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1 **Permutations uniquely identify states and unknown external**
2 **forces in non-autonomous dynamical systems**

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18 **Abstract**

19 It has been shown that a permutation can uniquely identify the joint set of an initial condition
20 and a non-autonomous external force realization added to the deterministic system in given time
21 series data. We demonstrate that our results can be applied to time series forecasting as well as
22 the estimation of common external forces. Thus, permutations provide a convenient description
23 for a time series dataset generated by non-autonomous dynamical systems.

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24 The symbolic method is a powerful tool for analyzing time series data by
25 coarse-graining. When the underlying dynamics is deterministic, a generating
26 partition and symbolic dynamics may be used to convert an initial condition
27 to an infinitely long symbolic sequence in a one-to-one manner. However, this
28 method fails for non-autonomous dynamics, e.g. deterministic dynamical sys-
29 tems in the presence of dynamical noise or external forces, because a partition
30 cannot remove the uncertainty in specifying an initial condition. Here, we show
31 that, unlike a generating partition for symbolic dynamics, permutations (ordinal
32 patterns) can represent a real-valued time series generated by non-autonomous
33 dynamical systems. We show that a permutation establishes a one-to-one cor-
34 respondence with a realized orbit based on the joint set of an initial condition
35 and external force realization added to the deterministic system, if the dynam-
36 ics under dynamical noise is topologically transitive. Thus, our results explain
37 why permutations, in some cases, can distinguish deterministic systems from
38 stochastic systems. In addition, we demonstrate that our findings can be ap-
39 plied to forecasting behavior as well as estimating common dynamical noise in
40 random dynamical systems.

41 I. INTRODUCTION

42 In the analysis of time series data generated by dynamical systems, coarse-graining a
43 state is one of the conventional approaches to describe dynamical systems [1–7]. This proce-
44 dure is the cornerstone of statistical mechanics and provides a framework to describe several
45 complex physical phenomena such as turbulent flow [8], molecular dynamics [9] among oth-
46 ers. For example, in a deterministic dynamical system, a generating partition helps us to
47 establish one-to-one correspondence between an initial condition and the orbit of symbolic
48 dynamics. Subsequently, we can provide rigorous foundations and/or calculations [3, 7] as
49 well as bridge the ideas coming from dynamical systems theory and information theory [10–
50 12]. However, this method fails for stochastic dynamics because any partition cannot remove
51 the uncertainty for a state in the system due to externally added noise [13, 14]. Recently,
52 stochastic chaos in random dynamical systems has been studied theoretically and experi-

53 mentally [15, 16]. The main finding is the possibility to describe the behavior of complex
 54 systems such as turbulent forced flows with simple dynamical systems where non-essential
 55 degrees of freedom are lumped in the random dynamics. These studies clearly show the
 56 existence of open problems on *nonlinear time series analysis for random / non-autonomous*
 57 *dynamical systems*.

58 Permutations (or ordinal patterns) or topological methods in nonlinear time series analy-
 59 sis, have been studied as an alternative analysis to coarse-grained dynamics [17]. It is known
 60 that we can estimate the Kolmogorov-Sinai entropy not only by generating partitions but
 61 also by permutations [18]. Distinguishing deterministic systems from stochastic systems is
 62 a recent trend in permutation studies. [19–25]. In physics, the interest is to understand
 63 whether a deterministic behavior can be separated from a stochastic dynamics, thus en-
 64 abling for simpler descriptions of complex systems, as in Ref. [16]. Here, we examine the
 65 hypothesis that a permutation can achieve a one-to-one correspondence with a joint set of
 66 an initial condition and a realization of the external force in random and non-autonomous
 67 dynamical systems, as the length of the permutations tends to infinity. The key idea is that
 68 the variety of permutations could grow super-exponentially as the size of permutations in-
 69 creases when the underlying dynamics is stochastic [24, 26]. This super-exponential growth
 70 can accommodate the information regarding the state space as well as a stochastic input
 71 series within a permutation.

72 II. OUR SETTINGS AND THEORETICAL RESULTS

73 We consider a non-autonomous dynamical system $f : X \times P \rightarrow X$

$$x_{t+1} = f(x_t, p_t), \quad x_t \in X, \quad p_t \in P, \quad (1)$$

74 Here x_t is a model of the state of the dynamical system, and p_t is a model of external force
 75 or noise, which drives the dynamical system at time t . Here we adopt both X and P as
 76 one-dimensional intervals.

77 We assume the following [27]:

- 78 1. The sequence $\{p_t\}_{t=0, \dots, n-1}$ is given beforehand as a hidden parameter to be estimated.
- 79 2. The function $f(x, p)$ is a continuous map and an embedding in terms of arbitrary p ;

80 Namely, under $x_{t+1} = f(x_t, p)$, the parameter p corresponds to x_{t+1} in a one-to-one
 81 manner when we fix x_t .

82 Our goal is to estimate both x_0 and $\{p_t\}_{t=0, \dots, n-1}$, based on the given time series data
 83 $\{x_t\}_{t=1, \dots, n}$. Here, as a shorthand, we write $x_{t+1} = f_{p_t}(x_t)$ and

$$x_{t+1} = f_{p_t}(f_{p_{t-1}}(\cdots f_{p_0}(x_0)\cdots)) = f_{p_0^t}(x_0), \quad (2)$$

84 where $p_0^t = \{p_\tau\}_{\tau=0, \dots, t}$.

85 We now introduce the permutations [17, 18]. Suppose that a scalar time series s_t ($t =$
 86 $1, 2, \dots, N$) is given. Now, consider n consecutive measurements $s_t, s_{t+1}, \dots, s_{t+n-1}$ starting
 87 from time t . If we sort these measurements in the ascending order, we have

$$s_{t+t_1} \leq s_{t+t_2} \leq \cdots \leq s_{t+t_n}, \quad (3)$$

88 where we define $s_{t+t_i} \leq s_{t+t_j}$, if $s_{t+t_i} = s_{t+t_j}$ and $t_i < t_j$. Then, the obtained series

$$\pi_t(n) = (t_1, t_2, \dots, t_n) \quad (4)$$

89 is called the *permutation* for time t with length n . It is known that the Kolmogorov-Sinai
 90 entropy can be obtained using permutations if the underlying dynamics is ergodic [18].

91 We especially consider the dynamics in a one-dimensional space X and introduce a natural
 92 measure μ if for all test functions $h : X \rightarrow \mathbf{R}$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^{N-1} h(f_{p_0^t}(x)) \rightarrow \int_A h d\mu \quad (5)$$

93 for almost all $x \in A \subset X$ and for almost all p_0^t [28]. We further assume that the non-
 94 autonomous dynamical system (2) has a natural measure.

95 We refer to $x_{t+1} = f_{p_0^t}(x_t)$ as topologically transitive if $\{x_t\}$ is dense in $A \subset X$. This
 96 definition can be equivalent to that if there exists $t > 0$ such that $f_{p_0^t}(U) \cap V \neq \phi$ for any
 97 pair of open sets $U, V \subset A \subset X$. Approximately, when one starts from an open set U ,
 98 we can visit the neighborhood of another open set V after finite iterations of f . Note that
 99 in general, the topological transitivity in random and non-autonomous dynamical systems
 100 depends on the given p_0^t . In other words, here we exclude the case where $f_{p_0^t}(x)$ only forms
 101 a finite periodic orbit because such an orbit is not dense on A . Our theoretical result is
 102 summarized as the following main theorem:

103 **Theorem 1.** *Suppose f on X has a natural measure. Let $[x_{i_0^n-1}, x_{i_0^n+1}]$ be an interval for*
104 *an initial condition x_0 specified by a permutation of length n . Similarly, $[\underline{p}_{t,n}, \bar{p}_{t,n}]$ be an*
105 *interval for the external force at time t specified by the same permutation. Then, each of*
106 *$[x_{i_0^n-1}, x_{i_0^n+1}]$ and $[\underline{p}_{t,n}, \bar{p}_{t,n}]$ for each t converges to a single point when the length n of the*
107 *permutation tends to infinity if and only if the dynamics f on X is topologically transitive.*

108 The following Lemma 1 and the contraposition of Lemma 2 lead to the above main
109 theorem.

110 **Lemma 1.** *Suppose that the dynamics f on X is topologically transitive and has a natural*
111 *measure. Then, a permutation can specify a joint set of an initial condition x_0 and a real-*
112 *ization of the external force $\{p_t\}$ as the length of the permutation goes to infinity. Namely,*
113 *$[x_{i_0^n-1}, x_{i_0^n+1}]$ and $[\underline{p}_{t,n}, \bar{p}_{t,n}]$ for each t converge to single points, respectively.*

114 *Proof.* Suppose that the current length of permutations is n . Additionally, let us assume
115 initially that x_0 is neither the minimum nor the maximum of $x \in X$. Then, if x_0 is the
116 i_0^n -th point from below, the initial value x_0 is sandwiched between the $i_0^n - 1$ -th point $x_{i_0^n-1}$
117 and the $i_0^n + 1$ -th point $x_{i_0^n+1}$, namely, we have $x_{i_0^n-1} \leq x_0 \leq x_{i_0^n+1}$, which is an interval
118 between the minimum and the maximum of x . Let us consider $x_{i_0^n-1}$ and $x_{i_0^n+1}$, separately.
119 First, we consider $x_{i_0^n-1}$. Because we assume that the dynamics is topologically transitive,
120 there is $m_L(1) > n$ such that $x_{i_0^{m_L(1)}-1} \in (x_{i_0^n-1}, x_0)$, implying that $x_{i_0^n-1} < x_{i_0^{m_L(1)}-1} < x_0$.
121 By applying the same logic repeatedly, we can choose a sequence $\{m_L(k) : k = 1, 2, \dots\}$
122 such that $x_{i_0^{m_L(k)}-1} < x_{i_0^{m_L(l)}-1} < x_0$ for $0 < k < l$. Since the sequence of $x_{i_0^{m_L(k)}-1}$ is always less
123 than x_0 and increasing monotonically, we have $x_{i_0^{m_L(k)}-1} \rightarrow x_0$ when $k \rightarrow \infty$. By using the
124 similar logic, there is a sequence $\{m_R(k) : k = 1, 2, \dots\}$ such that $x_{i_0^{m_R(k)}+1} > x_{i_0^{m_R(l)}+1} > x_0$ for
125 $0 < k < l$. Thus, $x_{i_0^{m_R(k)}+1} \rightarrow x_0$ when $k \rightarrow \infty$.

126 Both limits mean that the interval $[x_{i_0^n-1}, x_{i_0^n+1}]$ gets smaller and smaller, and converges
127 to a single point x_0 when the length of permutations tends to infinity. In addition, since
128 the i_0^n -th point among n points can be rephrased as a certain percentile point on μ as a
129 natural measure, the initial condition x_0 can be specified on X . Therefore, the theorem has
130 been proven. When x_0 is either the minimum or the maximum of $x \in X$, we can make
131 x_0 a sandwich within $[\min_{x \in I} x, x_{i_0^n+1}]$ or $[x_{i_0^n-1}, \max_{x \in I} x]$, respectively. Thus, the similar
132 monotonic convergence argument discussed above holds for both cases.

133 As it has been found that a series of states $\{x_t\}$ is identified by the corresponding per-
 134 mutation, each p_t can be inferred because now we assume that $f(x_t, p)$ is an embedding in
 135 terms of the arbitrary non-autonomous force p given x_t . Therefore, if f is known, we can
 136 identify the joint set of $\{x_t\}$ and $\{p_t\}$ by the corresponding permutation as the length of
 137 permutations tending to infinity. \square

138 **Lemma 2.** *Suppose that the dynamics f on a one-dimensional space X is not topologically*
 139 *transitive. Then, a permutation cannot specify all the initial conditions and realization of*
 140 *the external force $\{p_t\}$ when the length of the permutation tends to infinity.*

141 *Proof.* Assume that the underlying dynamics on a one-dimensional interval is not topo-
 142 logically transitive. Then, there is some interval $(a, b) \subset X$ such that any point for the
 143 underlying dynamics will not visit the interval (a, b) after any number of iterations, where a
 144 and b are some of the time points of the currently given time series up to some length $n > 2$.
 145 Therefore, if an initial condition starts within this interval, we cannot refine the uncertainty
 146 for such an initial condition even if we prolong the length for the permutation. Since $\{x_t\}$
 147 is not identified, there is no clue for identifying $\{p_t\}$ using the permutation. \square

148 Our theoretical foundation is based on topological transitivity, namely the denseness of
 149 orbits: If there is an interval within X , this interval is divided by future points obtained
 150 from the underlying dynamics. Thus, this interval is eventually narrowed to a point when
 151 the length of the permutations tends to infinity. Once each state is identified, we can also
 152 learn the value for dynamical noise because of the property of the embedding between p_t
 153 and x_{t+1} .

154 Moreover, we note that, using the same logic, we can also uniquely specify a point x_k for
 155 k ($0 < k < n$), which is demonstrated in numerical experiments in the next section.

156 III. NUMERICAL RESULTS

157 Here, we demonstrate how to use permutations for inferring a state as well as external
 158 noise. Our focus is on inferring information regarding the external force. For quantifying
 159 the external noise at a particular time, there are two approaches: (i) When we can access a
 160 time series of the external noise and (ii) when we cannot access a time series of the external
 161 noise.

162 When we can access a time series $\{p_t\}$ of the external noise p , we can estimate the mean
 163 state $M_\pi^p(\kappa)$ for the κ -th relative point of the external noise p by using time segments sharing
 164 the same permutation π obtained by time series x in the modelling part of the time series.
 165 When we encounter a permutation π in the validating part of the time series, we just need
 166 to recall the corresponding mean state $M_\pi^p(\kappa)$.

167 When we cannot access a time series of the external noise, we try to reproduce the
 168 underlying metric space from the generated permutations. We can subdivide this case into
 169 two sub-cases. If we assume that the external force is slow, we can directly apply the idea
 170 of Ref. [29] to remove the state space information and reproduce the information of the slow
 171 external force. If we cannot assume that the external force is slow, then we need to have
 172 multiple observation nodes [30, 31] which are subject to the same external force to remove
 173 the information of state spaces and reproduce this sudden external force. In either case, we
 174 use a recurrence plot [32, 33], which helps us to transform the binary information of whether
 175 the two states for the corresponding times are neighbors to a metric space, or a distance
 176 matrix.

177 A. Estimation of states

178 In addition to the theoretical proofs provided above, we tested our hypothesis numerically.
 179 For testing our idea, we used the logistic map [34] and the Hénon map [35] subject to
 180 dynamical noise. The logistic map we used is defined as follows:

$$x_{t+1} = (3.7 + \epsilon_t)x_t(1 - x_t), \quad (6)$$

181 where ϵ_t is a source of independent uniform noise on $[-0.1, 0.1]$. We chose the initial condition
 182 x_0 from a uniform distribution on $[0, 1]$ and observed the time evolution of the variable x_t .
 183 Similarly, the Hénon map we used is defined as follows:

$$\begin{aligned} x_{t+1} &= 1 - (1.2 + \eta_t)x_t^2 + 0.3y_t, \\ y_{t+1} &= x_t, \end{aligned} \quad (7)$$

184 where η_t independently follows a uniform distribution on $[-0.05, 0.05]$. In addition, we
 185 choose the initial conditions x_0 and y_0 from the uniform distribution on $[0, 1]$. Therefore,
 186 here we are considering dynamical systems subject to dynamical noise and discuss whether
 187 we can specify a state x_t by a permutation.

188 Although our theorem is restricted to one-dimensional interval dynamics, we use the
189 Hénon map to observe whether our claim holds for higher dimensional dynamical systems.
190 We adopt x_t as the observed time series. Our numerical approach aims to represent a time
191 series using a series of permutations (see Fig. 1 for the intuitive illustration). Thus, by spec-
192 ifying a permutation, we can eventually specify both series of states as well as stochastic
193 inputs, simultaneously. Since a permutation eventually specifies a state in the limit for the
194 length of permutation approaching infinity, representing points sharing the same permuta-
195 tion with a point becomes a reasonable approximation. We generated two time series x
196 and x' of the length N from the same system. Here we set $N = 1\,000\,000$. For each time
197 series x and x' , we also obtained a series of permutations $\{\pi_t(n)\}$ and $\{\pi'_t(n)\}$ by using
198 permutations of length n . Then, we use the first time series x and its series of permutations
199 $\{\pi_t(n)\}$ to compute the mean state

$$M_\pi^x(\kappa) = \frac{1}{|\#\{\pi_t = \pi | t = 1, 2, \dots, N\}|} \sum_{t:\pi_t=\pi} x_{t+\kappa} \quad (8)$$

200 for the κ -th point of each appearing permutation π . This step is similar to a step of the k-
201 means algorithm [36]. Especially, we define the middle point $K = \lfloor n/2 \rfloor$ of the permutation.
202 These mean states become our estimates for states corresponding to a permutation π since
203 each permutation corresponds to a single initial condition as well as a single series of the
204 external noise in the limit of the size of the permutation approaching to infinity. These means
205 enable us to represent the second time series x' by replacing each permutation $\pi'_t(n)$ with
206 the corresponding mean $M_{\pi'_t(n)}^x(\kappa)$ state of the κ -th point of permutation $\pi'_t(n)$ obtained
207 from the first time series x . (If $\kappa \geq n$, then such an estimation becomes a time series
208 forecast.) Lastly, we evaluate the estimation error

$$\varepsilon^x(\kappa - n + 1) = E_t [|x'_{t+\kappa} - M_{\pi'_t(n)}^x(\kappa)|] \quad (9)$$

209 by comparing the second time series x' and its representation $\{M_{\pi'_t(n)}^x(\kappa) | t = 1, 2, \dots, N -$
210 $n + 1\}$ constructed by permutations. This approach is called the mean representation [6].

211 We first estimated states using the mean representation method (Fig. 2). We found that
212 states corresponding to the time period of permutations were estimated more accurately
213 than the cases where we just used the mean states for all over the points of the entire time
214 series.

215 Moreover, we found a general tendency in the estimation error convergence to 0 with
216 increasing length of permutations (Figs. 3 (a) and (b), for the logistic map [34] and the Hénon

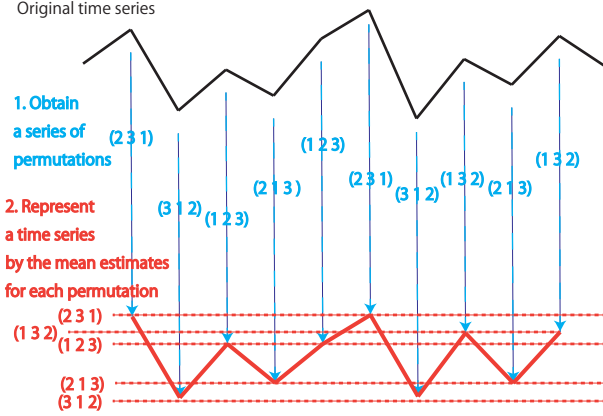


FIG. 1. Schematic for explaining the first numerical approach, the mean representation.

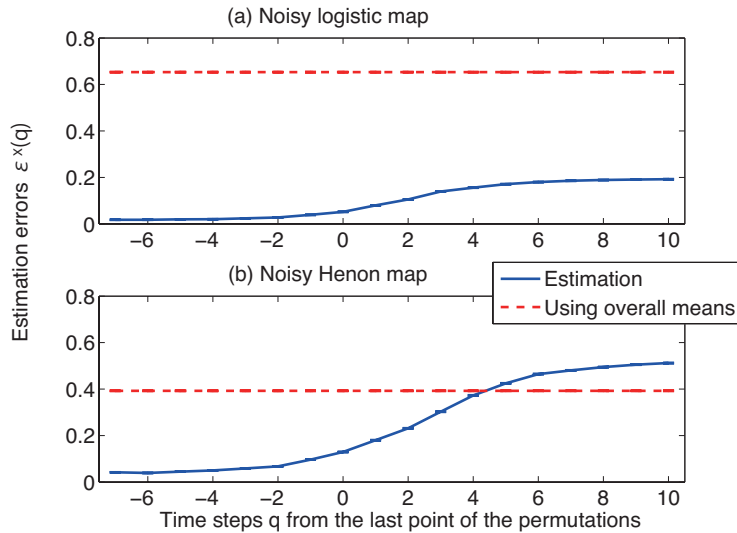


FIG. 2. Estimation errors for the current and future states. Panels (a) and (b) correspond to the cases for the logistic map and Hénon map subject to dynamical noise, respectively. For control, we showed the estimation errors by the overall means that do not depend on the position of the corresponding attractor. Each error bar is obtained from the mean and the standard deviation for ten simulations. Estimations for the future states can be rephrased as “forecasts” of the second time series given the first time series.

217 map [35], respectively). When we rigorously compared the model of exponential decrease
 218 converging to a constant in the limit of $n \rightarrow \infty$ with the model of exponential decrease
 219 converging to zero using the Akaike information criterion [37], the model of exponential
 220 decrease converging to zero was selected for both cases (see Fig. 4). Thus, these results

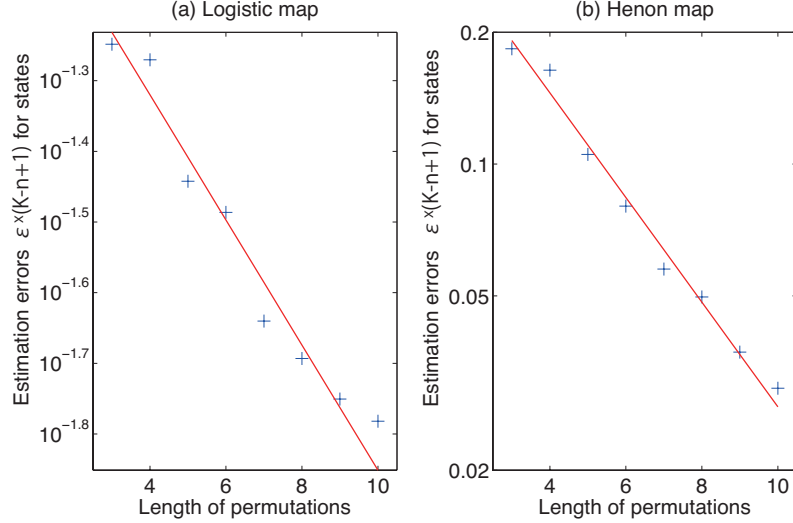


FIG. 3. Estimation error convergence by permutations in (a) the logistic map and (b) the Hénon map. For panels (a) and (b), we used 10 time series of length 1 000 000 to obtain the means for the estimation errors shown by +. The solid lines were obtained by fitting the model of exponential decrease.

221 imply that a permutation corresponds to an initial condition, or a state, for these models.

222 Furthermore, the mean representation method was extended so that we considered the
 223 means for q steps ahead by defining $\kappa = n - 1 + q$ in Eqs. (8) and (9) to make them
 224 “forecasts”. Then, we found that we could forecast short-term prediction horizons up to 10
 225 and 4 steps ahead better, in the noisy logistic map (Fig. 2(a)) and the noisy Hénon map
 226 (Fig. 2(b)), respectively, than the method of control where we considered the simple means
 227 over all the points of the attractor. The accuracy of these forecasts was achieved because the
 228 permutations could specify the past states and noise realization, even though uncertainty
 229 was generated due to the current and future parts of dynamical noise as well as the sensitive
 230 dependence on the initial conditions.

231 B. Estimation of realization of external force

232 Theorem 1 implies that we can estimate the realization of the external force when there
 233 is a mathematical model for the dynamical system. However, our results imply that even
 234 if such a model f is not available, we can estimate the realization of external force in the
 235 following two cases:

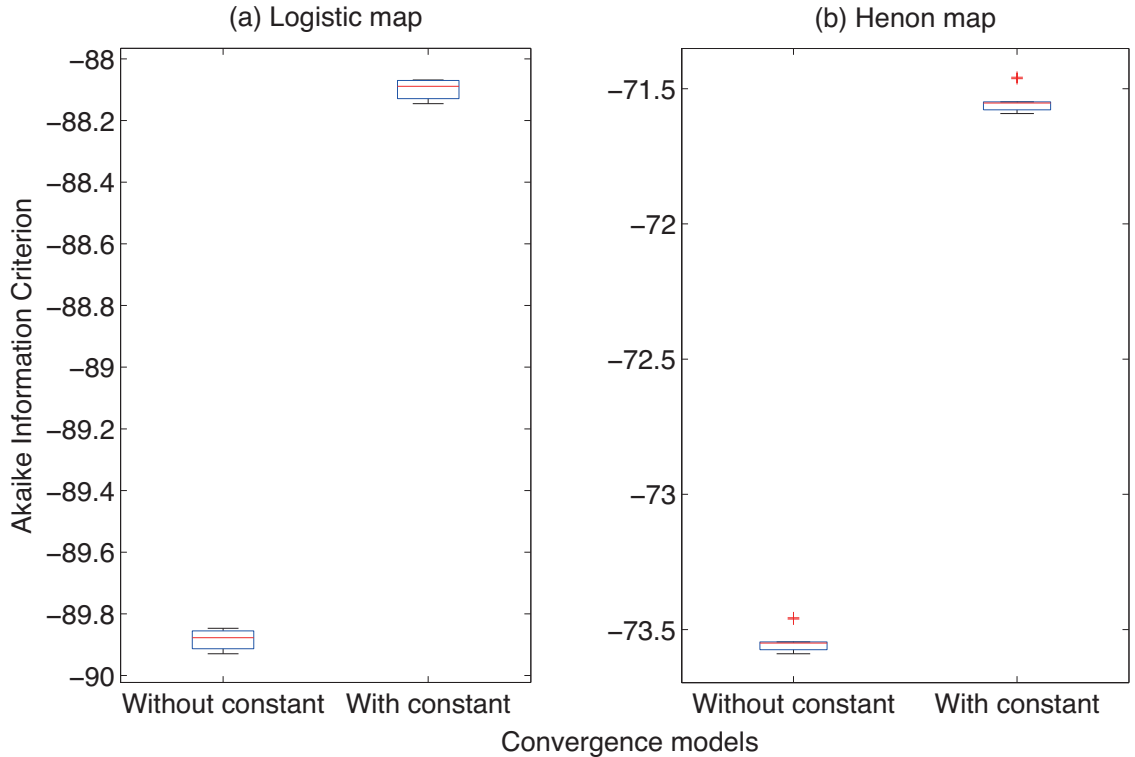


FIG. 4. Comparison of Akaike information criterion between the exponential convergence model without a constant term and with a constant term. Panels (a) and (b) correspond to the logistic map and the Hénon map, respectively. For the exponential convergence model without a constant term, we fitted the error model of $b \exp(-al)$, where l is the length of time series and $b > 0$. For the exponential convergence model with a constant term, we fitted the error model of $b \exp(-al) + c$, where $b, c > 0$.

- 236 1. Dynamics of the external force is gradual compared with the driven intrinsic dynamics;
- 237 2. We observe multiple time series whose intrinsic dynamics is governed by identical
- 238 dynamics and are subject to a common external force.

239 Therefore, Theorem 1 provides a way of obtaining the distribution of external noise as well.

240 To evaluate this possibility, we have estimated the dynamical noise by using the mean
 241 representation method and recurrence plot [32, 33] (See Appendix A). We found that the
 242 mean representation method achieved lower estimation errors than merely using the same
 243 means over all the permutations when the time steps are those corresponding to the positions
 244 of permutations (Fig. 5). In addition, the estimation errors for the noise realization also tend

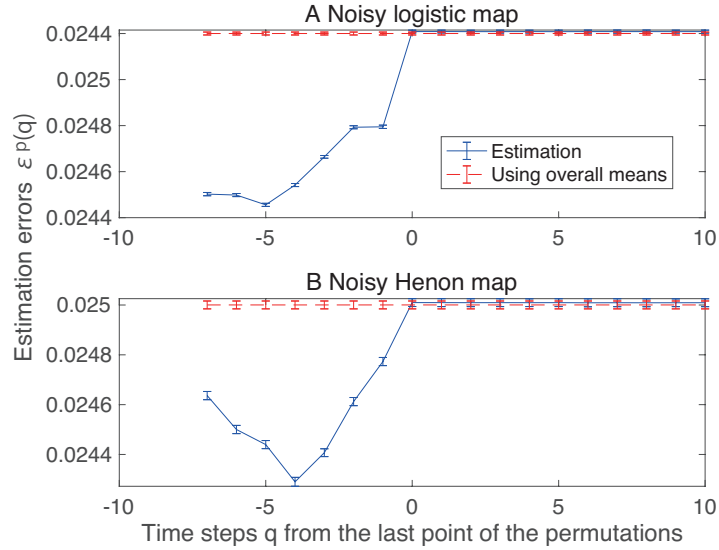


FIG. 5. Estimation errors for the current and future dynamical noise. See the caption of Fig. 2 to interpret the results.

245 to decrease as the length of permutations increases (Fig. 6). Although we assumed here that
 246 a time series of noise realization for modeling is available, this result implies that we could
 247 narrow down the possible realization of dynamical noise using permutations. This direction
 248 enables us to construct a random dynamical system model.

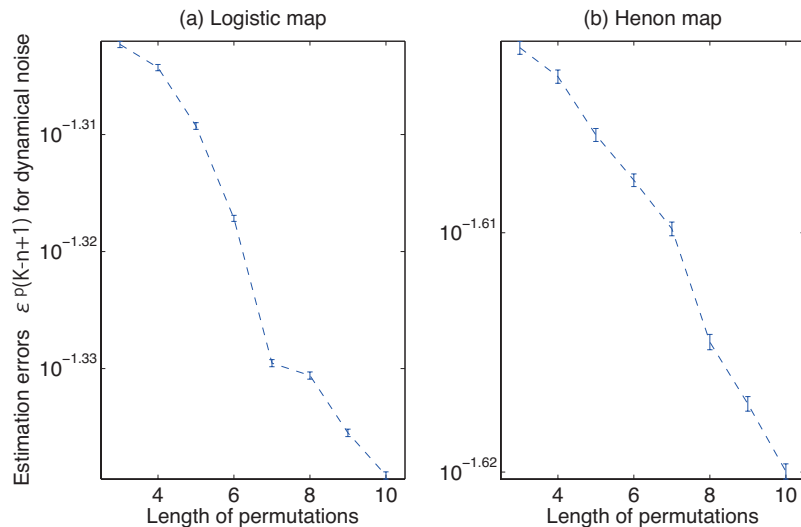


FIG. 6. Estimation error for noise realization in (a) the logistic map and (b) the Hénon map. The error bars show the means and standard deviations for the estimation errors obtained from 10 realizations of time series for each model. The rest of the conditions is the same as Fig. 3.

249 When we used recurrence plots, we tested with three systems: the logistic maps with the
 250 common dynamical noise η_t : For an additive noise case, we use

$$x_{i,t+1} = 3.7x_{i,t}(1 - x_{i,t}) + \eta_t, \quad (10)$$

251 for $\eta_t \in [0, 0.05]$; for a multiplicative noise case, we use

$$x_{i,t+1} = (3.7 + \eta_t)x_{i,t}(1 - x_{i,t}), \quad (11)$$

252 for $\eta_t \in [-0.2, 0.2]$.

253 In the Hénon maps, we use

$$y_{i,t+1} = 1 - 1.2y_{i,t}^2 + 0.3z_{i,t} + \zeta_t, \quad (12)$$

$$z_{i,t+1} = y_{i,t}, \quad (13)$$

254 for the common additive dynamical noise $\zeta_t \in [-0.1, 0.1]$, or

$$y_{i,t+1} = 1 - (1.2 + \zeta_t)y_{i,t}^2 + 0.3z_{i,t}, \quad (14)$$

$$z_{i,t+1} = y_{i,t}, \quad (15)$$

255 for the common multiplicative dynamical noise $\zeta_t \in [-0.1, 0.1]$.

256 Here, we also use a chaotic neuron model [38] to examine a more complicated situation.
 257 A chaotic neuron model is an extension of the Nagumo and Sato's neuron model [39] by
 258 replacing the Heaviside function with the sigmoid function, which defines whether or not a
 259 neuron fires. In a chaotic neuron model [38], we use

$$w_{i,t+1} = 0.5w_{i,t} - \frac{1}{1 + e^{-w_{i,t}/0.04}} + 0.24 + 0.02\sigma_t, \quad (16)$$

260 for the common additive dynamical noise $\sigma_t \in [-0.02, 0.02]$, or

$$w_{i,t+1} = 0.5w_{i,t} - \frac{1}{1 + e^{-w_{i,t}/(0.04+0.01\sigma_t)}} + 0.24, \quad (17)$$

261 for the common multiplicative dynamical noise $\sigma_t \in [-0.01, 0.01]$. Namely, states $w_{i,t}$ for
 262 multiple neurons at time t are forced by the common dynamical noise σ_t . Here, we assume
 263 that each of η_t , ζ_t and σ_t follows the independent and identical uniform noise, respectively.
 264 The number L of the maps were decided as the minimum number in the form of 10×2^n
 265 with whose corresponding network, each time point is connected with all the other time
 266 points within 10 steps. We assigned each of the initial conditions $x_{i,0}, y_{i,0}, z_{i,0}, w_{i,0}$ by the

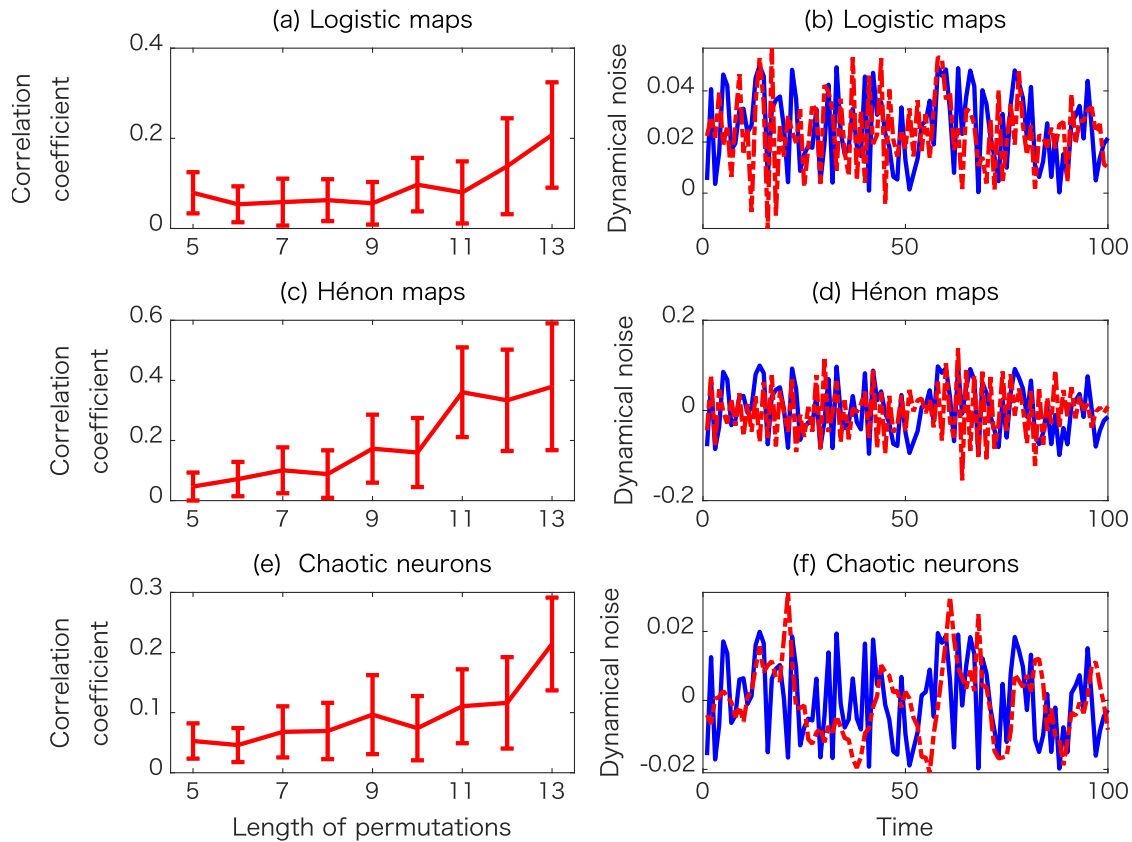


FIG. 7. Estimating the common additive dynamical noise. Panels (a), (c), and (e) show the means and standard deviations for the correlation coefficients between the estimated dynamical noise and its truth calculated over 30 time series, while panels (b), (d), and (f) are examples of estimated dynamical noise estimated for permutations of length 13. In each of the panels (b), (d), and (f), the blue solid line corresponds to the true dynamical noise, while the red dashed line corresponds to one of the reconstructed dynamical noise. Panels (a) and (b) show the examples for the logistic maps, Panels (c) and (d) show the examples for the Hénon maps, and Panels (e) and (f) show the examples for the chaotic neurons. In each simulation, we adjusted the minimum number L of maps as 10×2^n that each time point is connected with all the other time points within 10 steps when we regard the final recurrence plot as a network [31]. Note that there is a degree of freedom for the scaling the estimated dynamical noise. Thus, in Panels (b) ,(d), and (f), we adjusted the reconstructed dynamical noise so that the mean and standard deviation were the same as the actual truth as well as the direction for the reconstructed dynamical noise matches the truth.

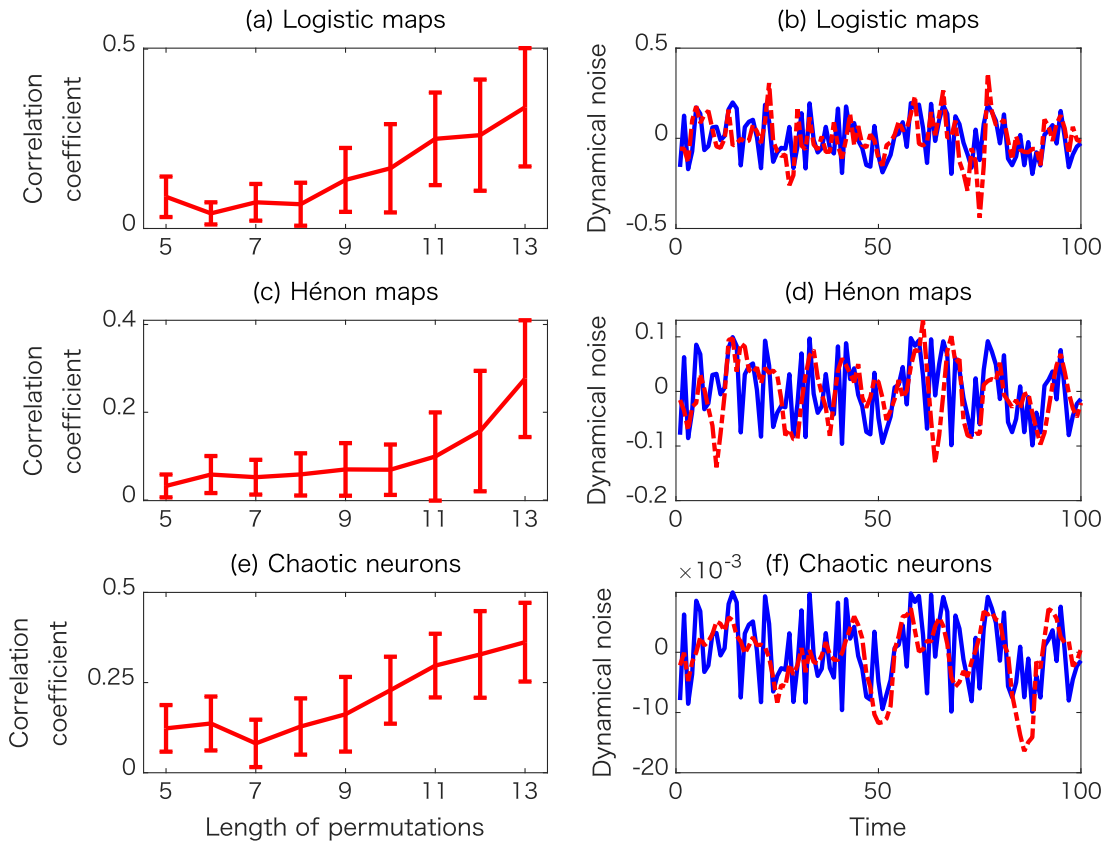


FIG. 8. Same as Fig. 7 but for multiplicative dynamical noise.

267 uniform distribution between 0 and 1, and generated a time series of length 500 each. We
 268 repeated this simulation 30 times to examine the robustness of our findings. We identified the
 269 tendency for longer permutations to perform more effectively in estimating the underlying
 270 dynamical noise (see panels (a), (c), and (e) of Figs. 7-8) for the additive dynamical noise
 271 and multiplicative dynamical noise, respectively. Examples shown in panels (b), (d), and (f)
 272 demonstrate that the dynamical noise reconstructed via recurrence plots agreed well with
 273 the true dynamical noise.

274 Overall, these numerical simulations confirmed that a permutation can specify a realiza-
 275 tion of the external forces, especially dynamical noise, which can be fast in its time scale.

276 IV. DISCUSSIONS

277 The goal of this study was to show theoretically and numerically that permutations can
278 uniquely characterize the dynamics and the external forcings in non-autonomous dynamical
279 systems. There are several related studies, and modeling of nonlinear stochastic systems is
280 not new, for example, [40]. Güttler et al. [41] reconstructed the parameter space from the
281 measured time series observed for fixed or slowly varying parameters. Recently, Hamilton
282 et al. [42] proposed a filtering technique with dynamical noise without explicitly modeling
283 the underlying system. The closest study is the one by Seifert et al. [43], where a Langevin
284 equation is assumed for inferring external forces from a measured time series. However,
285 because this work assumes a Langevin equation, the underlying dynamics should be a flow
286 and thus, this technique cannot be applied to a time-discontinuous system. In this sense,
287 the current results could work in a more general setting and provide a rigorous approach for
288 analyzing a non-autonomous system, while its target system could be a map.

289 Since the number of possible states in permutations increases in a combinatorial manner
290 rather than in an exponential manner, a typical example of which is symbolic dynamics
291 obtained by a generating partition, permutations can overcome dynamical noise by their
292 redundancy [44]. This super-exponential growth is important in enabling a permutation to
293 retain the information regarding both the state space as well as a stochastic input series.
294 If we consider this kind of redundancy, recurrence plots [31–33, 45, 46] can also provide
295 one-to-one correspondence between a time series generated from a stochastic system and its
296 representations.

297 The current work can be regarded as related to the fundamentals of the mechanism
298 by which permutations can distinguish deterministic systems from stochastic systems [19–
299 25, 47], and validate the analysis of transition matrices [23, 48] induced by a permutation
300 for a stochastic system.

301 In summary, a permutation can uniquely identify a state for the underlying dynamics
302 even if the dynamics is subject to realization of external force. We provided the mathemat-
303 ical proof (Theorem 1) that the unique specification of a joint set of an initial condition and
304 a realization of the external force by a permutation is equivalent to the condition for the
305 topological transitivity of the given dynamics. We also presented numerical demonstrations
306 using the mean representation as well as the recurrence plots to show that estimating the

307 realization of the unknown external force is possible. By specifying a permutation, we can
308 ultimately uniquely identify series of states as well as stochastic inputs. Thus, a permu-
309 tation can be used for time series forecasts of random dynamical systems. From another
310 viewpoint, topological transitivity is a good criterion for evaluating the chaotic nature for
311 the underlying dynamics even when it is generated by random and non-autonomous dy-
312 namical systems. Although this point will be further examined in our upcoming research,
313 permutations are certainly expected to provide a rigorous platform for analyzing random as
314 well as deterministic dynamical systems.

315 **SUPPLEMENTARY MATERIAL**

316 The supplementary material includes codes for reproducing numerical calculations and
317 plots presented in Figs. 2-8. Once you unzip the supplementary material, such codes for
318 each figure can be found separated into each folder.

319 **DATA AVAILABILITY STATEMENT**

320 The data that support the findings of this study are available within the article and its
321 supplementary material.

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327 **Appendix A: Estimation of realization of external force**

328 We estimate dynamical noise from observations from multiple maps by extending the
329 method of Ref. [31] (see Fig. 9). First, we obtained an order recurrence plot [49] using
330 permutations of the same length l (Each permutation $\pi_{i,t}(l)$ corresponds to the information
331 for the joint set $(x_{i,t}, p_t, p_{t+1}, \dots, p_{t+l-2})$). An order recurrence plot is a two-dimensional

332 figure and can be defined as follow:

$$R_i(j, k) = \begin{cases} 1, & \text{if } \pi_{i,j}(l) = \pi_{i,k}(l), \\ 0, & \text{otherwise.} \end{cases} \quad (\text{A1})$$

333 Then, we applied the OR operations for the order recurrence plots to obtain the resulting
 334 recurrence plot [31], which corresponds to the information of $p_t, p_{t+1}, \dots, p_{t+l-2}$. Namely,
 335 we define

$$R(j, k) = \begin{cases} 1, & \text{if } R_i(j, k) = 1 \text{ for some } i = 1, 2, \dots, I, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{A2})$$

336 Furthermore, we took the AND operations for $R(j+m, k+m)$ components for each (j, k) for
 337 $m = 0, 1, \dots, l-2$ by duplicating the resulting recurrence plots and applying time delays to
 338 obtain the final components $\tilde{R}(j, k)$ for the final recurrence plot. In mathematical language,
 339 we define

$$\tilde{R}(j, k) = \begin{cases} 1, & \text{if } R(j+m, k+m) = 1 \text{ for all } m = 0, 1, 2, \dots, l-2, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{A3})$$

340 With these AND operations, we could narrow down the information for p_t represented in
 341 the final recurrence plot. Lastly, we converted the final recurrence plot to a time series by
 342 the method of Ref. [31], which has mathematical support [45, 46]. On this process, first we
 343 regard a recurrence plot as a graph. In this graph, a time point corresponds to a node, and
 344 points plotted (j, k) and (k, j) correspond to an edge between j and k . Then, we assign to
 345 each edge the following local distance d :

$$d(j, k) = 1 - \frac{\sum_l \tilde{R}(j, l) \tilde{R}(k, l)}{\sum_l \tilde{R}(j, l) + \sum_l \tilde{R}(k, l) - \sum_l \tilde{R}(j, l) \tilde{R}(k, l)}. \quad (\text{A4})$$

346 Second, we obtain the shortest distance between every pair of nodes on this graph, con-
 347 structing a distance matrix of global distances. For this procedure, we may use Dijkstra's
 348 algorithm or Johnson's algorithm [50]. Third, we use multidimensional scaling for convert-
 349 ing the distance matrix to a time series. If we extract the most significant component,
 350 this component corresponds to the common dynamical noise. Thus, we can transform the
 351 information of the recurrence matrix, or the adjacency matrix, to that of the corresponding
 352 metric space, resulting in the estimated time series for the common dynamical noise.

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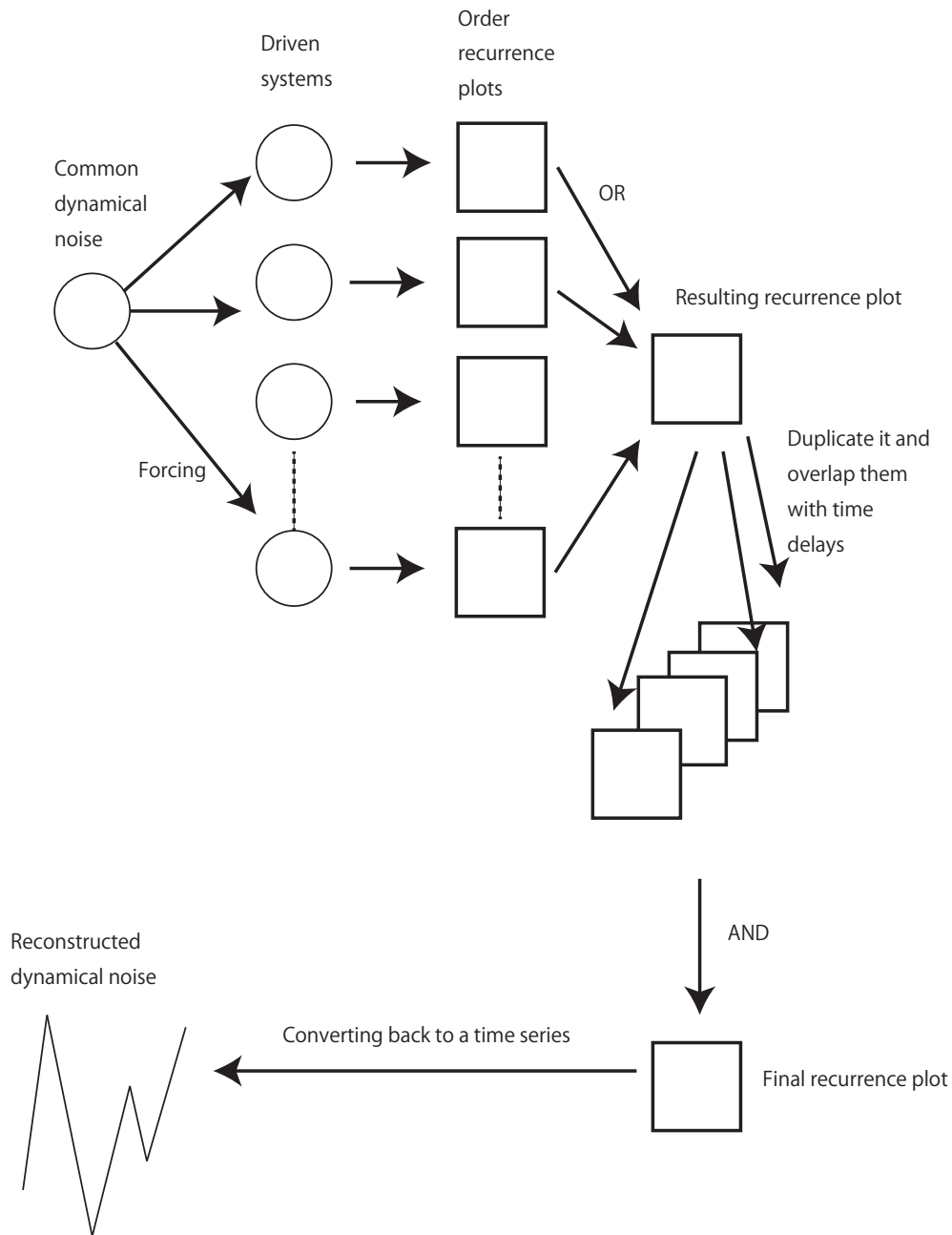


FIG. 9. Schematic for estimating the common dynamical noise using the observations of the driven systems.

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