Nonlinear control

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Chapter 12

Nonlinear Control

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12.1 Introduction

Nonlinear behavior is the general rule in physics and nature. Linear models, obtained by linearization or identification, are in general crude approximations of nonlinear behaviors of plants in the neighbourhood of an operating point. However, in many cases such as startup, shutdown, or important transient regimes, study of batch and fed-batch processes, a linear model is insufficient to correctly reproduce the reality, and the resulting linear controller cannot guarantee stability and performance. Yet, because of the difficulty to cope with nonlinear control, linear models and linear controllers are by far dominant.

Nevertheless, efficient methods exist that can be used with nonlinear models provided the end-users are willing to carry out some effort.

Among existing theories, can be found backstepping, sliding mode control, [Khalil, 1996, Krstić et al., 1995, Slotine and Li, 1991], flatness based control [Fliess et al., 1995, 1997], and methods based on Lyapunov stability, nonlinear model predictive control [Alamir, 2006, Allgöwer and Zheng, 2000, Rawlings et al., 1994]. These methods are powerful and would deserve a long development.

In this chapter, a particular method of nonlinear control, often called nonlinear geometric control [Isidori, 1989, 1995, Jurdjevic, 1997, Khalil, 1996] will be presented and discussed. It is based on differential geometry but can be understood in simpler words. Differential geometry is devoted in particular to the theory of differential equations in relation with geometry, surfaces, manifolds.

12.2 Some mathematical notions useful in nonlinear control

The theory of linear control was developed long before nonlinear control and some tools available in linear control can be adapted to nonlinear control without performing the usual linear approximation of dynamics by calculation of the Jacobian. Several textbooks are devoted to nonlinear systems, analysis and control, among which [Fossard and Normand-Cyrot, 1993, Isidori, 1989, 1995, Nijmeijer and VanderSchaft, 1990, Slotine and Li, 1991, Vidyasagar, 1993].

The important first point about nonlinear geometric control is that it can be used for systems that are affine with respect to the manipulated input, i.e. that can be described like the following single-input single-output plant as

$$\left\{ \begin{array}{ll} \dot{\boldsymbol{x}} & = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\, u \\ y & = h(\boldsymbol{x}) \end{array} \right. \tag{12.1}$$

where $\boldsymbol{x}$ is the state vector of dimension n, u the control input and y the controlled output. This might seem a severe restriction, but in chemical engineering, most systems are of the form (12.1). This is because the manipulated input is in general a flow rate, a position of valve, which appears linearly in nonlinear models. $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{g}(\boldsymbol{x})$ are respectively called vector fields of the dynamics and the control. They are assumed smooth mappings and $h(\boldsymbol{x})$ is a smooth function.

The parallel between the system (12.1) and the linear state-space model

$$\left\{ \begin{array}{ll} \dot{\boldsymbol{x}} & = \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{B}\,u \\ y & = \boldsymbol{C}\,\boldsymbol{x} \end{array} \right. \tag{12.2}$$

is obvious. The notions of linear control that will be developed and applied to nonlinear control can be found in [Corriou, 2004, Isidori, 1995]. The presentation will here deal only with nonlinear systems and control related notions.

12.2.1 Notions of differential geometry

The derivative of a function $\lambda(\boldsymbol{x})$ in the direction of the field $\boldsymbol{f}$ (directional derivative) is called the Lie derivative and is defined by

$$L_f \lambda(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i}\, f_i(\boldsymbol{x}) = < \frac{\partial \lambda}{\partial \boldsymbol{x}}, \boldsymbol{f}(\boldsymbol{x}) > \tag{12.3}$$

It plays a very important role in nonlinear control. Indeed, for the system (12.1)

$$\begin{array}{rl} \dfrac{dy}{dt} & = \displaystyle\sum_{i=1}^{n} \frac{\partial h}{\partial x_i}\, \frac{dx_i}{dt} \\ & = \displaystyle\sum_{i=1}^{n} \frac{\partial h}{\partial x_i}\, (f_i(\boldsymbol{x}) + g_i(\boldsymbol{x})\, u) \\ & = L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\, u \end{array} \tag{12.4}$$

thus the time derivative of the output is simply expressed with respect to the Lie derivatives. The Lie derivative $L_f\lambda(\boldsymbol{x})$ is the derivative of λ along the integral curves of the vector field $\boldsymbol{f}$. The integral curves are the curves of the solution $\boldsymbol{x}(t)$ passing by $\boldsymbol{x}^\circ$ for the state-space system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) \qquad ; \quad \boldsymbol{x}(0) = \boldsymbol{x}^\circ$$

Consider successive differentiations, such as the differentiation of λ in the direction of $\boldsymbol{f}$, then in the direction of $\boldsymbol{g}$, i.e.

$$L_g L_f \lambda(\boldsymbol{x}) = \frac{\partial L_f \lambda}{\partial \boldsymbol{x}}\, \boldsymbol{g}(\boldsymbol{x}) \tag{12.5}$$

or further, to differentiate λ, k times in the direction of $\boldsymbol{f}$

$$L_f^k \lambda(\boldsymbol{x}) = \frac{\partial L_f^{k-1} \lambda}{\partial \boldsymbol{x}}\, \boldsymbol{f}(\boldsymbol{x}) \qquad \text{with: } L_f^0 \lambda(\boldsymbol{x}) = \lambda(\boldsymbol{x}) \tag{12.6}$$

The Lie bracket is defined by

$$[\boldsymbol{f}, \boldsymbol{g}](\boldsymbol{x}) = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\, \boldsymbol{f}(\boldsymbol{x}) - \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\, \boldsymbol{g}(\boldsymbol{x}) \tag{12.7}$$

where $\partial f / \partial \boldsymbol{x}$ is the Jacobian matrix of $\boldsymbol{f}$ equal to (same for $\partial \boldsymbol{g}/\partial \boldsymbol{x}$, the Jacobian matrix of $\boldsymbol{g}$)

$$D\boldsymbol{f}(\boldsymbol{x}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{12.8}$$

The operation on the Lie bracket of $\boldsymbol{g}$ by iterating on $\boldsymbol{f}$ can be repeated and the following notation is adopted

$$\mathrm{ad}_f^k \boldsymbol{g}(\boldsymbol{x}) = [\boldsymbol{f}, \mathrm{ad}_f^{k-1} \boldsymbol{g}](\boldsymbol{x}) \qquad \text{for } \; k > 1 \quad \text{with: } \mathrm{ad}_f^0 \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) \tag{12.9}$$

The Lie bracket is a bilinear, skew-symmetric mapping and satisfies the Jacobi identity

$$[\boldsymbol{f}, [\boldsymbol{g}, \boldsymbol{p}]] + [\boldsymbol{g}, [\boldsymbol{p}, \boldsymbol{f}]] + [\boldsymbol{p}, [\boldsymbol{f}, \boldsymbol{g}]] = 0 \tag{12.10}$$

where $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{p}$ are vector fields.

12.2.2 Relative degree of a monovariable nonlinear system

For a linear transfer function

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_0 + b_1\, s + \cdots + b_m\, s^m}{a_0 + a_1\, s + \cdots + a_n\, s^n} \tag{12.11}$$

that has no common poles and zeros, the roots of the denominator are called the poles, the roots of the numerator the transmission zeros and the difference $n - m$ is the relative degree of the transfer function.

Considering the single-input single-output nonlinear system (12.1), the relative degree, or relative order, or characteristic index, stems from the following definition [Hirschorn, 1979]:
the relative degree of the nonlinear system (12.1) over a domain U is the smallest integer r for which

$$L_g L_f^{r-1} h(\boldsymbol{x}) \neq 0 \qquad \text{for all } \boldsymbol{x} \text{ in } U \tag{12.12}$$

For the linear system (12.2), it would yield

$$L_g L_f^{r-1} h(\boldsymbol{x}) = CA^{r-1}B \neq 0 \tag{12.13}$$

and is consistent with the definition of the relative degree for linear systems.

Thus, the nonlinear system (12.1) possesses a relative degree r equal to

$$\begin{array}{ll} r = 1 & \text{if } \; L_g h(\boldsymbol{x}) \neq 0 \\ r = 2 & \text{if } \; L_g h(\boldsymbol{x}) = 0 \;\text{ and }\; L_g L_f h(\boldsymbol{x}) \neq 0 \\ r = 3 & \text{if } \; L_g h(\boldsymbol{x}) = L_g L_f h(\boldsymbol{x}) = 0 \;\text{ and }\; L_g L_f^2 h(\boldsymbol{x}) \neq 0 \\ \ldots & \end{array} \tag{12.14}$$

Using this definition, the relative degree r can be obtained from the successive time derivatives of the output y as

$$\begin{array}{lll} \dfrac{dy}{dt} & = L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\, u & \\ & = L_f h(\boldsymbol{x}) & \text{if } \; 1 < r \\ & \vdots & \\ \dfrac{d^k y}{dt^k} & = L_f^k h(\boldsymbol{x}) + L_g L_f^{k-1} h(\boldsymbol{x})\, u & \\ & = L_f^k h(\boldsymbol{x}) & \text{if } k < r \\ \dfrac{d^r y}{dt^r} & = L_f^r h(\boldsymbol{x}) + L_g L_f^{r-1} h(\boldsymbol{x})\, u & \text{as } \; L_g L_f^{r-1} h(\boldsymbol{x}) \neq 0 \end{array} \tag{12.15}$$

Thus, the relative degree is the smallest degree of differentiation of the output y which depends explicitly on the input u. This can be a convenient way to find the relative degree of a single-input single-output system.

It may happen that, for example, the first Lie derivative $L_g h(\boldsymbol{x})$ of the sequence $L_g L_f^{k-1} h(\boldsymbol{x})$ is zero at a given point. In that case, according to Isidori [1995], the relative degree cannot be defined strictly at $\boldsymbol{x}^\circ$, but will be defined in the neighbourhood U (notion of dense open subset). This will be accepted in the following.

If it happens that

$$L_g L_f^k h(\boldsymbol{x}) = 0 \qquad \text{for all } k \qquad , \quad \text{for all } \boldsymbol{x} \text{ in } U \tag{12.16}$$

the relative degree cannot be defined in the neighbourhood of $\boldsymbol{x}^\circ$ and the output is not affected by the input u.

It can be shown [Isidori, 1995] that the matrix

$$\begin{bmatrix} Dh(\boldsymbol{x}) \\ DL_f h(\boldsymbol{x}) \\ \vdots \\ DL_f^{r-1} h(\boldsymbol{x}) \end{bmatrix} \begin{bmatrix} \boldsymbol{g}(\boldsymbol{x}) & \mathrm{ad}_f \boldsymbol{g}(\boldsymbol{x}) & \dots & \mathrm{ad}_f^{r-1} \boldsymbol{g}(\boldsymbol{x}) \end{bmatrix} \tag{12.17}$$

has rank r. This implies that the row vectors $Dh(\boldsymbol{x}), DL_f h(\boldsymbol{x}), \dots, DL_f^{r-1} h(\boldsymbol{x})$ are linearly independent. Thus, the r functions $h(\boldsymbol{x})$, $L_f h(\boldsymbol{x})$, ..., $L_f^{r-1} h(\boldsymbol{x})$ can form a new set of coordinates in the neighbourhood of point $\boldsymbol{x}^\circ$.

12.2.3 Frobenius Theorem

The Frobenius theorem gives a necessary and sufficient condition of integrability of a system of first-order partial differential equations whose right member depends only on variables or unknowns but not on partial derivatives of the unknowns. It is also called a Pfaff system.

It will be presented according to [Isidori, 1995].
a) First, let us consider d smooth vector fields $\boldsymbol{f}_i(\boldsymbol{x})$, defined on Ω°, which span a distribution Δ, denoted by

$$\Delta = \mathrm{span}\{\boldsymbol{f}_1(\boldsymbol{x}), \dots, \boldsymbol{f}_d(\boldsymbol{x})\} \tag{12.18}$$

To define a distribution, consider smooth vector fields $\boldsymbol{f}_1(\boldsymbol{x}), \dots, \boldsymbol{f}_d(\boldsymbol{x})$ that span at a point $\boldsymbol{x}$ of U a vector space dependent on $\boldsymbol{x}$ that can be denoted by $\Delta(\boldsymbol{x})$. The mapping assigning this vector space to any point $\boldsymbol{x}$ is called a smooth distribution.

In the same neighbourhood Ω°, the codistribution W of dimension $n-d$ is spanned by $n-d$ covector fields, $\boldsymbol{w}_1, \dots, \boldsymbol{w}_{n-d}$, such that

$$< \boldsymbol{w}_j(\boldsymbol{x}), \boldsymbol{f}_i(\boldsymbol{x}) >= 0 \qquad \forall\, 1 \le i \le d\,,\ 1 \le j \le n-d \tag{12.19}$$

Due to that property, the codistribution is denoted as: $W = \Delta^\perp$, and $\boldsymbol{w}_j$ is the solution of the equation

$$\boldsymbol{w}_j(\boldsymbol{x}) \boldsymbol{F}(\boldsymbol{x}) = 0 \tag{12.20}$$

where $\boldsymbol{F}(\boldsymbol{x})$ is the matrix of dimension $n \times d$, of rank d, equal to

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{f}_1(\boldsymbol{x}) & \dots & \boldsymbol{f}_d(\boldsymbol{x}) \end{bmatrix} \tag{12.21}$$

The row vectors $\boldsymbol{w}_j$ form a basis of the space of the solutions of Eq.(12.20).
b) We look for solutions such that

$$\boldsymbol{w}_j = \frac{\partial \lambda_j}{\partial \boldsymbol{x}} \tag{12.22}$$

correspond to smooth functions λ_j, i.e. we look for $n-d$ independent solutions (the row vectors $\partial\lambda_1/\partial\boldsymbol{x}, \ldots, \partial\lambda_{n-d}/\partial\boldsymbol{x}$ are independent) of the following differential equation

$$\frac{\partial\lambda_j}{\partial\boldsymbol{x}}\boldsymbol{F}(\boldsymbol{x}) = \frac{\partial\lambda_j}{\partial\boldsymbol{x}}\left[\begin{array}{ccc}\boldsymbol{f}_1(\boldsymbol{x}) & \ldots & \boldsymbol{f}_d(\boldsymbol{x})\end{array}\right] = 0 \tag{12.23}$$

c) We search the condition of existence of $n-d$ independent solutions of differential Eq.(12.23), which amounts to seeking the integrability of the distribution Δ: a distribution of dimension d, defined on an open domain U of $\mathbb{R}^n$, is completely integrable if, for any point $\boldsymbol{x}^\circ$ of U, there exist $n-d$ smooth functions, taking real values, defined on a neighbourhood of $\boldsymbol{x}^\circ$, such that

$$\text{span}\{\frac{\partial\lambda_1}{\partial\boldsymbol{x}}, \ldots, \frac{\partial\lambda_{n-d}}{\partial\boldsymbol{x}}\} = \Delta^\perp \tag{12.24}$$

The condition of existence is produced by the Frobenius theorem.
d) Frobenius theorem: a distribution is nonsingular if and only if it is involutive. A distribution Δ is defined as involutive if the Lie bracket of any couple of vector fields belonging to Δ belongs to Δ

$$\begin{array}{l}\boldsymbol{f}_1 \text{ and } \boldsymbol{f}_2 \in \Delta \Longrightarrow [\boldsymbol{f}_1, \boldsymbol{f}_2] \in \Delta \Longleftrightarrow \\ [\boldsymbol{f}_i, \boldsymbol{f}_j](\boldsymbol{x}) = \displaystyle\sum_{k=1}^{m} \alpha_{ijk}\boldsymbol{f}_k(\boldsymbol{x}) \qquad \forall i, j\end{array}$$

In the case where $\boldsymbol{F}$ is reduced to only a vector field $\boldsymbol{f}_1$ ($d=1$), Eq.(12.23) can be geometrically interpreted as:

- the gradient of λ is orthogonal to $\boldsymbol{f}_1$.
- the vector $\boldsymbol{f}_1$ is tangent to the surface $\lambda =$ constant passing by this point.
- the integral curve of $\boldsymbol{f}_1$ passing by this point is entirely on the surface $\lambda =$ constant.

12.2.4 Coordinates transformation

The objective of the change of coordinates is to present the system in a simpler form in the new coordinates.

A function Φ of $\mathbb{R}^n$ in $\mathbb{R}^n$, defined in a domain U, is called a diffeomorphism if it is smooth and if its inverse Φ^{-1} exists and is smooth. If the domain U is the whole space, the diffeomorphism is global; otherwise, it is local. The diffeomorphism is thus a nonlinear coordinate change possessing the previous properties.

Consider a function Φ defined in a domain U of $\mathbb{R}^n$. $\Phi(\boldsymbol{x})$ defines a local diffeomorphism on a sub-domain Ω° of Ω, if and only if the Jacobian matrix $\partial\Phi/\partial\boldsymbol{x}$ is nonsingular at $\boldsymbol{x}^\circ$ belonging to Ω,

A diffeomorphism allows us to transform a nonlinear system into another nonlinear system defined with regard to new states.

Given the single-input single-output nonlinear system (12.1) of relative degree r at $\boldsymbol{x}^\circ$, set the r first functions

$$\begin{array}{ll} \phi_1(\boldsymbol{x}) & = h(\boldsymbol{x}) \\ \phi_2(\boldsymbol{x}) & = L_f h(\boldsymbol{x}) \\ & \vdots \\ \phi_r(\boldsymbol{x}) & = L_f^{r-1} h(\boldsymbol{x}) \end{array} \tag{12.25}$$

If $r < n$, it is possible to find $n - r$ functions $\phi_{r+1}(\boldsymbol{x}), \ldots, \phi_n(\boldsymbol{x})$ such that the mapping

$$\Phi(\boldsymbol{x}) = \begin{bmatrix} \phi_1(\boldsymbol{x}) \\ \vdots \\ \phi_n(\boldsymbol{x}) \end{bmatrix} \tag{12.26}$$

has its Jacobian matrix nonsingular and thus constitutes a possible coordinate change at $\boldsymbol{x}^\circ$.

The value taken by the additional functions $\phi_{r+1}(\boldsymbol{x}), \ldots, \phi_n(\boldsymbol{x})$ at $\boldsymbol{x}^\circ$ is not important and these functions can be chosen such that

$$< D\phi_i(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{x}) >= L_g \phi_i(\boldsymbol{x}) = 0 \quad \text{for all } r+1 \leq i \leq n \quad \text{for all } \boldsymbol{x} \text{ in } \Omega \tag{12.27}$$

The demonstration makes use of the Frobenius theorem [Isidori, 1995].

12.2.5 Normal Form

Given the vector $\boldsymbol{z} = \Phi(\boldsymbol{x})$, making use of the r first new coordinates z_i defined by $z_i = y^{(i-1)} = \phi_i(\boldsymbol{x}), (i = 1, \ldots, r)$, defined according to the relations (12.25), the nonlinear system (12.1) can be described as

$$\begin{array}{ll} \dfrac{dz_1}{dt} & = \dfrac{\partial \phi_1}{\partial \boldsymbol{x}} \dfrac{d\boldsymbol{x}}{dt} = \dfrac{\partial h}{\partial \boldsymbol{x}} \dfrac{d\boldsymbol{x}}{dt} = L_f h(\boldsymbol{x}(t)) = \phi_2(\boldsymbol{x}(t)) = z_2(t) \\ & \vdots \\ \dfrac{dz_{r-1}}{dt} & = \dfrac{\partial \phi_{r-1}}{\partial \boldsymbol{x}} \dfrac{d\boldsymbol{x}}{dt} = \dfrac{\partial L_f^{r-2} h}{\partial \boldsymbol{x}} \dfrac{d\boldsymbol{x}}{dt} = L_f^{r-1} h(\boldsymbol{x}(t)) = \phi_r(\boldsymbol{x}(t)) = z_r(t) \\ \dfrac{dz_r}{dt} & = \dfrac{\partial \phi_r}{\partial \boldsymbol{x}} \dfrac{d\boldsymbol{x}}{dt} = \dfrac{\partial L_f^{r-1} h}{\partial \boldsymbol{x}} \dfrac{d\boldsymbol{x}}{dt} = L_f^r h(\boldsymbol{x}(t)) + L_g L_f^{r-1} h(\boldsymbol{x}(t))\, u(t) \end{array} \tag{12.28}$$

The expression of $\dot{z}_r(t)$ must be transformed with respect to $\boldsymbol{z}(t)$ by using the inverse relation $\boldsymbol{x}(t) = \Phi^{-1}(\boldsymbol{z}(t))$, yielding

$$\begin{array}{ll} \dfrac{dz_r}{dt} & = L_f^r h(\Phi^{-1}(\boldsymbol{z}(t))) + L_g L_f^{r-1} h(\Phi^{-1}(\boldsymbol{z}(t)))\, u(t) \\ & = b(\boldsymbol{z}(t)) + a(\boldsymbol{z}(t))\, u(t) \end{array} \tag{12.29}$$

by setting

$$a(\boldsymbol{z}(t)) = L_g L_f^{r-1} h(\Phi^{-1}(\boldsymbol{z}(t))) \quad ; \quad b(\boldsymbol{z}(t)) = L_f^r h(\Phi^{-1}(\boldsymbol{z}(t))) \tag{12.30}$$

and by noticing that, by definition of the relative degree, $a(z^\circ) \neq 0$ at $z^\circ = \Phi(\boldsymbol{x}^\circ)$.

The coordinates $z_i,\ r < i \leq n$, can be chosen according to Eq.(12.27) so that $L_g\phi_i(\boldsymbol{x}) = 0$, which gives

$$\begin{array}{ll} \dfrac{dz_i}{dt} & = \dfrac{\partial \phi_i}{\partial \boldsymbol{x}} \dfrac{d\boldsymbol{x}}{dt} = \dfrac{\partial \phi_i}{\partial \boldsymbol{x}} \left(\boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(\boldsymbol{x}(t))\, u(t)\right) \qquad ; \quad r < i \leq n \\ & = L_f\phi_i(\boldsymbol{x}(t)) + L_g\phi_i(\boldsymbol{x}(t))\, u(t)) = L_f\phi_i(\boldsymbol{x}(t)) \\ & = L_f\phi_i(\Phi^{-1}(\boldsymbol{z}(t))) \end{array} \tag{12.31}$$

Set

$$q_i(\boldsymbol{z}(t)) = L_f\phi_i(\Phi^{-1}(\boldsymbol{z}(t))) \qquad ; \quad r < i \leq n \tag{12.32}$$

Taking into account the previous equations, the normal form [Isidori, 1995, Kang and Krener, 2006] results

$$\begin{array}{ll} \dot{z}_1 & = z_2 \\ & \vdots \\ \dot{z}_{r-1} & = z_r \\ \dot{z}_r & = b(\boldsymbol{z}) + a(\boldsymbol{z})\, u(t) \\ \dot{z}_{r+1} & = q_{r+1}(\boldsymbol{z}) \\ \dot{z}_n & = q_n(\boldsymbol{z}) \end{array} \tag{12.33}$$

to which the equation of the output must be added

$$y = h(\boldsymbol{x}) = z_1 \tag{12.34}$$

This result can be symbolized in a block diagram (Fig. 12.1) using the chain of r integrators necessary to go from the control input to the output.

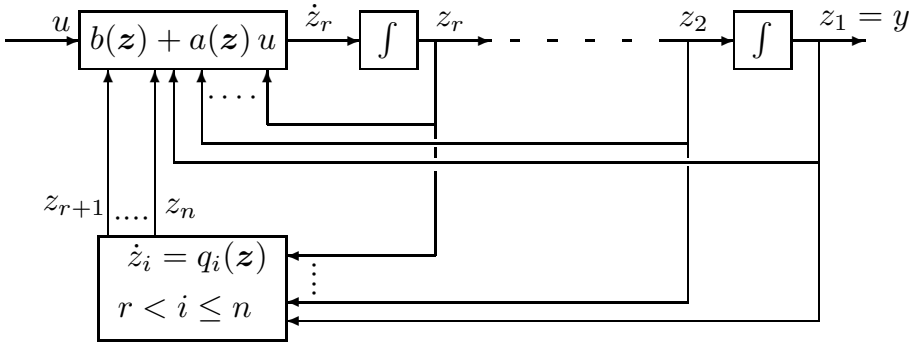

Figure 12.1: Description of the normal form

The condition $L_g\phi_i(\boldsymbol{x}) = 0$, fundamental for seeking the functions $\phi_i,\ r < i \leq n$, can be difficult to fulfill, as it corresponds to the solving of a system of $n - r$ partial differential equations. To define a coordinate change, it can be sufficient to find these functions so that the matrix Φ is simply nonsingular.

12.2.6 Controllability and Observability

Consider the nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\, u \tag{12.35}$$

defined in a domain U.

The system (12.35) is controllable if, given two arbitrary states $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$, there exists an admissible input $u(t)$ such that the system can be steered from the state $\boldsymbol{x}_0$ to the desired state $\boldsymbol{x}_1$ in finite time T.

The controllability of this nonlinear system can be studied by proceeding to a linearization of the system

$$\dot{\boldsymbol{z}} = \frac{\partial f}{\partial \boldsymbol{x}}\, \boldsymbol{z} + \boldsymbol{g}(\boldsymbol{x})\, v \tag{12.36}$$

and by studying the controllability matrix, in a way similar (except that it deals with distributions, see (12.54)) to the controllability of linear systems where it is defined as the rank of the controllability matrix

$$\boldsymbol{\mathcal{C}} = [\boldsymbol{B} \quad \boldsymbol{A}\,\boldsymbol{B} \quad \ldots \quad \boldsymbol{A}^{n-2}\,\boldsymbol{B} \quad \boldsymbol{A}^{n-1}\,\boldsymbol{B}] \tag{12.37}$$

which should be of rank n, given the model of the linear state-space system

$$\begin{array}{ll} \dot{\boldsymbol{x}}(t) & = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) & = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{array} \tag{12.38}$$

with the state $\boldsymbol{x}$ of dimension n.

However, this approach is not always satisfying; actually, a nonlinear system can be controllable whereas its linear approximation is not. It is necessary to introduce the notion of reachability [Isidori, 1995, Nijmeijer and VanderSchaft, 1990], a weaker form of controllability.

For observability that also requires complex topological notions, both previous books are recommended. Observability can be defined in an approximate manner by:
In parallel to the definition of controllability, if, given two different initial conditions $\boldsymbol{x}_1(0)$ and $\boldsymbol{x}_2(0)$, there exists a control input $u(t)$ defined in $[0, T]$ such that the corresponding outputs $y_1(\boldsymbol{x}, u, t)$ and $y_2(\boldsymbol{x}, u, t)$ are not totally similar in $[0, T]$, the system is observable. It means that, given the measurable input and output, it is possible to determine the state. The input $u(t)$ distinguishes the initial conditions $\boldsymbol{x}_1(0)$ and $\boldsymbol{x}_2(0)$ in $[0, T]$. If $u(t)$ distinguishes any pair $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ in $[0, T]$, the input $u(t)$ is universal.

12.2.7 Principle of Feedback Linearization

The objective is to design a control law that is a function of the states so that, to the resulting linearized system, efficient methods of linear control can be then applied. To perform this linearization, two types of feedback can be used, state feedback or output feedback, corresponding to input-state or input-output

linearization respectively. The states are assumed to be known. In the case where all the states are not known, it is necessary to couple a state estimator, called an observer (for linear systems, the linear Kalman filter is an optimal observer) to the control system. When the state feedback control law depends only on the values of the states $\boldsymbol{x}$ and the external input v, it is a static state feedback. If the control law corresponds to the output of a dynamic system, itself depending on the states $\boldsymbol{x}$ and on the external input v, it is a dynamic state feedback.

Consider the single-input single-output nonlinear system affine with respect to the input

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\, u \tag{12.39}$$

defined in a neighbourhood U of $\boldsymbol{x}^\circ$ and such that: $\boldsymbol{f}(\boldsymbol{x}^\circ) = 0$. The problem of feedback linearization is to find smooth functions p and q with $q(\boldsymbol{x}^\circ) \neq 0$, and a diffeomorphism Φ with $\Phi(\boldsymbol{x}^\circ) = 0$ such that by defining:

- an external input $v = p(\boldsymbol{x}) + q(\boldsymbol{x})\, u$,
- the transformed variables $\boldsymbol{z} = \Phi(\boldsymbol{x})$,

the resulting system is linear under the form

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\,\boldsymbol{z} + \boldsymbol{B}\, v \tag{12.40}$$

where the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable. The new state $\boldsymbol{z}$ is called a linearizing state and the control law is a linearizing control law.

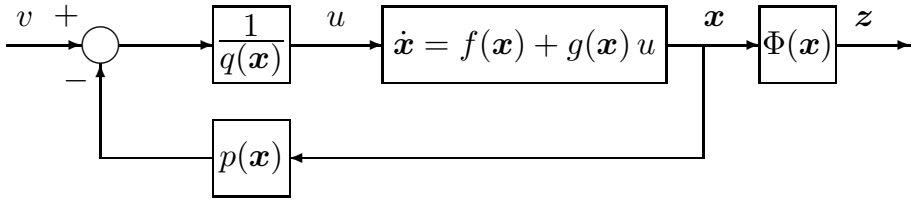

Figure 12.2: Linearizing feedback

Given the state $\boldsymbol{x}$, the control law (Fig. 12.2) is

$$u(t) = \frac{-p(\boldsymbol{x})}{q(\boldsymbol{x})} + \frac{v}{q(\boldsymbol{x})} = \alpha(\boldsymbol{x}) + \beta(\boldsymbol{x})\, v \tag{12.41}$$

12.2.8 Exact Input-State Linearization for a System of Relative Degree Equal to n

The input-state linearization is often called exact linearization [Isidori, 1995]. First, consider the system (12.39) possessing a relative degree $r = n$, thus equal to the dimension of the state vector, at a point $\boldsymbol{x}^\circ$.

In this case, the coordinate change necessary to obtain the normal form (section 12.2.5) is

$$\Phi(\boldsymbol{x}) = \begin{bmatrix} \phi_1(\boldsymbol{x}) \\ \vdots \\ \phi_n(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} h(\boldsymbol{x}) \\ \vdots \\ L_f^{n-1} h(\boldsymbol{x}) \end{bmatrix} \tag{12.42}$$

and the resulting normal form is

$$\begin{array}{ll} \dot{z}_1 & = z_2 \\ & \vdots \\ \dot{z}_{n-1} & = z_n \\ \dot{z}_n & = b(\boldsymbol{z}) + a(\boldsymbol{z})\, u(t) \end{array} \tag{12.43}$$

with $a(z^\circ) \neq 0$ because of the definition of the relative degree.

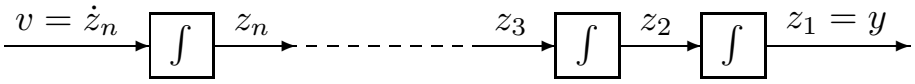

Figure 12.3: Exactly linearized system by static state feedback

The state feedback control law can be chosen as

$$u(t) = -\frac{b(\boldsymbol{z})}{a(\boldsymbol{z})} + \frac{v}{a(\boldsymbol{z})} \tag{12.44}$$

hence the resulting closed-loop system (Fig. 12.2 with its chain of integrators)

$$\begin{array}{ll} \dot{z}_1 & = z_2 \\ & \vdots \\ \dot{z}_{n-1} & = z_n \\ \dot{z}_n & = v \end{array} \tag{12.45}$$

that is a linear and controllable system expressed in the Brunovsky canonical form [Brunovsky, 1970]

$$\dot{\boldsymbol{z}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & & \dots & 0 \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} v \tag{12.46}$$

To obtain the control law (12.44), a coordinate change and a state feedback taken into any order, thus exchangeable, are used. If the state feedback is first

used and then the coordinate change, the following control law results

$$\begin{aligned} u(t) &= -\frac{b(\Phi(\boldsymbol{x}))}{a(\Phi(\boldsymbol{x}))} + \frac{v}{a(\Phi(\boldsymbol{x}))} \\ &= \frac{-L_f^n h(\boldsymbol{x}) + v}{L_g L_f^{n-1} h(\boldsymbol{x})} \end{aligned} \tag{12.47}$$

often used under this form that corresponds to the same controllable linear system (12.45). This control law is called linearizing state feedback and the coordinates Φ are the linearizing coordinates.

Two remarks [Isidori, 1995] are particularly important:

– It was assumed that $\boldsymbol{x}^\circ$ is a stationary point for the system (12.1), meaning that $\boldsymbol{f}(\boldsymbol{x}^\circ) = 0$ and $h(\boldsymbol{x}^\circ) = 0$, hence

$$\begin{aligned} \phi_1(\boldsymbol{x}^\circ) &= h(\boldsymbol{x}^\circ) = 0 \\ \phi_i(\boldsymbol{x}^\circ) &= \frac{\partial L_f^{i-2} h}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}^\circ) = 0 \qquad , \quad 1 < i \leq n \end{aligned} \tag{12.48}$$

or $\boldsymbol{z}^\circ = 0$. It is always possible to come back to $h(\boldsymbol{x}^\circ) = 0$ by an appropriate translation.

– It is possible to perform a pole placement or to satisfy an optimality criterion, by imposing a feedback (Fig. 12.4) as

$$v_2 = \boldsymbol{K}\,\boldsymbol{z} \qquad , \quad \text{with the gain vector: } \boldsymbol{K} = [c_0 \dots c_{n-1}] \tag{12.49}$$

equivalent to

$$v_2 = c_0 h(\boldsymbol{x}) + c_1 L_f h(\boldsymbol{x}) + \cdots + c_{n-1} L_f^{n-1} h(\boldsymbol{x}) \tag{12.50}$$

which is a nonlinear state feedback with respect to $\boldsymbol{x}$. The state feedback control law becomes, in this case,

$$u(t) = \frac{-L_f^n h(\boldsymbol{x}) + \dot{z}_n}{L_g L_f^{n-1} h(\boldsymbol{x})} = \frac{-L_f^n h(\boldsymbol{x}) - \sum_{i=0}^{n-1} c_i L_f^i h(\boldsymbol{x}) + v}{L_g L_f^{n-1} h(\boldsymbol{x})} \tag{12.51}$$

and appears as an extension of (12.47). The external input v can be chosen in many different ways. For example, it could be equal to the set point y_{ref}. When $v = 0$, this corresponds to the local asymptotic equilibrium $\boldsymbol{z} = 0$ which is preserved.

By expressing the transfer function $Y(s)/V(s)$, the characteristic polynomial associated to (12.51) then equal to

$$c_0 + c_1\, s + \cdots + c_{n-1}\, s^{n-1} + s^n \tag{12.52}$$

whose coefficients can be chosen so as to realize the adequate pole placement.

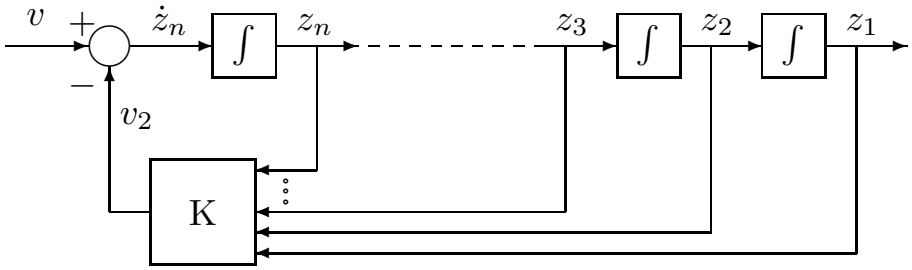

Figure 12.4: Nonlinear control with pole placement for a system of relative degree equal to n (exact input-state linearization)

By simply considering the system (12.39) a priori defined without output, Isidori [1995] shows that the exact input-state linearization is possible in a neighbourhood U of $\boldsymbol{x}^\circ$ if and only if there exists a scalar function $\lambda(\boldsymbol{x})$ such that the system with the "output" redefined

$$\begin{array}{ll} \dot{\boldsymbol{x}} & = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\, u \\ y & = \lambda(\boldsymbol{x}) \end{array} \tag{12.53}$$

has a relative degree equal to n at $\boldsymbol{x}^\circ$. Referring to (12.46) and Figure 12.2, the function $\lambda(\boldsymbol{x})$ is equal to $z_1(\boldsymbol{x})$.

The following general theorem [Hunt et al., 1983, Su, 1982] is particularly important:
Theorem:
the system (12.39) is exactly linearizable in state space (input-state linearization) in a neighbourhood U of $\boldsymbol{x}^\circ$ if and only if the following conditions are satisfied:

1. The vector fields $\{\boldsymbol{g}(\boldsymbol{x}^\circ), ad_f\boldsymbol{g}(\boldsymbol{x}^\circ), \ldots, ad_f^{n-1}\boldsymbol{g}(\boldsymbol{x}^\circ)\}$ are linearly independent.
2. The distribution span$\{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is involutive in U.

Condition 1 can be written as "the following matrix

$$[\boldsymbol{g}(\boldsymbol{x}^\circ), ad_f\boldsymbol{g}(\boldsymbol{x}^\circ), \ldots, ad_f^{n-1}\boldsymbol{g}(\boldsymbol{x}^\circ)] \tag{12.54}$$

has rank n". It is a controllability condition of the nonlinear system. This matrix must be invertible. For linear state-space systems, this matrix is the controllability matrix

$$[\boldsymbol{B}, \boldsymbol{AB}, \ldots, \boldsymbol{A}^{n-1}\boldsymbol{B}]$$

To realize the exact input-state linearization, it must be proceeded to the following stages:

- Build the vector fields $\boldsymbol{g}(\boldsymbol{x}^\circ), ad_f\boldsymbol{g}(\boldsymbol{x}^\circ), \ldots, ad_f^{n-1}\boldsymbol{g}(\boldsymbol{x}^\circ)$.
- Check if the conditions of controllability and involutivity are verified.
- If these conditions are verified, find the function $\lambda(\boldsymbol{x})$ from the equations

$$L_g\lambda(\boldsymbol{x}^\circ) = L_gL_f\lambda(\boldsymbol{x}^\circ) = \cdots = L_gL_f^{n-2}\lambda(\boldsymbol{x}^\circ) = 0$$
$$L_gL_f^{n-1}\lambda(\boldsymbol{x}^\circ) \neq 0$$

It is often mentioned that finding this function $\lambda(\boldsymbol{x})$ is a difficult task [Slotine and Li, 1991].

- Calculate the coordinate change

$$\Phi(\boldsymbol{x}) = [\lambda(\boldsymbol{x}), L_f\lambda(\boldsymbol{x}), \ldots, L_f^{n-1}\lambda(\boldsymbol{x})] \tag{12.55}$$

12.2.9 Input-Output Linearization of a System with Relative Degree r Lower than or Equal to n

Two cases are distinguished:

- [Isidori, 1995] notes that the nonlinear system affine with respect to the input

$$\begin{cases} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\, u \\ y &= h(\boldsymbol{x}) \end{cases} \tag{12.56}$$

  having a relative degree $r < n$ can satisfy the conditions of Hunt-Su-Meyer theorem. However, in this case, it was shown that there exists a different "output" λ such that the system has a relative degree equal to n. Thus, the newly defined system satisfies the previous theorem; by using a feedback $u = \alpha(\boldsymbol{x}) + \beta(\boldsymbol{x})\, v$ and a coordinate change $\Phi(\boldsymbol{x})$, it is transformed into a controllable linear system, but the real output, in general, is not linear with respect to the new one

$$y = h(\Phi^{-1}(\boldsymbol{z})) \tag{12.57}$$

- If the output y is fixed by $y = h(\boldsymbol{x})$ and if the system possesses a relative degree r lower than or equal to n, by using the coordinate change $\Phi(\boldsymbol{x})$ as in (12.28), it is possible to transform the system into the normal form (12.33) and to set $v = \dot{z}_r$ so that the system is simply expressed in the transformed coordinates in Byrnes-Isidori canonical form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= v = b(\boldsymbol{z}) + a(\boldsymbol{z})\, u(t) \\ \dot{z}_{r+1} &= q_{r+1}(\boldsymbol{z}) \\ \dot{z}_n &= q_n(\boldsymbol{z}) \\ y &= z_1 \end{aligned} \tag{12.58}$$

  with

$$b(\boldsymbol{z}) = L_f^r h(\boldsymbol{x}) \quad , \quad a(\boldsymbol{z}) = L_gL_f^{r-1}h(\boldsymbol{x}) \tag{12.59}$$

  The control law is deduced

$$\begin{aligned} u(t) &= -\frac{b(\boldsymbol{z})}{a(\boldsymbol{z})} + \frac{v}{a(\boldsymbol{z})} \\ &= \frac{-L_f^r h(\boldsymbol{x}) + v}{L_gL_f^{r-1}h(\boldsymbol{x})} \end{aligned} \tag{12.60}$$

The resulting system is only partially linear, but the output is influenced by the external input v only through a chain of r integrators (Fig. 12.1) related to the new states $z_1, \ldots, z_r$

$$y^{(r)} = L_f^r h(\boldsymbol{x}) + L_g L_f^{r-1} h(\boldsymbol{x})\, u = v \tag{12.61}$$

The new states $z_{r+1}, \ldots, z_n$ which constitute the nonlinear part of the system do not influence the output y.

Following the second case of this description of input-output linearization, the control law (12.60) can be used for many single-input single-output nonlinear plants affine with respect to the input, for which the relative degree is lower than n.

12.2.10 Zero Dynamics

For a linear system having a strictly proper transfer function (12.11), i.e. the degree of the numerator is strictly lower than that of the denominator, when positive zeros are present, the system is called non-minimum phase. It must be recalled that positive zeros becomes unstable poles for the inverse of the transfer function that can constitute the ideal controller. Note that, for this reason, positive zeros are considered apart in internal model control [Morari and Zafiriou, 1989].

For a single-input single-output system, the zero dynamics amounts to find an input u and initial conditions $\boldsymbol{x}_0$ such that $y(t) = 0,\ \forall t$. This implies that not only $y = 0$, but also its derivatives $y^{(i)} = 0,\ i = 0, \ldots, r$. The dynamics of the system corresponding to these conditions is called zero dynamics.

In the case of a linear time-invariant system of relative degree $r = n$, the numerator of the system transfer function is reduced to a constant, the transfer function has no zeros and the system does not have zero dynamics. Consequently, when $r < n$, the study of zero dynamics is important.

Similarly, for a nonlinear system, to study the zero dynamics, only the case where the relative degree r is lower than n is considered. The vector is represented in normal form (12.33) by separating the linear part of dimension r and the nonlinear part of dimension $n - r$ as

$$\boldsymbol{\xi} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_r \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(r-1)} \end{bmatrix} \quad ; \quad \boldsymbol{\eta} = \begin{bmatrix} z_{r+1} \\ \vdots \\ z_n \end{bmatrix} \tag{12.62}$$

so that the system can be rewritten as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= b(\boldsymbol{\xi}, \boldsymbol{\eta}) + a(\boldsymbol{\xi}, \boldsymbol{\eta})\, u(t) \\ \dot{\boldsymbol{\eta}} &= q(\boldsymbol{\xi}, \boldsymbol{\eta}) \end{aligned} \tag{12.63}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ constitute the normal coordinates or normal states.

The dynamics of the nonlinear system is thus decomposed into an external input-output part and an internal unobservable part. Whereas the external part is simple to design, there remains the problem of the internal stability corresponding to the last $(n-r)$ equations: $\dot{\boldsymbol{\eta}} = q(\boldsymbol{\xi}, \boldsymbol{\eta})$.

Considering $\boldsymbol{x}^\circ$ as an equilibrium point of the system, it results $\boldsymbol{f}(\boldsymbol{x}^\circ) = 0$ and it is possible to choose $h(\boldsymbol{x}^\circ) = 0$. In the normal coordinates $(\boldsymbol{\xi}, \boldsymbol{\eta})$, it can be assumed that the point $(0,0)$ is the equilibrium point, hence $b(0,0) = 0$ and $q(0,0) = 0$.

The aim is to make the output zero for all t in the neighbourhood of $t = 0$. In the normal form, this would amount to imposing

$$z_1 = \cdots = z_r = 0 \Longleftrightarrow \dot{z}_1 = \cdots = \dot{z}_r = 0 \Longleftrightarrow \boldsymbol{\xi} = 0 \qquad \text{for all } t \tag{12.64}$$

as, moreover, the output is imposed $y = z_1 = 0$. The input u results such that

$$0 = b(0, \boldsymbol{\eta}) + a(0, \boldsymbol{\eta})\, u(t) \tag{12.65}$$

with $a(0, \boldsymbol{\eta}) \neq 0$ still in the neighbourhood of $t = 0$. Moreover, the variable $\boldsymbol{\eta}$ is such that

$$\dot{\boldsymbol{\eta}} = q(0, \boldsymbol{\eta}) \qquad , \quad \text{with: } \boldsymbol{\eta}(0) = \boldsymbol{\eta}^\circ \tag{12.66}$$

which is an autonomous system of differential equations whose solution is the variable $\boldsymbol{\eta}(t)$. It yields the unique input that imposes a zero output in the neighbourhood of $t = 0$

$$u(t) = -\frac{b(0, \boldsymbol{\eta}(t))}{a(0, \boldsymbol{\eta}(t))} \tag{12.67}$$

The dynamics of Eq.(12.66), which results from the condition of zero output, is called zero dynamics or unforced zero dynamics, it describes the internal behaviour of the system. The zero dynamics is the dynamics of the inverse of the system.

The search of the zero output could have been realized in the original state space, by setting

$$y(t) = \dot{y}(t) = \cdots = y^{(r-1)}(t) = y^{(r)}(t) = 0 \qquad \text{for all } t \tag{12.68}$$

or further, in the neighbourhood of $\boldsymbol{x}^\circ$

$$\begin{aligned} &h(\boldsymbol{x}) = L_f h(\boldsymbol{x}) = \cdots = L_f^{r-1} h(\boldsymbol{x}) = 0 \\ &L_f^r h(\boldsymbol{x}) + L_g L_f^{r-1} h(\boldsymbol{x})\, u(t) = 0 \end{aligned} \tag{12.69}$$

The case of tracking a reference output y_{ref} is deducted by translation from the previous case of a zero output. In the neighbourhood of $t = 0$, the output is imposed

$$y(t) = y_{\text{ref}}(t) \tag{12.70}$$

giving in the new coordinates

$$z_i(t) = y_{\text{ref}}^{(i-1)}(t) \qquad , \quad 1 \leq i \leq r \tag{12.71}$$

By analogy with the previous case, we set

$$\boldsymbol{\xi}_{\text{ref}} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_r \end{bmatrix} = \begin{bmatrix} y_{\text{ref}} \\ y_{\text{ref}}^{(1)} \\ \vdots \\ y_{\text{ref}}^{(r-1)} \end{bmatrix} \tag{12.72}$$

The equation that imposes the control results

$$y_{\text{ref}}^{(r)}(t) = b(\boldsymbol{\xi}_{\text{ref}}(t), \boldsymbol{\eta}(t)) + a(\boldsymbol{\xi}_{\text{ref}}(t), \boldsymbol{\eta}(t))\, u(t) \tag{12.73}$$

where $\boldsymbol{\eta}$ is the solution of the following autonomous differential system

$$\dot{\boldsymbol{\eta}}(t) = q(\boldsymbol{\xi}_{\text{ref}}(t), \boldsymbol{\eta}(t)) \quad , \quad \text{with: } \boldsymbol{\eta}(0) = \boldsymbol{\eta}^{\circ} \tag{12.74}$$

From (12.73), results the equation of the unique control imposing on the output to exactly track the reference

$$u(t) = \frac{y_{\text{ref}}^{(r)}(t) - b(\boldsymbol{\xi}_{\text{ref}}(t), \boldsymbol{\eta}(t))}{a(\boldsymbol{\xi}_{\text{ref}}(t), \boldsymbol{\eta}(t))} \tag{12.75}$$

The system of differential Eqs. (12.74) coupled with Eq.(12.75), yields the forced zero dynamics or dynamics of the inverse of the system (12.56), corresponding to a control such that the output exactly tracks the reference. $\boldsymbol{\eta}$ is the state of the dynamics of the inverse, $\boldsymbol{\xi}_{\text{ref}}$ its control input and u its output.

12.2.11 Asymptotic Stability

Consider the system under its normal form

$$\begin{array}{ll} \dot{z}_1 & = z_2 \\ & \vdots \\ \dot{z}_{r-1} & = z_r \\ \dot{z}_r & = b(\boldsymbol{\xi}, \boldsymbol{\eta}) + a(\boldsymbol{\xi}, \boldsymbol{\eta})\, u(t) \\ \dot{\boldsymbol{\eta}} & = q(\boldsymbol{\xi}, \boldsymbol{\eta}) \end{array} \tag{12.76}$$

assuming as previously that $(\boldsymbol{\xi}, \boldsymbol{\eta}) = (0, 0)$ is an equilibrium point. In parallel to the state feedback (12.51), consider the external input

$$v = -\boldsymbol{K}\boldsymbol{z} \quad , \quad \text{with the gain vector: } \boldsymbol{K} = [c_0 \dots c_{r-1}] \tag{12.77}$$

so that the state feedback becomes

$$\begin{aligned} u(t) &= \frac{-b(\boldsymbol{\xi}, \boldsymbol{\eta}) - \sum_{i=0}^{r-1} c_i z_{i+1}}{a(\boldsymbol{\xi}, \boldsymbol{\eta})} \\ &= \frac{-L_f^r h(\boldsymbol{x}) - \sum_{i=0}^{r-1} c_i L_f^i h(\boldsymbol{x})}{L_g L_f^{r-1} h(\boldsymbol{x})} \end{aligned} \tag{12.78}$$

giving the closed-loop system

$$\begin{array}{ll} \dot{\boldsymbol{\xi}} & = A\boldsymbol{\xi} \\ \dot{\boldsymbol{\eta}} & = q(\boldsymbol{\xi},\boldsymbol{\eta}) \end{array} \tag{12.79}$$

where $\boldsymbol{A}$ is a companion controllability matrix equal to

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & & 0 & 1 \\ -c_0 & -c_1 & \dots & \dots & -c_{r-1} \end{bmatrix} \tag{12.80}$$

which has the characteristic polynomial

$$c_0 + c_1\, s + \cdots + c_{r-1}\, s^{r-1} + s^r \tag{12.81}$$

If, first the coefficients are chosen so that the roots of this polynomial have a negative real part and, secondly the zero dynamics corresponding to $\dot{\boldsymbol{\eta}} = q(0,\boldsymbol{\eta})$ is asymptotically locally stable, then the state feedback (12.78) stabilizes asymptotically locally the system (12.79) in the neighbourhood of the equilibrium $(\boldsymbol{\xi},\boldsymbol{\eta}) = (0,0)$.

The role and the importance of zero dynamics thus clearly appear at this level. If the linear approximation of the system possesses uncontrollable modes, the latter necessarily correspond to eigenvalues of the linear approximation Q of the zero dynamics. The linear approximation of the system is given by

$$\begin{array}{ll} \dot{z}_1 & = z_2 \\ & \vdots \\ \dot{z}_{r-1} & = z_r \\ \dot{z}_r & = b(\boldsymbol{\xi},\boldsymbol{\eta}) + a(\boldsymbol{\xi},\boldsymbol{\eta})\, u(t) \approx \boldsymbol{R}\,\boldsymbol{\xi} + \boldsymbol{S}\,\boldsymbol{\eta} + K\, u \\ \dot{\boldsymbol{\eta}} & = q(\boldsymbol{\xi},\boldsymbol{\eta}) \approx \boldsymbol{P}\,\boldsymbol{\xi} + \boldsymbol{Q}\,\boldsymbol{\eta} \end{array} \tag{12.82}$$

with the partial derivative matrices considered at $(\boldsymbol{\xi},\boldsymbol{\eta}) = (0,0)$

$$\boldsymbol{R} = \left[\frac{\partial b}{\partial \boldsymbol{\xi}}\right] \quad , \quad \boldsymbol{S} = \left[\frac{\partial b}{\partial \boldsymbol{\eta}}\right] \quad , \quad \boldsymbol{P} = \left[\frac{\partial q}{\partial \boldsymbol{\xi}}\right] \quad , \quad \boldsymbol{Q} = \left[\frac{\partial q}{\partial \boldsymbol{\eta}}\right] \tag{12.83}$$

Note that it is not necessary that the linear approximation be asymptotically stable for the nonlinear system to be stable.
As already realized for a system of relative degree n with the state feedback (12.51), an external input v (Fig. 12.5) can be taken into account as

$$u(t) = \frac{-L_f^r h(\boldsymbol{x}) - \displaystyle\sum_{i=0}^{r-1} c_i L_f^i h(\boldsymbol{x}) + v}{L_g L_f^{r-1} h(\boldsymbol{x})} \tag{12.84}$$

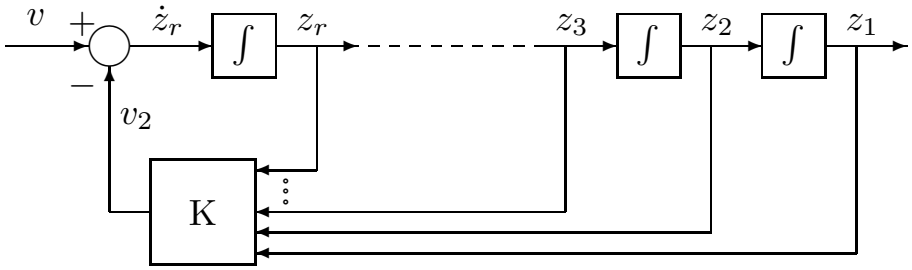

Figure 12.5: Nonlinear control with pole placement for a system of relative degree $r \leq n$

so that the system (12.76) is transformed into

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{B}\, v \\ \dot{\boldsymbol{\eta}} &= q(\boldsymbol{\xi}, \boldsymbol{\eta}) \end{aligned} \tag{12.85}$$

with: $\boldsymbol{B} = [0 \ \ldots \ 0 \ 1]^T$. Provided that the zero dynamics is stable, the stability will depend on the characteristic polynomial

$$c_0 + c_1\, s + \cdots + c_{r-1}\, s^{r-1} + s^r \tag{12.86}$$

whose coefficients can be imposed so as to achieve the desired pole placement.

By analogy with linear systems, a nonlinear system is called minimum-phase if its unforced zero dynamics is asymptotically locally stable at $(0, 0)$.

12.2.12 Tracking of a Reference Trajectory

To make the output $y = z_1$ converge asymptotically towards a reference trajectory y_{ref}, it suffices to gather the elements of both previous sections, the forced zero dynamics with reference trajectory and the asymptotic stability.

For the system in its normal form (12.76), consider the state feedback

$$\begin{aligned} u(t) &= \frac{-b(\boldsymbol{\xi}, \boldsymbol{\eta}) + y_{\text{ref}}^{(r)} - \displaystyle\sum_{i=0}^{r-1} c_i (z_{i+1} - y_{\text{ref}}^{(i)})}{a(\boldsymbol{\xi}, \boldsymbol{\eta})} \\ &= \frac{-L_f^r h(\boldsymbol{x}) + y_{\text{ref}}^{(r)} - \displaystyle\sum_{i=0}^{r-1} c_i (L_f^i h(\boldsymbol{x}) - y_{\text{ref}}^{(i)})}{L_g L_f^{r-1} h(\boldsymbol{x})} \end{aligned} \tag{12.87}$$

The error defined by

$$e(t) = y(t) - y_{\text{ref}}(t) \tag{12.88}$$

is the solution of the following differential equation

$$e^{(r)} + c_{r-1}\, e^{(r)} + \cdots + c_1\, e^{(1)} + c_0\, e = 0 \tag{12.89}$$

whose parallel with the characteristic polynomial (12.86) is obvious. By choosing adequately the coefficients c_i according to a pole placement strategy, the exponential convergence of the error towards 0 when $t \to \infty$ can be guaranteed. In the same manner as in the previous section, it is necessary that the zero dynamics (here forced) corresponding to

$$\dot{\boldsymbol{\eta}} = q(\boldsymbol{\xi}_{\text{ref}}, \boldsymbol{\eta}) \qquad , \quad \text{with: } \boldsymbol{\eta}_{\text{ref}}(0) = 0 \tag{12.90}$$

is stable (where $\boldsymbol{\eta}_{\text{ref}}(t)$ is the solution of this differential system) with the vector

$$\boldsymbol{\xi}_{\text{ref}} = [y_{\text{ref}}(t), y_{\text{ref}}^{(1)}(t), \ldots, y_{\text{ref}}^{(r-1)}(t)]^T \tag{12.91}$$

Isidori [1995] studied the particular case where the reference y_{ref} is defined as a linear reference model. In the case where an external input is taken into account, the state feedback (12.87) becomes

$$u(t) = \frac{v - L_f^r h(\boldsymbol{x}) + y_{\text{ref}}^{(r)} - \sum_{i=0}^{r-1} c_i (L_f^i h(\boldsymbol{x}) - y_{\text{ref}}^{(i)})}{L_g L_f^{r-1} h(\boldsymbol{x})} \tag{12.92}$$

Two single-input single-output applications of nonlinear geometric control performed in simulation with reference trajectories for realistic models are discussed in [Corriou, 2004]. The first one concerns a realistic continuous stirred tank reactor where a chemical reaction takes place. The second one concerns a fed-batch biological reactor. These reactors present different relative degrees. For each of them, an extended Kalman filter is used to estimate the states. An example of pilot application to a copolymerization reactor preceded by a study of dynamic optimization in order to obtain the optimal profile temperature is given by Gentric et al. [1999].

12.2.13 Decoupling with Respect to a Disturbance

We assume that a modelable disturbance d acts in an affine manner on the system thus reformulated

$$\begin{cases} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\, u + \boldsymbol{w}(\boldsymbol{x})\, d \\ y &= h(\boldsymbol{x}) \end{cases} \tag{12.93}$$

The aim is to define an input u by static state feedback such that the output y does not depend (is decoupled) on the disturbance d, at least theoretically. Considering $z_1 = y$, the derivative of the output can be expressed with the normal coordinates following (12.28)

$$\dot{z}_1 = L_f h(\boldsymbol{x}(t)) + L_g h(\boldsymbol{x}(t))\, u(t) + L_w h(\boldsymbol{x}(t))\, d(t) \tag{12.94}$$

Using the relative degree r of the system gives $L_g h = 0$, if $r > 1$, and it can be imposed that $L_w h = 0$ so that the disturbance has no influence. For the following coordinates, a similar condition is used

$$L_w L_f^{i-1} h = 0 \qquad , \quad 1 \leq i \leq r \tag{12.95}$$

hence the system in the normal form including the influence of the disturbance from rank $r+1$

$$\begin{array}{ll} \dot{z}_1 & = z_2 \\ & \vdots \\ \dot{z}_{r-1} & = z_r \\ \dot{z}_r & = L_f^r h(\boldsymbol{x}(t)) + L_g L_f^{r-1} h(\boldsymbol{x}(t))\, u(t) = b(\boldsymbol{\xi},\boldsymbol{\eta}) + a(\boldsymbol{\xi},\boldsymbol{\eta})\, u \\ \dot{\boldsymbol{\eta}} & = q(\boldsymbol{\xi},\boldsymbol{\eta}) + r(\boldsymbol{\xi},\boldsymbol{\eta})\, d \end{array} \tag{12.96}$$

with, moreover $y = z_1$. Clearly from the normal form (12.96), it follows that the input defined by the static state feedback

$$u = \frac{-b(\boldsymbol{\xi},\boldsymbol{\eta}) + v}{a(\boldsymbol{\xi},\boldsymbol{\eta})} \tag{12.97}$$

which gives

$$\dot{z}_r = v \tag{12.98}$$

perfectly decouples the output $y = z_1$ from the disturbance d.

But the decoupling condition is contained in the system of equations (12.95) which can be expressed as

$$\begin{array}{l} < L_f^{i-1} h, \boldsymbol{w} >= 0 \qquad , \quad 1 \le i \le r \Longleftrightarrow \\ \boldsymbol{w}(\boldsymbol{x}) \text{ belongs to the codistribution } \Delta^{\perp} \text{ in the neighbourhood of } \boldsymbol{x}^{\circ} \end{array} \tag{12.99}$$

where the distribution Δ is equal to

$$\Delta = \text{span}\{Dh, DL_f h, \ldots, DL_f^{r-1} h\} \tag{12.100}$$

The previous decoupling has been realized by state feedback. In some cases, the disturbance can be measured, therefore it is attractive to consider a feed-forward term in the control law, now formulated as

$$u(t) = \alpha(\boldsymbol{x}) + \beta(\boldsymbol{x})\, v + \gamma(\boldsymbol{x})\, d \tag{12.101}$$

hence the closed-loop system equations

$$\begin{array}{ll} \dot{\boldsymbol{x}} & = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})[\alpha(\boldsymbol{x}) + \beta(\boldsymbol{x})\, v] + [\boldsymbol{w}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\gamma(\boldsymbol{x})]\, d \\ y & = h(\boldsymbol{x}) \end{array} \tag{12.102}$$

The parallel to the decoupling by state feedback can be noticed. It suffices that

$$\begin{array}{l} < L_f^{i-1} h, \boldsymbol{w} + \boldsymbol{g}\gamma >= 0 \qquad , \quad 1 \le i \le r \qquad \Longleftrightarrow \\ [\boldsymbol{w}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\gamma(\boldsymbol{x})] \text{ belongs to the codistribution } \Delta^{\perp} \text{ in the neighbourhood of } \boldsymbol{x}^{\circ} \end{array} \tag{12.103}$$

This condition can be simplified by expressing the Lie bracket

$$\begin{array}{ll} < L_f^{i-1} h, \boldsymbol{w} + \boldsymbol{g}\gamma > & = L_{w+g\gamma} L_f^{i-1} h(\boldsymbol{x}) \\ & = L_w L_f^{i-1} h(\boldsymbol{x}) + L_g L_f^{i-1} h(\boldsymbol{x})\gamma(\boldsymbol{x}) = 0 \quad , \quad 1 \le i \le r \end{array} \tag{12.104}$$

yielding the expression of the function γ

$$\gamma(\boldsymbol{x}) = -\frac{L_w L_f^{r-1} h(\boldsymbol{x})}{L_g L_f^{r-1} h(\boldsymbol{x})} \tag{12.105}$$

which must be reinjected in control law (12.101).

12.2.14 Case of Nonminimum-Phase Systems

As previously described, the nonlinear control law (12.60) consisted essentially in inverting the nonlinear model of the system in a close way to the ideal controller of a linear system. However, for a linear system, the positive zeros become unstable poles and must be discarded from the effective controller like in internal model control. Rigorously and similarly, the nonlinear control law (12.60) cannot be applied to nonminimum-phase systems, as they are not invertible. Nevertheless, it is possible [Slotine and Li, 1991] to apply to them a control law that gives a small tracking error or to redefine the output so that the modified zero dynamics is stable. It requires that the perfect tracking of the newly defined output also leads to a good tracking of the actual controlled output. Some rare examples of control laws for unstable systems are available in the literature where [Kravaris and Daoutidis, 1990] study the case of second-order non-minimum phase systems or [Engell and Klatt, 1993] study an unstable CSTR. Clearly, nonlinear geometric control is not well adapted for these cases.

12.2.15 Globally Linearizing Control

This linearizing control of an input-output type proposed [Kravaris, 1988, Kravaris and Chung, 1987, Kravaris and Kantor, 1990a,b, Kravaris and Soroush, 1990] is based on the same concepts of differential geometry as those previously developed and can be considered as an extremely close variant. The control law applied to a minimum-phase system of relative degree r is

$$u = \frac{v - L_f^r h(\boldsymbol{x}) - \beta_1 L_f^{r-1} h(\boldsymbol{x}) - \cdots - \beta_{r-1} L_f h(\boldsymbol{x}) - \beta_r h(\boldsymbol{x})}{L_g L_f^{r-1} h(\boldsymbol{x})} \tag{12.106}$$

This is exactly Eq.(12.78) where the c_i's are replaced by β_i's. It gives the following input-output linear dynamics

$$y^{(r)} + \beta_1 \, y^{(r-1)} + \cdots + \beta_{r-1} \, y^{(1)} + \beta_r \, y = v \tag{12.107}$$

To guarantee a zero asymptotic error and improve robustness, i.e. in the presence of modelling errors and step disturbances (for a PI), the external input can be supplied by the following controller

$$v(t) = \int_0^t c(t-\tau) \, \left[y_{\text{ref}}(\tau) - y(\tau) \right] \, d\tau \tag{12.108}$$

where the function $c(t)$, for example, can be chosen as the inverse of a given transfer function. Frequently, v will be a PI controller (Fig. 12.6), e.g.

$$v(t) = K_c \left[y_{\text{ref}}(t) - y(t) + \frac{1}{\tau_i} \int_0^t (y_{\text{ref}}(\tau) - y(\tau))\, d\tau \right] \tag{12.109}$$

In this case, the system stability is conditioned by the roots of the characteristic polynomial

$$s^{r+1} + \beta_1\, s^r + \cdots + \beta_{r-1}\, s^2 + (\beta_r + K_c)\, s + \frac{K_c}{\tau_I} = 0 \tag{12.110}$$

where the roles of the nonlinear part and linear PI can be weighted in the choice of the parameters in the term $(\beta_r + K_c)$. This control has been tried out and extended under different forms such as combination with a feedforward controller [Daoutidis and Kravaris, 1989] to take into account disturbances, combination of a Smith predictor and a state observer to take into account time delays [Kravaris and Wright, 1989], with modified application to nonminimum-phase second-order systems [Kravaris and Daoutidis, 1990], application to a polymerization reactor [Soroush and Kravaris, 1992], with a robustness study [Kravaris and Palanki, 1988].

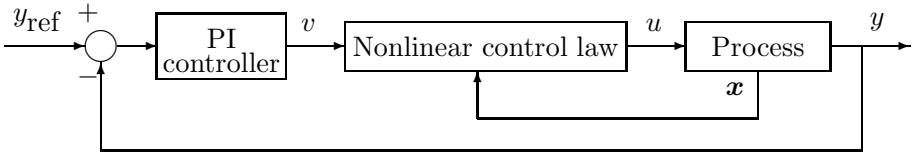

Figure 12.6: Globally Linearizing Control with PI controller

12.2.16 Generic model control

Generic model control [Lee, 1993] proposed by Lee and Sullivan [1988] has not been formulated by using the concepts of differential geometry. For a multi-input multi-output plant, it uses a different and general nonlinear model

$$\begin{cases} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}, t) \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}) \end{cases} \tag{12.111}$$

where the input does not appear any more in a linear way. It uses the expression of the derivative of the output as

$$\dot{\boldsymbol{y}} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\, \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}, t) = \boldsymbol{H}_{\boldsymbol{x}}\, \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}, t) \tag{12.112}$$

This derivative is compared to a reference that is an arbitrary function $\boldsymbol{r}^*$ so that $\dot{\boldsymbol{y}}^* = \boldsymbol{r}^*$ and

$$\dot{\boldsymbol{y}}^* = \boldsymbol{K}_1(\boldsymbol{y}^* - \boldsymbol{y}) + \boldsymbol{K}_2 \int (\boldsymbol{y}^* - \boldsymbol{y}) dt \tag{12.113}$$

where $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are constant matrices. After posing the control problem in an optimization setting, finally, using an approximate model $\tilde{\boldsymbol{f}}$ and $\tilde{\boldsymbol{h}}$, the problem is set as a system of equations

$$\tilde{\boldsymbol{H}}_{\boldsymbol{x}}\ \tilde{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d},t) - \boldsymbol{K}_1(\boldsymbol{y}^* - \boldsymbol{y}) - \boldsymbol{K}_2 \int (\boldsymbol{y}^* - \boldsymbol{y})dt = 0 \tag{12.114}$$

that [Lee and Sullivan, 1988] solve for a few cases.

Even if the system is not affine with respect to the input, it can be considered in the framework of nonlinear geometric control. [Lee and Sullivan, 1988] mention that when the relative degree is 1, an explicit solution is found. Indeed, it is based on a realization of the inverse of the model. For this reason, its applicability is limited [Henson and Seborg, 1990a], as it is reserved to systems of relative degree equal to 1.

Some authors [Bequette, 1991, Henson and Seborg, 1989, 1990b] compare globally linearizing control (GLC) and generic model control (GMC) to nonlinear control developed by differential geometry, as has been discussed by Isidori [1995]. GLC differs by nonlinear control only by the expression of the external input. GMC is much more different but presents some similarities.

12.3 Multivariable Nonlinear Control

Geometric nonlinear control up to now was presented for single-input single-output systems but it can be extended to multi-input multi-output systems. It will be presented only for square systems, i.e. with same number of inputs and outputs. Detailed demonstrations can be found, in particular, in the textbook by [Isidori, 1995] and only the main points will be cited here.

Multivariable nonlinear systems, affine with respect to the inputs, are modelled as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \sum_{i=1}^{m} g_i(\boldsymbol{x})\, u_i \\ y_i &= h_i(\boldsymbol{x}) \quad , \quad i = 1,\ldots,m \end{aligned} \tag{12.115}$$

with

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \quad ; \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \tag{12.116}$$

where $\boldsymbol{f}(\boldsymbol{x}), g_i(\boldsymbol{x}), h_i(\boldsymbol{x})$ are smooth vector fields. In a more compact form, the system is modelled by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\, \boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}) \end{aligned} \tag{12.117}$$

12.3.1 Relative Degree

The concepts of nonlinear geometric control for multivariable systems are an extension of the single-input single-output case. However, some of them are specific. In the multivariable case, the notion of relative degree is extended as

– The vector of relative degrees $[r_1, \dots r_m]^T$ defined in a neighbourhood of $\boldsymbol{x}^\circ$ by

$$L_{g_j} L_f^k h_i(\boldsymbol{x}) = 0 \quad , \quad \forall j = 1, \dots, m \quad , \quad \forall k < r_i - 1 \quad , \quad \forall i = 1, \dots, m \tag{12.118}$$

– The matrix $\boldsymbol{A}(\boldsymbol{x})$ defined by

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{bmatrix} L_{g_1} L_f^{r_1 - 1} h_1(\boldsymbol{x}) & \dots & L_{g_m} L_f^{r_1 - 1} h_1(\boldsymbol{x}) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m - 1} h_m(\boldsymbol{x}) & \dots & L_{g_m} L_f^{r_m - 1} h_m(\boldsymbol{x}) \end{bmatrix} \tag{12.119}$$

must be nonsingular at $\boldsymbol{x}^\circ$.

Given an output of subscript i, the vector of the Lie derivatives verifies

$$\begin{aligned} &\begin{bmatrix} L_{g_1} L_f^k h_i(\boldsymbol{x}) & \dots & L_{g_m} L_f^k h_i(\boldsymbol{x}) \end{bmatrix} = 0 \quad \forall k < r_i - 1 \\ &\begin{bmatrix} L_{g_1} L_f^{r_i - 1} h_i(\boldsymbol{x}) & \dots & L_{g_m} L_f^{r_i - 1} h_i(\boldsymbol{x}) \end{bmatrix} \neq 0 \end{aligned} \tag{12.120}$$

where r_i is the number of times that the output $y_i(t)$ must be differentiated to make at least one component of the vector $\boldsymbol{u}(t)$ appear. There exists at least one couple (u_j, y_i) having r_i as the relative degree.

Consider a system having $[r_1, \dots r_m]^T$ as the vector of relative degrees. The vectors

$$\begin{bmatrix} Dh_1(\boldsymbol{x}^\circ), & DL_f h_1(\boldsymbol{x}^\circ), & \dots, & DL_f^{r_1 - 1} h_1(\boldsymbol{x}^\circ) \end{bmatrix} \\ \vdots \\ \begin{bmatrix} Dh_m(\boldsymbol{x}^\circ), & DL_f h_m(\boldsymbol{x}^\circ), & \dots, & DL_f^{r_m - 1} h_m(\boldsymbol{x}^\circ) \end{bmatrix} \tag{12.121}$$

are linearly independent.

12.3.2 Coordinate Change

The proposed coordinate change for a multivariable system is exactly analogous to the single-input single-output case. Consider a system of relative degree vector $[r_1, \dots r_m]^T$. Let the coordinate change

$$\forall\, 1 \leq i \leq m \qquad \begin{aligned} \phi_1^i(\boldsymbol{x}) &= h_i(\boldsymbol{x}) \\ \phi_2^i(\boldsymbol{x}) &= L_f h_i(\boldsymbol{x}) \\ &\vdots \\ \phi_{r_i}^i(\boldsymbol{x}) &= L_f^{r_i - 1} h_i(\boldsymbol{x}) \end{aligned} \tag{12.122}$$

The total relative degree is denoted by $r = r_1 + \dots + r_m$.

If $r < n$, it is possible to find $(n - r)$ additional functions $\phi_{r+1}(\boldsymbol{x}), \dots, \phi_n(\boldsymbol{x})$ such that the mapping

$$\Phi(\boldsymbol{x}) = \left[\phi_1^1(\boldsymbol{x}), \dots, \phi_{r_1}^1(\boldsymbol{x}), \dots, \phi_1^m(\boldsymbol{x}), \dots, \phi_{r_m}^m(\boldsymbol{x}), \phi_{r+1}(\boldsymbol{x}), \dots, \phi_n(\boldsymbol{x})\right]^T \tag{12.123}$$

has its Jacobian matrix nonsingular at $\boldsymbol{x}^\circ$ and is a potential coordinate change in a neighbourhood U of $\boldsymbol{x}^\circ$.

Moreover, if the distribution

$$G = \text{span}\,\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_m\} \tag{12.124}$$

is involutive in U, the additional functions: $\phi_{r+1}(\boldsymbol{x}), \ldots, \phi_m(\boldsymbol{x})$ can be chosen such that

$$L_{g_j}\phi_i(\boldsymbol{x}) = 0 \quad , \quad r+1 \le i \le n \quad , \quad 1 \le j \le m \tag{12.125}$$

The condition of nonsingularity of matrix $\boldsymbol{A}(\boldsymbol{x})$ can be extended to a system having more inputs than outputs. It becomes a rank condition: the rank of the matrix must be equal to the number of its rows, and, furthermore, the system must have more inputs than outputs, which is usual.

12.3.3 Normal Form

The coordinate change gives for ϕ_i^1 (similarly for others ϕ_i^j)

$$\begin{aligned}
\dot{\phi}_1^1 &= \phi_2^1(\boldsymbol{x}) \\
&\vdots \\
\dot{\phi}_{r_1-1}^1 &= \phi_{r_1}^1(\boldsymbol{x}) \\
\dot{\phi}_{r_1-1}^1 &= L_f^{r_1} h_1(\boldsymbol{x}) + \sum_{j=1}^{m} L_{g_j} L_f^{r_1-1} h_1(\boldsymbol{x})\, u_j(t)
\end{aligned} \tag{12.126}$$

The normal coordinates are given by

$$\begin{aligned}
&\boldsymbol{\xi} = [\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^m] \quad \text{with:} \quad \boldsymbol{\xi}^i = \begin{bmatrix} \xi_1^i \\ \vdots \\ \xi_{r_i}^i \end{bmatrix} = \begin{bmatrix} \phi_1^i(\boldsymbol{x}) \\ \vdots \\ \phi_{r_i}^i(\boldsymbol{x}) \end{bmatrix}, i = 1, \ldots, m \\
&\boldsymbol{\eta} = \begin{bmatrix} \boldsymbol{\eta}_1 \\ \vdots \\ \boldsymbol{\eta}_{n-r} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\phi}_{r+1}(\boldsymbol{x}) \\ \vdots \\ \boldsymbol{\phi}_n(\boldsymbol{x}) \end{bmatrix}
\end{aligned} \tag{12.127}$$

which gives the normal form

$$\begin{aligned}
\dot{\xi}_1^i &= \xi_2^i \\
&\vdots \\
\dot{\xi}_{r_i-1}^i &= \xi_{r_i}^i \\
\dot{\xi}_{r_i}^i &= b_i(\boldsymbol{\xi}, \boldsymbol{\eta}) + \sum_{j=1}^{m} a_{ij}(\boldsymbol{\xi}, \boldsymbol{\eta})\, u_j \\
\dot{\boldsymbol{\eta}} &= q(\boldsymbol{\xi}, \boldsymbol{\eta}) + p(\boldsymbol{\xi}, \boldsymbol{\eta})\, u \\
y &= \xi_1^i
\end{aligned} \tag{12.128}$$

with

$$\begin{array}{ll} a_{ij}(\boldsymbol{\xi},\boldsymbol{\eta}) & = L_{g_j} L_f^{r_i-1} h_i(\Phi^{-1}(\boldsymbol{\xi},\boldsymbol{\eta})) \quad , \; i,j = 1,\ldots,m \\ b_i(\boldsymbol{\xi},\boldsymbol{\eta}) & = L_f^{r_i} h_i(\Phi^{-1}(\boldsymbol{\xi},\boldsymbol{\eta})) \quad , \; i = 1,\ldots,m \end{array} \tag{12.129}$$

where a_{ij} are the coefficients of matrix $\boldsymbol{A}(\boldsymbol{x})$.

12.3.4 Zero Dynamics

The zero dynamics is defined in the same way as for single-input single-output systems (section 12.2.10). The inputs and the initial conditions are sought so that the outputs are identically zero in a neighbourhood U of $\boldsymbol{x}^\circ$, implying $\boldsymbol{\xi} = 0$. The unique control vector results

$$\boldsymbol{u}(t) = -\boldsymbol{A}^{-1}(0,\boldsymbol{\eta}(t))\, \boldsymbol{b}(0,\boldsymbol{\eta}(t)) \tag{12.130}$$

and the zero dynamics or unforced dynamics is solution of the differential equations

$$\dot{\boldsymbol{\eta}}(t) = \boldsymbol{q}(\boldsymbol{\xi},\boldsymbol{\eta}) - \boldsymbol{p}(\boldsymbol{\xi},\boldsymbol{\eta})\, \boldsymbol{A}^{-1}(\boldsymbol{\xi},\boldsymbol{\eta})\, \boldsymbol{b}(\boldsymbol{\xi},\boldsymbol{\eta}) \qquad , \quad \boldsymbol{\eta}(0) = \boldsymbol{\eta}^\circ \tag{12.131}$$

The control (12.130) can also be expressed in the original state space as

$$\boldsymbol{u}^*(\boldsymbol{x}) = -\boldsymbol{A}^{-1}(\boldsymbol{x})\, \boldsymbol{b}(\boldsymbol{x}) \tag{12.132}$$

The change from the unforced dynamics to the forced dynamics would be realized in the same manner as for single-input single-output systems.

12.3.5 Exact Linearization by State Feedback and Diffeomorphism

Let the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\, \boldsymbol{u} \tag{12.133}$$

without considering the outputs.

The distributions are defined as

$$\begin{array}{ll} G_0 & = \text{span}\,\{g_1,\ldots,g_m\} \\ G_1 & = \text{span}\,\{g_1,\ldots,g_m, \text{ad}_f g_1,\ldots,\text{ad}_f g_m\} \\ & \vdots \\ G_i & = \text{span}\left\{\text{ad}_f^k g_1,\ldots,\text{ad}_f^k g_m \; ; 0 \le k \le i\right\} \qquad ; \quad i = 0,\ldots,n-1 \end{array} \tag{12.134}$$

The matrix $\boldsymbol{g}(\boldsymbol{x}^\circ)$ is assumed of rank m. The exact linearization is possible if and only if:

- The distribution G_i , $(i = 0,\ldots,n-1)$ has a constant dimension in the neighbourhood of $\boldsymbol{x}^\circ$.
- The distribution G_{n-1} has a dimension equal to n.
- The distribution G_i , $(i = 0,\ldots,n-2)$ is involutive.

It is necessary to check:

- the non-singularity of matrix $\boldsymbol{A}(\boldsymbol{x})$ (Eq. 12.119).
- the total relative degree r must be equal to n.
- as for single-input single-output systems, the outputs y_i are given by the solutions $\lambda_i(\boldsymbol{x})$, $(j = 1, \ldots, m)$ of the equations

$$L_{g_j} L_f^k \lambda_i(\boldsymbol{x}) = 0 \qquad , \quad 0 \leq k \leq r_i - 2 \ , \ j = 1, \ldots, m \tag{12.135}$$

Refering to (12.119) and (12.129), the linearizing state feedback is

$$\boldsymbol{u} = \boldsymbol{A}^{-1}(\boldsymbol{x}) \left(-\boldsymbol{b}(\boldsymbol{x}) + \boldsymbol{v} \right) \tag{12.136}$$

and the linearizing normal coordinates are

$$\xi_k^i(\boldsymbol{x}) = L_f^{k-1} h_i(\boldsymbol{x}) \qquad , \quad 1 \leq k \leq r_i \ , \ i = 1, \ldots, m \tag{12.137}$$

12.3.6 Nonlinear Control Perfectly Decoupled by Static State Feedback

The decoupling for the system (12.117) will be perfectly realized when any output y_i $(1 \leq i \leq m)$ is influenced only by the corresponding input v_i. This problem has a solution only if the decoupling matrix $\boldsymbol{A}(\boldsymbol{x})$ is nonsingular at $\boldsymbol{x}^\circ$, i.e. if the system possesses a vector of relative degrees.

The static state feedback is given by (12.136). To perform the decoupling, it suffices to consider the system in its normal form (12.128) and to propose the following control law

$$\boldsymbol{u} = -\boldsymbol{A}^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta}) \, \boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \boldsymbol{A}^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta}) \, \boldsymbol{v}) \tag{12.138}$$

which transforms the system into

$$\begin{aligned}
\dot{\xi}_1^i \quad &= \xi_2^i \\
&\vdots \\
\dot{\xi}_{r_i-1}^i \quad &= \xi_{r_i}^i \\
\dot{\xi}_{r_i}^i \quad &= b_i(\boldsymbol{\xi}, \boldsymbol{\eta}) + \sum_{j=1}^{m} a_{ij}(\boldsymbol{\xi}, \boldsymbol{\eta}) \, u_j \\
&= L_f^{r_i} h_i(\Phi^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})) + \sum_{j=1}^{m} L_{g_j} L_f^{r_i - 1} h_i(\Phi^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})) \, u_j \\
&= v_i \\
\dot{\boldsymbol{\eta}} \quad &= \boldsymbol{q}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \boldsymbol{p}(\boldsymbol{\xi}, \boldsymbol{\eta}) \, \boldsymbol{u} \\
y \quad &= \xi_1^i
\end{aligned} \tag{12.139}$$

as we could write

$$\begin{bmatrix} \dot{\xi}_{r_1}^1 \\ \cdots \\ \dot{\xi}_{r_m}^m \end{bmatrix} = \boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \boldsymbol{A}(\boldsymbol{\xi}, \boldsymbol{\eta}) \, \boldsymbol{u} = \boldsymbol{v} \tag{12.140}$$

from Eq. (12.129), giving the coefficients a_{ij}, the control law (12.138) and the normal form.

Two cases can take place:

- the total relative degree r is lower than n. In that case, there exists an unobservable part in the system, influenced by the inputs and the states, but not affecting the outputs.
- the total relative degree r is equal to n. Then, the system can be decoupled into m chains, each one composed of r_i integrators. The system is thus transformed into a completely linear and controllable system.

Evidently, it must be verified that the decoupling matrix $\boldsymbol{A}(\boldsymbol{x})$ is nonsingular at $\boldsymbol{x}^\circ$.

In the same way as for single-input single-output systems, pole placements can be performed by adding state feedbacks as

$$v_i = -c_0^i \xi_1^i - \cdots - c_{r_i-1}^i \xi_{r_i}^i \tag{12.141}$$

The different items treated for single-input single-output systems, i.e. asymptotic stability of the zero dynamics, disturbance rejection, reference model tracking, are studied in a very similar manner [Isidori, 1995].

Examples of application of multivariable nonlinear geometric control to plants are not many in the literature [Kravaris and Soroush, 1990, Soroush and Kravaris, 1992, 1993]. Among them, [To et al., 1995] describe nonlinear control of a simulated industrial evaporator by means of different techniques, including input-output linearization, Su-Hunt-Meyer transformation and generic model control. The process model possesses three states, two manipulated inputs and two controlled outputs.

12.3.7 Obtaining a Relative Degree by Dynamic Extension

It happens that some multivariable systems possess no relative degree vector. In this case, no static state feedback can change this result, as the relative degree property is independent of it.

Suppose that the nonlinear system (12.117) possesses no total relative degree, as the rank of matrix $\boldsymbol{A}(\boldsymbol{x})$ is smaller than m, number of inputs and outputs, for this square system. To obtain the relative degree, a dynamic part is added [Isidori, 1995] between the old inputs $\boldsymbol{u}$ and the new inputs $\boldsymbol{v}$, modelled by

$$\begin{aligned} \boldsymbol{u} &= \alpha(\boldsymbol{x}, \boldsymbol{\zeta}) + \beta(\boldsymbol{x}, \boldsymbol{\zeta})\, \boldsymbol{v} \\ \dot{\boldsymbol{\zeta}} &= \gamma(\boldsymbol{x}, \boldsymbol{\zeta}) + \delta(\boldsymbol{x}, \boldsymbol{\zeta})\, \boldsymbol{v} \end{aligned} \tag{12.142}$$

A simple type of dynamics used is the interposition of integrators between an input v_i and the input u_i (Fig. 12.7). In the case of two integrators, it is modelled by

$$\begin{aligned} u_i &= \zeta_1 \\ \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= v_i \end{aligned} \tag{12.143}$$

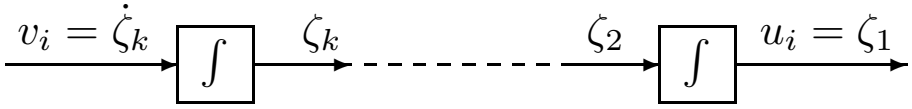

Figure 12.7: Dynamic extension

The modified system will be

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\,\alpha(\boldsymbol{x},\boldsymbol{\zeta}) + \boldsymbol{g}(\boldsymbol{x})\,\beta(\boldsymbol{x},\boldsymbol{\zeta})\,\boldsymbol{v} \\ \dot{\boldsymbol{\zeta}} &= \gamma(\boldsymbol{x},(\boldsymbol{\zeta}) + \delta(\boldsymbol{x},\boldsymbol{\zeta})\,\boldsymbol{v} \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}) \end{aligned} \tag{12.144}$$

The dynamic extension [Isidori, 1995] is performed according to an iterative procedure increasing the rank of matrix $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{\zeta})$ corresponding to the modified system until it is equal to m. This specific algorithm contains a procedure of identification of the inputs on which it is necessary to act. Thus, if the relative degree is obtained by dynamic extension and if the total relative degree of the extended system is equal to n, it is possible to assure that the original system can be transformed into a completely linear and controllable system by a dynamic state feedback and a coordinate change.

Such cases are rare in the literature. Soroush and Kravaris [1994] describe a continuous polymerization reactor where conversion and temperature are controlled by manipulating two coordinated flow rates and two coordinated heat inputs. According to this model, the characteristic matrix of the system is singular. The authors use a dynamic input-output linearizing state feedback by redefining the second input simply using the rate of change, i.e. the derivative, of monomer flow rate instead of the original monomer flow rate. This amounts to adding an integrator.

12.3.8 Nonlinear Adaptive Control

Frequently, the process model is uncertain or time-varying. Adaptive control was first developed for linear systems and the subject of many books [Aström, 1983, Aström and Wittenmark, 1989, Sastry and Bodson, 1989, Tao, 2003], in particular for generalized predictive control [Bitmead et al., 1990]. In most cases, the identification deals with parameters of the discrete-time models, often transfer functions, without physical signification [Goodwin and Sin, 1984, Haykin, 1991, Landau, 1990, Watanabe, 1992], posing serious problems of robustness [Ioannou and Sun, 1996]. However, it may occur that on-line identification of physical parameters influencing the model is necessary [Gustafsson and Waller, 1992, Seborg et al., 1986, Sung et al., 1998].

It has been extended to the nonlinear systems and nonlinear adaptive control is described in dedicated books [Krstić et al., 1995, Sastry and Bodson, 1989] and papers such as [Kosanovich et al., 1995, Wang et al., 1997, 1995].

A short description of a chemical process example is given in the following. The used technique is inspired from procedures of recursive identification pre-

sented by Sastry and Bodson [1989]. Wang et al. [1995] studied a batch styrene polymerization reactor. At the beginning of reaction, the extent of polymerization is still low and the viscosity of the reactor contents is close to that of the solvent. When the monomer conversion increases, the viscosity strongly increases because of the gel effect, and the heat transfer coefficient decreases significantly because the stirring progressively moves from a turbulent regime to a laminar one, creating reactor runaway hazard. The nonlinear model [Wang et al., 1995] describes these phenomena, which have been taken into account in the identification and nonlinear geometric control by using an augmented state vector. Besides the traditional states, which are concentrations and temperatures, the gel effect coefficient and the heat transfer coefficient are estimated during the reaction.

12.4 Nonlinear Multivariable Control of a Chemical Reactor

A continuous perfectly stirred chemical reactor [Corriou, 2004] is modelled with four states (x_1, concentration of A, x_2, reactor temperature, x_3, jacket temperature, x_4, liquid volume), two manipulated inputs corresponding to valve positions that allow to control the inlet temperature in the jacket $T_{j,in}$ and the feed concentration $C_{A,f}$ respectively, two controlled outputs, the concentration of A and the reactor temperature respectively. The nonlinear model of the reactor is:

$$\begin{aligned}
\dot{x}_1 &= \frac{F_f}{x_4}\left(C_{A,f} - x_1\right) - k\, x_1 \\
\dot{x}_2 &= \frac{F_f}{x_4}\left(T_f - x_2\right) - \frac{\Delta H\, k\, x_1}{\rho\, C_p} - \frac{UA(x_2 - x_3)}{\rho\, C_p\, x_4} \\
\dot{x}_3 &= \frac{F_j}{V_j}(T_{j,in} - x_2) + \frac{UA(x_2 - x_3)}{\rho_j\, C_{pj}\, V_j} \\
\dot{x}_4 &= F_f - F_{sp} + K_r\left(V_{sp} - x_4\right) \\
y_1 &= x_1 \\
y_2 &= x_2
\end{aligned} \tag{12.145}$$

with:

$$k = k_0\, \exp\left(-\frac{E_a}{R\, x_2}\right)\ ;\ C_{A,f} = u_1\, C_{A,0}\ ;\ T_{j,in} = u_2\, T_{hot} + (1 - u_2)\, T_{cold} \tag{12.146}$$

where T_{cold} and T_{hot} are the temperatures of two cold and hot heat exchangers assumed constant. The volume x_4 is controlled independently by a proportional controller. Both manipulated inputs are bounded in the interval $[0, 1]$.

The physical parameters of the chemical reactor are given in Table 12.1.

The nonlinear state-space model (12.145) is affine with respect to the inputs.

Table 12.1: Initial variables and main parameters of the CSTR

Flow rate of the feed	$F_f = F_{sp} = 3 \times 10^{-3}$ m^3.s^{-1}
Concentration of reactant A before the feed	$C_{A,0} = 3900$ mol.m^{-3}
Temperature of the feed	$T_f = 295$ K
Volume of reactor	$V(t=0) = V_{sp} = 1.5$ m^3
Kinetic constant	$k_0 = 2 \times 10^7$.s^{-1}
Activation energy	$E = 7 \times 10^4$ J.mol^{-1}
Heat of reaction	$\Delta H = -7 \times 10^4$ J.mol^{-1}
Density of reactor contents	$\rho = 1000$ kg.m^{-3}
Heat capacity of reactor contents	$C_p = 3000$ J.kg^{-1}.K^{-1}
Temperature of the cold heat exchanger	$T_c = 280$ K
Temperature of hot heat exchanger	$T_h = 360$ K
Flow rate of the heat-conducting fluid	$F_j = 5 \times 10^{-2}$ m^3.s^{-1}
Volume of jacket	$V_j = 0.1$ m^3
Heat transfer coefficient between the jacket and the reactor contents	$U = 900$ W.m^{-2}.K^{-1}
Heat-exchange area	$A = 20$ m^2
Density of the heat-conducting fluid	$\rho_j = 1000$ kg.m^{-3}
Heat capacity of the heat-conducting fluid	$C_{pj} = 4200$ J.kg^{-1}.K^{-1}
Proportional gain of the level controller	$K_r = 0.05$ s^{-1}

It is a square system that can be written as (12.147) exactly as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \sum_{i=1}^{2} \boldsymbol{g}_i(\boldsymbol{x})\, u_i \\ y_i &= h_i(\boldsymbol{x}) \quad , \quad i = 1, \ldots, 2 \end{aligned} \tag{12.147}$$

with the vector fields

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} -\dfrac{F_f}{x_4}\, x_1 - k\, x_1 \\ \dfrac{F_f}{x_4}\,(T_f - x_2) - \dfrac{\Delta H\, k\, x_1}{\rho\, C_p} - \dfrac{U A (x_2 - x_3)}{\rho\, C_p\, x_4} \\ \dfrac{F_j}{V_j}(T_{cold} - x_2) + \dfrac{U A (x_2 - x_3)}{\rho_j\, C_{pj}\, V_j} \\ F_f - F_{sp} + K_r\,(V_{sp} - x_4) \end{bmatrix} \tag{12.148}$$

$$\boldsymbol{g}_1(\boldsymbol{x}) = \begin{bmatrix} \dfrac{F_f}{x_4}\, C_{A,f} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad \boldsymbol{g}_2(\boldsymbol{x}) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{F_j}{V_j}(T_{hot} - T_{cold}) \\ 0 \end{bmatrix}$$

First, the relative degree of this system is to be determined. For that, the following Lie derivatives are calculated, beginning by considering the first output.

– The Lie derivative of $h_1(\boldsymbol{x})$ in the direction of the vector field f

$$L_f h_1(\boldsymbol{x}) = \sum_i \frac{\partial h_1}{\partial x_i} f_i(\boldsymbol{x}) = f_1(\boldsymbol{x}) \tag{12.149}$$

– Then $L_{g_1} L_f h_1$ is calculated

$$L_{g_1} L_f h_1 = L_{g_1} f_1 = \sum_i \frac{\partial f_1}{\partial x_i} g_{1,i} = \frac{\partial f_1}{\partial x_1} g_{1,1} \neq 0 \tag{12.150}$$

– Then $L_{g_2} L_f h_1$ is calculated

$$L_{g_2} L_f h_1 = L_{g_2} f_1 = \sum_i \frac{\partial f_1}{\partial x_i} g_{2,i} = \frac{\partial f_1}{\partial x_3} g_{2,3} = 0 \tag{12.151}$$

thus the vector

$$[L_{g_1} L_f h_1 \ , \ L_{g_2} L_f h_1] \neq 0 \tag{12.152}$$

hence $r_1 = 2$.

Now, the second output is concerned.

– The Lie derivative of $h_2(\boldsymbol{x})$ in the direction of the vector field f

$$L_f h_2(\boldsymbol{x}) = \sum_i \frac{\partial h_2}{\partial x_i} f_i(\boldsymbol{x}) = f_2(\boldsymbol{x}) \tag{12.153}$$

– Then $L_{g_1} L_f h_2$ is calculated

$$L_{g_1} L_f h_2 = L_{g_1} f_2 = \sum_i \frac{\partial f_2}{\partial x_i} g_{1,i} = \frac{\partial f_2}{\partial x_1} g_{1,1} \neq 0 \tag{12.154}$$

– Then $L_{g_2} L_f h_2$ is calculated

$$L_{g_2} L_f h_2 = L_{g_2} f_2 = \sum_i \frac{\partial f_2}{\partial x_i} g_{2,i} = \frac{\partial f_2}{\partial x_3} g_{2,3} \neq 0 \tag{12.155}$$

This implies that

$$[L_{g_1} L_f h_2 \ , \ L_{g_2} L_f h_2] \neq 0 \tag{12.156}$$

hence $r_2 = 2$.

The matrix $\boldsymbol{A}(\boldsymbol{\Phi}(x))$ is then equal to

$$\boldsymbol{A}(\boldsymbol{\Phi}(x)) = \begin{bmatrix} L_{g_1} L_f h_1 & L_{g_2} L_f h_1 \\ L_{g_1} L_f h_2 & L_{g_2} L_f h_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} g_{1,1} & 0 \\ \dfrac{\partial f_2}{\partial x_1} g_{1,1} & \dfrac{\partial f_2}{\partial x_3} g_{2,3} \end{bmatrix} \tag{12.157}$$

which is nonsingular for most $\boldsymbol{x}$ in the physical domain. The total relative degree is $r = r_1 + r_2 = 4$. Note that $r = n$, dimension of the system.

The proposed change of coordinates is

$$\begin{bmatrix} \phi_1^1(\boldsymbol{x}) = h_1(\boldsymbol{x}) = x_1 & \phi_1^2(\boldsymbol{x}) = h_2(\boldsymbol{x}) = x_2 \\ \phi_2^1(\boldsymbol{x}) = L_f h_1(\boldsymbol{x}) = f_1 & \phi_2^2(\boldsymbol{x}) = L_f h_2(\boldsymbol{x}) = f_2 \end{bmatrix} \tag{12.158}$$

giving

$$\Phi(\boldsymbol{x}) = [\phi_1^1\ ,\ \phi_2^1\ ,\ \phi_1^2\ ,\ \phi_2^2] \tag{12.159}$$

must have its Jacobian matrix non singular. It can be verified as this Jacobian matrix is equal to

$$\begin{bmatrix} 1 & \dfrac{\partial f_1}{\partial x_1} & 0 & \dfrac{\partial f_2}{\partial x_1} \\ 0 & \dfrac{\partial f_1}{\partial x_2} & 1 & \dfrac{\partial f_2}{\partial x_2} \\ 0 & 0 & 0 & \dfrac{\partial f_2}{\partial x_3} \\ 0 & \dfrac{\partial f_1}{\partial x_4} & 0 & \dfrac{\partial f_2}{\partial x_4} \end{bmatrix} \tag{12.160}$$

where the present partial derivatives are not zero.

The normal coordinates result

$$\boldsymbol{\xi}^1 = \begin{bmatrix} x_1 \\ L_f h_1 = f_1 \end{bmatrix} \quad ; \quad \boldsymbol{\xi}^2 = \begin{bmatrix} x_2 \\ L_f h_2 = f_2 \end{bmatrix} \tag{12.161}$$

and $\boldsymbol{\eta}$ has a null dimension. The normal form is then

$$\begin{aligned} \dot{\phi}_1^1(\boldsymbol{x}) &= \phi_2^1(\boldsymbol{x}) \\ \dot{\phi}_2^1(\boldsymbol{x}) &= \sum_i \frac{\partial f_1}{\partial x_i}\dot{x}_i = \frac{\partial f_1}{\partial x_1}(f_1 + g_{11}u_1) + \frac{\partial f_1}{\partial x_2}f_2 + \frac{\partial f_1}{\partial x_4}f_4 \\ \dot{\phi}_2^2(\boldsymbol{x}) &= \phi_2^2(\boldsymbol{x}) \\ \dot{\phi}_2^2(\boldsymbol{x}) &= \sum_i \frac{\partial f_2}{\partial x_i}\dot{x}_i = \frac{\partial f_2}{\partial x_1}(f_1 + g_{11}u_1) + \frac{\partial f_2}{\partial x_2}f_2 + \frac{\partial f_2}{\partial x_3}(f_3 + g_{23}u_2) + \frac{\partial f_2}{\partial x_4}f_4 \end{aligned} \tag{12.162}$$

Clearly, it gives

$$\begin{aligned} \begin{bmatrix} \dot{\phi}_2^1(\boldsymbol{x}) \\ \dot{\phi}_2^2(\boldsymbol{x}) \end{bmatrix} &= \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}f_1 + \dfrac{\partial f_1}{\partial x_2}f_2 + \dfrac{\partial f_1}{\partial x_4}f_4 \\ \dfrac{\partial f_2}{\partial x_1}f_1 + \dfrac{\partial f_2}{\partial x_2}f_2 + \dfrac{\partial f_2}{\partial x_3}f_3 + \dfrac{\partial f_2}{\partial x_4}f_4 \end{bmatrix} + \boldsymbol{A}(\boldsymbol{\Phi}(x)) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} L_f^2 h_1 \\ L_f^2 h_2 \end{bmatrix} + \boldsymbol{A}(\boldsymbol{\Phi}(x)) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \boldsymbol{b}(\boldsymbol{\Phi}(x)) + \boldsymbol{A}(\boldsymbol{\Phi}(x))\, \boldsymbol{u} \end{aligned} \tag{12.163}$$

where the fact that f_1 does not depend on x_3 is used. It results in agreement with Eq.(12.140) that the control vector law is equal to

$$\boldsymbol{u} = \boldsymbol{A}^{-1}(\boldsymbol{x})(-\boldsymbol{b} + \boldsymbol{v}) \tag{12.164}$$

where $\boldsymbol{v}$ is the external input vector which can be chosen according to pole placement for example or according to a PI vector. Taking into account reference trajectories, the multivariable control law becomes

$$\boldsymbol{u} = \boldsymbol{A}^{-1}(\boldsymbol{\xi}_{ref}(t)) \left[\boldsymbol{y}_{ref}^{(2)}(t) - \boldsymbol{b}(\boldsymbol{\xi}_{ref}(t)) \right] \tag{12.165}$$

Note that this control law does not depend on $\boldsymbol{\eta}$.
Numerical application:
A pole placement has been performed on each manipulated input to set the corresponding external input as a discretized PI controller under the velocity form

$$\begin{aligned} \boldsymbol{v}_{k+1} &= \boldsymbol{v}_k + K_c \left[(\boldsymbol{y}_{k+1}^{ref} - \boldsymbol{y}_{k+1}) - (\boldsymbol{y}_k^{ref} - \boldsymbol{y}_k) \right] \\ &+ \frac{K_c}{\tau_I} \frac{T_s}{2} \left[(\boldsymbol{y}_{k+1}^{ref} - \boldsymbol{y}_{k+1}) + (\boldsymbol{y}_k^{ref} - \boldsymbol{y}_k) \right] \end{aligned} \tag{12.166}$$

where k is the sampling instant (sampling period $T_s = 5$s), $\boldsymbol{y}^{ref}$ the reference trajectory, K_c the proportional gain, τ_I the integral time constant. To perform the pole placement [Corriou, 2004], the control law (12.165) was modified by taking the vector $\boldsymbol{b}$ as

$$\boldsymbol{b} = \begin{bmatrix} L_f^2 h_1 + c_1 L_f h_1 + c_2 (y_1 - y_1^{ref}) \\ L_f^2 h_2 + c_1 L_f h_2 + c_2 (y_2 - y_2^{ref}) \end{bmatrix} \tag{12.167}$$

Approximating the discretized PI controller as a continuous one results approximately in the following input-output dynamics

$$y_i^{(r)} + c_1 y_i^{(r-1)} + c_2 y_i = v_i \tag{12.168}$$

with a relative degree $r = 2$. This is only approximate as the control law (12.165) is indeed multivariable and the previous equation would only be valid for a SISO system. Taking into account the continuous PI controller, the corresponding characteristic equation for each input-output couple is

$$s^3 + c_1 s^2 + (c_2 + K_c)s + \frac{K_c}{\tau_I} = 0 \tag{12.169}$$

The parameters were chosen to satisfy the ITAE criterion as $K_c = 2.12\ 10^{-4}$, $\tau_I = 6.84$s, $c_1 = 0.055$, $c_2 = 1.91\ 10^{-3}$.

Recall that both manipulated inputs are constrained in the domain $[0, 1]$. By applying the modified control law (12.165), it appeared that the manipulated input u_1 corresponding mainly to the concentration control rapidly reached its constraint and remained at this value. Nonlinear geometric control is not designed for handling the constraints, opposite to model predictive control, so that, in actual problems, some adaptations must be considered. When looking numerically at the elements a' of matrix $\boldsymbol{A}^{-1}$, the element a'_{11} is very large, a'_{12} is zero, a'_{21} is small, a'_{22} is medium. The multiplication of a'_{11} by b results in a large numerical value, explaining that the contraint for u_1 is hit. Physically,

this is due to the strong influence of temperature on the concentration in particular because of the Arrhenius term in the kinetic constant and also the heat of reaction. This is not the case for u_2 for which the nonlinear control law (12.165) can be implemented with only the introduction of c_i's. Thus, a modification of u_1 is performed by maintaining the term b_1 of (12.167), but changing the factor a'_{11} in a way close to an anti-windup to avoid saturation. This resulted in the controlled concentration of Figure 12.8. The control of temperature where coupling intervenes posed little problem. The manipulated inputs (Figure 12.9) are not saturated, although the input u_1 changes abruptly after $t = 2000$s when a concentration set point variation is imposed. The nonlinear control law (12.165) perfectly decoupled the output y_1 from the influence of u_2 but the output y_2 is influenced by both manipulated inputs. This coupling is hardly visible on Figure 12.9 only because of the values of the respective elements a'_{21} and a'_{22} where a'_{22} is much larger than a'_{21}. The previous study was performed in simulation and all the states were assumed to be known. With little additional difficulty, it would be possible to estimate the states using a nonlinear observer such as the extended Kalman filter [Corriou, 2004] or another one, assuming that the reactor temperature and the concentration are measured. Then the estimated states are used in the nonlinear control law. Furthermore, a robustness study taking into account the unavoidable uncertainty of some physical parameters would be useful.

In conclusion, the nonlinear multivariable control of the chemical reactor is possible but some adaptations were necessary because of the physical constraints of the manipulated inputs. Compared to model predictive control, the multivariable nonlinear geometric control appears more difficult to implement, but shows real efficiency. For SISO plants, nonlinear geometric control is generally easier to use.

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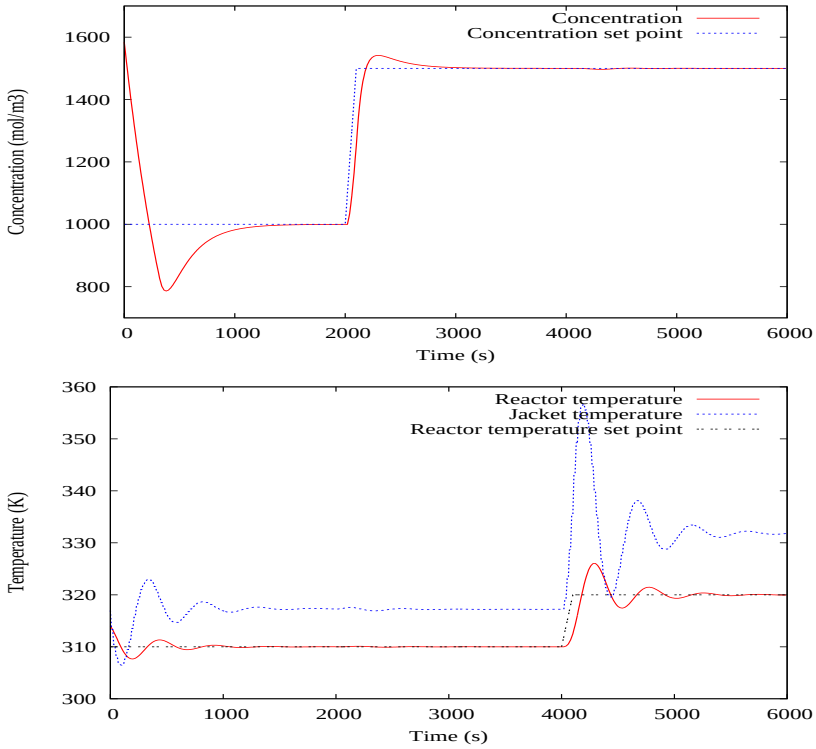

Figure 12.8: Nonlinear geometric control of chemical reactor. Top: Concentration. Bottom: Reactor temperature.

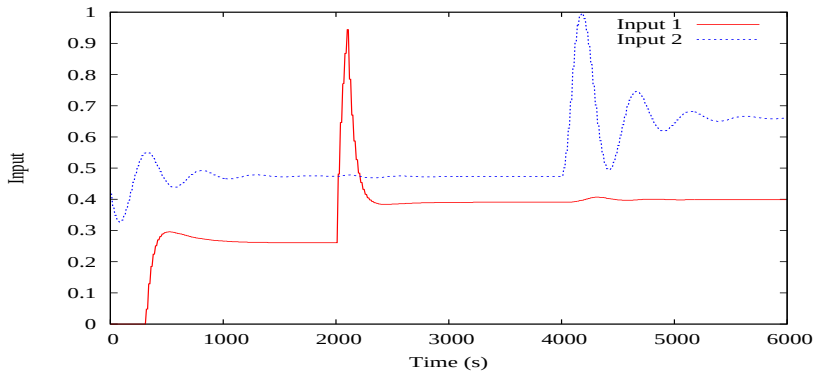

Figure 12.9: Nonlinear geometric control of chemical reactor. Manipulated inputs.

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