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# A note on the lifted Miller-Tucker-Zemlin subtour elimination constraints for routing problems with time windows

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**Abstract:** We propose lifted versions of the Miller-Tucker-Zemlin subtour elimination constraints for routing problems with time windows (TW). The constraints are valid for problems such as the travelling salesman problem with TW, the vehicle routing problem with TW, the generalized travelling salesman problem with TW, and the general vehicle routing problem with TW. They are corrected versions of the constraints proposed by Desrochers and Laporte (1991).

**Keywords:** Miller-Tucker-Zemlin; subtour elimination constraints; routing problems; time windows.

## 1 Introduction

The travelling salesman problem with time windows (TSPTW) is defined on a graph  $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ , where  $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$  is the vertex set and  $\mathcal{A} = \{(v_i, v_j) : v_i, v_j \in \mathcal{V}, i \neq j\}$  is the arc set.  $v_1$  is the depot, from where the salesman starts and ends the visiting tour.

Each vertex  $v_i \in \mathcal{V} \setminus \{v_1\}$  must be visited within a specified time window  $[a_i, b_i]$  and waiting time is allowed, i.e., the salesman can arrive at  $v_i$  before  $a_i$  and wait until  $a_i$  to visit vertex  $v_i$ . The time window of the depot is  $[a_1, b_1]$ : the salesman leaves the depot after  $a_1$  and returns to the depot before  $b_1$ . Each arc  $(v_i, v_j)$  is associated with a travel cost  $c_{ij} \geq 0$  and a travel time  $t_{ij} \geq 0$ . The TSPTW consists of determining a tour such that the total travel cost is minimized and every vertex  $v_i \in \mathcal{V}$  is visited exactly once within its time window  $[a_i, b_i]$  (Dumas et al. (1995), Gendreau et al. (1998)).

The TSPTW can be formulated using two types of variables.  $x_{ij}$  is a binary variable that equals to 1 if and only if arc  $(v_i, v_j) \in \mathcal{A}$  is used in the solution.  $u_i$  specifies the service time at vertex  $v_i \in \mathcal{V} \setminus \{v_1\}$ , and  $u_1$  is the departure time from the depot  $v_1$ . Let  $\mathcal{N} = \{1, \dots, n\}$  be the set of vertex indices. The TSPTW can be formulated as follows:

$$\text{minimize } \sum_{i \in \mathcal{N}} \sum_{\substack{j \in \mathcal{N} \\ j \neq i}} c_{ij} x_{ij} \tag{1}$$

$$\text{s.t. } \sum_{\substack{j \in \mathcal{N} \\ j \neq i}} x_{ij} = 1, \quad \forall i \in \mathcal{N}, \tag{2}$$

$$\sum_{\substack{i \in \mathcal{N} \\ j \neq i}} x_{ij} = 1, \quad \forall j \in \mathcal{N}, \quad (3)$$

$$u_i - u_j + Mx_{ij} \leq M - t_{ij}, \quad \forall i \in \mathcal{N}, j \in \mathcal{N} \setminus \{1\}, i \neq j, \quad (4)$$

$$a_i \leq u_i \leq b_i, \quad \forall i \in \mathcal{N}, \quad (5)$$

$$u_i + t_{i1}x_{i1} \leq b_1, \quad \forall i \in \mathcal{N} \setminus \{1\}, \quad (6)$$

$$x_{ij} \in \{0, 1\}, \quad \forall i, j \in \mathcal{N}, i \neq j, \quad (7)$$

$$u_i \geq 0, \quad \forall i \in \mathcal{N}. \quad (8)$$

The formulation adapts the formulation proposed by Dantzig et al. (1954) for the travelling salesman problem where the subtour elimination constraints are written in a Miller-Tucker-Zemlin (MTZ) fashion. The objective function (1) minimizes the total cost. Constraints (2) ensure that each vertex is visited exactly once. Constraints (3) are flow conservation constraints. Constraints (4) guarantee the feasibility of the tour with respect to timing constraints. These constraints are also known as MTZ subtour elimination constraints since they ensure that the solution does not contain subtours disconnected from the depot.  $M$  is a large constant such that  $M \geq \max_{i,j \in \mathcal{N}} \{b_i - a_j + t_{ij}\}$ . Constraints (5) ensure that each vertex is visited during its time window, and the salesman leaves the depot during its time window. Constraints (6) ensure that the salesman is back at the depot before  $b_1$ . Constraints (7) and (8) define the variables.

In this note, we discuss on the lifting of the Constraints (4) for the TSPTW. In the paper by Desrochers and Laporte (1991), the authors proposed a lifted version of these constraints, which we find and prove is wrong.

The remainder of this note is organized as follows. In Section 2, we propose two lifted versions of the MTZ subtour elimination constraints, where we correct the ones proposed by Desrochers and Laporte (1991). Conclusions are drawn in Section 3.

## 2 Lifting the MTZ subtour elimination constraints for the TSPTW

In the TSPTW, because of time window constraints, some arcs  $(v_i, v_j)$  are *infeasible*. Indeed, if  $b_j < a_i + t_{ij}$ , then it is not possible to visit vertex  $v_j$  right after visiting vertex  $v_i$ . According to Desrochers and Laporte (1991), if arcs  $(v_i, v_j)$  and  $(v_j, v_i)$  are both *feasible*, by taking into account the inverse arcs  $(v_j, v_i)$ , Constraints (4) can be strengthened as:

$$u_i - u_j + Mx_{ij} + (M - t_{ij} + \min\{-t_{ji}, b_j - a_i\})x_{ji} \leq M - t_{ij}, \quad \forall i \in \mathcal{N}, j \in \mathcal{N} \setminus \{1\}, i \neq j. \quad (9)$$

However, we found that these valid inequalities are incorrectly written since they can eliminate feasible or optimal solutions.

First, we provide an example, depicted in Figure 1, to illustrate the situation. Let us consider an instance with four vertices  $v_1, v_2, v_3$ , and  $v_4$  in the graph and vertex  $v_1$  represents the depot. Vertices  $v_1, v_2, v_3$  and  $v_4$  are associated with time windows  $[0, 60]$ ,  $[20, 25]$ ,  $[10, 45]$ , and  $[40, 50]$  respectively. Vertices  $v_1, v_2, v_3$ , and  $v_4$  are located at coordinates  $(0, 0)$ ,  $(5, 0)$ ,  $(5, 5)$  and  $(0, 5)$  respectively. For each

arc  $(v_i, v_j)$ , we consider that  $c_{ij}$  and  $t_{ij}$  are equal to the Euclidean distance between  $v_i$  and  $v_j$ . It is clear that the tour  $v_1 - v_2 - v_3 - v_4 - v_1$  is feasible and optimal. Feasible values for  $u$  variables are:  $u_1 = 15, u_2 = 20, u_3 = 25, u_4 = 40$ .

When  $i = 3$  and  $j = 2$ , Constraints (9) are:

$$u_3 - u_2 + Mx_{32} + (M - t_{32} + \min\{-t_{23}, b_2 - a_3\})x_{23} \leq M - t_{32}.$$

Thus, when Constraints (9) for  $i = 3$  and  $j = 2$  are applied to this solution they will provide the following equation:

$$u_3 - u_2 + 0 + (M - 5 + \min\{-5, 25 - 10\}) \leq M - 5;$$

$$u_3 \leq u_2 + 5.$$

Thus, the constraint  $u_3 \leq u_2 + 5$  is imposed. Similarly, when  $i = 4$  and  $j = 3$ , Constraints (9) are  $u_4 \leq u_3 + 5$ . These two constraints lead to  $u_4 \leq u_2 + 10$ . However,  $u_2 \leq 25$  and  $u_4 \geq 40$ . Hence, route  $v_1 - v_2 - v_3 - v_4 - v_1$  is not feasible when considering Constraints (9).

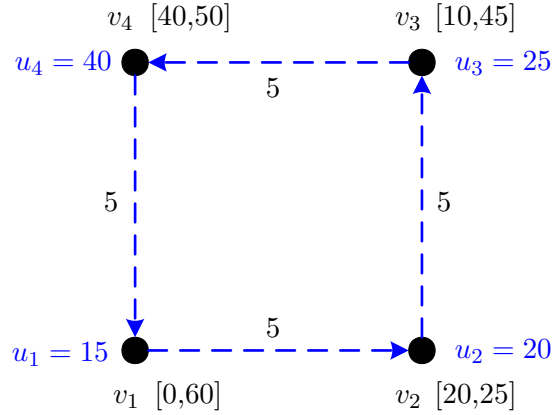


Figure 1: Example where the optimal solution is cut off by Constraints (9).

Therefore, we correct the Constraints (9) as follows.

**Proposition 1.** *The constraints*

$$u_i - u_j + Mx_{ij} + (M - t_{ij} + a_j - b_i)x_{ji} \leq M - t_{ij} \quad \forall i \in \mathcal{N}, j \in \mathcal{N} \setminus \{1\}, i \neq j \quad (10)$$

are valid inequalities for the TSPTW.

*Proof.* Consider the general constraints:

$$u_i - u_j + Mx_{ij} + (M - t_{ij} + \alpha_{ji})x_{ji} \leq M - t_{ij} \quad \forall i \in \mathcal{N}, j \in \mathcal{N} \setminus \{1\}, i \neq j. \quad (11)$$

We seek for the largest value of  $\alpha_{ji}$  such that Constraints (11) are valid.

If  $x_{ji} = 0$ , these constraints are obviously satisfied for any value of  $\alpha_{ji}$ . If  $x_{ji} = 1$ , (vertex  $v_i$  is visited right after visiting vertex  $v_j$ ), then  $x_{ij} = 0$ , and we obtain the following constraints:

$$u_i - u_j + \alpha_{ji} \leq 0. \quad (12)$$

We are thus seeking for a value of  $\alpha_{ji}$  such that  $\alpha_{ji} \leq u_j - u_i$ . From Constraints (5), we have:  $u_i \leq b_i$  and  $u_j \geq a_j$ . Thus, we obtain:

$$a_j - b_i \leq u_j - u_i. \quad (13)$$

Hence, we set  $\alpha_{ji} = a_j - b_i$ . □

Let us continue to consider the example discussed above in Figure 1. When  $i = 3$  and  $j = 2$ , Constraints (10) impose  $u_3 \leq u_2 + 25$ . When  $i = 4$  and  $j = 3$ , Constraints (10) impose  $u_4 \leq u_3 + 40$ . These two constraints lead to  $u_4 \leq u_2 + 65$ , which means route  $v_1 - v_2 - v_3 - v_4 - v_1$  is feasible.

Moreover, it can be noticed that when the optimal solution contains waiting times, the formulation (1)-(8) may have multiple optimal solutions because of the timing variables  $u_i$ . Indeed, given optimal values for the  $x_{ij}$  variables, there may be several values for the  $u_i$  variables that satisfy Constraints (4) and (5). Moreover, given optimal values for the  $x_{ij}$  variables, there are always feasible values for the  $u_i$  variables such that each vertex (except the depot) is visited as early as possible, namely minimizing waiting times (see for example the solution in Figure 1).

Thus, in the following, we propose supervalid inequalities for the TSPTW. An inequality is *supervalid* if it does not cut off all optimal solutions. This concept is a generalization of the concept of valid inequalities and has been introduced by Israeli and Wood (2002).

**Proposition 2.** *The constraints*

$$u_i - u_j + Mx_{ij} + (M - t_{ij} + \min\{-t_{ji}, a_j - a_i\})x_{ji} \leq M - t_{ij} \quad \forall i \in \mathcal{N}, j \in \mathcal{N} \setminus \{1\}, i \neq j \quad (14)$$

are supervalid inequalities for the TSPTW.

*Proof.* Consider the general constraints:

$$u_i - u_j + Mx_{ij} + (M - t_{ij} + \alpha_{ji})x_{ji} \leq M - t_{ij} \quad \forall i \in \mathcal{N}, j \in \mathcal{N} \setminus \{1\}, i \neq j. \quad (15)$$

We seek for the largest value of  $\alpha_{ji}$  such that Constraints (15) are supervalid.

If  $x_{ji} = 0$ , these constraints are obviously satisfied for any value of  $\alpha_{ji}$ . If  $x_{ji} = 1$  (vertex  $v_i$  is visited right after visiting vertex  $v_j$ ), then  $x_{ij} = 0$ , and we obtain the following constraints:

$$u_i - u_j + \alpha_{ji} \leq 0 \quad \forall i \in \mathcal{N}, j \in \mathcal{N} \setminus \{1\}, i \neq j. \quad (16)$$

Thus, we seek for a value of  $\alpha_{ji}$  such that  $\alpha_{ji} \leq u_j - u_i$ . In order to provide such a value, we consider an optimal solution where each vertex is visited as soon as possible.

Two cases may be considered. In the first case,  $a_j + t_{ji} \geq a_i$ . This means that vertex  $v_i$  can be visited right after vertex  $v_j$  without any waiting time. Hence, we have  $u_i = u_j + t_{ji}$ . We then obtain:

$$-t_{ji} = u_j - u_i \quad (17)$$

Hence,  $\alpha_{ji} = -t_{ji}$  is valid for this first case.

In the second case,  $a_j + t_{ji} < a_i$ . This means that a waiting time may be required before visiting vertex  $v_i$ . Let us suppose that the value of  $u_j$  has been fixed. The value of  $u_i$  will then be chosen such

that vertex  $v_i$  is visited as soon as possible. If  $u_j + t_{ji} \geq a_i$ , then  $v_i$  can be visited right after  $v_j$  without waiting time. Then, similarly to the first case,  $\alpha_{ji} = -t_{ji}$  is valid. If  $u_j + t_{ji} < a_i$ , then a waiting time is required, and vertex  $v_i$  is visited as soon as possible, i.e.  $u_i = a_i$ . From Constraints (5), we have  $u_j \geq a_j$ . We then obtain:

$$a_j - a_i \leq u_j - u_i \tag{18}$$

Hence,  $\alpha_{ji} = a_j - a_i$  is valid in this case.

In all cases, we can set  $\alpha_{ji} = \min \{-t_{ji}, a_j - a_i\}$ . □

Using the example introduced previously and depicted in Figure 1, it can be observed that Constraints (14) do not cut the solution proposed in Figure 1. However, other optimal solutions are eliminated. Let us consider another solution  $s'$  with the same values for  $x_{ij}$  variables and  $u_1 = 15, u_2 = 20, u_3 = 35$ , and  $u_4 = 40$ . This solution is valid and optimal with respect to the formulation of the TSPTW. However, Constraints (14) applied to  $i = 3$  and  $j = 2$  give  $u_3 - u_2 + 0 + (M - 5 + \min \{-5, 20 - 10\}) \leq M - 5$ . Thus the constraint is  $u_3 \leq u_2 + 5$ , and the solution  $s'$  would be cut off by Constraints (14).

Note that this result can be extended to routing problems with time windows (TW) as the generalized TSPTW (Yuan et al. (2018)), the vehicle routing problem with TW (VRPTW) (Pecin et al. (2017)), the generalized VRPTW (Yuan et al. (2019)). Readers are referred to Toth and Vigo (2014) for an overview on routing problems.

### 3 Conclusions

In this note we provide two lifted inequalities from the Miller-Tucker-Zemlin subtour elimination constraints for routing problems with time windows. We provide an example on which the constraints proposed by Desrochers and Laporte (1991) cut off the optimal solution, and we propose a correct lifting of Miller-Tucker-Zemlin constraints. We also propose a family of supervalid inequalities in the sense that they cut off some feasible solutions but guarantee to not cut off at least one optimal solution.

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