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EXPONENTIAL RAREFACTION OF MAXIMAL REAL ALGEBRAIC HYPERSURFACES

MICHELE ANCONA

ABSTRACT. Given an ample real Hermitian holomorphic line bundle L over a real algebraic variety X, the space of real holomorphic sections of $L^{\otimes d}$ inherits a natural Gaussian probability measure. We prove that the probability that the zero locus of a real holomorphic section s of $L^{\otimes d}$ defines a maximal hypersurface tends to 0 exponentially fast as d goes to infinity. This extends to any dimension a result of Gayet and Welschinger [10] valid for maximal real algebraic curves inside a real algebraic surface.

The starting point is a low degree approximation property which relates the topology of the real vanishing locus of a real holomorphic section of $L^{\otimes d}$ with the topology of the real vanishing locus a real holomorphic section of $L^{\otimes d'}$ for a sufficiently smaller d' < d. Such a statement is inspired by the recent work of Diatta and Lerario [9].

1. INTRODUCTION

1.1. Real algebraic varieties and maximal hypersurfaces. Let (X, c_X) be a real algebraic variety, that is a complex (smooth, projective and connected) algebraic variety equipped with an antiholomorphic involution $c_X : X \to X$, called the real structure. For example, the projective space $\mathbb{C}P^n$ equipped with the standard conjugaison *conj* is a real algebraic variety. More generally, the solutions of a system of homogeneous real polynomial equations in n + 1 variables define a real algebraic variety X inside $\mathbb{C}P^n$, whose real structure is the restriction of *conj* to X. The real locus $\mathbb{R}X$ of a real algebraic variety is the set of fixed points of the real structure, that is $\mathbb{R}X = \operatorname{Fix}(c_X)$. The real locus $\mathbb{R}X$ is either empty, or a finite union of n dimensional \mathcal{C}^{∞} -manifolds, where n is the (complex) dimension of X. The study of the topology of real algebraic varieties is a central topic in real algebraic geometry since the works of Harnack and Klein on the topology of real algebraic curves [14, 18] and the famous Hilbert's sixteenth problem [27]. A fundamental restriction on the topology of the real locus of a real algebraic variety is given by the Smith-Thom inequality [24] which asserts that the total Betti number of the real locus $\mathbb{R}X$ of a real algebraic variety is bounded from above by the total Betti number of the complex locus:

(1)
$$\sum_{i=0}^{n} \dim H_i(\mathbb{R}X, \mathbb{Z}/2) \le \sum_{i=0}^{2n} \dim H_i(X, \mathbb{Z}/2).$$

We will more compactly write $b_*(\mathbb{R}X) \leq b_*(X)$, where b_* denotes the total Betti number (i.e. the sum of the $\mathbb{Z}/2$ -Betti numbers). For a real algebraic curve C, the inequality (1) is known as Harnack's inequality [14, 18] and reads $b_0(\mathbb{R}C) \leq g+1$, where g is the genus of C. A real algebraic variety which realizes the equality in (1) is called *maximal*.

In this paper, we will study maximal real algebraic hypersurfaces inside a fixed real algebraic variety X. More precisely, we are interested in the following question: given a real linear system of divisors in X, what is the probability to find a divisor defining a real algebraic maximal hypersurface? The goal of the paper is to show that real algebraic maximal hypersurfaces are exponentially rare inside their linear system.

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1.2. Real Hermitian line bundles and Gaussian measures. In order to answer the previous question, let $\pi : (L, c_L) \to (X, c_X)$ be an ample real holomorphic line bundle over X, that is, an ample holomorphic line bundle L over X equipped with a real structure c_L such that $\pi \circ c_L = c_X \circ \pi$. We equip L with a smooth real Hermitian metric h of positive curvature ω (we recall that real means $c_L^* h = \bar{h}$). We denote by $\mathbb{R}H^0(X, L^d)$ the space of real global sections of $L^{\otimes d} = L^d$, that is the space of holomorphic sections $s \in H^0(X, L^d)$ such that $s \circ c_X = c_{L^d} \circ s$. This space is naturally equipped with a \mathcal{L}^2 -scalar product defined by

(2)
$$\langle s_1, s_2 \rangle_{\mathcal{L}^2} = \int_X h^d(s_1, s_2) \frac{\omega^{\wedge n}}{n!}$$

for any pair of real global sections $s_1, s_2 \in \mathbb{R}H^0(X, L^d)$, where h^d is the real Hermitian metric on L^d induced by h. In turn, the \mathcal{L}^2 -scalar product (2) naturally induces a Gaussian probability measure μ_d defined by

(3)
$$\mu_d(A) = \frac{1}{\sqrt{\pi^{N_d}}} \int_{s \in A} e^{-\|s\|_{\mathcal{L}^2}^2} \mathrm{d}s$$

for any open set $A \subset \mathbb{R}H^0(X, L^d)$, where N_d is the dimension of $\mathbb{R}H^0(X, L^d)$ and ds the Lebesgue measure induced by the \mathcal{L}^2 -scalar product (2). The probability space we will consider is then $(\mathbb{R}H^0(X, L^d), \mu_d)$.

Example 1.1. When (X, c_X) is the n-dimensional projective space and (L, c_L, h) is the degree 1 real holomorphic line bundle equipped with the standard Fubini-Study metric, then the vector space $\mathbb{R}H^0(X, L^d)$ is isomorphic to the space $\mathbb{R}_d^{hom}[X_0, \ldots, X_n]$ of degree d homogeneous real polynomials in n + 1 variables and the \mathcal{L}^2 -scalar product is the one which makes the family of monomials $\{\sqrt{\binom{(n+d)!}{n!\alpha_0!\cdots\alpha_n!}}X_0^{\alpha_0}\cdots X_n^{\alpha_n}\}_{\alpha_0+\cdots+\alpha_n=d}$ an orthonormal basis. A random polynomial with respect to the Gaussian probability measure induced by this scalar product is called a Kostlan polynomial.

1.3. Statement of the main results. The zero locus of a real global section s_d of L^d is denoted by Z_{s_d} and its real locus by $\mathbb{R}Z_{s_d}$. The total Betti number $b_*(Z_{s_d})$ of Z_{s_d} verifies the asymptotic $b_*(Z_{s_d}) = v(L)d^n + O(d^{n-1})$, as $d \to \infty$, where $v(L) := \int_X c_1(L)^n$ is the so-called volume of the line bundle L (see, for example, [12, Lemma 3]). If the total Betti number $b_*(\mathbb{R}Z_{s_d})$ of the real locus of a sequence of real algebraic hypersurfaces verifies the same asymptotics, then the hypersurfaces are called *asymptotically maximal*. The existence of asymptotically maximal hypersurfaces is known in many cases, for example for real algebraic surfaces [10, Theorem 5], for projective spaces [17] and for toric varieties [4, Theorem 1.3]. The first main result of the paper shows that asymptotically maximal hypersurfaces are very rare in their linear system. More precisely:

Theorem 1.2. Let (X, c_X) be a real algebraic variety of dimension n and (L, c_L) be a real Hermitian line bundle of positive curvature. Then, there exists a positive $a_0 < v(L)$, such that, for any $a > a_0$, we have

$$\mu_d \left\{ s \in \mathbb{R}H^0(X, L^d), b_*(\mathbb{R}Z_s) \ge ad^n \right\} \le O(d^{-\infty})$$

as $d \to \infty$.

The notation $O(d^{-\infty})$ stands for $O(d^{-k})$ for any $k \in \mathbb{N}$ and the measure μ_d is the Gaussian measure defined in Equation (3). Note that actually we have more than "asymptotically maximal hypersurfaces are very rare". Indeed, asymptotically maximal hypersurfaces corresponds to the asymptotics $b_*(\mathbb{R}Z_s) = v(L)d^n + O(d^{n-1})$, while in Theorem 1.2 we consider bigger subsets of $\mathbb{R}H^0(X, L^d)$ of the form $b_*(\mathbb{R}Z_s) \geq ad^n$, for $v(L) > a > a_0$.

For maximal real algebraic hypersurfaces, the rarefaction is even exponential. More precisely: **Theorem 1.3.** Let (X, c_X) be a real algebraic variety of dimension n and (L, c_L) be a real Hermitian line bundle of positive curvature. For any a > 0 there exists c > 0 such that

$$\mu_d \{ s \in \mathbb{R}H^0(X, L^d), b_*(\mathbb{R}Z_s) \ge b_*(Z_s) - ad^{n-1} \} \le O(e^{-c\sqrt{d\log d}})$$

as $d \to \infty$. Moreover, if the real Hermitian metric on L is analytic, one has the estimate

$$\mu_d \{ s \in \mathbb{R}H^0(X, L^d), b_*(\mathbb{R}Z_s) \ge b_*(Z_s) - ad^{n-1} \} \le O(e^{-cd})$$

Theorems 1.3 extends to any dimension a result of Gayet and Welschinger [10], in which the authors prove, using the theory of laminary currents, that maximal real curves are exponentially rare in a real algebraic surface. We stress that our techniques are different from those of [10].

Remark 1.4. The \mathcal{L}^2 -scalar product on $\mathbb{R}H^0(X, L^d)$ also induces a Fubini-Study volume on the linear system $P(\mathbb{R}H^0(X, L^d))$. Both sets considered in Theorems 1.2 and 1.3 are cones in $\mathbb{R}H^0(X, L^d)$ and the volume (with respect to the Fubini-Study form) of their projectivization coincides with the Gaussian measures estimated in Theorems 1.2 and 1.3.

Theorems 1.2 and 1.3 are consequence of a more general "approximation theorem" which states that, for some b < 1, with very high probability, the zero locus of a real section of L^d is diffeomorphic to the zero locus of a real section of $L^{\lfloor bd \rfloor}$, where $\lfloor bd \rfloor$ is the greatest integer less than or equal to bd. More precisely we have:

Theorem 1.5. Let (X, c_X) be a real algebraic variety and (L, c_L) be a real Hermitian line bundle of positive curvature.

- (1) There exists a positive $b_0 < 1$ such that for any $b_0 < b < 1$ the following happens: the probability that, for a real section s of L^d , there exists a real section s' of $L^{\lfloor bd \rfloor}$ such that the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s'})$ are isotopic, is at least $1 - O(d^{-\infty})$, as $d \to \infty$.
- (2) For any $k \in \mathbb{N}$ there exists c > 0 such that the following happens: the probability that, for a real section s of L^d , there exists a real section s' of L^{d-k} such that the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s'})$ are isotopic, is at least $1 - O(e^{-c\sqrt{d}\log d})$, as $d \to \infty$. If moreover the real Hermitian metric on L is analytic, this probability is at least $1 - O(e^{-cd})$, as $d \to \infty$

Assertion (1) (resp. assertion (2)) of Theorem 1.5, together with the Smith-Thom inequality (1) and of the asymptotics $b_*(Z_{s_d}) = v(L)d^n + O(d^{n-1})$, for $s_d \in \mathbb{R}H^0(X, L^d)$, will imply Theorem 1.2 (resp. Theorem 1.3). It is also worth pointing out that the assertions (1) and (2) of Theorem 1.5 are independent of each other, although their proofs, which will be sketched in Section 4, are similar.

Remark 1.6. Theorem 1.5 implies not only that maximal hypersurfaces are rare, but that "maximal configurations" are. For instance, we will show in Section 5 that in some suitable real algebraic surfaces the probability that a real algebraic curve has a deep nest of ovals is exponentially small (roughly speaking a nest of ovals means severals ovals one inside the other), see Theorems 5.3 and 5.7.

When $(X, c_X) = (\mathbb{C}P^n, conj)$, $L = \mathcal{O}(1)$ and the Hermitian metric on L equals the Fubini-Study metric (that is for the case of the so-called Kostlan polynomials, see Example 1.1), a low degree approximation property was recently proved by Diatta and Lerario [9] (see also [5]), so that Theorem 1.5 is a natural generalization of their result. Actually, in [9] the authors proves a general low degree approximation property for Kostlan polynomials: for instance, they have proved that a degree d Kostlan polynomial can be approximated (in the sense of Theorem 1.5) by a degree $b\sqrt{d \log d}$ Kostlan polynomial, b > 0, with probability $1 - O(d^{-a})$, with a > 0 depending on b.

We stress that in [9], it is essential that the considered real algebraic variety is the projective space and that the metric on $\mathcal{O}(1)$ is the Fubini-Study one (and not, for instance, a small perturbation of it). Indeed, in this situation:

- (1) the induced \mathcal{L}^2 -scalar product on $\mathbb{R}^{hom}_d[X_0, \ldots, X_n]$ is invariant under the action of the orthogonal group O(n+1), which acts on the variables X_0, \ldots, X_n (equivalently, the group O(n+1) acts by real holomorphic isometries on $(\mathbb{C}P^n, conj)$);
- (2) there exists a canonical O(n+1)-invariant decomposition $\mathbb{R}_d^{hom}[X_0, \ldots, X_n] = \bigoplus_{d-\ell \in 2\mathbb{N}} V_{d,\ell}$, where $V_{d,\ell}$ is the space of homogeneous harmonic polynomials of degree ℓ , thanks to which it is possible to define projections of degree d polynomials into lower degree ones.

These two properties, together with the classification [19] of the O(n + 1)-invariant scalar products on $\mathbb{R}^{hom}_d[X_0, \ldots, X_n]$, are fundamental for the proof of the results in [9]. This is a very special feature of Kostlan polynomials and the reason why in our general case some of the approximations of [9] cannot be obtained. Indeed,

- (1) on a general real algebraic variety equipped with a Kähler metric ω the group of holomorphic isometries is trivial;
- (2) given a real Hermitian holomorphic line bundle $L \to X$ there is no canonical decomposition of $\mathbb{R}H^0(X, L^d)$.

Hence, in order to obtain an approximation property for sections of line bundles on a general real algebraic variety, we have to use a different strategy, that we will explain in more details in Section 1.4. In particular, in our proof, in constrast to the case of polynomials [9], the complex locus of the variety X (and not only the real one) plays a fundamental role. Indeed, we will consider real subvarieties of X with empty real loci and we will study the real sections of L^d that vanish along these subvarieties. These real sections are the fundamental tool for our low degree approximation property.

During the proof of Theorem 1.5, we will also need to understand how much we can perturb a real section $s \in \mathbb{R}H^0(X, L^d)$ without changing the topology of its real locus. This leads us to study two quantities related to the discriminant $\mathbb{R}\Delta_d \subset \mathbb{R}H^0(X, L^d)$, that is the subset of sections which do not vanish transversally along $\mathbb{R}X$. More precisely, we will consider the volume (with respect to the Gaussian measure μ_d) of tubular neighborhoods of $\mathbb{R}\Delta_d$ and the function "distance to the discriminant". Such quantities have already been observed to be important in the case of Kostlan polynomials in [9, Section 4], however in our general framework the lack of symmetries makes the computation of those more delicate and requires the use of the Bergman kernel's estimates along the diagonal [3, 28, 20].

In particular, denoting by $\operatorname{dist}_{\mathbb{R}\Delta_d}(s)$ the distance (induced by the \mathcal{L}^2 -scalar product (2)) from a section s to the discriminant $\mathbb{R}\Delta_d$, we obtain the uniform estimate (see Lemma 3.1):

$$\operatorname{dist}_{\mathbb{R}\Delta_d}(s) = \min_{x \in \mathbb{R}X} \left(\frac{\|s(x)\|_{h^d}^2}{d^n} + \frac{\|\nabla s(x)\|_{h^d}^2}{d^{n+1}} \right)^{1/2} \left(\pi^{n/2} + O(1/d) \right)$$

as $d \to \infty$. We believe that this formula has an independent interest.

Finally, let us conclude this section by pointing out that, while in the present paper we are interested in some rare events (that is, real algebraic hypersurfaces with rich topology), the expected value $\mathbb{E}[b_i(\mathbb{R}Z_{s_d})]$ of the *i*-th Betti number of $\mathbb{R}Z_{s_d}$ is bounded from below and from above by respectively $c_{i,n}d^{n/2}$ and $C_{i,n}d^{n/2}$, for some positive constants $0 < c_{i,n} \leq C_{i,n}$, see [13, 11]. Moreover, in the case of the 0-th Betti number (that is, for the number of connected components), it is known that $\lim_{d\to\infty} \frac{1}{d^{n/2}}\mathbb{E}[b_0(\mathbb{R}Z_{s_d})]$ exists and is positive [21].

1.4. Idea of the proof of Theorem 1.5. In this section, we sketch the proof of Assertion (2) of Theorem 1.5, the proof of Theorem 1.5 (1) is similar.

We want to prove that, with very high probability, the real vanishing locus of a real section s of L^d is ambient isotopic to the real vanishing locus of a real section s' of L^{d-k} .

The first fact we will use is the existence of a real section σ of L^k , for some suitable even $k \in 2\mathbb{N}$ large enough, with the properties that σ vanishes transversally and $\mathbb{R}Z_{\sigma} = \emptyset$ (see Proposition 2.1). In order to obtain such section σ , we consider an integer m such that L^m is

very ample and a basis $\{s_1, \ldots, s_N\}$ is a basis of $\mathbb{R}H^0(X, L^m)$. Then, σ is any general small perturbation of the section $\sum_{i=1}^N s_i^{\otimes 2}$ of L^{2m} . Let us define $\mathbb{R}H_{d,\sigma}$ to be the vector space of real global sections $s \in \mathbb{R}H^0(X, L^d)$ such

Let us define $\mathbb{R}H_{d,\sigma}$ to be the vector space of real global sections $s \in \mathbb{R}H^0(X, L^d)$ such that s vanishes along the vanishing locus Z_{σ} of σ . We also denote by $\mathbb{R}H_{d,\sigma}^{\perp}$ the orthogonal complement of $\mathbb{R}H_{d,\sigma}$ with respect to the \mathcal{L}^2 -scalar product defined in Equation (2). Then, for any section $s \in \mathbb{R}H^0(X, L^d)$, there exists an unique decomposition $s = s_{\sigma}^{\perp} + s_{\sigma}^0$ with $s_{\sigma}^0 \in \mathbb{R}H_{d,\sigma}$ and $s_{\sigma}^{\perp} \in \mathbb{R}H_{d,\sigma}^{\perp}$.

The fundamental point, that we will prove in Section 2.2 using the theory of logarithmic and partial Bergman kernels [7, 23], is that the "orthogonal component" s_{σ}^{\perp} of s has a very small \mathcal{C}^1 -norm along the real locus $\mathbb{R}X$. A geometric reason why this is true is that the space $\mathbb{R}H_{d,\sigma}^{\perp}$ is generated by the peak sections [25] which have a peak on Z_{σ} and, as $Z_{\sigma} \cap \mathbb{R}X = \emptyset$, the pointwise \mathcal{C}^1 -norm of these peak sections is very small along $\mathbb{R}X$ (indeed, a peak section has a very small \mathcal{C}^1 -norm on any compact disjoint from its peak).

From the fact that s_{σ}^{\perp} has a very small \mathcal{C}^1 -norm along the real locus $\mathbb{R}X$, we deduce that s is a "small pertubation" of s_{σ}^0 : Thom's Isotopy Lemma would therefore imply that the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s_{\sigma}^0})$ are isotopic if s_{σ}^0 has a big enough \mathcal{C}^1 -norm along $\mathbb{R}X$. This last implication can be translated in terms of distance from s_{σ}^0 to the discriminant $\mathbb{R}\Delta_d$, that is the space of real sections which do not vanish transversally along $\mathbb{R}X$: if s_{σ}^0 is far enough away from the discriminant, then the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s_{\sigma}^0})$ are isotopic.

Using the Bergman kernel, we are able to estimate the function "distance to the discriminant" (Lemma 3.1). These estimates, together with an approach similar to [9, Section 4], allows us to prove that, with very high probability, s_{σ}^{0} is far enough away from the discriminant, so that, with very high probability, the pairs ($\mathbb{R}X, \mathbb{R}Z_s$) and ($\mathbb{R}X, \mathbb{R}Z_{s_{\sigma}^{0}}$) are isotopic (Lemmas 4.3 and 4.4).

Finally, we prove that the section s_{σ}^{0} can be written as $\sigma \otimes s'$, for some real section s' of L^{d-k} . In particular, as σ does not have any real zero, we have the equality $\mathbb{R}Z_{s_{\sigma}^{0}} = \mathbb{R}Z_{s'}$, which proves Theorem 1.5.

1.5. Organization of the paper. The paper is organized as follows.

In Section 2, we prove the existence of a real section σ of L^k with empty real vanishing locus and then we will study the real sections of L^d which vanish along Z_{σ} . This leads us to consider logarithmic and partial Bergman kernels in Section 2.2.

In Section 3, we study the geometry of the discriminant $\mathbb{R}\Delta_d \subset \mathbb{R}H^0(X, L^d)$ and compute several quantities related to it: its degree, the volume (with respect to the Gaussian measure μ_d) of tubular neighborhoods and the function "distance to the discriminant".

In Section 4, we prove our main results, namely Theorems 1.2, 1.3 and 1.5.

Finally, in Section 5, we study the depth of the nests of real algebraic curves inside some real algebraic surfaces, namely Hirzebruch surfaces (Theorem 5.3) and Del Pezzo surfaces (Theorem 5.7).

2. Sections vanishing along a fixed hypersurface

In this section, we prove the existence of a real global section σ of L^k with empty real vanishing locus and we study the real sections of L^d which vanish along Z_{σ} .

2.1. A global section with empty real vanishing locus. Let (L, c_L) be an ample real holomorphic line bundle over a real algebraic variety (X, c_X) . A real section of L is a global holomorphic section s of L such that $s \circ c_X = c_L \circ s$.

Proposition 2.1. There exists an even positive integer k_0 such that for any even $k \ge k_0$ there exists a real section σ of L^k with the following properties: (i) σ vanishes transversally and (ii) $\mathbb{R}Z_{\sigma}$ is empty.

Proof. Let m_0 be the smallest integer such that L^{m_0} is very ample and set $k_0 = 2m_0$. For any integer $m \ge m_0$, fix a basis s_1, \ldots, s_{N_m} of $\mathbb{R}H^0(X, L^m)$ and consider the real section $s = \sum_{i=1}^{N_m} s_i^{\otimes 2}$, which is a real section of L^{2m} whose real vanishing locus is empty. Remark that any small perturbation of s inside $\mathbb{R}H^0(X, L^{2m})$ will have empty real vanishing locus and that the discriminant (i.e. the sections which do not vanish transversally) is an algebraic hypersurface of $\mathbb{R}H^0(X, L^{2m})$. We can then find a small perturbation σ of s which has the desired properties, namely σ is a real global section of L^{2m} vanishing transversally and whose real vanishing locus is empty.

Definition 2.2. Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1, where k is a fixed large enough integer, and let Z_{σ} be its vanishing locus. For any integer ℓ , we will use the notation σ^{ℓ} to designate the section $\sigma^{\otimes \ell}$ of $L^{k\ell}$. For any pair of integers d and ℓ , let us define the space $H_{d,\sigma^{\ell}}$ to be the subspace of $H^0(X, L^d)$ consisting of sections which vanish along Z_{σ} with order at least ℓ . Similarly, the vector space $\mathbb{R}H_{d,\sigma^{\ell}}$ denotes the subspace of $\mathbb{R}H^0(X, L^d)$ of real sections which vanish along Z_{σ} with order at least ℓ .

Proposition 2.3. Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1, for some fixed k large enough. For any pair of integers d and ℓ , the space $H_{d,\sigma^{\ell}}$ coincides with the space of section $s \in H^0(X, L^d)$ such that $s = \sigma^{\ell} \otimes s'$, for some $s' \in H^0(X, L^{d-k\ell})$. Similarly, the space $\mathbb{R}H_{d,\sigma^{\ell}}$ coincides with the space of real section $s \in \mathbb{R}H^0(X, L^d)$ such that $s = \sigma^{\ell} \otimes s'$, for some $s' \in \mathbb{R}H^0(X, L^{d-k\ell})$.

Proof. If a section s of $H^0(X, L^d)$ is of the form $s = \sigma^\ell \otimes s'$, for some $s' \in \mathbb{R}H^0(X, L^{d-k\ell})$, then $s \in H_{d,\sigma^\ell}$ (that is, s vanishes along Z_σ with order at least ℓ). Moreover if such section is real then $s \in \mathbb{R}H_{d,\sigma^\ell}$. Let us then prove the other inclusion. Let $s \in H^0(X, L^d)$ be such that svanishes along Z_σ with order at least ℓ . We want to prove that there exists $s' \in H^0(X, L^{d-k})$ such that $s = \sigma^\ell \otimes s'$. For this, let $\{U_i\}_i$ be a cover of X by open subsets such that the line bundle L is obtained by gluing together the local models $\{U_i \times \mathbb{C}\}_i$ using the maps $f_{ij} : (x, v) \in (U_i \cap U_j) \times \mathbb{C} \mapsto (x, g_{ij}(x)v) \in (U_i \cap U_j) \times \mathbb{C}$, where $g_{ij} : U_i \cap U_j \to \mathbb{C}^*$ are holomorphic maps. Remark that, for any integer d, the line bundle L^d is obtained by gluing together the same local models $\{U_i \times \mathbb{C}\}_i$ using the maps $f_{ij}^d : (x, v) \in (U_i \cap U_j) \times \mathbb{C} \mapsto$ $(x, g_{ij}^d(x)v) \in (U_i \cap U_j) \times \mathbb{C}$. Using these trivializations, a global section s of L^d is equivalent to the data of local holomorphic functions s_i on U_i such that $s_i = g_{ij}^d s_j$ on $U_i \cap U_j$. We can then define locally the section s' we are looking for by setting $s'_i = \frac{s_i}{\sigma_i^\ell}$. Indeed, these are holomorphic functions because s_i vanishes along the zero set of σ_i with order bigger or equal than ℓ . With this definition, it is straightforward to check that the family $\{s'_i\}_i$ glues together and define a global section s' of $L^{d-k\ell}$ with the property that $s = \sigma^\ell \otimes s'$.

The proof for the real case follows from the complex case and from that fact that if s and σ are real sections, then s' is also real.

2.2. \mathcal{L}^2 -orthogonal complement to $\mathbb{R}H_{d,\sigma^\ell}$ and \mathcal{C}^1 -estimates. In this section, we equip L with a real Hermitian metric h with positive curvature and consider the induced \mathcal{L}^2 -scalar product on $\mathbb{R}H^0(X, L^d)$ given by Equation (2). The main goal is to study the real sections of L^d which are \mathcal{L}^2 -orthogonal to the space $\mathbb{R}H_{d,\sigma^\ell}$ defined in Definition 2.2.

Definition 2.4 (\mathcal{C}^1 -norm). Let $K \subset X$ be a compact set. We define the $\mathcal{C}^1(K)$ -norm of a global section s of L^d to be $\|s\|_{\mathcal{C}^1(K)} = \max_{x \in K} \|s(x)\|_{h^d} + \max_{x \in K} \|\nabla s(x)\|_{h^d}$, where $\|\cdot\|_{h^d}$ is the norm induced by the Hermitian metric h^d , ∇ is the Chern connection on L^d induced by h and $\|\nabla s(x)\|_{h^d}^2 = \sum_{i=1}^n \|\nabla_{v_i} s(x)\|_{h^d}^2$, with $\{v_1, \ldots, v_n\}$ an orthonormal basis of $T_x X$.

Remark 2.5. Recall that $\mathbb{R}H_{d,\sigma^{\ell}} = H_{d,\sigma^{\ell}} \cap \mathbb{R}H^{0}(X, L^{d})$. The orthogonal complement (with respect to the \mathcal{L}^{2} -scalar product given by (2)) of $\mathbb{R}H_{d,\sigma^{\ell}}$ inside $\mathbb{R}H^{0}(X, L^{d})$ coincides with $H_{d,\sigma^{\ell}}^{\perp} \cap \mathbb{R}H^{0}(X, L^{d})$. Here, $H_{d,\sigma^{\ell}}^{\perp}$ is the orthogonal complement of $H_{d,\sigma^{\ell}}$ with respect to the \mathcal{L}^{2} -Hermitian product defined by

(4)
$$\langle s_1, s_2 \rangle_{\mathcal{L}^2} = \int_X h^d(s_1, s_2) \frac{\omega^{\wedge n}}{n!}$$

for any pair of global sections $s_1, s_2 \in H^0(X, L^d)$.

Proposition 2.6. Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1, for some fixed k large enough. There exists a positive real number t_0 such that, for any $t \in (0, t_0)$, we have the uniform estimate $\|\tau\|_{\mathcal{C}^1(\mathbb{R}X)} = O(d^{-\infty})$ for any real section $\tau \in \mathbb{R}H_{d,\sigma^{\lfloor td \rfloor}}^{\perp}$ with $\|\tau\|_{\mathcal{L}^2} = 1$, as $d \to \infty$. Here, $\mathbb{R}H_{d,\sigma^{\lfloor td \rfloor}}$ is as in Definition 2.2 and $\lfloor td \rfloor$ is the greatest integer less than or equal to td.

Proof. Let $\tau_1, \ldots, \tau_{m_d}$ be an orthonormal basis of $H_{d,\sigma^{\lfloor td \rfloor}}^{\perp}$ and $s_1, \ldots, s_{N_d-m_d}$ be an orthonormal basis of $H_{d,\sigma^{\lfloor td \rfloor}}$ (the Hermitian products are the ones induced by Equation (4)). We will denote by $P_d^{\perp}(x) = \sum_{i=1}^{m_d} \|\tau_i(x)\|_{h^d}^2$ and by $P_d^0(x) = \sum_{i=1}^{N_d-m_d} \|s_i(x)\|_{h^d}^2$, so that $P_d^{\perp} + P_d^0$ equals the Bergman function P_d (i.e. the value of the Bergman kernel on the diagonal). The function P_d^0 equals the *partial Bergman kernel* of order $\lfloor td \rfloor$ associated with the subvariety Z_{σ} , see [7]. By [7, Theorem 1.3], for any compact subset K of X which is disjoint from Z_{σ} and for any $r \in \mathbb{N}$, there exists a positive real number $t_0(K)$ such that, for any $t < t_0(K)$, one has $\|P_d - P_d^0\|_{\mathcal{C}^r(K)} = O(d^{-\infty})$. Now, we have $P_d^{\perp} = P_d - P_d^0$, so that, by choosing $K = \mathbb{R}X$, which is disjoint from Z_{σ} by construction of σ , we obtain $\|P_d^{\perp}\|_{\mathcal{C}^r(\mathbb{R}X)} = O(d^{-\infty})$, as soon as $t < t_0 = t_0(\mathbb{R}X)$. This implies that for any $t < t_0$ and any $\tau \in H_{d,\sigma^{\lfloor td \rfloor}}^{\perp}$, with $\|\tau\|_{\mathcal{L}^2} = 1$, one has $\|\tau\|_{\mathcal{C}^1(\mathbb{R}X)} = O(d^{-\infty})$ and, in particular, this happens if $\tau \in \mathbb{R}H_{d,\sigma^{\lfloor td \rfloor}}^{\perp}$.

Proposition 2.7. Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1, for some fixed k large enough. There exists c > 0 (depending on k) such that, we have the uniform estimate $\|\tau\|_{\mathcal{C}^1(\mathbb{R}X)} \leq O(e^{-c\sqrt{d}\log d})$ for any real section $\tau \in \mathbb{R}H_{d,\sigma}^{\perp}$ with $\|\tau\|_{\mathcal{L}^2} = 1$. If the real Hermitian metric on L is analytic, then we have the uniform estimate $\|\tau\|_{\mathcal{C}^1(\mathbb{R}X)} \leq O(e^{-cd})$ for any real section $\tau \in \mathbb{R}H_{d,\sigma}^{\perp}$ with $\|\tau\|_{\mathcal{L}^2} = 1$.

Proof. The proof follows the same idea of Proposition 2.6. Let $\tau_1, \ldots, \tau_{m_d}$ be an orthonormal basis of $H_{d,\sigma}^{\perp}$ and $s_1, \ldots, s_{N_d-m_d}$ be an orthonormal basis of $H_{d,\sigma}$ (the Hermitian products are the ones induced by Equation (4)). We will denote by $P_d^{\perp}(x) = \sum_{i=1}^{m_d} \|\tau_i(x)\|_{h^d}^2$ and by $P_d^0(x) = \sum_{i=1}^{N_d - m_d} \|s_i(x)\|_{h^d}^2$, so that $P_d^\perp + P_d^0$ equals the Bergman function P_d . The function P_d^0 equals the logarithmic Bergman kernel associated with the subvariety Z_{σ} , see [23]. In [23, Theorem 3.4], it is proved that for any sequence of compact sets K_d whose distance from Z_{σ} is bigger than $\frac{\log d}{\sqrt{d}}$, one has $\|P_d - P_d^0\|_{\mathcal{C}^r(K_d)}^1 = O(d^{-\infty})$. The proof uses the fact that the \mathcal{C}^r -norm of a peak section [25] at a point z_d (depending on d) is $O(d^{-\infty})$, as soon as the distance between the peak and the point z_d is bigger than $\frac{\log d}{\sqrt{d}}$. This is a direct consequence of the fact that the \mathcal{C}^r -norm of the Bergman kernel at a pair of points (x_d, y_d) with dist $(x_d, y_d) > \frac{\log d}{\sqrt{d}}$ is $O(d^{-\infty})$. Now, the Bergman kernel decreases exponentially fast outside any fixed neighborhood of the diagonal, that is its C^r -norm at a *fixed* pair of distinct points (x, y) is $O(e^{-c\sqrt{d}\log d})$, in the general case, and $O(e^{-cd})$, if the metric on L is analytic (see, for instance, [15, Theorem 1.1 and Corollary 1.4]). This implies that the \mathcal{C}^r -norm of a peak section at a fixed, not depending on d, point z is $O(e^{-c\sqrt{d\log d}})$, in the general case, and $O(e^{-cd})$, if the metric on L is analytic.

Using this fact, the same proof as in [23, Theorem 3.4] gives us $||P_d - P_d^0||_{\mathcal{C}^r(K)} \leq O(e^{-c\sqrt{d}\log d})$ (and $||P_d - P_d^0||_{\mathcal{C}^r(K)} \leq O(e^{-cd})$, in the analytic case), where K is any compact disjoint from Z_{σ} and c is a positive constant depending on K and $r \in \mathbb{N}$.

Now, we have $P_d^{\perp} = P_d - P_d^0$, so that, by choosing $K = \mathbb{R}X$, which is disjoint from Z_{σ} by construction of σ , we obtain $\|P_d^{\perp}\|_{\mathcal{C}^r(\mathbb{R}X)} \leq O(e^{-c\sqrt{d}\log d})$. This implies that for any $\tau \in H_{d,\sigma}^{\perp}$, with $\|\tau\|_{\mathcal{L}^2} = 1$, one has $\|\tau\|_{\mathcal{C}^1(\mathbb{R}X)} \leq O(e^{-c\sqrt{d}\log d})$ and, in particular, this happens if $\tau \in \mathbb{R}H_{d,\sigma}^{\perp}$. Similarly, for the analytic case, we have $\|\tau\|_{\mathcal{C}^1(\mathbb{R}X)} \leq O(e^{-cd})$ for any $\tau \in \mathbb{R}H_{d,\sigma}^{\perp}$ with $\|\tau\|_{\mathcal{L}^2} = 1$.

3. Geometry of the discriminant: distance, degree and volume

Let $\mathbb{R}\Delta_d \subset \mathbb{R}H^0(X, L^d)$ denote the discriminant, that is the subset of sections which do not vanish transversally along $\mathbb{R}X$. In this section, we will study and compute several quantities related to the discriminant: its degree, the volume (with respect to the Gaussian measure μ_d) of tubular conical neighborhoods (see Definition 3.3) and the function "distance to the discriminant".

As in the previous section, we equip L with a real Hermitian metric h with positive curvature and consider the induced \mathcal{L}^2 -scalar product on $\mathbb{R}H^0(X, L^d)$ given by Equation (2). Given $s \in \mathbb{R}H^0(X, L^d)$, the next lemma gives us an explicit formula for computing the distance from s to the discriminant $\mathbb{R}\Delta_d$. This distance is computed with respect to the \mathcal{L}^2 -scalar product, that is

$$\operatorname{dist}_{\mathbb{R}\Delta_d}(s) := \min_{\tau \in \mathbb{R}\Delta_d} \|s - \tau\|_{\mathcal{L}^2}.$$

Lemma 3.1 (Distance to the discriminant). Let (L, c_L) be a real Hermitian ample line bundle over a real algebraic variety X of dimension n. Let denote by $\|\cdot\|_{h^d}$ the Hermitian metric on L^d induced by a Hermitian metric h on L with positive curvature. Then, as $d \to \infty$, we have the uniform estimate

$$\operatorname{dist}_{\mathbb{R}\Delta_d}(s) = \min_{x \in \mathbb{R}X} \left(\frac{\|s(x)\|_{h^d}^2}{d^n} + \frac{\|\nabla s(x)\|_{h^d}^2}{d^{n+1}} \right)^{1/2} \left(\pi^{n/2} + O(1/d) \right).$$

Proof. We will follow the approach of [22, Section 3] which we combine with Bergman kernel estimates along the diagonal [8, 20, 3, 28]. Let us define $\mathbb{R}\Delta_{d,x}$ to be the linear subspace of real sections which do not vanish transversally to x, that is

$$\mathbb{R}\Delta_{d,x} = \{ s \in \mathbb{R}H^0(X, L^d), s(x) = 0 \text{ and } \nabla s(x) = 0 \}.$$

Let us fix some notations:

- We fix once for all an orthonormal basis s_1, \ldots, s_{N_d} of $\mathbb{R}H^0(X, L^d)$ (with respect to the \mathcal{L}^2 -scalar product (2)) and a trivialization e_L of L around x whose norm equals 1 at x.
- Any section s_i of the basis can be written as $s_i = f_i e_L^d$. We will denote by f(x) the line vector $(f_1(x), \ldots, f_{N_d}(x)) \in \mathbb{R}^{N_d}$.
- We will also fix an orthonormal basis $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ of $T_x \mathbb{R}X$ (the Riemannian metric is the one induced by the curvature form ω of the real Hermitian line bundle (L, h)). • We denote by ∇_j the covariant derivative along $\frac{\partial}{\partial x_j}$ on L^d induced by the metric h^d .
- We denote by ∇_j the covariant derivative along $\frac{\partial}{\partial x_j}$ on L^d induced by the metric h^d . Locally, the covariant derivative along $\frac{\partial}{\partial x_j}$ can be written as $\partial_j := \frac{\partial}{\partial x_j} + \lambda_j$ (that is the sum of the derivative along $\frac{\partial}{\partial x_i}$ plus a term of order 0).
- We will denote by M the $(n+1) \times N_d$ matrix whose n+1 lines are $f(x), \partial_1 f(x), \ldots, \partial_n f(x)$, where $\partial_j f(x) = (\partial_j f_1(x), \ldots, \partial_j f_{N_d}(x)) \in \mathbb{R}^{N_d}$.

Using these notations, a real holomorphic section $s = \sum_{i} a_{i}s_{i}$ lies in $\mathbb{R}\Delta_{d,x}$ if and only if s(x) = 0 and $\nabla_{j}s(x) = 0$ for any $j \in \{1, \ldots, n\}$. Using the local picture, these conditions can be written as $\sum_{i=1}^{N_{d}} a_{i}f_{i}(x) = 0$ and $\sum_{i=1}^{N_{d}} a_{i}\partial_{j}f_{i}(x) = 0$ for any $j \in \{1, \ldots, n\}$. We now compute dist $(s, \mathbb{R}\Delta_{d,x})$, for any $s \in \mathbb{R}H^{0}(X, L^{d})$. By definition, the distance

We now compute dist $(s, \mathbb{R}\Delta_{d,x})$, for any $s \in \mathbb{R}H^0(X, L^d)$. By definition, the distance dist $(s, \mathbb{R}\Delta_{d,x})$ equals $\min_{s+\tau \in \mathbb{R}\Delta_{d,x}} \|\tau\|_{\mathcal{L}^2}$, that is, it equals the \mathcal{L}^2 -norm of a section τ which is orthogonal to $\mathbb{R}\Delta_{d,x}$ and such that $s + \tau \in \mathbb{R}\Delta_{d,x}$. Writing $s = \sum_i a_i s_i$ and $\tau = \sum_i b_i s_i$, the last condition reads M(a + b) = 0, where $a = (a_1, \ldots, a_{N_d})^t$, $b = (b_1, \ldots, b_{N_d})^t$ and the multiplication is the standard matrix multiplication. The condition " τ is orthogonal to $\mathbb{R}\Delta_{d,x}$ " can be written as "there exists $\alpha = (\alpha_0, \ldots, \alpha_n)^t \in \mathbb{R}^{n+1}$ such that $b = M^t \alpha$ ". Putting together these two conditions, we obtain $M(a + M^t \alpha) = Ma + MM^t \alpha = 0$. Now, denoting $A = MM^t$, we have $\alpha = -A^{-1}Ma$, so that $b = -M^t A^{-1}Ma$. We then obtain

(5) dist²(s,
$$\mathbb{R}\Delta_{d,x}$$
) = $\|\tau\|_{\mathcal{L}^2}^2 = \|b\|^2 = \|M^t A^{-1} M a\|^2$
= $a^t M^t (A^{-1})^t M M^t A^{-1} M a = a^t M^t (A^{-1})^t M a.$

Remark now that A is a $(n+1) \times (n+1)$ -symmetric matrix of the following form

$$\begin{bmatrix} \langle f(x), f(x) \rangle & \langle \partial_1 f(x), f(x) \rangle & \cdots & \langle \partial_n f(x), f(x) \rangle \\ \langle f(x), \partial_1 f(x) \rangle & \langle \partial_1 f(x), \partial_1 f(x) \rangle & \cdots & \langle \partial_n f(x), \partial_1 f(x) \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f(x), \partial_n f(x) \rangle & \langle \partial_1 f(x), \partial_n f(x) \rangle & \cdots & \langle \partial_n f(x), \partial_n f(x) \rangle \end{bmatrix}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on \mathbb{R}^{N_d} .

We can see that the quantities $\langle f(x), f(x) \rangle$, $\langle \partial_i f(x), f(x) \rangle$ and $\langle \partial_i f(x), \partial_j f(x) \rangle$ equal the value of the Bergman kernel and of its first derivatives at (x, x). The asymptotics of these quantities are well-known (see, for instance, [8, 20, 3, 28]). In particular, we have $\langle f(x), f(x) \rangle = \frac{d^n}{\pi^n} + O(d^{n-1}), \langle \partial_i f(x), f(x) \rangle = O(d^{n-1}), \langle \partial_i f(x), \partial_j f(x) \rangle = O(d^n)$ for $i \neq j$ and $\langle \partial_i f(x), \partial_i f(x) \rangle = \frac{d^{n+1}}{\pi^n} + O(d^n)$, as $d \to \infty$. As a consequence, as $d \to \infty$, the matrix Aequals

$$\begin{bmatrix} \frac{d^n}{\pi^n} + O(d^{n-1}) & O(d^{n-1}) & \cdots & O(d^{n-1}) \\ O(d^{n-1}) & \frac{d^{n+1}}{\pi^n} + O(d^n) & \cdots & O(d^n) \\ \vdots & \vdots & \ddots & \vdots \\ O(d^{n-1}) & O(d^n) & \cdots & \frac{d^{n+1}}{\pi^n} + O(d^n) \end{bmatrix} = \begin{bmatrix} \frac{d^n}{\pi^n} & 0 & \cdots & 0 \\ 0 & \frac{d^{n+1}}{\pi^n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{d^{n+1}}{\pi^n} \end{bmatrix} \left(\text{Id} + O(d^{-1}) \right)$$

so that A^{-1} equals

$$A^{-1} = \begin{bmatrix} d^{-n} & 0 & \cdots & 0\\ 0 & d^{-n-1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & d^{-n-1} \end{bmatrix} \left(\pi^n \mathrm{Id} + O(d^{-1}) \right).$$

In particular, recalling the definition of the matrix M and the vector a given above, we get that (5) equals $\left(\frac{\|s(x)\|_{h^d}^2}{d^n} + \frac{\|\nabla s(x)\|_{h^d}^2}{d^{n+1}}\right) (\pi^n + O(1/d))$. Taking the minimum over $x \in \mathbb{R}X$, we obtain the result.

Lemma 3.2 (Degree of the discriminant). Let (L, c_L) be a real ample line bundle over a real algebraic variety X of dimension n and denote by $\mathbb{R}\Delta_d$ the discriminant in $\mathbb{R}H^0(X, L^d)$. Then, there exists an homogeneous real polynomial Q_d vanishing on $\mathbb{R}\Delta_d$ and such that $\deg(Q_d) = (n+1) \int_X c_1(X)^n d^n + O(d^{n-1}).$

Proof. Let $\Delta_d \subset H^0(X, L^d)$ be the (complex) discriminant. First, remark that if $s \in \mathbb{R}\Delta_d$, then $s \in \Delta_d$, so that, if we find a real polynomial vanishing on Δ_d , then it will vanish also on $\mathbb{R}\Delta_d$. We will then find a polynomial vanishing along Δ_d and estimate its degree. Remark that Δ_d is a cone in $H^0(X, L^d)$, so that the degree of Δ_d equals the number of intersection points of a generic line γ in $\mathbb{P}(H^0(X, L^d))$ with $\mathbb{P}(\Delta_d)$. We remark that a line γ in $\mathbb{P}(H^0(X, L^d))$ induces a Lefschetz pencil $u : X \dashrightarrow \mathbb{C}P^1$, by sending $x \in X$ to $[s_1(x) : s_2(x)] \in \mathbb{C}P^1$, where s_1 and s_2 are two distinct sections lying on γ . Now, the cardinality of the intersection $\gamma \cap \mathbb{P}(\Delta_d)$ corresponds exactly to the number of singular fibers of the Lefschetz pencil u. By [1, Proposition 2.3], we then have deg $(\mathbb{P}(\Delta_d)) = (n+1) \int_X c_1(X)^n d^n + O(d^{n-1})$. In particular, there exists an homogenous polynomial Q_d of degree deg $(\mathbb{P}(\Delta_d))$ vanishing on Δ_d . Also, if $s \in \Delta_d$, then we have $c_L \circ s \circ c_X \in \Delta_d$, so that Δ_d is a real algebraic hypersurface in $H^0(X, L^d)$ (with respect to the natural real structure $s \mapsto c_{L^d} \circ s \circ c_X$). It is then possible to choose the polynomial Q_d to be real.

Definition 3.3. A tubular conical neighborhood is a tubular neighborhood which is a cone (that is, if s is in the neighborhood, then λs is also in the neighborhood, for any $\lambda \in \mathbb{R}^*$).

To define a tubular conical neighborhood of $\mathbb{R}\Delta_d$, it is then enough to define its trace on the unit sphere $S_d := \{s \in \mathbb{R}H^0(X, L^d), \|s\|_{\mathcal{L}^2} = 1\}$. We also denote by $S\Delta_d$ the trace of the discriminant on S_d , that is $S\Delta_d = S_d \cap \mathbb{R}\Delta_d$. The next lemma estimates the measure of very small tubular conical neighborhoods of the discriminant $\mathbb{R}\Delta_d$, that are tubular conical neighborhoods whose trace on S_d is of the form $\{s \in S_d, \operatorname{dist}(s, S\Delta_d) < r_d\}$, for r_d a sequence of real numbers which goes to zero at least as d^{-2n} .

Lemma 3.4 (Volume of tubular conical neighborhoods). Let (L, c_L) be a real Hermitian ample line bundle over a real algebraic variety (X, c_X) of dimension n. Then there exists a positive constant c, such that, for any sequence r_d verifying $r_d \leq cd^{-2n}$, one has

$$\mu_d\{s \in \mathbb{R}H^0(X, L^d), \operatorname{dist}_{\mathbb{R}\Delta_d}(s) \le r_d \|s\|_{\mathcal{L}^2}\} \le O(r_d d^{2n}).$$

Here, μ_d is the Gaussian probability measure defined in Equation (3).

Proof. We start by a standard remark about Gaussian measures on Euclidian spaces: let us denote by S_d the unit sphere in $\mathbb{R}H^0(X, L^d)$, and by ν_d the probability measure induced by its volume form (i.e. for any $A \subset S_d$, $\nu_d(A) = \operatorname{Vol}(A)\operatorname{Vol}(S_d)^{-1}$), then the Gaussian measure of every cone C_d in $\mathbb{R}H^0(X, L^d)$ equals $\nu_d(C_d \cap S_d)$. We apply this remark to the tubular conical neighborhood $C_d = \{\operatorname{dist}_{\mathbb{R}\Delta_d}(s) \leq r_d \|s\|_{\mathcal{L}^2}\}$ and obtain that its Gaussian measure equals $\nu_d \{s \in S_d, \operatorname{dist}(s, S\Delta_d) \leq r_d\}$, where $S\Delta_d$ denote the trace of the discriminant on S_d . By Lemma 3.2, we have that, for d large enough, there exists a polynomial of degree bounded by $2(n+1) \int_X c_1(X)^n d^n$ whose zero locus contains $S\Delta_d$. We are then in the hypothesis of [6, Theorem 21.1] which gives us

(6)
$$\nu_d \{ s \in S_d, \operatorname{dist}(s, S\Delta_d) \le r_d \} \le cN_d d^n r_d$$

for some constant c > 0, where $N_d = \dim \mathbb{R}H^0(X, L^d)$. By Riemann-Roch Theorem, we have that $N_d = O(d^n)$, so that the right-hand side of (6) is $O(r_d d^{2n})$. The lemma then follows from the fact that the ν_d -measure of the set $\{s \in S_d, \operatorname{dist}(s, S\Delta_d) \leq r_d\}$ appearing in (6) equals the Gaussian measure of its cone. This implies

$$\mu_d\{s \in \mathbb{R}H^0(X, L^d), \operatorname{dist}_{\mathbb{R}\Delta_d}(s) \le r_d \|s\|_{\mathcal{L}^2}\} \le O(r_d d^{2n}),$$

which concludes the proof.

4. Proof of the main results

In this section, we prove our main results, namely Theorems 1.2, 1.3 and 1.5. We will use the notations of the previous sections, in particular we consider an ample Hermitian real holomorphic line bundle (L, c_L, h) over a real algebraic variety (X, c_X) and we denote by μ_d the Gaussian measure on $\mathbb{R}H^0(X, L^d)$ defined in Equation (3).

Notation 4.1. Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1, for some fixed k large enough. Let d and ℓ be two positive integers and let $\mathbb{R}H_{d,\sigma^\ell}$ be as in Definition 2.2. For any real section $s \in \mathbb{R}H^0(X, L^d)$ there exists an unique decomposition $s = s_{\sigma^\ell}^{\perp} + s_{\sigma^\ell}^0$ with $s_{\sigma^\ell}^0 \in \mathbb{R}H_{d,\sigma^\ell}$ and $s_{\sigma^\ell}^{\perp} \in \mathbb{R}H_{d,\sigma^\ell}^{\perp}$ (the orthogonal is with respect to the \mathcal{L}^2 -scalar product defined in Equation (2)).

Proposition 4.2. Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1, for some fixed k large enough. The following holds.

(1) There exists a positive real number t_0 such that for any $t \in (0, t_0)$ the following happens. Let C > 0 and $r \in \mathbb{N}$. For any $w_d \ge Cd^{-r}$, there exists $d_0 \in \mathbb{N}$, such that for any $d \ge d_0$, we have

$$\mu_d\{s \in \mathbb{R}H^0(X, L^d), \left\|s_{\sigma^{\lfloor td \rfloor}}^{\perp}\right\|_{\mathcal{C}^1(\mathbb{R}X)} \ge w_d \|s\|_{\mathcal{L}^2}\} = 0.$$

(2) There exist two positive constants c_1 and c_2 such that, for any sequence $w_d \ge c_1 e^{-c_2\sqrt{d}\log d}$, we have

$$\mu_d\{s \in \mathbb{R}H^0(X, L^d), \left\|s_{\sigma}^{\perp}\right\|_{\mathcal{C}^1(\mathbb{R}X)} \ge w_d \|s\|_{\mathcal{L}^2}\} = 0$$

If the real Hermitian metric on L is analytic, then the last estimate is true for any sequence $w_d \ge c_1 e^{-c_2 d}$.

Here, s_{σ}^{\perp} and $s_{\sigma^{\lfloor td \rfloor}}^{\perp}$ are given by Notation 4.1.

Proof. Let us prove point (1). Let us fix t_0 given by Proposition 2.6 and take $t < t_0$. Let us denote by S_d the unit sphere in $\mathbb{R}H^0(X, L^d)$, and by ν_d the probability measure induced by its volume form (i.e. for any $A \subset S_d$, $\nu_d(A) = \operatorname{Vol}(A)\operatorname{Vol}(S_d)^{-1}$). Then, the Gaussian measure of every cone C_d in $\mathbb{R}H^0(X, L^d)$ equals $\nu_d(C_d \cap S_d)$. We apply this remark to the cone we are interested in, that is $C_d = \{s \in \mathbb{R}H^0(X, L^d), \|s_{\sigma^{\lfloor td \rfloor}}^{\perp}\|_{\mathcal{C}^1(\mathbb{R}X)} \geq w_d \|s\|_{\mathcal{L}^2}\}$, and obtain

(7)
$$\mu_d \left\{ s \in \mathbb{R}H^0(X, L^d), \left\| s_{\sigma^{\lfloor td \rfloor}}^{\perp} \right\|_{\mathcal{C}^1(\mathbb{R}X)} \ge w_d \| s \|_{\mathcal{L}^2} \right\} = \nu_d \left\{ s \in S_d, \left\| s_{\sigma^{\lfloor td \rfloor}}^{\perp} \right\|_{\mathcal{C}^1(\mathbb{R}X)} \ge w_d \right\}.$$

Now, by Proposition 2.6, as $\left\|s_{\sigma^{\lfloor td \rfloor}}^{\perp}\right\|_{\mathcal{L}^2} \leq 1$, there exists a constant $c_r > 0$ such that $\left\|s_{\sigma^{\lfloor td \rfloor}}^{\perp}\right\|_{\mathcal{C}^1(\mathbb{R}X)} \leq c_r d^{-r-1}$, which is strictly smaller than w_d , for d large enough. We then obtain

$$\nu_d \{ s \in S_d, \left\| s_{\sigma \lfloor td \rfloor}^{\perp} \right\|_{\mathcal{C}^1(\mathbb{R}X)} \ge w_d \} = 0,$$

which, together with (7), proves point (1) of the result. The proof of point (2) follows the same lines, using Proposition 2.7 instead of Proposition 2.6. \Box

Lemma 4.3. There exists a positive integer d_0 such that for any $d \ge d_0$ and any real section $s \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_d$, the following happens. For any real global section $s' \in \mathbb{R}H^0(X, L^d)$ such that

$$\left\|s-s'\right\|_{\mathcal{C}^1(\mathbb{R}X)} < \frac{d^{n/2}}{4\pi^{n/2}} \mathrm{dist}_{\mathbb{R}\Delta_d}(s),$$

we have that the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s'})$ are isotopic.

Proof. By Lemma 3.1, there exists d_0 such that for any $d \ge d_0$ one has

$$\operatorname{dist}_{\mathbb{R}\Delta_d}(s) < 2\pi^{n/2} \min_{x \in \mathbb{R}X} \left(\frac{\|s(x)\|_{h^d}^2}{d^n} + \frac{\|\nabla s(x)\|_{h^d}^2}{d^{n+1}} \right)^{1/2}$$

In particular, for any $d \ge d_0$, the inequality $||s - s'||_{\mathcal{C}^1(\mathbb{R}X)} < \frac{d^{n/2}}{4\pi^{n/2}} \operatorname{dist}_{\mathbb{R}\Delta_d}(s)$ implies

(8)
$$\|s - s'\|_{\mathcal{C}^1(\mathbb{R}X)} < \frac{1}{2} \min_{x \in \mathbb{R}X} \left(\|s(x)\|_{h^d}^2 + \frac{\|\nabla s(x)\|_{h^d}^2}{d} \right)^{1/2}$$

Now, denoting $\delta(s) := \min_{x \in \mathbb{R}X} \left(\|s(x)\|_{h^d}^2 + \frac{\|\nabla s(x)\|_{h^d}^2}{d} \right)^{1/2}$ and following the lines of [9, Proposition 3], we have that the inequality (8) implies that the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s'})$ are isotopic.

Lemma 4.4. Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1, for some fixed k large enough. Then, we have the following estimates as $d \to \infty$.

(1) There exists $t_0 > 0$ such that, for any $t \in (0, t_0)$, we have

$$\mu_d \left\{ s \in \mathbb{R}H^0(X, L^d), \left\| s_{\sigma^{\lfloor td \rfloor}}^{\perp} \right\|_{\mathcal{C}^1(\mathbb{R}X)} < \frac{d^{n/2}}{4\pi^{n/2}} \mathrm{dist}_{\mathbb{R}\Delta_d}(s) \right\} \ge 1 - O(d^{-\infty}).$$

(2) There exists a positive c > 0 such that

$$\mu_d \left\{ s \in \mathbb{R}H^0(X, L^d), \left\| s_{\sigma}^{\perp} \right\|_{\mathcal{C}^1(\mathbb{R}X)} < \frac{d^{n/2}}{4\pi^{n/2}} \operatorname{dist}_{\mathbb{R}\Delta_d}(s) \right\} \ge 1 - O(e^{-c\sqrt{d}\log d}).$$

Moreover, if the real Hermitian metric on L is analytic, then the last measure is even bigger then $1 - O(e^{-cd})$.

Proof. First, remark that, by Proposition 3.4, for any $m \in \mathbb{N}$, setting $r_d = C_1 d^{-2n-m}$, one has

(9)
$$\mu_d \{ s \in \mathbb{R}H^0(X, L^d), \operatorname{dist}_{\mathbb{R}\Delta_d}(s) > r_d \|s\|_{\mathcal{L}^2} \} \ge 1 - O(d^{-m}).$$

Also, by point (1) of Proposition 4.2, for any $t < t_0$, any integer r, any sequence w_d of the form $C_2 d^{-r}$ and any d large enough, we have

(10)
$$\mu_d \left\{ s \in \mathbb{R}H^0(X, L^d), \left\| s_{\sigma^{\lfloor td \rfloor}}^\perp \right\|_{\mathcal{C}^1(\mathbb{R}X)} < w_d \| s \|_{\mathcal{L}^2} \right\} = 1.$$

Putting together (9) and (10), we have that, for any such sequences r_d and w_d ,

(11)
$$\mu_d \left\{ s \in \mathbb{R}H^0(X, L^d), \left\| s_{\sigma^{\lfloor td \rfloor}}^{\perp} \right\|_{\mathcal{C}^1(\mathbb{R}X)} < \frac{w_d}{r_d} \text{dist}_{\mathbb{R}\Delta_d}(s) \right\} \ge 1 - O(d^{-m})$$

By choosing $w_d = \frac{d^{n/2}}{4\pi^{n/2}}r_d$, we then obtain that for any $m \in \mathbb{N}$

(12)
$$\mu_d \left\{ s \in \mathbb{R}H^0(X, L^d), \left\| s_{\sigma^{\lfloor td \rfloor}}^{\perp} \right\|_{\mathcal{C}^1(\mathbb{R}X)} < \frac{d^{n/2}}{4\pi^{n/2}} \mathrm{dist}_{\mathbb{R}\Delta_d}(s) \right\} \ge 1 - O(d^{-m})$$

which proves the point (1) of the proposition. Point (2) of the proposition follows the same lines, using point (2) of Proposition 4.2 and setting $r_d = d^{-2n}e^{-c\sqrt{d}}$, where c is given by point (2) of Proposition 4.2.

We now prove Theorems 1.5, 1.2 and 1.3.

Proof of Theorem 1.5. Let us start with the proof of point (1) of the theorem. Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1, for some fixed k large enough. We want to prove that there exists $b_0 < 1$ such that for any $b > b_0$, the measure

(13)
$$\mu_d \left\{ s \in \mathbb{R}H^0(X, L^d), \exists s' \in \mathbb{R}H^0(X, L^{\lfloor bd \rfloor}) \text{ such that } (\mathbb{R}X, \mathbb{R}Z_s) \sim (\mathbb{R}X, \mathbb{R}Z_{s'}) \right\}$$

equals $1 - O(d^{-\infty})$, as $d \to \infty$, where $(\mathbb{R}X, \mathbb{R}Z_s) \sim (\mathbb{R}X, \mathbb{R}Z_{s'})$ means there the two pairs are isotopic. Let us consider $t_0 > 0$ given by point (1) of Proposition 4.4 and set $b_0 = 1 - kt_0$. Then, for any $b > b_0$, there exists $t < t_0$, such that $\lfloor bd \rfloor \ge d - k \lfloor td \rfloor$ holds. Let us fix such b and t. For any $s \in \mathbb{R}H^0(X, L^d)$, let us write the orthogonal decomposition $s = s_{\sigma \lfloor td \rfloor}^\perp + s_{\sigma \lfloor td \rfloor}^0$ given by Notation 4.1. By Lemma 4.3, if the $\mathcal{C}^1(\mathbb{R}X)$ -norm of $s_{\sigma \lfloor td \rfloor}$ is smaller than $\frac{d^{n/2}}{4\pi^{n/2}} \operatorname{dist}_{\mathbb{R}\Delta_d}(s)$, then the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s_{\sigma \lfloor td \rfloor}^0})$ are isotopic. This implies that the measure (13) is bigger than the Gaussian measure of the set

(14)
$$\left\{s \in \mathbb{R}H^0(X, L^d), \left\|s_{\sigma^{\lfloor td \rfloor}}^{\perp}\right\|_{\mathcal{C}^1(\mathbb{R}X)} < \frac{d^{n/2}}{4\pi^{n/2}} \mathrm{dist}_{\mathbb{R}\Delta_d}(s)\right\}$$

which, in turn, by the point (1) of Proposition 4.4, is bigger than $1 - O(d^{-\infty})$. We have then proved that the pair $(\mathbb{R}X, \mathbb{R}Z_s)$ is isotopic to the pair $(\mathbb{R}X, \mathbb{R}Z_{s_{\sigma^{\lfloor td \rfloor}}^0})$ with probability $1 - O(d^{-\infty})$. Now, $s_{\sigma^{\lfloor td \rfloor}}^0 \in \mathbb{R}H_{d,\sigma^{\lfloor td \rfloor}}$ which implies, by Proposition 2.3, that there exists $s' \in \mathbb{R}H^0(X, L^{d-k \lfloor td \rfloor})$ such that $s_{\sigma^{\lfloor td \rfloor}}^0 = \sigma^{\lfloor td \rfloor} \otimes s'$. Assertion (1) of the theorem then follows from the fact that the real zero locus of $s_{\sigma^{\lfloor td \rfloor}}^0$ coincides with the real zero locus of s'. Indeed, the real zero locus of $s_{\sigma^{\lfloor td \rfloor}}^0 \in \mathbb{R}Z_{s'}$ and this is equal to $\mathbb{R}Z_{s'}$, because $\mathbb{R}Z_{\sigma} = \emptyset$.

Assertion (2) of the theorem follows the same lines, using the orthogonal decomposition $s = s_{\sigma}^{\perp} + s_{\sigma}^{0}$ and Assertion (2) of Proposition 4.4.

Proof of Theorems 1.2. Recall that we want to prove that there exists $a_0 < v(L)$ such that, for any $a > a_0$, we have

 $\mu_d \{ s \in \mathbb{R}H^0(X, L^d), b_*(\mathbb{R}Z_s) < ad^n \} = 1 - O(d^{-\infty})$

as $d \to \infty$. Let $a_0 := \int_X c_1(L)^n b_0^n$, where b_0 is given by point (1) of Theorem 1.5 and let $a > a_0$.

By point (1) of Theorem 1.5, for any $b > b_0$, the real zero locus of a global section s is diffeomorphic to the real zero locus of a global section s' of $L^{\lfloor bd \rfloor}$ with probability $1 - O(d^{-\infty})$.

Now, by Smith-Thom inequality (see Equation (1)), the total Betti number $b_*(\mathbb{R}Z_{s'})$ of the real zero locus a generic section s' of $L^{\lfloor bd \rfloor}$ is smaller or equal than $b_*(Z_{s'})$, which, by [12, Lemma 3], has the asymptotic $b_*(Z_{s'}) = \int_X c_1(L)^n (\lfloor bd \rfloor)^n + O(d^{n-1})$. In particular, with probability $1 - O(d^{-\infty})$, the total Betti number $b_*(\mathbb{R}Z_s)$ of the real zero locus of a section s of L^d is smaller than $\int_X c_1(L)^n (\lfloor bd \rfloor)^n + \epsilon d^n$, for any $\epsilon > 0$, as $d \to \infty$. Choosing ϵ small enough, we can find $1 > b > b_0$ such that $\int_X c_1(L)^n b^n + \epsilon < a$, which implies the result. \Box

Proof of Theorem 1.3. We want to prove that, for any a > 0, there exists c > 0 such that

$$\mu_d \{ s \in \mathbb{R}H^0(X, L^d), b_*(\mathbb{R}Z_s) < b_*(Z_s) - ad^{n-1} \} \ge 1 - O(e^{-c\sqrt{d}\log d})$$

as $d \to \infty$ (and also the similar estimate with $1 - O(e^{-cd})$ in the right-hand side, if the real Hermitian metric on L is analytic).

Remark that the total Betti number of the zero locus Z_s a generic section s of L^d has the asymptotic

(15)
$$b_*(Z_s) = \int_X c_1(L)^n d^n - \int_X c_1(L)^{n-1} \wedge c_1(X) d^{n-1} + O(d^{n-2}),$$

as $d \to \infty$, where $c_1(X)$ is the first Chern class of X. (This is a classical estimate which follows from the adjunction formula and Lefschetz hyperplane section theorem. A proof of this estimates follows the lines of [12, Lemma 3], by just taking care of the second term of the asymptotic which can be derived using [12, Lemma 2]).

Similarly, for any $k \in \mathbb{N}$, the zero locus of a generic section s' of L^{d-k} is such that

(16)
$$b_*(Z_{s'}) = \int_X c_1(L)^n d^n - \left(\int_X c_1(L)^{n-1} \wedge c_1(X) + nk \int_X c_1(L)^n\right) d^{n-1} + O(d^{n-2}),$$

as $d \to \infty$. Let us choose once for all an integer k big enough such that $nk \int_X c_1(L)^n > a$. In particular, by (15) and (16), the zero loci of generic sections s and s' of L^d and L^{d-k} are such that

(17)
$$b_*(Z_{s'}) < b_*(Z_s) - ad^{n-1}$$

 $d \to \infty$. Now, by the Assertion (2) of Theorem 1.5, the probability that the real zero locus of a section s of L^d is diffeomorphic to the real zero locus of a section s' of L^{d-k} equals $1 - O(e^{-c\sqrt{d}\log d})$ (or $1 - O(e^{-cd})$, if the real Hermitian metric on L is analytic). By Smith-Thom inequality and by (17), this implies that the probability of the event "the total Betti number of the real zero locus of a section s of L^d is smaller than $b_*(Z_s) - ad^{n-1}$ " is bigger or equal than $1 - O(e^{-c\sqrt{d}\log d})$ (or $1 - O(e^{-cd})$, if the real Hermitian metric on L is analytic), which proves the theorem.

5. Real curves in Algebraic surfaces and depth of nests

In this section, we study the nests defined by a random real algebraic curve inside a real algebraic surface. We will focus in particular on two examples: the Hirzebruch surfaces (Section 5.1) and the Del Pezzo surfaces (Section 5.2). The main results of the section (Theorems 5.3 and 5.7) say that, inside these surfaces, real algebraic curves with deep nests are rare, in the same spirit of Theorems 1.2 and 1.3. In order to define what "deep nests" means in this context, we need to prove some bounds on the depth of nests (Propositions 5.2 and 5.5) which are well known to experts. These bounds play the same role that Smith-Thom inequality (1) plays for Theorems 1.2 and 1.3. Theorems 5.3 and 5.7 will also follows from our low degree approximation (Theorem 1.5).

Definition 5.1. Let S be a real algebraic surface and let M be a connected component of the real part $\mathbb{R}S$.

• An oval in M is an embedded circle $\mathbb{S}^1 \hookrightarrow M$ which bounds a disk.

- Suppose that M is not diffeomorphic to the sphere \mathbb{S}^2 , then two ovals form an *injective* pair if one of the two oval is contained in the disk bounded by the other one. A nest N in M is a collection of ovals with the property that each pair of ovals in the collection is an injective pair.
- Suppose that M is diffeomorphic to \mathbb{S}^2 , then a *nest* N is a collection of ovals with the property that each connected component of $M \setminus N$ is either a disk or an annulus.
- The number of ovals which form a nest is called *depth* of the nest.

5.1. Nests in Hirzebruch surfaces. A Hirzebruch surface S is a compact complex surface which admits a holomorphic fibration over $\mathbb{C}P^1$ with fiber $\mathbb{C}P^1$. Given a Hirzebruch surface S, then there exists a positive integer n such that S is isomorphic to the projective bundle $P(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathcal{O}_{\mathbb{C}P^1})$ over $\mathbb{C}P^1$ (see for example [2]). We will denote such a Hirzebruch surface by Σ_n . Remark that a Hirzebruch surface is simply connected, that Σ_0 is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ and Σ_1 to $\mathbb{C}P^2$ blown-up at a point.

The Hirzebruch surface Σ_n admits a natural real structure c_n , which we wix from now on, such that $\mathbb{R}\Sigma_n$ is diffeomorphic to a torus if n is even and to a Klein bottle if n is odd (see, for instance, [26, Page 7]).

We denote by F_n any fiber of the natural fibration $\Sigma_n \to \mathbb{C}P^1$ and by B_n the section $P(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \{0\})$ of this fibration. Remark that B_n is a real algebraic curve and F_n can be chosen to be a real algebraic curve. The homology classes $[F_n]$ and $[B_n]$ form a basis of $H_2(\Sigma_n, \mathbb{Z})$, with $[F_n] \cdot [F_n] = 0$, $[F_n] \cdot [B_n] = 1$ and $[B_n] \cdot [B_n] = n$, where "." denotes the intersection product of $H_2(\Sigma_n, \mathbb{Z})$.

The second homology group $H_2(\Sigma_n, \mathbb{Z})$ can be identified with the Picard group of Σ_n . We say the an algebraic curve in Σ_n realizes the class (a, b) (or it is of bidegree (a, b)) if its homology class equals $a[F_n] + b[B_n]$. If a, b > 0, then the divisor associated with the class (a, b) is an ample divisor.

Proposition 5.2. Let (Σ_n, c_n) be a real Hirzebruch surface and let a, b be two positive integers. Then, the depth of a nest of a real algebraic curve in the class (a, b) is smaller or equal than b/2.

Proof. Let C be a real curve of bidegree (a, b), with a, b > 0. Let N be a nest of $\mathbb{R}C$ of depth land choose a point p inside the innermost oval and a point q outside the outer oval of the nest. We can choose p and q so that they lie in the same fiber F_n of the fibration $\Sigma_n \to \mathbb{C}P^1$ and such that real locus of the fiber $\mathbb{R}F_n$ intersects the nest transversally. The real locus of the fiber $\mathbb{R}F_n$ intersects each of the l ovals of the nest in at least 2 points, so that $\mathbb{R}F_n \cap N \ge 2l$. On the other hand, by Bezout's theorem, we have that $\mathbb{R}F_n \cap N \le [F_n] \cdot [C] \le b$, hence the result.

Theorem 5.3. Let (Σ_n, c_n) be a real Hirzebruch surface and a and b be two positive integers.

- (1) There exists a positive $\beta_0 < \lfloor b/2 \rfloor$ such that for any $\beta \in (\beta_0, \lfloor b/2 \rfloor)$, the probability that a real algebraic curve of bidegree (da, db) has a nest of depth $\geq \beta d$ is smaller or equal than $O(d^{-\infty})$.
- (2) For any $k \in \mathbb{N}$ there exists a positive constant c such that the probability that a real algebraic curve of bidegree (da, db) has a nest of depth $\geq \lfloor db/2 \rfloor k$ is smaller or equal than $O(e^{-c\sqrt{d}\log d})$.

Proof. Notice that as a probability space we are considering the linear system associated with the divisor of bidegree (a, b), rather than the space of real sections of the associated line bundle. As noticed in Remark 1.4, these two probability spaces are equivalent for our purpose.

By Assertion (1) of Theorem 1.5, the real locus of a real algebraic curve of bidegree (da, db) is ambient isotopic to the real locus of a real algebraic curve of bidegree $(\lfloor dta \rfloor, \lfloor dtb \rfloor)$ with probability bigger or equal than $1 - O(d^{-\infty})$, where $t_0 < t < 1$ for some t_0 . From this, using Proposition 5.2, we obtain that with probability bigger or equal than $1 - O(d^{-\infty})$, the depth of a nest of a real algebraic curve of bidegree (da, db) is smaller or equal than $\lfloor dtb/2 \rfloor$. Setting $\beta_0 = \lfloor t_0 b/2 \rfloor$, we obtain Assertion (1).

The proof of the Assertion (2) follows the same lines, by using Assertion (2) of Theorem 1.5 instead of Assertion (1). \Box

5.2. Nests in Del Pezzo surfaces. An algebraic surface S is called a *Del Pezzo surface*, if the anticanonical bundle K_S^* is ample. We will say that a real algebraic curve inside S is of class d if it belongs to the real linear system defined by the real divisor $-dK_S$, where $d \in \mathbb{N}$. The *degree* of a Del Pezzo surface is the self-intersection of the canonical class K_S .

Notation 5.4. We denote by m_S the smallest value of $-H \cdot K_S$, where H is a real divisor such that $-H \cdot K_S - 1 \ge 2$.

When S has degree bigger or equal than 3, choosing $H = -K_S$, we can see that m_S is smaller or equal than the degree of S.

Proposition 5.5. Let (S, c_S) be a real Del Pezzo surface of degree bigger or equal than 3, with non empty real locus. Then, the depth of a nest of a real algebraic curve of class d is smaller or equal than $dm_S/2$.

Proof. Let m_S be the smallest value of $-H \cdot K_S$, where H is a real divisor with the property that $-H \cdot K_S - 1 \ge 2$. Fix such a real divisor H.

Let Z a real algebraic curve of class d and let N be a nest of $\mathbb{R}Z$ of depth l. Choose generic points p and q such that p is inside the innermost oval and q is outside the outer oval of the nest (if the connected component M where the nest is located is a sphere, then $M \setminus N$ as exactly two disks, and one of the ovals bordering one of these two disks is the innermost oval and the other is the outer one). By [16], there exists a real rational curve C of class H whose real locus passes through p and q. This curve intersects each oval of the nest in at least two points, so that $N \cap \mathbb{R}C \geq 2l$. On the other hand, by Bezout's theorem, we have that $\mathbb{R}C \cap N \leq -dK_s \cdot H = dm_S$, hence the result. \Box

Remark 5.6. The same statement and proof also works for many degree 2 Del Pezzo surfaces (always using a real rational curves passing through 2 points, whose existence is ensured by the positivity of certain Welschinger invariants [16]). However, there exist degree 2 Del Pezzo surfaces for which such curves may not exist, as explained in [16, Section 2.3]. To avoid explaining these differences, we have preferred to restrict ourselves only to Del Pezzo surfaces of degree greater than or equal to 3.

Theorem 5.7. Let (S, c_S) be a Del Pezzo surface of degree bigger or equal than 3, with non empty real locus. Then the following happens.

- (1) There exists a positive $\alpha_0 < \lfloor m_S/2 \rfloor$ such that for any $\alpha \in (\alpha_0, \lfloor m_S/2 \rfloor)$, the probability that a real curve of class d has a nest of depth $\geq \alpha d$ is smaller or equal than $O(d^{-\infty})$.
- (2) For any $k \in \mathbb{N}$ there exists a positive constant c such that the probability that a real curve of class d has a nest of depth $\geq \lfloor dm_S/2 \rfloor k$ is smaller or equal than $O(e^{-c\sqrt{d}\log d})$.

Remark 5.8. The degree of a Del Pezzo surface is at least 1 and at most 9. The only degree 9 Del Pezzo surface is the projective plane $\mathbb{C}P^2$ and, in this particular case, Theorem 5.7 follows from [9, Theorem 9].

Proof of Theorem 5.7. The proof of the follows the same lines of the proof of Theorem 5.3, using Proposition 5.5 instead of Proposition 5.2. \Box

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