu \in \mathcal{J}_b$  for the flow  $X$  satisfying

$$a = \int_{Y_d} b(y) \cdot \xi \, d\mu(y),$$

which by hypothesis implies that  $a = \zeta \cdot \xi$ . Hence,  $\zeta \cdot \xi$  is the unique limit point of the bounded sequence  $(u_n)_{n \in \mathbb{N}}$  which thus converges to  $\zeta \cdot \xi$ . Therefore, due to the arbitrariness of the sequence  $(t_n)_{n \in \mathbb{N}}$  we obtain that

$$\forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \lim_{t \rightarrow \infty} \left( \frac{1}{t} \int_0^t b(X(s, x)) \, ds \right) = \zeta.$$

Conversely, assume that the right-hand side of (2.1) holds, which implies that

$$\forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \left( \frac{1}{t} \int_0^t b(X(s, x)) \, ds \right) = \zeta.$$

Then, integrating over  $Y_d$  the former equality with respect to any probability measure  $\mu \in \mathcal{J}_b$ , then applying successively Lebesgue's dominated convergence theorem and Fubini's theorem, we get that

$$\begin{aligned} \zeta &= \lim_{t \rightarrow \infty} \int_{Y_d} \left( \frac{1}{t} \int_0^t b(X(s, x)) \, ds \right) d\mu(x) \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left( \int_{Y_d} b(X(s, x)) \, d\mu(x) \right) ds = \int_{Y_d} b(x) \, d\mu(x), \end{aligned}$$

which shows that  $\mathbf{C}_b = \{\zeta\}$ . This concludes the proof of (2.1).  $\square$

**Remark 2.1** *Stability of the singleton condition by a diffeomorphism on the torus.*

A mapping  $\Psi \in C^1(\mathbb{R}^d)^d$  is said to be a  $C^1$ -diffeomorphism on  $Y_d$  if  $\Psi$  satisfies the following conditions:

- $\det(\nabla \Psi(x)) \neq 0$  for any  $x \in \mathbb{R}^d$ ,
- there exist a matrix  $A \in \mathbb{Z}^{d \times d}$  with  $|\det(A)| = 1$ , and a mapping  $\Psi_\# \in C^1_\#(Y_d)^d$  such that

$$\forall x \in \mathbb{R}^d, \quad \Psi(x) = Ax + \Psi_\#(x). \quad (2.2)$$

Note that the invertibility of  $A$  and the  $\mathbb{Z}^d$ -periodicity of  $\Psi_\#$  in (2.2) imply that  $\Psi$  is a proper function (i.e., the inverse image by the function of any compact set in  $\mathbb{R}^d$  is a compact set). Hence, by virtue of Hadamard-Caccioppoli's theorem [12] (also called Hadamard-Lévy's theorem) the mapping  $\Phi$  is actually a  $C^1$ -diffeomorphism on  $\mathbb{R}^d$ . Also note that due to  $A^{-1} \in \mathbb{Z}^{d \times d}$ , we have

$$\forall \kappa \in \mathbb{Z}^d, \quad \forall x \in Y_d, \quad \begin{cases} \Psi(x + \kappa) - \Psi(x) = A\kappa & \in \mathbb{Z}^d \\ \Psi^{-1}(x + \kappa) - \Psi^{-1}(x) = A^{-1}\kappa & \in \mathbb{Z}^d, \end{cases}$$



hence  $\Psi$  well defines an isomorphism on the torus.

Now, let  $b$  be a vector field in  $C_{\#}^1(Y_d)^d$  and let  $\Psi$  be a  $C^2$ -diffeomorphism on  $Y_d$ . Define the flow  $\tilde{X}$  obtained from  $\Psi$  by

$$\tilde{X}(t, x) := \Psi(X(t, \Psi^{-1}(x))) \quad \text{for } (t, x) \in \mathbb{R} \times Y_d. \quad (2.3)$$

Using the chain rule it is easy to check that the mapping  $\tilde{X}$  is the flow associated with the vector field  $\tilde{b} \in C_{\#}^1(Y_d)^d$  defined by

$$\tilde{b}(x) = \nabla \Psi(\Psi^{-1}(x)) b(\Psi^{-1}(x)) \quad \text{for } x \in Y_d. \quad (2.4)$$

Combining (2.2) and (2.3) we clearly have

$$\forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \frac{\tilde{X}(t, x)}{t} \text{ exists} \Leftrightarrow \lim_{t \rightarrow \infty} \frac{X(t, \Psi^{-1}(x))}{t} \text{ exists},$$

and in the case of existence of the limit for a given  $x \in Y_d$ , we get the equality

$$\lim_{t \rightarrow \infty} \frac{\tilde{X}(t, x)}{t} = A \left( \lim_{t \rightarrow \infty} \frac{X(t, \Psi^{-1}(x))}{t} \right).$$

This combined with equivalence (2.1) implies that

$$\#\mathbf{C}_{\tilde{b}} = 1 \Leftrightarrow \#\mathbf{C}_b = 1, \quad (2.5)$$

and in this case we obtain the equality  $\mathbf{C}_{\tilde{b}} = A \mathbf{C}_b$ . Therefore, the singleton condition is stable by any  $C^2$ -diffeomorphism on  $Y_d$ .

In particular, such a diffeomorphism has been used by Tassa [26] in dimension two, assuming that the first coordinate  $b_1$  of  $b$  does not vanish in  $Y_2$  and that there exists an invariant probability measure for the flow associated with  $b$  having a density  $\sigma \in C_{\#}^1(Y_2)$  with respect to Lebesgue's measure. In this case a variant [26, Theorem 2.3] of Kolmogorov's theorem [21] (which holds under the weaker assumption that  $b$  is non vanishing) provides a diffeomorphism on  $Y_2$  which rectifies the vector field  $b$  to a vector field  $\tilde{b} = a\xi$  with a positive function  $a \in C_{\#}^1(Y_2)$  and a fixed direction  $\xi \in \mathbb{R}^d$ . Under the additional ergodic assumption that the coordinates of  $\xi$  are rationally independent, the singleton condition is shown to be satisfied [26, Theorem 4.2]. Corollary 3.4 below provides a more general result in dimension two with a vanishing vector field  $b$ , without using a rectification of the field  $b$ .

## 2.2 A divergence-curl result

Liouville's theorem provides a criterium for a probability measure on a smooth compact manifold in  $\mathbb{R}^d$  (see, e.g., [13, Theorem 1, Section 2.2]) to be invariant for the flow. The next result revisits this theorem in  $\mathcal{M}_p(Y_d)$  in association with a divergence-curl result on the torus.

**Proposition 2.2** *Let  $b \in C_{\#}^1(Y_d)^d$  and let  $\mu \in \mathcal{M}_p(Y_d)$ . We define the Radon measure  $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^d)$  on  $\mathbb{R}^d$  by*

$$\int_{\mathbb{R}^d} \varphi(x) d\tilde{\mu}(x) := \int_{Y_d} \varphi_{\#}(y) d\mu(y) \quad \text{where} \quad \varphi_{\#}(\cdot) := \sum_{\kappa \in \mathbb{Z}^d} \varphi(\cdot + \kappa) \quad \text{for } \varphi \in C_c^0(\mathbb{R}^d). \quad (2.6)$$

*Then, the three following assertions are equivalent:*



(i)  $\mu$  is invariant for the flow  $X$ , i.e. (1.2) holds,

(ii)  $\tilde{\mu}b$  is divergence free in  $\mathbb{R}^d$ , i.e.

$$\operatorname{div}(\tilde{\mu}b) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \quad (2.7)$$

(iii)  $\mu b$  is divergence free in  $Y_d$ , i.e.

$$\forall \psi \in C_{\#}^1(Y_d), \quad \int_{Y_d} b(y) \cdot \nabla \psi(y) d\mu(y) = 0. \quad (2.8)$$

*Proof.*

*Proof of (i)  $\Rightarrow$  (ii).* Assume that  $\mu$  is invariant for the flow, i.e. (1.2). Let  $\varphi \in C_c^1(\mathbb{R}^d)$ . Since by (1.17) we have for any  $t \in \mathbb{R}$  and  $y \in \mathbb{R}^d$ ,

$$[\varphi(X(t, \cdot))]_{\#}(y) = \sum_{\kappa \in \mathbb{Z}^d} \varphi(X(t, y + \kappa)) = \sum_{\kappa \in \mathbb{Z}^d} \varphi(X(t, y) + \kappa) = \varphi_{\#}(X(t, y)), \quad (2.9)$$

it follows from (2.6) and the invariance of  $\mu$  that

$$\begin{aligned} \forall t \in \mathbb{R}, \quad \int_{\mathbb{R}^d} \varphi(X(t, x)) d\tilde{\mu}(x) &= \int_{Y_d} [\varphi(X(t, \cdot))]_{\#}(y) d\mu(y) = \int_{Y_d} \varphi_{\#}(X(t, y)) d\mu(y) = \\ &= \int_{Y_d} \varphi_{\#}(y) d\mu(y) = \int_{\mathbb{R}^d} \varphi(x) d\tilde{\mu}(x). \end{aligned}$$

Taking the derivative of the former expression with respect to  $t$ , we get that

$$\forall t \in \mathbb{R}, \quad \int_{\mathbb{R}^d} b(X(t, x)) \cdot \nabla \varphi(X(t, x)) d\tilde{\mu}(x) = 0,$$

which at  $t = 0$  yields

$$\forall \varphi \in C_c^1(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} b(x) \cdot \nabla \varphi(x) d\tilde{\mu}(x) = 0, \quad (2.10)$$

namely the variational formulation of the distributional equation (2.7).

*Proof of (ii)  $\Rightarrow$  (i).* Conversely, assume that equation (2.7) holds true, and let us prove that  $\mu$  is invariant for the flow  $X$ . Let  $\varphi \in C_c^1(\mathbb{R}^d)$  and define the function  $\phi \in C^1(\mathbb{R} \times \mathbb{R}^d)$  by  $\phi(t, x) := \varphi(X(t, x))$ . By the semi-group property (1.16) we have for any  $s, t \in \mathbb{R}$  and  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} \frac{\partial}{\partial s}(\phi(s+t, X(-s, x))) &= \frac{\partial}{\partial s}(\phi(t, x)) = 0 \\ &= \frac{\partial \phi}{\partial s}(s+t, X(-s, x)) - b(X(-s, x)) \cdot \nabla_x \phi(s+t, X(-s, x)), \end{aligned}$$

which at  $s = 0$  gives the classical transport equation

$$\forall t \in \mathbb{R}, \quad \forall x \in Y_d, \quad \frac{\partial \phi}{\partial t}(t, x) = b(x) \cdot \nabla_x \phi(t, x). \quad (2.11)$$

Hence, since  $\varphi(X(t, \cdot))$  is in  $C^1(\mathbb{R}^d)$  and has a compact support independent of  $t$  when  $t$  lies in a compact set of  $\mathbb{R}$ , we deduce from (2.11) and (2.7) that

$$\forall t \in \mathbb{R}, \quad \frac{d}{dt} \left( \int_{\mathbb{R}^d} \varphi(X(t, x)) d\tilde{\mu}(x) \right) = \int_{\mathbb{R}^d} b(x) \cdot \nabla_x (\varphi(X(t, x))) d\tilde{\mu}(x) = 0,$$



or equivalently,

$$\forall t \in \mathbb{R}, \quad \int_{\mathbb{R}^d} \varphi(X(t, x)) d\tilde{\mu}(x) = \int_{\mathbb{R}^d} \varphi(x) d\tilde{\mu}(x).$$

On the other hand, we have the following result.

**Lemma 2.3** ([7], Lemma 3.5) *For any smooth function  $\psi \in C^\infty_\#(Y_d)$  defined in  $Y_d$ , there exists a smooth function  $\varphi \in C^\infty_c(\mathbb{R}^d)$  with compact support in  $\mathbb{R}^d$  such that  $\psi = \varphi_\#$ .*

Hence, using relation (2.9) and definition (2.6) we get that for any  $\psi \in C^\infty_\#(Y)$ ,

$$\begin{aligned} \forall t \in \mathbb{R}, \quad \int_{Y_d} \psi(X(t, y)) d\mu(y) &= \int_{Y_d} \varphi_\#(X(t, y)) d\mu(y) = \int_{\mathbb{R}^d} [\varphi \circ X(t, \cdot)]_\#(y) d\mu(y) = \\ &= \int_{\mathbb{R}^d} \varphi(X(t, x)) d\tilde{\mu}(x) = \int_{\mathbb{R}^d} \varphi(x) d\tilde{\mu}(x) = \int_{Y_d} \varphi_\#(y) d\mu(y) = \int_{Y_d} \psi(y) d\mu(y), \end{aligned}$$

which shows that  $\mu$  is invariant for the flow  $X$ . We have just proved the equivalence between the invariance of  $\mu$  for the flow and the distributional equation (2.7) satisfied by  $\tilde{\mu}$ .

*Proof of (ii)  $\Leftrightarrow$  (iii).* The equivalence between (2.7), or equivalently (2.10), and (2.8) is a straightforward consequence of the following relation (which is deduced from  $[b \cdot \nabla \varphi]_\# = b \cdot \nabla \varphi_\#$  and (2.6))

$$\forall \varphi \in C^1_c(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} b(x) \cdot \nabla \varphi(x) d\tilde{\mu}(x) = \int_{Y_d} b(y) \cdot \nabla \varphi_\#(y) d\mu(y),$$

combined with Lemma 2.3. This concludes the proof of Proposition 2.2.  $\square$

**Remark 2.2** *Equation (2.8) can be considered as the divergence free of the vector-valued measure  $\mu b$  in the torus  $Y_d$ , while equation (2.7) is exactly the divergence free of the vector-valued measure  $\tilde{\mu} b$  in the space  $\mathbb{R}^d$ . Equation (2.8) is also equivalent to*

$$\forall (\nabla \psi) \in C^0_\#(Y_d)^d, \quad \int_{Y_d} b(y) \cdot \nabla \psi(y) d\mu(y) = \left( \int_{Y_d} b(y) d\mu(y) \right) \cdot \left( \int_{Y_d} \nabla \psi(y) dy \right), \quad (2.12)$$

since

$$\nabla \psi \in C^0_\#(Y_d)^d \Leftrightarrow \left( x \mapsto \psi(x) - x \cdot \int_{Y_d} \nabla \psi(y) dy \right) \in C^1_\#(Y_d).$$

So, condition (2.12) may be regarded as a divergence-curl result involving the divergence free vector field  $\mu b$  with the invariant probability measure  $\mu$  and the gradient field  $\nabla \psi$  with Lebesgue's measure.

## 2.3 A first application

As a direct consequence of Proposition 2.1 and Proposition 2.2, the following result gives the asymptotics of the ODE's flow (1.14) when  $b$  is a (not necessarily divergence free) current field.

**Proposition 2.4** *Assume that*

$$b = A \nabla v \text{ in } Y_d \quad \text{with} \quad \begin{cases} A \in C^1_\#(Y_d)^{d \times d}, \quad A = A^T \geq 0 \text{ in } Y_d, \\ v \in C^2_\#(Y_d)^d. \end{cases} \quad (2.13)$$

*Then, the flow  $X$  associated with  $b$  satisfies the nul asymptotics*

$$\forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = 0. \quad (2.14)$$



*Proof.* Let  $\mu \in \mathcal{M}_p(Y_d)$  be invariant for the flow  $X$ . By the divergence-curl relation (2.8) we have

$$\int_{Y_d} \underbrace{A(y) \nabla v(y) \cdot \nabla v(y)}_{\geq 0} d\mu(y) = \int_{Y_d} b(y) \cdot \nabla v(y) d\mu(y) = 0,$$

which implies that  $A \nabla v \cdot \nabla v = 0$   $\mu$ -a.e. in  $Y_d$ . Since the matrix-valued  $A$  is symmetric and non negative, from the Cauchy-Schwarz inequality we deduce that  $A \nabla v = 0$   $\mu$ -a.e. in  $Y_d$ , and thus

$$\int_{Y_d} b(y) d\mu(y) = \int_{Y_d} A(y) \nabla v(y) d\mu(y) = 0.$$

Therefore, we get that  $C_b = \{0\}$  which combined with (2.1) implies asymptotics (2.14).  $\square$

### 3 Some new results involving the singleton condition

#### 3.1 A perturbation result

The main result of the paper is the following.

**Theorem 3.1** *Let  $b \in C_{\sharp}^1(Y_d)^d$  be such that  $b = \rho \Phi$ , where  $\rho$  is a non negative not null function in  $C_{\sharp}^1(Y_d)$  and  $\Phi$  is a non vanishing vector field in  $C_{\sharp}^1(Y_d)^d$ . Assume that there exists a sequence  $(\rho_n)_{n \in \mathbb{N}} \in C_{\sharp}^1(Y_d)^{\mathbb{N}}$  such that*

- (i) *for any  $n \in \mathbb{N}$ ,  $\rho_n > 0$ , and  $(\rho_n)_{n \in \mathbb{N}}$  converges uniformly to  $\rho$  on  $Y_d$ ,*
- (ii) *for any  $n \in \mathbb{N}$ ,  $C_{b_n} = \{\zeta_n\}$  for some  $\zeta_n \in \mathbb{R}^d$ , where  $b_n := \rho_n \Phi$ .*

*Then, the sequence  $(\zeta_n)_{n \in \mathbb{N}}$  converges to some  $\zeta \in \mathbb{R}^d$ . Moreover, we have the following alternative:*

- *If  $\rho$  is positive in  $Y_d$ , then  $C_b = \{\zeta\}$  with  $\zeta \neq 0$ .*
- *If  $\rho$  vanishes in  $Y_d$ , then  $C_b = [0, \zeta]$ . Moreover, we have  $\{0, \zeta\} \subset A_b$ .*

*Proof.* First of all, assume that  $C_b \neq \{0\}$ , namely there exists  $\mu \in \mathcal{J}_b$  satisfying

$$\int_{Y_d} b(x) d\mu(x) \neq 0.$$

This implies that

$$\int_{Y_d} \rho(x) d\mu(x) = \int_{Y_d} \frac{|b(x)|}{|\Phi(x)|} d\mu(x) > 0. \quad (3.1)$$

For every  $n \in \mathbb{N}$ , define the probability measure  $\mu_n$  on  $Y_d$  by

$$d\mu_n(x) := C_n \frac{\rho(x)}{\rho_n(x)} d\mu(x), \quad \text{where} \quad C_n := \left( \int_{Y_d} \frac{\rho(y)}{\rho_n(y)} d\mu(y) \right)^{-1} \in (0, \infty) \quad (3.2)$$

due to  $\rho_n > 0$  and (3.1). Since  $\mu \in \mathcal{J}_b$ , it follows from equality (2.8) that

$$\forall \varphi \in C_{\sharp}^1(Y_d), \quad \int_{Y_d} \rho_n(x) \Phi(x) \cdot \nabla \varphi(x) d\mu_n(x) = C_n \int_{Y_d} \rho(x) \Phi(x) \cdot \nabla \varphi(x) d\mu(x) = 0,$$



hence by the equivalence (i)-(iii) of Proposition 2.2,  $\mu_n \in \mathcal{J}_{b_n}$ . Then, from assumption  $\mathbf{C}_{b_n} = \{\zeta_n\}$  we deduce that

$$\zeta_n = \int_{Y_d} b_n(x) d\mu_n(x) = C_n \int_{Y_d} \rho_n(x) \Phi(x) \frac{\rho(x)}{\rho_n(x)} d\mu(x) = C_n \int_{Y_d} b(x) d\mu(x).$$

Moreover, by Lebesgue's theorem and (3.1) we have

$$\lim_{n \rightarrow \infty} C_n = c_\mu := \left( \int_{\{\rho > 0\}} d\mu(x) \right)^{-1} \in [1, \infty). \quad (3.3)$$

Therefore, the sequence  $(\zeta_n)_{n \in \mathbb{N}}$  converges to some  $\zeta \in \mathbb{R}^d$  which is independent of  $\mu$  and satisfies the equality

$$\zeta = c_\mu \int_{Y_d} b(x) d\mu(x), \quad \text{for any } \mu \in \mathcal{J}_b \text{ with } \int_{Y_d} b(x) d\mu(x) \neq 0. \quad (3.4)$$

- If  $\rho$  is positive in  $Y_d$ , then (3.1) and  $c_\mu = 1$  hold for any  $\mu \in \mathcal{J}_b$ . This combined with (3.4) implies that  $\mathbf{C}_b \setminus \{0\} = \{\zeta\}$ . Therefore, due to the convexity of  $\mathbf{C}_b$  we obtain that  $\mathbf{C}_b = \{\zeta\}$ .
- On the contrary, assume that there exists  $\alpha \in Y_d$  such that  $\rho(\alpha) = 0$ . Since the Dirac distribution  $\delta_\alpha$  at  $\alpha$  is invariant for the flow associated with  $b$ , we have

$$0 = \int_{Y_d} b(x) d\delta_\alpha(x) \in \mathbf{C}_b.$$

If the sequence  $(\zeta_n)_{n \in \mathbb{N}}$  converges to 0, then we have  $\mathbf{C}_b = \{0\}$ . Indeed, if  $\mathbf{C}_b \neq \{0\}$ , then the first step of the proof implies (3.3) and (3.4) for some measure  $\mu \in \mathcal{J}_b$  with  $\zeta = \lim_n \zeta_n \neq 0$ , which yields a contradiction.

Now, assume that the sequence  $(\zeta_n)_{n \in \mathbb{N}}$  does not converge to 0, and consider a subsequence  $(\zeta_{n_k})_{k \in \mathbb{N}}$  which converges to some  $\hat{\zeta} \neq 0$ . Up to extract a new subsequence, we may assume that the sequence  $(\mu_{n_k})_{k \in \mathbb{N}}$  converges weakly  $*$  to some probability measure  $\hat{\mu}$  on  $Y_d$ . Passing to the limit in the divergence-curl relation (2.8) satisfied by  $\mu_{n_k} \in \mathcal{J}_{b_{n_k}}$ :

$$\forall \varphi \in C_\#^1(Y_d), \quad \int_{Y_d} b_{n_k}(x) \cdot \nabla \varphi(x) d\mu_{n_k}(x) = 0, \quad (3.5)$$

we get that

$$\forall \varphi \in C_\#^1(Y_d), \quad \int_{Y_d} b(x) \cdot \nabla \varphi(x) d\hat{\mu}(x) = 0, \quad (3.6)$$

so that  $\hat{\mu} \in \mathcal{J}_b$  again by the equivalence (i)-(iii) of Proposition 2.2. Moreover, by the uniform convergence of the sequence  $(b_n)_{n \in \mathbb{N}}$  to  $b$  we have

$$0 \neq \hat{\zeta} = \lim_{k \rightarrow \infty} \zeta_{n_k} = \lim_{k \rightarrow \infty} \int_{Y_d} b_{n_k}(x) d\mu_{n_k}(x) = \int_{Y_d} b(x) d\hat{\mu}(x) \in \mathbf{C}_b \setminus \{0\}. \quad (3.7)$$

Therefore, we may apply the first step of the proof with  $\mu = \hat{\mu}$ . Hence, the whole sequence  $(\zeta_n)_{n \in \mathbb{N}}$  converges to  $\zeta = \hat{\zeta} \in \mathbf{C}_b \setminus \{0\}$ . Moreover, by (3.3) equality (3.4) also reads as

$$\int_{Y_d} b(x) d\mu(x) = \frac{1}{c_\mu} \zeta \in [0, \zeta], \quad \text{for any } \mu \in \mathcal{J}_b \text{ with } \int_{Y_d} b(x) d\mu(x) \neq 0. \quad (3.8)$$

This combined with  $0 \in \mathbf{C}_b$ ,  $\zeta \in \mathbf{C}_b \setminus \{0\}$  and the convexity of  $\mathbf{C}_b$  implies that  $\mathbf{C}_b = [0, \zeta]$ .



Now, let us prove that  $\{0, \zeta\} \subset \mathbf{A}_b$ . On the one hand, since  $X(\cdot, \alpha) = \alpha$ , we have

$$\lim_{t \rightarrow \infty} \frac{X(t, \alpha)}{t} = 0,$$

hence  $0 \in \mathbf{A}_b$ . On the other hand, by equality (3.9) in Lemma 3.1 below (based on the above proved formulas (3.3) and (3.8)), we get immediately that  $\zeta \in \mathbf{A}_b$ .

Finally, it remains to deal with the case  $\mathbf{C}_b = \{0\}$ . We have just to prove that the sequence  $(\zeta_n)_{n \in \mathbb{N}}$  converges to 0. Repeating the argument between (3.5) and (3.7) for any subsequence  $(\zeta_{n_k})_{k \in \mathbb{N}}$  which converges to  $\hat{\zeta} \in \mathbb{R}^d$  and any subsequence  $(\mu_{n_k})_{k \in \mathbb{N}}$  which converges weakly  $*$  to  $\hat{\mu}$  in  $\mathcal{M}_p(Y_d)$ , we get that  $\hat{\mu} \in \mathcal{J}_b$  and

$$\hat{\zeta} = \lim_{k \rightarrow \infty} \zeta_{n_k} = \lim_{k \rightarrow \infty} \int_{Y_d} b_{n_k}(x) d\mu_{n_k}(x) = \int_{Y_d} b(x) d\hat{\mu}(x) = 0.$$

Therefore, the whole sequence  $(\zeta_n)_{n \in \mathbb{N}}$  converges to 0. The proof of Theorem 3.1 is now complete.  $\square$

**Remark 3.1** *In the setting of Theorem 3.1, Example 4.2 below provides a two-dimensional case of vector field  $b = \rho \Phi$  with a vanishing function  $\rho$ , in which  $\mathbf{C}_b$  is a closed line segment of  $\mathbb{R}^2$  not reduced to a singleton and thus  $\#\mathbf{A}_b \geq 2$ .*

**Lemma 3.1** *Let  $b$  be a vector field satisfying the assumptions of Theorem 3.1 with a vanishing function  $\rho$ . Then, for any  $\mu \in \mathcal{J}_b$  with  $\int_{Y_d} b(x) d\mu(x) \neq 0$ , we have*

$$\lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \zeta \quad \text{for } \mu\text{-a.e. } x \in \{\rho > 0\}. \quad (3.9)$$

*Proof.* Let  $\mu \in \mathcal{J}_b$  with  $\int_{Y_d} b(x) d\mu(x) \neq 0$ . By Birkhoff's theorem, there exists a measurable function  $x \mapsto g(x)$  such that by (1.8) and (1.10)

$$\lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = g(x) \in \mathbf{A}_b \subset \mathbf{C}_b \quad \text{for } \mu\text{-a.e. } x \in Y_d.$$

(Also see Proposition A.1 in the Appendix to get that  $g(x) \in \mathbf{C}_b$ ). Hence, by virtue of Theorem 3.1 we have

$$\lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \Delta(x) \zeta \quad \text{for } \mu\text{-a.e. } x \in Y_d, \quad (3.10)$$

where  $\Delta : Y_d \rightarrow [0, 1]$  is a measurable function such that  $\Delta(x) = 0$  if  $\rho(x) = 0$ . Applying successively convergence (3.10), Lebesgue's theorem, Fubini's theorem and the invariance of the measure  $\mu$ , we get that

$$\begin{aligned} \left( \int_{Y_d} \Delta(x) d\mu(x) \right) \zeta &= \lim_{t \rightarrow \infty} \int_{Y_d} \frac{X(t, x)}{t} d\mu(x) = \lim_{t \rightarrow \infty} \int_{Y_d} \left( \frac{1}{t} \int_0^t b(X(s, x)) ds \right) d\mu(x) \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left( \int_{Y_d} b(X(s, x)) d\mu(x) \right) ds = \int_{Y_d} b(x) d\mu(x), \end{aligned}$$

or equivalently,

$$\int_{\{\rho > 0\}} b(x) d\mu(x) = \left( \int_{\{\rho > 0\}} \Delta(x) d\mu(x) \right) \zeta.$$



Moreover, since  $\int_{Y_d} b(x) d\mu(x) \neq 0$ , from the formulas (3.3) and (3.8) in the proof of Theorem 3.1 we deduce that

$$\int_{Y_d} b(x) d\mu(x) = \int_{\{\rho>0\}} b(x) d\mu(x) = \beta_\mu \zeta, \quad \text{where} \quad \beta_\mu := \int_{\{\rho>0\}} d\mu(x),$$

so that

$$\int_{\{\rho>0\}} d\mu(x) = \int_{\{\rho>0\}} \Delta(x) d\mu(x),$$

or equivalently,

$$\int_{\{\rho>0\}} (1 - \Delta(x)) d\mu(x) = 0.$$

Since the function  $\Delta$  takes values in  $[0, 1]$ , the former equality implies that  $\Delta = 1$  for  $\mu$ -a.e. in  $\{\rho > 0\}$ , which combined with (3.10) yields the desired limit (3.9).  $\square$

**Remark 3.2** *Theorem 3.1 cannot be extended to the more general case where the direction  $\Phi$  of  $b_n$  also depends on  $n$ , as shown in Example 4.1 below. More precisely, the independence of  $\Phi$  with respect to  $n$  is crucial for the proof of Theorem 3.1 to build in (3.2) the invariant probability measure  $\mu_n \in \mathcal{J}_{b_n}$  from a given invariant probability measure  $\mu \in \mathcal{J}_b$ .*

**Corollary 3.1** *Let  $\Phi$  be a non vanishing vector field in  $C_\#^1(Y_d)^d$ . Then, we have for the uniform convergence topology in  $C_\#^0(Y_d)$ ,*

$$\begin{aligned} \{\rho \in C_\#^1(Y_d) : \rho > 0 \text{ and } \#C_{\rho\Phi} = 1\} &= \{\rho \in C_\#^1(Y_d) : \rho > 0 \text{ and } \#A_{\rho\Phi} = 1\} \\ &\text{is a closed subset of } \{\rho \in C_\#^1(Y_d) : \rho > 0\}. \end{aligned} \quad (3.11)$$

*Proof.* The equality of the sets in (3.11) follows directly from (1.10). Moreover, the fact that the first set is closed is a straightforward consequence of the first case of Theorem 3.1.  $\square$

**Remark 3.3** *Example 1 of [22] provides a two-dimensional case where the field  $b^* = \Phi^*$ , i.e.  $\rho^* = 1$ , is such that  $\#A_{\Phi^*} = 2$ . Therefore, by (3.11) there exists an open ball  $B^*$  centered at  $\rho^* = 1$  such that  $\#A_{\rho\Phi^*} > 1$  for any  $\rho \in B^*$ .*

*On the contrary, the assertion (3.11) of Corollary 3.1 does not hold in general when condition  $\rho > 0$  is enlarged to condition  $\rho \geq 0$ . Indeed, in the setting of Theorem 3.1 the two-dimensional Example 4.2 provides a sequence of fields  $(b_n = \rho_n \Phi)_{n \geq 1}$  which converges uniformly in  $Y_2$  to some field  $b = \rho \Phi$ , so that  $C_b$  is not reduced to a singleton while  $C_{b_n}$  is a singleton for any  $n \geq 1$ . Therefore, in this case the set*

$$\{r \in C_\#^1(Y_d) : r \geq 0 \text{ and } \#C_{r\Phi} = 1\} = \{r \in C_\#^1(Y_d) : r \geq 0 \text{ and } \#A_{r\Phi} = 1\}$$

*is not closed in  $\{r \in C_\#^1(Y_d) : r \geq 0\}$ .*

## 3.2 Applications to the asymptotics of the ODE's flow

The first result provides the asymptotics of the flow  $X(\cdot, x)$  at any point  $x$  whose orbit does not meet the set  $\{\rho = 0\}$  in  $Y_d$ .

**Corollary 3.2** *Assume that the conditions of Theorem 3.1 hold with a vanishing function  $\rho$ . Denote by  $\pi$  the canonical projection from  $\mathbb{R}^d$  on the torus  $Y_d$ , and define for  $x \in \mathbb{R}^d$ ,*

$$F_x := \overline{\pi(X(\mathbb{R}, x))}^{Y_d}, \quad (3.12)$$

*i.e. the closure in  $Y_d$  of the projection on  $Y_d$  of the orbit  $X(\mathbb{R}, x)$  of  $x$ . Then, we have*

$$\forall x \in Y_d, \quad F_x \cap \{\rho = 0\} = \emptyset \Rightarrow \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \zeta. \quad (3.13)$$



*Proof.* Let  $x \in Y_d$  be such that  $F_x \cap \{\rho = 0\} = \emptyset$ . First, by virtue of Proposition A.1 with  $g = b_i$  and  $x_n = x$ , any limit point derived from the asymptotics  $X(t, x)/t$  as  $t \rightarrow \infty$ , is of the form

$$\int_{Y_d} b(y) d\mu(y) \quad \text{for some } \mu \in \mathcal{I}_b.$$

Let us prove that the support of such an invariant probability measure  $\mu$  is contained in the closed set  $F_x$ . By Lemma A.2  $\mu$  is a limit point for the weak-\* of some sequence  $(\nu_n)_{n \in \mathbb{N}}$  in  $\mathcal{M}_p(Y_d)$  given by

$$\int_{Y_d} f(y) d\nu_n(y) = \frac{1}{r_n} \int_0^{r_n} f(X(s, x)) ds \quad \text{for } f \in C_{\#}^0(Y_d).$$

Let  $f \in C_{\#}^0(Y_d)$  which is nul in  $F_x$ . Then, since  $\pi(X(s, x)) \in F_x$  for any  $s \in \mathbb{R}$ , we have

$$\int_{Y_d} f(y) d\nu_n(y) = 0.$$

Passing to the limit in the previous equality we get that for any  $f \in C_{\#}^0(Y_d)$  nul in  $F_x$ ,

$$\int_{Y_d} f(y) d\mu(y) = 0,$$

which means that the support of  $\mu$  is contained in  $F_x$ .

Now, let  $a$  be a limit point of  $X(t, x)/t$  as  $t \rightarrow \infty$ . Then, we have

$$a = \int_{Y_d} f(y) d\mu(y) \quad \text{with} \quad \text{Supp}(\mu) \subset F_x \subset \{\rho > 0\}.$$

Hence, the constant  $c_{\mu}$  of (3.3) is equal to 1, so that by (3.4) we get that

$$\int_{Y_d} b(y) d\mu(y) = \zeta.$$

Therefore, any limit point  $a$  is equal to the constant  $\zeta$  obtained in Theorem 3.1, which implies that

$$\lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \zeta.$$

The desired implication (3.13) is thus established.  $\square$

The following result provides some condition on the direction  $\Phi$  of the vector field  $b = \rho \Phi$ , which allows us to specify the result of Theorem 3.1 when the function  $\rho$  vanishes.

**Corollary 3.3** *Consider a vector field  $b = \rho \Phi \in C_{\#}^1(Y_d)^d$ , a sequence  $(\rho_n)_{n \in \mathbb{N}} \in C_{\#}^1(Y_d)^{\mathbb{N}}$  and  $b_n = \rho_n \Phi$  satisfying the assumptions of Theorem 3.1. Also assume that there exists a positive function  $\sigma \in C_{\#}^1(Y_d)$  such that  $\sigma \Phi$  is divergence free in  $\mathbb{R}^d$ , and that  $\rho$  vanishes in  $Y_d$ .*

*We have the following alternative involving the harmonic mean  $\underline{\rho/\sigma}$  of  $\rho/\sigma$ :*

- *If  $\underline{\rho/\sigma} = 0$ , then the flow  $X$  associated with  $b$  satisfies the asymptotics*

$$\forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = 0. \quad (3.14)$$



- If  $\underline{\rho/\sigma} > 0$ , then we have

$$\mathbf{C}_b = [0, \zeta] \quad \text{with} \quad \zeta := \underline{\rho/\sigma} \int_{Y_d} \Phi(y) \sigma(y) dy. \quad (3.15)$$

*Proof.* Since  $\sigma \Phi$  is divergence free in  $\mathbb{R}^d$  and  $\sigma$  is  $\mathbb{Z}^d$ -periodic, by virtue of Proposition 2.2 the probability measure on  $Y_d$ :  $\sigma(x)/\bar{\sigma} dx$ , where  $\bar{\sigma} > 0$  is the arithmetic mean of  $\sigma$ , is an invariant probability measure for the flow associated with the vector field  $\Phi$ , i.e.  $\sigma(x)/\bar{\sigma} dx \in \mathcal{I}_\Phi$ .

For every  $n \in \mathbb{N}$ , define the probability measure  $\mu_n$  on  $Y_d$  by

$$d\mu_n(x) := \frac{C_n}{\rho_n(x)} \sigma(x) dx \quad \text{where} \quad C_n := \left( \int_{Y_d} \frac{1}{\rho_n(y)} \sigma(y) dy \right)^{-1}. \quad (3.16)$$

Due to  $\sigma(x)/\bar{\sigma} dx \in \mathcal{I}_\Phi$  we have by Proposition 2.2

$$\forall \psi \in C_\#^1(Y_d), \quad \int_{Y_d} b_n(x) \cdot \nabla \psi(x) d\mu_n(x) = C_n \int_{Y_d} \Phi(x) \cdot \nabla \psi(x) \sigma(x) dx = 0,$$

which again by Proposition 2.2 implies that  $\mu_n \in \mathcal{I}_{b_n}$ . This combined with the singleton assumption  $\mathbf{C}_{b_n} = \{\zeta_n\}$  yields

$$\zeta_n = \int_{Y_d} b_n(x) d\mu_n(x) = C_n \int_{Y_d} \Phi(x) \sigma(x) dx. \quad (3.17)$$

- If  $\underline{\rho/\sigma} = 0$ , then by Fatou's lemma we get that

$$\infty = \int_{Y_d} \frac{\sigma(y)}{\rho(y)} dy \leq \liminf_{n \rightarrow \infty} \left( \int_{Y_d} \frac{\sigma(y)}{\rho_n(y)} dy \right),$$

which implies that the sequence  $(C_n)_{n \in \mathbb{N}}$  tends to 0. Hence, by (3.17) the sequence  $(\zeta_n)_{n \in \mathbb{N}}$  converges to  $\zeta = 0$ . Therefore, by the second case of Theorem 3.1 we have  $\mathbf{C}_b = \{0\}$ , or equivalently by (2.1), the nul asymptotics (3.14) holds.

- If  $\underline{\rho/\sigma} > 0$ , then from the convergence of  $\rho_n$  to  $\rho$ , the inequality  $\rho_n \geq \rho$  and Lebesgue's theorem we deduce that

$$\lim_{n \rightarrow \infty} \int_{Y_d} \frac{\sigma(y)}{\rho_n(y)} dy = \int_{Y_d} \frac{\sigma(y)}{\rho(y)} dy = \frac{1}{\underline{\rho/\sigma}} < \infty,$$

which by (3.16), (3.17) implies that

$$\zeta = \lim_{n \rightarrow \infty} \zeta_n = \underline{\rho/\sigma} \int_{Y_d} \Phi(y) \sigma(y) dy.$$

Therefore, again by the second case of Theorem 3.1 we obtain the set  $\mathbf{C}_b$  (3.15). □

The following result uses both Theorem 3.1, Corollary 3.3 and the two-dimensional ergodic approach of [24].



**Corollary 3.4** *Let  $b$  be a two-dimensional vector field in  $C_{\sharp}^1(Y_2)^2$  such that  $b = \rho \Phi$ , with  $\rho$  a non nul non negative function in  $C_{\sharp}^1(Y_2)$ , and*

$$\Phi = a R_{\perp} \nabla u \text{ in } Y_2, \quad (3.18)$$

*where  $a$  is a positive function in  $C_{\sharp}^1(Y_2)$ ,  $\nabla u$  is a non vanishing gradient field in  $C_{\sharp}^1(Y_2)^2$  and  $R_{\perp}$  is the  $-\pi/2$  rotation matrix of  $\mathbb{R}^{2 \times 2}$ . Also assume that*

$$\forall \kappa \in \mathbb{Z}^2 \setminus \{0\}, \quad \left( \int_{Y_2} \nabla u(y) dy \right) \cdot \kappa \neq 0. \quad (3.19)$$

*We have the following cases:*

- *If  $\rho$  is positive in  $Y_2$ , then the flow  $X$  associated with  $b$  satisfies the asymptotics*

$$\forall x \in Y_2, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \underline{a\rho} \int_{Y_2} R_{\perp} \nabla u(y) dy. \quad (3.20)$$

- *If  $\rho$  vanishes in  $Y_2$  and  $\underline{a\rho} = 0$ , then the flow  $X$  satisfies the nul asymptotics*

$$\forall x \in Y_2, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = 0. \quad (3.21)$$

- *If  $\rho$  vanishes in  $Y_2$  and  $\underline{a\rho} > 0$ , then the set  $\mathbb{C}_b$  is given by*

$$\mathbb{C}_b = [0, \zeta] \quad \text{with} \quad \zeta := \underline{a\rho} \int_{Y_2} R_{\perp} \nabla u(y) dy \neq 0. \quad (3.22)$$

*Proof.* Consider the sequence  $(\sigma_n)_{n \in \mathbb{N}}$  defined by

$$\sigma_n := \frac{C_n}{a\rho_n} \quad \text{where} \quad C_n = \underline{a\rho_n} := \left( \int_{Y_2} \frac{dy}{(a\rho_n)(y)} \right)^{-1}, \quad (3.23)$$

where  $(\rho_n)_{n \in \mathbb{N}}$  is a sequence in  $C_{\sharp}^1(Y_2)^{\mathbb{N}}$  satisfying the condition (i) of Theorem 3.1.

First, note that by definition (3.23) the vector field  $b_n := \rho_n \Phi$  satisfies

$$\sigma_n b_n = \frac{C_n}{a} \Phi = C_n R_{\perp} \nabla u \quad (3.24)$$

(recall that  $C_n$  is a positive constant) so that  $b_n$  is orthogonal to  $\nabla u$ . Hence, the function  $u$  is invariant by the flow  $X_n$  associated with  $b_n$ , *i.e.*

$$\forall x \in Y_2, \forall t \in \mathbb{R}, \quad u(X_n(t, x)) = u(x). \quad (3.25)$$

This combined with the irrationality (or ergodicity) condition (3.19) implies that the flow  $X_n$  has no periodic solution in  $Y_2$  according to (1.18). Otherwise, there exists  $x \in Y_2$ ,  $T > 0$ , and  $\kappa \in \mathbb{Z}^2$  such that  $X_n(T, x) = x + \kappa$ , hence it follows that

$$u(x) = u(X_n(T, x)) = u(x + \kappa). \quad (3.26)$$

Moreover, since  $\nabla u$  is  $\mathbb{Z}^2$ -periodic, the function

$$z \longmapsto u(z) - \left( \int_{Y_2} \nabla u(y) dy \right) \cdot z \quad \text{is } \mathbb{Z}^2\text{-periodic.}$$



This combined with (3.26) yields

$$\left( \int_{Y_2} \nabla u(y) dy \right) \cdot \kappa = 0.$$

Hence, from the irrationality condition of (3.19) we deduce that  $\kappa = 0$ . The flow  $X_n(\cdot, x)$  is thus  $T$ -periodic, namely  $X_n(\mathbb{R}, x)$  is a closed orbit. However, by virtue of the preliminary remark of the proof of [24, Theorem 3.1] this leads us to a contradiction since  $b_n = \rho_n \Phi$  is non vanishing. Therefore, the flow  $X_n$  associated with  $b_n$  has no periodic solution in  $Y_2$ . Then, using the second step of the proof of [24, Theorem 3.1] we get the existence of a vector  $\zeta_n \in \mathbb{R}^2$  such that

$$\forall x \in Y_2, \quad \lim_{t \rightarrow \infty} \frac{X_n(t, x)}{t} = \zeta_n, \quad (3.27)$$

or equivalently by (2.1),  $\mathbf{C}_{b_n} = \{\zeta_n\}$ , namely the condition (ii) of Theorem 3.1 holds. Moreover, due to (3.24)  $\sigma_n b_n$  is divergence free in  $\mathbb{R}^2$ , or equivalently, in the torus sense (2.8)

$$\forall \psi \in C_{\#}^1(Y_d), \quad \int_{Y_d} b_n(x) \cdot \nabla \psi(x) \sigma_n(x) dx = 0.$$

Hence, by virtue of Proposition 2.2  $\sigma_n(x) dx$  is an invariant probability measure for the flow  $X_n$ , which combined with (3.24) implies that

$$\int_{Y_2} \sigma_n(y) b_n(y) dy = C_n \int_{Y_2} R_{\perp} \nabla u(y) dy \in \mathbf{C}_{b_n} = \{\zeta_n\}. \quad (3.28)$$

Let us conclude:

- If  $\rho$  is positive in  $Y_2$ , then from equality (3.28) and the uniform convergence of  $\rho_n$  to  $\rho > 0$  we deduce that

$$\lim_{n \rightarrow \infty} \zeta_n = \zeta := C \int_{Y_2} R_{\perp} \nabla u(y) dy, \quad \text{where} \quad C := \lim_{n \rightarrow \infty} C_n = \underline{a\rho}.$$

Therefore, by the first result of Theorem 3.1 we get that  $\mathbf{C}_b = \{\zeta\}$ , or equivalently by (2.1), asymptotics (3.20) is satisfied.

- Otherwise,  $\rho$  vanishes in  $Y_2$ . Moreover, by (3.18) the vector field  $a^{-1}\Phi$  is clearly divergence free in  $\mathbb{R}^2$ . Therefore, by virtue of Corollary 3.3 with  $\sigma = a^{-1}$ , we deduce the nul asymptotic (3.21) if  $\underline{a\rho} = 0$ , and the set  $\mathbf{C}_b$  (3.22) if  $\underline{a\rho} > 0$ . The fact that  $\zeta \neq 0$  in (3.22) follows immediately from the ergodic condition (3.19).

The proof is now complete. □

**Remark 3.4** In Corollary 3.4 if  $\rho$  is in  $C_{\#}^2(Y_2)$  and vanishes at some point  $x_0 \in Y_2$ , then the harmonic mean  $\underline{a\rho}$  is 0. Indeed, since  $\rho$  is non negative,  $x_0$  is a critical point of  $\rho$ . Hence, we get that for any  $x$  close to  $x_0$ ,

$$\rho(x) = \frac{1}{2} \nabla^2 \rho(x_0) (x - x_0) \cdot (x - x_0) + o(|x - x_0|^2) \quad \text{and thus} \quad \frac{1}{a(x)\rho(x)} \geq \frac{C}{|x - x_0|^2}.$$

which implies that  $(a\rho)^{-1} \notin L^1(Y_2)$  and  $\underline{a\rho} = 0$ . Therefore, the nul asymptotics (3.20) holds. Otherwise, if

$$\#\{x \in Y_2 : \rho(x) = 0\} \in (0, \infty),$$



and if for any  $x_0 \in Y_2$  with  $\rho(x_0) = 0$  we have for any  $x$  close to  $x_0$ ,

$$\rho(x) \geq c_0 |x - x_0|^{\alpha_0}, \quad \text{for some } \alpha_0 \in (1/2, 1) \text{ and } c_0 > 0,$$

(note that the former condition remains compatible with  $\rho \in C^1_{\sharp}(Y_2)$ ), then  $\underline{a\rho} > 0$ . Indeed, we are led by a translation to  $x_0 = 0$ , and passing to polar coordinates we deduce that any  $r_0 \in (0, 1/2)$ ,

$$\int_{\{x \in Y_2: |x| < r_0\}} \frac{dx}{|x|^{\alpha_0}} = 2\pi \int_0^{r_0} \frac{dr}{r^{2\alpha_0-1}} < \infty.$$

Therefore, we get the full closed line segment (3.15) for the limit set  $C_b$ .

In Example 4.2 below we will provide a two-dimensional example of such an enlarged limit set  $C_b$  obtained from a sequence of singleton sets  $(C_{b_n})_{n \in \mathbb{N}}$  where  $b_n = \rho_n \Phi$  has a fixed direction  $\Phi$ .

**Remark 3.5** The condition (3.18) on the direction  $\Phi$  of the field  $b = \rho \Phi$  may seem to be quite restrictive at the first glance. Actually, we can deduce (3.18) from the existence of a function  $v \in C^2(Y_2)$  and a constant  $c > 0$  satisfying the inequality

$$\Phi \cdot \nabla v \geq c \text{ in } \mathbb{R}^2. \quad (3.29)$$

This inequality means that the equipotential  $\{v = 0\}$  (or any equipotential of  $v$ ) is transverse to each orbit  $Y(\mathbb{R}, x)$ ,  $x \in \mathbb{R}^2$ , of the flow  $Y$  associated with  $\Phi$ . In other words, the equipotential  $\{v = 0\}$  can be regarded as a Siegel's curve [27, Lemma 3] for the flow  $Y$  in  $\mathbb{R}^2$  rather than in the torus  $Y_2$ . Assuming that  $b$  is non vanishing and that the flow  $X$  associated with  $b$  has no periodic trajectory in  $Y_2$  according to (1.18) (which is an ergodic type condition) and using a Siegel's curve in the torus, Peirone [24, Theorem 3.1] has proved that the asymptotics of the flow  $X(\cdot, x)$  does exist for any  $x \in Y_2$  and is independent of  $x$ , or equivalently by (2.1), that  $C_b$  is a singleton. Therefore, condition (3.18) and the ergodic condition (3.19) play the same role for the vector field  $\Phi$  than Peirone's conditions for the vector field  $b$  through a similar Siegel's curve approach. However, working with the non vanishing vector field  $\Phi$  rather than  $b$  allows us to obtain some new asymptotics when the vector field  $b = \rho \Phi$  does vanish with the non negative function  $\rho$ .

Now, let us check that condition (3.29) implies condition (3.18) under some (minor) additional assumptions. To this end, we will follow the same procedure as [11, Theorem 2.15] derived for a gradient field. Since we have for any  $x \in Y_2$ ,

$$\forall t \in \mathbb{R}, \quad \frac{\partial}{\partial t}(v(Y(t, x))) = (\Phi \cdot \nabla v)(Y(t, x)) \geq c > 0,$$

the mapping  $t \mapsto v(Y(t, x))$  is a  $C^1$ -diffeomorphism on  $\mathbb{R}$ , hence there exists a unique  $\tau(x) \in \mathbb{R}$  such that

$$v(Y(\tau(x), x)) = 0,$$

namely the trajectory  $Y(\cdot, x)$  reaches the equipotential  $\{v = 0\}$  at time  $\tau(x)$ . The uniqueness of  $\tau$  combined with the semi-group property of the flow  $Y$  easily implies that

$$\forall x \in Y_2, \quad \forall t \in \mathbb{R}, \quad \tau(Y(t, x)) = \tau(x) - t. \quad (3.30)$$

Moreover, by the implicit functions theorem and the  $C^1(\mathbb{R} \times \mathbb{R}^2)$  regularity of the flow  $Y$ , the function  $\tau$  belongs to  $C^1(\mathbb{R}^2)$ . Then, define the positive function  $\sigma_0 \in C^1(\mathbb{R}^2)$  by

$$\sigma_0(x) := \exp \left( \int_0^{\tau(x)} (\operatorname{div} \Phi)(Y(s, x)) ds \right) \quad \text{for } x \in \mathbb{R}^2. \quad (3.31)$$



From now, assume that  $\operatorname{div}(\Phi) \in C^1(\mathbb{R}^2)$ . Then, the function  $\sigma_0$  belongs to  $C^1(\mathbb{R}^2)$ . By using (3.30) and the semi-group property of the flow  $Y$ , then making the change of variable  $r = s + t$ , we get that

$$\forall x \in Y_2, \forall t \in \mathbb{R}, \quad \sigma_0(Y(t, x)) = \exp \left( \int_t^{\tau(x)} (\operatorname{div} \Phi)(Y(r, x)) dr \right).$$

Next, taking the derivative of the previous equality at  $t = 0$ , we obtain that

$$\nabla \sigma_0 \cdot \Phi + \sigma_0 \operatorname{div}(\Phi) = \operatorname{div}(\sigma_0 \Phi) = 0 \quad \text{in } \mathbb{R}^2,$$

or equivalently, there exists a function  $u_0 \in C^1(\mathbb{R}^2)$  such that

$$\Phi = \frac{1}{\sigma_0} R_\perp \nabla u_0 \quad \text{in } \mathbb{R}^2.$$

This is nearly the desired condition (3.18) except that the function  $\sigma_0$  is not necessarily  $\mathbb{Z}^2$ -periodic. However, also assuming that the function  $\sigma_0$  is bounded from below and above by positive constants, the averaging procedure of [11, Theorem 2.17] allows us to build a positive periodic function  $\sigma \in L^\infty_\#(Y_2)$  satisfying

$$\operatorname{div}(\sigma \Phi) = 0 \quad \text{in } \mathbb{R}^2,$$

which is equivalent to condition (3.18) with  $a := 1/\sigma$ . But the regularity of  $\sigma$  is not ensured. Finally, to get the regularity  $\sigma \in C^1_\#(Y_2)$ , it is enough to assume in addition that  $\sigma_0$  and  $\nabla \sigma_0$  are uniformly continuous in  $\mathbb{R}^2$  (see [11, Remark 2.19]).

In contrast with Corollary 3.4 the following result uses both Theorem 3.1 and the non-ergodic approach of [8] in any dimension, and provides an alternative approach to the two-dimensional Corollary 3.4.

**Corollary 3.5** *For  $d \geq 2$  and  $n \in \mathbb{N}^*$ , let  $U_n = (u_1^n, u_2, \dots, u_d)$  be a sequence of vector fields in  $C^2(Y_d)^d$  with  $\nabla U_n \in C^1_\#(Y_d)^{d \times d}$ , and let  $a$  be a positive function in  $C^1_\#(Y_d)$  such that*

$$\begin{cases} \det(\nabla U_n) > 0 & \text{in } Y_d \\ \rho_n := \frac{1}{a \det(\nabla U_n)} & \text{converges uniformly in } Y_d \text{ to some } \rho \in C^1_\#(Y_d) \\ \rho \leq \rho_n & \text{in } Y_d. \end{cases} \quad (3.32)$$

Then, the sequence of vector fields  $(b_n)_{n \in \mathbb{N}}$  defined by

$$b_n = \rho_n \Phi, \quad \text{where} \quad \Phi := \begin{cases} a R_\perp \nabla u_2 & \text{if } d = 2 \\ a (\nabla u_2 \times \dots \times \nabla u_d) & \text{if } d > 2 \end{cases} \quad (3.33)$$

converges uniformly in  $Y_d$  to the vector field  $b = \rho \Phi$ . Moreover, we have:

- If  $\rho$  is positive in  $Y_d$ , then the flow  $X$  associated with  $b$  satisfies the asymptotics

$$\forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \begin{cases} \frac{a\rho}{t} \int_{Y_2} R_\perp \nabla u_2(y) dy & \text{if } d = 2 \\ \frac{a\rho}{t} \int_{Y_d} (\nabla u_2 \times \dots \times \nabla u_d)(y) dy & \text{if } d > 2, \end{cases} \quad (3.34)$$

where  $\underline{a\rho}$  is the harmonic mean of  $\rho$ .



- If  $\rho$  vanishes in  $Y_d$  and  $\underline{a\rho} = 0$ , then the flow  $X$  associated with  $b$  satisfies the asymptotics

$$\forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = 0. \quad (3.35)$$

- If  $\rho$  vanishes in  $Y_d$  and  $\underline{a\rho} > 0$ , then we get the set

$$\mathbf{C}_b = [0, \zeta] \quad \text{with} \quad \zeta := \begin{cases} \underline{a\rho} \int_{Y_2} R_\perp \nabla u_2(y) dy \neq 0 & \text{if } d = 2 \\ \underline{a\rho} \int_{Y_d} (\nabla u_2 \times \cdots \times \nabla u_d)(y) dy \neq 0 & \text{if } d > 2. \end{cases} \quad (3.36)$$

*Proof.* We have

$$b_n \cdot \nabla u_1^n = \begin{cases} a\rho_n \nabla u_1^n \cdot R_\perp \nabla u_2 & \text{if } d = 2 \\ a\rho_n \nabla u_1^n \cdot (\nabla u_2 \times \cdots \times \nabla u_d) & \text{if } d > 2 \end{cases} = a\rho_n \det(\nabla U_n) = 1 \quad \text{in } Y_d, \quad (3.37)$$

and

$$\frac{1}{a\rho_n} b_n = \begin{cases} R_\perp \nabla u_2 & \text{if } d = 2 \\ \nabla u_2 \times \cdots \times \nabla u_d & \text{if } d > 2 \end{cases}$$

is divergence free in  $\mathbb{R}^d$ . Hence, from [8, Corollary 4.1] we deduce that  $\mathbf{C}_{b_n}$  is the singleton  $\{\zeta_n\}$  with

$$\zeta_n := \frac{\int_{Y_d} (a\rho_n)^{-1}(y) b_n(y) dy}{\int_{Y_d} (a\rho_n)^{-1}(y) dy} = \begin{cases} \underline{a\rho_n} \int_{Y_2} R_\perp \nabla u_2(y) dy & \text{if } d = 2 \\ \underline{a\rho_n} \int_{Y_d} (\nabla u_2 \times \cdots \times \nabla u_d)(y) dy & \text{if } d > 2, \end{cases} \quad (3.38)$$

where  $\underline{a\rho_n}$  is the harmonic mean of  $a\rho_n$ .

Let us conclude:

- If  $\rho > 0$  in  $Y_d$ , then the sequence  $(a\rho_n)^{-1}$  converges uniformly to  $(a\rho)^{-1}$  in  $Y_d$ . Therefore, by the first case of Theorem 3.1 combined with (3.38) we get that  $\mathbf{C}_b = \{\zeta\}$ , or equivalently by (2.1), asymptotics (3.34) holds.
- Otherwise,  $\rho$  vanishes in  $Y_2$ . Moreover, by (3.33) the vector field  $a^{-1}\Phi$  is clearly divergence free in  $\mathbb{R}^2$ . Therefore, by virtue of Corollary 3.3 with  $\sigma = a^{-1}$ , we deduce the nul asymptotics (3.35) if  $\underline{a\rho} = 0$ , and the set  $\mathbf{C}_b$  (3.36) if  $\underline{a\rho} > 0$ . It remains to prove that  $\zeta \neq 0$  in (3.36). By the definition of  $U_n$  and (3.32) we have

$$\int_{Y_2} \underbrace{\det(\nabla U_n)(y)}_{>0} dy = \begin{cases} \int_{Y_2} \nabla u_1^n(y) \cdot R_\perp \nabla u_2(y) dy > 0 & \text{if } d = 2 \\ \int_{Y_d} \nabla u_1^n(y) \cdot (\nabla u_2 \times \cdots \times \nabla u_d)(y) dy > 0 & \text{if } d > 2. \end{cases}$$

Hence, from the quasi-affinity of the determinant (see, e.g., [14, Section 4.3.2]), namely:

$$\det \left( \int_{Y_2} \nabla U_n(y) dy \right) = \int_{Y_2} \det(\nabla U_n)(y) dy > 0,$$



we deduce that

$$\begin{cases} \int_{Y_2} \nabla u_1^n \cdot \left( \int_{Y_2} R_\perp \nabla u_2 \right) = \int_{Y_2} \nabla u_1^n \cdot R_\perp \nabla u_2 > 0 & \text{if } d = 2 \\ \int_{Y_d} \nabla u_1^n \cdot \left( \int_{Y_d} \nabla u_2 \times \cdots \times \int_{Y_d} \nabla u_d \right) = \int_{Y_d} \nabla u_1^n \cdot (\nabla u_2 \times \cdots \times \nabla u_d) > 0 & \text{if } d > 2. \end{cases}$$

Therefore, again using the quasi-affinity of the determinant (multiplying the second equality by any constant vector of  $\mathbb{R}^d$  to get a determinant) we get that

$$\begin{cases} \int_{Y_2} R_\perp \nabla u_2(y) dy \neq 0 & \text{if } d = 2 \\ \int_{Y_d} (\nabla u_2 \times \cdots \times \nabla u_d)(y) dy = \int_{Y_d} \nabla u_2(y) dy \times \cdots \times \int_{Y_d} \nabla u_d(y) dy \neq 0 & \text{if } d > 2, \end{cases}$$

which implies that  $\zeta \neq 0$  in (3.36). □

**Remark 3.6** In Corollary 3.4 and in the two-dimensional case of Corollary 3.5, the vector field  $b_n$  has the same form  $b_n = \rho_n a R_\perp \nabla u$ . In Corollary 3.4 the function  $\rho_n$  is arbitrary, while  $\nabla u$  satisfies the ergodic condition (3.19). On the contrary, in the two dimensional case of Corollary 3.5  $\rho_n$  does depend on the functions  $a$  and  $\nabla u$  by (3.32), while  $\nabla u$  is arbitrary. Therefore, these results provide two quite different approaches on the asymptotics of the flow: an ergodic one using [24] and a non-ergodic one using [8].

## 4 Examples

The following counter-example shows that Theorem 3.1 does not extend to a vector-valued perturbation.

**Exemple 4.1** Consider the fields  $b$  and  $b_n$  defined by

$$b(x) = a(x) e_1 \quad \text{and} \quad b_n(x) = a_n(x) (e_1 + \gamma_n e_2), \quad \text{for } x \in Y_2 \text{ and } n \in \mathbb{N} \setminus \{0\},$$

where  $a$  is a non negative function in  $C_\#^1(Y_2)$ ,  $a_n := a + 1/n$ , and  $(\gamma_n)_{n \in \mathbb{N}}$  is a positive sequence in  $\mathbb{R} \setminus \mathbb{Q}$  which converges to 0. In this case, the flows associated with the vector fields  $b$  and  $b_n$  can be computed explicitly.

On the one hand, by the ergodic case of [26, Section 3.1] with the irrational rotation number  $\gamma_n$ , the flow  $X_n$  associated with the field  $b_n$  satisfies the asymptotics

$$\forall x \in Y_2, \quad \lim_{t \rightarrow \infty} \frac{X_n(t, x)}{t} = \underline{a}_n (e_1 + \gamma_n e_2),$$

where  $\underline{a}_n$  is the harmonic mean of  $a_n$ , or equivalently by (2.1),

$$C_{b_n} = \{\zeta_n\} \quad \text{with} \quad \zeta_n := \underline{a}_n (e_1 + \gamma_n e_2). \quad (4.1)$$

On the other hand, by the non-ergodic case of [26, Section 3.1] with the rational rotation number 0 and its extension when  $a$  vanishes in  $Y_2$ , the flow  $X$  associated with the field  $b$  satisfies the asymptotics

$$\forall x \in Y_2, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \begin{cases} \frac{a(\cdot e_1 + x) e_1}{0} & \text{if } a(\cdot e_1 + x) \text{ is positive in } Y_2 \\ 0 & \text{if } a(\cdot e_1 + x) \text{ vanishes in } Y_2, \end{cases}$$



where  $\underline{a(\cdot e_1 + x)}$  is the harmonic mean defined by

$$\underline{a(\cdot e_1 + x)} := \left( \int_{Y_1} \frac{dt}{a(te_1 + x)} \right)^{-1} \quad \text{for } x \in Y_2,$$

or equivalently, the set  $\mathbf{A}_b$  (1.8) is given by

$$\mathbf{A}_b = \left\{ \underline{a(\cdot e_1 + x)} e_1 : x \in Y_2 \right\}.$$

Moreover, the function  $(x \mapsto \underline{a(\cdot e_1 + x)})$  is continuous on the compact set  $Y_2$ . It is clear at any point  $x \in Y_2$  such that  $\underline{a(\cdot e_1 + x)}$  is positive. Otherwise, if  $\underline{a(\cdot e_1 + x)} \in C_{\sharp}^1(Y_1)$  vanishes in  $Y_1$ , by Fatou's lemma we get that for any sequence  $(x_n)_{n \in \mathbb{N}}$  converging to  $x$ ,

$$\infty = \frac{1}{\underline{a(\cdot e_1 + x)}} \leq \liminf_{n \rightarrow \infty} \left( \frac{1}{\underline{a(\cdot e_1 + x_n)}} \right) = \lim_{n \rightarrow \infty} \left( \frac{1}{\underline{a(\cdot e_1 + x_n)}} \right) = \infty.$$

Hence, the set  $\mathbf{A}_b$  is actually a closed line segment of  $\mathbb{R}^2$ , which by (1.10) implies that

$$\mathbf{C}_b = \text{conv}(\mathbf{A}_b) = \mathbf{A}_b = \left\{ \underline{a(\cdot e_1 + x)} e_1 : x \in Y_2 \right\}. \quad (4.2)$$

In particular, when  $a$  vanishes in  $Y_2$ , we get that

$$\mathbf{C}_b = [0, \zeta] \quad \text{with} \quad \zeta := \left( \max_{x \in Y_2} \underline{a(\cdot e_1 + x)} \right) e_1.$$

Therefore, taking into account (4.1), contrary to the second case of Theorem 3.1 we may have

$$\underline{a} e_1 = \lim_{n \rightarrow \infty} \zeta_n \neq \zeta = \left( \max_{x \in Y_2} \underline{a(\cdot e_1 + x)} \right) e_1.$$

For example, take

$$a(x) := \sin^2(\pi x_1) + \sin^2(\pi x_2) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.$$

Then, it follows that

$$\begin{aligned} \frac{1}{\underline{a}} &= \int_{Y_1} \left( \int_{Y_1} \frac{dx_2}{\sin^2(\pi x_1) + \sin^2(\pi x_2)} \right) dx_1 > \int_{Y_1} \frac{dx_1}{\sin^2(\pi x_1) + 1} \\ & \quad (\text{for any } x_1 \in Y_1 \text{ and } x_2 = \tfrac{1}{2}) = \min_{x \in Y_2} \left( \int_{Y_1} \frac{dt}{\sin^2(\pi t + \pi x_1) + \sin^2(\pi x_2)} \right) \\ & = \min_{x \in Y_2} \left( \frac{1}{\underline{a(\cdot e_1 + x)}} \right), \end{aligned}$$

which implies that

$$\underline{a} < \max_{x \in Y_2} \underline{a(\cdot e_1 + x)}.$$

Therefore, Theorem 3.1 does not extend in general to the case where the direction  $\Phi$  of the field  $b_n = \rho_n \Phi$  also depends on  $n$ .

Finally, note that the inclusion  $\{0, \zeta\} \subset \mathbf{A}_b$  of the second case of Theorem 3.1 is not an equality in general, since in the particular case (4.2)  $\mathbf{A}_b$  is the closed line segment  $[0, \zeta]$ .



The second example shows that the singleton condition is not in general asymptotically preserved under the assumptions of Theorem 3.1.

**Example 4.2** Let  $\nabla u \in C_{\sharp}^1(Y_2)^2$  be satisfying the ergodic condition (3.19), let  $(\rho_n)_{n \in \mathbb{N}}$  be the sequence of positive functions in  $C_{\sharp}^1(Y_2)$  defined by

$$\rho_n(x) := (\sin^2(\pi x_1) + \sin^2(\pi x_2) + 1/n)^\alpha \quad \text{for } x \in Y_2, \quad \text{with } \alpha \in (1/2, 1).$$

and let  $(b_n)_{n \in \mathbb{N}}$  be the sequence of vector fields defined by  $b_n = \rho_n R_{\perp} \nabla u$ . Since the function  $(t \mapsto t^\alpha)$  is uniformly continuous in  $[0, \infty)$ , the sequence  $(\rho_n)_{n \in \mathbb{N}}$  converges uniformly in  $Y_d$  to the function

$$\rho(x) := (\sin^2(\pi x_1) + \sin^2(\pi x_2))^\alpha \quad \text{for } x = (x_1, x_2) \in Y_2,$$

which belongs to  $C_{\sharp}^1(Y_2)$  due to  $\alpha > 1/2$ , and vanishes at the sole point  $(0, 0)$  in the torus  $Y_2$ . Moreover, we have for any  $x$  close to  $(0, 0)$ ,

$$\frac{c^{-1}}{|x|^{2\alpha}} \leq \frac{1}{\rho(x)} \leq \frac{c}{|x|^{2\alpha}}, \quad \text{for some } c > 1,$$

so that  $\underline{\rho} > 0$  due to  $\alpha < 1$  (see Remark 3.4). Therefore, by the asymptotics (3.22) of Corollary 3.4 and condition (3.19) we get that

$$\mathbf{C}_b = [0, \zeta] \quad \text{with} \quad \zeta := \left( \int_{Y_2} \frac{dx}{(\sin^2(\pi x_1) + \sin^2(\pi x_2))^\alpha} \right)^{-1} \int_{Y_d} R_{\perp} \nabla u(y) dy \neq 0.$$

Note that, if  $\alpha \geq 1$ , then  $\underline{\rho} = 0$ . Therefore, by Corollary 3.4 we get that  $\mathbf{C}_b = \{0\}$ , and the flow  $X$  satisfies the nul asymptotics (3.21).

The third example illustrates Corollary 3.5.

**Example 4.3** Let  $U_n = (u_1^n, u_2, \dots, u_d) \in C^2(Y_d)^d$ ,  $d \geq 2$  and  $n \geq 1$ , be such that  $\nabla U_n$  is  $\mathbb{Z}^d$ -periodic, the functions  $u_2, \dots, u_d$  only depend on the variables  $x' = (x_2, \dots, x_d)$ ,

$$u_1^n(x) := \int_0^{x_1} \frac{dt}{f_n(t, x')} \quad \text{and} \quad \Delta(x') := \det \left( \left[ \frac{\partial u_i}{\partial x_j}(x') \right]_{2 \leq i, j \leq d} \right) > 0 \quad \text{for } x \in Y_d,$$

where  $(f_n)_{n \in \mathbb{N}}$  is a positive sequence in  $C_{\sharp}^1(Y_d)^{\mathbb{N}}$  which converges uniformly to  $f \leq f_n$  in  $Y_d$ . Expanding the determinant with respect to its first column we have

$$\forall x \in Y_d, \quad \det(\nabla U_n)(x) = \frac{\Delta(x')}{f_n(x)} > 0 \quad \text{and} \quad \rho_n(x) := \frac{1}{\det(\nabla U_n)(x)} \rightarrow \rho(x) := \frac{f(x)}{\Delta(x')} \leq \rho_n(x)$$

uniformly in  $Y_d$ , so that condition (3.32) is fulfilled with  $a = 1$ . Define the vector field  $b_n$  in  $C_{\sharp}^1(Y_d)^d$  by (3.33). Therefore, the sequence  $(b_n)_{n \in \mathbb{N}}$  converges uniformly in  $Y_d$  to the function  $b$  given by

$$b(x) = \begin{cases} \frac{f(x)}{\Delta(x')} R_{\perp} \nabla u_2(x) & \text{if } d = 2 \\ \frac{f(x)}{\Delta(x')} (\nabla u_2 \times \dots \times \nabla u_d)(x) & \text{if } d > 2, \end{cases} \quad \text{for } x \in Y_d.$$

Moreover, due to the 1-periodicity of  $\nabla_{x'} u_1^n$  with respect to the variable  $x_1$ , we have

$$\forall x' \in \mathbb{R}^{d-1}, \quad \int_0^1 \nabla_{x'} \left( \frac{1}{f_n(t, x')} \right) dt = 0,$$



which implies the existence of a positive constant  $c_n$  such that

$$\forall x' \in \mathbb{R}^{d-1}, \quad \int_0^1 \frac{dt}{f_n(t, x')} = c_n. \quad (4.3)$$

Hence, from inequality  $f \leq f_n$  and Fatou's lemma we deduce that

$$\forall x' \in \mathbb{R}^{d-1}, \quad \limsup_{n \rightarrow \infty} c_n \leq \int_0^1 \frac{dt}{f(t, x')} \leq \liminf_{n \rightarrow \infty} \int_0^1 \frac{dt}{f_n(t, x')} = \liminf_{n \rightarrow \infty} c_n,$$

which implies that

$$\forall x' \in \mathbb{R}^{d-1}, \quad \int_0^1 \frac{dt}{f(t, x')} = \lim_{n \rightarrow \infty} c_n. \quad (4.4)$$

Then, we have the following alternative:

- If  $f$  is positive in  $Y_d$ , then by virtue of the first case of Corollary 3.5 we obtain the asymptotics of the flow associated with  $b$

$$\forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \begin{cases} \underline{\rho} \int_{Y_2} R_{\perp} \nabla u_2(y) dy & \text{if } d = 2 \\ \underline{\rho} \int_{Y_d} (\nabla u_2 \times \cdots \times \nabla u_d)(y) dy & \text{if } d > 2, \end{cases}$$

where

$$\underline{\rho} = \left( \int_{Y_d} \frac{\Delta(x')}{f(x)} dx \right)^{-1}.$$

- If  $f$  vanishes at some point  $x_0 \in Y_d$ , then since  $f(\cdot, x'_0)$  is in  $C^1_{\#}(Y_1)$  and vanishes at  $t = (x_0)_1$ , we have by (4.4)

$$\forall x' \in \mathbb{R}^{d-1}, \quad \lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} \int_0^1 \frac{dt}{f_n(t, x')} = \int_0^1 \frac{dt}{f(t, x'_0)} = \infty. \quad (4.5)$$

Assume by contradiction that  $\underline{f} > 0$ . Then, using successively Lebesgue's theorem with inequality  $f \leq f_n$ , Fubini's theorem and equality (4.3), we get that

$$\infty > \int_{Y_d} \frac{dx}{f(x)} = \lim_{n \rightarrow \infty} \int_{Y_d} \frac{dx}{f_n(x)} = \lim_{n \rightarrow \infty} \left[ \int_{Y_{d-1}} dx' \left( \int_0^1 \frac{dt}{f_n(t, x')} \right) \right] = \lim_{n \rightarrow \infty} c_n,$$

which contradicts (4.5). Hence, we deduce that  $\underline{f} = 0$ , and due to the positivity of  $\Delta$  we get that

$$\underline{\rho} = \left( \int_{Y_d} \frac{\Delta(x')}{f(x)} dx \right)^{-1} = 0.$$

Therefore, by virtue of the second case of Corollary 3.5 we obtain the nul asymptotics (3.35).

Note that the third case of Corollary 3.5 cannot arise when the functions  $u_2, \dots, u_d$  are independent of the variable  $x_1$ .

The fourth example deals with the case of an electric field. It is based on the divergence-curl Proposition 2.2, and illustrates the framework of Theorem 3.1. We cannot characterize precisely the set  $C_b$  except in the two-dimensional ergodic case. However, the two-dimensional case and the three-dimensional case are shown to be quite different.



**Example 4.4** Let  $\sigma \in C_{\sharp}^3(Y_d)$  be a positive function with  $\int_{Y_d} \sigma(y) dy = 1$ . Consider the vector-valued function  $U \in C^2(\mathbb{R}^d)^d$  (see, e.g., [15, Theorem 8.13]) unique solution (up to an additive constant vector) to the conductivity problem

$$\begin{cases} \operatorname{Div}(\sigma DU) = 0 & \text{in } \mathbb{R}^d \\ y \mapsto U(y) - y & \text{is } \mathbb{Z}^d\text{-periodic,} \end{cases} \quad (4.6)$$

where  $DU = (\nabla U)^T$  and the vector-valued operator  $\operatorname{Div}$  consists in the divergence of the columns of  $\sigma DU$ . The variational formulation of (4.6) reads as

$$DU \in C_{\sharp}^1(Y_d)^d \quad \text{and} \quad \forall \Psi \in C_{\sharp}^1(Y_d)^d, \quad \int_{Y_d} \sigma(y) DU(y) : D\Psi(y) dy = 0. \quad (4.7)$$

In (4.7) “:” denotes the scalar product in  $\mathbb{R}^{d \times d}$  defined by

$$M : N := \operatorname{tr}(M^T N) \quad \text{for } M, N \in \mathbb{R}^{d \times d}.$$

The so-called homogenized matrix (see, e.g., [5, Chapter I, Section 2.3]) associated with the conductivity  $\sigma$  is defined by

$$A^* := \int_{Y_d} \sigma(y) DU(y) dy, \quad (4.8)$$

which is known to be symmetric positive definite. Also define the associated electric field

$$b_{\lambda} := \nabla u_{\lambda} = \nabla(U\lambda) = DU\lambda \quad \text{for } \lambda \in \mathbb{R}^d \setminus \{0\}. \quad (4.9)$$

Case  $d = 2$ : Alessandrini and Nesi [1, Theorems 1,2] have proved that  $U$  is a  $C^1$ -diffeomorphism of  $\mathbb{R}^2$  with

$$\det(DU) > 0 \quad \text{in } Y_2. \quad (4.10)$$

Let  $\lambda \in \mathbb{R}^2 \setminus \{0\}$ . As a consequence (see [1, Proposition 2]) the gradient field  $b_{\lambda}$  defined by (4.9) does not vanish in  $Y_2$ . Hence, the set of invariant probability measures  $\mathcal{I}_{b_{\lambda}}$  for the flow associated with  $b_{\lambda}$  does not contain any Dirac measure.

Moreover, by (4.7) and (4.9) we have

$$\forall \psi \in C_{\sharp}^1(Y_2), \quad \int_{Y_2} b_{\lambda}(y) \cdot \nabla \psi(y) \sigma(y) dy = \int_{Y_2} \sigma(y) \nabla u_{\lambda}(y) \cdot \nabla \psi(y) dy = 0, \quad (4.11)$$

which by virtue of the equivalence (i)-(iii) of Proposition 2.2 implies that  $\sigma(x) dx$  is an invariant probability measure with positive density. Therefore, by definition (4.8) we can only conclude that (recall that  $A^*$  is positive definite and  $\lambda$  is non zero)

$$A^* \lambda = \int_{Y_2} b_{\lambda}(y) \sigma(y) dy \in \mathcal{C}_{b_{\lambda}}. \quad (4.12)$$

On the other hand, let  $\mu \in \mathcal{I}_{b_{\lambda}}$  be an invariant probability measure for the flow  $X_{\lambda}$  associated with  $b_{\lambda}$ . By the divergence-curl relation (2.12) and (4.6) we have

$$\int_{Y_d} |b_{\lambda}|^2 \mu(dy) = \int_{Y_d} b_{\lambda}(y) \cdot \nabla u_{\lambda} d\mu(y) = \left( \int_{Y_d} b_{\lambda}(y) d\mu(y) \right) \cdot \lambda. \quad (4.13)$$

Due to inequality (4.10) the gradient field  $b_{\lambda} = DU\lambda$  does not vanish in  $Y_2$ . Hence, since  $\mu$  is a probability measure, we deduce that  $\mu(\{b_{\lambda} \neq 0\}) > 0$ , which implies that the first term of (4.13) is positive. Therefore, we get that

$$0 \neq \int_{Y_d} b_{\lambda}(y) d\mu(y) \in \mathcal{C}_{b_{\lambda}}.$$



This combined with (4.12) yields

$$\{A^*\lambda\} \subset \mathbb{C}_{b_\lambda} \subset \mathbb{R}^d \setminus \{0\}. \quad (4.14)$$

We may improve the former result under the extra ergodic condition

$$\forall \kappa \in \mathbb{Z}^2 \setminus \{0\}, \quad (A^*\lambda) \cdot \kappa \neq 0. \quad (4.15)$$

Indeed, since by (4.11)  $\sigma b_\lambda$  is divergence free, by a classical duality argument there exists a potential  $v_\lambda$  with  $\nabla v_\lambda \in C_\#^1(Y_2)^2$ , such that  $\sigma b_\lambda = R_\perp \nabla v_\lambda$  in  $Y_2$ . Moreover, since by (4.12)

$$\int_{Y_2} \nabla v_\lambda(y) dy = -R_\perp \int_{Y_2} \sigma(y) b_\lambda(y) dy = -R_\perp A^*\lambda,$$

condition (4.15) means that the gradient field  $\nabla v_\lambda$  satisfies the ergodic condition (3.19). Hence, the vector field  $b_\lambda = 1/\sigma R_\perp \nabla v_\lambda$  satisfies the first result of Corollary 3.4 with  $u := v_\lambda$ ,  $a := 1/\sigma$  and  $\rho := 1$ . Therefore, the flow  $X_\lambda$  associated with  $b_\lambda$  satisfies asymptotics (3.20) which reads as

$$\forall x \in Y_2, \quad \lim_{t \rightarrow \infty} \frac{X_\lambda(t, x)}{t} = \frac{1}{\sigma} \int_{Y_2} R_\perp \nabla v(y) dy = A^*\lambda,$$

or equivalently by (2.1), we obtain that

$$\mathbb{C}_{b_\lambda} = \{A^*\lambda\}. \quad (4.16)$$

Case  $d = 3$ : Contrary to the two-dimensional case with the positivity (4.10), by virtue of [10, Theorem 4.1] there exists a positive conductivity  $\sigma_\gamma \in L_\#^\infty(Y_3)$  such that the vector-valued function  $U_\gamma$  solution to the equation (4.6) with  $\sigma_\gamma$  has a determinant which changes sign.

More precisely, the conductivity of [10] rescaled by its  $Y_3$ -average value denoted by  $\sigma_\gamma$  (recall that  $\sigma_\gamma(x) dx$  has to be a probability measure) takes two values:  $\sigma_\gamma = 1$  in a cubic symmetric lattice of interlocking rings which do not intersect, and  $\sigma_\gamma = \gamma \ll 1$  elsewhere in  $\mathbb{R}^3$ . The conductivity  $\sigma_\gamma$  is thus not regular. However, enlarging each ring with a width  $d_\gamma \ll 1$ , we can build a new conductivity  $\sigma \in C_\#^3(Y_3)$  (also depending on  $\gamma$ ) whose values pass from 1 on the boundary of each ring to  $\gamma$  on the boundary of the corresponding enlarged ring. Then, it is easy to check that for  $\gamma$  and  $d_\gamma$  small enough, the matrix-valued function  $DU$  defined by (4.6) with the regular conductivity  $\sigma$  has a determinant which also changes sign. Thus, by a continuity argument we get that

$$\exists y_0 \in Y_3, \quad \det(DU)(y_0) = 0, \quad (4.17)$$

which implies that there exists  $\lambda \in \mathbb{R}^3 \setminus \{0\}$  such that  $DU(y_0)\lambda = 0$ . Hence, the gradient field  $b_\lambda$  defined by (4.9) vanishes at point  $y_0$ . Therefore, in contrast with the two-dimensional result (4.12) we obtain the more complete result

$$A^*\lambda = \int_{Y_3} b_\lambda(y) \sigma(y) dy \in \mathbb{C}_{b_\lambda} \setminus \{0\} \quad \text{and} \quad [0, A^*\lambda] \subset \mathbb{C}_{b_\lambda}, \quad (4.18)$$

since the Dirac mass  $\delta_{y_0}$  belongs to  $\mathcal{S}_{b_\delta}$  and

$$\int_{Y_3} b_\lambda(y) d\delta_{y_0}(dy) = b_\lambda(y_0) = 0.$$

Result (4.18) corresponds to the second case of Theorem 3.1. In contrast with the result (4.14) of the two-dimensional case, we obtain that

$$[0, A^*\lambda] \subset \mathbb{C}_{b_\lambda}. \quad (4.19)$$



## 5 Homogenization of linear transport equations

The following theorem is an extension of various homogenization results [6, 16, 17, 20, 26] (and the references therein) of linear transport equations with an oscillating velocity, which are based on the classical ergodic approach. Here, in a regular and periodic framework the ergodic approach is replaced by the singleton approach of Section 2.1, whose a very particular case has been first obtained in [7, Corollary 4.4].

**Theorem 5.1** *Let  $b$  be a vector field in  $C^1_{\sharp}(Y_d)^d$  and let  $u_0 \in C^1(\mathbb{R}^d)$ . Consider the transport equation with the oscillating velocity  $b(x/\varepsilon)$ :*

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} - b(x/\varepsilon) \cdot \nabla u_\varepsilon = 0 & \text{in } (0, \infty) \times \mathbb{R}^d \\ u_\varepsilon(0, x) = u_0(x) & \text{for } x \in \mathbb{R}^d. \end{cases} \quad (5.1)$$

Assume that there exists a vector  $\zeta \in \mathbb{R}^d$  such that

$$\forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \zeta, \quad (5.2)$$

where  $X$  is the flow (1.15) associated with the vector field  $b$ . Then, the solution  $u_\varepsilon$  to transport equation (5.1) converges strongly in  $L^p_{\text{loc}}([0, \infty) \times \mathbb{R}^d)$  for any  $p \in [1, \infty)$ , to  $u_0(x + t\zeta)$  which is solution to the transport equation (5.1) with the constant velocity  $\zeta$  in place of  $b(x/\varepsilon)$ .

*Proof.* Let  $X$  be the flow associated to the vector field  $b$ . By  $\varepsilon$ -rescaling the flow  $X$ , let us define the flow  $X_\varepsilon$  associated with the oscillating vector field  $b(x/\varepsilon)$  by

$$\forall (t, x) \in (0, \infty) \times Y_d, \quad X_\varepsilon(t, x) := \varepsilon X\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) = x + \varepsilon \int_0^{\frac{t}{\varepsilon}} b\left(X(s, \frac{x}{\varepsilon})\right) ds. \quad (5.3)$$

Taking into account the regularity conditions the characteristics method induced by the flow  $X_\varepsilon$  implies that the solution  $u_\varepsilon$  to (5.1) is given by

$$\forall (t, x) \in [0, \infty) \times Y_d, \quad u_\varepsilon(t, x) = u_0(X_\varepsilon(t, x)) = u_0\left(x + \varepsilon \int_0^{\frac{t}{\varepsilon}} b\left(X(s, \frac{x}{\varepsilon})\right) ds\right). \quad (5.4)$$

On the other hand, let  $(\varepsilon_n)_{n \in \mathbb{N}}$  be a positive sequence converging to 0. Let  $(t, x) \in (0, \infty) \times Y_d$ , set  $t_n := t/\varepsilon_n$  and  $x_n := x/\varepsilon_n$ . Then, by virtue of Proposition A.1 the limit points of the sequence

$$v_n := \varepsilon_n \int_0^{\frac{t}{\varepsilon_n}} b\left(X(s, \frac{x}{\varepsilon_n})\right) ds = t \times \frac{1}{t_n} \int_0^{t_n} b(X(s, x_n)) ds$$

belong to  $t\mathbb{C}_b$ . However, by (5.2) combined with equivalence (2.1) we have  $\mathbb{C}_b = \{\zeta\}$ . Hence, for any positive sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  converging to 0, the whole sequence  $(v_n)_{n \in \mathbb{N}}$  converges to  $t\zeta$ , which combined with (5.3) implies that

$$\forall (t, x) \in (0, \infty) \times Y_d, \quad \lim_{\varepsilon \rightarrow 0} X_\varepsilon(t, x) = x + t\zeta. \quad (5.5)$$

Moreover, making the change of variable  $r = \varepsilon s$  in (5.3) we have

$$\forall (t, x) \in [0, \infty) \times Y_d, \quad X_\varepsilon(t, x) = x + \int_0^t b\left(X\left(\frac{r}{\varepsilon}, \frac{x}{\varepsilon}\right)\right) dr.$$

This combined with the boundedness of  $b$  and Lebesgue's theorem implies that the pointwise convergence (5.5) of  $X_\varepsilon$  holds actually in  $L^p_{\text{loc}}([0, \infty) \times \mathbb{R}^d)$  for any  $p \in [1, \infty)$ . Therefore, by the expression (5.4) with  $u_0 \in C^1(\mathbb{R}^d)$ ,  $u_\varepsilon(t, x)$  converges strongly in  $L^p_{\text{loc}}([0, \infty) \times \mathbb{R}^d)$  for any  $p \in [1, \infty)$ , to the function  $u_0(x + t\zeta)$ , which concludes the proof.  $\square$



## A Derivation of invariant probability measures

Let  $T_t$  for  $t \in \mathbb{R}$ , be the mapping from  $C_\#^0(Y_d)$  into itself defined by

$$(T_t f)(x) := f(X(t, x)) \quad \text{for } f \in C_\#^0(Y_d) \text{ and } x \in Y_d. \quad (\text{A.1})$$

When a flow preserves the set of the continuous functions on a compact metric space, the existence of an invariant probability measure for the flow is a classical statement which can be derived thanks to a weak compactness argument applied to sequences of probability measures defined from the Birkhoff time averages in (1.5) (see, *e.g.*, [13, Theorem 1, Section 1.8] in the discrete time case). The following result adapts this statement restricting it to the limit points of the Birkhoff time averages for a given fixed function, adding possible variations of the spatial parameter  $x$  in the averages.

**Proposition A.1** *Let  $b \in C_\#^1(Y_d)^d$ . There exists an invariant probability measure on  $Y_d$  for the flow  $X$  (1.15) associated with  $b$ . Moreover, let  $g \in C_\#^0(Y_d)$ , let  $(x_n)_{n \in \mathbb{N}} \in (\mathbb{R}^d)^\mathbb{N}$ , and let  $(t_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}$  be such that  $\lim_n t_n = \infty$ . Then, for any limit point  $a$  of the sequence  $(u_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}$  defined by*

$$u_n := \frac{1}{t_n} \int_0^{t_n} g(X(s, x_n)) ds, \quad n \in \mathbb{N}, \quad (\text{A.2})$$

*there exists a probability measure  $\mu \in \mathcal{M}_p(Y_d)$  (depending a priori on  $(x_n)_{n \in \mathbb{N}}$  and  $g$ ) which is invariant for the flow  $X$  and which satisfies*

$$a = \int_{Y_d} g(y) d\mu(y). \quad (\text{A.3})$$

*Proof.* We have the following result which is proved below.

**Lemma A.2** *Let  $(y_n)_{n \in \mathbb{N}} \in (\mathbb{R}^d)^\mathbb{N}$ , let  $(r_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}$  be such that  $\lim_n r_n = \infty$ , and let  $\nu_n$ ,  $n \in \mathbb{N}$ , be the probability measure defined by*

$$\int_{Y_d} f(y) d\nu_n(y) = \frac{1}{r_n} \int_0^{r_n} f(X(s, y_n)) ds \quad \text{for } f \in C_\#^0(Y_d). \quad (\text{A.4})$$

*Then, there exists a subsequence  $(\nu_{n_k})_{k \in \mathbb{N}}$  of  $(\nu_n)_{n \in \mathbb{N}}$  which converges weakly  $*$  to some probability measure  $\mu \in \mathcal{M}_p(Y_d)$  which is invariant for the flow  $X$ .*

Let  $a$  be a limit point of the sequence  $(u_n)_{n \in \mathbb{N}}$  (A.2), namely

$$a = \lim_{n \rightarrow \infty} \frac{1}{t_{\theta(n)}} \int_0^{t_{\theta(n)}} g(X(t, x_{\theta(n)})) dt,$$

for some strictly increasing sequence  $(\theta(n))_{n \in \mathbb{N}}$  of integer numbers. Set  $r_n := t_{\theta(n)}$ ,  $y_n := x_{\theta(n)}$ , and consider the associated sequence  $(\nu_n)_{n \in \mathbb{N}}$  of probability measures on  $Y_d$  given by (A.4). By Lemma A.2 we can extract a subsequence  $(\nu_{n_k})_{k \in \mathbb{N}}$  which converges weakly  $*$  to some invariant probability measure  $\mu \in \mathcal{I}_b$  for the flow  $X$ . We thus have

$$\forall f \in C_\#^0(Y_d), \quad \lim_{k \rightarrow \infty} \int_{Y_d} f(y) d\nu_{n_k}(y) = \int_{Y_d} f(y) d\mu(y),$$

which implies in particular that

$$a = \lim_{k \rightarrow \infty} \frac{1}{t_{\theta(n_k)}} \int_0^{t_{\theta(n_k)}} g(X(s, x_{\theta(n_k)})) ds = \lim_{k \rightarrow \infty} \int_{Y_d} g(y) d\nu_{n_k}(y) = \int_{Y_d} g(y) d\mu(y).$$

□



*Proof of Lemma A.2.* Since  $Y_d$  is a compact metrizable space, there exists a subsequence  $(\nu_{n_k})_{k \in \mathbb{N}}$  of  $(\nu_n)_{n \in \mathbb{N}}$  which converges weakly  $*$  to some probability measure  $\mu \in \mathcal{M}_p(Y_d)$ , namely for any  $f \in C_\#^0(Y_d)$ ,

$$\int_{Y_d} f(y) d\nu_{n_k}(y) = \frac{1}{r_{n_k}} \int_0^{r_{n_k}} f(X(s, y_{n_k})) ds \xrightarrow{k \rightarrow \infty} \int_{Y_d} f(y) d\mu(y). \quad (\text{A.5})$$

Let us prove that  $\mu$  is invariant for the flow  $X$ . For the sake of simplicity denote  $\tau_k := r_{n_k}$ ,  $z_k := y_{n_k}$  and  $\mu_k := \nu_{n_k}$ . Let  $t \in \mathbb{R}$  and  $f \in C_\#^0(Y_d)$ . By the semi-group property of the flow (1.16) we have

$$\int_{Y_d} (T_t f)(y) d\mu_k(y) = \frac{1}{\tau_k} \int_0^{\tau_k} f(X(s+t, z_k)) ds.$$

By the change of variable  $r = s + t$ , it follows that

$$\begin{aligned} \int_{Y_d} (T_t f)(y) d\mu_k(y) &= \frac{1}{\tau_k} \int_t^{t+\tau_k} f(X(r, z_k)) dr \\ &= \frac{1}{\tau_k} \int_0^{\tau_k} f(X(r, z_k)) dr + \frac{1}{\tau_k} \int_{\tau_k}^{t+\tau_k} f(X(r, z_k)) dr - \frac{1}{\tau_k} \int_0^t f(X(r, z_k)) dr \end{aligned}$$

Since  $f$  is bounded and  $t \in \mathbb{R}$  is fixed, we deduce from (A.5) that

$$\lim_{k \rightarrow \infty} \int_{Y_d} (T_t f)(y) d\mu_k(y) = \int_{Y_d} f(y) d\mu(y).$$

However, by the definition of  $\mu$  we also have

$$\lim_{k \rightarrow \infty} \int_{Y_d} (T_t f)(y) d\mu_k(y) = \int_{Y_d} (T_t f)(y) d\mu(y).$$

Hence, we get that

$$\forall t \in \mathbb{R}, \forall f \in C_\#^0(Y_d), \quad \int_{Y_d} (T_t f)(y) d\mu(y) = \int_{Y_d} f(y) d\mu(y),$$

which implies that  $\mu$  is invariant for the flow  $X$ . □

## References

- [1] G. ALESSANDRINI & V. NESI: "Univalent  $\sigma$ -harmonic mappings", *Arch. Rational Mech. Anal.*, **158** (2001), 155-171.
- [2] Y. AMIRAT, K. HAMDACHE & A. ZIANI: "Homogénéisation d'équations hyperboliques du premier ordre et application aux écoulements miscibles en milieu poreux" (French) [Homogenization of a system of first-order hyperbolic equations and application to miscible flows in a porous medium], *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **6** (5) (1989), 397-417.
- [3] Y. AMIRAT, K. HAMDACHE & A. ZIANI: "Homogénéisation par décomposition en fréquences d'une équation de transport dans  $\mathbb{R}^n$ " (French) [Homogenization by decomposition into frequencies of a transport equation in  $\mathbb{R}^n$ ], *C. R. Acad. Sci. Paris Sér. I Math.*, **312** (1) (1991), 37-40.



- [4] Y. AMIRAT, K. HAMDACHE & A. ZIANI: "Homogenisation of parametrised families of hyperbolic problems", *Proc. Roy. Soc. Edinburgh Sect. A*, **120** 3-4 (1992), 199-221.
- [5] A. BENSOUSSAN, A., J.-L LIONS & G. PAPANICOLAOU: *Asymptotic Analysis for Periodic Structures*, Studies in Mathematics and its Applications **5**, North-Holland Publishing Co., Amsterdam-New York, 1978, 700 pp.
- [6] Y. BRENIER: "Remarks on some linear hyperbolic equations with oscillatory coefficients", *Proceedings of the Third International Conference on Hyperbolic Problems* (Uppsala 1990) Vol. I, II, Studentlitteratur, Lund (1991), 119-130.
- [7] M. BRIANE: "Isotropic realizability of a strain field for the two-dimensional incompressible elasticity system", *Inverse Problems*, **32** (6) (2016), 22 pp.
- [8] M. BRIANE: "Isotropic realizability of fields and reconstruction of invariant measures under positivity properties. Asymptotics of the flow by a nonergodic approach", *SIAM J. App. Dyn. Sys.*, **18** (4) (2019), 1846-1866.
- [9] M. BRIANE: "Homogenization of linear transport equations. A new approach," *J. École Polytechnique - Mathématiques*, **7** (2020), 479-495.
- [10] M. BRIANE, G.W. MILTON & V. NESI : "Change of sign of the corrector's determinant for homogenization in three-dimensional conductivity", *Arch. Rat. Mech. Anal.*, **173** (1) (2004), 133-150.
- [11] M. BRIANE, G.W. MILTON ET A. TREIBERGS : "Which electric fields are realizable in conducting materials?", *ESAIM: Math. Model. Numer. Anal.*, **48** (2) (2014), 307-323.
- [12] R. CACCIOPPOLI: "Sugli elementi uniti delle trasformazioni funzionali: un teorema di esistenza e unicit?ed alcune sue applicazioni", *Rend. Sem. Mat. Padova*, **3** (1932), 1-15.
- [13] I.P. CORNFELD, S.V. FOMIN & YA.G. SINAI: *Ergodic Theory*, translated from the Russian by A.B. Sosinskii, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **245**, Springer-Verlag, New York, 1982, 486 pp.
- [14] B. DACOROGNA: *Direct Methods in the Calculus of Variations*, Second Edition, Applied Mathematical Sciences **78**, Springer, New York, 2008, 619 pp.
- [15] D. GILBARG & N.S. TRUDINGER: *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001, 531 pp.
- [16] F. GOLSE: "Moyennisation des champs de vecteurs et EDP" (French), [The averaging of vector fields and PDEs], *Journées Équations aux Dérivées Partielles*, Saint Jean de Monts 1990, Exp. no. **XVI**, École Polytech. Palaiseau, 1990, 17 pp.
- [17] F. GOLSE: "Perturbations de systèmes dynamiques et moyennisation en vitesse des EDP" (French), [On perturbations of dynamical systems and the velocity averaging method for PDEs], *C. R. Acad. Sci. Paris Sér. I Math.*, **314** (2) (1992), 115-120.
- [18] M.R. HERMAN: "Existence et non existence de tores invariants par des difféomorphismes symplectiques" (French), [Existence and nonexistence of tori invariant under symplectic diffeomorphisms], *Séminaire sur les Équations aux Dérivées Partielles 1987-1988*, **XIV**, École Polytech. Palaiseau, 1988, 24 pp.



- [19] M.W. HIRSCH, S. SMALE & R.L. DEVANEY: *Differential equations, Dynamical Systems, and an Introduction to Chaos*, Second edition, *Pure and Applied Mathematics* **60**, Elsevier Academic Press, Amsterdam, 2004, 417 pp.
- [20] T.Y. HOU & X. XIN: "Homogenization of linear transport equations with oscillatory vector fields", *SIAM J. Appl. Math.*, **52** (1) (1992), 34-45.
- [21] A.N. KOLMOGOROV: "On dynamical systems with an integral invariant on the torus" (Russian), *Doklady Akad. Nauk SSSR (N.S.)*, **93** (1953), 763-766.
- [22] J. LLIBRE & R.S. MACKAY: "Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity", *Ergodic Theory Dynam. Systems*, **11** (1) (1991), 115-128.
- [23] M. MISIUREWICZ & K. ZIEMIAN: "Rotation sets for maps of tori", *J. London Math. Soc. (2)*, **40** (3) (1989), 490-506.
- [24] R. PEIRONE: *Convergence of solutions of linear transport equations*, *Ergodic Theory Dynam. Systems*, **23** (3) (2003), 919-933.
- [25] *Dynamical Systems II, Ergodic Theory with Applications to Dynamical Systems and Statistical Mechanics* (Translated from the Russian), Edited by E.Ya Sinai, Encyclopaedia of Mathematical Sciences **2**, Springer-Verlag Berlin 1989, 281 pp.
- [26] T. TASSA: "Homogenization of two-dimensional linear flows with integral invariance", *SIAM J. Appl. Math.*, **57** (5) (1997), 1390-1405.
- [27] C.L. SIEGEL: "Note on Differential Equations on the Torus", *Annals of Mathematics*, **46** (3) (1945), pp. 423-428.
- [28] L. TARTAR: "Nonlocal effects induced by homogenization", *Partial Differential Equations and the Calculus of Variations Vol. II*, F. Colombini et al. (eds.), 925-938, Progr. Nonlinear Differential Equations Appl. **2**, Birkhäuser Boston, Boston, MA, 1989.