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## ► To cite this version:

Roman Bezrukavnikov, Simon Riche. Hecke action on the principal block. *Compositio Mathematica*, In press. hal-02946323v1

**HAL Id: hal-02946323**

**<https://hal.science/hal-02946323v1>**

Submitted on 23 Sep 2020 (v1), last revised 2 Jun 2022 (v2)

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# HECKE ACTION ON THE PRINCIPAL BLOCK

ROMAN BEZRUKAVNIKOV AND SIMON RICHE

ABSTRACT. In this paper we construct an action of the affine Hecke category on the principal block of representations of a simply-connected semisimple algebraic group over an algebraically closed field of characteristic bigger than the Coxeter number. This confirms a conjecture of G. Williamson and the second author, and provides a new proof of the tilting character formula in terms of antispherical  $p$ -Kazhdan–Lusztig polynomials.

## 1. INTRODUCTION

**1.1. Representation theory of reductive algebraic groups and the Hecke category.** Let  $G$  be a connected reductive algebraic group over an algebraically closed field  $\mathbb{k}$  of characteristic  $p$  (assumed to be larger than the Coxeter number of  $G$ ). In [RW1], G. Williamson and the second author of the present paper started advocating the idea that the combinatorics of the category  $\text{Rep}(G)$  of finite-dimensional algebraic  $G$ -modules (e.g. character formulas for simple and indecomposable tilting  $G$ -modules) should be expressible in terms of the  $p$ -Kazhdan–Lusztig combinatorics, introduced a few years before by G. Williamson (partly in collaboration, see [JMW, JeW]). In this paper it was in particular observed that a concrete incarnation of this idea (a character formula for indecomposable tilting modules in the principal block, in terms of antispherical  $p$ -Kazhdan–Lusztig polynomials) was a consequence of a simple “categorical” conjecture stating that the wall-crossing functors define an action of the Hecke category of the associated affine Weyl group on the principal block of  $G$ . This conjecture was motivated in particular by the philosophy of categorical action of Lie algebras; it was proved in [RW1, Part II] in the special case when  $G = \text{GL}_n(\mathbb{k})$  for some  $n$ , using the machinery of 2-Kac–Moody algebras [Ro].

Later the “combinatorial” consequence of this conjecture (the tilting character formula) was proved for general  $G$  (in fact, in two very different ways, see [AMRW2] and [RW2]), but these proofs use other tools, and none of them imply the original categorical conjecture. The main result of the present paper is a construction of this Hecke category action. In particular, this provides a third general proof of the tilting character formula. This new proof does not involve constructible sheaves in any way; it uses coherent sheaves, but mostly over affine schemes, and hence can be considered essentially algebraic.

**1.2. Localization for Harish-Chandra bimodules.** The category  $\text{Rep}(G)$  can be naturally seen as a full subcategory of the category of  $G$ -equivariant  $\mathcal{U}\mathfrak{g}$ -modules via differentiation, where  $\mathcal{U}\mathfrak{g}$  is the enveloping algebra of the Lie algebra  $\mathfrak{g}$  of  $G$ . As such it admits an action of the monoidal category of Harish-Chandra  $\mathcal{U}\mathfrak{g}$ -bimodules, and the wall-crossing functors (and, more generally, the translation functors) can be described as the action of some specific (completed) such bimodules. To construct

the desired action it therefore suffices to construct a monoidal functor from the Hecke category to an appropriate category of completed Harish-Chandra bimodules.

The main tool we will use for this is a localization theory for Harish-Chandra bimodules. Recall that  $\mathcal{U}\mathfrak{g}$  possesses a large central subalgebra  $Z_{\text{Fr}}$  called the *Frobenius center* and isomorphic to  $\mathcal{O}(\mathfrak{g}^{*(1)})$  (where the superscript (1) denotes Frobenius twist). Even though in the end we are interested in  $G$ -modules, which when seen as  $\mathcal{U}\mathfrak{g}$ -modules have a *trivial* Frobenius central character, we will localize our bimodules on the *regular part* of  $Z_{\text{Fr}}$ , and more precisely on a Kostant section  $\mathcal{S}^{*(1)} \subset \mathfrak{g}^{*(1)}$  to the (co-)adjoint quotient. This is made possible by the fact that the appropriate restriction to such a Kostant section is fully faithful on the “diagonally induced” Harish-Chandra bimodules that describe translation functors. Hence the bimodules we are interested in can be studied using bimodules over  $\mathcal{U}\mathfrak{S}\mathfrak{g} := \mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{O}(\mathcal{S}^{*(1)})$ , equivariant with respect to a group scheme over  $\mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)$  constructed out of the universal centralizer group scheme over  $\mathfrak{t}^{*(1)}/W$ . (Here  $\mathfrak{t}$  is the Lie algebra of a maximal torus in  $G$ ,  $W$  is the Weyl group, and the map  $\mathfrak{t}^*/(W, \bullet) \rightarrow \mathfrak{t}^{*(1)}/W$  is induced by the Artin–Schreier map.)

It is a classical observation that  $\mathcal{U}\mathfrak{S}\mathfrak{g} := \mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{O}(\mathcal{S}^{*(1)})$  is an Azumaya algebra over its center. This property is not sufficient for our purposes since the two actions of this center on Harish-Chandra bimodules do *not* coincide; however by using bimodules realizing translation to and from the “most singular” Harish-Chandra character (namely,  $-\rho$ ), we construct in Section 3 an equivariant splitting bundle for each completion of the algebra  $\mathcal{U}\mathfrak{S}\mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} \mathcal{U}\mathfrak{S}\mathfrak{g}^{\text{op}}$  at an ideal corresponding to a pair of weights in the lower closure of the fundamental alcove. As a consequence, we obtain equivalences of categories between equivariant modules over these algebras and representations of certain completions of the involved group scheme.

A general theory of localization for modules over  $\mathcal{U}\mathfrak{g}$  has been developed by the first author with Mirković and Rumynin, see [BMR1, BMR2, BM]. The localization that we require is however slightly different, and the present paper does not rely on the results of [BMR1, BMR2, BM]. One difference is that we are interested in *bimodules* and not modules, which are in addition equivariant for the diagonal  $G$ -action. Some of the constructions in [BMR1, BMR2, BM] (in particular, the non-canonicity of the choice of splitting bundle) make it difficult to use in an equivariant setting, and our construction is different. Finally, as explained above we only need to consider the *regular part* of the Frobenius center, which simplifies the situation a lot, and in particular allows us to work completely at the level of *abelian* categories, without having to consider the more involved *derived* categories.

### 1.3. The Hecke category and representations of the universal centralizer.

The other crucial ingredient of our proof is a new incarnation of the Hecke category (for any Coxeter system  $(W, S)$ ) recently found by Abe [Ab1]. The Hecke category is a categorification of the Hecke algebra of  $(W, S)$ , depending on a choice of extra data (comprising a representation  $V$  of  $W$ ). Its original definition in terms of Soergel bimodules [S3] applies to “reflection faithful” representations of Coxeter systems, which include natural examples of representations over fields of characteristic 0 (e.g. geometric representations of finite Coxeter systems and representations appearing in the theory of Kac–Moody Lie algebras for crystallographic Coxeter systems), but does *not* include important examples over fields of positive characteristic (e.g. some natural representations of affine Weyl groups of reductive groups). Under this assumption Soergel bimodules can be defined as a full subcategory of

the category of graded bimodules over the polynomial algebra  $\mathcal{O}(V)$ . More recently Elias and Williamson [EW] have proposed a definition of the Hecke category in terms of generators and relations which applies (and behaves as one might expect) in a much greater generality, encompassing the representation of the affine Weyl group that we require. It is in terms of this construction that the conjecture in [RW1] was stated. (For more on the Hecke category, see also [Wi, JeW].)

*Remark 1.1.* In the case of Coxeter systems associated with Kac–Moody groups (i.e. the *crystallographic* Coxeter systems), and for representations appearing naturally in this theory over a field of characteristic 0, it is well known that the category of Soergel bimodules can also be described in terms of constructible sheaves on the corresponding flag variety. (The fact that Kazhdan–Lusztig combinatorics in this case is related to constructible sheaves on the flag variety is a fundamental observation of Kazhdan–Lusztig [KL1]. The equivalence of categories with Soergel bimodules is due to Soergel [S2] in the case of finite crystallographic groups—using the earlier definition of Soergel bimodules in this case in [S1]—and to Härterich for Kac–Moody groups [Hä].) A similar result relating the Hecke category of [EW] to constructible sheaves on the associated flag variety (which holds also for coefficients in a field of positive characteristic) can be found in [RW1, Part III]. Although important for some other purposes, this realization of the Hecke category will not play any role in the present paper.

The main drawback of this construction, however, is that it is much less concrete than Soergel’s original definition, and does not involve  $\mathcal{O}(V)$ -bimodules. This drawback is exactly compensated by Abe’s work; under a minor technical assumption he proves in [Ab1] that the category of Elias and Williamson identifies with a category of “enhanced Soergel bimodules,” i.e. graded bimodules over  $\mathcal{O}(V)$  together with a decomposition of its tensor product (on the left) with  $\text{Frac}(\mathcal{O}(V))$  parametrized by  $W$ .

Based on Abe’s work, in the case of the affine Weyl group acting on  $\mathbb{X} \otimes_{\mathbb{Z}} \mathbb{k}$  (where  $\mathbb{X}$  is the character lattice) through the natural action of the finite Weyl group, we realize the Hecke category as a full subcategory in coherent sheaves on the preimage of the Kostant slice in the (Frobenius twist of the) Steinberg variety of triples,<sup>1</sup> identified with  $\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)}$ , equivariant with respect to the pullback of the universal centralizer. This construction allows us to define a monoidal functor from the Hecke category to the category of representations considered in §1.2, and then to the category of completed equivariant  $\mathcal{U}_{\mathfrak{S}\mathfrak{g}}$ -bimodules. (This construction applies when the central character corresponds to a point  $(\lambda, \lambda)$  with  $\lambda$  in the fundamental alcove; in this case natural étale maps allow us to identify the completions of the schemes  $\mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)$  and  $\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)}$  at the images of  $(\lambda, \lambda)$ .)

*Remark 1.2.* Although the concrete incarnation of this idea that is relevant in the present paper is new, the fact that affine Soergel bimodules are closely related with representations of the universal centralizer was already known: it dates back (at least) to [Do]; see also [MR] for an adaptation of these ideas to positive characteristic coefficients.

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<sup>1</sup>Here, by Steinberg variety of triples we mean the fiber product of two copies of the Grothendieck resolution over the dual of the Lie algebra, and not the version involving the Springer resolution.

At this point, to conclude our proof it only remains to show that our functor sends the objects of the Hecke category labelled by simple reflections to the bimodules realizing the wall-crossing functors. In the case when the simple reflection belongs to the finite Weyl group  $W$ , this can be checked explicitly, using localization at a character involving a weight on the corresponding wall of the fundamental alcove. The general case is reduced to this one using a standard trick (used e.g. in [R2, BM]), based on the observation that in the *extended* affine Weyl group each simple reflection is conjugate to a simple reflection which belongs to  $W$ . The concrete proof involves the study of an analogue of the affine braid group action from [BR] in our present context; in this case the situation simplifies however (once again because we work over the regular part of the Frobenius center) and this action in fact factors through an action of the extended affine Weyl group.

*Remark 1.3.* One of the motivations for Abe’s work [Ab1] was an attempt to prove the conjecture in [RW1]. What he was actually able to construct is rather an action on the principal block of the category of  $G_1T$ -modules, which is less useful; see [Ab2].

**1.4. Towards a coherent realization of the Hecke category.** Thanks to work of Kazhdan–Lusztig [KL2] and Ginzburg [CG], it is known that the Hecke algebra of the affine Weyl group identifies with the Grothendieck group of the category of equivariant coherent sheaves on the Steinberg variety of triples. The construction outlined in §1.3 can be seen to provide a fully faithful monoidal functor from the Hecke category to the category of equivariant coherent sheaves on the *regular part* of the Steinberg variety. In later work we will upgrade this construction to a fully faithful monoidal functor to the category of equivariant coherent sheaves on the whole Steinberg variety. This construction will be part of our project (joint with L. Rider) of constructing a modular version of the equivalence constructed by the first author in [Be]; see [BRR] for a first step towards this goal.

**1.5. Contents.** In Section 2 we introduce the categories of completed Harish-Chandra bimodules we will work with, and prove that restriction to a Kostant section is fully faithful on an appropriate subcategory. In Section 3 we develop our localization theory for Harish-Chandra bimodules. In Section 4 we prove (for later use) some technical results using the relation between  $\mathcal{U}\mathfrak{g}$  and differential operators on the flag variety. In Section 5 we recall Abe’s results, and use them to construct our monoidal functor from the Hecke category to the appropriate category of representations of the universal centralizer. Finally, in Section 6 we prove the main result of the paper, i.e. we construct the Hecke action on the principal block and prove that objects associated with simple reflections act via wall-crossing functors.

**1.6. Acknowledgements.** We are given to understand that J. Ciappara has recently proved a result similar to our main theorem via a completely different method. His proof relies on the Smith–Treumann theory of [RW2], and is still in preparation. We thank G. Williamson for keeping us informed of this work, and for useful comments.

R.B. was supported by NSF Grant No. DMS-1601953. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (S.R., grant agreement No. 677147).

The project realized in this paper was conceived during the workshop *Modular Representation Theory* organized in Oxford by the Clay Mathematical Institute. We thank the organizers of this conference (G. Williamson, I. Losev and M. Emerton) for the fruitful working atmosphere they have created, and the participants for the inspiring talks. A later part of this work was accomplished at the Institut Henri Poincaré, as part of the thematic program *Representation Theory* organized by D. Hernandez, N. Jacon, E. Letellier, S. R., and É. Vasserot.

## 2. SOME CATEGORIES OF EQUIVARIANT $\mathcal{U}\mathfrak{g}$ -BIMODULES

**2.1. Weights.** We fix an algebraically closed field  $\mathbb{k}$  of characteristic  $p > 0$ , and a simply connected semisimple algebraic group  $G$  over  $\mathbb{k}$ . We will denote by  $\mathfrak{g}$  the Lie algebra of  $G$ . We also choose a Borel subgroup  $B \subset G$  and a maximal torus  $T \subset B$ , and denote by  $\mathfrak{b}$  and  $\mathfrak{t}$  their respective Lie algebras. Let  $U$  be the unipotent radical of  $B$ ,  $\mathfrak{n}$  be its Lie algebra, and  $W$  be the Weyl group of  $(G, T)$ . We set  $\mathbb{X} := X^*(T)$ , and denote by  $\mathfrak{R} \subset \mathbb{X}$  the root system of  $(G, T)$ . The choice of  $B$  determines a system of positive roots  $\mathfrak{R}^+ \subset \mathfrak{R}$ , chosen as the  $T$ -weights in  $\mathfrak{g}/\mathfrak{b}$ . We will denote by  $\mathfrak{R}^s \subset \mathfrak{R}$  the corresponding subset of simple roots, and by  $\rho \in \mathbb{X}$  the halfsum of the positive roots. We also set  $\mathbb{X}^\vee := X_*(T)$ , and denote by  $\mathfrak{R}^\vee \subset \mathbb{X}^\vee$  the coroot system. The canonical bijection  $\mathfrak{R} \xrightarrow{\sim} \mathfrak{R}^\vee$  will be denoted as usual  $\alpha \mapsto \alpha^\vee$ .

The *affine Weyl group* associated with  $G$  is the semi-direct product

$$W_{\text{aff}} := W \ltimes p\mathbb{Z}\mathfrak{R}$$

where  $\mathbb{Z}\mathfrak{R} \subset \mathbb{X}$  is the lattice generated by  $\mathfrak{R}$ , and the  $W$ -action on  $p\mathbb{Z}\mathfrak{R}$  is induced by the natural action on  $\mathbb{X}$ . The group  $W_{\text{aff}}$  is a normal subgroup in the *extended affine Weyl group*

$$W_{\text{ext}} := W \ltimes p\mathbb{X}.$$

Given  $\mu \in p\mathbb{X}$ , we will denote by  $t_\mu$  the associated element of  $W_{\text{ext}}$ . It is well known that the group  $W_{\text{aff}}$  is generated by the subset  $S_{\text{aff}}$  consisting of the reflections  $s_\alpha$  with  $\alpha \in \mathfrak{R}^s$ , together with the products  $t_{p\beta}s_\beta$  where  $\beta \in \mathfrak{R}$  is such that  $\beta^\vee$  is a maximal coroot. Moreover, the pair  $(W_{\text{aff}}, S_{\text{aff}})$  is a Coxeter system, see [J2, §II.6.3]. We will consider the “dot” action of  $W_{\text{ext}}$  (or its subgroup  $W_{\text{aff}}$ ) on  $\mathbb{X}$  defined by

$$(t_\mu w) \bullet \lambda = w(\lambda + \rho) - \rho + \mu$$

for  $\mu \in p\mathbb{X}$ ,  $w \in W$  and  $\lambda \in \mathbb{X}$ .

Given a character  $\lambda \in \mathbb{X}$ , we will denote by  $\bar{\lambda} \in \mathfrak{t}^*$  the differential of  $\lambda$ . We will also set

$$\mathfrak{t}_{\mathbb{Z}}^* := \{\bar{\lambda} : \lambda \in \mathbb{X}\} \subset \mathfrak{t}^*.$$

In this way, the map  $\lambda \mapsto \bar{\lambda}$  induces an isomorphism of abelian groups

$$\mathbb{X}/p\mathbb{X} \xrightarrow{\sim} \mathfrak{t}_{\mathbb{Z}}^*.$$

(In particular,  $\mathfrak{t}_{\mathbb{Z}}^*$  is finite.)

The group  $W$  naturally acts on  $\mathfrak{t}^*$ . We also have a “dot” action of  $W$  on  $\mathfrak{t}^*$ , defined by

$$w \bullet \xi := w(\xi + \bar{\rho}) - \bar{\rho}.$$

With this definition the map  $\mathbb{X} \rightarrow \mathfrak{t}^*$  sending  $\lambda$  to  $\bar{\lambda}$  is  $W_{\text{ext}}$ -equivariant, where  $W_{\text{ext}}$  acts on  $\mathbb{X}$  via the dot-action and on  $\mathfrak{t}^*$  via the projection  $W_{\text{ext}} \rightarrow W$  and the dot-action of  $W$  on  $\mathfrak{t}^*$ . This observation legitimates the use of the same notation for these actions. It also shows that the subset  $\mathfrak{t}_{\mathbb{Z}}^* \subset \mathfrak{t}^*$  is stable under the dot-action

of  $W$ . Below we will also consider the quotient  $\mathfrak{t}^*/(W, \bullet)$  of the dot-action of  $W$  on  $\mathfrak{t}^*$ . For  $\lambda \in \mathbb{X}$ , we will denote by  $\bar{\lambda}$  the image of  $\lambda$  in  $\mathfrak{t}^*/(W, \bullet)$ .

We will assume throughout the paper that  $p$  is very good for  $G$ . This assumption implies that the quotient  $\mathbb{X}/\mathbb{Z}\mathfrak{R}$  has no  $p$ -torsion, or in other words that

$$(2.1) \quad \mathbb{Z}\mathfrak{R} \cap p\mathbb{X} = p\mathbb{Z}\mathfrak{R}.$$

This equality has the following consequences.

**Lemma 2.1.** *Let  $\lambda \in \mathbb{X}$ .*

(1) *We have*

$$W_{\text{aff}} \bullet \lambda = (W_{\text{ext}} \bullet \lambda) \cap (\lambda + \mathbb{Z}\mathfrak{R}).$$

(2) *The stabilizer of  $\bar{\lambda}$  for the dot-action of  $W$  on  $\mathfrak{t}^*$  is the image under the natural surjection  $W_{\text{aff}} \rightarrow W$  of the stabilizer of  $\lambda$  for the dot-action of  $W_{\text{aff}}$  on  $\mathbb{X}$ .*

*Proof.* (1) Since  $W \bullet \lambda \subset \lambda + \mathbb{Z}\mathfrak{R}$ , we have

$$(W_{\text{ext}} \bullet \lambda) \cap (\lambda + \mathbb{Z}\mathfrak{R}) = (W \bullet \lambda + p\mathbb{X}) \cap (\lambda + \mathbb{Z}\mathfrak{R}) = W \bullet \lambda + (\mathbb{Z}\mathfrak{R} \cap p\mathbb{X}).$$

The claim follows, in view of (2.1).

(2) For  $w \in W$  we have

$$w \bullet \bar{\lambda} = \overline{w \bullet \lambda},$$

so that  $w \bullet \bar{\lambda} = \bar{\lambda}$  iff  $w \bullet \lambda \in \lambda + p\mathbb{X}$ . Since  $w \bullet \lambda \in \lambda + \mathbb{Z}\mathfrak{R}$ , as above this condition is equivalent to  $w \bullet \lambda \in \lambda + p\mathbb{Z}\mathfrak{R}$ , i.e. to the existence of  $\mu \in p\mathbb{Z}\mathfrak{R}$  such that  $t_\mu w \in W_{\text{aff}}$  stabilizes  $\lambda$ .  $\square$

For any subset  $I \subset \mathfrak{R}^s$ , we will denote by  $W_I \subset W$  the subgroup generated by the reflections  $\{s_\alpha : \alpha \in I\}$ . Recall that an element of  $\mathbb{X}$  is called *regular* if its stabilizer in  $W_{\text{aff}}$  (for the dot-action) is trivial. As a consequence of Lemma 2.1, we obtain in particular the following claim.

**Lemma 2.2.** *Let  $\lambda \in \mathbb{X}$ , and assume that the stabilizer of  $\lambda$  for the dot-action of  $W_{\text{aff}}$  is  $W_I$ . Then the morphism*

$$\mathfrak{t}^*/(W_I, \bullet) \rightarrow \mathfrak{t}^*/(W, \bullet)$$

*induced by the quotient morphism  $\mathfrak{t}^* \rightarrow \mathfrak{t}^*/(W, \bullet)$  is étale at the image of  $\bar{\lambda}$ . In particular, if  $\lambda$  is regular then the quotient morphism  $\mathfrak{t}^* \rightarrow \mathfrak{t}^*/(W, \bullet)$  is étale at  $\bar{\lambda}$ .*

*Proof.* By Lemma 2.1(2), the stabilizer of  $\bar{\lambda}$  for the dot-action of  $W$  on  $\mathfrak{t}^*$  is  $W_I$ . Hence the claim follows from the general criterion [SGA1, Exp. V, Proposition 2.2].  $\square$

**2.2. The center of the enveloping algebra.** Consider the universal enveloping algebra  $\mathcal{U}\mathfrak{g}$  of  $\mathfrak{g}$ . Its center  $Z(\mathcal{U}\mathfrak{g})$  can be described as follows. First we set

$$Z_{\text{HC}} := (\mathcal{U}\mathfrak{g})^G.$$

(Here, the subscript “HC” stands for Harish-Chandra.) Next, as the Lie algebra of an algebraic group over a field of characteristic  $p$ ,  $\mathfrak{g}$  admits a “restricted  $p$ -th power” operation  $x \mapsto x^{[p]}$ , which stabilizes the Lie algebra of any algebraic subgroup of  $G$ . We will denote by

$$Z_{\text{Fr}}$$

the  $\mathbb{k}$ -subalgebra of  $\mathcal{U}\mathfrak{g}$  generated by the elements of the form  $x^p - x^{[p]}$  for  $x \in \mathfrak{g}$ . Then by [MR, Theorem 2] multiplication induces an isomorphism

$$Z_{\text{Fr}} \otimes_{Z_{\text{Fr}} \cap Z_{\text{HC}}} Z_{\text{HC}} \xrightarrow{\sim} Z(\mathcal{U}\mathfrak{g}).$$

It is well known that  $\mathcal{U}\mathfrak{g}$  is finite as a  $Z_{\text{Fr}}$ -algebra (hence a fortiori as a  $Z(\mathcal{U}\mathfrak{g})$ -algebra).

These central subalgebras can be described geometrically. Namely, for a  $\mathbb{k}$ -scheme  $X$  we will denote by  $X^{(1)}$  the associated Frobenius twist, defined as the fiber product  $X^{(1)} := X \times_{\text{Spec}(\mathbb{k})} \text{Spec}(\mathbb{k})$ , where the morphism  $\text{Spec}(\mathbb{k}) \rightarrow \text{Spec}(\mathbb{k})$  is associated with the map  $x \mapsto x^p$ . (The projection  $X^{(1)} \rightarrow X$  is an isomorphism of abstract schemes, but not of  $\mathbb{k}$ -schemes.) With this notation, it is well known that the map  $x \mapsto x^p - x^{[p]}$  induces a  $\mathbb{k}$ -algebra isomorphism

$$(2.2) \quad \mathcal{O}(\mathfrak{g}^{*(1)}) \xrightarrow{\sim} Z_{\text{Fr}}.$$

We also have  $Z_{\text{Fr}} \cap Z_{\text{HC}} = (Z_{\text{Fr}})^G$ , and the  $G$ -action on  $\mathfrak{g}^{*(1)}$  factors through the Frobenius morphism  $G \rightarrow G^{(1)}$ , so that we obtain an isomorphism

$$\mathcal{O}(\mathfrak{g}^{*(1)}/G^{(1)}) \xrightarrow{\sim} Z_{\text{Fr}} \cap Z_{\text{HC}}.$$

On the other hand, the ‘‘Harish-Chandra isomorphism’’ provides a  $\mathbb{k}$ -algebra isomorphism

$$(2.3) \quad \mathcal{O}(\mathfrak{t}^*/(W, \bullet)) \xrightarrow{\sim} Z_{\text{HC}},$$

see [MR, Theorem 1(2)].

The *Artin-Schreier morphism*

$$\text{AS} : \mathfrak{t}^* \rightarrow \mathfrak{t}^{*(1)}$$

is the morphism associated with the algebra map  $\mathcal{O}(\mathfrak{t}^{*(1)}) \rightarrow \mathcal{O}(\mathfrak{t}^*)$  defined by  $h \mapsto h^p - h^{[p]}$  for  $h \in \mathfrak{t}$ . It is well known that AS is a Galois covering with Galois group  $\mathbb{Z}$  (acting on  $\mathfrak{t}^*$  via addition). The morphism AS is  $W$ -equivariant, where  $W$  acts on  $\mathfrak{t}^*$  via the dot-action and on  $\mathfrak{t}^{*(1)}$  via the natural action. It therefore induces a morphism

$$\mathfrak{t}^*/(W, \bullet) \rightarrow \mathfrak{t}^{*(1)}/W.$$

Recall the Chevalley isomorphism

$$\mathfrak{t}^{*(1)}/W \xrightarrow{\sim} \mathfrak{g}^{*(1)}/G^{(1)},$$

see [MR, Theorem 1(3)]. Under this identification, the embedding  $Z_{\text{Fr}} \cap Z_{\text{HC}} \hookrightarrow Z_{\text{HC}}$  is induced by the morphism  $\mathfrak{t}^*/(W, \bullet) \rightarrow \mathfrak{t}^{*(1)}/W$  considered above.

Combining all these descriptions, and setting

$$\mathfrak{C} := \mathfrak{g}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet),$$

we therefore obtain a  $\mathbb{k}$ -algebra isomorphism

$$\mathcal{O}(\mathfrak{C}) \xrightarrow{\sim} Z(\mathcal{U}\mathfrak{g}),$$

see [MR, Corollary 3].

Using this identification one can consider  $\mathcal{U}\mathfrak{g}$  as an  $\mathcal{O}(\mathfrak{C})$ -algebra. The  $G$ -action on  $\mathfrak{C}$  induced by the adjoint  $G$ -action on  $\mathcal{U}\mathfrak{g}$  is the action obtained by pullback via the Frobenius morphism  $G \rightarrow G^{(1)}$  of the  $G^{(1)}$ -action on  $\mathfrak{C}$  induced by the coadjoint  $G^{(1)}$ -action on  $\mathfrak{g}^{*(1)}$ . Using this action, one can therefore see  $\mathcal{U}\mathfrak{g}$  as a  $G$ -equivariant  $\mathcal{O}(\mathfrak{C})$ -algebra.



**2.3. Central reductions.** In view of (2.2), the maximal ideals in  $Z_{\text{Fr}}$  are in a canonical bijection with elements in  $\mathfrak{g}^{*(1)}$ . Given  $\eta \in \mathfrak{g}^{*(1)}$ , we will denote by  $\mathfrak{m}_\eta \subset Z_{\text{Fr}}$  the corresponding maximal ideal, and set

$$\mathcal{U}_\eta \mathfrak{g} := \mathcal{U} \mathfrak{g} / \mathfrak{m}_\eta \cdot \mathcal{U} \mathfrak{g}.$$

Similarly, in view of (2.3) the maximal ideals in  $Z_{\text{HC}}$  are in a canonical bijection with closed points in  $\mathfrak{t}^*/(W, \bullet)$ , i.e. with  $(W, \bullet)$ -orbits in  $\mathfrak{t}^*$ . Given a closed point  $\xi \in \mathfrak{t}^*/(W, \bullet)$ , we will denote by  $\mathfrak{m}_\xi \subset Z_{\text{HC}}$  the corresponding maximal ideal, and set

$$\mathcal{U}^\xi \mathfrak{g} := \mathcal{U} \mathfrak{g} / \mathfrak{m}_\xi \cdot \mathcal{U} \mathfrak{g}.$$

If  $\eta$  and  $\xi$  have the same image in  $\mathfrak{t}^{*(1)}/W$ , then  $\mathfrak{m}_\eta \cdot Z(\mathcal{U} \mathfrak{g}) + \mathfrak{m}_\xi \cdot Z(\mathcal{U} \mathfrak{g})$  is a maximal ideal in  $Z(\mathcal{U} \mathfrak{g})$ , and we can also set

$$\mathcal{U}_\eta^\xi \mathfrak{g} := \mathcal{U} \mathfrak{g} / (\mathfrak{m}_\eta \cdot \mathcal{U} \mathfrak{g} + \mathfrak{m}_\xi \cdot \mathcal{U} \mathfrak{g}).$$

In the cases we will encounter more specifically below, the point  $\xi$  will often be the image  $\tilde{\lambda}$  of the differential of a character  $\lambda \in \mathbb{X}$ . In this setting we will write  $\mathfrak{m}^\lambda$ ,  $\mathcal{U}^\lambda \mathfrak{g}$  and  $\mathcal{U}_\eta^\lambda \mathfrak{g}$  instead of  $\mathfrak{m}^{\tilde{\lambda}}$ ,  $\mathcal{U}^{\tilde{\lambda}} \mathfrak{g}$  and  $\mathcal{U}_\eta^{\tilde{\lambda}} \mathfrak{g}$ . The image of any element of  $\mathfrak{t}_\mathbb{Z}^*$  under the Artin–Schreier map is 0; therefore, if we denote by

$$\mathcal{N}^* \subset \mathfrak{g}^*$$

the preimage of the image of 0 under the coadjoint morphism  $\mathfrak{g}^* \rightarrow \mathfrak{t}^*/W$ , then given any  $\lambda \in \mathbb{X}$  the elements  $\eta \in \mathfrak{g}^{*(1)}$  whose image in  $\mathfrak{t}^{*(1)}/W$  coincides with that of  $\tilde{\lambda}$  are exactly those in  $\mathcal{N}^{*(1)}$ .

**2.4. Harish-Chandra bimodules.** We will denote by  $\text{HC}$  the category whose objects are the  $\mathcal{U} \mathfrak{g}$ -bimodules  $V$  endowed with an (algebraic) action of  $G$  which satisfy the following conditions:

- (1) the action morphisms  $\mathcal{U} \mathfrak{g} \otimes V \rightarrow V$  and  $V \otimes \mathcal{U} \mathfrak{g} \rightarrow V$  are  $G$ -equivariant (for the diagonal actions on  $\mathcal{U} \mathfrak{g} \otimes V$  and  $V \otimes \mathcal{U} \mathfrak{g}$ );
- (2) the  $\mathfrak{g}$ -action on  $V$  obtained by differentiating the  $G$ -action is given by  $(x, v) \mapsto x \cdot v - v \cdot x$ ;
- (3)  $V$  is finitely generated both as a left and as a right  $\mathcal{U} \mathfrak{g}$ -module.

Morphisms in the category  $\text{HC}$  are morphisms of bimodules which also commute with the  $G$ -actions. Objects in this category are called Harish-Chandra bimodules. It is easily seen that the tensor product  $\otimes_{\mathcal{U} \mathfrak{g}}$  of bimodules endows  $\text{HC}$  with the structure of a monoidal category, where the  $G$ -action on the tensor product is the diagonal action.

If  $M$  belongs to  $\text{HC}$ , the  $\mathcal{U} \mathfrak{g}$ -action obtained by differentiating the  $G$ -action must vanish on  $Z_{\text{Fr}} \cap (\mathfrak{g} \cdot \mathcal{U} \mathfrak{g})$ . This implies that the two actions of  $Z_{\text{Fr}}$  on  $M$  obtained by restriction of the left and right  $\mathcal{U} \mathfrak{g}$ -actions coincide; in other words, the action of  $\mathcal{U} \mathfrak{g} \otimes_{\mathbb{k}} \mathcal{U} \mathfrak{g}^{\text{op}}$  on  $M$  must factor through an action of  $\mathcal{U} \mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U} \mathfrak{g}^{\text{op}}$ . However, the two actions of  $Z_{\text{HC}}$  on a Harish-Chandra bimodule might differ. Note that  $\mathcal{U} \mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U} \mathfrak{g}^{\text{op}}$  is in a natural way a finite algebra over the commutative ring

$$\mathcal{Z} := Z(\mathcal{U} \mathfrak{g}) \otimes_{Z_{\text{Fr}}} Z(\mathcal{U} \mathfrak{g}) = Z_{\text{HC}} \otimes_{Z_{\text{Fr}} \cap Z_{\text{HC}}} Z_{\text{Fr}} \otimes_{Z_{\text{Fr}} \cap Z_{\text{HC}}} Z_{\text{HC}} \cong \mathcal{O}(\mathfrak{C} \times_{\mathfrak{g}^{*(1)}} \mathfrak{C}).$$

Note also that since  $\mathcal{U} \mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U} \mathfrak{g}^{\text{op}}$  is finitely generated both as a left and as a right  $\mathcal{U} \mathfrak{g}$ -module, the third condition in the definition of Harish-Chandra bimodules can be equivalently replaced by the condition that the object is finitely generated as a  $\mathcal{U} \mathfrak{g}$ -bimodule.

Let us denote by  $\text{Mod}_{\text{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}})$  the category of  $G$ -equivariant finitely generated (left) modules over  $\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}$ . As above, since  $\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}$  is finitely generated both as a left and as a right  $\mathcal{U}\mathfrak{g}$ -module, the tensor product  $\otimes_{\mathcal{U}\mathfrak{g}}$  endows this category with a monoidal structure. In this way we obtain a fully faithful monoidal functor

$$\text{HC} \rightarrow \text{Mod}_{\text{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}).$$

One can construct interesting objects in  $\text{HC}$  from  $G$ -modules as follows. Let us denote by  $\text{Rep}(G)$  the category of finite-dimensional algebraic  $G$ -modules. Then given  $V$  in  $\text{Rep}(G)$  we consider the Harish-Chandra bimodule

$$V \otimes \mathcal{U}\mathfrak{g},$$

where the left  $\mathcal{U}\mathfrak{g}$ -action is diagonal (with respect to the action on  $V$  obtained by differentiation, and the action on  $\mathcal{U}\mathfrak{g}$  by left multiplication), the right  $\mathcal{U}\mathfrak{g}$ -action is induced by right multiplication on  $\mathcal{U}\mathfrak{g}$ , and the  $G$ -action is diagonal (with respect to the given action on  $V$  and the adjoint action on  $\mathcal{U}\mathfrak{g}$ ). In particular, for  $x, y, z \in \mathcal{U}\mathfrak{g}$  and  $v \in V$  we have

$$x \cdot (v \otimes z) \cdot y = (x_{(1)} \cdot v) \otimes (x_{(2)}zy),$$

where we use Sweedler's notation for the comultiplication in the Hopf algebra  $\mathcal{U}\mathfrak{g}$ . It is easily seen that the map  $(x \otimes y) \otimes v \mapsto (x_{(1)} \cdot v) \otimes (x_{(2)}y)$  induces an isomorphism

$$(\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}^{\text{op}}) \otimes_{\mathcal{U}\mathfrak{g}} V \xrightarrow{\sim} V \otimes \mathcal{U}\mathfrak{g},$$

where the tensor product on the left-hand side is taken with respect to the morphism  $\mathcal{U}\mathfrak{g} \rightarrow \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}^{\text{op}}$  defined by  $x \mapsto x_{(1)} \otimes S(x_{(2)})$ , where  $S$  is the antipode. In particular, the modules  $V \otimes \mathcal{U}\mathfrak{g}$  are "induced from the diagonal." For  $V, V'$  in  $\text{Rep}(G)$ , we have a canonical isomorphism

$$(2.4) \quad (V \otimes \mathcal{U}\mathfrak{g}) \otimes_{\mathcal{U}\mathfrak{g}} (V' \otimes \mathcal{U}\mathfrak{g}) \xrightarrow{\sim} (V \otimes V') \otimes \mathcal{U}\mathfrak{g}.$$

One can similarly consider, again for  $V$  in  $\text{Rep}(G)$ , the Harish-Chandra bimodule

$$\mathcal{U}\mathfrak{g} \otimes V$$

where now the actions of  $\mathcal{U}\mathfrak{g}$  are defined by

$$x \cdot (z \otimes v) \cdot y = (xzy_{(1)}) \otimes (S(y_{(2)}) \cdot v)$$

for  $x, y, z \in \mathcal{U}\mathfrak{g}$  and  $v \in V$  (and the  $G$ -action is still diagonal). As above we have an isomorphism

$$(\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}^{\text{op}}) \otimes_{\mathcal{U}\mathfrak{g}} V \xrightarrow{\sim} \mathcal{U}\mathfrak{g} \otimes V,$$

now given by  $(x \otimes y) \otimes v \mapsto (xy_{(1)}) \otimes (S(y_{(2)}) \cdot v)$ . In particular, the objects  $V \otimes \mathcal{U}\mathfrak{g}$  and  $\mathcal{U}\mathfrak{g} \otimes V$  are isomorphic; explicitly the isomorphism is given by

$$v \otimes x \mapsto x \otimes (S(x) \cdot v).$$

**2.5. Completed Harish-Chandra bimodules.** Now, we need to adapt the considerations of §2.4 to the setting of completed Harish-Chandra characters.

Recall from §2.3 that to each  $\lambda \in \mathbb{X}$  we have associated a maximal ideal  $\mathfrak{m}^\lambda \subset Z_{\text{HC}}$ . For  $\lambda, \mu \in \mathbb{X}$ , we will denote by

$$\mathcal{U}^{\hat{\lambda}, \hat{\mu}}$$

the completion of the  $\mathcal{Z}$ -algebra  $\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}$  with respect to the ideal

$$\mathcal{I}^{\lambda, \mu} := (\mathfrak{m}^\lambda \cdot Z(\mathcal{U}\mathfrak{g})) \otimes_{Z_{\text{Fr}}} Z(\mathcal{U}\mathfrak{g}) + Z(\mathcal{U}\mathfrak{g}) \otimes_{Z_{\text{Fr}}} (\mathfrak{m}^\mu \cdot Z(\mathcal{U}\mathfrak{g})) \subset \mathcal{Z};$$

in other words we have

$$\mathcal{U}^{\hat{\lambda}, \hat{\mu}} = \varprojlim_{n \geq 1} (\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}) / ((\mathcal{I}^{\lambda, \mu})^n \cdot (\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}})).$$

The natural (algebraic) action of  $G$  on  $\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}$  induces an (algebraic) action on each quotient

$$\mathcal{U}^{\hat{\lambda}, \hat{\mu}} / (\mathcal{I}^{\lambda, \mu})^n \cdot \mathcal{U}^{\hat{\lambda}, \hat{\mu}} = (\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}) / ((\mathcal{I}^{\lambda, \mu})^n \cdot (\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}})).$$

If we denote by  $\mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$  the completion of  $\mathcal{Z}$  at  $\mathcal{I}^{\lambda, \mu}$  then  $\mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$  is a Noetherian ring, complete with respect to the ideal  $\mathcal{I}^{\lambda, \mu} \cdot \mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$ , see [SP, Tag 05GH]. Since  $\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}$  is finite as a  $\mathcal{Z}$ -module the natural morphism

$$(2.5) \quad \mathcal{Z}^{\hat{\lambda}, \hat{\mu}} \otimes_{\mathcal{Z}} (\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}) \rightarrow \mathcal{U}^{\hat{\lambda}, \hat{\mu}}$$

is an isomorphism, see [SP, Tag 00MA]. In particular, the  $\mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$ -algebra  $\mathcal{U}^{\hat{\lambda}, \hat{\mu}}$  is finite; therefore any finitely generated  $\mathcal{U}^{\hat{\lambda}, \hat{\mu}}$ -module is also finitely generated as a  $\mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$ -module, which implies that the natural morphism

$$M \rightarrow \varprojlim_{n \geq 1} M / ((\mathcal{I}^{\lambda, \mu})^n \cdot M)$$

is an isomorphism, see again [SP, Tag 00MA]. Adapting the definition above, we define a  $G$ -equivariant finitely generated  $\mathcal{U}^{\hat{\lambda}, \hat{\mu}}$ -module as a finitely generated  $\mathcal{U}^{\hat{\lambda}, \hat{\mu}}$ -module  $M$  together with the datum of an algebraic  $G$ -module structure on each quotient  $M / ((\mathcal{I}^{\lambda, \mu})^n \cdot M)$  such that the natural surjection

$$M / ((\mathcal{I}^{\lambda, \mu})^{n+1} \cdot M) \rightarrow M / ((\mathcal{I}^{\lambda, \mu})^n \cdot M)$$

and the action map

$$(\mathcal{U}^{\hat{\lambda}, \hat{\mu}} / (\mathcal{I}^{\lambda, \mu})^n \cdot \mathcal{U}^{\hat{\lambda}, \hat{\mu}}) \otimes (M / ((\mathcal{I}^{\lambda, \mu})^n \cdot M)) \rightarrow M / ((\mathcal{I}^{\lambda, \mu})^n \cdot M)$$

are morphisms of  $G$ -modules for any  $n \geq 1$ . These modules are naturally objects of an abelian category, which will be denoted  $\mathbf{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}})$ . More specifically, in view of [EGA1, Chap. 0, Corollaire 7.2.10], for any finitely generated  $\mathcal{U}^{\hat{\lambda}, \hat{\mu}}$ -modules  $M, N$  we have

$$\text{Hom}_{\mathcal{U}^{\hat{\lambda}, \hat{\mu}}}(M, N) \cong \varprojlim_{n \geq 1} \text{Hom}_{\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}}(M / (\mathcal{I}^{\lambda, \mu})^n \cdot M, N / (\mathcal{I}^{\lambda, \mu})^n \cdot N).$$

If  $M, N$  are  $G$ -equivariant finitely generated  $\mathcal{U}^{\hat{\lambda}, \hat{\mu}}$ -modules, then we set

$$(2.6) \quad \text{Hom}_{\mathbf{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}})}(M, N) = \varprojlim_{n \geq 1} \text{Hom}_{\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}}(M / (\mathcal{I}^{\lambda, \mu})^n \cdot M, N / (\mathcal{I}^{\lambda, \mu})^n \cdot N)^G,$$

which we endow with the obvious composition law.

From this definition we see that the forgetful functor from  $\mathbf{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}})$  to the category of finitely generated  $\mathcal{U}^{\hat{\lambda}, \hat{\mu}}$ -modules is faithful. This functor is exact. More precisely, if  $M, N$  belong to  $\mathbf{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}})$  and if  $f \in \text{Hom}_{\mathbf{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}})}(M, N)$ , then the cokernel  $C$  of  $f$  (seen as a morphism of  $\mathcal{U}^{\hat{\lambda}, \hat{\mu}}$ -module) is such that  $C / (\mathcal{I}^{\lambda, \mu})^n \cdot C$  is the cokernel of the morphism  $M / (\mathcal{I}^{\lambda, \mu})^n \cdot M \rightarrow N / (\mathcal{I}^{\lambda, \mu})^n \cdot N$  induced by  $f$ , hence admits a natural structure of algebraic  $G$ -module. On the other hand, the kernel  $K$

of  $f$  admits an action of  $G$ , seen as an abstract group, since we have such actions on  $M$  and  $N$  obtained by taking the projective limits of the actions on  $M/(\mathcal{I}^{\lambda,\mu})^n \cdot M$  and  $N/(\mathcal{I}^{\lambda,\mu})^n \cdot N$ . By the Artin–Rees lemma, given  $n \geq 1$ , for  $m \gg n$  we have a ( $G$ -equivariant) surjection

$$K/(K \cap ((\mathcal{I}^{\lambda,\mu})^m \cdot M)) \rightarrow K/(\mathcal{I}^{\lambda,\mu})^n \cdot K,$$

which shows that the action on  $K/(\mathcal{I}^{\lambda,\mu})^n \cdot K$  is algebraic, since  $K/(K \cap ((\mathcal{I}^{\lambda,\mu})^m \cdot M))$  is a submodule of the algebraic  $G$ -module  $M/((\mathcal{I}^{\lambda,\mu})^m \cdot M)$ .

The full subcategory of  $\text{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda},\hat{\mu}})$  whose objects are the modules  $M$  such that  $M/(\mathcal{I}^{\lambda,\mu})^n \cdot M$  is a Harish-Chandra bimodule for any  $n$  will be denoted  $\text{HC}^{\hat{\lambda},\hat{\mu}}$ . Objects of this subcategory will be called Harish-Chandra  $\mathcal{U}^{\hat{\lambda},\hat{\mu}}$ -modules.

**2.6. Completion of diagonally induced bimodules.** Given  $\lambda, \mu \in \mathbb{X}$ , we have a natural exact functor

$$\mathcal{C}^{\lambda,\mu} : \text{Mod}_{\text{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}) \rightarrow \text{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda},\hat{\mu}}),$$

which sends a  $\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}$ -module  $M$  to its completion with respect to  $\mathcal{I}^{\lambda,\mu}$ . This functor restricts to a functor from  $\text{HC}$  to  $\text{HC}^{\hat{\lambda},\hat{\mu}}$ . We will denote by

$$\text{HC}_{\text{diag}}^{\hat{\lambda},\hat{\mu}}$$

the full additive subcategory of  $\text{HC}^{\hat{\lambda},\hat{\mu}}$  whose objects are direct summands of objects of the form  $\mathcal{C}^{\lambda,\mu}(V \otimes \mathcal{U}\mathfrak{g})$  with  $V$  in  $\text{Rep}(G)$ .

In case  $\lambda = \mu$ , we will set

$$\mathcal{U}^{\hat{\lambda}} = \mathcal{C}^{\lambda,\lambda}(\mathbb{k} \otimes \mathcal{U}\mathfrak{g}),$$

where here  $\mathbb{k}$  is the trivial  $G$ -module.

For later use, we also introduce some completed bimodules which are closely related to the translation functors for  $G$ -modules (see §6.3 below for details). Recall that a weight  $\lambda \in \mathbb{X}$  is said to belong to the *fundamental alcove*, resp. to the *closure of the fundamental alcove*, if it satisfies

$$0 < \langle \lambda + \rho, \alpha^\vee \rangle < p, \quad \text{resp.} \quad 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq p,$$

for any positive root  $\alpha$ . With this notation, the set of weights which belong to the closure of the fundamental alcove is a fundamental domain for the  $(W_{\text{aff}}, \bullet)$ -action on  $\mathbb{X}$ . Moreover, if  $\lambda \in \mathbb{X}$  belongs to the closure of the fundamental alcove, then its stabilizer in  $W_{\text{aff}}$  is the parabolic subgroup generated by the elements  $s \in S_{\text{aff}}$  such that  $s \bullet \lambda = \lambda$ ; see [J2, §II.6.3].

Let  $\mathbb{X}^+ \subset \mathbb{X}$  be the subset of dominant weights determined by  $\mathfrak{R}^+$ . For any  $\nu \in \mathbb{X}^+$ , we will denote by  $\text{L}(\nu)$  the simple  $G$ -module with highest weight  $\nu$ .

Given two weights  $\lambda, \mu \in \mathbb{X}$  which belong to the closure of the fundamental alcove, we set

$$\mathbb{P}^{\lambda,\mu} := \mathcal{C}^{\lambda,\mu}(\text{L}(\nu) \otimes \mathcal{U}\mathfrak{g}) \in \text{HC}_{\text{diag}}^{\hat{\lambda},\hat{\mu}},$$

where  $\nu$  is the unique dominant  $W$ -translate of  $\lambda - \mu$ .

**2.7. Comparison of completions.** For notational simplicity, let us now fix a subset  $\Lambda \subset \mathbb{X}$  such that the map  $\lambda \mapsto \tilde{\lambda}$  restricts to a bijection  $\Lambda \xrightarrow{\sim} \mathfrak{t}_{\mathbb{Z}}^*(W, \bullet)$ .

We will denote by  $\mathcal{I} \subset \mathcal{O}(\mathfrak{t}^{*(1)}/W) = Z_{\text{HC}} \cap Z_{\text{Fr}}$  the maximal ideal corresponding to the image of  $0 \in \mathfrak{t}^{*(1)}$ . Then  $\mathcal{I} \cdot Z_{\text{Fr}}$  is the ideal of definition of  $\mathcal{N}^{*(1)} \subset \mathfrak{g}^{*(1)}$ , and each ideal  $\mathcal{I}^{\lambda, \mu}$  contains  $\mathcal{I} \cdot \mathcal{Z}$ . We will denote by  $\mathcal{Z}^\wedge$  the completion of  $\mathcal{Z}$  with respect to the ideal  $\mathcal{I} \cdot \mathcal{Z}$ .

**Lemma 2.3.** *The natural morphism*

$$\mathcal{Z}^\wedge \rightarrow \prod_{\lambda, \mu \in \Lambda} \mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$$

*is a ring isomorphism.*

*Proof.* The morphism considered in this statement is the product of the morphisms  $\mathcal{Z}^\wedge \rightarrow \mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$  induced by the natural morphisms  $\mathcal{Z}/(\mathcal{I}^n \cdot \mathcal{Z}) \rightarrow \mathcal{Z}/(\mathcal{I}^{\lambda, \mu})^n$ . This morphism is clearly a ring morphism; to prove that it is invertible we will construct its inverse.

Let us fix some  $n \geq 1$ , and consider the quotient

$$(Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}})/\mathcal{I}^n \cdot (Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}}).$$

Here  $Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}}$  is a finite  $\mathcal{O}(\mathfrak{t}^{*(1)}/W)$ -module (since  $Z_{\text{HC}}$  is finite, as a submodule of the finite module  $\mathcal{O}(\mathfrak{t}^*)$ ); therefore this algebra is finite-dimensional. Its maximal ideals are in bijection with the maximal ideals of  $Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}}$  containing  $\mathcal{I} \cdot (Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}})$ , hence with  $\Lambda \times \Lambda$  through

$$(\lambda, \mu) \mapsto (\mathfrak{m}^\lambda \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}} + Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} \mathfrak{m}^\mu)/\mathcal{I}^n \cdot (Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}}).$$

In view of the general theory of Artin rings (see e.g. [AM, Chap. 8]), for any  $\lambda, \mu \in \Lambda$  the quotient

$$(Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}})/(\mathcal{I}^n \cdot (Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}}) + (\mathfrak{m}^\lambda \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}} + Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} \mathfrak{m}^\mu)^m)$$

does not depend on  $m$  for  $m \gg 0$ , and the natural morphism from  $(Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}})/\mathcal{I}^n \cdot (Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}})$  to the product of these rings is an isomorphism. Now we have

$$\mathcal{Z} = (Z_{\text{HC}} \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{HC}}) \otimes_{Z_{\text{HC}} \cap Z_{\text{Fr}}} Z_{\text{Fr}}.$$

From the preceding considerations we deduce that for  $m \gg 0$  the natural morphism

$$\mathcal{Z}/(\mathcal{I}^n \cdot \mathcal{Z}) \rightarrow \prod_{\lambda, \mu \in \Lambda} \mathcal{Z}/(\mathcal{I}^n \cdot \mathcal{Z} + (\mathcal{I}^{\lambda, \mu})^m)$$

is a ring isomorphism.

Now we are ready to define the wished-for inverse morphism

$$\prod_{\lambda, \mu \in \Lambda} \mathcal{Z}^{\hat{\lambda}, \hat{\mu}} \rightarrow \mathcal{Z}^\wedge.$$

For this it suffices to define, for any  $n \geq 1$ , a ring morphism

$$(2.7) \quad \prod_{\lambda, \mu \in \Lambda} \mathcal{Z}^{\hat{\lambda}, \hat{\mu}} \rightarrow \mathcal{Z}/(\mathcal{I}^n \cdot \mathcal{Z}).$$

For this we fix  $m$  such that the natural morphism

$$\mathcal{Z}/(\mathcal{I}^n \cdot \mathcal{Z}) \rightarrow \prod_{\lambda, \mu \in \Lambda} \mathcal{Z}/(\mathcal{I}^n \cdot \mathcal{Z} + (\mathcal{I}^{\lambda, \mu})^m)$$

is an isomorphism. Then we have natural ring morphisms

$$\prod_{\lambda, \mu \in \Lambda} \mathcal{Z}^{\hat{\lambda}, \hat{\mu}} \rightarrow \prod_{\lambda, \mu \in \Lambda} \mathcal{Z}/(\mathcal{I}^{\lambda, \mu})^m \rightarrow \prod_{\lambda, \mu \in \Lambda} \mathcal{Z}/(\mathcal{I}^n \cdot \mathcal{Z} + (\mathcal{I}^{\lambda, \mu})^m).$$

Composing with the inverse of the preceding isomorphism we deduce the desired map (2.7).

It is easy (and left to the reader) to check that the two morphisms considered above are inverse to each other.  $\square$

*Remark 2.4.* Another interpretation of the considerations in the proof above (in case  $n = 1$ ) is that given  $\lambda, \mu \in \mathbb{X}$  there exists  $N \in \mathbb{Z}_{\geq 0}$  such that for any  $m \geq N$  we have  $(\mathcal{I}^{\lambda, \mu})^N \subset \mathcal{I} \cdot \mathcal{Z} + (\mathcal{I}^{\lambda, \mu})^m$ . As a consequence we have  $(\mathcal{I}^{\lambda, \mu})^N \cdot \mathcal{Z}^{\hat{\lambda}, \hat{\mu}} \subset \mathcal{I} \cdot \mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$ . We also have  $\mathcal{I} \cdot \mathcal{Z}^{\hat{\lambda}, \hat{\mu}} \subset \mathcal{I}^{\lambda, \mu} \cdot \mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$ ; we deduce that for any finitely generated  $\mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$ -module  $M$  the natural morphism

$$M \rightarrow \varprojlim_{n \geq 1} M/\mathcal{I}^n \cdot M$$

is an isomorphism.

We will also denote by  $\mathcal{U}^\wedge$  the completion of the  $\mathcal{Z}$ -algebra  $\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}$  with respect to the ideal  $\mathcal{I} \cdot \mathcal{Z}$ . By the same considerations as in §2.5, since  $\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}$  is finite as a  $\mathcal{Z}$ -module, we have a canonical isomorphism

$$\mathcal{Z}^\wedge \otimes_{\mathcal{Z}} (\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}) \xrightarrow{\sim} \mathcal{U}^\wedge.$$

Comparing with (2.5) and using Lemma 2.3, we deduce that the natural morphism

$$(2.8) \quad \mathcal{U}^\wedge \rightarrow \prod_{\lambda, \mu \in \Lambda} \mathcal{U}^{\hat{\lambda}, \hat{\mu}}$$

is an algebra isomorphism.

Following the same procedure as for the categories  $\text{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}})$  and  $\text{HC}^{\hat{\lambda}, \hat{\mu}}$ , we can define the categories  $\text{Mod}_{\text{fg}}^G(\mathcal{U}^\wedge)$  and  $\text{HC}^\wedge$  of  $G$ -equivariant finitely generated  $\mathcal{U}^\wedge$ -modules and Harish-Chandra  $\mathcal{U}^\wedge$ -modules. In view of (2.8), we then have canonical equivalences of categories

$$(2.9) \quad \text{Mod}_{\text{fg}}^G(\mathcal{U}^\wedge) \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}}), \quad \text{HC}^\wedge \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{HC}^{\hat{\lambda}, \hat{\mu}}.$$

We also have a canonical functor

$$(2.10) \quad \text{C}^\wedge : \text{Mod}_{\text{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}) \rightarrow \text{Mod}_{\text{fg}}^G(\mathcal{U}^\wedge)$$

sending a module to its completion with respect to  $\mathcal{I} \cdot \mathcal{Z}$ , or in other words to its tensor product with  $\mathcal{Z}^\wedge$  over  $\mathcal{Z}$ , which restricts to a functor from  $\text{HC}$  to  $\text{HC}^\wedge$ . For the same reasons as above, for any  $M$  in  $\text{Mod}_{\text{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}})$  we have a canonical isomorphism

$$\text{C}^\wedge(M) \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{C}^{\lambda, \mu}(M).$$

We will denote by  $\mathrm{HC}_{\mathrm{diag}}^\wedge$  the full additive subcategory of  $\mathrm{HC}^\wedge$  whose objects are the direct summands of objects of the form  $\mathcal{C}^\wedge(V \otimes \mathcal{U}\mathfrak{g})$  with  $V$  in  $\mathrm{Rep}(G)$ . With this definition, we have a canonical equivalence of categories

$$\mathrm{HC}_{\mathrm{diag}}^\wedge \cong \bigoplus_{\lambda, \mu \in \Lambda} \mathrm{HC}_{\mathrm{diag}}^{\hat{\lambda}, \hat{\mu}}.$$

**2.8. Monoidal structure.** We now want to define some analogue of the monoidal structure on  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}})$  for the categories  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}})$ . More specifically, given  $\lambda, \mu, \nu$  in  $\mathbb{X}$  we want to define a canonical right exact bifunctor

$$(2.11) \quad (-) \hat{\otimes}_{\mathcal{U}\mathfrak{g}} (-) : \mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}}) \times \mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\mu}, \hat{\nu}}) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\nu}}),$$

these bifunctors satisfying obvious unit and associativity axioms. Explicitly, we require that:

- in case  $\mu = \lambda$  we have a canonical isomorphism of functors

$$\mathcal{U}^{\hat{\lambda}} \hat{\otimes}_{\mathcal{U}\mathfrak{g}} (-) \cong \mathrm{id},$$

and in case  $\nu = \mu$  we have a canonical isomorphism

$$(-) \hat{\otimes}_{\mathcal{U}\mathfrak{g}} \mathcal{U}^{\hat{\mu}} \cong \mathrm{id};$$

- for four weights  $\lambda, \mu, \nu, \eta \in \mathbb{X}$  we have an isomorphism

$$((-) \hat{\otimes}_{\mathcal{U}\mathfrak{g}} (-)) \hat{\otimes}_{\mathcal{U}\mathfrak{g}} (-) \xrightarrow{\sim} (-) \hat{\otimes}_{\mathcal{U}\mathfrak{g}} ((-) \hat{\otimes}_{\mathcal{U}\mathfrak{g}} (-))$$

of functors from

$$\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}}) \times \mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\mu}, \hat{\nu}}) \times \mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\nu}, \hat{\eta}})$$

to  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\eta}})$ .

In particular, in case  $\lambda = \mu = \nu$ , this construction will equip  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\lambda}})$  with the structure of a monoidal category.

For this we can assume that all the weights involved belong to the subset  $\Lambda$  chosen in §2.7. It therefore suffices to construct a monoidal structure of the category  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^\wedge)$ , with monoidal unit  $\mathcal{C}^\wedge(\mathbb{k} \otimes \mathcal{U}\mathfrak{g})$ ; the bifunctor (2.11) will then be deduced by restriction to the factor  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}}) \times \mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\mu}, \hat{\nu}})$  in the decomposition (2.9).

The corresponding product is defined by setting, for  $M, N$  in  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^\wedge)$ ,

$$M \hat{\otimes}_{\mathcal{U}\mathfrak{g}} N = \varprojlim_{n \geq 1} (M/\mathcal{I}^n \cdot M) \otimes_{\mathcal{U}\mathfrak{g}} (N/\mathcal{I}^n \cdot N).$$

Here, for any  $n \geq 1$  the transition morphism

$$(M/\mathcal{I}^{n+1} \cdot M) \otimes_{\mathcal{U}\mathfrak{g}} (N/\mathcal{I}^{n+1} \cdot N) \rightarrow (M/\mathcal{I}^n \cdot M) \otimes_{\mathcal{U}\mathfrak{g}} (N/\mathcal{I}^n \cdot N)$$

is surjective, with kernel  $\mathcal{I}^n \cdot ((M/\mathcal{I}^{n+1} \cdot M) \otimes_{\mathcal{U}\mathfrak{g}} (N/\mathcal{I}^{n+1} \cdot N))$ ; hence we are in the setting of [EGA1, Chap. 0, Proposition 7.2.9], which guarantees that  $M \hat{\otimes}_{\mathcal{U}\mathfrak{g}} N$  is a finitely generated  $\mathcal{U}^\wedge$ -module which satisfies

$$(M \hat{\otimes}_{\mathcal{U}\mathfrak{g}} N)/\mathcal{I}^n \cdot (M \hat{\otimes}_{\mathcal{U}\mathfrak{g}} N) = (M/\mathcal{I}^n \cdot M) \otimes_{\mathcal{U}\mathfrak{g}} (N/\mathcal{I}^n \cdot N).$$

By definition of the category  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^\wedge)$  the quotients  $M/\mathcal{I}^n \cdot M$  and  $N/\mathcal{I}^n \cdot N$  admit  $G$ -module structures; we deduce a (diagonal)  $G$ -module structure on  $(M \hat{\otimes}_{\mathcal{U}\mathfrak{g}} N)/\mathcal{I}^n \cdot (M \hat{\otimes}_{\mathcal{U}\mathfrak{g}} N)$ , so that  $M \hat{\otimes}_{\mathcal{U}\mathfrak{g}} N$  indeed admits a natural structure of  $G$ -equivariant

finitely generated  $\mathcal{U}^\wedge$ -module. It is clear from the definition that the full subcategory  $\mathrm{HC}^\wedge$  of  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^\wedge)$  is a monoidal subcategory. It is clear also that, with this monoidal structure, the functor  $\mathcal{C}^\wedge$  of (2.10) has a canonical structure of monoidal functor. In particular, using (2.4) we deduce that the full subcategory  $\mathrm{HC}_{\mathrm{diag}}^\wedge$  is also a monoidal subcategory.

It is easily seen on the definition that if  $\lambda, \mu, \nu \in \mathbb{X}$  and if  $M$  belongs to  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}})$  and  $N$  belongs to  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\mu}, \hat{\nu}})$ , then seeing  $M$  and  $N$  as objects in  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^\wedge)$  via (2.9) the product  $M \hat{\otimes}_{\mathcal{U}_{\mathfrak{g}}} N$  belongs to the factor  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\nu}})$ , which provides the desired bifunctor (2.11). From the corresponding properties for  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}^\wedge)$  we deduce that the subcategories  $\mathrm{HC}^{\hat{\lambda}, \hat{\mu}}$  and  $\mathrm{HC}_{\mathrm{diag}}^{\hat{\lambda}, \hat{\mu}}$  are stable under these bifunctors. In this setting the functor  $\mathcal{C}^{\hat{\lambda}, \hat{\mu}}$  satisfies

$$\mathcal{C}^{\hat{\lambda}, \hat{\mu}}(M \otimes_{\mathcal{U}_{\mathfrak{g}}} N) \cong \bigoplus_{\nu \in \Lambda} \mathcal{C}^{\hat{\lambda}, \hat{\nu}}(M) \hat{\otimes}_{\mathcal{U}_{\mathfrak{g}}} \mathcal{C}^{\hat{\nu}, \hat{\mu}}(N)$$

for any  $M, N$  in  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}_{\mathfrak{g}} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}_{\mathfrak{g}}^{\mathrm{op}})$ .

**2.9. Universal centralizer and Kostant section.** From now on we fix once and for all a  $G$ -equivariant isomorphism  $\varkappa : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ . Note that the composition  $\mathfrak{t} \hookrightarrow \mathfrak{g} \xrightarrow{\varkappa} \mathfrak{g}^* \twoheadrightarrow \mathfrak{t}^*$  is then a  $W$ -equivariant isomorphism. This allows in particular to identify the coadjoint quotient  $\mathfrak{g}^*/G$  with  $\mathfrak{t}^*/W$ .

We will denote by  $\mathfrak{g}_{\mathrm{reg}} \subset \mathfrak{g}$  the open subsets consisting of regular elements, i.e. elements whose centralizer has dimension  $\dim(T)$ . The “regular universal centralizer” is the affine group scheme

$$\mathbb{J}_{\mathrm{reg}} := \mathfrak{g}_{\mathrm{reg}} \times_{\mathfrak{g}_{\mathrm{reg}} \times \mathfrak{g}_{\mathrm{reg}}} (G \times \mathfrak{g}_{\mathrm{reg}})$$

over  $\mathfrak{g}_{\mathrm{reg}}$ , where the morphism  $\mathfrak{g}_{\mathrm{reg}} \rightarrow \mathfrak{g}_{\mathrm{reg}} \times \mathfrak{g}_{\mathrm{reg}}$  is the diagonal embedding, and the map  $G \times \mathfrak{g}_{\mathrm{reg}} \rightarrow \mathfrak{g}_{\mathrm{reg}}$  sends  $(g, x)$  to  $(g \cdot x, x)$ . For any  $x \in \mathfrak{g}_{\mathrm{reg}}$ , the fiber of  $\mathbb{J}_{\mathrm{reg}}$  over  $x$  is the scheme-theoretic centralizer of  $x$  for the adjoint  $G$ -action. By construction  $\mathbb{J}_{\mathrm{reg}}$  is a closed subgroup scheme in  $G \times \mathfrak{g}_{\mathrm{reg}}$ , and as explained in [R3, Corollary 3.3.6] it is smooth over  $\mathfrak{g}_{\mathrm{reg}}$ . We will also denote by  $\mathfrak{g}_{\mathrm{reg}}^*$  the image of  $\mathfrak{g}_{\mathrm{reg}}$  under  $\varkappa$ , and by  $\mathbb{J}_{\mathrm{reg}}^*$  the smooth affine group scheme over  $\mathfrak{g}_{\mathrm{reg}}^*$  obtained by pushforward from  $\mathbb{J}_{\mathrm{reg}}$ . (It is easily seen that these objects do not depend on the choice of  $\varkappa$ .)

Let us note for later use that there exists a canonical morphism

$$(2.12) \quad \mathfrak{t}^* \times_{\mathfrak{t}^*/W} \mathbb{J}_{\mathrm{reg}}^* \rightarrow (\mathfrak{t}^* \times_{\mathfrak{t}^*/W} \mathfrak{g}_{\mathrm{reg}}^*) \times T$$

of group schemes over  $\mathfrak{t}^* \times_{\mathfrak{t}^*/W} \mathfrak{g}_{\mathrm{reg}}^*$ , whose construction we now explain. Recall that the *Grothendieck resolution* is the  $G$ -equivariant vector bundle over  $G/B$  given by

$$\tilde{\mathfrak{g}} := \{(\xi, gB) \in \mathfrak{g}^* \times G/B \mid \xi|_{\mathfrak{g} \cdot \mathfrak{n}} = 0\}.$$

We have natural maps

$$\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}^*, \quad \vartheta : \tilde{\mathfrak{g}} \rightarrow \mathfrak{t}^*.$$

(The morphism  $\pi$  is induced by the first projection. The morphism  $\vartheta$  sends a pair  $(\xi, gB)$  to  $\xi|_{\mathfrak{g} \cdot \mathfrak{b}}$ , seen as an element in  $(g \cdot \mathfrak{b}/g \cdot \mathfrak{n})^* \cong (\mathfrak{b}/\mathfrak{n})^* \cong \mathfrak{t}^*$ , where the first isomorphism is induced by conjugation by the inverse of any representative for the coset  $gB$ .) If we denote by  $\tilde{\mathfrak{g}}_{\mathrm{reg}}$  the preimage of  $\mathfrak{g}_{\mathrm{reg}}^*$  in  $\tilde{\mathfrak{g}}$ , then these maps induce an isomorphism of schemes

$$(2.13) \quad \tilde{\mathfrak{g}}_{\mathrm{reg}} \xrightarrow{\sim} \mathfrak{g}_{\mathrm{reg}}^* \times_{\mathfrak{t}^*/W} \mathfrak{t}^*,$$



see [R3, Lemma 3.5.3]. Moreover, under this identification, by [R3, Proposition 3.5.6] the group scheme  $\mathfrak{t}^* \times_{\mathfrak{t}^*/W} \mathbb{J}_{\text{reg}}$  identifies with the universal centralizer associated with the action of  $G$  on  $\mathfrak{g}_{\text{reg}}$ , which is such that the fiber over  $(\xi, gB)$  is the scheme-theoretic stabilizer of  $\xi$  for the action of  $gBg^{-1}$ . Now as above in the definition of  $\vartheta$ , there exists for any  $g \in G$  a canonical isomorphism  $gBg^{-1}/gUg^{-1} \cong T$ , which allows us to define the wished-for morphism (2.12).

Let us choose a regular nilpotent element  $e \in \mathfrak{n}$  and an affine subspace  $\mathcal{S} \subset \mathfrak{g}$  as in [R3, §3.1] (in other words, a “Kostant section” for the adjoint quotient). Such a subspace is automatically contained in  $\mathfrak{g}_{\text{reg}}$ , see [R3, Equation (3.1.1)]. Moreover, the composition  $\mathcal{S} \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{g}/G$  (where the second map is the adjoint quotient morphism) is an isomorphism, see [R3, Theorem 3.2.2]. We will denote by  $\mathcal{S}^*$  the image of  $\mathcal{S}$  under  $\varkappa$ , by  $\mathbb{J}_{\mathcal{S}}^*$  the restriction of  $\mathbb{J}_{\text{reg}}^*$  to  $\mathcal{S}^*$  (a closed subgroup scheme of  $G \times \mathcal{S}^*$ , smooth over  $\mathcal{S}^*$ ), and set

$$\mathbb{I}_{\mathcal{S}}^* := (G \times \mathcal{S}^{*(1)}) \times_{G^{(1)} \times \mathcal{S}^{*(1)}} (\mathbb{J}_{\mathcal{S}}^*)^{(1)},$$

where the map  $G \times \mathcal{S}^{*(1)} \rightarrow G^{(1)} \times \mathcal{S}^{*(1)}$  is the product of the Frobenius morphism of  $G$  and the identity of  $\mathcal{S}^{*(1)}$ . Since  $G$  is smooth its Frobenius morphism is flat, and therefore  $\mathbb{I}_{\mathcal{S}}^*$  is a flat affine group scheme over  $\mathcal{S}^{*(1)}$ . Note also that the morphism (2.12) induces a group-scheme morphism

$$(2.14) \quad \mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}_{\text{reg}}^* \rightarrow (\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathcal{S}^{*(1)}) \times T^{(1)}.$$

Finally, we set

$$\mathcal{U}_{\mathcal{S}}\mathfrak{g} := \mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{O}(\mathcal{S}^{*(1)}),$$

where  $\mathcal{O}(\mathcal{S}^{*(1)})$  is seen as a  $Z_{\text{Fr}}$ -algebra via the identification (2.2). If we set

$$\mathfrak{C}_{\mathcal{S}} := \mathcal{S}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet),$$

then  $\mathcal{U}_{\mathcal{S}}\mathfrak{g}$  is an  $\mathcal{O}(\mathfrak{C}_{\mathcal{S}})$ -algebra. Recall that the algebra  $\mathcal{U}\mathfrak{g}$  can be seen as a  $G$ -equivariant  $\mathcal{O}(\mathfrak{C})$ -algebra (see §2.2). Using the general construction recalled in [MR, §2.2], from this we deduce on  $\mathcal{U}_{\mathcal{S}}\mathfrak{g}$  a natural structure of module for the group scheme  $\mathfrak{C}_{\mathcal{S}} \times_{\mathcal{S}^{*(1)}} \mathbb{I}_{\mathcal{S}}^*$ , such that the multiplication morphism is equivariant.

**2.10. (Completed) Harish-Chandra bimodules for  $\mathcal{U}_{\mathcal{S}}\mathfrak{g}$ .** We now want to define, given  $\lambda, \mu \in \mathbb{X}$ , a category analogous to  $\text{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}})$  but for the algebra  $\mathcal{U}_{\mathcal{S}}\mathfrak{g}$  in place of  $\mathcal{U}\mathfrak{g}$ . We start with the non-completed version.

First we consider the category  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathcal{S}}\mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} (\mathcal{U}_{\mathcal{S}}\mathfrak{g})^{\text{op}})$  of finitely generated  $\mathcal{U}_{\mathcal{S}}\mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} (\mathcal{U}_{\mathcal{S}}\mathfrak{g})^{\text{op}}$ -modules endowed with a compatible structure of  $\mathbb{I}_{\mathcal{S}}^*$ -module. Since  $\mathbb{I}_{\mathcal{S}}^*$  is flat over  $\mathcal{S}^{*(1)}$ , this category is abelian. Here  $\mathcal{U}_{\mathcal{S}}\mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} (\mathcal{U}_{\mathcal{S}}\mathfrak{g})^{\text{op}}$  is an algebra over

$$\begin{aligned} \mathcal{Z}_{\mathcal{S}} &:= \mathcal{O}(\mathfrak{t}^*/(W, \bullet)) \otimes_{\mathcal{O}(\mathfrak{t}^{*(1)}/W)} \mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{t}^{*(1)}/W)} \mathcal{O}(\mathfrak{t}^*/(W, \bullet)) \\ &\cong \mathcal{O}(\mathfrak{t}^*/(W, \bullet)) \otimes_{\mathcal{O}(\mathfrak{t}^{*(1)}/W)} \mathcal{O}(\mathfrak{t}^*/(W, \bullet)) = \mathcal{O}(\mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)). \end{aligned}$$

As in §2.4 the tensor product  $\otimes_{\mathcal{O}(\mathcal{S}^{*(1)})}$  defines a monoidal structure on this category, and using the construction of [MR, §2.2] considered above the functor  $\mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{Z_{\text{Fr}}} (-)$  defines a monoidal functor

$$\text{Mod}_{\text{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}}) \rightarrow \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathcal{S}}\mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} (\mathcal{U}_{\mathcal{S}}\mathfrak{g})^{\text{op}}).$$

Now we add completions to the picture. Given  $\lambda, \mu \in \mathbb{X}$ , one can define as in §2.5 the algebra

$$\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}}$$

as the completion of  $\mathcal{U}_S \mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} (\mathcal{U}_S \mathfrak{g})^{\text{op}}$  with respect to the (maximal) ideal

$$\mathcal{I}_S^{\lambda, \mu} := \mathcal{I}^{\lambda, \mu} \cdot \mathcal{Z}_S = \mathfrak{m}^\lambda \otimes_{\mathcal{O}(\mathfrak{t}^{*(1)}/W)} \mathcal{O}(\mathfrak{t}^*/(W, \bullet)) + \mathcal{O}(\mathfrak{t}^*/(W, \bullet)) \otimes_{\mathcal{O}(\mathfrak{t}^{*(1)}/W)} \mathfrak{m}^\mu$$

in  $\mathcal{Z}_S$ . The category  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}})$  can also be defined by the same procedure as for  $\text{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}})$ , namely as the category of finitely generated  $\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}}$ -modules  $M$  together with a  $(\mathfrak{C}_S \times_{\mathcal{S}^{*(1)}} \mathfrak{C}_S) \times_{\mathcal{S}^{*(1)}} \mathbb{I}_S^*$ -module structure on each quotient  $M/(\mathcal{I}_S^{\lambda, \mu})^n \cdot M$ , such that the quotient morphism

$$M/(\mathcal{I}_S^{\lambda, \mu})^{n+1} \cdot M \rightarrow M/(\mathcal{I}_S^{\lambda, \mu})^n \cdot M$$

is equivariant for any  $n \geq 1$ , and similarly for the action morphism.

By exactness of completion we have

$$\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}} = \mathcal{U}^{\hat{\lambda}, \hat{\mu}} \otimes_{Z_{\text{Fr}}} \mathcal{O}(\mathcal{S}^{*(1)}).$$

In particular, the functor  $\mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{Z_{\text{Fr}}} (-)$  defines a natural functor

$$(2.15) \quad \text{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}}) \rightarrow \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}}).$$

Similarly, if we set

$$\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}} := \mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} \mathcal{Z}^{\hat{\lambda}, \hat{\mu}},$$

then  $\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}}$  is a  $\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}$ -algebra, and  $\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}$  identifies with the completion of the algebra  $\mathcal{Z}_S$  with respect to  $\mathcal{I}_S^{\lambda, \mu}$ . If  $\lambda, \mu \in \mathbb{X}$  belong to the closure of the fundamental alcove, we will also set

$$\mathbb{P}_S^{\lambda, \mu} := \mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} \mathbb{P}^{\lambda, \mu}.$$

Using considerations similar to those of §2.8 one constructs, again for  $\lambda, \mu, \nu \in \mathbb{X}$ , a canonical bifunctor

$$(2.16) \quad (-) \hat{\otimes}_{\mathcal{U}_S \mathfrak{g}} (-) : \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}}) \times \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\mu}, \hat{\nu}}) \rightarrow \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\nu}}),$$

this construction being unital, associative, and compatible in the natural way with the bifunctors  $(-) \hat{\otimes}_{\mathcal{U}_{\mathfrak{g}}} (-)$  via the functors (2.15). More explicitly, one remarks that if  $\mathcal{Z}_S^{\hat{\lambda}}$  is the completion of  $\mathcal{Z}_S$  with respect to the ideal  $\mathcal{I} \cdot \mathcal{Z}_S$ , then as in Lemma 2.3 we have a canonical isomorphism

$$\mathcal{Z}_S^{\hat{\lambda}} \xrightarrow{\sim} \prod_{\lambda, \mu \in \Lambda} \mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}.$$

One then defines in the natural way the category  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}})$ , where

$$\mathcal{U}_S^{\hat{\lambda}} = \mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} \mathcal{U}^{\hat{\lambda}}$$

is the completion of the  $\mathcal{Z}_S$ -algebra  $\mathcal{U}_S \mathfrak{g}$  with respect to the ideal  $\mathcal{I} \cdot \mathcal{Z}_S$ , and the bifunctor

$$(-) \hat{\otimes}_{\mathcal{U}_S \mathfrak{g}} (-) : \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}}) \times \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}}) \rightarrow \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}})$$

by

$$M \hat{\otimes}_{\mathcal{U}_S \mathfrak{g}} N = \varprojlim_{n \geq 1} (M/I^n \cdot M) \otimes_{\mathcal{U}_S \mathfrak{g}} (N/I^n \cdot N).$$

As in (2.9) we have

$$(2.17) \quad \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^\wedge) \cong \bigoplus_{\lambda, \mu \in \Lambda} \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}}),$$

and the bifunctor (2.16) is then obtained by restriction to the appropriate summands. In case  $\lambda = \mu = \nu$ , this bifunctor equips  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\lambda}})$  with a structure of monoidal category, with unit object

$$\mathcal{U}_S^{\hat{\lambda}} := \mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} \mathcal{U}^{\hat{\lambda}}.$$

**Lemma 2.5.** *For any  $\lambda, \mu \in \mathbb{X}$  which belong to the closure of the fundamental alcove and any  $\nu \in \mathbb{X}$ , the functor*

$$\mathbb{P}_S^{\lambda, \mu} \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} (-) : \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\mu}, \hat{\nu}}) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\nu}})$$

is both left and right adjoint to the functor

$$\mathbb{P}_S^{\mu, \lambda} \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} (-) : \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\nu}}) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\mu}, \hat{\nu}}).$$

A similar property holds for the functors  $(-) \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{P}_S^{\lambda, \mu}$  and  $(-) \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{P}_S^{\mu, \lambda}$ .

*Proof.* We prove the case of convolution on the left; convolution on the right can be treated similarly. We remark that for any  $V \in \mathrm{Rep}(G)$ , the functor

$$\left( \mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} \mathcal{C}^\wedge(V) \right) \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} (-) : \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^\wedge) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^\wedge)$$

is both left and right adjoint to the functor

$$\left( \mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} \mathcal{C}^\wedge(V^*) \right) \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} (-) : \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^\wedge) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^\wedge).$$

(In fact, these functors can be realized more concretely as tensor product with  $V$  and  $V^*$  respectively.) On the other hand, the inclusion functor

$$\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\nu}}) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^\wedge)$$

(see (2.17)) is both left and right adjoint to the corresponding projection functor

$$\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^\wedge) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\nu}}),$$

and similarly for  $\mu$  in place of  $\lambda$ . The desired claim follows, since the functors  $\mathbb{P}_S^{\lambda, \mu} \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} (-)$  and  $\mathbb{P}_S^{\mu, \lambda} \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} (-)$  are isomorphic to compositions of functors of this form.  $\square$

**2.11. Restriction to the Kostant section for diagonally induced bimodules.** In this subsection we aim at proving the following claim.

**Proposition 2.6.** *For any  $\lambda, \mu \in \mathbb{X}$ , the functor (2.15) is fully faithful on the subcategory  $\mathrm{HC}_{\mathrm{diag}}^{\hat{\lambda}, \hat{\mu}}$ .*

The proof of this proposition will use a preliminary lemma. For this we denote by  $j : \mathfrak{g}_{\mathrm{reg}}^{*(1)} \hookrightarrow \mathfrak{g}^{*(1)}$  the open embedding, and identify in the natural way the category of quasi-coherent sheaves on the affine scheme  $\mathfrak{g}^{*(1)}$  with the category of modules over the ring  $\mathcal{O}(\mathfrak{g}^{*(1)})$ .

**Lemma 2.7.** *For any  $n \geq 1$ , the adjunction morphism*

$$\mathcal{O}(\mathfrak{g}^{*(1)})/(\mathcal{I}^n \cdot \mathcal{O}(\mathfrak{g}^{*(1)})) \rightarrow j_* j^* (\mathcal{O}(\mathfrak{g}^{*(1)})/(\mathcal{I}^n \cdot \mathcal{O}(\mathfrak{g}^{*(1)})))$$

is an isomorphism.

*Proof.* We proceed by induction on  $n$ . The case  $n = 1$  follows from the fact that the subscheme  $\mathcal{N}^*$  of  $\mathfrak{g}^*$  considered in §2.3 is normal (in fact, it is isomorphic to the unipotent cone of  $G$ , which is known to be normal; see [BK, §5.3] for references) and that the open subset  $\mathfrak{g}_{\text{reg}}^* \cap \mathcal{N}^*$  has a complement of codimension 2 (since  $\mathcal{N}^*$  is a union of finitely many  $G$ -orbits, which all have even dimension).

Now, assume the claim is known for some  $n \geq 1$ . Since  $\mathfrak{g}^*$  is isomorphic to  $\mathfrak{g}$  as a  $G$ -module, the coadjoint quotient morphism  $\mathfrak{g}^{*(1)} \rightarrow \mathfrak{t}^{*(1)}/W$  is flat (see [SL, §3.14]), and  $\mathfrak{t}^{*(1)}/W$  is isomorphic to an affine space (see [SL, §3.12]). Hence there exists  $m \in \mathbb{Z}_{\geq 1}$  and an exact sequence of  $\mathcal{O}(\mathfrak{g}^{*(1)})$ -modules

$$\mathcal{O}(\mathfrak{g}^{*(1)})/(\mathcal{I} \cdot \mathcal{O}(\mathfrak{g}^{*(1)}))^{\oplus m} \hookrightarrow \mathcal{O}(\mathfrak{g}^{*(1)})/(\mathcal{I}^{n+1} \cdot \mathcal{O}(\mathfrak{g}^{*(1)})) \twoheadrightarrow \mathcal{O}(\mathfrak{g}^{*(1)})/(\mathcal{I}^n \cdot \mathcal{O}(\mathfrak{g}^{*(1)})).$$

Applying the left exact functor  $j_*j^*$  we obtain an exact sequence

$$\begin{aligned} 0 \rightarrow j_*j^*(\mathcal{O}(\mathfrak{g}^{*(1)})/(\mathcal{I} \cdot \mathcal{O}(\mathfrak{g}^{*(1)}))^{\oplus m}) &\rightarrow j_*j^*(\mathcal{O}(\mathfrak{g}^{*(1)})/(\mathcal{I}^{n+1} \cdot \mathcal{O}(\mathfrak{g}^{*(1)}))) \\ &\rightarrow j_*j^*(\mathcal{O}(\mathfrak{g}^{*(1)})/(\mathcal{I}^n \cdot \mathcal{O}(\mathfrak{g}^{*(1)}))). \end{aligned}$$

Here the rightmost morphism is surjective, since the composition

$$\begin{aligned} \mathcal{O}(\mathfrak{g}^{*(1)})/(\mathcal{I}^{n+1} \cdot \mathcal{O}(\mathfrak{g}^{*(1)})) &\rightarrow j_*j^*(\mathcal{O}(\mathfrak{g}^{*(1)})/(\mathcal{I}^{n+1} \cdot \mathcal{O}(\mathfrak{g}^{*(1)}))) \\ &\rightarrow j_*j^*(\mathcal{O}(\mathfrak{g}^{*(1)})/(\mathcal{I}^n \cdot \mathcal{O}(\mathfrak{g}^{*(1)}))) \end{aligned}$$

is surjective, as follows from the induction hypothesis. The desired claim for  $n + 1$  follows, using the 5-lemma.  $\square$

*Proof of Proposition 2.6.* We remark that the functor  $\mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} (-)$  induces a monoidal functor

$$(2.18) \quad \text{Mod}_{\text{fg}}^G(\mathcal{U}^\wedge) \rightarrow \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^\wedge),$$

from which the functor (2.15) is obtained by restriction to the direct summand  $\text{Mod}_{\text{fg}}^G(\mathcal{U}^{\hat{\lambda}, \hat{\mu}})$ . To prove the proposition, it therefore suffices to prove that (2.18) is fully faithful on the subcategory  $\text{HC}_{\text{diag}}^\wedge$ . In fact, what we will prove is that for any  $M$  in  $\text{Mod}_{\text{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}})$  and any  $V$  in  $\text{Rep}(G)$ , our functor induces an isomorphism

$$\begin{aligned} \text{Hom}_{\text{Mod}_{\text{fg}}^G(\mathcal{U}^\wedge)}(C^\wedge(M), C^\wedge(V \otimes \mathcal{U}\mathfrak{g})) &\xrightarrow{\sim} \\ \text{Hom}_{\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^\wedge)}(\mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} C^\wedge(M), \mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} C^\wedge(V \otimes \mathcal{U}\mathfrak{g})). \end{aligned}$$

Let us fix  $M$  and  $V$  as above. In view of (the analogue for  $\mathcal{U}^\wedge$  of) (2.6), we have

$$\begin{aligned} \text{Hom}_{\text{Mod}_{\text{fg}}^G(\mathcal{U}^\wedge)}(C^\wedge(M), C^\wedge(V \otimes \mathcal{U}\mathfrak{g})) &= \\ \varprojlim_{n \geq 1} \text{Hom}_{\text{Mod}_{\text{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{Fr}}} \mathcal{U}\mathfrak{g}^{\text{op}})}(M/(\mathcal{I}^n \cdot M), V \otimes (\mathcal{U}\mathfrak{g}/\mathcal{I}^n \cdot \mathcal{U}\mathfrak{g})), \end{aligned}$$

and similarly we have

$$\begin{aligned} \text{Hom}_{\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^\wedge)}(\mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} C^\wedge(M), \mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} C^\wedge(V \otimes \mathcal{U}\mathfrak{g})) &= \\ \varprojlim_{n \geq 1} \text{Hom}_{\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S \mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} (\mathcal{U}_S \mathfrak{g})^{\text{op}})}(M_S/(\mathcal{I}^n \cdot M_S), V \otimes (\mathcal{U}_S \mathfrak{g}/\mathcal{I}^n \cdot \mathcal{U}_S \mathfrak{g})), \end{aligned}$$

where  $M_S := \mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} M$ . To complete the proof, it therefore suffices to prove that for any  $n \geq 1$  the functor  $\mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} (-)$  induces an isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}})}(M/(\mathcal{I}^n \cdot M), V \otimes (\mathcal{U}\mathfrak{g}/\mathcal{I}^n \cdot \mathcal{U}\mathfrak{g})) &\xrightarrow{\sim} \\ \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}_S \mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} (\mathcal{U}_S \mathfrak{g})^{\mathrm{op}})}(M_S/(\mathcal{I}^n \cdot M_S), V \otimes (\mathcal{U}_S \mathfrak{g}/\mathcal{I}^n \cdot \mathcal{U}_S \mathfrak{g})). \end{aligned}$$

The algebra  $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$  is a  $G$ -equivariant finite  $\mathcal{O}(\mathfrak{g}^{*(1)})$ -algebra. Therefore, it identifies with the global sections of a  $G$ -equivariant coherent sheaf of  $\mathcal{O}_{\mathfrak{g}^{*(1)}}$ -algebras  $\mathcal{U}$  on  $\mathfrak{g}^{*(1)}$ . Moreover, the restriction  $\mathcal{U}_S$  of  $\mathcal{U}$  to  $\mathcal{S}^{*(1)}$  is an  $\mathbb{I}_S^*$ -equivariant sheaf of  $\mathcal{O}_{\mathcal{S}^{*(1)}}$ -algebras on  $\mathcal{S}^{*(1)}$ , whose global sections are  $\mathcal{U}_S \mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} (\mathcal{U}_S \mathfrak{g})^{\mathrm{op}}$ . Consider now the open embedding  $j$  as in Lemma 2.7, and set  $\mathcal{U}_{\mathrm{reg}} := j^*(\mathcal{U})$ . Let us denote by  $\mathrm{Coh}^G(\mathfrak{g}_{\mathrm{reg}}^{*(1)}, \mathcal{U}_{\mathrm{reg}})$  the category of  $G$ -equivariant coherent sheaves on  $\mathfrak{g}_{\mathrm{reg}}^{*(1)}$  equipped with a structure of  $\mathcal{U}_{\mathrm{reg}}$ -module, compatible with the  $G$ -equivariant structure in the natural way. Then we have a natural restriction functor

$$j^* : \mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \rightarrow \mathrm{Coh}^G(\mathfrak{g}_{\mathrm{reg}}^{*(1)}, \mathcal{U}_{\mathrm{reg}}).$$

We claim that this functor induces an isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}})}(M/(\mathcal{I}^n \cdot M), V \otimes (\mathcal{U}\mathfrak{g}/\mathcal{I}^n \cdot \mathcal{U}\mathfrak{g})) &\xrightarrow{\sim} \\ \mathrm{Hom}_{\mathrm{Coh}^G(\mathfrak{g}_{\mathrm{reg}}^{*(1)}, \mathcal{U}_{\mathrm{reg}})}(j^*(M/(\mathcal{I}^n \cdot M)), j^*(V \otimes (\mathcal{U}\mathfrak{g}/\mathcal{I}^n \cdot \mathcal{U}\mathfrak{g}))) \end{aligned}$$

for any  $n \geq 1$ . In fact, consider the category  $\mathrm{Mod}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}})$  of all (non necessarily finitely generated)  $G$ -equivariant  $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ -modules, and the category  $\mathrm{QCoh}^G(\mathfrak{g}_{\mathrm{reg}}^{*(1)}, \mathcal{U}_{\mathrm{reg}})$  of  $G$ -equivariant quasi-coherent sheaves of  $\mathcal{U}_{\mathrm{reg}}$ -modules. The functor  $j^*$  considered above is the restriction of a functor

$$j^* : \mathrm{Mod}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \rightarrow \mathrm{QCoh}^G(\mathfrak{g}_{\mathrm{reg}}^{*(1)}, \mathcal{U}_{\mathrm{reg}}).$$

which admits a right adjoint

$$j_* : \mathrm{QCoh}^G(\mathfrak{g}_{\mathrm{reg}}^{*(1)}, \mathcal{U}_{\mathrm{reg}}) \rightarrow \mathrm{Mod}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}),$$

which coincides with the usual pushforward functor at the level of quasi-coherent sheaves on  $\mathfrak{g}_{\mathrm{reg}}^{*(1)}$  and  $\mathfrak{g}^{*(1)}$ . We deduce a canonical isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Coh}^G(\mathfrak{g}_{\mathrm{reg}}^{*(1)}, \mathcal{U}_{\mathrm{reg}})}(j^*(M/(\mathcal{I}^n \cdot M)), j^*(V \otimes (\mathcal{U}\mathfrak{g}/\mathcal{I}^n \cdot \mathcal{U}\mathfrak{g}))) &\cong \\ \mathrm{Hom}_{\mathrm{Mod}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g})}(M/(\mathcal{I}^n \cdot M), j_* j^*(V \otimes (\mathcal{U}\mathfrak{g}/\mathcal{I}^n \cdot \mathcal{U}\mathfrak{g}))). \end{aligned}$$

Now since  $\mathcal{U}\mathfrak{g}$  is free of finite rank as an  $\mathcal{O}(\mathfrak{g}^{*(1)})$ -module Lemma 2.7 guarantees that the adjunction morphism

$$V \otimes (\mathcal{U}\mathfrak{g}/\mathcal{I}^n \cdot \mathcal{U}\mathfrak{g}) \rightarrow j_* j^*(V \otimes (\mathcal{U}\mathfrak{g}/\mathcal{I}^n \cdot \mathcal{U}\mathfrak{g}))$$

is an isomorphism, which finishes the proof of our claim.

Now we can conclude as follows. By [R3, Proposition 3.3.11], restriction to  $\mathcal{S}^{*(1)}$  induces an equivalence of abelian categories

$$\mathrm{Coh}^{G^{(1)}}(\mathfrak{g}_{\mathrm{reg}}^{*(1)}) \xrightarrow{\sim} \mathrm{Rep}((\mathbb{J}_S^*)^{(1)}),$$

where the right-hand side denotes the category of representations of the affine group scheme  $(\mathbb{J}_S^*)^{(1)}$  (see §2.9) on coherent  $\mathcal{O}_{\mathcal{S}^{*(1)}}$ -modules. The same considerations

provide an equivalence of categories

$$\mathrm{Coh}^G(\mathfrak{g}_{\mathrm{reg}}^{*(1)}) \xrightarrow{\sim} \mathrm{Rep}(\mathbb{I}_S^*).$$

(Here we use the fact that the Frobenius morphism of  $G$  is flat and surjective, hence faithfully flat.) This equivalence is monoidal with respect to the natural tensor product on each side, and the image of the algebra  $\mathcal{U}_{\mathrm{reg}}$  is  $\mathcal{U}_S$ ; therefore it induces an equivalence of abelian categories

$$\mathrm{Coh}^G(\mathfrak{g}_{\mathrm{reg}}^{*(1)}, \mathcal{U}_{\mathrm{reg}}) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S \mathfrak{g} \otimes_{\mathcal{O}(S^{*(1)})} \mathcal{U}_S \mathfrak{g}),$$

which provides for any  $n$  an isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Coh}^G(\mathfrak{g}_{\mathrm{reg}}^{*(1)}, \mathcal{U}_{\mathrm{reg}})}(j^*(M/(\mathcal{I}^n \cdot M)), j^*(V \otimes (\mathcal{U} \mathfrak{g}/\mathcal{I}^n \cdot \mathcal{U} \mathfrak{g}))) &\xrightarrow{\sim} \\ \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S \mathfrak{g} \otimes_{\mathcal{O}(S^{*(1)})} \mathcal{U}_S \mathfrak{g})}(M_S/(\mathcal{I}^n \cdot M_S), V \otimes (\mathcal{U}_S \mathfrak{g}/\mathcal{I}^n \cdot \mathcal{U}_S \mathfrak{g})). \end{aligned}$$

This concludes the proof.  $\square$

### 3. LOCALIZATION FOR HARISH-CHANDRA BIMODULES

**3.1. Azumaya algebras.** We start by recalling the basic theory of Azumaya algebras.

Let  $R$  be a commutative ring. Recall that an  $R$ -module  $P$  is called *faithfully projective* if it is projective of finite type and if moreover the only  $R$ -module  $M$  such that  $P \otimes_R M = 0$  is  $M = 0$ . By [KO, Chap. I, Lemme 6.2] this condition is equivalent to requiring that  $P$  is projective of finite type and faithful (i.e. its annihilator in  $R$  is trivial). An  $R$ -module  $P$  is finitely generated and projective iff it is finitely presented and moreover the localization  $P_{\mathfrak{p}}$  is free over  $R_{\mathfrak{p}}$  for any  $\mathfrak{p} \in \mathrm{Spec}(R)$ , see [KO, Chap. I, Lemme 5.2] or [SP, Tag 00NX]. In this setting,  $P$  is faithful iff the rank of  $P_{\mathfrak{p}}$  is positive for any  $\mathfrak{p}$ , see [KO, Chap. I, Lemme 6.1]. This notion is important in Morita theory since if  $P$  is a faithfully projective  $R$ -module, then we obtain quasi-inverse equivalences of categories

$$\mathrm{Mod}(R) \xLeftrightarrow{\sim} \mathrm{Mod}(\mathrm{End}_R(P))$$

given by  $M \mapsto P \otimes_R M$  and  $N \mapsto \mathrm{Hom}_R(P, R) \otimes_{\mathrm{End}_R(P)} N$  where  $\mathrm{Mod}(A)$  is the category of left  $A$ -modules for any ring  $A$ ; see [KO, Chap. I, Lemme 7.2]. In case  $R$  is Noetherian, the ring  $\mathrm{End}_R(P)$  is left Noetherian (as a noncommutative ring), and these equivalences restrict to equivalences

$$(3.1) \quad \mathrm{Mod}_{\mathrm{fg}}(R) \xLeftrightarrow{\sim} \mathrm{Mod}_{\mathrm{fg}}(\mathrm{End}_R(P))$$

between subcategories of finitely generated modules. (Here, a left  $\mathrm{End}_R(P)$ -module is finitely generated iff it is finitely generated as an  $R$ -module.)

Let  $A$  be an  $R$ -algebra. (By this mean we mean that we are given a ring morphism from  $R$  to the *center* of the nonnecessarily commutative algebra  $A$ .) Recall (see [KO, §III.5]) that  $A$  is called an Azumaya  $R$ -algebra if it satisfies one of the following equivalent conditions:

- $A$  is faithfully projective as an  $R$ -module, and the morphism sending  $a \otimes b$  to the map  $x \mapsto axb$  induces an isomorphism of  $R$ -algebras

$$A \otimes_R A^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_R(A);$$

- $A$  is finite as an  $R$ -module, the ring morphism  $R \rightarrow A$  is injective, and moreover for any maximal ideal  $\mathfrak{m} \subset R$  the finite dimensional  $R/\mathfrak{m}$ -algebra  $A/\mathfrak{m}A$  is a central simple algebra.

In particular, the first characterization shows that in this case we have canonical equivalences of categories

$$\mathrm{Mod}(R) \xLeftrightarrow{\quad} \mathrm{Mod}(A \otimes_R A^{\mathrm{op}}).$$

**3.2. Azumaya property of  $\mathcal{U}\mathfrak{g}$ .** The following property is standard (see [BG, BGR]); we recall its proof for the reader's convenience.

**Proposition 3.1.** *The  $\mathcal{O}(\mathfrak{C}_S)$ -algebra  $\mathcal{U}\mathfrak{g}$  is Azumaya.*

*Proof.* What we have to prove is that if  $\mathfrak{m} \subset Z(\mathcal{U}\mathfrak{g})$  is a maximal ideal which belongs to  $\mathfrak{C}_S$ , then  $\mathcal{U}\mathfrak{g}/\mathfrak{m}\mathcal{U}\mathfrak{g}$  is a central simple algebra. In fact, this property holds more generally if  $\mathfrak{m}$  belongs to  $\mathfrak{C}_{\mathrm{reg}} := \mathfrak{g}_{\mathrm{reg}}^{*(1)} \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet)$ . Indeed, let  $M(\mathfrak{g})$  be the maximal dimension of a simple  $\mathcal{U}\mathfrak{g}$ -module. By [BG, Proposition 3.1], if  $\mathfrak{m} \subset Z(\mathcal{U}\mathfrak{g})$  is a maximal ideal such that  $\mathcal{U}\mathfrak{g}/\mathfrak{m}\mathcal{U}\mathfrak{g}$  admits a simple module  $V$  of dimension  $M(\mathfrak{g})$ , then  $\mathcal{U}\mathfrak{g}/\mathfrak{m}\mathcal{U}\mathfrak{g}$  is a central simple algebra; more specifically, the algebra morphism  $\mathcal{U}\mathfrak{g}/\mathfrak{m}\mathcal{U}\mathfrak{g} \rightarrow \mathrm{End}_{\mathbb{k}}(V)$  is an isomorphism. Now by [PS, Theorem 4.4] we have  $M(\mathfrak{g}) = p^{\dim(\mathfrak{n})}$ . And by [PS, Theorem 5.6], if  $\mathfrak{m}$  belongs to  $\mathfrak{C}_{\mathrm{reg}}$  then any simple  $\mathcal{U}\mathfrak{g}/\mathfrak{m}\mathcal{U}\mathfrak{g}$ -module has dimension divisible by  $p^{\dim(\mathfrak{n})}$  hence equal to  $M(\mathfrak{g})$ .  $\square$

It follows in particular from Proposition 3.1 that  $\mathcal{U}\mathfrak{g}$  is faithfully projective as an  $\mathcal{O}(\mathfrak{C}_S)$ -module.

One can make the Azumaya property from Proposition 3.1 a bit more concrete as follows. First we need to recall the definition of baby Verma modules. Given a Borel subgroup  $B' \subset G$ , we will denote by  $r_u(B')$  its unipotent radical. Consider some element  $\eta \in \mathfrak{g}^{*(1)}$ , and some Borel subgroup  $B' \subset G$  such that  $\eta$  vanishes on  $\mathrm{Lie}(r_u(B'))^{(1)}$ . (Such a Borel subgroup exists for any  $\eta$ , see [J1, Lemma 6.6].) Then  $\eta$  defines an element in  $(\mathrm{Lie}(B')/\mathrm{Lie}(r_u(B')))^{*(1)}$ . Let  $\xi \in \mathfrak{t}^*$  be an element whose image under the map

$$\mathfrak{t}^* \cong (\mathrm{Lie}(B)/\mathrm{Lie}(U))^* \xrightarrow{\sim} (\mathrm{Lie}(B')/\mathrm{Lie}(r_u(B')))^* \rightarrow (\mathrm{Lie}(B')/\mathrm{Lie}(r_u(B')))^{*(1)}$$

is  $\eta$ , where the second map is induced by conjugation by an element  $g \in G$  such that  $gBg^{-1} = B'$  (it is well known that the isomorphism does not depend on the choice of  $g$ ), and the second one is the Artin–Schreier map associated with the torus  $B'/r_u(B')$ . Then we can consider the associated *baby Verma module*

$$Z_{\eta, B'}(\xi) := \mathcal{U}_{\eta}\mathfrak{g} \otimes_{\mathcal{U}_{\eta}\mathrm{Lie}(B')} \mathbb{k}_{\xi},$$

where  $\mathcal{U}_{\eta}\mathrm{Lie}(B')$  is the central reduction of the enveloping algebra of  $\mathrm{Lie}(B')$  at the image of  $\eta$  in  $\mathrm{Lie}(B')^{*(1)}$ , and  $\mathbb{k}_{\xi}$  is its 1-dimensional module defined by the image of  $\xi$  in  $(\mathrm{Lie}(B')/\mathrm{Lie}(U'))^*$ . This module has dimension  $p^{\dim(\mathfrak{n})}$ ; if we assume furthermore that  $\eta \in \mathfrak{g}_{\mathrm{reg}}^{*(1)}$ , then the considerations in the proof of Proposition 3.1 therefore imply that this module is simple, and that the algebra morphism

$$(3.2) \quad \mathcal{U}_{\eta}^{\xi'}\mathfrak{g} \rightarrow \mathrm{End}_{\mathbb{k}}(Z_{\eta, B'}(\xi))$$

is an isomorphism, where we denote by  $\xi'$  the image of  $\xi$  in  $\mathfrak{t}^*/(W, \bullet)$ .

**3.3. Some categories of coherent sheaves.** Below we will construct some tools that will allow us to study the categories  $\mathrm{HC}^{\hat{\lambda}, \hat{\mu}}$  via geometric methods. In this subsection we introduce the categories of sheaves that will be involved in these constructions. Our model will be the category  $\mathrm{Coh}^G(\mathfrak{C} \times_{\mathfrak{g}^{*(1)}} \mathfrak{C})$  of  $G$ -equivariant coherent sheaves on  $\mathfrak{C} \times_{\mathfrak{g}^{*(1)}} \mathfrak{C}$ , or in other words of  $G$ -equivariant finitely generated  $\mathcal{Z}$ -modules, which is a monoidal category for the operation sending a pair  $(M, N)$  to

$$M \otimes_{Z(\mathcal{U}\mathfrak{g})} N,$$

where in the tensor product  $Z(\mathcal{U}\mathfrak{g})$  acts on  $M$  via the right action and on  $N$  via the left action. The  $\mathcal{Z}$ -action on  $M \otimes_{Z(\mathcal{U}\mathfrak{g})} N$  comes from the left action of  $Z(\mathcal{U}\mathfrak{g})$  on  $M$  and the right action on  $N$ . In practice however, we will have to add generalized characters to this picture.

Recall the isomorphism

$$\mathcal{Z}^\wedge \xrightarrow{\sim} \prod_{\lambda, \mu \in \Lambda} \mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$$

of Lemma 2.3. If we define the notion of a  $G$ -equivariant finitely generated  $\mathcal{Z}^\wedge$ -module, resp.  $\mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$ -module, as a finitely generated  $\mathcal{Z}^\wedge$ -module, resp.  $\mathcal{Z}^{\hat{\lambda}, \hat{\mu}}$ -module,  $M$  together with an algebraic  $G$ -module structure on each quotient  $M/\mathcal{I}^n \cdot M$ , resp.  $M/(\mathcal{I}^{\lambda, \mu})^n \cdot M$ , compatible with the natural projections and action maps, then one obtains in a natural way abelian categories  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}^\wedge)$  and  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}^{\hat{\lambda}, \hat{\mu}})$  whose objects are these equivariant modules. The isomorphism above provides a canonical equivalence of categories

$$(3.3) \quad \mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}^\wedge) \cong \bigoplus_{\lambda, \mu \in \Lambda} \mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}^{\hat{\lambda}, \hat{\mu}}).$$

Following the same pattern as in §2.8, one obtains a monoidal structure on the category  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}^\wedge)$  by setting, for any  $M, N$  in  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}^\wedge)$ ,

$$M \hat{\star} N = \varprojlim_{n \geq 1} (M/\mathcal{I}^n \cdot M) \otimes_{Z(\mathcal{U}\mathfrak{g})} (N/\mathcal{I}^n \cdot N).$$

For any  $\lambda, \mu, \nu \in \mathbb{X}$ , the bifunctor  $\hat{\star}$  restricts to a bifunctor

$$\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}^{\hat{\lambda}, \hat{\mu}}) \times \mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}^{\hat{\mu}, \hat{\nu}}) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}^{\hat{\lambda}, \hat{\nu}}),$$

which will again be denoted  $\hat{\star}$ . (Here we see each of the categories involved as a direct summand in  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}^\wedge)$  via (3.3).) In particular, when  $\lambda = \mu = \nu$  we obtain a monoidal structure on the category  $\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}^{\hat{\lambda}, \hat{\lambda}})$ . The unit object in this category is the completion  $\mathcal{Z}^{\hat{\lambda}}$  of the diagonal  $\mathcal{Z}$ -module  $Z(\mathcal{U}\mathfrak{g})$  with respect to the ideal  $\mathcal{I}^{\lambda, \lambda}$ .

This construction also has natural analogues in the setting of §2.10. As above we have abelian categories  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^\wedge)$  and  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}})$  (where we replace the datum of a  $G$ -module structure by that of an  $\mathbb{I}_S^*$ -module structure), and a canonical equivalence

$$\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^\wedge) \cong \bigoplus_{\lambda, \mu \in \Lambda} \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}).$$

The category  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^\wedge)$  admits a natural monoidal structure, with monoidal product denoted

$$(-) \hat{\star}_S (-) : \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^\wedge) \times \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^\wedge) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^\wedge),$$



which restricts to a bifunctor

$$\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}) \times \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\mu}, \hat{\nu}}) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\lambda}, \hat{\nu}})$$

for any  $\lambda, \mu, \nu \in \mathbb{X}$ . In particular, for any  $\lambda \in \mathbb{X}$  we obtain a monoidal category  $(\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\lambda}, \hat{\lambda}}), \hat{\star}_S)$  with unit object  $\mathcal{Z}_S^{\hat{\lambda}} = \mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} \mathcal{Z}_S^{\hat{\lambda}}$ , which identifies with the completion of  $\mathcal{O}(\mathfrak{t}^*/(W, \bullet))$  with respect to  $\mathfrak{m}^{\lambda}$ .

With these definitions, the functor  $\mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} (-)$  induces functors

$$\mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}^{\wedge}) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\wedge}), \quad \mathrm{Mod}_{\mathrm{fg}}^G(\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}})$$

for  $\lambda, \mu \in \mathbb{X}$  which are compatible with the bifunctors  $\hat{\star}$  and  $\hat{\star}_S$  in the natural way.

**3.4. Splitting bundles for the algebras  $\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}}$ .** Recall that a weight  $\lambda \in \mathbb{X}$  is said to belong to the lower closure of the fundamental alcove if it satisfies

$$0 \leq \langle \lambda + \rho, \alpha^{\vee} \rangle < p$$

for any positive root  $\alpha$ . Recall also the completed bimodules introduced in §2.6. In particular, given  $\lambda, \mu \in \mathbb{X}$  which belong to the lower closure of the fundamental alcove, we have the objects

$$\begin{aligned} \mathbb{P}^{\lambda, -\rho} &= \mathcal{C}^{\lambda, -\rho}(\mathcal{L}(\lambda + \rho) \otimes \mathcal{U}_{\mathfrak{g}}) \in \mathrm{HC}_{\mathrm{diag}}^{\hat{\lambda}, -\rho}, \\ \mathbb{P}^{-\rho, \mu} &= \mathcal{C}^{-\rho, \mu}(\mathcal{L}(-w_0\mu + \rho) \otimes \mathcal{U}_{\mathfrak{g}}) \in \mathrm{HC}_{\mathrm{diag}}^{-\rho, \hat{\mu}}. \end{aligned}$$

We set

$$\mathbb{M}^{\lambda, \mu} := \mathbb{P}^{\lambda, -\rho} \hat{\otimes}_{\mathcal{U}_{\mathfrak{g}}} \mathbb{P}^{-\rho, \mu} \in \mathrm{HC}_{\mathrm{diag}}^{\hat{\lambda}, \hat{\mu}}.$$

We also set  $\mathbb{M}_S^{\lambda, \mu} := \mathcal{O}(\mathcal{S}^{*(1)}) \otimes_{\mathcal{O}(\mathfrak{g}^{*(1)})} \mathbb{M}^{\lambda, \mu}$ , so that

$$\mathbb{M}_S^{\lambda, \mu} = \mathbb{P}_S^{\lambda, -\rho} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{P}_S^{-\rho, \mu}$$

where we use the notation of §2.10.

The main technical result of this section is the following theorem. Its proof will be given in §3.6, after some preliminaries treated in §3.5.

**Theorem 3.2.** *For any  $\lambda, \mu \in \mathbb{X}$  in the lower closure of the fundamental alcove, the  $\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}$ -module  $\mathbb{M}_S^{\lambda, \mu}$  is faithfully projective, and the natural algebra morphism*

$$\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}} \rightarrow \mathrm{End}_{\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}}(\mathbb{M}_S^{\lambda, \mu})$$

*is an isomorphism.*

**3.5. Study of some fibers.** Let us set

$$\mathfrak{Z}_S := \mathrm{Spec}(\mathcal{Z}_S),$$

which we identify with

$$\mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet),$$

see §3.6. We also set

$$\tilde{\mathfrak{Z}}_S := \mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*.$$

Since the Artin–Schreier map  $\mathfrak{t}^* \rightarrow \mathfrak{t}^{*(1)}$  is a Galois covering with Galois group  $\mathfrak{t}_{\mathbb{Z}}^*$ , we have a canonical isomorphism

$$\mathfrak{t}_{\mathbb{Z}}^* \times \mathfrak{t}^* \xrightarrow{\sim} \tilde{\mathfrak{Z}}_S$$

defined by  $(\eta, \xi) \mapsto (\eta + \xi, \xi)$ . For  $\lambda \in \mathbb{X}$  we will denote by  $\tilde{\mathfrak{Z}}_{\mathcal{S}}(\lambda)$  the image of  $\{\overline{\lambda + \rho}\} \times \mathfrak{t}^*$  in  $\tilde{\mathfrak{Z}}_{\mathcal{S}}$ ; if  $\tilde{\Lambda} \subset \mathbb{X}$  is a subset of representatives for the quotient  $\mathfrak{t}_{\mathbb{Z}}^*$ , we then have

$$\tilde{\mathfrak{Z}}_{\mathcal{S}} = \bigsqcup_{\lambda \in \tilde{\Lambda}} \tilde{\mathfrak{Z}}_{\mathcal{S}}(\lambda).$$

We have a natural finite morphism

$$\tilde{\mathfrak{Z}}_{\mathcal{S}} \rightarrow \mathfrak{Z}_{\mathcal{S}}.$$

It is easily seen that the image of the latter morphism contains all the closed points of  $\mathfrak{Z}_{\mathcal{S}}$ , so that this morphism is surjective since closed points are dense in  $\mathfrak{Z}_{\mathcal{S}}$  (see [SP, Tag 02J6]). For any  $\lambda \in \mathbb{X}$  we denote by  $\mathfrak{Z}_{\mathcal{S}}(\lambda)$  the scheme-theoretic image of  $\tilde{\mathfrak{Z}}_{\mathcal{S}}(\lambda)$  in  $\mathfrak{Z}_{\mathcal{S}}$ . Since  $\tilde{\mathfrak{Z}}_{\mathcal{S}}(\lambda)$  is integral, so is  $\mathfrak{Z}_{\mathcal{S}}(\lambda)$ . Moreover, we have

$$\mathfrak{Z}_{\mathcal{S}}(\lambda) = \mathfrak{Z}_{\mathcal{S}}(\mu) \quad \text{iff} \quad \tilde{\lambda} = \tilde{\mu}.$$

If  $\Lambda \subset \mathbb{X}$  is (as in §2.7) a subset of representatives for  $\mathfrak{t}_{\mathbb{Z}}^*/(W, \bullet)$ , we therefore have

$$\mathfrak{Z}_{\mathcal{S}} = \bigcup_{\lambda \in \Lambda} \mathfrak{Z}_{\mathcal{S}}(\lambda),$$

and this constitutes the decomposition of  $\mathfrak{Z}_{\mathcal{S}}$  into its irreducible components.

Let us consider the open subset

$$\mathfrak{t}_{\circ}^* := \{\xi \in \mathfrak{t}^* \mid \forall w \in W, w \bullet \xi - \xi \notin \mathfrak{t}_{\mathbb{Z}}^* \setminus \{0\}\} \subset \mathfrak{t}^*.$$

Then  $\mathfrak{t}_{\circ}^*$  is stable under the  $(W, \bullet)$ -action, and is in fact the pullback of an open subset of  $\mathfrak{t}^*/(W, \bullet)$ , which therefore identifies with the quotient  $\mathfrak{t}_{\circ}^*/(W, \bullet)$ .

Recall the Grothendieck resolution  $\tilde{\mathfrak{g}}$  and the morphism  $\vartheta : \tilde{\mathfrak{g}} \rightarrow \mathfrak{t}^*$  introduced in §2.9. If we denote by  $\tilde{\mathcal{S}}^*$  the (scheme-theoretic) preimage of  $\mathcal{S}^*$  in  $\tilde{\mathfrak{g}}$ , then by [R3, Proposition 3.5.5] the morphism  $\vartheta$  restricts to an isomorphism  $\tilde{\mathcal{S}}^* \xrightarrow{\sim} \mathfrak{t}^*$ . In concrete terms, this means that given  $\zeta \in \mathfrak{t}^{*(1)}/W$  identified with an element in  $\mathcal{S}^{*(1)}$ , the datum of a preimage of  $\zeta$  in  $\mathfrak{t}^{*(1)}$  is equivalent to the datum of a Borel subgroup  $B' \subset G$  such that  $\zeta_{|\text{Lie}(\mathfrak{r}_{\mathfrak{u}}(B'))} = 0$ .

**Proposition 3.3.** *Let  $\lambda \in \mathbb{X}$  be a weight which belongs to the lower closure of the fundamental alcove. Consider some element  $\xi \in \mathfrak{t}_{\circ}^*$ , and denote by  $(\zeta_1, \zeta_2) \in \mathfrak{Z}_{\mathcal{S}}(\lambda)$  the image of  $(\xi + \overline{\lambda + \rho}, \xi) \in \tilde{\mathfrak{Z}}_{\mathcal{S}}(\lambda)$  in  $\mathfrak{Z}_{\mathcal{S}}$ . Let also  $\eta \in \mathcal{S}^{*(1)}$  be the element corresponding to the images of  $\zeta_1$  and  $\zeta_2$  in  $\mathfrak{t}^{*(1)}/W$ , and let  $B' \subset G$  be the Borel subgroup such that  $\eta_{|\text{Lie}(\mathfrak{r}_{\mathfrak{u}}(B'))} = 0$  corresponding to the image of  $\xi$  in  $\mathfrak{t}^{*(1)}$ .*

*If we denote by  $i : \text{Spec}(\mathbb{k}) \rightarrow \mathfrak{Z}_{\mathcal{S}}$  the morphism defined by  $(\zeta_1, \zeta_2)$ , there exists an isomorphism of  $\mathcal{U}_{\eta}^{\zeta_1} \mathfrak{g} \otimes (\mathcal{U}_{\eta}^{\zeta_2} \mathfrak{g})^{\text{op}}$ -modules*

$$i^*(\mathbb{L}(\lambda + \rho) \otimes \mathcal{U}_{\mathcal{S}} \mathfrak{g}) \cong Z_{\eta, B'}(\xi + \overline{\lambda + \rho}) \otimes Z_{\eta, B'}(\xi)^*.$$

*Proof.* By definition we have

$$i^*(\mathbb{L}(\lambda + \rho) \otimes \mathcal{U}_{\mathcal{S}} \mathfrak{g}) \cong \mathbb{k}_{\zeta_1} \otimes_{Z_{\text{HC}}} (\mathbb{L}(\lambda + \rho) \otimes \mathcal{U}_{\eta}^{\zeta_2} \mathfrak{g}).$$

By construction, the image of  $\xi$  in  $\mathfrak{t}^{*(1)}$  corresponds to the element in the space  $(\text{Lie}(B')/\text{Lie}(\mathfrak{r}_{\mathfrak{u}}(B')))^{*(1)}$  defined by  $\eta$ ; by (3.2), we therefore have a canonical isomorphism

$$\mathcal{U}_{\eta}^{\zeta_2} \mathfrak{g} \xrightarrow{\sim} \text{End}_{\mathbb{k}}(Z_{\eta, B'}(\xi)) \cong Z_{\eta, B'}(\xi) \otimes Z_{\eta, B'}(\xi)^*,$$

under which the action of  $\mathcal{U}\mathfrak{g}$  induced by left multiplication on the left-hand side corresponds to the natural action on  $Z_{\eta,B'}(\xi)$ . We deduce an isomorphism

$$i^*(\mathbb{L}(\lambda + \rho) \otimes \mathcal{U}\mathfrak{g}) \cong \mathbb{k}_{\zeta_1} \otimes_{Z_{\text{HC}}} (\mathbb{L}(\lambda + \rho) \otimes Z_{\eta,B'}(\xi)) \otimes Z_{\eta,B'}(\xi)^*,$$

which shows that to conclude the proof it suffices to construct an isomorphism of  $\mathcal{U}_{\eta}^{\zeta_1}$ -modules

$$(3.4) \quad \mathbb{k}_{\zeta_1} \otimes_{Z_{\text{HC}}} (\mathbb{L}(\lambda + \rho) \otimes Z_{\eta,B'}(\xi)) \cong Z_{\eta,B'}(\xi + \overline{\lambda + \rho}).$$

As above we have a canonical isomorphism

$$\mathcal{U}_{\eta}^{\zeta_1} \mathfrak{g} \xrightarrow{\sim} \text{End}_{\mathbb{k}}(Z_{\eta,B'}(\xi + \overline{\lambda + \rho}));$$

therefore, any  $\mathcal{U}_{\eta}^{\zeta_1} \mathfrak{g}$ -module is isomorphic to a direct sum of copies of  $Z_{\eta,B'}(\xi + \overline{\lambda + \rho})$ . To analyze how many copies we have for the specific module in the left-hand side of (3.4), we observe that

$$\begin{aligned} \text{Hom}_{\mathcal{U}_{\eta}^{\zeta_1} \mathfrak{g}}(\mathbb{k}_{\zeta_1} \otimes_{Z_{\text{HC}}} (\mathbb{L}(\lambda + \rho) \otimes Z_{\eta,B'}(\xi)), Z_{\eta,B'}(\xi + \overline{\lambda + \rho})) &= \\ \text{Hom}_{\mathcal{U}_{\eta} \mathfrak{g}}(\mathbb{L}(\lambda + \rho) \otimes Z_{\eta,B'}(\xi), Z_{\eta,B'}(\xi + \overline{\lambda + \rho})) &\cong \\ \text{Hom}_{\mathcal{U}_{\eta} \mathfrak{g}}(Z_{\eta,B'}(\xi), \mathbb{L}(-w_0\lambda + \rho) \otimes Z_{\eta,B'}(\xi + \overline{\lambda + \rho})). \end{aligned}$$

We now consider the module  $\mathbb{L}(-w_0\lambda + \rho) \otimes Z_{\eta,B'}(\xi + \overline{\lambda + \rho})$ , and more specifically the direct summand on which  $Z_{\text{HC}}$  acts with a generalized character corresponding to  $\zeta_2$ . We have a canonical isomorphism of  $\mathcal{U}_{\eta} \mathfrak{g}$ -modules

$$\mathbb{L}(-w_0\lambda + \rho) \otimes Z_{\eta,B'}(\xi + \overline{\lambda + \rho}) \cong \mathcal{U}_{\eta} \mathfrak{g} \otimes_{\mathcal{U}_{\eta} \text{Lie}(B')} (\mathbb{L}(-w_0\lambda + \rho)|_{B'} \otimes \mathbb{k}_{\xi + \overline{\lambda + \rho}}).$$

The  $B'$ -module  $\mathbb{L}(-w_0\lambda + \rho)|_{B'}$  admits a filtration

$$0 \subset M_1 \subset \cdots \subset M_n = \mathbb{L}(-w_0\lambda + \rho)|_{B'}$$

where each  $M_i/M_{i-1}$  is 1-dimensional; moreover these modules are associated with the characters of  $B'/r_u(B') \cong B/U \cong T$  corresponding to the  $T$ -weights of  $\mathbb{L}(-w_0\lambda + \rho)$ , counted with multiplicities. This filtration induces a filtration of  $\mathbb{L}(-w_0\lambda + \rho)|_{B'} \otimes \mathbb{k}_{\xi + \overline{\lambda + \rho}}$ , and then of  $\mathbb{L}(-w_0\lambda + \rho) \otimes Z_{\eta,B'}(\xi + \overline{\lambda + \rho})$ , whose subquotients are of the form  $Z_{\eta,B'}(\xi + \overline{\lambda + \rho + \mu})$ , where  $\mu$  runs over the  $T$ -weights of  $\mathbb{L}(-w_0\lambda + \rho)$ , counted with multiplicities.

We claim that there exists exactly one subquotient in this filtration on which  $Z_{\text{HC}}$  acts via the character  $\zeta_2$ , corresponding to the multiplicity-1 weight  $-\lambda - \rho$  of  $\mathbb{L}(-w_0\lambda + \rho)$ . Indeed, assume that  $Z_{\text{HC}}$  acts with character  $\zeta_2$  on  $Z_{\eta,B'}(\xi + \overline{\lambda + \rho + \mu})$ . Then there exists  $w \in W$  such that  $\xi + \overline{\lambda + \rho + \mu} = w \bullet \xi$ . Since  $\xi$  belongs to  $\mathfrak{t}_*^*$ , this condition implies that  $\xi + \overline{\lambda + \rho + \mu} = \xi$ , hence that  $\lambda + \rho + \mu \in p\mathbb{X}$ . On the other hand,  $\mu$  is a weight of  $\mathbb{L}(-w_0\lambda + \rho)$ , hence it belongs to  $-w_0\lambda + \rho + \mathbb{Z}\mathfrak{R} = -\lambda - \rho + \mathbb{Z}\mathfrak{R}$ . In view of (2.1) these conditions imply that  $\lambda + \rho + \mu \in p\mathbb{Z}\mathfrak{R}$ , i.e. that  $\lambda + \mu \in -\rho + p\mathbb{Z}\mathfrak{R} = W_{\text{aff}} \bullet (-\rho)$ . By [J2, Lemma II.7.7] (applied to the pair of elements  $(\lambda, -\rho)$ ), there must then exist  $w \in W_{\text{aff}}$  such that  $w \bullet \lambda = \lambda$  and  $\lambda + \mu = w \bullet (-\rho)$ . Here, since  $\lambda$  belongs to the lower closure of the fundamental alcove, the first condition implies that  $w \in W$  (see §2.6); it follows that  $w \bullet (-\rho) = -\rho$ , hence that  $\lambda + \mu = -\rho$ , which finishes the proof of our claim.

This claim implies that the direct summand of  $\mathbb{L}(-w_0\lambda + \rho) \otimes Z_{\eta,B'}(\xi + \overline{\lambda + \rho})$  corresponding to the generalized character of  $Z_{\text{HC}}$  given by  $\zeta_2$  is isomorphic to

$Z_{\eta, B'}(\xi)$ ; it follows that

$$\mathrm{Hom}_{\mathcal{U}_{\eta}^{\zeta_1} \mathfrak{g}} \left( \mathbb{k}_{\zeta_1} \otimes_{Z_{\mathrm{HC}}} (\mathbb{L}(\lambda + \rho) \otimes Z_{\eta, B'}(\xi)), Z_{\eta, B'}(\xi + \overline{\lambda + \rho}) \right)$$

is 1-dimensional, which finally proves (3.4).  $\square$

The statement of Proposition 3.3 is not symmetric, in that the conditions we impose imply that  $\zeta_2$  necessarily belongs to  $\mathfrak{t}_{\circ}^*/(W, \bullet)$ , whereas  $\zeta_1$  might not. Below we will also need the other variant of this statement, in which the *first* component has to belong to  $\mathfrak{t}_{\circ}^*/(W, \bullet)$ . Its proof is analogous to that of Proposition 3.3. (More precisely, in this case the counterpart of (3.4) can be obtained directly, without recourse to the computation in the paragraph following this equation.)

**Proposition 3.4.** *Let  $\mu \in \mathbb{X}$  be a weight which belongs to the lower closure of the fundamental alcove. Consider some element  $\xi \in \mathfrak{t}_{\circ}^*$ , and denote by  $(\zeta_1, \zeta_2) \in \mathfrak{Z}_{\mathcal{S}}(-w_0\mu)$  the image of  $(\xi, \xi + \overline{\mu + \rho}) \in \mathfrak{Z}_{\mathcal{S}}(-w_0\mu)$  in  $\mathfrak{Z}_{\mathcal{S}}$ . Let also  $\eta \in \mathcal{S}^{*(1)}$  be the element corresponding to the images of  $\zeta_1$  and  $\zeta_2$  in  $\mathfrak{t}^{*(1)}/W$ , and let  $B' \subset G$  be the Borel subgroup such that  $\eta|_{\mathrm{Lie}(\mathfrak{r}_{\mathrm{u}}(B'))} = 0$  corresponding to the image of  $\xi$  in  $\mathfrak{t}^{*(1)}$ .*

*If we denote by  $i : \mathrm{Spec}(\mathbb{k}) \rightarrow \mathfrak{Z}_{\mathcal{S}}$  the morphism defined by  $(\zeta_1, \zeta_2)$ , there exists an isomorphism of  $\mathcal{U}_{\eta}^{\zeta_1} \mathfrak{g} \otimes (\mathcal{U}_{\eta}^{\zeta_2} \mathfrak{g})^{\mathrm{op}}$ -modules*

$$i^*(\mathbb{L}(-w_0\mu + \rho) \otimes \mathcal{U}_{\mathcal{S}} \mathfrak{g}) \cong Z_{\eta, B'}(\xi) \otimes Z_{\eta, B'}(\xi + \overline{\mu + \rho})^*.$$

**3.6. Proof of Theorem 3.2.** The proof of Theorem 3.2 will require two more preliminary lemmas.

**Lemma 3.5.** *Let  $X$  be a reduced scheme locally of finite type over  $\mathbb{k}$ , and let  $\mathcal{F}$  be a coherent sheaf on  $X$ . Assume that there exists  $d \geq 0$  such that for any morphism  $i : \mathrm{Spec}(\mathbb{k}) \rightarrow X$  the pullback  $i^*(\mathcal{F}) \in \mathrm{Coh}(\mathrm{Spec}(\mathbb{k})) = \mathrm{Vect}_{\mathbb{k}}$  has dimension  $d$ . Then  $\mathcal{F}$  is a locally free  $\mathcal{O}_X$ -module of rank  $d$ .*

*Proof.* Of course we can assume that  $X$  is also affine and of finite type, i.e. that  $X = \mathrm{Spec}(A)$  for some finitely generated reduced  $\mathbb{k}$ -algebra  $A$ . Let us denote by  $M$  the  $A$ -module corresponding to  $\mathcal{F}$ . In this setting the datum of a morphism  $i : \mathrm{Spec}(\mathbb{k}) \rightarrow X$  is equivalent to the datum of a maximal ideal  $\mathfrak{m} \subset A$ , and we have  $i^*(\mathcal{F}) = M/\mathfrak{m} \cdot M$ . In view of [SP, Tag 0FWG], to show that  $M$  is locally free of rank  $d$  it suffices to prove that for any  $\mathfrak{p} \in \mathrm{Spec}(A)$  we have

$$\dim_{A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}} (M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}) = d.$$

Now by [Pe, Theorem 7.33], the function

$$\mathfrak{p} \mapsto \dim_{A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}} (M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}})$$

is upper semi-continuous. By assumption, this function is constant (equal to  $d$ ) on the subset of  $\mathrm{Spec}(A)$  consisting of maximal ideals, i.e. of closed points. Hence the open subset

$$\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \dim_{A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}} (M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}) \leq d\}$$

contains all closed points. Since  $\mathrm{Spec}(A)$  is Jacobson (see [SP, Tag 02J6]) this implies that this open subset is the whole of  $\mathrm{Spec}(A)$ . On the other hand the open subset

$$\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \dim_{A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}} (M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}) \leq d - 1\}$$

does not contain any closed point, hence it is empty.  $\square$

**Lemma 3.6.** *The morphism*

$$\mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet) \rightarrow \mathfrak{Z}_S$$

*induced by projection on the first and third factors is étale at any point of the form  $(\widetilde{\lambda}, \widetilde{-\rho}, \widetilde{\mu})$  with  $\lambda, \mu \in \mathbb{X}$ .*

*Proof.* To prove this claim it suffices to prove that the morphism  $\mathfrak{t}^*/(W, \bullet) \rightarrow \mathfrak{t}^{*(1)}/W$  is étale at  $\widetilde{-\rho}$ . The dot-action of  $W$  and the natural action of  $\mathfrak{t}_{\mathbb{Z}}^*$  on  $\mathfrak{t}^*$  combine to provide an action of the semi-direct product  $\mathfrak{t}_{\mathbb{Z}}^* \rtimes W$  (where  $W$  acts on  $\mathfrak{t}_{\mathbb{Z}}^*$  through the natural, unshifted, action) defined by  $(\bar{\lambda}w) \bullet \xi = w(\xi + \bar{\rho}) - \bar{\rho} + \bar{\lambda}$  for  $\bar{\lambda} \in \mathfrak{t}_{\mathbb{Z}}^*$  and  $w \in W$ . Moreover, the composition  $\mathfrak{t}^* \rightarrow \mathfrak{t}^*/(W, \bullet) \rightarrow \mathfrak{t}^{*(1)}/W$  is the quotient morphism for this action. Since  $\widetilde{-\rho}$  is stabilized by  $W$ , the claim then follows from [SGA1, Exp. V, Proposition 2.2].  $\square$

For  $\lambda \in \mathbb{X}$ , whose image in  $\mathfrak{t}_{\mathbb{Z}}^*/(W, \bullet)$  is that of  $\lambda' \in \Lambda$ , we set

$$\mathfrak{Z}_S^0(\lambda) := \left( \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}_o^*/(W, \bullet) \right) \setminus \left( \bigcup_{\mu \in \Lambda \setminus \{\lambda'\}} \mathfrak{Z}_S(\mu) \right).$$

Then  $\mathfrak{Z}_S^0(\lambda)$  is an open subset of  $\mathfrak{Z}_S$ , contained in  $\mathfrak{Z}_S(\lambda)$ . We will denote by  $j_\lambda : \mathfrak{Z}_S^0(\lambda) \rightarrow \mathfrak{Z}_S$  the embedding.

Continuing with the same notation, we also set

$$\mathfrak{Z}_S^{0'}(\lambda) := \left( \mathfrak{t}_o^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet) \right) \setminus \left( \bigcup_{\mu \in \Lambda \setminus \{\lambda'\}} \mathfrak{Z}_S(\mu) \right),$$

and we denote by  $j'_\lambda : \mathfrak{Z}_S^{0'}(\lambda) \rightarrow \mathfrak{Z}_S$  the open embedding.

*Proof of Theorem 3.2.* Let  $\lambda, \mu \in \mathbb{X}$  which belong to the lower closure of the fundamental alcove. Note that  $(\widetilde{\lambda}, \widetilde{-\rho}) \in \mathfrak{Z}_S^0(\lambda)$ ; in fact,  $\widetilde{-\rho}$  belongs to  $\mathfrak{t}_o^*$  since this point is stable under the dot-action of  $W$ . On the other hand, if  $\nu \in \mathbb{X}$  is such that  $(\widetilde{\lambda}, \widetilde{-\rho}) \in \mathfrak{Z}_S(\nu)$ , then there exists  $\xi \in \mathfrak{t}^*$  such that the point  $(\xi + \bar{\nu} + \bar{\rho}, \xi) \in \mathfrak{Z}_S$  has image  $(\widetilde{\lambda}, \widetilde{-\rho})$  in  $\mathfrak{Z}_S$ ; we then have  $\xi \in W \bullet \widetilde{-\rho} = \{\widetilde{-\rho}\}$  and  $\xi + \bar{\nu} + \bar{\rho} \in W \bullet \widetilde{\lambda}$ , so that  $\widetilde{\lambda} = \widetilde{\nu}$ . Similarly, we have  $(\widetilde{-\rho}, \widetilde{\mu}) \in \mathfrak{Z}_S^{0'}(-w_0\mu)$ .

Consider the morphism

$$f : \mathfrak{Z}_S^0(\lambda) \times_{\mathfrak{t}^*/(W, \bullet)} \mathfrak{Z}_S^{0'}(-w_0\mu) \rightarrow \mathfrak{t}^*/(W, \bullet)$$

induced by projection on the middle summand. The algebra  $\mathcal{U}_S \mathfrak{g}$  is an  $\mathcal{O}(\mathfrak{t}^*/(W, \bullet))$ -algebra; it therefore defines a coherent sheaf of  $\mathcal{O}_{\mathfrak{t}^*/(W, \bullet)}$ -algebras  $\mathcal{A}$  on  $\mathfrak{t}^*/(W, \bullet)$ . Consider also the projections

$$\begin{aligned} p : \mathfrak{Z}_S^0(\lambda) \times_{\mathfrak{t}^*/(W, \bullet)} \mathfrak{Z}_S^{0'}(-w_0\mu) &\rightarrow \mathfrak{Z}_S^0(\lambda), \\ q : \mathfrak{Z}_S^0(\lambda) \times_{\mathfrak{t}^*/(W, \bullet)} \mathfrak{Z}_S^{0'}(-w_0\mu) &\rightarrow \mathfrak{Z}_S^{0'}(-w_0\mu). \end{aligned}$$

The sheaves  $p^* j_\lambda^*(\mathcal{L}(\lambda + \rho) \otimes \mathcal{U}_S \mathfrak{g})$  and  $q^*(j'_{-w_0\mu})^*(\mathcal{L}(-w_0\mu + \rho) \otimes \mathcal{U}_S \mathfrak{g})$  are naturally sheaves of modules for  $f^* \mathcal{A}$ , so that we can consider the tensor product

$$(3.5) \quad p^* j_\lambda^*(\mathcal{L}(\lambda + \rho) \otimes \mathcal{U}_S \mathfrak{g}) \otimes_{f^* \mathcal{A}} q^*(j'_{-w_0\mu})^*(\mathcal{L}(-w_0\mu + \rho) \otimes \mathcal{U}_S \mathfrak{g}).$$

We claim that this sheaf is a locally free  $\mathcal{O}_{\mathfrak{Z}_S^0(\lambda) \times_{\mathfrak{t}^*/(W, \bullet)} \mathfrak{Z}_S^{0'}(-w_0\mu)}$ -module, of rank  $p^{2 \dim(\mathfrak{n})}$ . In fact, by Lemma 3.5, to prove this it suffices to prove that for any

closed point  $(\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{Z}_S^0(\lambda) \times_{\mathfrak{t}^*/(W, \bullet)} \mathfrak{Z}_S^{0'}(-w_0\mu)$ , denoting by  $i : \text{Spec}(\mathbb{k}) \rightarrow \mathfrak{Z}_S^0(\lambda) \times_{\mathfrak{t}^*/(W, \bullet)} \mathfrak{Z}_S^{0'}(-w_0\mu)$  the corresponding morphism, the vector space

$$(3.6) \quad i^*(p^*j_\lambda^*(\mathbb{L}(\lambda + \rho) \otimes \mathcal{U}_S \mathfrak{g}) \otimes_{f^* \mathcal{A}} q^*(j'_{-w_0\mu})^*(\mathbb{L}(-w_0\mu + \rho) \otimes \mathcal{U}_S \mathfrak{g}))$$

has dimension  $p^{2 \dim(\mathfrak{n})}$ . If we denote by  $i_1 : \text{Spec}(\mathbb{k}) \rightarrow \mathfrak{Z}_S$  and  $i_2 : \text{Spec}(\mathbb{k}) \rightarrow \mathfrak{Z}_S$  the embeddings of the points  $(\zeta_1, \zeta_2)$  and  $(\zeta_2, \zeta_3)$  respectively, then this vector space can be written as

$$i_1^*(\mathbb{L}(\lambda + \rho) \otimes \mathcal{U}_S \mathfrak{g}) \otimes_{\mathcal{U}_\eta^{\zeta_2} \mathfrak{g}} i_2^*(\mathbb{L}(-w_0\mu + \rho) \otimes \mathcal{U}_S \mathfrak{g}),$$

where  $\eta \in \mathcal{S}^{*(1)}$  is the image of the  $\zeta_i$ 's. Let  $\xi \in \mathfrak{t}^*$  such that  $(\zeta_1, \zeta_2)$  is the image of  $(\xi + \overline{\lambda + \rho}, \xi)$ , and let  $B' \subset G$  be the Borel subgroup such that  $\eta|_{\text{Lie}(\mathfrak{r}_u(B'))} = 0$  corresponding to the image of  $\xi$  in  $\mathfrak{t}^{*(1)}$ . By Proposition 3.3 we have

$$i_1^*(\mathbb{L}(\lambda + \rho) \otimes \mathcal{U}_S \mathfrak{g}) \cong Z_{\eta, B'}(\xi + \overline{\lambda + \rho}) \otimes Z_{\eta, B'}(\xi)^*.$$

Similarly, if  $\xi' \in \mathfrak{t}^*$  is such that  $(\zeta_2, \zeta_3)$  is the image of  $(\xi', \xi' + \overline{\mu + \rho})$ , and if  $B'' \subset G$  is the Borel subgroup such that  $\eta|_{\text{Lie}(\mathfrak{r}_u(B''))} = 0$  corresponding to the image of  $\xi'$  in  $\mathfrak{t}^{*(1)}$ , then we have

$$i_2^*(\mathbb{L}(-w_0\mu + \rho) \otimes \mathcal{U}_S \mathfrak{g}) \cong Z_{\eta, B''}(\xi') \otimes Z_{\eta, B''}(\xi' + \overline{\mu + \rho})^*.$$

Here  $Z_{\eta, B'}(\xi)$  and  $Z_{\eta, B''}(\xi')$  are two simple modules over the matrix algebra  $\mathcal{U}_\eta^{\zeta_2} \mathfrak{g}$ , see §3.2; they must therefore be isomorphic. Fixing an isomorphism  $\varphi : Z_{\eta, B'}(\xi) \xrightarrow{\sim} Z_{\eta, B''}(\xi')$ , we obtain a pairing

$$Z_{\eta, B'}(\xi)^* \otimes Z_{\eta, B''}(\xi') \rightarrow \mathbb{k}$$

defined by  $f \otimes v \mapsto f(\varphi^{-1}(v))$ , which induces an isomorphism

$$Z_{\eta, B'}(\xi)^* \otimes_{\mathcal{U}_\eta^{\zeta_2} \mathfrak{g}} Z_{\eta, B''}(\xi') \xrightarrow{\sim} \mathbb{k}.$$

Combining these observations we obtain that the vector space in (3.6) is isomorphic to

$$Z_{\eta, B'}(\xi + \overline{\lambda + \rho}) \otimes Z_{\eta, B''}(\xi' + \overline{\mu + \rho})^*,$$

hence has dimension  $p^{2 \dim(\mathfrak{n})}$ , as desired.

Now we consider the morphism

$$\mathfrak{Z}_S^0(\lambda) \times_{\mathfrak{t}^*/(W, \bullet)} \mathfrak{Z}_S^{0'}(-w_0\mu) \rightarrow \mathfrak{Z}_S$$

obtained from that of Lemma 3.6 by restriction to the open subset

$$\mathfrak{Z}_S^0(\lambda) \times_{\mathfrak{t}^*/(W, \bullet)} \mathfrak{Z}_S^{0'}(-w_0\mu) \subset \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet).$$

This lemma ensures that this morphism is étale at  $(\tilde{\lambda}, \tilde{-\rho}, \tilde{\mu})$ ; it therefore identifies the completion of  $\mathfrak{Z}_S^0(\lambda) \times_{\mathfrak{t}^*/(W, \bullet)} \mathfrak{Z}_S^{0'}(-w_0\mu)$  at  $(\tilde{\lambda}, \tilde{-\rho}, \tilde{\mu})$  with the completion of  $\mathfrak{Z}_S$  at  $(\tilde{\lambda}, \tilde{\mu})$ , i.e. with the spectrum of  $\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}$ . By construction the completion of the sheaf (3.5) at  $(\tilde{\lambda}, \tilde{-\rho}, \tilde{\mu})$  is  $\mathbb{M}_S^{\lambda, \mu}$ ; since this sheaf is locally free this proves that  $\mathbb{M}_S^{\lambda, \mu}$  is faithfully projective. In fact, since the ring  $\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}$  is local, this module is even free (of rank  $p^{2 \dim(\mathfrak{n})}$ ).

Finally we consider the natural morphism

$$\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}} \rightarrow \text{End}_{\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}}(\mathbb{M}_S^{\lambda, \mu}).$$

Here, both sides are free as modules over  $\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}$ . In fact, for the ring-hand side this follows from the same property for the module  $\mathbb{M}_S^{\lambda, \mu}$ , which we have seen above. For the left-hand side, we observe that  $\mathcal{U}_S \mathfrak{g}$  is projective over  $\mathcal{O}(\mathfrak{C}_S)$  by Lemma 3.1; it follows that  $\mathcal{U}_S \mathfrak{g} \otimes_{\mathcal{O}(\mathfrak{S}^{*(1)})} \mathcal{U}_S \mathfrak{g}^{\text{op}}$  is projective over  $\mathcal{Z}_S$ , and finally that  $\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}}$  is projective, hence free, over the local ring  $\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}$ . Given this property, to prove that our morphism is an isomorphism it suffices to prove that it is invertible after application of the functor  $\mathbb{k} \otimes_{\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}} (-)$ . Now we have

$$\mathbb{k} \otimes_{\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}} \mathcal{U}_S^{\hat{\lambda}, \hat{\mu}} = \mathcal{U}_\chi^\lambda \mathfrak{g} \otimes (\mathcal{U}_\chi^\mu \mathfrak{g})^{\text{op}},$$

where  $\chi := \varkappa(e)$ , seen as a point in  $\mathfrak{g}^{*(1)}$ . (See §2.9 for the definitions of  $\varkappa$  and  $e$ .) On the other hand, since  $\mathbb{M}_S^{\lambda, \mu}$  is a free module we have

$$\mathbb{k} \otimes_{\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}} \text{End}_{\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}}(\mathbb{M}_S^{\lambda, \mu}) \cong \text{End}_{\mathbb{k}}(\mathbb{k} \otimes_{\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}} \mathbb{M}_S^{\lambda, \mu}),$$

and applying the considerations above with  $\xi = \xi' = -\bar{\rho}$  we have

$$(3.7) \quad \mathbb{k} \otimes_{\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}} \mathbb{M}_S^{\lambda, \mu} \cong Z_{\chi, B'}(\bar{\lambda}) \otimes Z_{\chi, B'}(\bar{\mu})^*,$$

where  $B' \subset G$  is the unique Borel subgroup such that  $\chi|_{\text{Lie}(\mathfrak{r}_u(B'))} = 0$ . By (3.2) our morphism is indeed an isomorphism, which finishes the proof.  $\square$

**3.7. Localization for Harish-Chandra bimodules.** The main consequence of Theorem 3.2 that will be used below is the following statement.

**Corollary 3.7.** *For any  $\lambda, \mu \in \mathbb{X}$  in the lower closure of the fundamental alcove, the functor  $\mathbb{M}_S^{\lambda, \mu} \otimes_{\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}} (-)$  induces an equivalence of abelian categories*

$$\mathcal{L}_{\lambda, \mu} : \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}) \xrightarrow{\sim} \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}}).$$

Moreover, in case  $\lambda = \mu$ , there exists a canonical isomorphism

$$(3.8) \quad \mathcal{L}_{\lambda, \lambda}(\mathcal{Z}_S^{\hat{\lambda}}) \cong \mathcal{U}_S^{\hat{\lambda}}.$$

*Proof.* The properties stated in Theorem 3.2 ensure that the functor

$$\mathbb{M}_S^{\lambda, \mu} \otimes_{\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}} (-)$$

induces an equivalence of abelian categories

$$\text{Mod}_{\text{fg}}(\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}) \xrightarrow{\sim} \text{Mod}_{\text{fg}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}}),$$

see (3.1). Adding the  $\mathbb{I}_S^*$ -actions in the picture we obtain the desired equivalence

$$\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\lambda}, \hat{\mu}}) \xrightarrow{\sim} \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}}).$$

Finally, we consider the special case  $\lambda = \mu$ , and construct a canonical isomorphism  $\mathcal{L}_{\lambda, \lambda}(\mathcal{Z}_S^{\hat{\lambda}}) \cong \mathcal{U}_S^{\hat{\lambda}}$ . Adjunction (see Lemma 2.5) provides a canonical morphism

$$\mathbb{P}_S^{\lambda, -\rho} \hat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{-\rho, \lambda} \rightarrow \mathcal{U}_S^{\hat{\lambda}},$$

which factors through a morphism

$$\mathcal{L}_{\lambda, \lambda}(\mathcal{Z}_S^{\hat{\lambda}}) = \mathbb{M}_S^{\lambda, \lambda} \otimes_{\mathcal{Z}_S^{\hat{\lambda}, \hat{\lambda}}} \mathcal{Z}_S^{\hat{\lambda}} \rightarrow \mathcal{U}_S^{\hat{\lambda}}.$$

Here both sides are finite free modules over the local ring  $\widehat{\mathcal{Z}}_S$ ; to prove that this morphism is an isomorphism it therefore suffices to check that the induced morphism

$$\left( \mathbb{M}_S^{\lambda, \lambda} \otimes_{\widehat{\mathcal{Z}}_S^{\lambda, \lambda}} \widehat{\mathcal{Z}}_S^{\lambda} \right) \otimes_{\widehat{\mathcal{Z}}_S^{\lambda}} \mathbb{k} \rightarrow \mathcal{U}_S^{\lambda} \otimes_{\widehat{\mathcal{Z}}_S^{\lambda}} \mathbb{k}$$

is invertible. The right-hand side identifies with  $\mathcal{U}_\chi^\lambda \mathfrak{g}$ , and by (3.7) the left-hand side identifies with  $Z_{\chi, B'}(\bar{\lambda}) \otimes Z_{\chi, B'}(\bar{\lambda})^*$ , where  $B' \subset G$  is the unique Borel subgroup such that  $\chi|_{\text{Lie}(\text{ru}(B'))} = 0$ ; the desired claim is therefore clear from the isomorphism (3.2).  $\square$

*Remark 3.8.* We will prove later (at least in the special case when  $\mu$  belongs to the fundamental alcove, see §4.5) that the equivalences of Corollary 3.7 intertwine the bifunctors

$$\widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} : \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\lambda, \hat{\mu}}) \times \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\mu}, \hat{\nu}}) \rightarrow \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\lambda, \hat{\nu}})$$

and

$$\widehat{\star}_S : \text{Mod}_{\text{fg}}^{\mathbb{I}}(\widehat{\mathcal{Z}}_S^{\lambda, \hat{\mu}}) \times \text{Mod}_{\text{fg}}^{\mathbb{I}}(\widehat{\mathcal{Z}}_S^{\hat{\mu}, \hat{\nu}}) \rightarrow \text{Mod}_{\text{fg}}^{\mathbb{I}}(\widehat{\mathcal{Z}}_S^{\lambda, \hat{\nu}})$$

for any  $\lambda, \nu \in \mathbb{X}$ .

#### 4. $\mathcal{U}\mathfrak{g}$ AND DIFFERENTIAL OPERATORS ON THE FLAG VARIETY

In this section we study the equivalences  $\mathcal{L}_{\lambda, \mu}$  of Corollary 3.7 further, using the relation between the algebra  $\mathcal{U}\mathfrak{g}$  and differential operators on the flag variety of  $G$ .

**4.1. Universal twisted differential operators.** Set  $\mathcal{B} := G/B$ , and consider the natural projection morphism

$$\omega : G/U \rightarrow \mathcal{B}.$$

Here  $G/U$  admits a natural action of  $T$  induced by multiplication on the right on  $G$ , and  $\omega$  is a (Zariski locally trivial)  $T$ -torsor. The sheaf of universal twisted differential operators on  $\mathcal{B}$  is the quasi-coherent sheaf of algebras

$$\widetilde{\mathcal{D}} := \omega_*(\mathcal{D}_{G/U})^T,$$

where the exponent means  $T$ -invariants. The actions of  $G$  and  $T$  on  $G/U$  induce a canonical algebra morphism

$$(4.1) \quad \mathcal{U}\mathfrak{g} \otimes_{Z_{\text{HC}}} \mathcal{O}(\mathfrak{t}^*) \rightarrow \Gamma(\mathcal{B}, \widetilde{\mathcal{D}}),$$

see [BMR1, Lemma 3.1.5].

Recall the Grothendieck resolution  $\widetilde{\mathfrak{g}}$  introduced in §2.9. Consider the Frobenius morphism  $\text{Fr}_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}^{(1)}$  and the natural morphism  $f : \widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^* \rightarrow \mathcal{B}^{(1)}$ . As explained in [BMR1, §2.3], there exists a canonical algebra morphism

$$f_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*} \rightarrow (\text{Fr}_{\mathcal{B}})_* \widetilde{\mathcal{D}}$$

which takes values in the center of  $(\text{Fr}_{\mathcal{B}})_* \widetilde{\mathcal{D}}$ , and which makes  $(\text{Fr}_{\mathcal{B}})_* \widetilde{\mathcal{D}}$  a locally finitely generated  $f_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}$ -module. Since all the morphisms involved in this construction are affine, using this morphism one can consider  $\widetilde{\mathcal{D}}$  as a coherent sheaf of  $\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}$ -algebras on  $\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$ . (We will not introduce a different notation for this sheaf of algebras.)



Recall also (see §3.5) that we denote by  $\tilde{\mathcal{S}}^*$  the preimage of  $\mathcal{S}^*$  under the natural morphism  $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}^*$ , and that the morphism  $\vartheta$  restricts to an isomorphism  $\tilde{\mathcal{S}}^* \xrightarrow{\sim} \mathfrak{t}^*$ ; in particular,  $\tilde{\mathcal{S}}^*$  is an affine scheme. We set

$$\tilde{\mathcal{D}}_{\mathcal{S}} := \tilde{\mathcal{D}}|_{\tilde{\mathcal{S}}^{*(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}, \quad \mathbb{D}_{\mathcal{S}} := \Gamma(\tilde{\mathcal{S}}^{*(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*, \tilde{\mathcal{D}}_{\mathcal{S}}).$$

We will also set

$$\tilde{\mathcal{U}}_{\mathcal{S}\mathfrak{g}} := \mathcal{U}_{\mathcal{S}\mathfrak{g}} \otimes_{Z_{\text{HC}}} \mathcal{O}(\mathfrak{t}^*).$$

**Lemma 4.1.** *The morphism (4.1) induces an algebra isomorphism*

$$\tilde{\mathcal{U}}_{\mathcal{S}\mathfrak{g}} \xrightarrow{\sim} \mathbb{D}_{\mathcal{S}}.$$

*Proof.* Consider the natural morphism

$$h : \tilde{\mathcal{S}}^{*(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^* \rightarrow \mathcal{S}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet).$$

If we still denote by  $\mathcal{U}_{\mathcal{S}\mathfrak{g}}$  the sheaf of  $\mathcal{O}_{\mathcal{S}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)}$ -algebras associated with this  $\mathcal{O}(\mathcal{S}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet))$ -algebra, then as in [BMR1, Proposition 5.2.1] the morphism (4.1) induces a canonical isomorphism of sheaves of algebras

$$h^*(\mathcal{U}_{\mathcal{S}\mathfrak{g}}) \xrightarrow{\sim} \tilde{\mathcal{D}}_{\mathcal{S}}.$$

Now  $h$  induces an isomorphism

$$\tilde{\mathcal{S}}^{*(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^* \rightarrow (\mathcal{S}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)) \times_{\mathfrak{t}^*/(W, \bullet)} \mathfrak{t}^*$$

(in fact, both sides identify canonically with  $\mathfrak{t}^*$ ) so that the claim follows by taking global sections.  $\square$

*Remark 4.2.* One can give a different proof of Lemma 4.1 as follows. By [BMR1, Proposition 3.4.1], the morphism (4.1) is an isomorphism; in other words, identifying quasi-coherent sheaves on  $\mathfrak{g}^{*(1)}$  and  $\mathcal{O}(\mathfrak{g}^{*(1)})$ -modules, we have a canonical isomorphism of sheaves of  $\mathcal{O}(\mathfrak{g}^{*(1)})$ -algebras

$$g_* \tilde{\mathcal{D}} \cong \mathcal{U}_{\mathfrak{g}} \otimes_{Z_{\text{HC}}} \mathcal{O}(\mathfrak{t}^*),$$

where  $g : \tilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^* \rightarrow \mathfrak{g}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$  is the morphism induced by  $\pi$ . Restricting this isomorphism first to  $\mathfrak{g}_{\text{reg}}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$  and then to  $\mathcal{S}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$  we deduce the isomorphism of the lemma, since  $g$  restricts to an isomorphism on the preimage of  $\mathfrak{g}_{\text{reg}}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$  (see (2.13)).

**4.2. Study of some equivariant  $\mathcal{U}_{\mathcal{S}\mathfrak{g}}$ -bimodules.** Given any  $\lambda \in \mathbb{X}$ , we have a line bundle  $\mathcal{O}_{\mathcal{B}}(\lambda)$  on  $\mathcal{B}$  attached naturally to  $\lambda$ . (Our normalization is that of [J2], so that line bundles attached to dominant weights are ample.) This line bundle identifies with the direct summand of  $\omega_* \mathcal{O}_{G/U}$  consisting of sections which have weight  $\lambda$  for the  $T$ -action induced by right multiplication on  $G$ ; it therefore admits a natural action of the sheaf of algebras  $\tilde{\mathcal{D}}$ . Using this action and the natural action on  $\tilde{\mathcal{D}}$ , we obtain a left action of  $\tilde{\mathcal{D}}$  on the tensor product

$$\mathcal{O}_{\mathcal{B}}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{D}}.$$

As for  $\tilde{\mathcal{D}}$  itself, this module can be also considered as a sheaf of modules on  $\tilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$ . We set

$$\mathbb{D}_{\mathcal{S}, \lambda} := \Gamma(\tilde{\mathcal{S}}^{*(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*, (\mathcal{O}_{\mathcal{B}}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{D}})|_{\tilde{\mathcal{S}}^{*(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}),$$

which therefore admits a natural left action of  $\mathbb{D}_{\mathcal{S}}$ , hence of  $\tilde{\mathcal{U}}_{\mathcal{S}}\mathfrak{g}$  (see Lemma 4.1). The tensor product  $\mathcal{O}_{\mathcal{B}}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{G}}$  also admits a natural right action of  $\tilde{\mathcal{G}}$ , induced by right multiplication on the second factor. The action of  $\pi_*^{(1)} \mathcal{O}_{\tilde{\mathfrak{g}}^{(1)}}$  on  $(\mathrm{Fr}_{\mathcal{B}})_* \mathcal{O}_{\mathcal{B}}(\lambda)$  being trivial, the two actions of this subalgebra of  $(\mathrm{Fr}_{\mathcal{B}})_* (\mathcal{O}_{\mathcal{B}}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{G}})$  coincide, and  $\mathbb{D}_{\mathcal{S},\lambda}$  therefore also admits a right action of  $\tilde{\mathcal{U}}_{\mathcal{S}}\mathfrak{g}$ ; moreover these actions combine to provide an action of  $\tilde{\mathcal{U}}_{\mathcal{S}}\mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} (\tilde{\mathcal{U}}_{\mathcal{S}}\mathfrak{g})^{\mathrm{op}}$ . By construction the action of the central subalgebra

$$\mathcal{O}(\mathfrak{t}^*) \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} \mathcal{O}(\mathfrak{t}^*) \cong \mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*)$$

factors through an action of the image of the closed embedding  $\mathfrak{t}^* \rightarrow \mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$  given by

$$\xi \mapsto (\xi + \bar{\lambda}, \xi).$$

The object  $\mathcal{O}_{\mathcal{B}}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{G}}$  also admits a natural structure of  $G$ -equivariant quasi-coherent sheaf, compatible with the actions considered above. The module  $\mathbb{D}_{\mathcal{S},\lambda}$  therefore also admits a natural and compatible structure of module for the group scheme

$$\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}_{\mathcal{S}}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*,$$

see §2.9.

For  $\lambda, \mu \in \mathbb{X}$ , we will denote by

$$\tilde{\mathcal{U}}_{\mathcal{S}}^{\hat{\lambda}, \hat{\mu}}$$

the completion of the  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*)$ -algebra  $\tilde{\mathcal{U}}_{\mathcal{S}}\mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} (\tilde{\mathcal{U}}_{\mathcal{S}}\mathfrak{g})^{\mathrm{op}}$  at the ideal corresponding to the point  $(\bar{\lambda}, \bar{\mu}) \in \mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$ . Copying the constructions in §2.10 (replacing  $\mathcal{U}_{\mathcal{S}}^{\hat{\lambda}, \hat{\mu}}$  by  $\tilde{\mathcal{U}}_{\mathcal{S}}^{\hat{\lambda}, \hat{\mu}}$  and  $\mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}_{\mathcal{S}}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)$  by  $\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}_{\mathcal{S}}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$ ) we define the category  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathcal{S}}^{\hat{\lambda}, \hat{\mu}})$ . Copying the definition of  $\widehat{\otimes}_{\mathcal{U}_{\mathcal{S}}\mathfrak{g}}$  we obtain, for  $\lambda, \mu, \nu \in \mathbb{X}$ , a bifunctor

$$(-) \widehat{\otimes}_{\tilde{\mathcal{U}}_{\mathcal{S}}\mathfrak{g}} (-) : \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathcal{S}}^{\hat{\lambda}, \hat{\mu}}) \times \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathcal{S}}^{\hat{\mu}, \hat{\nu}}) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathcal{S}}^{\hat{\lambda}, \hat{\nu}}).$$

For any  $\lambda, \mu \in \mathbb{X}$  we have a natural “forgetful” functor

$$\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathcal{S}}^{\hat{\lambda}, \hat{\mu}}) \rightarrow \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathcal{S}}^{\hat{\lambda}, \hat{\mu}}),$$

which we will usually omit from notation. In case  $\lambda$  and  $\mu$  are regular, this functor is an equivalence by Lemma 2.2. In case  $\mu$  is regular, for  $M \in \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathcal{S}}^{\hat{\lambda}, \hat{\mu}})$  and  $N \in \mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathcal{S}}^{\hat{\mu}, \hat{\nu}})$  we also have a canonical identification

$$M \widehat{\otimes}_{\mathcal{U}_{\mathcal{S}}\mathfrak{g}} N \xrightarrow{\sim} M \widehat{\otimes}_{\tilde{\mathcal{U}}_{\mathcal{S}}\mathfrak{g}} N.$$

For  $\lambda, \mu \in \mathbb{X}$ , we will denote by  $\mathbb{B}_{\lambda, \mu}$  the completion of the module  $\mathbb{D}_{\mathcal{S}, \lambda - \mu}$  at the ideal of  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*)$  corresponding to the element  $(\bar{\lambda}, \bar{\mu})$ . In view of the remarks above, this object can equivalently be obtained by completing  $\mathbb{D}_{\mathcal{S}, \lambda - \mu}$  at the ideal of  $\mathcal{O}(\mathfrak{t}^*)$  corresponding to  $\bar{\lambda}$  for the left action, or by completing  $\mathbb{D}_{\mathcal{S}, \lambda - \mu}$  at the ideal of  $\mathcal{O}(\mathfrak{t}^*)$  corresponding to  $\bar{\mu}$  for the right action. This construction provides an object in  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathcal{S}}^{\hat{\lambda}, \hat{\mu}})$ , hence a fortiori in  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathcal{S}}^{\hat{\lambda}, \hat{\mu}})$ .

**Lemma 4.3.** *For  $\lambda, \mu, \nu \in \mathbb{X}$ , there exists a canonical isomorphism*

$$\mathbb{B}_{\lambda, \mu} \widehat{\otimes}_{\tilde{\mathcal{U}}_{\mathcal{S}}\mathfrak{g}} \mathbb{B}_{\mu, \nu} \xrightarrow{\sim} \mathbb{B}_{\lambda, \nu}$$

in  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_S^{\hat{\lambda}, \hat{\nu}})$ . In particular, in case  $\mu$  is regular there exists a canonical isomorphism

$$\mathbb{B}_{\lambda, \mu} \hat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{B}_{\mu, \nu} \xrightarrow{\sim} \mathbb{B}_{\lambda, \nu}$$

in  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\nu}})$ .

*Proof.* There exist canonical isomorphisms

$$\begin{aligned} (\mathcal{O}_{\mathcal{B}}(\lambda - \mu) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{D}}) \otimes_{\tilde{\mathcal{D}}} (\mathcal{O}_{\mathcal{B}}(\mu - \nu) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{D}}) &\xrightarrow{\sim} \\ \mathcal{O}_{\mathcal{B}}(\lambda - \mu) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_{\mathcal{B}}(\mu - \nu) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{D}} &\cong \mathcal{O}_{\mathcal{B}}(\lambda - \nu) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{D}}, \end{aligned}$$

where the first map is given locally by  $f \otimes \partial \otimes f' \otimes \partial' \mapsto f \otimes (\partial \cdot f') \otimes (\partial \partial')$ . The desired isomorphism follows by restriction to  $\tilde{\mathcal{S}}^{*(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$  and then completion at  $(\lambda, \nu)$ .  $\square$

If  $\lambda \in \mathbb{X}$  is regular, Lemma 2.2 and Lemma 4.1 imply that we have  $\mathcal{U}_S^{\hat{\lambda}} \cong \mathbb{B}_{\lambda, \lambda}$ . Hence the functor of convolution on the left, resp. right, with  $\mathbb{B}_{\lambda, \lambda}$  is isomorphic to the identity of  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\mu}})$ , resp.  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\mu}, \hat{\lambda}})$ , for any  $\mu \in \mathbb{X}$ . Combining this observation with Lemma 4.3, we see that if  $\lambda \in \mathbb{X}$  belongs to the fundamental alcove, then for any  $w \in W_{\mathrm{ext}}$  the object  $\mathbb{B}_{\lambda, w \bullet \lambda}$  is invertible in the monoidal category  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\lambda}})$ , with inverse  $\mathbb{B}_{w \bullet \lambda, \lambda}$ .

Recall the morphism of group schemes (2.14). Restricting to  $\mathcal{S}^{*(1)}$  and then taking the fiber product with the morphism  $\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^* \rightarrow \mathfrak{t}^* \xrightarrow{\mathrm{AS}} \mathfrak{t}^{*(1)}$  (where the first morphism is the first projection) we obtain a morphism of groups schemes

$$\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}_S^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^* \rightarrow (\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*) \times T^{(1)}.$$

Using this morphism, for any character  $\eta$  of  $T^{(1)}$  we obtain a structure of representation of  $\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}_S^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$  on  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*)$  defined by this character. Tensoring with this representation we obtain an autoequivalence of  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_S^{\hat{\lambda}, \hat{\mu}})$ , which we denote  $M \mapsto M(\eta)$ .

Note that the morphism from  $X^*(T^{(1)})$  to  $\mathbb{X}$  induced by the Frobenius morphism  $T \rightarrow T^{(1)}$  is injective, and that its image is  $p \cdot \mathbb{X}$ . We will therefore identify  $X^*(T^{(1)})$  with  $p \cdot \mathbb{X}$  via this morphism.

**Lemma 4.4.** *For any  $\lambda, \nu \in \mathbb{X}$ , there exists a canonical isomorphism*

$$\mathbb{B}_{\lambda + p\nu, \lambda} \cong \mathbb{B}_{\lambda, \lambda}(p\nu)$$

in  $\mathrm{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_S^{\hat{\lambda}, \hat{\lambda}})$ .

*Proof.* By definition,  $\mathbb{B}_{\lambda + p\nu, \lambda}$  is the completion at the ideal corresponding to  $(\bar{\lambda}, \bar{\lambda})$  of the  $\tilde{\mathcal{U}}_S \mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} (\tilde{\mathcal{U}}_S \mathfrak{g})^{\mathrm{op}}$ -module  $\mathbb{D}_{S, p\nu}$ . If we denote by  $U^+$  the unipotent radical of the Borel subgroup opposite to  $B$ , then  $U^+B/B \subset \mathcal{B}$  is an open subvariety isomorphic to  $U^+$ , and the projection  $\tilde{\mathcal{S}}^* \rightarrow \mathcal{B}$  factors through a morphism  $\tilde{\mathcal{S}}^* \rightarrow U^+B/B$ , see [MR, Lemma 4.8]. As a consequence, the sheaf  $(\mathcal{O}_{\mathcal{B}}(p\nu) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{D}})_{|\tilde{\mathcal{S}}^{*(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}$  can be obtained as a further restriction of  $(\mathcal{O}_{\mathcal{B}}(p\nu) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{D}})_{|U^+B/B}$ .

Since  $\tilde{\mathcal{D}}$  acts on  $\mathcal{O}_{\mathcal{B}}(p\nu)$ , we have an action of the algebra  $\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{HC}}} \mathcal{O}(\mathfrak{t}^*)$  on  $\Gamma(U^+B/B, \mathcal{O}_{\mathcal{B}}(p\nu))$ , see (4.1). We have

$$\Gamma(U^+B/B, \mathcal{O}_{\mathcal{B}}(p\nu)) = \{f : U^+B \rightarrow \mathbb{k} \mid \forall b \in B, x \in U^+B, f(xb^{-1}) = (p\nu)(b) \cdot f(x)\}.$$

In this space we have a canonical vector, namely the function  $f : U^+B \rightarrow \mathbb{k}$  defined by  $f(u_1tu_2) = (p\nu)^{-1}(t)$  for all  $u_1 \in U^+$ ,  $t \in T$  and  $u_2 \in U$ . This section does not vanish on  $U^+B/B$ , hence induces an isomorphism of line bundles  $\mathcal{O}_{U^+B/B} \xrightarrow{\sim} \mathcal{O}_{\mathcal{B}}(p\nu)|_{U^+B/B}$ . We claim that it is furthermore annihilated by the action of  $\mathfrak{g} \subset \mathcal{U}\mathfrak{g}$  and  $\mathfrak{t} \subset \mathcal{O}(\mathfrak{t}^*)$ . In fact, the second case is clear. For the action of  $\mathfrak{g}$ , in case  $\nu \in \mathbb{X}^+$  the claim follows from the fact that our vector is the restriction of the unique (up to scalar) vector of weight  $p\nu$  in  $\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(p\nu))$  (see [J2, Proof of Proposition II.2.6]), and that this vector belongs to the  $G$ -submodule  $L(p\nu)$ , on which the action of  $\mathfrak{g}$  is well known to vanish. From this we deduce the general case by using the Leibniz rule for the action on tensor products of line bundles.

Tensoring this section with the unit in  $\tilde{\mathcal{D}}$  we obtain a section of  $(\mathcal{O}_{\mathcal{B}}(p\nu) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{D}})|_{U^+B/B}$ . The right action on this section provides an isomorphism

$$\tilde{\mathcal{D}}|_{U^+B/B} \rightarrow (\mathcal{O}_{\mathcal{B}}(p\nu) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{D}})|_{U^+B/B},$$

which commutes with the natural left and right actions of  $\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{HC}}} \mathcal{O}(\mathfrak{t}^*)$ . Restricting further we obtain an isomorphism

$$\tilde{\mathcal{D}}|_{\tilde{\mathcal{S}}^*(1) \times_{\mathfrak{t}^*(1)} \mathfrak{t}^*} \xrightarrow{\sim} (\mathcal{O}_{\mathcal{B}}(p\nu) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{D}})|_{\tilde{\mathcal{S}}^*(1) \times_{\mathfrak{t}^*(1)} \mathfrak{t}^*},$$

and then taking global sections and completing an isomorphism of  $\tilde{\mathcal{U}}_{\mathcal{S}}^{\hat{\lambda}, \hat{\lambda}}$ -modules  $\mathbb{B}_{\lambda, \lambda} \xrightarrow{\sim} \mathbb{B}_{\lambda+p\nu, \lambda}$ . Taking the action of  $\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathbb{I}_{\mathcal{S}}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*$  into account, this provides the desired isomorphism  $\mathbb{B}_{\lambda, \lambda}(p\nu) \xrightarrow{\sim} \mathbb{B}_{\lambda+p\nu, \lambda}$ .  $\square$

**4.3. Relation with translation bimodules.** We now explain the relation between the objects  $\mathbb{B}_{\lambda, \mu}$  and the “translation bimodules” introduced in §2.6.

**Lemma 4.5.** *Let  $\lambda, \mu \in \mathbb{X}$ , with  $\lambda$  belonging to the fundamental alcove and  $\mu$  belonging to the closure of the fundamental alcove. Then for any  $w \in W_{\text{ext}}$  there exist isomorphisms*

$$\mathbb{P}_{\mathcal{S}}^{\mu, \lambda} \cong \mathbb{B}_{w \bullet \mu, w \bullet \lambda}, \quad \mathbb{P}_{\mathcal{S}}^{\lambda, \mu} \cong \mathbb{B}_{w \bullet \lambda, w \bullet \mu}$$

in  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathcal{S}}^{\hat{\mu}, \hat{\lambda}})$  and  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathcal{S}}^{\hat{\lambda}, \hat{\mu}})$  respectively.

*Proof.* We prove the first isomorphism; the second one can be obtained similarly. It is clear that we can assume that  $w \in W$ . Let  $\nu \in \mathbb{X}$  be the unique dominant weight which belongs to  $W(\mu - \lambda)$ . Then by definition,  $\mathbb{P}_{\mathcal{S}}^{\mu, \lambda}$  is the completion of the module

$$L(\nu) \otimes \mathcal{U}_{\mathcal{S}}\mathfrak{g}$$

at the ideal corresponding to the point  $(\tilde{\mu}, \tilde{\lambda}) \in \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet)$ . Now by Lemma 2.2 the quotient morphism  $\mathfrak{t}^* \rightarrow \mathfrak{t}^*/(W, \bullet)$  is étale at  $w \bullet \bar{\lambda}$ . It follows that  $\mathbb{P}_{\mathcal{S}}^{\mu, \lambda}$  can also be obtained as the completion of the  $\mathcal{U}_{\mathcal{S}}\mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^*(1))} (\tilde{\mathcal{U}}_{\mathcal{S}}\mathfrak{g})^{\text{op}}$ -module

$$L(\nu) \otimes \tilde{\mathcal{U}}_{\mathcal{S}}\mathfrak{g}$$

with respect to the ideal of  $\mathcal{O}(\mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*)$  corresponding to  $(\tilde{\mu}, w \bullet \bar{\lambda})$ .

By Lemma 4.1 we have canonical isomorphisms

$$\mathbf{L}(\nu) \otimes \tilde{\mathcal{U}}_{\mathcal{S}} \mathfrak{g} \cong \mathbf{L}(\nu) \otimes \mathbb{D}_{\mathcal{S}} \cong \Gamma(\tilde{\mathcal{S}}^{*(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*, \mathbf{L}(\nu) \otimes \tilde{\mathcal{G}}_{\mathcal{S}}).$$

It is a classical fact that the coherent sheaf  $\mathbf{L}(\nu) \otimes \mathcal{O}_{\mathcal{B}}$  on  $\mathcal{B}$  admits a filtration whose subquotients have the form  $\mathcal{O}_{\mathcal{B}}(\eta)$  where  $\eta$  runs over the weights of  $\mathbf{L}(\nu)$  (counted with multiplicities). We deduce a similar filtration for the sheaf  $\mathbf{L}(\nu) \otimes \tilde{\mathcal{G}}$ , and then for its restriction to  $\tilde{\mathcal{S}}^{*(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$ . (Here we use the fact that restriction along the closed embedding  $\tilde{\mathcal{S}}^{*(1)} \hookrightarrow \tilde{\mathfrak{g}}^{(1)}$  is exact on the category  $\mathrm{QCoh}^G(\tilde{\mathfrak{g}}^{(1)})$ , since it identifies with the composition of pullback along the flat morphism  $G \times \tilde{\mathcal{S}}^{*(1)} \rightarrow \tilde{\mathfrak{g}}^{(1)}$  composed with the obvious equivalence  $\mathrm{QCoh}^G(G \times \tilde{\mathcal{S}}^{*(1)}) \cong \mathrm{QCoh}(\tilde{\mathcal{S}}^{*(1)})$ .) In other words, we have obtained a filtration of  $\mathbf{L}(\nu) \otimes \tilde{\mathcal{U}}_{\mathcal{S}} \mathfrak{g}$  with subquotients  $\mathbb{D}_{\mathcal{S}, \eta}$  where  $\eta$  runs over the weights of  $\mathbf{L}(\nu)$  (counted with multiplicities). This filtration is clearly compatible with the action of  $\mathcal{U}_{\mathcal{S}} \mathfrak{g} \otimes_{\mathcal{O}(\mathcal{S}^{*(1)})} (\tilde{\mathcal{U}}_{\mathcal{S}} \mathfrak{g})^{\mathrm{op}}$  and the natural structure of module over the group scheme  $\mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}_{\mathcal{S}}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$ .

Let us denote by  $\varpi : \mathfrak{t}^* \rightarrow \mathfrak{t}^*/(W, \bullet)$  the quotient morphism. The irreducible components of the scheme

$$\mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$$

are parametrized by  $\mathfrak{t}_{\mathbb{Z}}^*$ , with the component corresponding to  $\bar{\gamma}$  being the image of the closed embedding  $\mathfrak{t}^* \rightarrow \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$  given by  $\xi \mapsto (\varpi(\xi + \bar{\gamma}), \xi)$ . The components containing the point  $(\tilde{\mu}, w \bullet \bar{\lambda})$  correspond to the elements  $\bar{\gamma} \in \mathfrak{t}_{\mathbb{Z}}^*$  such that  $w \bullet \bar{\lambda} + \bar{\gamma} \in W \bullet \bar{\mu}$ , i.e.  $\bar{\lambda} + w^{-1}\bar{\gamma} \in W \bullet \bar{\mu}$ . On the other hand, the module  $\mathbb{D}_{\mathcal{S}, \eta}$  is supported on the component corresponding to  $\bar{\eta}$ . Hence, after completion at  $(\tilde{\mu}, w \bullet \bar{\lambda})$ , the only subquotients that survive are those corresponding to the weight  $\eta$  such that  $\bar{\lambda} + w^{-1}\bar{\eta} \in W \bullet \bar{\mu}$ , i.e.  $\lambda + w^{-1}\eta \in W_{\mathrm{ext}} \bullet \mu$ . Since  $\eta$  is a weight of  $\mathbf{L}(\nu)$ , it belongs to  $\mu - \lambda + \mathbb{Z}\mathfrak{R}$ , so that  $\lambda + w^{-1}\eta \in \mu + \mathbb{Z}\mathfrak{R}$ . By Lemma 2.1(1) the condition that  $\lambda + w^{-1}\eta \in W_{\mathrm{ext}} \bullet \mu$  is therefore equivalent to  $\lambda + w^{-1}\eta \in W_{\mathrm{aff}} \bullet \mu$ . Now by [J2, Lemma II.7.7] this condition is satisfied only when  $\lambda + w^{-1}\eta = \mu$ , i.e.  $\eta = w(\mu - \lambda)$ . We deduce the desired isomorphism, since  $w \bullet \mu - w \bullet \lambda = w(\mu - \lambda)$ .  $\square$

*Remark 4.6.* Let  $\lambda, \mu \in \mathbb{X}$  belonging to the closure of the fundamental alcove, and assume that the stabilizer of  $\lambda$  for the dot-action of  $W_{\mathrm{aff}}$  is contained in the stabilizer of  $\mu$ . Then, if we denote by  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet))^{\hat{\lambda}, \hat{\mu}}$  the completion of  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet))$  at the ideal corresponding to  $(\bar{\lambda}, \bar{\mu})$ , the same considerations as in the proof of Lemma 4.5 show that there exists an isomorphism

$$\mathbb{B}_{\lambda, \mu} \cong \mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet))^{\hat{\lambda}, \hat{\mu}} \otimes_{\mathbb{Z}_{\hat{\lambda}, \hat{\mu}}} \mathbb{P}_{\mathcal{S}}^{\lambda, \mu}.$$

Recall that given a simple reflection  $s \in S_{\mathrm{aff}}$ , a weight  $\lambda \in \mathbb{X}$  belonging to the closure of the fundamental alcove is said to be on the *wall* corresponding to  $s$  if  $s \bullet \lambda = \lambda$ .

**Lemma 4.7.** *Let  $\lambda, \mu \in \mathbb{X}$ , with  $\lambda$  belonging to the fundamental alcove and  $\mu$  on exactly one wall of the fundamental alcove, attached to the simple reflection  $s$ . Let also  $w \in W$ .*

*If  $ws \bullet \lambda > w \bullet \lambda$ , then there exists an exact sequence*

$$\mathbb{B}_{w \bullet \lambda, w \bullet \lambda} \hookrightarrow \mathbb{P}_{\mathcal{S}}^{\lambda, \mu} \hat{\otimes}_{\mathcal{U}_{\mathcal{S}} \mathfrak{g}} \mathbb{P}_{\mathcal{S}}^{\mu, \lambda} \twoheadrightarrow \mathbb{B}_{ws \bullet \lambda, w \bullet \lambda}$$

in  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\lambda}})$ . If  $ws \bullet \lambda < w \bullet \lambda$ , then there exists an exact sequence

$$\mathbb{B}_{ws \bullet \lambda, w \bullet \lambda} \hookrightarrow \mathbb{P}_S^{\lambda, \mu} \hat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu, \lambda} \twoheadrightarrow \mathbb{B}_{w \bullet \lambda, w \bullet \lambda}$$

in  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\lambda}})$ .

*Proof.* By Lemma 4.5 we have

$$\mathbb{P}_S^{\lambda, \mu} \hat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu, \lambda} \cong \mathbb{P}_S^{\lambda, \mu} \hat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{B}_{w \bullet \mu, w \bullet \lambda}.$$

Hence, if we denote by  $\nu$  the unique dominant weight in  $W(\lambda - \mu)$ , this object can be obtained by completing the bimodule

$$\mathbb{L}(\nu) \otimes \mathbb{D}_{S, w \bullet \mu - w \bullet \lambda}$$

with respect to the ideal of  $\mathcal{O}(\mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*)$  corresponding to  $(\tilde{\lambda}, w \bullet \bar{\lambda})$ . Now we have

$$\mathbb{L}(\nu) \otimes \mathbb{D}_{S, w \bullet \mu - w \bullet \lambda} \cong \Gamma(\tilde{\mathcal{S}}^{*(1)} \times_{\mathfrak{t}^*(1)} \mathfrak{t}^*, \mathbb{L}(\nu) \otimes (\mathcal{O}_{\mathcal{B}}(w \bullet \mu - w \bullet \lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \tilde{\mathcal{D}})|_{\tilde{\mathcal{S}}^{*(1)} \times_{\mathfrak{t}^*(1)} \mathfrak{t}^*}).$$

Hence, as in the proof of Lemma 4.5, if we choose an enumeration  $\eta_1, \dots, \eta_n$  of the  $T$ -weights of  $\mathbb{L}(\nu)$  (counted with multiplicities) such that  $\eta_i < \eta_j$  implies  $i < j$ , then this bimodule admits a filtration

$$\{0\} = M_0 \subset M_1 \subset \dots \subset M_n = \mathbb{L}(\nu) \otimes \mathbb{D}_{S, w \bullet \lambda - w \bullet \mu}$$

such that  $M_i/M_{i-1} \cong \mathbb{D}_{S, w \bullet \mu - w \bullet \lambda + \eta_i}$  for any  $i$ . The subquotient  $\mathbb{D}_{S, w \bullet \mu - w \bullet \lambda + \eta_i}$  survives after completion at the ideal corresponding to  $(\tilde{\lambda}, w \bullet \bar{\lambda})$  iff

$$w \bullet \bar{\mu} - w \bullet \bar{\lambda} + \bar{\eta}_i \in W \bullet \bar{\lambda} - w \bullet \bar{\lambda},$$

i.e. iff

$$\mu + w^{-1}\eta_i \in W_{\text{ext}} \bullet \lambda.$$

Here  $w^{-1}\eta_i$  is a weight of  $\mathbb{L}(\nu)$ , hence  $\mu + w^{-1}\eta_i$  belongs to  $\lambda + \mathbb{Z}\mathfrak{R}$ ; in view of Lemma 2.1(1), this condition is therefore equivalent to  $\mu + w^{-1}\eta_i \in W_{\text{aff}} \bullet \lambda$ . Since the stabilizer of  $\mu$  for the dot-action of  $W_{\text{aff}}$  is  $\{e, s\}$ , by [J2, Lemma II.7.7] this condition is satisfied for two values of  $\eta_i$ , corresponding to

$$\mu + w^{-1}\eta_i = \lambda \quad \text{and} \quad \mu + w^{-1}\eta_i = s \bullet \lambda,$$

i.e.

$$w \bullet \mu + \eta_i = w \bullet \lambda \quad \text{and} \quad w \bullet \mu + \eta_i = ws \bullet \lambda.$$

Hence  $\mathbb{P}_S^{\lambda, \mu} \hat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu, \lambda}$  admits a filtration with two subquotients, isomorphic respectively to  $\mathbb{B}_{w \bullet \lambda, w \bullet \lambda}$  and  $\mathbb{B}_{ws \bullet \lambda, w \bullet \lambda}$ . The order in which these subquotients appear depends on whether  $ws \bullet \lambda > w \bullet \lambda$  or  $ws \bullet \lambda < w \bullet \lambda$ , and are as indicated in the statement.  $\square$

**4.4. Convolution with translation bimodules.** Let  $\lambda, \mu \in \mathbb{X}$ , and assume that  $\lambda$  is regular. Then there exists a canonical algebra morphism

$$(4.2) \quad \mathcal{Z}_S^{\hat{\mu}, \hat{\lambda}} \rightarrow \mathcal{Z}_S^{\hat{\lambda}, \hat{\lambda}}$$

which can be defined as follows. The algebra  $\mathcal{Z}_S^{\hat{\mu}, \hat{\lambda}}$  is by definition the completion of  $\mathcal{O}(\mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))$  at the ideal corresponding to  $(\tilde{\mu}, \tilde{\lambda})$ . Hence it admits a canonical morphism to the completion  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))^{\hat{\mu}, \hat{\lambda}}$  of  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W}$

$\mathfrak{t}^*/(W, \bullet)$  at the ideal corresponding to  $(\bar{\mu}, \tilde{\lambda})$ . Now the morphism  $\mathfrak{t}^* \rightarrow \mathfrak{t}^*$  defined by  $\xi \mapsto \xi + \bar{\mu} - \bar{\lambda}$  provides an isomorphism

$$\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet) \xrightarrow{\sim} \mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet)$$

sending  $(\bar{\lambda}, \tilde{\lambda})$  to  $(\bar{\mu}, \tilde{\lambda})$ , which therefore induces an isomorphism

$$\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))^{\hat{\mu}, \hat{\lambda}} \xrightarrow{\sim} \mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))^{\hat{\lambda}, \hat{\lambda}},$$

where the right-hand side is the completion of  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))$  at the ideal corresponding to  $(\bar{\lambda}, \tilde{\lambda})$ . Finally, the natural morphism

$$\mathcal{Z}_S^{\hat{\lambda}, \hat{\lambda}} \rightarrow \mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))^{\hat{\lambda}, \hat{\lambda}}$$

is an isomorphism by Lemma 2.2; combining these constructions we obtain the wished-for morphism (4.2).

Our goal in this subsection is to prove the following claim.

**Proposition 4.8.** *Assume that  $p \neq 2$ . Let  $\lambda, \mu \in \mathbb{X}$ , with  $\lambda$  belonging to the fundamental alcove and  $\mu$  on exactly one wall of the fundamental alcove, attached to a simple reflection  $s$  which belongs to  $W$ . Then there exists an isomorphism*

$$\mathbb{P}_S^{\lambda, \mu} \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{P}_S^{\mu, \lambda} \cong \mathcal{L}_{\lambda, \lambda} \left( \mathcal{Z}_S^{\hat{\lambda}, \hat{\lambda}} \otimes_{\mathcal{Z}_S^{\hat{\mu}, \hat{\lambda}}} \mathcal{Z}_S^{\hat{\lambda}} \right)$$

in  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\lambda}})$ , where  $\mathcal{Z}_S^{\hat{\lambda}}$  is regarded as a  $\mathcal{Z}_S^{\hat{\mu}, \hat{\lambda}}$ -module via the morphism (4.2).

This proposition will be deduced from the following claim.

**Lemma 4.9.** *Assume that  $p \neq 2$ . Let  $\lambda, \mu \in \mathbb{X}$ , with  $\lambda$  belonging to the fundamental alcove and  $\mu$  on exactly one wall of the fundamental alcove, attached to a simple reflection  $s$  which belongs to  $W$ . Then there exist isomorphisms of functors which make the diagrams*

$$\begin{array}{ccc} \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\lambda}, \hat{\lambda}}) & \xrightarrow{\quad} & \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\mu}, \hat{\lambda}}) \\ \mathcal{L}_{\lambda, \lambda} \downarrow & & \downarrow \mathcal{L}_{\mu, \lambda} \\ \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\lambda}}) & \xrightarrow{\mathbb{P}_S^{\mu, \lambda} \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} (-)} & \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\mu}, \hat{\lambda}}) \end{array}$$

(where the upper horizontal arrow is the restriction-of-scalars functor associated with the morphism (4.2)) and

$$\begin{array}{ccc} \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\mu}, \hat{\lambda}}) & \xrightarrow{\mathcal{Z}_S^{\hat{\lambda}, \hat{\lambda}} \otimes_{\mathcal{Z}_S^{\hat{\mu}, \hat{\lambda}}} (-)} & \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\hat{\lambda}, \hat{\lambda}}) \\ \mathcal{L}_{\mu, \lambda} \downarrow & & \downarrow \mathcal{L}_{\lambda, \lambda} \\ \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\mu}, \hat{\lambda}}) & \xrightarrow{\mathbb{P}_S^{\lambda, \mu} \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} (-)} & \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\lambda}}) \end{array}$$

commutative.

*Proof.* By definition we have

$$\mathbb{P}_S^{\mu, \lambda} \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathcal{L}_{\lambda, \lambda}(-) \cong \left( \mathbb{P}_S^{\mu, \lambda} \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{P}_S^{\lambda, -\rho} \widehat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{P}_S^{-\rho, \lambda} \right) \otimes_{\mathcal{Z}_S^{\hat{\lambda}, \hat{\lambda}}} (-).$$

Using Lemma 4.3 and Lemma 4.5, we deduce that

$$\mathbb{P}_S^{\mu,\lambda} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathcal{L}_{\lambda,\lambda}(-) \cong \left( \mathbb{B}_{\mu,-\rho} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{-\rho,\lambda} \right) \otimes_{\mathcal{Z}_S^{\widehat{\lambda},\widehat{\lambda}}} (-).$$

Hence to prove the commutativity of the first diagram it suffices to construct an isomorphism

$$\left( \mathbb{P}_S^{\mu,-\rho} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{-\rho,\lambda} \right) \otimes_{\mathcal{Z}_S^{\widehat{\mu},\widehat{\lambda}}} \mathcal{Z}_S^{\widehat{\lambda},\widehat{\lambda}} \xrightarrow{\sim} \mathbb{B}_{\mu,-\rho} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{-\rho,\lambda}$$

or in other words an isomorphism

$$(4.3) \quad \left( \mathbb{P}_S^{\mu,-\rho} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{-\rho,\lambda} \right) \otimes_{\mathcal{Z}_S^{\widehat{\mu},\widehat{\lambda}}} \mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))^{\widehat{\mu},\widehat{\lambda}} \xrightarrow{\sim} \mathbb{B}_{\mu,-\rho} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{-\rho,\lambda}.$$

To construct such a morphism it suffices to construct a morphism

$$(4.4) \quad \mathbb{P}_S^{\mu,-\rho} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{-\rho,\lambda} \rightarrow \mathbb{B}_{\mu,-\rho} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{-\rho,\lambda}$$

in  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\widehat{\mu},\widehat{\lambda}})$ . By Remark 4.6 we have

$$\mathbb{B}_{\mu,-\rho} \cong \mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))^{\widehat{\mu},\widehat{\rho}} \otimes_{\mathcal{Z}_S^{\widehat{\mu},-\rho}} \mathbb{P}_S^{\mu,-\rho};$$

in particular there exists a natural morphism  $\mathbb{P}_S^{\mu,-\rho} \rightarrow \mathbb{B}_{\mu,-\rho}$ , which allows to define the wished-for morphism (4.4), hence the morphism (4.3).

Now we claim that  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))^{\widehat{\mu},\widehat{\lambda}}$ , resp.  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))^{\widehat{\mu},\widehat{\rho}}$ , is free of rank 2 over  $\mathcal{Z}_S^{\widehat{\mu},\widehat{\lambda}}$ , resp.  $\mathcal{Z}_S^{\widehat{\mu},-\rho}$ , which will imply that the morphism (4.3) is an isomorphism. The two cases are similar, so that we only consider  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))^{\widehat{\mu},\widehat{\lambda}}$ . It follows from Lemma 2.2 that  $\mathcal{Z}_S^{\widehat{\mu},\widehat{\lambda}}$  identifies canonically with the completion  $\mathcal{O}(\mathfrak{t}^*/(\{e, s\}, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))^{\widehat{\mu},\widehat{\lambda}}$  of  $\mathcal{O}(\mathfrak{t}^*/(\{e, s\}, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))$  with respect to the ideal corresponding to the image of  $(\widetilde{\mu}, \widetilde{\lambda})$ . Now  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))$  is free of rank 2 as a module over  $\mathcal{O}(\mathfrak{t}^*/(\{e, s\}, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))$ , and its completion with respect to the ideal corresponding to  $(\widetilde{\mu}, \widetilde{\lambda})$  coincides with its completion with respect to the ideal of  $\mathcal{O}(\mathfrak{t}^*/(\{e, s\}, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet))$  corresponding to the image of  $(\widetilde{\mu}, \widetilde{\lambda})$  (because  $(\widetilde{\mu}, \widetilde{\lambda})$  is the only closed point in the fiber over its image in  $\mathfrak{t}^*/(\{e, s\}, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet)$ ). The desired claim follows.

We have finally proved the commutativity of the first diagram of the lemma. The commutativity of the second diagram follows from that of the first one by adjunction, in view of Lemma 2.5.  $\square$

*Proof of Proposition 4.8.* Lemma 4.9 provides isomorphisms

$$\mathbb{P}_S^{\lambda,\mu} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu,\lambda} \cong \mathcal{L}_{\lambda,\lambda}(\mathcal{Z}_S^{\widehat{\lambda},\widehat{\lambda}} \otimes_{\mathcal{Z}_S^{\widehat{\mu},\widehat{\lambda}}} \mathcal{L}_{\mu,\lambda}^{-1}(\mathbb{P}_S^{\mu,\lambda})) \cong \mathcal{L}_{\lambda,\lambda}(\mathcal{Z}_S^{\widehat{\lambda},\widehat{\lambda}} \otimes_{\mathcal{Z}_S^{\widehat{\mu},\widehat{\lambda}}} \mathcal{L}_{\lambda,\lambda}^{-1}(\mathcal{U}_S^{\widehat{\lambda}})).$$

The desired claim follows, using the isomorphism (3.8).  $\square$

**4.5. Monoidality of the functors  $\mathcal{L}_{\lambda,\lambda}$ .** Our goal in this subsection is to prove the following claim, announced in Remark 3.8.

**Proposition 4.10.** *Let  $\lambda, \nu \in \mathbb{X}$  in the lower closure of the fundamental alcove, and let  $\mu \in \mathbb{X}$  be in the fundamental alcove. Then for  $M \in \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\widehat{\lambda},\widehat{\mu}})$  and  $N \in \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{Z}_S^{\widehat{\mu},\widehat{\nu}})$  there exists a canonical (in particular, bifunctorial) isomorphism*

$$\mathcal{L}_{\lambda,\nu}(M \widehat{\star}_S N) \cong \mathcal{L}_{\lambda,\mu}(M) \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathcal{L}_{\mu,\nu}(N).$$



In case  $\lambda = \mu = \nu$ , this isomorphism and (3.8) define on  $\mathcal{L}_{\lambda,\lambda}$  the structure of a monoidal functor.

*Proof.* By definition we have

$$\mathcal{L}_{\lambda,\mu}(M) = \mathbb{M}_S^{\lambda,\mu} \otimes_{\mathcal{Z}_S^{\hat{\lambda},\hat{\mu}}} M = \left( \mathbb{P}_S^{\lambda,-\rho} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{-\rho,\mu} \right) \otimes_{\mathcal{Z}_S^{\hat{\lambda},\hat{\mu}}} M$$

and

$$\mathcal{L}_{\mu,\nu}(N) = \mathbb{M}_S^{\mu,\nu} \otimes_{\mathcal{Z}_S^{\hat{\mu},\hat{\nu}}} N = \left( \mathbb{P}_S^{\mu,-\rho} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{-\rho,\nu} \right) \otimes_{\mathcal{Z}_S^{\hat{\mu},\hat{\nu}}} N.$$

The object  $\mathcal{L}_{\lambda,\mu}(M) \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathcal{L}_{\mu,\nu}(N)$  is therefore the projective limit (over  $n$ ) of the objects

$$\begin{aligned} & \left( \mathbb{P}_S^{\lambda,-\rho} / (\mathcal{I}^n \cdot \mathbb{P}_S^{\lambda,-\rho}) \otimes_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{-\rho,\mu} / (\mathcal{I}^n \cdot \mathbb{P}_S^{-\rho,\mu}) \otimes_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu,-\rho} / (\mathcal{I}^n \cdot \mathbb{P}_S^{\mu,-\rho}) \right. \\ & \left. \otimes_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{-\rho,\nu} / (\mathcal{I}^n \cdot \mathbb{P}_S^{-\rho,\nu}) \right) \otimes_{\mathcal{O}(\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet))} \\ & (M / (\mathcal{I}^n \cdot M) \otimes_{\mathcal{O}(\mathfrak{t}^*/(W,\bullet))} N / (\mathcal{I}^n \cdot N)). \end{aligned}$$

By Lemma 4.5 and Lemma 4.3 we have

$$\mathbb{P}_S^{-\rho,\mu} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu,-\rho} \cong \mathbb{B}_{-\rho,\mu} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{B}_{\mu,-\rho} \cong \mathbb{B}_{-\rho,-\rho}.$$

By Lemma 4.1,  $\mathbb{B}_{-\rho,-\rho}$  identifies with the completion of  $\widetilde{\mathcal{U}}_S \mathfrak{g}$  with respect to the ideal of  $\mathcal{O}(\mathfrak{t}^*)$  corresponding to  $-\rho$ . Recall that the projection  $\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet) \rightarrow \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet)$  on the first and third summands is étale at  $(\widetilde{\lambda}, -\rho, \widetilde{\nu})$ , see Lemma 3.6. From these remarks we deduce that  $\mathcal{L}_{\lambda,\mu}(M) \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathcal{L}_{\mu,\nu}(N)$  is the projective limit of the objects

$$\begin{aligned} & \left( \mathbb{M}_S^{\lambda,\nu} / (\mathcal{I}^n \cdot \mathbb{M}_S^{\lambda,\nu}) \right) \otimes_{\mathcal{O}(\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet))} \\ & \mathcal{O}(\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet)) \\ & \otimes_{\mathcal{O}(\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet))} (M / (\mathcal{I}^n \cdot M) \otimes_{\mathcal{O}(\mathfrak{t}^*/(W,\bullet))} N / (\mathcal{I}^n \cdot N)), \end{aligned}$$

where the morphism  $\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet) \rightarrow \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet)$  is the projection on the first and third summands. Now since the natural morphism

$$\begin{aligned} & \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet) \rightarrow \\ & \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet) \end{aligned}$$

is étale at  $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu})$  by Lemma 2.2,  $\mathcal{L}_{\lambda,\mu}(M) \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathcal{L}_{\mu,\nu}(N)$  identifies with the projective limit of the objects

$$\left( \mathbb{M}_S^{\lambda,\nu} / (\mathcal{I}^n \cdot \mathbb{M}_S^{\lambda,\nu}) \right) \otimes_{\mathcal{O}(\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W,\bullet))} (M / (\mathcal{I}^n \cdot M) \otimes_{\mathcal{O}(\mathfrak{t}^*/(W,\bullet))} N / (\mathcal{I}^n \cdot N)),$$

i.e. with  $\mathcal{L}_{\lambda,\nu}(M \widehat{\star}_S N)$ .

In case  $\lambda = \mu = \nu$ , the fact that the relevant isomorphisms define a monoidal structure on  $\mathcal{L}_{\lambda,\lambda}$  is clear from constructions.  $\square$

*Remark 4.11.* Proposition 4.10 also holds in case  $\mu$  is singular (in the lower closure of the fundamental alcove). This case can be treated using the methods of §4.6 below; since it is not needed in this paper, we omit the details.

**4.6. Singular analogues.** Let  $I \subset \mathfrak{R}^s$  be a subset, and let  $P_I \subset G$  be the associated standard (i.e., containing  $B$ ) parabolic subgroup of  $G$ . (In practice, only the case  $\#I = 1$  will be considered below.) Let  $U_I \subset P_I$  be the unipotent radical of  $P_I$ , and let  $L_I$  be the Levi factor containing  $T$ , so that  $P_I \cong L_I \ltimes U_I$ . Let  $\mathcal{P}_I := G/P_I$ , and consider the natural projection

$$\omega_I : G/U_I \rightarrow G/P_I.$$

The group  $L_I$  acts naturally on  $G/U_I$  on the right, via the action induced by multiplication on the right on  $G$ ; this action makes  $\omega_I$  a (Zariski locally trivial)  $L_I$ -torsor. We set

$$\tilde{\mathcal{D}}_I := (\omega_I)_*(\mathcal{D}_{G/U_I})^{L_I},$$

where the exponent means  $L_I$ -invariants. The actions of  $G$  and  $L_I$  on  $G/U_I$  induce a canonical algebra morphism

$$(4.5) \quad \mathcal{U}\mathfrak{g} \otimes_{Z_{\text{HC}}} \mathcal{O}(\mathfrak{t}^*/(W_I, \bullet)) \rightarrow \Gamma(\mathcal{P}_I, \tilde{\mathcal{D}}_I),$$

see [BMR2, Proposition 1.2.3].

Let  $\tilde{\mathfrak{g}}_I$  be the parabolic Grothendieck resolution associated with  $I$ , defined as

$$\tilde{\mathfrak{g}}_I := G \times^{P_I} (\mathfrak{g}/\text{Lie}(U_I))^*.$$

Here  $\tilde{\mathfrak{g}}_I$  is a vector bundle over  $\mathcal{P}_I$ , and there is a natural morphism

$$\tilde{\mathfrak{g}}_I \rightarrow \text{Lie}(L_I)^*/L_I \cong \mathfrak{t}^*/W_I,$$

where  $W_I \subset W$  is as in §2.1 (or, in other words, the Weyl group of  $(L_I, T)$ ). Consider the induced morphism  $f_I : \tilde{\mathfrak{g}}_I^{(1)} \times_{\mathfrak{t}^*(1)/W_I} \mathfrak{t}^*/(W_I, \bullet) \rightarrow \mathcal{P}_I^{(1)}$ , and the Frobenius morphism  $\text{Fr}_{\mathcal{P}_I} : \mathcal{P}_I \rightarrow \mathcal{P}_I^{(1)}$ . As explained in [BMR2, §1.2.1], there exists a canonical algebra morphism

$$(f_I)_* \mathcal{O}_{\tilde{\mathfrak{g}}_I^{(1)} \times_{\mathfrak{t}^*(1)/W_I} \mathfrak{t}^*/(W_I, \bullet)} \rightarrow (\text{Fr}_{\mathcal{P}_I})_* \tilde{\mathcal{D}}_I,$$

where the morphism  $\mathfrak{t}^*/(W_I, \bullet) \rightarrow \mathfrak{t}^*(1)/W_I$  is induced by the Artin–Schreier map. This morphism takes values in the center of  $(\text{Fr}_{\mathcal{P}_I})_* \tilde{\mathcal{D}}_I$ , and makes  $(\text{Fr}_{\mathcal{P}_I})_* \tilde{\mathcal{D}}_I$  a locally finitely generated  $(f_I)_* \mathcal{O}_{\tilde{\mathfrak{g}}_I^{(1)} \times_{\mathfrak{t}^*(1)/W_I} \mathfrak{t}^*/(W_I, \bullet)}$ -module. Since all the morphisms involved in this construction are affine, using this morphism one can consider  $\tilde{\mathcal{D}}_I$  as a coherent sheaf of  $\mathcal{O}_{\tilde{\mathfrak{g}}_I^{(1)} \times_{\mathfrak{t}^*(1)/W_I} \mathfrak{t}^*/(W_I, \bullet)}$ -algebras on  $\tilde{\mathfrak{g}}_I^{(1)} \times_{\mathfrak{t}^*(1)/W_I} \mathfrak{t}^*/(W_I, \bullet)$ . (We will not introduce a different notation for this sheaf of algebras.)

We also have a canonical morphism  $\tilde{\mathfrak{g}}_I \rightarrow \mathfrak{g}^*$ , and we denote by  $\tilde{\mathcal{S}}_I^*$  the (scheme-theoretic) inverse image of  $\mathcal{S}^*$  under this morphism. As in the case  $I = \emptyset$ , using [R3, Remark 3.5.4] one can check that the morphism  $\tilde{\mathfrak{g}}_I \rightarrow \mathfrak{t}^*/W_I$  considered above restricts to an isomorphism  $\tilde{\mathcal{S}}_I^* \xrightarrow{\sim} \mathfrak{t}^*/W_I$ ; in particular, this scheme is affine. We set

$$\tilde{\mathcal{D}}_{I, \mathcal{S}} := (\tilde{\mathcal{D}}_I)|_{\tilde{\mathcal{S}}_I^{*(1)} \times_{\mathfrak{t}^*(1)/W_I} \mathfrak{t}^*/(W_I, \bullet)}, \quad \mathbb{D}_{I, \mathcal{S}} = \Gamma(\tilde{\mathcal{S}}_I^{*(1)} \times_{\mathfrak{t}^*(1)/W_I} \mathfrak{t}^*/(W_I, \bullet), \tilde{\mathcal{D}}_{I, \mathcal{S}}).$$

The following lemma is a parabolic analogue of Lemma 4.1, for which the same proof applies.

**Lemma 4.12.** *The morphism (4.5) induces an algebra isomorphism*

$$\mathcal{U}\mathfrak{g} \otimes_{Z_{\text{HC}}} \mathcal{O}(\mathfrak{t}^*/(W_I, \bullet)) \xrightarrow{\sim} \mathbb{D}_{I, \mathcal{S}}.$$

Let

$$\mathbb{X}_I := \{\lambda \in \mathbb{X} \mid \forall \alpha \in I, \langle \lambda, \alpha^\vee \rangle = 0\}.$$

Then any  $\lambda \in \mathbb{X}_I$  defines a line bundle  $\mathcal{O}_{\mathcal{P}_I}(\lambda)$  on  $\mathcal{P}_I$ , from which one can define the object

$$\mathbb{D}_{I,S,\lambda} := \Gamma(\tilde{\mathcal{S}}_I^{*(1)} \times_{\mathfrak{t}^*(1)/W_I} \mathfrak{t}^*/(W_I, \bullet), (\mathcal{O}_{\mathcal{P}_I}(\lambda) \otimes \tilde{\mathcal{D}}_I)_{|\tilde{\mathcal{S}}_I^{*(1)} \times_{\mathfrak{t}^*(1)/W_I} \mathfrak{t}^*/(W_I, \bullet)}).$$

This object admits a natural action of the algebra

$$(\mathcal{U}_{\mathcal{S}\mathfrak{g}} \otimes_{Z_{\text{HC}}} \mathcal{O}(\mathfrak{t}^*/(W_I, \bullet))) \otimes_{\mathcal{O}(\mathcal{S}^*(1))} (\mathcal{U}_{\mathcal{S}\mathfrak{g}}^{\text{op}} \otimes_{Z_{\text{HC}}} \mathcal{O}(\mathfrak{t}^*/(W_I, \bullet)))$$

and of the group scheme

$$\mathfrak{t}^*/(W_I, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathbb{I}_{\mathcal{S}}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W_I, \bullet).$$

Since  $\lambda$  is  $W_I$ -invariant, the map  $\xi \mapsto \bar{\lambda} + \xi$  factors through an isomorphism  $\tau_\lambda^I : \mathfrak{t}^*/(W_I, \bullet) \xrightarrow{\sim} \mathfrak{t}^*/(W_I, \bullet)$ , and the action of the subalgebra  $\mathcal{O}(\mathfrak{t}^*/(W_I, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W_I, \bullet))$  on  $\mathbb{D}_{I,S,\lambda}$  factors through an action of the spectrum of the image of the closed embedding

$$\tau_\lambda^I \times \text{id} : \mathfrak{t}^*/(W_I, \bullet) \rightarrow \mathfrak{t}^*/(W_I, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W_I, \bullet).$$

Given  $\lambda, \mu \in \mathbb{X}$  such that  $\lambda - \mu \in \mathbb{X}_I$ , one can then define the object

$$\mathbb{B}_{\lambda,\mu}^I \in \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathcal{S}}^{\hat{\lambda}, \hat{\mu}})$$

as the completion of  $\mathbb{D}_{I,S,\lambda-\mu}$  at the ideal of  $\mathcal{O}(\mathfrak{t}^*/(W_I, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W_I, \bullet))$  corresponding to the image of  $(\bar{\lambda}, \bar{\mu})$ . As for  $\mathbb{B}_{\lambda,\mu}$ , this object can be obtained by completing  $\mathbb{D}_{I,S,\lambda-\mu}$  at the ideal of  $\mathcal{O}(\mathfrak{t}^*/(W_I, \bullet))$  corresponding to the image of  $\bar{\lambda}$  with respect to the left action, or by completing  $\mathbb{D}_{I,S,\lambda-\mu}$  at the ideal of  $\mathcal{O}(\mathfrak{t}^*/(W_I, \bullet))$  corresponding to the image of  $\bar{\mu}$  with respect to the right action.

**Lemma 4.13.** *Let  $\lambda, \mu, \nu \in \mathbb{X}$ .*

- (1) *Assume that the stabilizer of  $\mu$  for the dot-action of  $W_{\text{aff}}$  is  $W_I$ , and that  $\nu \in -\rho + \mathbb{X}_I$ . Then there exists a canonical isomorphism*

$$\mathbb{B}_{\lambda,\mu}^I \hat{\otimes}_{\mathcal{U}_{\mathcal{S}\mathfrak{g}}} \mathbb{B}_{\mu,\nu}^I \xrightarrow{\sim} \mathbb{B}_{\lambda,\nu}$$

*in  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathcal{S}}^{\hat{\lambda}, \hat{\nu}})$ . Similarly, if the stabilizer of  $\mu$  for the dot-action of  $W_{\text{aff}}$  is  $W_I$ , and  $\lambda \in -\rho + \mathbb{X}_I$ , then there exists a canonical isomorphism*

$$\mathbb{B}_{\lambda,\mu}^I \hat{\otimes}_{\mathcal{U}_{\mathcal{S}\mathfrak{g}}} \mathbb{B}_{\mu,\nu} \xrightarrow{\sim} \mathbb{B}_{\lambda,\nu}$$

*in  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathcal{S}}^{\hat{\lambda}, \hat{\nu}})$ .*

- (2) *Assume that the stabilizer of  $\mu$  for the dot-action of  $W_{\text{aff}}$  is  $W_I$ , and that  $\lambda, \nu \in -\rho + \mathbb{X}_I$ . Then there exists a canonical isomorphism*

$$\mathbb{B}_{\lambda,\mu}^I \hat{\otimes}_{\mathcal{U}_{\mathcal{S}\mathfrak{g}}} \mathbb{B}_{\mu,\nu}^I \xrightarrow{\sim} \mathbb{B}_{\lambda,\nu}^I$$

*in  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathcal{S}}^{\hat{\lambda}, \hat{\nu}})$ .*

*Proof.* (1) We only prove the first isomorphism; the proof of the second one is similar. Our assumptions ensure that  $\mu - \nu \in \mathbb{X}_I$ , so that the object  $\mathbb{B}_{\mu,\nu}^I$  is well defined. Consider the natural morphism  $a : \mathcal{B} \rightarrow \mathcal{P}_I$ . By [BMR2, Proposition 1.2.3] there exists a canonical morphism of sheaves of algebras

$$\tilde{\mathcal{D}}_I \rightarrow a_* \tilde{\mathcal{D}}.$$

By the projection formula, and since  $a_*\mathcal{O}_{\mathcal{B}} \cong \mathcal{O}_{\mathcal{P}_I}$ , we also have

$$a_*\mathcal{O}_{\mathcal{B}}(\mu - \nu) \cong \mathcal{O}_{\mathcal{P}_I}(\mu - \nu),$$

and via this isomorphism the action of  $\tilde{\mathcal{G}}_I$  on  $\mathcal{O}_{\mathcal{P}_I}(\mu - \nu)$  is obtained by restriction of scalars along the morphism above from the natural action of  $a_*\tilde{\mathcal{G}}$  on  $a_*\mathcal{O}_{\mathcal{B}}(\mu - \nu)$ . We deduce a natural isomorphism

$$\begin{aligned} (a_*\mathcal{O}_{\mathcal{B}}(\lambda - \mu) \otimes_{a_*\mathcal{O}_{\mathcal{B}}} a_*\tilde{\mathcal{G}}) \otimes_{\tilde{\mathcal{G}}_I} (\mathcal{O}_{\mathcal{P}_I}(\mu - \nu) \otimes_{\mathcal{O}_{\mathcal{P}_I}} \tilde{\mathcal{G}}_I) &\xrightarrow{\sim} \\ a_*\mathcal{O}_{\mathcal{B}}(\lambda - \mu) \otimes_{\mathcal{O}_{\mathcal{P}_I}} \mathcal{O}_{\mathcal{P}_I}(\mu - \nu) \otimes_{\mathcal{O}_{\mathcal{P}_I}} a_*\tilde{\mathcal{G}} &\cong a_*\mathcal{O}_{\mathcal{B}}(\lambda - \nu) \otimes_{\mathcal{O}_{\mathcal{P}_I}} a_*\tilde{\mathcal{G}}, \end{aligned}$$

where the first map is similar to that considered in the proof of Lemma 4.3. We deduce the desired isomorphism by restricting to  $\tilde{\mathcal{S}}_I^{*(1)} \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I, \bullet)$  and then completing, using Lemma 4.12 and the fact that the natural morphism  $\mathfrak{t}^*/(W_I, \bullet) \rightarrow \mathfrak{t}^*/(W, \bullet)$  is étale at the image of  $\bar{\mu}$ , see Lemma 2.2.

(2) The proof is similar to that of Lemma 4.3.  $\square$

**4.7. Conjugation of wall-crossing bimodules.** The following proposition will eventually reduce the question of the description of the bimodules realizing wall-crossing functors for  $G$  to the case of wall-crossing functors attached to simple reflections in  $W$ .

**Proposition 4.14.** *Let  $\lambda, \mu, \mu' \in \mathbb{X}$ , let  $s \in S_{\text{aff}}$ , and let  $s' \in S_{\text{aff}} \cap W$ . Assume that  $\lambda$  belongs to the fundamental alcove, and that  $\mu$ , resp.  $\mu'$ , belongs to the wall of the fundamental alcove attached to  $s$ , resp.  $s'$ , and on no other wall. Let also  $w \in W_{\text{ext}}$  be such that  $s' = wsw^{-1}$ . Then there exists an isomorphism*

$$\mathbb{P}_S^{\lambda, \mu'} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{P}_S^{\mu', \lambda} \cong \mathbb{B}_{\lambda, w \bullet \lambda} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} (\mathbb{P}_S^{\lambda, \mu} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{P}_S^{\mu, \lambda}) \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{w \bullet \lambda, \lambda}$$

in  $\text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_S^{\hat{\lambda}, \hat{\lambda}})$ .

*Proof.* By Lemma 4.5 we have isomorphisms

$$\mathbb{P}_S^{\lambda, \mu} \cong \mathbb{B}_{w \bullet \lambda, w \bullet \mu}, \quad \mathbb{P}_S^{\mu, \lambda} \cong \mathbb{B}_{w \bullet \mu, w \bullet \lambda}.$$

Using Lemma 4.3, we deduce isomorphisms

$$\begin{aligned} \mathbb{B}_{\lambda, w \bullet \lambda} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} (\mathbb{P}_S^{\lambda, \mu} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{P}_S^{\mu, \lambda}) \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{w \bullet \lambda, \lambda} \\ \cong \mathbb{B}_{\lambda, w \bullet \lambda} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{w \bullet \lambda, w \bullet \mu} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{w \bullet \mu, w \bullet \lambda} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{w \bullet \lambda, \lambda} \cong \mathbb{B}_{\lambda, w \bullet \mu} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{w \bullet \mu, \lambda}. \end{aligned}$$

Now the stabilizers of both  $\mu'$  and  $w \bullet \mu$  for the dot-action of  $W_{\text{aff}}$  is  $W_{\{\alpha\}}$ , where  $\alpha \in \mathfrak{R}^s$  is the simple reflection such that  $s = s_{\alpha}$ . By Lemma 4.13(1), it follows that we have isomorphisms

$$\mathbb{B}_{\lambda, w \bullet \mu} \cong \mathbb{B}_{\lambda, \mu'} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{\mu', w \bullet \mu}^{\{\alpha\}}, \quad \mathbb{B}_{w \bullet \mu, \lambda} \cong \mathbb{B}_{w \bullet \mu, \mu'}^{\{\alpha\}} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{\mu', \lambda},$$

from which we obtain an isomorphism

$$\mathbb{B}_{\lambda, w \bullet \mu} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{w \bullet \mu, \lambda} \cong \mathbb{B}_{\lambda, \mu'} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{\mu', w \bullet \mu}^{\{\alpha\}} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{w \bullet \mu, \mu'}^{\{\alpha\}} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{\mu', \lambda}.$$

Then by Lemma 4.13(2) we have

$$\mathbb{B}_{\mu', w \bullet \mu}^{\{\alpha\}} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{w \bullet \mu, \mu'}^{\{\alpha\}} \cong \mathbb{B}_{\mu', \mu'}^{\{\alpha\}},$$

which implies (using again Lemma 4.13(1)) that

$$\mathbb{B}_{\lambda, w \bullet \mu} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{w \bullet \mu, \lambda} \cong \mathbb{B}_{\lambda, \mu'} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{\mu', \lambda}.$$

The desired claim follows, in view of Lemma 4.5.  $\square$

## 5. THE AFFINE HECKE CATEGORY AND REPRESENTATIONS OF THE REGULAR CENTRALIZER

This section is independent of the previous ones, and applies in a slightly more general setting. Here we explain that the category of “enhanced Soergel bimodules” recently investigated by Abe [Ab1] can be interpreted as a category of representations of (a pullback of) the universal centralizer.

**5.1. The affine Weyl group and the associated Hecke category.** In this section we denote by  $\mathbf{G}$  a connected reductive group over  $\mathbb{k}$ . We fix a Borel subgroup  $\mathbf{B} \subset \mathbf{G}$  and a maximal torus  $\mathbf{T} \subset \mathbf{B}$ . The Lie algebras of  $\mathbf{G}$ ,  $\mathbf{B}$ ,  $\mathbf{T}$  will be denoted  $\mathfrak{g}$ ,  $\mathfrak{b}$  and  $\mathfrak{t}$  respectively. (We use notations different from those of previous sections since in our later applications the reductive group  $\mathbf{G}$  will not be  $G$ , but rather its Frobenius twist.) We set  $\mathbf{X} := X^*(\mathbf{T})$ , resp.  $\mathbf{X}^\vee := X_*(\mathbf{T})$ , and denote by  $\Phi \subset \mathbf{X}$ , resp.  $\Phi^\vee \subset \mathbf{X}^\vee$ , the root system, resp. coroot system, of  $(\mathbf{G}, \mathbf{T})$ . The canonical bijection  $\Phi \xrightarrow{\sim} \Phi^\vee$  will be denoted  $\alpha \mapsto \alpha^\vee$ . The choice of  $\mathbf{B}$  determines a subset  $\Phi^+ \subset \Phi$  of positive roots, consisting of the  $\mathbf{T}$ -weights in  $\mathfrak{g}/\mathfrak{b}$ ; the corresponding basis of  $\Phi$  will be denoted  $\Phi^s$ . In this section we will make the following assumptions:

- $\ell$  is good for  $\mathbf{G}$ , and  $\ell \geq 5$ ;
- neither  $\mathbf{X}/\mathbb{Z}\Phi$  nor  $\mathbf{X}^\vee/\mathbb{Z}\Phi^\vee$  has  $p$ -torsion;
- there exists a  $\mathbf{G}$ -equivariant isomorphism  $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ .

For simplicity we will fix once and for all a  $\mathbf{G}$ -equivariant isomorphism  $\kappa : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ . As in §2.9, this choice also provides an identification of  $\mathfrak{t}$  and  $\mathfrak{t}^*$ .

Let  $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$  be the Weyl group of  $(\mathbf{G}, \mathbf{T})$ . The associated *affine Weyl group* is the semi-direct product

$$\mathbf{W}_{\text{aff}} := \mathbf{W} \ltimes \mathbb{Z}\Phi$$

where  $\mathbb{Z}\Phi \subset \mathbf{X}$  is the lattice generated by the roots. For  $\lambda \in \mathbb{Z}\Phi$  we will denote by  $t_\lambda$  the image of  $\lambda \in \mathbf{W}_{\text{aff}}$ . It is well known that  $\mathbf{W}_{\text{aff}}$  is generated by the subset  $\mathbf{S}_{\text{aff}}$  consisting of the reflections  $s_\alpha$  with  $\alpha \in \Phi^s$ , together with the products  $t_\beta s_\beta$  where  $\beta \in \Phi$  is such that  $\beta^\vee$  is a maximal coroot. Moreover, the pair  $(\mathbf{W}_{\text{aff}}, \mathbf{S}_{\text{aff}})$  is a Coxeter system, see [J2, §II.6.3].

As in §2.1 we will “enlarge” this group by considering translations by all elements of  $\mathbf{X}$ . Namely, the *extended affine Weyl group* is the semi-direct product

$$\mathbf{W}_{\text{ext}} := \mathbf{W} \ltimes \mathbf{X}.$$

Then  $\mathbf{W}_{\text{aff}}$  is a normal subgroup in  $\mathbf{W}_{\text{ext}}$ .

We will consider the balanced “realization” of  $\mathbf{W}_{\text{aff}}$  over  $\mathbb{k}$  (in the sense of [EW]) defined as follows:

- the underlying  $\mathbb{k}$ -vector space is  $\mathfrak{t}^*$ ;
- if  $\alpha \in \Phi^s$  and  $s = s_\alpha$ , then the “root”  $\alpha_s \in \mathfrak{t}$  (resp. “coroot”  $\alpha_s^\vee \in \mathfrak{t}^*$ ) associated with  $s$  is the differential of  $\alpha^\vee$  (resp. of  $\alpha$ );
- if  $\beta \in \Phi^+$  is such that  $\beta^\vee$  is a maximal coroot and  $s = t_\beta s_\beta$  then the “root”  $\alpha_s$  (resp. “coroot”  $\alpha_s^\vee$ ) associated with  $s$  is the differential of  $-\beta^\vee$  (resp. of  $-\beta$ ).

This realization is an example of a Cartan realization in the sense of [AMRW1, §10.1]. There is an associated action of  $\mathbf{W}_{\text{aff}}$  on  $\mathfrak{t}^*$ , which simply is the natural action of  $\mathbf{W}$ , seen as an action of  $\mathbf{W}_{\text{aff}}$  via the projection  $\mathbf{W}_{\text{aff}} \rightarrow \mathbf{W}$ .

**Lemma 5.1.** *For any distinct  $s, t \in \mathbf{S}_{\text{aff}}$  such that  $st$  has finite order, the action of the subgroup  $\mathbf{W}_{s,t}$  generated by  $\{s, t\}$  on  $\mathfrak{t}$  is reflection faithful in the sense of [S3].*

*Proof.* By definition, what we have to prove is that the representation is faithful, and that the elements having a fixed points set of codimension 1 are exactly the conjugates of  $s$  or  $t$ . Here, all the conjugates of  $s$  or  $t$  act as reflections on  $\mathfrak{t}$ ; in particular their fixed-point sets indeed are hyperplanes. If  $m \in \{2, 3, 4, 6\}$  is the order of  $st$  in  $\mathbf{W}_{\text{aff}}$ , then the other elements of  $\mathbf{W}_{s,t}$  are the elements  $(st)^k$  and  $(ts)^k$  with  $k \in \{0, \dots, m-1\}$ . The Cartan matrix  $\begin{pmatrix} \langle \alpha_s, \alpha_s^\vee \rangle & \langle \alpha_s, \alpha_t^\vee \rangle \\ \langle \alpha_t, \alpha_s^\vee \rangle & \langle \alpha_t, \alpha_t^\vee \rangle \end{pmatrix}$  being invertible since  $\ell \geq 5$ , the subspace of  $\mathfrak{t}$  generated by  $\alpha_s^\vee$  and  $\alpha_t^\vee$  is 2-dimensional. Hence to conclude it is enough to check that  $(st)^{m-1}$  has no nontrivial fixed point on this subspace, which can be easily done by hands in each case.  $\square$

We will denote by  $\mathbf{D}_{\text{BS}}$  the “diagrammatic Hecke category” defined by Elias–Williamson [EW] for the Coxeter system  $(\mathbf{W}_{\text{aff}}, \mathbf{S}_{\text{aff}})$  and this choice of realization. (For a discussion of this definition, see also [AMRW1, Chap. 2].) By definition the objects in this category are pairs  $(\underline{w}, n)$  where  $\underline{w}$  is a word in  $\mathbf{S}_{\text{aff}}$  and  $n \in \mathbb{Z}$ , and for any words  $\underline{w}, \underline{w}'$  the direct sum of morphism spaces

$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathbf{D}_{\text{BS}}}((\underline{w}, 0), (\underline{w}', n))$$

is a graded bimodule over  $R := \mathcal{O}(\mathfrak{t}^*) = \text{Sym}(\mathfrak{t})$  (where the grading is such that elements in  $\mathfrak{t}$  have degree 2). Following usual conventions, the object  $(\underline{w}, n)$  will rather be denoted  $B_{\underline{w}}(n)$ . Then there exists a natural “grading shift” autoequivalence of  $\mathbf{D}_{\text{BS}}$  such that  $(B_{\underline{w}}(n))(1) = B_{\underline{w}}(n+1)$  for any  $\underline{w}$  and any  $n \in \mathbb{Z}$ .

**5.2. Abe’s incarnation of the Hecke category.** The proof of Theorem 6.3 will use a description of  $\mathbf{D}_{\text{BS}}$  in terms of  $R$ -bimodules due to Abe [Ab1], which is close to the definition of Soergel bimodules [S3], and which we now recall.

We will denote by  $Q$  the fraction field of  $R$ . Following [Ab1], we denote by  $\mathbf{C}'$  the category defined as follows. Objects are pairs consisting of a graded  $R$ -bimodule  $M$  together with a decomposition

$$(5.1) \quad Q \otimes_R M = \bigoplus_{w \in \mathbf{W}_{\text{aff}}} M_Q^w$$

such that:

- there exist only finitely many  $w$ ’s such that  $M_Q^w \neq 0$ ;
- for any  $w \in \mathbf{W}_{\text{aff}}$ ,  $r \in R$  and  $m \in M_Q^w$  we have

$$(5.2) \quad m \cdot r = w(r) \cdot m.$$

Morphisms in  $\mathbf{C}'$  are defined in the obvious way, as morphisms of graded bimodules compatible with the decompositions (5.1). We also denote by  $\mathbf{C}$  the full subcategory of  $\mathbf{C}'$  whose objects are those whose underlying graded  $R$ -bimodule  $M$  is finitely generated as an  $R$ -bimodule and flat as a left  $R$ -module. As explained in [Ab1, Lemma 2.6], the underlying  $R$ -bimodule of any object in  $\mathbf{C}$  is in fact finitely generated as a left and as a right  $R$ -module; this property shows that the tensor product over  $R$  induces in a natural way a monoidal product on  $\mathbf{C}$ . We also have a “grading shift” autoequivalence of  $\mathbf{C}$ , which only changes the grading of the underlying graded  $R$ -bimodule in such a way that  $M(1)^i = M^{i+1}$ .

For  $s \in \mathbf{S}_{\text{aff}}$ , we consider the  $s$ -invariants  $R^s \subset R$ , and the graded  $R$ -bimodule  $B_s^{\text{Bim}} := R \otimes_{R^s} R(1)$ . This object has a natural “lift” as an object in  $\mathbf{C}$ , which will also be denoted  $B_s^{\text{Bim}}$  (see [Ab1, §2.4]).

The following result follows from [Ab1, Theorem 5.6]. (The assumptions of this theorem hold in our present setting by Lemma 5.1.)

**Theorem 5.2.** *There exists a canonical fully-faithful monoidal functor*

$$\mathbf{D}_{\text{BS}} \rightarrow \mathbf{C}$$

*sending  $B_s$  to  $B_s^{\text{Bim}}$  for any  $s \in \mathbf{S}_{\text{aff}}$  and intertwining the grading shifts (1).*

Below we will have to consider a slight extension of the category  $\mathbf{C}$ , adapted to the group  $\mathbf{W}_{\text{ext}}$ . Namely, the action of  $\mathbf{W}_{\text{aff}}$  on  $\mathbf{t}^*$  extends in a natural way to  $\mathbf{W}_{\text{ext}}$  (using now the projection  $\mathbf{W}_{\text{ext}} \rightarrow \mathbf{W}$ ). We will denote by  $\mathbf{C}'_{\text{ext}}$  the category whose objects are pairs consisting a graded  $R$ -bimodule  $M$  together with a decomposition

$$Q \otimes_R M = \bigoplus_{w \in \mathbf{W}_{\text{ext}}} M_Q^w$$

such that:

- there exist only finitely many  $w$ ’s such that  $M_Q^w \neq 0$ ;
- for any  $w \in \mathbf{W}_{\text{ext}}$ ,  $r \in R$  and  $m \in M_Q^w$  we have  $m \cdot r = w(r) \cdot m$ ,

and where morphisms are defined in the obvious way. We will also denote by  $\mathbf{C}_{\text{ext}}$  the full subcategory of  $\mathbf{C}'_{\text{ext}}$  whose objects are those whose underlying graded  $R$ -bimodule  $M$  is finitely generated as an  $R$ -bimodule and flat as a left  $R$ -module. It is clear that  $\mathbf{C}'$  is a full subcategory in  $\mathbf{C}'_{\text{ext}}$ , that  $\mathbf{C}$  is a full subcategory in  $\mathbf{C}_{\text{ext}}$ , and that the tensor product  $\otimes_R$  defines a monoidal structure on  $\mathbf{C}_{\text{ext}}$ .

In addition to the objects  $B_s^{\text{Bim}}$  considered above, the category  $\mathbf{C}_{\text{ext}}$  also possesses “standard” objects  $(\Delta_x : x \in \mathbf{W}_{\text{ext}})$  defined as follows. For any  $x \in \mathbf{W}_{\text{ext}}$ ,  $\Delta_x$  is isomorphic to  $R$  as a graded vector space, and the structure of  $R$ -bimodule is given by

$$r \cdot m \cdot r' = rmx(r')$$

for  $r, r' \in R$  and  $m \in \Delta_x$ . The decomposition of  $Q \otimes_R \Delta_x$  is defined so that this object is concentrated in degree  $x$ . For any  $x, y \in \mathbf{W}_{\text{ext}}$  we have a canonical isomorphism

$$(5.3) \quad \Delta_x \otimes_R \Delta_y \xrightarrow{\sim} \Delta_{xy}$$

in  $\mathbf{C}_{\text{ext}}$ , defined by  $m \otimes m' \mapsto mx(m')$ .

**Lemma 5.3.** *Let  $s, t \in \mathbf{S}_{\text{aff}}$  and  $x \in \mathbf{W}_{\text{ext}}$  be such that  $s = txt^{-1}$ . Then there exists a canonical isomorphism*

$$B_s^{\text{Bim}} \cong \Delta_x \otimes_R B_t^{\text{Bim}} \otimes_R \Delta_{x^{-1}}.$$

*Proof.* The isomorphism of  $R$ -bimodules

$$\Delta_x \otimes_R B_t^{\text{Bim}} \otimes_R \Delta_{x^{-1}} \xrightarrow{\sim} B_s^{\text{Bim}}$$

is defined by

$$r_1 \otimes (r_2 \otimes r_3) \otimes r_4 \mapsto (r_1 x(r_2)) \otimes (x(r_3) x(r_4)).$$

We leave it to the reader to check that this morphism is well defined, and indeed defines an isomorphism in  $\mathbf{C}_{\text{ext}}$ .  $\square$

In Section 6 we will also need the following standard claim, for which we refer to [EW, §3.4].

**Lemma 5.4.** *Assume that  $s \in \mathbf{S}_{\text{aff}} \cap \mathbf{W}$ . Then there exist exact sequences of  $R$ -bimodules*

$$\Delta_s \hookrightarrow R \otimes_{R^s} R \twoheadrightarrow \Delta_e, \quad \Delta_e \hookrightarrow R \otimes_{R^s} R \twoheadrightarrow \Delta_s.$$

**5.3. The universal centralizer and restriction to a Kostant section.** We now repeat the considerations of §2.9 in the present setting. We will denote by  $\mathfrak{g}_{\text{reg}}^* \subset \mathfrak{g}^*$  the open subset consisting of regular elements, i.e. elements whose centralizer has minimal dimension (namely,  $\dim(\mathbf{T})$ ). The “regular universal centralizer” is the affine group scheme

$$\mathbf{J}_{\text{reg}}^* := \mathfrak{g}_{\text{reg}}^* \times_{\mathfrak{g}_{\text{reg}}^* \times \mathfrak{g}_{\text{reg}}^*} (\mathbf{G} \times \mathfrak{g}_{\text{reg}}^*)$$

over  $\mathfrak{g}_{\text{reg}}^*$ , where the morphism  $\mathfrak{g}_{\text{reg}}^* \rightarrow \mathfrak{g}_{\text{reg}}^* \times \mathfrak{g}_{\text{reg}}^*$  is the diagonal embedding, and the map  $\mathbf{G} \times \mathfrak{g}_{\text{reg}}^* \rightarrow \mathfrak{g}_{\text{reg}}^*$  sends  $(g, x)$  to  $(g \cdot x, x)$ . As explained in [R3, Corollary 3.3.6],  $\mathbf{J}_{\text{reg}}^*$  is smooth over  $\mathfrak{g}_{\text{reg}}^*$ . As in §2.9, we also have a canonical morphism of group schemes

$$(5.4) \quad \mathbf{J}_{\text{reg}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^* \rightarrow \mathbf{T} \times (\mathfrak{g}_{\text{reg}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*).$$

Let now  $\mathfrak{g}_{\text{rs}} \subset \mathfrak{g}$  denote the open subset of semisimple regular elements, and set  $\mathfrak{g}_{\text{rs}}^* := \kappa(\mathfrak{g}_{\text{rs}})$ . We will denote by  $\mathbf{J}_{\text{rs}}^*$  the restriction of  $\mathbf{J}_{\text{reg}}^*$  to  $\mathfrak{g}_{\text{rs}}^*$ . Finally, we consider the adjoint quotient  $\mathfrak{g}^*/\mathbf{G}$ , which we identify with  $\mathbf{t}^*/\mathbf{W}$  in the standard way, see e.g. [R3, Proposition 2.3.2].

**Lemma 5.5.** *The morphism (5.4) restricts to an isomorphism*

$$\mathbf{J}_{\text{rs}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^* \xrightarrow{\sim} \mathbf{T} \times (\mathfrak{g}_{\text{rs}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$$

*of group schemes over  $\mathfrak{g}_{\text{rs}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$ .*

*Proof.* The lemma is obvious from the construction of the morphism (5.4) and the fact that the centralizer of any regular semisimple element is a maximal torus (see [R3, Lemma 2.3.3]).  $\square$

As in §2.9 we choose an affine subspace  $\mathbf{S} \subset \mathfrak{g}$  as in [R3, §3.1] (in other words, a “Kostant section” for the adjoint quotient), and set  $\mathbf{S}^* := \kappa(\mathbf{S})$ . We then have  $\mathbf{S}^* \subset \mathfrak{g}_{\text{reg}}^*$ , see [R3, Equation (3.1.1)], and the composition  $\mathbf{S}^* \hookrightarrow \mathfrak{g}^* \rightarrow \mathfrak{g}^*/\mathbf{G}$  (where the second map is the adjoint quotient) is an isomorphism, see [R3, Theorem 3.2.2]. As explained e.g. in [MR, §4.4], there exists a natural action of the multiplicative group  $\mathbb{G}_m$  on  $\mathbf{S}^*$  such that this map is  $\mathbb{G}_m$ -equivariant, where  $t \in \mathbb{k}^\times$  acts on  $\mathfrak{g}^*$  by multiplication by  $t^{-2}$ , and on  $\mathfrak{g}^*/\mathbf{G}$  by the induced action. We also set

$$\mathbf{J}_{\mathbf{S}}^* := \mathbf{S}^* \times_{\mathfrak{g}_{\text{reg}}^*} \mathbf{J}_{\text{reg}}^*.$$

Then  $\mathbf{J}_{\mathbf{S}}^*$  is a smooth affine group scheme over  $\mathbf{S}^*$ , and as explained in [MR, §4.5, p. 2302] there exists a natural  $\mathbb{G}_m$ -action on  $\mathbf{J}_{\mathbf{S}}^*$  such that the structure morphism  $\mathbf{J}_{\mathbf{S}}^* \rightarrow \mathbf{S}^*$ , the multiplication map  $\mathbf{J}_{\mathbf{S}}^* \times_{\mathbf{S}^*} \mathbf{J}_{\mathbf{S}}^* \rightarrow \mathbf{J}_{\mathbf{S}}^*$  and the inversion morphism  $\mathbf{J}_{\mathbf{S}}^* \rightarrow \mathbf{J}_{\mathbf{S}}^*$  are  $\mathbb{G}_m$ -equivariant.

Consider once again the isomorphism given by the composition

$$\mathbf{S}^* \hookrightarrow \mathfrak{g}^* \rightarrow \mathfrak{g}^*/\mathbf{G} \cong \mathbf{t}^*/\mathbf{W}.$$

Transporting the group scheme  $\mathbf{J}_{\mathbf{S}}^*$  along this isomorphism we obtain a group scheme over  $\mathbf{t}^*/\mathbf{W}$ , which we denote by  $\mathbf{J}_{\text{adj}}^*$ . The considerations above show that there



exists an action of  $\mathbb{G}_m$  on  $\mathbf{J}_{\text{adj}}^*$  such that the projection  $\mathbf{J}_{\text{adj}}^* \rightarrow \mathbf{t}^*/\mathbf{W}$  is  $\mathbb{G}_m$ -equivariant, where the action on  $\mathbf{t}^*/\mathbf{W}$  is induced by the action on  $\mathbf{t}^*$  where  $t \in \mathbb{k}^\times$  acts by multiplication by  $t^{-2}$ .

**5.4. Representations of the universal centralizer and Abe's category.** The actions of  $\mathbb{G}_m$  on  $\mathbf{t}^*$  and  $\mathbf{t}^*/\mathbf{W}$  considered in §5.3 provide an action on the fiber product  $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$ . Let us now consider the category

$$\text{Rep}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$$

of  $\mathbb{G}_m$ -equivariant coherent representations of the affine group scheme  $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$  over  $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$ , i.e.  $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$ -modules equipped with a structure of  $\mathbb{G}_m$ -equivariant coherent sheaf on  $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$ , such that the action map is  $\mathbb{G}_m$ -equivariant. This category admits a natural convolution product  $\star$ , such that the  $\mathcal{O}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ -module underlying the product  $M \star N$  is the tensor product  $M \otimes_{\mathcal{O}(\mathbf{t}^*)} N$  (where  $\mathcal{O}(\mathbf{t}^*)$  acts on  $M$  via the second projection  $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^* \rightarrow \mathbf{t}^*$  and on  $N$  via the first projection  $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^* \rightarrow \mathbf{t}^*$ ). In this way,  $(\text{Rep}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*), \star)$  is a monoidal category. We will denote by

$$\text{Rep}_{\text{fl}}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$$

the full subcategory of  $\text{Rep}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$  whose objects are the representations whose underlying coherent sheaves are flat with respect to the first projection  $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^* \rightarrow \mathbf{t}^*$ . It is not difficult to check that this subcategory is stable under  $\star$ , hence also admits a canonical structure of monoidal category.

**Proposition 5.6.** *There exists a canonical fully faithful monoidal functor*

$$(\text{Rep}_{\text{fl}}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*), \star) \rightarrow (\mathcal{C}_{\text{ext}}, \otimes_R).$$

*Proof.* We start by constructing a functor

$$(5.5) \quad \text{Rep}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*) \rightarrow \mathcal{C}'_{\text{ext}}.$$

By definition, any object in  $\text{Rep}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$  is in particular a  $\mathbb{G}_m$ -equivariant coherent sheaf on  $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$ , hence can be seen as a graded  $R$ -bimodule. To equip this graded bimodule with the structure of an object in  $\mathcal{C}'_{\text{ext}}$ , we must provide a decomposition of its tensor product with  $Q$  parametrized by  $\mathbf{W}_{\text{ext}}$ . In fact, we will provide such a decomposition for its tensor product with  $\mathcal{O}(\mathbf{t}_{\text{rs}}^*)$ , where  $\mathbf{t}_{\text{rs}}^* := \mathbf{t}^* \cap \mathbf{g}_{\text{rs}}^*$ .

First, the open subset  $\mathbf{t}_{\text{rs}}^* \subset \mathbf{t}^*$  is the complement of the kernels of the differentials of the coroots. This open subset is stable under the action of  $\mathbf{W}$ , and the restriction of this action is free, see [R3, Lemma 2.3.3]. In particular we have an open subset  $\mathbf{t}_{\text{rs}}^*/\mathbf{W} \subset \mathbf{t}^*/\mathbf{W}$ , the morphism  $\mathbf{t}_{\text{rs}}^* \rightarrow \mathbf{t}_{\text{rs}}^*/\mathbf{W}$  is étale, and the map  $(w, x) \mapsto (x, w(x))$  induces an isomorphism of schemes

$$\mathbf{W} \times \mathbf{t}_{\text{rs}}^* \xrightarrow{\sim} \mathbf{t}_{\text{rs}}^* \times_{\mathbf{t}_{\text{rs}}^*/\mathbf{W}} \mathbf{t}_{\text{rs}}^*,$$

see [SGA1, Exp. V, §2]. As a consequence, for any coherent sheaf  $\mathcal{F}$  on  $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$ , the tensor product

$$\mathcal{O}(\mathbf{t}_{\text{rs}}^*) \otimes_R \Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathcal{F})$$

admits a canonical decomposition (as an  $\mathcal{O}(\mathbf{t}_{\text{rs}}^*)$ -bimodule) parametrized by  $\mathbf{W}$ , such that the action on the factor corresponding to  $w \in \mathbf{W}$  factors through the quotient

$$\mathcal{O}(\mathbf{t}_{\text{rs}}^* \times \mathbf{t}_{\text{rs}}^*) \twoheadrightarrow \mathcal{O}(\text{Gr}(w, \mathbf{t}_{\text{rs}}^*))$$

(where in the right-hand side  $\text{Gr}(w, \mathbf{t}_{\text{rs}}^*)$  denotes the graph of  $w$  acting on  $\mathbf{t}_{\text{rs}}^*$ ), i.e. satisfies the condition in (5.2).

Next, let us explain how this decomposition can be refined if  $\mathcal{F}$  belongs to  $\text{Rep}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ . For this, we consider the restriction  $\mathbf{M}_w$  of  $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$  to  $\text{Gr}(w, \mathbf{t}_{\text{rs}}^*)$ . Identifying the latter subscheme with  $\mathbf{t}_{\text{rs}}^*$  via the first projection and using Lemma 5.5, we obtain a canonical isomorphism of group schemes

$$\mathbf{M}_w \xrightarrow{\sim} \mathbf{t}_{\text{rs}}^* \times \mathbf{T}.$$

This means that the category of representations of  $\mathbf{M}_w$  on coherent sheaves on  $\text{Gr}(w, \mathbf{t}_{\text{rs}}^*)$  is canonically equivalent to the category of  $\mathbf{X}$ -graded coherent sheaves on  $\mathbf{t}_{\text{rs}}^*$ . Starting with an object  $\mathcal{F}$  in  $\text{Rep}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ , we therefore obtain a decomposition of  $\mathcal{O}(\mathbf{t}_{\text{rs}}^*) \otimes_R \Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathcal{F})$  parametrized by  $\mathbf{W}_{\text{ext}}$  by defining, for  $\lambda \in \mathbf{X}$  and  $w \in \mathbf{W}$ , the summand associated with  $t_\lambda w$  as the  $\lambda$ -graded part in the summand associated with  $w$  (which is a representation of  $\mathbf{M}_w$ ). This finishes the description of the functor (5.5).

It is clear from construction that this functor sends objects in  $\text{Rep}_{\mathbb{H}}^{\mathbb{G}^{\text{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$  to objects in  $\mathbf{C}_{\text{ext}}$ , which therefore provides the functor of the statement. This functor is also easily seen to be monoidal. Let us now explain why it is fully faithful. Consider  $\mathcal{F}, \mathcal{G}$  in  $\text{Rep}_{\mathbb{H}}^{\mathbb{G}^{\text{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ , and denote their images by  $M, N$  (so that the underlying graded bimodule of  $M$ , resp.  $N$ , is  $\Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathcal{F})$ , resp.  $\Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathcal{G})$ ). By construction, morphisms in  $\mathbf{C}_{\text{ext}}$  from  $M$  to  $N$  are morphisms of graded bimodules from  $\Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathcal{F})$  to  $\Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathcal{G})$  whose restriction to  $\eta \times \mathbf{t}^*$  (where  $\eta$  is the generic point of  $\mathbf{t}^*$ ) commutes with the action of the restriction of  $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$ . Now, since by assumption  $\Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathcal{G})$  is flat as a left  $R$ -module and  $\mathcal{O}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$  is flat over  $\mathcal{O}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ , such a morphism is automatically a morphism of  $\mathcal{O}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ -comodules. This proves the desired fully faithfulness.  $\square$

**Lemma 5.7.** *For any  $w \in \mathbf{W}$ , the object  $\Delta_w$  belongs to the essential image of the functor of Proposition 5.6.*

*Proof.* The isomorphism (5.3) reduces the proof to the case  $w$  belongs either to  $\mathbf{W}$  or to  $\mathbf{X}$ . The case  $w \in \mathbf{W}$  is obvious: in this case  $\Delta_x$  is the image of its underlying graded  $R$ -bimodule, endowed with the trivial structure of representation. For the case  $w \in \mathbf{X}$ , in view of the construction of the functor in Proposition 5.6, the claim follows from the fact that the isomorphism of Lemma 5.5 is the restriction of the morphism (5.4).  $\square$

*Remark 5.8.* For  $w \in \mathbf{W}_{\text{ext}}$ , we will denote by  $\Delta_w^{\mathbf{J}}$  the unique object in the category  $\text{Rep}_{\mathbb{H}}^{\mathbb{G}^{\text{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$  which is sent to  $\Delta_w$ .

### 5.5. Representations of the universal centralizer and the Hecke category.

By Proposition 5.6 the category  $\text{Rep}_{\mathbb{H}}^{\mathbb{G}^{\text{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$  can be seen as a full monoidal subcategory in Abe's category  $\mathbf{C}_{\text{ext}}$ , and by Theorem 5.2 the same is true for the Hecke category  $\mathbf{D}_{\text{BS}}$ . We now investigate the relation between these two subcategories.

**Lemma 5.9.** *The essential image of the functor of Theorem 5.2 is contained in the essential image of the functor of Proposition 5.6.*

*Proof.* By definition, the category  $\mathbf{D}_{\text{BS}}$  is generated under convolution and grading shift by the objects  $(B_s : s \in \mathbf{S}_{\text{aff}})$ . Hence to prove the lemma it suffices to prove each  $B_s^{\text{Bim}}$  belongs to the essential image of the functor of Proposition 5.6.

If  $s = s_\alpha$  for some  $\alpha \in \Phi^s$ , then  $B_s^{\text{Bim}}$  is the image of the appropriate shift of  $\mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\{e,s\}} \mathfrak{t}^*)$ , endowed with the trivial structure of representation of  $\mathfrak{t}^* \times_{\mathfrak{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathfrak{t}^*/\mathbf{W}} \mathfrak{t}^*$ . If  $s \in \mathbf{S}_{\text{aff}}$  is not of this form, then there exist  $x \in \mathbf{W}_{\text{ext}}$  and  $t \in \mathbf{S}_{\text{aff}}$  such that  $t = s_\alpha$  for some  $\alpha \in \Phi^s$  and  $s = xtx^{-1}$ . (In fact, such a statement is even true in the braid group associated with  $\mathbf{W}_{\text{ext}}$ : see [R2, Lemma 6.1.2] or [BM, Lemma 2.1.1].) By Lemma 5.3 we then have  $B_s^{\text{Bim}} \cong \Delta_x \otimes_R B_t^{\text{Bim}} \otimes_R \Delta_{x^{-1}}$ ; since  $B_t^{\text{Bim}}$  is now known to belong to the essential image of our functor, and since  $\Delta_x$  also satisfies this property by Lemma 5.7, this finishes the proof.  $\square$

From this lemma we deduce the following claim.

**Theorem 5.10.** *There exists a canonical fully-faithful monoidal functor*

$$\mathbf{D}_{\text{BS}} \rightarrow \text{Rep}_{\mathfrak{h}}^{\mathbf{G}^{\text{m}}}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\mathbf{W}} \mathbf{J}_{\text{adj}}^* \times_{\mathfrak{t}^*/\mathbf{W}} \mathfrak{t}^*).$$

## 6. HECKE ACTION ON THE PRINCIPAL BLOCK

We come back to the setting and notation of Sections 2–3–4, assuming now that  $p > \max(h, 4)$ , where  $h$  is the Coxeter number of  $G$ . (In particular, this ensures that  $p$  is very good for  $G$ .)

**6.1. Categories of  $G$ -modules and  $G$ -equivariant  $\mathcal{U}\mathfrak{g}$ -modules.** We now take a closer look at the category  $\text{Rep}(G)$  of finite-dimensional algebraic  $G$ -modules, and review its decomposition into “blocks.” This will involve some of the constructions of Section 5 applied to the group  $\mathbf{G} = G^{(1)}$ , its Borel subgroup  $\mathbf{B} = B^{(1)}$ , and the maximal torus  $\mathbf{T} := T^{(1)}$ . Recall (see §4.2) that the morphism from  $\mathbf{X} = X^*(T^{(1)})$  to  $\mathbb{X}$  induced by the Frobenius morphism  $T \rightarrow T^{(1)}$  is injective, and that its image is  $p \cdot \mathbb{X}$ , which allows us to identify  $\mathbf{X}$  with  $p \cdot \mathbb{X}$ . The Frobenius morphism  $G \rightarrow G^{(1)}$  also induces an isomorphism

$$N_G(T)/T \xrightarrow{\sim} N_{G^{(1)}}(T^{(1)})/T^{(1)},$$

which allows us to identify the Weyl group  $\mathbf{W}$  of  $G^{(1)}$  with the Weyl group  $W$  of  $G$ . In this way, the identification  $\mathbf{X} = p \cdot \mathbb{X}$  is  $W$ -equivariant, and we have  $\Phi = \{p \cdot \alpha : \alpha \in \mathfrak{R}\}$ . In particular, the affine Weyl group  $W_{\text{aff}}$  of §2.1 identifies with the affine Weyl group  $\mathbf{W}_{\text{aff}}$  of §5.1. To make our notation more consistent, we will now denote by  $W_{\text{ext}}$  the extended affine Weyl group  $\mathbf{W}_{\text{ext}}$ , so that  $W_{\text{ext}} = W \ltimes (p \cdot \mathbb{X})$ . The dot-action of  $W_{\text{aff}}$  on  $\mathbb{X}$  considered in §2.1 extends in the obvious way to an action of  $W_{\text{ext}}$ , such that the map  $\mathbb{X} \rightarrow \mathfrak{t}^*$  sending a character to its differential is  $W_{\text{ext}}$ -equivariant, where  $W_{\text{ext}}$  acts on  $\mathbb{X}$  via the dot-action, and on  $\mathfrak{t}^*$  via the projection  $W_{\text{ext}} \rightarrow W$  and the action of  $W$  on  $\mathfrak{t}^*$  also denoted  $\bullet$  in §2.1.

Recall (see §2.6) that for any  $\lambda \in \mathbb{X}^+$  we have a simple  $G$ -module  $\mathbf{L}(\lambda)$  of highest weight  $\lambda$ , and that all simple  $G$ -modules are of this form. The *linkage principle* (see [J2, Corollary II.6.17]) states that for  $\lambda, \mu \in \mathbb{X}^+$  we have

$$\text{Ext}_{\text{Rep}(G)}^1(\mathbf{L}(\lambda), \mathbf{L}(\mu)) \neq 0 \quad \Rightarrow \quad W_{\text{aff}} \bullet \lambda = W_{\text{aff}} \bullet \mu.$$

As a consequence, if for a  $W_{\text{aff}}$ -orbit  $\mathbf{c} \subset \mathbb{X}$  we denote by  $\text{Rep}_{\mathbf{c}}(G)$  the Serre subcategory of  $\text{Rep}(G)$  generated by the simple objects  $\mathbf{L}(\lambda)$  with  $\lambda \in \mathbf{c} \cap \mathbb{X}$ , then we

have a direct sum decomposition

$$(6.1) \quad \text{Rep}(G) = \bigoplus_{\mathbf{c} \in \mathbb{X}/(W_{\text{aff}}, \bullet)} \text{Rep}_{\mathbf{c}}(G).$$

For  $\lambda \in \mathbb{X}$ , we will write  $[\lambda]$  for the  $W_{\text{aff}}$ -orbit of  $\lambda$ . We will also set

$$\text{Rep}_{[\lambda]}(G) = \bigoplus_{\substack{\mathbf{c} \in \mathbb{X}/(W_{\text{aff}}, \bullet) \\ \mathbf{c} \subset W_{\text{ext}} \bullet \lambda}} \text{Rep}_{\mathbf{c}}(G).$$

We will also denote by  $\text{Mod}_{\text{fg}}^G(\mathcal{U}\mathfrak{g})$  the category of  $G$ -equivariant finitely generated  $\mathcal{U}\mathfrak{g}$ -modules. For  $\xi \in \mathfrak{t}^*/(W, \bullet)$ , we will denote by

$$\text{Mod}_{\text{fg}}^{G, \xi}(\mathcal{U}\mathfrak{g})$$

the full subcategory of  $\text{Mod}_{\text{fg}}^G(\mathcal{U}\mathfrak{g})$  whose objects are the modules annihilated by a power of  $\mathfrak{m}^\xi$ . As for other similar notations, in case  $\xi = \tilde{\lambda}$  for some  $\lambda \in \mathbb{X}$ , we will write  $\text{Mod}_{\text{fg}}^{G, \lambda}(\mathcal{U}\mathfrak{g})$  for  $\text{Mod}_{\text{fg}}^{G, \tilde{\lambda}}(\mathcal{U}\mathfrak{g})$ . If we denote by  $\text{Mod}_{\text{fg}}^{G, \wedge}(\mathcal{U}\mathfrak{g})$  the category of  $G$ -equivariant finitely generated  $\mathcal{U}\mathfrak{g}$ -modules annihilated by a power of the ideal  $\mathcal{I}$  defined in §2.7, then as e.g. in (2.9) we have a canonical decomposition

$$\text{Mod}_{\text{fg}}^{G, \wedge}(\mathcal{U}\mathfrak{g}) \cong \bigoplus_{\lambda \in \Lambda} \text{Mod}_{\text{fg}}^{G, \lambda}(\mathcal{U}\mathfrak{g}),$$

where  $\Lambda \subset \mathbb{X}$  is as in (2.9).

There is a natural fully faithful functor

$$(6.2) \quad \text{Rep}(G) \rightarrow \text{Mod}_{\text{fg}}^G(\mathcal{U}\mathfrak{g})$$

sending a  $G$ -module  $V$  to itself, with its  $G$ -module structure, and with the  $\mathcal{U}\mathfrak{g}$ -module structure obtained by differentiating the  $G$ -action. The essential image of this functor consists of the finite-dimensional  $G$ -equivariant  $\mathcal{U}\mathfrak{g}$ -modules having the property that their  $\mathcal{U}\mathfrak{g}$ -module structure is obtained from their  $G$ -module structure by differentiation. Since, for any  $\lambda \in \mathbb{X}^+$ , the action of  $Z_{\text{HC}}$  on  $L(\lambda)$  factors through the quotient  $Z_{\text{HC}}/\mathfrak{m}^\lambda$ , the functor (6.2) restricts to a functor

$$\text{Rep}_{[\lambda]}(G) \rightarrow \text{Mod}_{\text{fg}}^{G, \lambda}(\mathcal{U}\mathfrak{g})$$

for any  $\lambda \in \mathbb{X}$ . Since  $\mathfrak{m}^\lambda$  only depends on the orbit  $W_{\text{ext}} \bullet \lambda$ , in this way we also obtain a fully faithful functor

$$\text{Rep}_{[\lambda]}(G) \rightarrow \text{Mod}_{\text{fg}}^{G, \lambda}(\mathcal{U}\mathfrak{g}).$$

**6.2. Action of completed bimodules.** There exists a canonical bifunctor

$$(-) \hat{\otimes}_{\mathcal{U}\mathfrak{g}} (-) : \text{Mod}_{\text{fg}}^G(\mathcal{U}^\wedge) \times \text{Mod}_{\text{fg}}^{G, \wedge}(\mathcal{U}\mathfrak{g}) \rightarrow \text{Mod}_{\text{fg}}^{G, \wedge}(\mathcal{U}\mathfrak{g})$$

which can be defined as follows. Consider some  $M$  in  $\text{Mod}_{\text{fg}}^G(\mathcal{U}^\wedge)$  and some  $V$  in  $\text{Mod}_{\text{fg}}^{G, \wedge}(\mathcal{U}\mathfrak{g})$ . By definition, there exists  $m \in \mathbb{Z}_{\geq 1}$  such that  $\mathcal{I}^m$  acts trivially on  $V$ . Then the tensor product

$$(M/\mathcal{I}^m \cdot M) \otimes_{\mathcal{U}\mathfrak{g}} V$$

is a finitely generated left  $\mathcal{U}\mathfrak{g}$ -module (where in the tensor product we consider the right  $\mathcal{U}\mathfrak{g}$ -action on  $M/\mathcal{I}^m \cdot M$ ), which does not depend on the choice of  $m$ , and which admits a natural (diagonal) structure of algebraic  $G$ -module. Moreover the action of  $\mathcal{I}$  on this module is nilpotent. We can therefore take this as the definition for  $M \hat{\otimes}_{\mathcal{U}\mathfrak{g}} V$ .

The bifunctor  $\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}$  defines on  $\text{Mod}_{\text{fg}}^{G,\wedge}(\mathcal{U}\mathfrak{g})$  a structure of module category for the monoidal category  $\text{Mod}_{\text{fg}}^G(\mathcal{U}^\wedge)$ . It is also easily seen that for  $\lambda, \mu \in \mathbb{X}$  this bifunctor restricts to a bifunctor

$$\text{Mod}_{\text{fg}}^G(\mathcal{U}^{\widehat{\lambda}, \widehat{\mu}}) \times \text{Mod}_{\text{fg}}^{G, \mu}(\mathcal{U}\mathfrak{g}) \rightarrow \text{Mod}_{\text{fg}}^{G, \lambda}(\mathcal{U}\mathfrak{g}),$$

which itself restricts to a bifunctor

$$\text{HC}^{\widehat{\lambda}, \widehat{\mu}} \times \text{Rep}_{\langle \mu \rangle}(G) \rightarrow \text{Rep}_{\langle \lambda \rangle}(G).$$

**6.3. Relation with translation functors.** Recall the definition of the translation functors for  $G$ -modules. Fix  $\lambda, \mu \in \mathbb{X}$ , and denote by  $\nu$  the only element in  $\mathbb{X}^+ \cap \{w(\lambda - \mu) : w \in W\}$ . Then the translation functor

$$T_\mu^\lambda : \text{Rep}_{[\mu]}(G) \rightarrow \text{Rep}_{[\lambda]}(G)$$

is the functor sending an object  $V$  to the direct summand of  $L(\nu) \otimes V$  which belongs to  $\text{Rep}_{[\lambda]}(G)$  in the decomposition provided by (6.1). We will consider these functors only in case  $\lambda$  and  $\mu$  both belong to the closure of the fundamental alcove. In this setting, we have defined in §2.6 an object  $\mathbb{P}^{\lambda, \mu} \in \text{HC}_{\text{diag}}^{\widehat{\lambda}, \widehat{\mu}}$ .

**Lemma 6.1.** *Let  $\lambda, \mu \in \mathbb{X}$  belonging to the closure of the fundamental alcove. The composition*

$$\text{Rep}_{[\mu]}(G) \rightarrow \text{Rep}_{\langle \mu \rangle}(G) \xrightarrow{\mathbb{P}^{\lambda, \mu} \widehat{\otimes}_{\mathcal{U}\mathfrak{g}} (-)} \text{Rep}_{\langle \lambda \rangle}(G)$$

*is canonically isomorphic to the composition*

$$\text{Rep}_{[\mu]}(G) \xrightarrow{T_\mu^\lambda} \text{Rep}_{[\lambda]}(G) \rightarrow \text{Rep}_{\langle \lambda \rangle}(G).$$

*Proof.* By definition, the first functor sends a module  $V$  in  $\text{Rep}_{[\mu]}(G)$  to the quotient

$$(L(\nu) \otimes V) / (\mathfrak{m}^\lambda)^n \cdot (L(\nu) \otimes V)$$

for  $n \gg 0$ , i.e. to the direct sum of the factors in  $L(\nu) \otimes V$  corresponding to orbits included in  $W_{\text{ext}} \bullet \lambda$  in the decomposition provided by (6.1). However, all the  $T$ -weights in  $L(\nu) \otimes V$  belong to  $\lambda + \mathbb{Z} \cdot \mathfrak{R}$ . In view of Lemma 2.1(1), this implies that  $[\lambda]$  is the only  $W_{\text{aff}}$ -orbit contained in  $W_{\text{ext}} \bullet \lambda$  that can contribute to the direct sum above.  $\square$

*Remark 6.2.* See [R2, Lemma 4.3.1] for a different proof of this claim, under more restrictive assumptions which would be sufficient for our present purposes.

**6.4. Main result.** We now consider the category  $\text{D}_{\text{BS}}$  of §5.1 associated with the group  $\mathbf{G} = G^{(1)}$ . We also fix a weight  $\lambda$  in the fundamental alcove. For any  $s \in S_{\text{aff}}$ , we choose a weight  $\mu_s \in \mathbb{X}$  in the closure of the fundamental alcove, which lies on the wall associated with  $s$  but on no other wall. (For the existence of such a wall, see [J2, §II.6.3])

The main result of the present section (and of this paper) is the following.

**Theorem 6.3.** *There exists a monoidal functor*

$$\text{D}_{\text{BS}} \rightarrow \text{HC}^{\widehat{\lambda}, \widehat{\lambda}}$$

*sending  $B_s$  to  $\mathbb{P}^{\lambda, \mu_s} \widehat{\otimes}_{\mathcal{U}\mathfrak{g}} \mathbb{P}^{\mu_s, \lambda}$ , for any  $s \in S_{\text{aff}}$ .*

The category  $\mathcal{D}_{\text{BS}}$  admits a canonical autoequivalence  $\iota$  which satisfies  $\iota(X \cdot Y) = \iota(Y) \cdot \iota(X)$  for any  $X, Y \in \mathcal{D}_{\text{BS}}$ , see e.g. [RW1, §4.2]. In view of Lemma 6.1, Theorem 6.3 therefore implies that there exists a right action of the monoidal category  $\mathcal{D}_{\text{BS}}$  on  $\text{Rep}_{[\lambda]}(G)$ , where  $B_s$  acts by the wall-crossing functor  $T_{\mu_s}^\lambda T_\lambda^{\mu_s}$  for any  $s \in S_{\text{aff}}$ . This confirms [RW1, Conjecture 5.1.1], or rather the slightly less precise statement considered in [RW1, Remark 5.1.2(3)]. (As explained in *loc. cit.*, this version is however sufficient to deduce all the applications considered in [RW1, Part I], and in particular the character formula for tilting modules in  $\text{Rep}_{[\lambda]}(G)$ .)

**6.5. Proof of Theorem 6.3.** Recall that by construction we have a canonical morphism  $\mathbb{I}_{\mathcal{S}}^* \rightarrow (\mathbb{J}_{\mathcal{S}}^*)^{(1)}$  of group schemes over  $\mathcal{S}^{*(1)}$ , see §2.9. Now if the Kostant section  $\mathbf{S} \subset \mathfrak{g} = \mathfrak{g}^{(1)}$  of §5.3 is chosen as  $\mathcal{S}^{(1)}$ , and if the isomorphism  $\kappa$  of §5.1 is chosen as  $\varkappa^{(1)}$ , then the group scheme  $(\mathbb{J}_{\mathcal{S}}^*)^{(1)}$  identifies with the group scheme  $\mathbf{J}_{\mathcal{S}}^*$  of §5.3. In this way, Theorem 5.10 provides a monoidal functor

$$\mathcal{D}_{\text{BS}} \rightarrow \text{Rep}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} (\mathbb{J}_{\mathcal{S}}^*)^{(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)}).$$

Pulling back under the morphism  $\mathbb{I}_{\mathcal{S}}^* \rightarrow (\mathbb{J}_{\mathcal{S}}^*)^{(1)}$  we deduce a monoidal functor

$$(6.3) \quad \mathcal{D}_{\text{BS}} \rightarrow \text{Rep}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}_{\mathcal{S}}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)}),$$

where the monoidal product in the right-hand side is defined by the same recipe as for  $\text{Rep}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} (\mathbb{J}_{\mathcal{S}}^*)^{(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})$ . For consistence of notation, in this section we will denote by  $(\Delta_w^{\mathbb{J}} : w \in W_{\text{ext}})$  the objects considered in Remark 5.8; we will also denote by  $(\Delta_w^{\mathbb{I}} : w \in W_{\text{ext}})$  their images in  $\text{Rep}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}_{\mathcal{S}}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})$ .

Now we observe that since  $\lambda$  belongs to the fundamental alcove, it is regular, so that the quotient morphism

$$\mathfrak{t}^* \rightarrow \mathfrak{t}^*/(W, \bullet)$$

is étale at  $\bar{\lambda}$ , see Lemma 2.2. Similarly, the Artin–Schreier map

$$\mathfrak{t}^* \rightarrow \mathfrak{t}^{*(1)}$$

is étale (everywhere, hence in particular at  $\bar{\lambda}$ ), and sends  $\bar{\lambda}$  to 0. Using these maps we obtain morphisms

$$\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)} \leftarrow \mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^* \rightarrow \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)$$

étale at  $(\bar{\lambda}, \bar{\lambda})$ , which identify the algebra  $\mathcal{Z}_{\mathcal{S}}^{\hat{\lambda}, \hat{\lambda}}$  with the completion  $\mathcal{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})^{\hat{0}, \hat{0}}$  of  $\mathcal{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})$  with respect to the maximal ideal of  $(0, 0)$ . Using this identification, the functor of completion with respect to this maximal ideal defines a monoidal functor

$$(6.4) \quad \text{Rep}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}_{\mathcal{S}}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)}) \rightarrow \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathcal{S}}^{\hat{\lambda}, \hat{\lambda}}).$$

Precomposing this functor with (6.3), and then composing with the equivalence of Corollary 3.7 we obtain a monoidal functor

$$(6.5) \quad \mathcal{D}_{\text{BS}} \rightarrow \text{Mod}_{\text{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathcal{S}}^{\hat{\lambda}, \hat{\lambda}}).$$

**Proposition 6.4.** *For any  $s \in S_{\text{aff}} \cap W$ , the functor (6.5) sends the object  $B_s$  to  $\mathbb{P}_{\mathcal{S}}^{\lambda, \mu_s} \hat{\otimes}_{\mathcal{U}_{\mathcal{S}}^{\hat{\lambda}, \hat{\lambda}}} \mathbb{P}_{\mathcal{S}}^{\mu_s, \lambda}$ .*

*Proof.* In the course of the proof of Lemma 5.9 we have seen that the image of  $B_s$  in  $\text{Rep}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} (\mathbb{J}_S^*)^{(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})$  is  $\mathcal{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/\{e,s\}} \mathfrak{t}^{*(1)})$ , endowed with the trivial module structure. It follows that the same is true for its image under (6.3). On the other hand, by Proposition 4.8 the wall-crossing bimodule  $\mathbb{P}_S^{\lambda, \mu_s} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{P}_S^{\mu_s, \lambda}$  corresponds to the object  $\mathcal{Z}_S^{\hat{\lambda}, \hat{\lambda}} \otimes_{\mathcal{Z}_S^{\hat{\mu}, \hat{\lambda}}} \mathcal{Z}_S^{\hat{\lambda}}$ . The considerations in the proof of Lemma 4.9 show that the algebra  $\mathcal{Z}_S^{\hat{\mu}, \hat{\lambda}}$  identifies with the completion of  $\mathcal{O}(\mathfrak{t}^*/(\{e, s\}, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*)$  at the ideal corresponding to  $(\bar{\mu}, \bar{\lambda})$ , and that via this identification the morphism  $\mathcal{Z}_S^{\hat{\mu}, \hat{\lambda}} \rightarrow \mathcal{Z}_S^{\hat{\lambda}, \hat{\lambda}}$  is induced by the natural morphism

$$\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^* \rightarrow \mathfrak{t}^*/(\{e, s\}, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$$

sending  $(\bar{\lambda}, \bar{\lambda})$  to the image of  $(\bar{\mu}, \bar{\lambda})$ . This morphism fits in a natural commutative diagram

$$\begin{array}{ccccc} \mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)} & \longleftarrow & \mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^* & \longrightarrow & \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet) \\ \downarrow & & \downarrow & & \parallel \\ \mathfrak{t}^{*(1)}/\{e, s\} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)} & \longleftarrow & \mathfrak{t}^*/(\{e, s\}, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^* & \longrightarrow & \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet). \end{array}$$

Here the morphisms on the upper row are étale at  $(\bar{\lambda}, \bar{\lambda})$ , and those on the lower row are étale at the image of  $(\bar{\mu}, \bar{\lambda})$ . (In fact, for the right-hand side this has already been observed in the course of the proof of Lemma 4.9, and for the left-hand side the proof is similar to that of Lemma 3.6.) This observation shows that  $\mathcal{Z}_S^{\hat{\lambda}, \hat{\lambda}} \otimes_{\mathcal{Z}_S^{\hat{\mu}, \hat{\lambda}}} \mathcal{Z}_S^{\hat{\lambda}}$  identifies with the  $\mathcal{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})^{\hat{0}, \hat{0}}$ -module

$$\mathcal{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})^{\hat{0}, \hat{0}} \otimes_{\mathcal{O}(\mathfrak{t}^{*(1)}/\{e, s\} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})^{\hat{0}, \hat{0}}} \mathcal{O}(\mathfrak{t}^{*(1)})^{\hat{0}},$$

where  $\mathcal{O}(\mathfrak{t}^{*(1)}/\{e, s\} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})^{\hat{0}, \hat{0}}$  is the completion of  $\mathcal{O}(\mathfrak{t}^{*(1)}/\{e, s\} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})$  at the ideal corresponding to the image of  $(0, 0)$ , and  $\mathcal{O}(\mathfrak{t}^{*(1)})^{\hat{0}}$  is the completion of  $\mathcal{O}(\mathfrak{t}^{*(1)})$  (seen as an  $\mathcal{O}(\mathfrak{t}^{*(1)}/\{e, s\} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})$ -module in the natural way) at the ideal corresponding to 0. Using the same considerations as in the proof of Lemma 4.9, it is easily seen that this module identifies with the completion of  $\mathcal{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/\{e, s\}} \mathfrak{t}^{*(1)})$ , which finishes the proof of our claim.  $\square$

**Lemma 6.5.** *For any  $w \in W_{\text{ext}}$ , the object  $\mathcal{L}_{\lambda, \lambda}^{-1}(\mathbb{B}_{\lambda, w \bullet \lambda})$  is the image of  $\Delta_w^{\mathbb{I}}$  under the functor (6.4).*

*Proof.* Write  $w = t_\nu x$  with  $\nu \in p\mathbb{X}$  and  $x \in W$ . Then by Lemma 4.3 we have

$$\mathbb{B}_{\lambda, w \bullet \lambda} = \mathbb{B}_{\lambda, x \bullet \lambda + p\nu} \cong \mathbb{B}_{\lambda, x \bullet \lambda} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{x \bullet \lambda, x \bullet \lambda + p\nu}.$$

It is easily seen from Lemma 4.4 that  $\mathcal{L}_{\lambda, \lambda}^{-1}(\mathbb{B}_{x \bullet \lambda, x \bullet \lambda + p\nu})$  is the image of  $\Delta_{t_\nu}^{\mathbb{I}}$ . In view of (5.3), to conclude it therefore suffices to prove that  $\mathcal{L}_{\lambda, \lambda}^{-1}(\mathbb{B}_{\lambda, x \bullet \lambda})$  is the image of  $\Delta_x^{\mathbb{I}}$ . In turn, if  $x = s_1 \cdots s_r$  is a reduced expression (with each  $s_i$  in  $W \cap S_{\text{aff}}$ ) then we have

$$\mathbb{B}_{\lambda, x \bullet \lambda} \cong \mathbb{B}_{\lambda, s_1 \bullet \lambda} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{s_1 \bullet \lambda, s_1 s_2 \bullet \lambda} \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \cdots \hat{\otimes}_{\mathcal{U}_{S\mathfrak{g}}} \mathbb{B}_{(s_1 \cdots s_{r-1}) \bullet \lambda, x \bullet \lambda}.$$

If we write  $y_j = s_1 \cdots s_j$  for any  $j$ , then by monoidality of  $\mathcal{L}_{\lambda, \lambda}$  (see Proposition 4.10), to conclude it suffices to prove that  $\mathcal{L}_{\lambda, \lambda}^{-1}(\mathbb{B}_{y_{i-1} \bullet \lambda, y_i \bullet \lambda})$  is the image of  $\Delta_{s_i}^{\mathbb{I}}$  for any  $i$ .



Here we have  $y_i \bullet \lambda < y_{i-1} \bullet \lambda$ . By Lemma 4.7 we therefore have an exact sequence

$$\mathbb{B}_{y_i \bullet \lambda, y_{i-1} \bullet \lambda} \hookrightarrow \mathbb{P}_S^{\lambda, \mu_{s_i}} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu_{s_i}, \lambda} \twoheadrightarrow \mathbb{B}_{y_{i-1} \bullet \lambda, y_{i-1} \bullet \lambda}.$$

As seen in the course of the proof of Proposition 6.4,  $\mathcal{L}_{\lambda, \lambda}^{-1}(\mathbb{P}_S^{\lambda, \mu_{s_i}} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu_{s_i}, \lambda})$  is the completion of  $\mathcal{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/\{e, s_i\}} \mathfrak{t}^{*(1)})$  (with the trivial action of the group scheme  $\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}_S^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)}$ ), and it is clear that  $\mathcal{L}_{\lambda, \lambda}^{-1}(\mathbb{B}_{y_{i-1} \bullet \lambda, y_{i-1} \bullet \lambda})$  is the completion of  $\mathcal{O}(\mathfrak{t}^{*(1)})$ . The object  $\mathcal{L}_{\lambda, \lambda}^{-1}(\mathbb{B}_{y_i \bullet \lambda, y_{i-1} \bullet \lambda})$  is therefore the kernel of a surjection from the former completion to the latter completion. However, up to an automorphism of the completion of  $\mathcal{O}(\mathfrak{t}^{*(1)})$  there exists only one such surjection, and its kernel is isomorphic to the completion of  $\Delta_{s_i}^{\mathbb{I}}$  by Lemma 5.4.  $\square$

We can finally complete the proof of Theorem 6.3.

*Proof of Theorem 6.3.* Consider the functor (6.5). For any  $s \in S_{\text{aff}}$ , this functor sends  $B_s$  to  $\mathbb{P}_S^{\lambda, \mu_s} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu_s, \lambda}$ . In fact, if  $s \in W$  this claim is the content of Proposition 6.4. Otherwise, as already seen in the course of the proof of Lemma 5.9, there exist  $x \in W_{\text{ext}}$  and  $t \in S_{\text{aff}} \cap W$  such that  $s = xtx^{-1}$ . By Lemma 5.3 and Lemma 6.5, the image of  $B_s$  is then

$$\mathbb{B}_{\lambda, w \bullet \lambda} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \left( \mathbb{P}_S^{\lambda, \mu_t} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu_t, \lambda} \right) \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{B}_{w \bullet \lambda, \lambda},$$

which by Proposition 4.14 is isomorphic to  $\mathbb{P}_S^{\lambda, \mu_s} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu_s, \lambda}$ .

Since each object of  $D_{\text{BS}}$  is isomorphic to a shift of a product of objects  $B_s$ , and since both of the involved functors are monoidal, our claim implies that the functor (6.5) takes values in the essential image of the fully faithful functor of Proposition 2.6. Composing with the inverse of the latter functor, we therefore obtain a monoidal functor

$$D_{\text{BS}} \rightarrow \text{Mod}_{\text{fg}}^G(\mathcal{U}^{\widehat{\lambda}, \widehat{\lambda}})$$

sending  $B_s$  to  $\mathbb{P}_S^{\lambda, \mu_s} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu_s, \lambda}$  for any  $s \in S_{\text{aff}}$ . Since each object  $\mathbb{P}_S^{\lambda, \mu_s} \widehat{\otimes}_{\mathcal{U}_S \mathfrak{g}} \mathbb{P}_S^{\mu_s, \lambda}$  belongs to  $\text{HC}^{\widehat{\lambda}, \widehat{\lambda}}$ , and since this subcategory is monoidal, our functor factors through a functor  $D_{\text{BS}} \rightarrow \text{HC}^{\widehat{\lambda}, \widehat{\lambda}}$ , which finishes the proof.  $\square$

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