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Abstract. We consider Mean Field Games without idiosyncratic but with Brownian type common noise. We introduce a notion of solutions of the associated backward-forward system of stochastic partial differential equations. We show that the solution exists and is unique for monotone coupling functions. This the first general result for solutions of the Mean Field Games system with common and no idiosynctratic noise. We also use the solution to find approximate optimal strategies (Nash equilibria) for N-player differential games with common but no idiosyncratic noise. An important step in the analysis is the study of the well-posedness of a stochastic backward Hamilton-Jacobi equation.

1. Introduction

We consider Mean Field Games (MFG for short) without idiosyncratic but with Brownian-type common noise described by the backward-forward system of stochastic partial differential equations (SPDEs for short)

$$\begin{cases} d_t u_t = \big[-\beta\Delta u_t + H(Du_t,x) - F(x,m_t) - \sqrt{2\beta}\mathrm{div}(v_t)\big]dt \\ \qquad\qquad\qquad\qquad\qquad\qquad + v_t\cdot\sqrt{2\beta}dW_t \ \text{ in } \ \mathbb{R}^d\times(0,T), \\ d_t m_t = \big[\beta\Delta m_t + \mathrm{div}\big(m_t D_p H(Du_t,x)\big)\big]dt - \mathrm{div}(m_t\sqrt{2\beta}dW_t) \ \text{ in } \ \mathbb{R}^d\times(0,T), \\ u_T(x) = G(x,m_T) \quad m_0 = \overline{m}_0 \ \text{ in } \ \mathbb{R}^d. \end{cases} \tag{1.1}$$

We introduce a notion of solution of (1.1), which is adapted to the common noise W, and prove existence and uniqueness when the couplings F and G are nonlocal and satisfy the well-known Lasry-Lions monotonicity condition introduced in [30]. Exact assumptions are stated later. To the best of our knowledge, this is the first general result for solutions of the MFG-system with common and no idiosynctratic noise. We also use the solution to derive approximate Nash equilibria for N-player differential games with common but no idiosyncratic noise assuming a structure condition on H.

An important step in our analysis is the study of the well-posedness of strong, that is, a.e. in space-time and semiconcave in space, solutions of the backward stochastic Hamilton-Jacobi (HJ for short) equation

$$\begin{cases} d_t u_t = \big[-\beta\Delta u_t + H_t(Du_t,x) - \sqrt{2\beta}\mathrm{div}(v_t)\big]dt + v_t\cdot\sqrt{2\beta}dW_t \ \text{ in } \ \mathbb{R}^d\times(0,T), \\ u_T = G \ \text{ in } \ \mathbb{R}^d, \end{cases} \tag{1.2}$$

with H uniformly convex; again exact assumptions are stated later.

MFG with common noise describe optimal control problems with infinitely many small and interacting controllers whose dynamics are subjected to common noise. Such models appear

often in macroeconomics under the name "heterogenous agent models"; see, for instance, the work of Krusell and Smith [24].

The mathematical description of MFG with common noise, which was introduced by Lions [31] and then discussed, at an informal level, in Bensoussan, Frehse and Yam [5] and Carmona and Delarue [13], is either probabilistic or analytic.

The probabilistic formulation takes the form of an optimal stochastic control problem involving a random distribution of the agents, which is the conditional law, given the common noise, of the optimal trajectory of the agents. In this setting, the existence of generalized solutions, that is, solutions adapted to a larger filtration than the one generated by the common noise, has been established by Carmona, Delarue and Lacker [15] under very general assumptions; see also Ahuja [1] and Lacker and Webster [27]. The former reference also establishes the existence and uniqueness of a strong solution, which is adapted to the filtration of the common noise, under the Lasry-Lions monotonicity condition and an assumption on the uniqueness of the optimal solution (with relaxed controls) of the stochastic control problem. Although the probabilistic formulation allows for general existence and uniqueness results, it requires the resolution of an optimal stochastic control problem, which is not convenient for the numerical approximation or the explicit analysis. Moreover, the condition on the uniqueness of the optimal solution, which is quite demanding, has been shown to be satisfied only under the strong assumption that either there is a nondegenerate idiosyncratic noise, or that the value function is convex in space. The latter is known to hold only for dynamics which are linear in space and for cost functions which are strictly convex in space and control.

The analytic approach to study MFG problems with common noise involves the value function and the partial differential equation (PDE for short) it satisfies. There are two different but also very related formulations involving either stochastic MFG systems or the so-called master equation.

The former describes the problem as a coupled system of SPDEs known as the backward-forward stochastic MFG system. For problems without common noise, the system was introduced and studied by Lasry and Lions in [28, 29]. In the presence of both idiosyncratic and common noises the stochastic MFG system was first investigated by Cardaliaguet, Delarue, Lasry and Lions in [10].

The second analytic approach, which was introduced by Lasry and Lions and presented by Lions in [31], is based on the master equation, which is a deterministic infinite dimensional PDE set in the space of measures. The existence and uniqueness of solutions of the general infinite dimensional version of the master equation with idiosyncratic and common noise was shown in [10]; see also [14] for a generalization. Among other recent references about the master equation we point out the works of Cardaliaguet, Cirant and Porretta [9] who proposed a splitting method, Lions [31] who introduced the Hilbertian approach, in which the master equation is embedded in the space of square integrable random variables, and, finally, Bayraktar, Cecchin, Cohen and Delarue [4] and Bertucci, Lasry and Lions [6, 7] who investigated the existence and uniqueness for problems with common noise in finite state spaces.

Here we study the stochastic MFG system (1.1) which consists of a backward stochastic HJ-equation coupled with a forward stochastic Kolmogorov-Focker-Plank (KFP for short) PDE. In (1.1), the Brownian motion W is the common noise and the unknown is the triplet (u, m, v), consisting of the value function u of a small agent, which solves the backward HJ

SPDE with Hamiltonian of the form $H(Du_t, x) - F(x, m_t)$, an auxiliary function v which ensures that u is adapted to the filtration generated by the noise W, and the density m of the players which solves the forward stochastic KFP SPDE. The two equations are coupled through F and G, which depend on m in a nonlocal way.

The main difference with previous works and, in particular, [10] is that, due to the absence of idiosyncratic noise, the solution of (1.1) is not expected to be "smooth" since the HJ and KFP equations are only degenerate parabolic.

To explain this problem, we first consider the HJ equation separately, that is, we look at (1.2), which is a backward SPDE (BSPDE for short) associated with an optimal control problem with random coefficients. It follows from the work of Peng [32] that (1.2) has a unique solution provided that the noise satisfies a nondegeneracy assumption, which, roughly speaking, means that the β in front of Δu_t is greater than the β in front of the terms involving v. More recently, (1.2) was studied by Qiu [33] and Qiu and Wei [34], who introduced a notion of viscosity solution involving derivatives on the path space and proved its existence and uniqueness. The equations studied in the last references are more general than (1.2), in particular, the volatility is not constant, and require few conditions on the Hamiltonian other than the standard growth and regularity.

The study of MFG with common noise necessitates the use of a completely different approach, since the continuity equation for m involves the derivative of u in space and not a weak derivative in the path space.

To motivate the new approach we are putting forward here, we recall what happens for MFG problems without noise at all, that is, when $\beta = 0$. In this deterministic case, where one can take $v \equiv 0$, the natural concept of solution for (1.1) requires u to be Lipschitz continuous and to satisfy the HJ equation in the viscosity sense, while m has to be bounded and to satisfy the KFP equation in the sense of distributions; see [30] and Cardaliaguet and Hadikhanloo [11] for details. We note that, since m is absolutely continuous and bounded, the term $mD_pH(Du_t, x)$ is well-defined. However, it is also known that the boundedness condition on m can hold on large time intervals only if $H = H(p, x)$ is convex in p; see, for example, Golse and James [21] where, to study a forward-forward system with nonconvex H, it is necessary to consider a much more degenerate notion of solution. When H is convex in p, the solution of the HJ equation is naturally semiconcave. Then the notion of viscosity solution of HJ is equivalent to the one of the semiconcave a.e. solution studied by Kruzhkov [25]; see also Douglis [17], Evans [18], and Fleming [20].

The first contribution of the paper is to show that the notion of semiconcave a.e. solution can be adapted to the BSPDE (1.2) when reinterpreted in a suitable way. The starting point is the change of variable

$$\widetilde{u}_t(x) = u_t(x + \sqrt{2\beta}W_t),$$

which, using the Itô-Wentzell formula, leads, at least formally, to

$$d_t\widetilde{u}_t = \widetilde{H}_t(D\widetilde{u}_t, x)dt + d\widetilde{M}_t \ \text{ in } \ \mathbb{R}^d \times (0,T) \qquad \widetilde{u}_T = \widetilde{G} \ \text{ in } \ \mathbb{R}^d, \tag{1.3}$$

with

$$\widetilde{H}_t(p,x) = H_t(p, x + \sqrt{2\beta}W_t) \quad \text{and} \quad \widetilde{G}(x) = G(x + \sqrt{2\beta}W_T). \tag{1.4}$$

The problem then becomes to find a pair $(\widetilde{u}_t, \widetilde{M}_t)_{t\in[0,T]}$ adapted to the filtration of $(W_t)_{t\in[0,T]}$, where $\widetilde{M} = \widetilde{M}_t(x)$ is a globally bounded martingale, $\widetilde{u}$ is continuous and semiconcave in the space variable, and (1.3) is satisfied in an integrated form.

Theorem 2.3 and Proposition 2.5 establish respectively that (1.3) has a solution and a comparison principle is satisfied. In Proposition 2.7 we also provide a stochastic control representation of the solution. The optimality conditions and the existence and the uniqueness of optimal trajectories are respectively the topics of Theorem 2.8 and Proposition 2.13. However, this last point, which relies on the analysis of a continuity equation associated with the drift $-D_p\widetilde{H}(D\widetilde{u}_t(x),x)$ (see Proposition 2.11 and Proposition 2.12) requires a much stronger structure condition on the Hamiltonian, which we discuss later.

Next we apply this approach to (1.1). After the formal change of variables

$$\widetilde{u}_t(x) = u_t(x+\sqrt{2\beta}W_t,x) \ \text{ and } \ \widetilde{m}_t = (id-\sqrt{2\beta}W_t)\sharp m_t, \tag{1.5}$$

we obtain the new system

$$\begin{cases} d_t\widetilde{u}_t = \left[\widetilde{H}_t(D\widetilde{u}_t(x),x) - \widetilde{F}_t(x,\widetilde{m}_t)\right]dt + d\widetilde{M}_t \ \text{ in } \ \mathbb{R}^d\times(0,T), \\ \partial_t\widetilde{m}_t = \operatorname{div}(\widetilde{m}_t D_p\widetilde{H}(D\widetilde{u}_t(x),x))dt \ \text{ in } \ \mathbb{R}^d\times(0,T), \\ \widetilde{m}_0 = \overline{m}_0 \qquad \widetilde{u}_T = \widetilde{G}(\cdot,\widetilde{m}_T) \ \text{ in } \ \mathbb{R}^d \end{cases} \tag{1.6}$$

with $\widetilde{H}$ as in (1.4), and

$$\widetilde{F}_t(x,m) = F(x+\sqrt{2\beta}W_t,(id+\sqrt{2\beta}W_t)\sharp m) \ \text{ and } \widetilde{G}(x) = G(x-\sqrt{2\beta}W_T,(id+\sqrt{2\beta}W_T)\sharp m_T).$$

This transformation was used in [10] to prove the existence of a strong solution of the stochastic MFG system with common and idiosyncratic noises, in which case (1.6) is non degenerate and has a "smooth" solution.

Coming back to the degenerate system (1.6), the problem is to find a solution $(\widetilde{u}_t,\widetilde{M}_t,\widetilde{m}_t)_{t\in[0,T]}$ which is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by W and $\widetilde{M} = \widetilde{M}_t(x)$ is a $(\mathcal{F}_t)_{t\in[0,T]}$ martingale. By a solution, we mean that $\widetilde{u}$ is space-time continuous and semiconcave in space, while the martingale $\widetilde{M}$ is bounded, the HJ equation, once integrated in time, is satisfied $\mathbb{P}-$a.s. and a.e., and the random measure $\widetilde{m}$ has globally bounded density satisfying the continuity equation in the sense of distribution $\mathbb{P}-$a.s..

Our main result, Theorem 3.3, is that, under suitable structure and regularity assumptions on the data and assuming that F and G are strongly monotone in the Lasry-Lions sense, (1.6) has a unique solution $(\widetilde{u},\widetilde{M},\widetilde{m})$. This is the first existence and uniqueness result of a strong solution for MFG problems with a common and without idiosyncratic noise with value function that is neither smooth nor convex in space in contrast with [10, 15].

The final result concerning MFG is Proposition 3.9. It asserts that it is possible to use the solution of the stochastic MFG system (1.1) to derive approximate Nash equilibria in $N-$player differential games with a common noise. Such a statement is standard in the MFG literature. The first results in this direction go back to Huang, Caines and Malhamé [22], [23] for linear and nonlinear dynamics respectively. In both these references, the dynamics and payoff depend on the empirical measure through an average. Hence, the Central Limit Theorem implies that the error term is of order $N^{-1/2}$. The result for a genuinely non linear version of MFG problems without common noise was obtained by Carmona and Delarue [12]; see also [13], Section 6 in Vol. II. Since then, there have been many variations and extensions, and we we refer to [13] and the references therein. As far as we know, Proposition 3.9 is the first result for MFG problems with common and without idiosyncratic noises. The main reason for proving it is that it justifies the (somewhat formal) change of variables made

to pass from the original MFG system (1.1) to the transformed one (1.6). Indeed, we use (1.6) to solve a problem which should actually involve the solution of (1.1).

In contrast with the analysis of the MFG system, to find the approximate Nash equilibria we need to consider a special class the Hamiltonian $\widetilde{H}$. Indeed, we assume that, for some smooth and strictly positive coefficient $\widetilde{a}$ and a smooth and bounded vector field $\widetilde{b}$, $\widetilde{H}$ is of the form

$$\widetilde{H}_t(p,x) = \frac{1}{2}\widetilde{a}_t(x)|p|^2 + \widetilde{b}_t(x)\cdot p. \tag{1.7}$$

We suspect that (1.7) may not be necessary. The reason we have to require it is to obtain the uniqueness, for fixed ω and given the bounded variation vector field $-D_pH(D\widetilde{u}_t(x),x)$, of the solution of the continuity equation in (1.6).

The uniqueness of solutions of linear transport and continuity equations under weak assumptions on the vector field is a very intriguing problem. Its study goes back to DiPerna and Lions [16] and Ambrosio [2]. These results cannot be applied to the case at hand, since they require regularity which is not satisfied by $-D_pH(D\widetilde{u}_t(x),x)$. Instead, here we rely on a result of Bouchut, James and Mancini [8] which requires a half-Lipschitz condition on the vector field. To use it, however, here we need to assume (1.7).

Organization of the paper. The paper is organized in two parts. In the first, we study the backward HJ SPDE (1.2). We state the assumptions in subsection 2.1, show the existence of a solution in subsection 2.2, prove its uniqueness by a comparison principle in subsection 2.3, propose an optimal control representation and discuss a maximum principle in subsection 2.4. In order to prove the existence of optimal solutions in subsection 2.6, we first discuss in subsection 2.5 the continuity equation associated for the optimal drift. The second part is devoted to the stochastic MFG system (1.1). We state the assumptions in subsection 3.1, and the main existence and uniqueness result in subsection 3.2. In subsection 3.3 we recall the case without noise, for which we provide sharp estimates. We then construct approximate solutions of the stochastic MFG system in subsection 3.4 and, finally, pass to the limit to prove the main result in subsection 3.5. In subsection 3.6 we show the existence of the approximate Nash equilibria for the N-player game. Finally, in the appendix we revisit the result of [8] on the uniqueness of the solution to some continuity equations.

Notation. Throughout the paper $\mathcal{O}$ is an open subset of $\mathbb{R}^d$, and $C^2(\mathcal{O})$ is the space of C^2-maps on $\mathcal{O}$ with bounded derivatives endowed with the sup-norm

$$\|u\|_{C^2(\mathcal{O})} = \|u\|_{L^\infty(\mathcal{O})} + \|Du\|_{L^\infty(\mathcal{O})} + \|D^2u\|_{L^\infty(\mathcal{O})};$$

depending on the context, we often omit the subscript $L^\infty(\mathcal{O})$ and simply write $\|\cdot\|_\infty$. We work on a complete filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ carrying an d-dimensional Wiener process $W=(W_t)_{t\in[0,T]}$ such that $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by W augmented by all the $\mathbb{P}-$null sets in $\mathcal{F}$. We denote by $\mathcal{P}$ the $\sigma-$algebra of the predictable sets on $\Omega\times[0,T]$ associated with $(\mathcal{F}_t)_{t\geq 0}$. Given a complete metric space E and $p\geq 1$, $\mathcal{S}^p(E)$ is the space of continuous, E-valued, $\mathcal{P}-$measurable processes $X=(X_t)_{t\geq 0}$ such that, for some $x_0\in E$ and, therefore, any point in E,

$$\mathbb{E}[\sup_{t\in[0,T]} d_E(X_t,x_0)^p dt] < +\infty.$$

We set $\mathcal{S}^r(C^2_{loc}(\mathbb{R}^d)) = \bigcap_{n\geq 1}\mathcal{S}^r(C^2(B_n))$, where B_n is the open ball centered at 0 and of radius n, and define similarly $\mathcal{S}^r(L^1_{loc}(\mathbb{R}^d))$ and $\mathcal{S}^r(W^{1,1}_{loc}(\mathbb{R}^d))$. We write $L^\infty((\Omega\times\mathcal{F}_T);C^2(\mathbb{R}^d))$ for the the space of bounded $C^2(\mathbb{R}^d)$-valued and $\mathcal{F}_T$ maps on Ω. For $k\geq 1$, $\mathcal{P}_k(\mathbb{R}^d)$ denotes the set of Borel probability measures on $\mathbb{R}^d$ with finite $k-$th order moment $M_k(m) = \int_{\mathbb{R}^d}|x|^k m(dx)$, endowed with the Wasserstein distance $\mathbf{d}_k$. Finally, ν_y denotes the external normal vector to a ball B_r at $y\in\partial B_r$.

Some assumptions and terminology. To ease the presentation and avoid repetitions in the rest of the paper, we summarize here some of the terminology we use and the assumptions make.

A map $\mathcal{G}:\mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d)\to\mathbb{R}$ is called strongly monotone, if there exists $\alpha>0$ such that, for all $m_1,m_2\in\mathcal{P}_1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d}(\mathcal{G}(x,m_1)-\mathcal{G}(x,m_2))(m_1-m_2)(dx)\geq\alpha\int_{\mathbb{R}^d}(\mathcal{G}(x,m_1)-\mathcal{G}(x,m_2))^2dx. \tag{1.8}$$

A map $\mathcal{G}:\mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d)\to\mathbb{R}$ is called strictly monotone, if

$$\int_{\mathbb{R}^d}(\mathcal{G}(x,m_1)-\mathcal{G}(x,m_2))(m_1-m_2)(dx)\leq 0 \ \text{ implies } \ m_1=m_2. \tag{1.9}$$

The typical assumption required for Hamiltonians $\mathcal{H}:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ we consider in this paper is that

$$\begin{cases} (i)\ \mathcal{H}=\mathcal{H}(p,x) \text{ is convex in } p,\\ (ii) \text{ for any } R>0, \text{ there exists } C_R>0 \text{ such that,}\\ \quad \text{for all } x,p\in\mathbb{R}^d \text{ with } |p|\leq R,\\ \qquad |\mathcal{H}(p,x)|+|D_p\mathcal{H}(p,x)|+|D^2_{px}\mathcal{H}(p,x)|+|D^2_{pp}\mathcal{H}(p,x)|\leq C_R,\\ (iii)\ \text{ there exists } \lambda>0 \text{ and } C_0> \text{ such that,}\\ \text{for any } p,q,x,z\in\mathbb{R}^d \text{ with } |z|=1 \text{ and in the sense of distributions,}\\ \lambda(D_p\mathcal{H}(p,x)\cdot p-\mathcal{H}(p,x))+D^2_{pp}\mathcal{H}(p,x)q\cdot q\\ \qquad\qquad +2D^2_{px}\mathcal{H}(p,x)z\cdot q+D^2_{xx}\mathcal{H}p,x)\ z\cdot z\geq -C_0. \end{cases} \tag{1.10}$$

The typical regularity assumption we will need for maps $\mathcal{G}:\mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d)\to\mathbb{R}$ is

$$\begin{cases} \mathcal{G}\in C(\mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d);\mathbb{R}) \text{ and there exists } C_0>0 \text{ such that}\\ \sup\limits_{m\in\mathcal{P}_1(\mathbb{R}^d),\ t\in[0,T]}\left[\|\mathcal{G}(\cdot,m)\|_\infty+\|D\mathcal{G}(\cdot,m)\|_\infty+\|D^2\mathcal{G}(\cdot,m)\|_\infty\right]\leq C_0, \end{cases} \tag{1.11}$$

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2. The stochastic backward Hamilton-Jacobi equation

Following the discussion in the introduction, here we study the HJ SPDE (1.3) with $\widetilde{H}_t$ and $\widetilde{G}$ given by (1.4) and $\widetilde{M}$ an unknown martingale.

2.1. **Assumptions and the notion of solution.** We introduce the main assumptions about the continuity of $\widetilde{H}$ and $\widetilde{G}$, the convexity and coercivity in the gradient and the higher regularity in the space and gradient variables of $\widetilde{H}$ ((H1) and (H2)), and the continuity in time of $\widetilde{H}$ (H3). To simplify the notation, when possible, we omit the explicit dependence on ω.

We assume that

$$\widetilde{G} \in L^\infty(\Omega, \mathcal{F}_T, C^2(\mathbb{R}^d)) \quad \text{and} \quad \widetilde{H} \in \mathcal{S}^r(C^2_{loc}(\mathbb{R}^d \times \mathbb{R}^d)) \quad \text{for all} \ \ r \geq 1, \tag{H1}$$

and

$$\widetilde{H}_t \text{ satisfies (1.10) uniformly in } t \in [0,T] \text{ and in } \omega \in \Omega. \tag{H2}$$

To quantify the continuity of $\widetilde{H}$ in time, we define, for $R > 0$,

$$\omega_R^N = \sup_{|p|\leq R,\ |s-t|\leq 1/N,\ y\in\mathbb{R}^d} |\widetilde{H}_s(p,y) - \widetilde{H}_t(p,y)|. \tag{2.1}$$

We assume that, for all $R > 0$,

$$\lim_{N\to\infty} \mathbb{E}[\omega_R^N] = 0. \tag{H3}$$

The notion of solution of (1.3) is introduced next.

Definition 2.1. The couple $(\widetilde{u}, \widetilde{M}) : \mathbb{R}^d \times [0,T] \times \Omega \to \mathbb{R}^2$ is a solution of (1.3) if the following conditions hold:

(i) $\widetilde{u} \in \mathcal{S}^r(W^{1,1}_{loc}(\mathbb{R}^d))$ and $\widetilde{M} \in \mathcal{S}^r(L^1_{loc}(\mathbb{R}^d))$ for all $r \geq 1$,

(ii) there exists $C > 0$ such that, $\mathbb{P}-$a.s., for a.e. $t \in [0,T]$ and all $z \in \mathbb{R}^d$ such that $|z| \leq 1$ and in the sense of distributions,

$$\|\widetilde{u}_t\|_\infty + \|D\widetilde{u}_t\|_\infty + \|\widetilde{M}_t\|_\infty + D^2\widetilde{u}_t \ z \cdot z \leq C,$$

(iii) for a.e. $x \in \mathbb{R}^d$, the process $(\widetilde{M}_t(x))_{t\in[0,T]}$ is a continuous martingale,

and

(iv) for a.e. $(x,t) \in \mathbb{R}^d \times [0,T]$ and $\mathbb{P}-$a.s.,

$$\widetilde{u}_t(x) = \widetilde{G}(x) - \int_t^T \widetilde{H}_s(D\widetilde{u}_s(x), x)ds - \widetilde{M}_T(x) + \widetilde{M}_t(x). \tag{2.2}$$

Remark 2.2. As it is often the case in the literature, the martingale $\widetilde{M}_t(x)$ can be written as a stochastic integral of an adapted process $\widetilde{Z}_t(x)$.

2.2. **Existence of a solution.** We prove that (1.3) has a solution as in Definition 2.1.

Theorem 2.3. *Assume* (H1), (H2) *and* (H3). *Then there exists a solution of* (1.3).

Proof. The solution is obtained as the limit of solutions of a sequence of approximate problems which we introduce next.

For each $N \in \mathbb{N}$, we consider the partition of $(t_n^N)_{n\in\{0,\dots,N\}}$ of $(0,T)$ with $t_n^N = Tn/N$, set

$$\widetilde{H}_t^N(p,x) = \widetilde{H}_{t_n}(p,x) \ \text{ on } \ [t_n^N, t_{n+1}^N),$$

denote by $(\mathcal{F}_t^N)_{t\in[0,T]}$ the piecewise constant filtration given by

$$\mathcal{F}_t^N = \mathcal{F}_{t_n^N} \ \text{ if } \ t \in [t_n^N, t_{n+1}^N),$$

define using backward induction the càdlàg in time processes $\widetilde{u}^N = \widetilde{u}^N_t(x)$ and $\Delta M^N = \Delta M^N_{t_n}(x)$ by $\widetilde{u}^N_T = \widetilde{G}$ in $\mathbb{R}^d$, and, for $n = 0, \dots, N-1$,

$$-\partial_t \widetilde{u}^N_t + \widetilde{H}^N_t(D\widetilde{u}^N_t, x) = 0 \text{ in } \mathbb{R}^d \times (t^N_n, t^N_{n+1}),$$

and, for $x \in \mathbb{R}^d$,

$$u^N_{t^{N,-}_{n+1}}(x) = \mathbb{E}\left[\widetilde{u}^N_{t^{N,+}_{n+1}}(x) \mid \mathcal{F}_{t_n}\right] \text{ and } \Delta\widetilde{M}^N_{t^{N,-}_{n+1}}(x) = u^N_{t^{N,+}_{n+1}}(x) - \mathbb{E}\left[u^N_{t^{N,+}_{n+1}}(x) \mid \mathcal{F}_{t^N_n}\right],$$

and, finally, set

$$\widetilde{M}^N_t(x) = \sum_{t_n \le t} \Delta\widetilde{M}^N_{t_n}(x).$$

Using our assumptions, we have the following lemma. Its proof is presented after the end of the ongoing one.

Lemma 2.4. *Assume* (H1), (H2), *and* (H3). *There exists a $C > 0$ such that, $\mathbb{P}-$a.s., a.e. $t \in [0,T]$ and for all $z \in \mathbb{R}^d$ with $|z| \le 1$, and in the sense of distributions,*

$$\|\widetilde{u}^N\|_\infty + \|D\widetilde{u}^N\|_\infty + \|\widetilde{M}^N\|_\infty + D^2\widetilde{u}^N \; z \cdot z \le C.$$

Moreover, for any $x \in \mathbb{R}^d$, the process $(\widetilde{u}^N_t(x))_{t\in[0,T]}$ is adapted to the filtration $(\mathcal{F}^N_t)_{t\in[0,T]}$ and $(\widetilde{M}^N_{t_n}(x))_{t\in[0,T]}$ is a martingale with respect to the discrete filtration $(\mathcal{F}_{t_n})_{n=0,\dots,N}$.

It is immediate from the definition of the filtration $(\mathcal{F}^N_t)_{t\in[0,T]}$ that $(\widetilde{u}^N_t(x))_{t\in[0,T]}$ is also adapted to $(\mathcal{F}_t)_{t\in[0,T]}$.

Continuing with the ongoing proof we note that $(\widetilde{u}^N_t, \widetilde{M}^N_t)_{t\in[0,T]}$ solves the backward equation

$$d\widetilde{u}^N_t = \widetilde{H}^N_t(D\widetilde{u}^N_t, x)dt + d\widetilde{M}^N_t \text{ in } \mathbb{R}^d \times (0,T) \quad \widetilde{u}^N_T = \widetilde{G} \text{ in } \mathbb{R}^d, \tag{2.3}$$

in the sense that, $\mathbb{P}-$a.s. and for a.e. $(x,t) \in \mathbb{R}^d \times [0,T]$,

$$\widetilde{u}^N_t(x) = \widetilde{G}(x) - \int_t^T \widetilde{H}^N_s(D\widetilde{u}^N_s(x), x)ds - \widetilde{M}^N_T(x) + \widetilde{M}^N_t(x).$$

We show next that $(\widetilde{u}^N_t, \widetilde{M}^N_t)_{t\in[0,T]}$ is Cauchy sequence in a suitable space, and for this we follow Douglis' [17] uniqueness proof (see also [18]).

Fix $0 < N < K$, let $\phi : \mathbb{R} \to [0,\infty)$ be a smooth, Lipschitz continuous, convex and nonincreasing map, and set

$$w_t(x) = \phi(\widetilde{u}^N_t(x) - \widetilde{u}^K_t(x)).$$

Using induction and the convexity of ϕ to cancel the jump terms, which are martingales, we find that, for any $t \in [0,T)$ and $h \in (0, T-h)$,

$$\begin{cases} \mathbb{E}\Big[w_{t+h}(x) - w_t(x)\Big] \ge \\ \mathbb{E}\Big[\int_t^{t+h} \phi'(\widetilde{u}^N_s(x) - \widetilde{u}^K_s(x))(\widetilde{H}^N_s(D\widetilde{u}^N_s, x) - \widetilde{H}^K_s(D\widetilde{u}^K_s, x))ds\Big] \\ \qquad\qquad = \mathbb{E}\Big[\int_t^{t+h} (\mathbf{b}_s(x) \cdot Dw_s(x) + \zeta_s(x))ds\Big] \end{cases} \tag{2.4}$$

where

$$\zeta_s(x) = \phi'(\widetilde{u}^N_s(x) - \widetilde{u}^K_s(x))(\widetilde{H}^N_s(D\widetilde{u}^K_s(x), x) - \widetilde{H}^K_s(D\widetilde{u}^K_s(x), x)),$$

and

$$\mathbf{b}_s(x) = \int_0^1 D_p\widetilde{H}_s^N((1-\lambda)D\widetilde{u}_s^N(x) + \lambda D\widetilde{u}_s^K(x), x)d\lambda.$$

Let $\mathbf{b}^\varepsilon$ be a regularization of b to be specified below. Then (2.4) can be rearranged to read

$$\mathbb{E}\Big[w_{t+h}(x) - w_t(x)\Big] \geq \mathbb{E}\Big[\int_t^{t+h} \operatorname{div}(\mathbf{b}_s^\varepsilon w_s) - \operatorname{div}(\mathbf{b}_s^\varepsilon)w_s + (\mathbf{b}_s - \mathbf{b}_s^\varepsilon)\cdot Dw_s + \zeta_s ds\Big]. \quad (2.5)$$

For $\alpha, \beta > 0$ to be chosen, we consider the quantity

$$e_t = \mathbb{E}[\int_{B_{\alpha+\beta t}} w_t dx],$$

and claim that

$$\begin{cases} e_T - e_t \geq \mathbb{E}\Big[\beta \int_t^T \int_{\partial B_{\alpha+\beta s}} w_s(x)\ dxds\Big] \\ \qquad +\mathbb{E}\Big[\int_t^T \int_{B_{\alpha+\beta s}} \operatorname{div}(\mathbf{b}_s^\varepsilon w_s) - \operatorname{div}(\mathbf{b}_s^\varepsilon)w_s + (\mathbf{b}_s - \mathbf{b}_s^\varepsilon)\cdot Dw_s + \zeta_s dxds\Big]. \end{cases} \quad (2.6)$$

Indeed, let k be a large integer and set $\theta_r^k = t + \frac{r}{k}(T-t)$ for $r \in \{0, \dots, k\}$. Integrating (2.5) over $B_{\alpha+\beta\theta_r^k}$ with $t = \theta_r^k$ and $h = 1/k$ and summing over r we obtain

$$e_t - e_t - \mathbb{E}\Big[\sum_{r=0}^{k-2} \int_{B_{\alpha+\beta\theta_{r+1}^k}\setminus B_{\alpha+\beta\theta_r^k}} w_{\theta_{r+1}^k}(x)dx\Big] = \mathbb{E}\Big[\sum_{r=0}^{k-1} \int_{B_{\alpha+\beta\theta_r^k}} (w_{\theta_{r+1}^k}(x) - w_{\theta_r^k}(x))dx\Big]$$

$$\geq \mathbb{E}\Big[\sum_{r=0}^{k-1} \int_{\theta_r^k}^{\theta_{r+1}^k} \int_{B_{\alpha+\beta\theta_r^k}} (\operatorname{div}(\mathbf{b}_s^\varepsilon w_s) - \operatorname{div}(\mathbf{b}_s^\varepsilon)w_s + (\mathbf{b}_s - \mathbf{b}_s^\varepsilon)\cdot Dw_s + \zeta_s)dyds\Big],$$

and, after letting $k \to +\infty$, (2.6).

Next, rearranging (2.6) we find, for all $t \in (0,T)$.

$$\begin{cases} e_t \leq \mathbb{E}\Big[-(\int_t^T \int_{B_{\alpha+\beta s}} (\operatorname{div}(\mathbf{b}_s^\varepsilon w_s) - \operatorname{div}(\mathbf{b}_s^\varepsilon)w_s + (\mathbf{b}_s - \mathbf{b}_s^\varepsilon)\cdot Dw_s + \zeta_s)dyds \\ \qquad -\beta \int_t^T \int_{\partial B_{\alpha+\beta s}} w_s dxds\Big] + e_T, \end{cases}$$

and, after integrating by parts,

$$\begin{cases} e_t \leq \mathbb{E}\Big[-\int_t^T \int_{B_{\alpha+\beta s}} (-\operatorname{div}(\mathbf{b}_s^\varepsilon)w_s + (\mathbf{b}_s - \mathbf{b}_s^\varepsilon)\cdot Dw_s + \zeta_s)dyds \\ \qquad -\int_t^T \int_{\partial B_{\alpha+\beta s}} (\mathbf{b}_s^\varepsilon \cdot \nu_y + \beta)w_s dyds\Big] + e_T. \end{cases} \quad (2.7)$$

We return now to the choice of $\mathbf{b}^\varepsilon$. For this, let $\widetilde{u}^{N,\varepsilon} = \widetilde{u}^N * \xi^\varepsilon$ and $\widetilde{u}^{K,\varepsilon} = \widetilde{u}^K * \xi^\varepsilon$ be space-time regularizations of $\widetilde{u}^N$ and $\widetilde{u}^K$ with a smooth compactly supported kernel ξ^ε. Then, for all $(x,t) \in \mathbb{R}^d \times (0,T)$, $z \in \mathbb{R}^d$ such that $|z| \leq 1$ and in the sense of distributions,

$$\begin{cases} |\widetilde{u}^{N,\varepsilon}(x,t)| \leq \|\widetilde{u}^N\|_\infty, \quad |\widetilde{u}^{K,\varepsilon}(x,t)| \leq \|\widetilde{u}^K\|_\infty, \\ |D\widetilde{u}^{N,\varepsilon}(x,t)| \leq \|D\widetilde{u}^N\|_\infty, \quad |D\widetilde{u}^{K,\varepsilon}(x,t)| \leq \|D\widetilde{u}^K\|_\infty, \\ D^2\widetilde{u}^{N,\varepsilon}(x,t)\ z\cdot z \leq C, \quad D^2\widetilde{u}^{K,\varepsilon}(x,t)\ z\cdot z \leq C, \end{cases} \quad (2.8)$$

and, as $\varepsilon \to 0$ and for a.e. (x,t),

$$D\widetilde{u}^{N,\varepsilon}(x,t) \to Du^N(x,t) \ \text{ and } \ D\widetilde{u}^{K,\varepsilon}(x,t) \to Du^K(x,t). \quad (2.9)$$

Let

$$\mathbf{b}_s^\varepsilon(x) = \int_0^1 D_p\widetilde{H}_s^N((1-\lambda)D\widetilde{u}_s^{N,\varepsilon}(x) + \lambda D\widetilde{u}_s^{K,\varepsilon}(x), x)d\lambda.$$

It is immediate from the properties of $\widetilde{H}$, (2.8) and Lemma 2.4 that there exists $C_1 > 0$ such that, for all (x,t) and $\mathbb{P}-$a.s. in ω,

$$|\mathbf{b}_s^\varepsilon(x)| \leq \sup_{|p|\leq C,\ y\in\mathbb{R}^d,\ \omega\in\Omega} |D_p\widetilde{H}_s^N(p,y,\omega)| \leq C_1, \tag{2.10}$$

where C is the upper bound on $\|D\widetilde{u}^N\|_\infty$ and $\|D\widetilde{u}^K\|_\infty$ in Lemma 2.4.

Furthermore, as $\varepsilon \to 0$ and $\mathbb{P}-$a.s., $\mathbf{b}^\varepsilon \to \mathbf{b}$ for a.e. (x,t) and in any L^p_{loc}.

Finally, since

$$\begin{aligned}\operatorname{div}(\mathbf{b}_t^\varepsilon(x)) = &\int_0^1 \operatorname{Tr}(D^2_{pp}\widetilde{H}_t^N((1-\lambda)D\widetilde{u}^{N,\varepsilon} + \lambda D\widetilde{u}^{K,\varepsilon}, x)((1-\lambda)D^2\widetilde{u}^{N,\varepsilon} + \lambda D^2\widetilde{u}^{K,\varepsilon}))d\lambda\\ &+ \int_0^1 \operatorname{Tr}(D^2_{px}\widetilde{H}_t^N((1-\lambda)D\widetilde{u}^{N,\varepsilon} + \lambda D\widetilde{u}^{K,\varepsilon}, x))d\lambda,\end{aligned}$$

it follows from the convexity of $\widetilde{H}$, (H2) with $R = C$ from Lemma 2.4, and (2.8) that there exists $\widetilde{C} > 0$ such that

$$\operatorname{div}(\mathbf{b}_t^\varepsilon(x)) \leq \widetilde{C}.$$

We choose $\beta = C_1$ in (2.7). Recalling that $w \geq 0$, we find, for some other $C > 0$,

$$\begin{aligned}e_t &\leq \mathbb{E}\Big[\int_t^T \int_{B_{\alpha+\beta s}} (\widetilde{C}w_s - (\mathbf{b}_s - \mathbf{b}_s^\varepsilon)\cdot Dw_t - \zeta_s)dyds\Big] + e_T\\ &\leq C\int_t^T e_s ds + \mathbb{E}\Big[\int_t^T \int_{B_{\alpha+\beta s}} (-(\mathbf{b}_s - \mathbf{b}_s^\varepsilon)\cdot Dw_t - \zeta_s)dyds\Big] + e_T.\end{aligned}$$

Using Gronwall's inequality and letting $\varepsilon \to 0$, we get

$$e_t \leq C(e_T + \mathbb{E}\Big[\int_t^T \int_{B_{\alpha+\beta s}} \zeta_s(x)dxds\Big]). \tag{2.11}$$

Next we note that, since, in view of Lemma 2.4, $\|Du^N\| \leq C$, if ω_C^N is as in (2.1), we have

$$\|\zeta_s\|_\infty \leq \|\phi'\|_\infty \omega_C^N.$$

Assume next that ϕ is positive on $(-\infty, 0)$, vanishes on $(0, +\infty)$ and $\|\phi'\|_\infty \leq 2$. Since $\widetilde{u}_T^N = \widetilde{u}_T^K$, we have $e_T = 0$.

Therefore, it follows from (2.11) and the above that, for all $t \in [0,T]$,

$$e_t \leq C_{\alpha,\beta}\mathbb{E}[\omega_C^N].$$

Note that a standard approximation argument implies the same inequality for $\phi(s) = (-s)_+$.

Hence, for all $t \in [0,T]$, we have

$$\mathbb{E}\left[\int_{B_{\alpha+\beta t}} (-(\widetilde{u}_t^N(x) - \widetilde{u}_t^K(x))_+ dx\right] \leq C_{\alpha,\beta}\mathbb{E}[\omega_C^N],$$

and, after exchanging the roles of u^N and u^K, for all $t \in [0,T]$,

$$\mathbb{E}\left[\int_{B_{\alpha+\beta t}} |\widetilde{u}_t^N(x) - \widetilde{u}_t^K(x)|dx\right] \leq C_{\alpha,\beta}\mathbb{E}[\omega_C^N].$$

Since α is arbitrary and the $\widetilde{u}^N$'s are uniformly bounded and uniformly Lipschitz continuous in space, the inequality

$$\|u\|_{L^\infty(B_R)} \leq C_R \|Du\|_{L^\infty(B_R)}^{d/(d+1)} \|u\|_{L^1(B_R)}^{1/(d+1)}$$

yields that, for all $R > 0$,

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|\widetilde{u}_t^N - \widetilde{u}_t^K\|_{L^\infty(B_R)}^{d+1}\right] \leq C_R\mathbb{E}[\omega_C^N].$$

In view of (H3), it follows that $(\widetilde{u}^N)_{N\in\mathbb{N}}$ is a Cauchy sequence for the family of seminorms

$$(\sup_{t\in[0,T]} \mathbb{E}l[\|\widetilde{u}_t\|_{L^\infty(B_R)}^{d+1}])_{R>0}.$$

Thus, there is a subsequence, which we denote in the same way as the full sequence, along which the $\widetilde{u}^N$'s converge, in the seminorms above and for every $R > 0$, to a limit $\widetilde{u}$, which, in view of the uniform estimates in Lemma 2.4, is Lipschitz continuous and semiconcave in x. Moreover, the process $(\widetilde{u}_t(x))_{t\in[0,T]}$ is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Finally, up to a further subsequence, we can assume that $\widetilde{u}^N$ converge to $\widetilde{u}$ locally uniformly in x, $\mathbb{P}-$a.s. and a.e. $t \in [0,T]$. In view of the uniform semiconcavity, the last observation implies that, as $N \to \infty$ and $\mathbb{P}-$a.s. and for a.e. $(x,t) \in \mathbb{R}^d \times [0,T]$, $D\widetilde{u}_t^N(x) \to D\widetilde{u}_t(x)$.

Let $\omega \in \Omega$ and $t \in [0,T]$ be such that, as $N \to \infty$, $\widetilde{u}_t^N(\cdot,\omega)$ converges locally uniformly to $\widetilde{u}_t(\cdot,\omega)$. Integrating (2.3) over $[t,T]$, we then find, using Fubini's theorem, that, for a.e. $x \in \mathbb{R}^d$,

$$\widetilde{u}_t^N(x) = \widetilde{G}(x) - \int_t^T \widetilde{H}_s^N(D\widetilde{u}_s^N(x),x)ds + \widetilde{M}_t^N(x).$$

Since $D\widetilde{u}^N$ converges a.e. to $D\widetilde{u}$ on $\mathbb{R}^d \times (0,T)$ and is bounded, we can pass in the $N \to \infty$ limit in the equality above to get, for a.e. $x \in \mathbb{R}^d$,

$$\lim_{N\to+\infty} \widetilde{M}_t^N(x) = \widetilde{u}_t(x) - \widetilde{G}(x) + \int_t^T \widetilde{H}_s^N(D\widetilde{u}_s^N(x),x)ds. \tag{2.12}$$

Hence, the $\widetilde{M}^N$'s converge $\mathbb{P}-$a.s. and for a.e. $x \in \mathbb{R}^d$ to some bounded process denoted by $\widetilde{M}$ and we have, for a.e. $x \in \mathbb{R}^d$,

$$\widetilde{M}_t(x) = \widetilde{u}_t(x) - \widetilde{G}(x) + \int_t^T \widetilde{H}_s^N(D\widetilde{u}_s^N(x),x)ds.$$

Since $(\widetilde{M}_{t_n}^N(x))_{t\in[0,T]}$ is a martingale in the filtration $(\mathcal{F}_{t_n})_{n=0,\dots,N}$, it follows that $(\widetilde{M}_t(x))_{t\in[0,T]}$ is, for a.e. $x \in \mathbb{R}^d$, a martingale in the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. In particular, for a.e. $x \in \mathbb{R}^d$, $t \to \widetilde{M}_t(x)$ is continuous, which shows that $t \to \widetilde{u}_t(x)$ is continuous as well. Therefore $(\widetilde{u},\widetilde{M})$ is a solution to (1.3). □

We conclude this subsection with the proof of Lemma 2.4.

Proof of Lemma 2.4. Since the estimates are standard, here we explain only the formal ideas. As usual the computations can be justified by vanishing viscosity-type arguments.

The uniform L^∞-bound follows by backward induction of a straightforward application of the comparison principle, which implies that, for any $n \in \{0, \dots, N-1\}$,

$$\sup_{t\in[t_n^N,t_{n+1}^N)} \|\widetilde{u}_t^N\|_\infty \le \|\widetilde{u}^N_{t_{n+1}^{N,-}}\|_\infty + CN^{-1} \le \|\widetilde{u}^N_{t_{n+1}^{N,+}}\|_\infty + CN^{-1},$$

where the last inequality holds because the conditional expectation is a contraction in the L^∞-norm.

For the semiconcavity estimate, we note that, if v is a viscosity solution of the Hamilton-Jacobi equation

$$-\partial_t v + H_t(Dv, x) = 0 \ \text{ in } \ \mathbb{R}^d \times (0,T) \quad v_T(\cdot) = v_T \ \text{ in } \ \mathbb{R}^d,$$

with H satisfying (1.10), then there exist $c_0, c_1 > 0$ such that, for any $|z| \le 1$ and in the sense of distributions,

$$\text{if } \ D^2 v_T \ z\cdot z - \lambda v_T \le c_1, \ \text{ then } \ D^2 v_t \ z\cdot z - \lambda v_t \le c_1 + c_0 T. \tag{2.13}$$

Indeed, for any $z \in \mathbb{R}^d$ with $|z| \le 1$, the map $w_t(x) = D^2 v_t(x) z\cdot z - \lambda v_t(x)$ satisfies (formally)

$$\begin{aligned} &-\partial_t w_t + D_p H_t(Dv_t, x)\cdot Dw_t + D^2_{pp} H_t(Dv_t, x)\ Dv_{t,z}\cdot Dv_{t,z} + 2D^2_{pz} H_t(Dv_t, x)\cdot Dv_{t,z} \\ &\quad + D^2_{zz} H_t(Dv_t, x) - \lambda(H_t(Dv_t, x) - D_p H_t(Dv_t, x)\cdot Dv_t) = 0, \end{aligned}$$

and, hence, in view of given (1.10), (2.13) follows from the comparison principle.

Applying (2.13) to $\widetilde{u}^N$ provides the uniform semiconcavity estimate by a backward induction argument similar to the one for the L^∞-bound. The bounds above immediately imply the Lipschitz estimate of $D\widetilde{u}^N$.

Recall that, for each $x \in \mathbb{R}^d$, $(\widetilde{u}^N(x))_{t\in[0,T]}$ is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $(\widetilde{M}_t^N(x))_{t\in[0,T]}$ is a càdlàg martingale with respect to the discrete filtration $(\mathcal{F}_{t_n})$.

The bound on $\widetilde{M}^N$ follows from the observation that, since $\widetilde{M}_T = 0$, by induction we have, for $x \in \mathbb{R}^d$ and $\mathbb{P}-$a.s.,

$$\widetilde{M}_t^N(x) = \widetilde{u}_t^N(x) - \widetilde{G}(x) + \int_t^T \widetilde{H}_s^N(D\widetilde{u}_s^N(x), x)ds.$$

□

2.3. **Comparison and uniqueness.** We say that $(\widetilde{u}^1, \widetilde{M}^1)$ (resp. $(\widetilde{u}^2, \widetilde{M}^2)$) is a supersolution (resp. subsolution) of (1.3), if $\widetilde{u}^1$ (resp. $\widetilde{u}^2$) satisfies all the conditions of Definition 2.1 but (2.2) which is replaced by the requirement that, $\mathbb{P}-$a.s., and for a.e $(x,t,t') \in \mathbb{R}^d \times [0,T] \times [0,T]$ with $t < t'$,

$$\widetilde{u}_t^1(x) \ge \widetilde{u}_{t'}^1(x) - \int_t^{t'} \widetilde{H}_s(D\widetilde{u}_s^1(x), x)ds - \widetilde{M}_{t'}^1(x) + \widetilde{M}_t^1(x) \ \text{ and } \ u_T^1 \ge \widetilde{G} \ \text{ in } \ \mathbb{R}^d,$$

$\Big($resp.

$$\widetilde{u}_t^2(x) \le \widetilde{u}_{t'}^2(x) - \int_t^{t'} \widetilde{H}_s(D\widetilde{u}_s^2(x), x)ds - \widetilde{M}_{t'}^2(x) + \widetilde{M}_t^2(x) \ \text{ and } \ u_T^2 \le \widetilde{G} \ \text{ in } \ \mathbb{R}^d.\Big)$$

The comparison result between supersolutions and subsolutions is stated next.

Proposition 2.5 (Comparison)**.** *Assume* (H1), (H2), (H2) *and* (H3), *and let* $(\widetilde{u}^1, \widetilde{M}^1)$ *and* $(\widetilde{u}^2, \widetilde{M}^2)$ *be respectively a supersolution and a subsolution of* (1.3). *Then,* $\mathbb{P}-a.s.$, $\widetilde{u}^1 \geq \widetilde{u}^2$ *in* $\mathbb{R}^d \times [0,T]$.

The following uniqueness result follows immediately.

Corollary 2.6 (Uniqueness)**.** *Assume* (H1), (H2) *and* (H3). *Then there exists a unique solution to* (1.3).

Proof of Proposition 2.5. The proof follows again Douglis's uniqueness proof and is very similar with the proof of Theorem 2.3 with $\widetilde{u}^1$ and $\widetilde{u}^2$ in place of $\widetilde{u}^N$ and $\widetilde{u}^K$. Hence, in what follows we present a brief sketch.

Fix a smooth, convex and nonincreasing map $\phi : \mathbb{R} \to \mathbb{R}^+$, and let $w_t(x) = \phi(\widetilde{u}^1_t(x) - \widetilde{u}^2_t(x))$. Then by Itô's formula and the inequalities satisfied by the $\widetilde{u}^i$'s, we have

$$\begin{aligned} dw_t(x) &\geq \phi'(\widetilde{H}_t(D\widetilde{u}^1_t(x), x) - \widetilde{H}_t(D\widetilde{u}^2_t(x), x))dt + \frac{1}{2}\phi'' d < \widetilde{M}(x) >_t + \phi' d\widetilde{M}_t(x) \\ &\geq \mathbf{b}_t(x) \cdot Dw_t(x)dt + \phi' d\widetilde{M}_t(x), \end{aligned}$$

where ϕ and its derivatives are evaluated at $\widetilde{u}^1_t(x) - \widetilde{u}^2_t(x)$, $\widetilde{M}_t = \widetilde{M}^1_t - \widetilde{M}^2_t$ and

$$\mathbf{b}_t(x) = \int_0^1 D_p \widetilde{H}_t((1-s)D\widetilde{u}^1_t(x) + sD\widetilde{u}^2_t(x))ds.$$

The rest of the proof follows almost verbatim the arguments of the proof of Theorem 2.3. It consists of an appropriate regularization $\mathbf{b}^\varepsilon$ similar to the one in the aforementioned proof and a rewriting of the inequality satisfied by w_t as

$$dw_t(x) \geq (\mathrm{div}(\mathbf{b}^\varepsilon_t(x)w_t) - \mathrm{div}(\mathbf{b}^\varepsilon_t(x))w_t + (\mathbf{b}_t - \mathbf{b}^\varepsilon_t) \cdot Dw_t)dt + \phi' d\widetilde{M}_t.$$

Next we consider the quantity

$$e_t = \mathbb{E}\left[\int_{B_{\alpha+\beta t}} w_t(x)dx,\right]$$

and we find, as in the proof of Theorem 2.3, for $t_1 \in [0,T]$,

$$\begin{aligned} e_T - e_{t_1} \geq \mathbb{E}\Big[&\int_{t_1}^T \int_{B_{\alpha+\beta t}} (\mathrm{div}(\mathbf{b}^\varepsilon_t(x)w_t) - \mathrm{div}(\mathbf{b}^\varepsilon_t(x))w_t + (\mathbf{b}_t - \mathbf{b}^\varepsilon_t) \cdot Dw_t)dydt \\ &+ \beta \int_{t_1}^T \int_{\partial B_{\alpha+\beta t}} w_t(y)dydt\Big] \\ = \mathbb{E}\Big[&\int_{t_1}^T \int_{\partial B_{\alpha+\beta t}} (\mathbf{b}^\varepsilon_t(x) \cdot \nu_y + \beta)w_t dydt \\ &+ \int_{t_1}^T \int_{B_{\alpha+\beta t}} (-\mathrm{div}(\mathbf{b}^\varepsilon_t(x))w_t + (\mathbf{b}_t - \mathbf{b}^\varepsilon_t) \cdot Dw_t)dydt\Big]. \end{aligned}$$

The properties of $\mathbf{b}^\varepsilon$, a suitable choice of β and Grownwall's inequality lead after letting $\varepsilon \to$ to

$$e_{t_1} \leq e^{CT} e_T.$$

We choose (after approximation) $\phi(r) = (-r)_+$. Then $e_T = 0$ since $\widetilde{u}^1_T \geq \widetilde{u}^2_T$. Therefore $e_t = 0$ for any t, which shows that $\widetilde{u}^1 \geq \widetilde{u}^2$ since α is arbitrary.

□

2.4. **Optimal control representation.** We develop a stochastic optimal control formulation for $\widetilde{u}$ and present a stochastic maximum principle-type result.

In what follows, $\widetilde{L}$ is the Legendre transform of $\widetilde{H}$, that is, for $x, \alpha \in \mathbb{R}^d, t \in [0,T]$ and $\omega \in \Omega$,

$$\widetilde{L}_t(\alpha, x, \omega) = \sup_{p \in \mathbb{R}^d} [-p \cdot \alpha - \widetilde{H}_t(p, x, \omega)].$$

Moreover, for $x \in \mathbb{R}^d$ and $t \in [0,T]$, $\mathcal{A}_{t,x}$ is the set of admissible paths defined by

$$\mathcal{A}_{t,x} = \{\gamma \in \mathcal{S}^2(\mathbb{R}^d) : \ \gamma_t = x \text{ and } \gamma \in H^1([t,T];\mathbb{R}^d) \text{ a.s.}\}.$$

Proposition 2.7. *Assume* (H1), (H2), (H2) *and* (H3), *and let* $\widetilde{u}$ *be the solution of* (1.3). *Then*

$$\widetilde{u}_t(x) = \operatorname*{essinf}_{\gamma \in \mathcal{A}_{t,x}} \mathbb{E}\left[\int_t^T \widetilde{L}_s(\dot\gamma_s, \gamma_s) ds + \widetilde{G}(\gamma_T) \mid \mathcal{F}_t\right]. \tag{2.14}$$

Proof. To simplify the notation, we present the proof for $t = 0$. Let $(\widetilde{u}^N)_{N \geq 1}$ be as in the proof of Theorem 2.3. Since on each time interval (t_n^N, t_{n+1}^N), $\widetilde{u}^N$ is a viscosity solution of a standard HJ equation, for any fixed ω and $\widetilde{L}^N$ the Legendre transform of $\widetilde{H}^N$, we have

$$\widetilde{u}^N_{t_n^N}(x) = \inf_{\gamma \in \widetilde{\mathcal{A}}_{t_n^N, t_{n+1}^N, x}} \int_{t_n^N}^{t_{n+1}^N} \widetilde{L}_s^N(\dot\gamma_s, \gamma_s) ds + \widetilde{u}^{N,-}_{t_{n+1}^N}(\gamma_{t_{n+1}^N}),$$

where

$$\widetilde{\mathcal{A}}_{t,s,x} = \{\gamma \in H^1([t,s];\mathbb{R}^d) : \ \gamma_t = x\}.$$

The $\mathcal{F}_{t_n^N}-$measurability of $\widetilde{u}^N_{t_{n+1}^-}$ allows to find a $\mathcal{F}_{t_n}-$measurable selection $(\omega, x) \to (\widetilde{\gamma}_t^N(\omega, x))_{t \in [t_n^N, t_{n+1}^N]}$ of minimizers. Note that, since $D\widetilde{u}^N$ is uniformly bounded, $\dot{\widetilde{\gamma}}^N$ is uniformly bounded as well by some constant C.

Concatenating these minimizers, we find, for any $x \in \mathbb{R}^d$, a $(\mathcal{F}_{t_n^N})_{n=0,\dots,N}-$adapted path $\widetilde{\gamma}^N \in \mathcal{A}_{0,x}$ such that

$$\widetilde{u}_0^N(x) = \mathbb{E}\left[\int_0^T \widetilde{L}_s^N(\dot{\widetilde{\gamma}}_s^N, \widetilde{\gamma}_s^N) ds + \widetilde{G}(\widetilde{\gamma}_T^N)\right] = \inf_{\gamma \in \mathcal{A}_{0,x}} \mathbb{E}\left[\int_0^T \widetilde{L}_s^N(\dot\gamma_s, \gamma_s) ds + \widetilde{G}(\gamma_T)\right],$$

where the second equality can be proved by dynamic programming and induction.

Then, in view of the continuity of the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ and the definition of $\widetilde{L}^N$, we find

$$\widetilde{u}_0(x) = \lim_{N \to \infty} u_0^N(x) \leq \inf_{\gamma \in \mathcal{A}_{0,x}} \mathbb{E}\left[\int_0^T \widetilde{L}_s(\dot\gamma_s, \gamma_s) ds + \widetilde{G}(\gamma_T)\right].$$

On the other hand, the time regularity of $\widetilde{L}$ and the uniform in N bound on $\dot{\widetilde{\gamma}}^N$, which we denote by C, imply that

$$\widetilde{u}_0^N(x) \geq \mathbb{E}\left[\int_0^T \widetilde{L}_s(\dot{\widetilde{\gamma}}_s^N, \widetilde{\gamma}_s^N) ds + \widetilde{G}(\widetilde{\gamma}_T^N)\right] - \mathbb{E}[\omega_C^N].$$

It follows that

$$\widetilde{u}_0(x) = \lim_{N\to\infty} u_0^N(x) \geq \limsup_{N\to\infty} \Big[\inf_{\gamma\in\mathcal{A}_{0,x}} \mathbb{E}\left[\int_0^T \widetilde{L}_s(\dot\gamma_s,\gamma_s)ds + \widetilde{G}(\gamma_T)\right] - \mathbb{E}[\omega_C^N]\Big]$$
$$= \inf_{\gamma\in\mathcal{A}_{0,x}} \mathbb{E}\left[\int_0^T \widetilde{L}_s(\dot\gamma_s,\gamma_s)ds + \widetilde{G}(\gamma_T)\right].$$

□

We now discuss the maximum principle and the regularity of the value function along optimal solutions. We point, however, that we do not claim the existence of an optimal solution.

Theorem 2.8 (Maximum principle)**.** *Assume* (H1), (H2), (H2) *and* (H3), *let* $\overline{\gamma}\in\mathcal{A}_{0,x}$ *be optimal for* $\widetilde{u}_0(x)$ *and define, for* $t\in[0,T]$, *the* $(\mathcal{F}_t)_{t\in[0,T]}-$*adapted continuous process* $\overline{p}$ *by*

$$\overline{p}_t = \mathbb{E}\left[\int_t^T D_x\widetilde{L}(\dot{\overline{\gamma}}_s,\overline{\gamma}_s)ds + D\widetilde{G}(\overline{\gamma}_T) \;\Big|\; \mathcal{F}_t\right]. \tag{2.15}$$

Then, for $t\in[0,T]$,

$$\dot{\overline{\gamma}}_t = -D_p\widetilde{H}_t(\overline{p}_t,\overline{\gamma}_t) \tag{2.16}$$

and $\overline{p}$ *solves the BSDE*

$$d\overline{p}_t = D_x\widetilde{H}_t(\overline{p}_t,\overline{\gamma}_t)dt + d\overline{m}_t \quad in \quad [0,T], \quad \overline{p}_T = D\widetilde{G}(\overline{\gamma}_T), \tag{2.17}$$

where $(\overline{m}_t)_{t\in[0,T]}\in\mathcal{S}^r(\mathbb{R})$ *is a continuous martingale.*

Remark 2.9. The theorem implies that $\overline{\gamma}$ is of class C^1.

Proof. Fix $h>0$ small and $t\in[0,T)$ at which $\overline{\gamma}$ is differentiable $\mathbb{P}-$a.s., and let $v\in L^\infty(\Omega,\mathbb{R}^d)$ be $\mathcal{F}_t$-measuarable.
Define γ^h by $\gamma^h(0)=x$ and

$$\dot\gamma_s^h = \begin{cases} \dot{\overline{\gamma}}_s & \text{if } s\in[0,t]\cup[t+h,T],\\ v & \text{otherwise,}\end{cases}$$

note that, since for $s\geq t+h$,

$$\gamma_s^h = \overline{\gamma}_s + hv - (\overline{\gamma}_{t+h} - \overline{\gamma}_t), \tag{2.18}$$

γ^h is admissible, and use the dynamic programming principle to get

$$\mathbb{E}\left[\int_t^T \widetilde{L}_s(\dot\gamma_s^h,\gamma_s^h)ds + \widetilde{G}(\gamma_T^h) \mid \mathcal{F}_t\right] \geq \mathbb{E}\left[\int_t^T \widetilde{L}_s(\dot{\overline{\gamma}}_s,\overline{\gamma}_s)ds + \widetilde{G}(\overline{\gamma}_T) \mid \mathcal{F}_t\right].$$

Hence

$$\mathbb{E}\Big[\int_t^{t+h}(\widetilde{L}_s(v,\gamma_s^h) - \widetilde{L}_s(\dot{\overline{\gamma}}_s,\overline{\gamma}_s))ds$$
$$+ \int_{t+h}^T (\widetilde{L}_s(\dot{\overline{\gamma}}_s,\gamma_s^h) - \widetilde{L}_s(\dot{\overline{\gamma}}_s,\overline{\gamma}_s))ds + \widetilde{G}(\gamma_T^h) - \widetilde{G}(\overline{\gamma}_T) \mid \mathcal{F}_t\Big] \;\geq\; 0.$$

Then

$$\mathbb{E}\Big[\int_t^{t+h}(\widetilde{L}_s(v,\gamma_s^h)-\widetilde{L}_s(\dot{\overline{\gamma}}_s,\overline{\gamma}_s))ds$$
$$+\int_{t+h}^{T}(\widetilde{L}_s(\dot{\overline{\gamma}}_s,\gamma_s^h)-\widetilde{L}_s(\dot{\overline{\gamma}}_s,\overline{\gamma}_s))ds+D\widetilde{G}(\overline{\gamma}_T)\cdot(\gamma_T^h-\overline{\gamma}_T)+C|\gamma_T^h-\overline{\gamma}_T|^2\mid\mathcal{F}_t\Big]\ \geq\ 0.$$

Dividing by h, letting $h\to 0^+$ and using (2.18) we find

$$\widetilde{L}_t(v,\overline{\gamma}_t)-\widetilde{L}_t(\overline{\gamma}_t,\overline{\gamma}_t)+(v-\dot{\overline{\gamma}}_t)\cdot\mathbb{E}\left[\int_t^T D_x\widetilde{L}_s(\dot{\overline{\gamma}}_s,\overline{\gamma}_s)ds+D\widetilde{G}(\overline{\gamma}_T)\mid\mathcal{F}_t\right]\geq 0.$$

Since $v\in\mathcal{F}_t$ is arbitrary, we conclude that, $\mathbb{P}-$a.s.,

$$D_\alpha\widetilde{L}_t(\overline{\gamma}_t,\overline{\gamma}_t)+\mathbb{E}\left[\int_t^T D_x\widetilde{L}_s(\dot{\overline{\gamma}}_s,\overline{\gamma}_s)ds+D\widetilde{G}(\overline{\gamma}_T)\mid\mathcal{F}_t\right]=0,$$

and (2.16) holds with $\overline{p}_t$ is defined by (2.15).

To prove (2.17), we first note that the (standard) BSDE (2.17) has a unique solution and $D_x\widetilde{L}_t(\alpha,x)=-D_x\widetilde{H}_t(p,x)$ if $\alpha=-D_p\widetilde{H}_t(p,x)$. Thus, in view of (2.16), we have $D_x\widetilde{L}_t(\dot{\overline{\gamma}}_t,\overline{\gamma}_t)=-D_p\widetilde{H}_t(\overline{p}_t,\overline{\gamma}_t)$ and (2.15) can be written, for $t\in[0,T]$, as

$$\overline{p}_t=\mathbb{E}\left[-\int_t^T D_x\widetilde{H}(\overline{p}_s,\overline{\gamma}_s)ds+D\widetilde{G}(\overline{\gamma}_T)\ \Big|\ \mathcal{F}_t\right].$$

It follows that $t\to\overline{p}_t-\overline{p}_0-\int_0^t D_x\widetilde{H}(\overline{p}_s,\overline{\gamma}_s)ds$ is a martingale, which proves (2.17).

□

The next result is about the regularity of $\widetilde{u}$ along the optimal path.

Lemma 2.10. *Let $\overline{\gamma}$ and $\overline{p}$ be as in Theorem 2.8. Then, $\mathbb{P}-$a.s. and for any $t\in(0,T]$, $x\to\widetilde{u}_t(x)$ is differentiable at $\overline{\gamma}_t$ and $\overline{p}_t=D\widetilde{u}_t(\overline{\gamma}_t)$.*

Proof. It follows from the dynamic programming principle that, for any $h>0$ small, all $\mathcal{F}_t$ measurable and bounded v, and γ^h such that $\gamma^h_{t-h}=\overline{\gamma}_{t-h}$ and $\dot{\gamma}^h_s=v^h=\mathbb{E}[v\mid\mathcal{F}_{t-h}]$ on $[t-h,t]$,

$$\widetilde{u}_{t-h}(\overline{\gamma}_{t-h})=\mathbb{E}\Big[\int_{t-h}^t\widetilde{L}_s(\dot{\overline{\gamma}}_s,\overline{\gamma}_s)ds+\widetilde{u}_t(\overline{\gamma}_t)\ \Big|\ \mathcal{F}_{t-h}\Big]\leq\mathbb{E}\Big[\int_{t-h}^t\widetilde{L}_s(v^h,\gamma_s^h)ds+\widetilde{u}_t(\gamma_t^h)\ \Big|\ \mathcal{F}_{t-h}\Big].$$

Let $q=q(\omega)$ be a measurable selection of $D^+u_t(\overline{\gamma}_t,\omega)$. Then, using the semiconcavity of $\widetilde{u}_t$, we find

$$0\leq\mathbb{E}\Big[\int_{t-h}^t(\widetilde{L}_s(v^h,\gamma_s^h)-\widetilde{L}_s(\dot{\overline{\gamma}}_s,\overline{\gamma}_s))ds+q\cdot(\gamma_t^h-\overline{\gamma}_t)+C|\gamma_t^h-\overline{\gamma}_t|^2\ \Big|\ \mathcal{F}_{t-h}\Big].$$

Dividing by h, letting $h\to 0^+$ and using that the filtration $(\mathcal{F}_t)$ is continuous, we obtain

$$0\leq\widetilde{L}_t(v,\overline{\gamma}_t)-\widetilde{L}_t(\dot{\overline{\gamma}}_t,\overline{\gamma}_t)+q\cdot(v-\dot{\overline{\gamma}}_t).$$

Since $\dot{\overline{\gamma}}_t$ maximizes $v\to-q\cdot v-\widetilde{L}_t(v,\overline{\gamma}_t)$, it follows from (2.16), that

$$\dot{\overline{\gamma}}_t=-D_p\widetilde{H}_t(q,\overline{\gamma}_t)=-D_p\widetilde{H}_t(\overline{p}_t,\overline{\gamma}_t).$$

Thus $q=\overline{p}_t$ and $D^+\widetilde{u}_t(\gamma_t)$ is a singleton. In view of the semiconcavity of $\widetilde{u}_t$, the last fact implies that $\widetilde{u}_t$ is differentiable at $\overline{\gamma}_t$ and $\overline{p}_t=D\widetilde{u}_t(\overline{\gamma}_t)$. □

2.5. **The continuity equation.** We now investigate the continuity equation associated with the vector field $-D_p\widetilde{H}(D\widetilde{u}_t(x),x)$. As in the previous subsections, $(\widetilde{u},\widetilde{M})$ is the solution of (1.3).

Proposition 2.11. *Assume* (H1), (H2) *and* (H3). *Then, for each $\overline{m}_0 \in L^\infty(\mathbb{R}^d)\cap\mathcal{P}_2(\mathbb{R}^d)$, there exists a $(\mathcal{F}_t)_{t\in[0,T]}-$adapted process $\widetilde{m}\in\mathcal{S}^r(\mathcal{P}_1(\mathbb{R}^d))\cap L^\infty(\mathbb{R}^d\times(0,T))$ which solves, $\mathbb{P}-$a.s. and in the sense of distributions, the continuity equation (with random coefficients)*

$$\partial_t\widetilde{m}_t = \operatorname{div}(\widetilde{m}_t D_p\widetilde{H}_t(D\widetilde{u}_t(x),x))dt \ \text{ in } \ \mathbb{R}^d\times(0,T) \quad \widetilde{m}_0=\overline{m}_0 \ \text{ in } \ \mathbb{R}^d. \tag{2.19}$$

Proof. We use the discretization in the proof of Theorem 2.3 and consider the solution $(\widetilde{u}^N,\widetilde{M}^N)$ of the discretized problem defined there.

Let $\widetilde{m}^N\in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ be the $(\mathcal{F}_{t_n})_{n=0,\dots,N}-$adapted process that solves, in the sense of distributions,

$$\partial_t\widetilde{m}^N_t = \operatorname{div}(\widetilde{m}^N_t D_p\widetilde{H}^N_t(D\widetilde{u}^N_t,x))dt \ \text{ in } \ \mathbb{R}^d\times(0,T) \quad \widetilde{m}_0=\overline{m}_0 \ \text{ in } \ \mathbb{R}^d, \tag{2.20}$$

which, following the discussion in the appendix of [11], can be built by induction. Indeed, we can construct $\widetilde{m}^N$ on each time interval $[t^N_n,t^N_{n+1}]$, since on this interval $\widetilde{u}^N$ satisfies a standard HJ equation. In addition, for some $C>0$ and $\mathbb{P}-$a.s.,

$$\begin{aligned}
&(i) \quad \sup_{t\in[0,T]}\int_{\mathbb{R}^d}|x|^2\widetilde{m}^N_t(x)dx\le C,\\
&(ii) \quad \mathbf{d}_1(\widetilde{m}^N_t,\widetilde{m}^N_s)\le C|s-t| \ \text{ for all } \ s,t\in[0,T],\\
&(iii) \quad \|\widetilde{m}^N\|_\infty\le C.
\end{aligned}$$

Let $\widetilde{m}$ be an (up to a subsequence) limit of $(\widetilde{m}^N)_{N\in\mathbb{N}}$ in $L^\infty(\mathbb{R}^d\times[0,T]\times\Omega)-$weak*. Then, since, as $N\to\infty$ and $\mathbb{P}-$a.s. and (x,t) a.e., $D\widetilde{u}^N\to D\widetilde{u}$, we can pass to the limit in (2.20). The claim then follows. □

We now turn to the question of uniqueness, for which, unfortunately, we require a much stronger condition than the standing ones. Indeed, we need to assume that $\widetilde{H}$ is of the form

$$\widetilde{H}_t(p,x)=\frac{\widetilde{a}_t(x)}{2}|p|^2+\widetilde{B}_t(x)\cdot p+\widetilde{f}_t(x), \tag{2.21}$$

where, for some constant $C_0>0$,

$$\widetilde{a},\widetilde{f}\in\mathcal{S}^2(C^2(\mathbb{R}^d)) \text{ and } \widetilde{B}\in\mathcal{S}^2(C^2(\mathbb{R}^d;\mathbb{R}^d)) \quad \text{with } C_0^{-1}\le\widetilde{a}_t(x)\le C_0. \tag{2.22}$$

We note that (2.21) and (2.22) yield that $\widetilde{H}$ satisfies (H1), (H2), and (H3).

The following result is a variation of the one in [7].

Proposition 2.12. *Assume* (H1), (2.21) *and* (2.22). *Then, for each $\overline{m}_0\in L^\infty(\mathbb{R}^d)\cap\mathcal{P}_1(\mathbb{R}^d)$, there exists a unique $(\mathcal{F}_t)_{t\in[0,T]}-$adapted process $\widetilde{m}\in\mathcal{S}^2(\mathcal{P}_1(\mathbb{R}^d))$ with bounded density in $\mathbb{R}^d\times(0,T))$ which solves* (2.20), *$\mathbb{P}-$a.s. and in the sense of distributions.*

In the deterministic case considered in the appendix of [11], the uniqueness of a solution does not require an addition structure assumption on $\widetilde{H}$. Instead, it relies on the fact that the forward-forward Hamiltonian system (2.16)-(2.17) has a unique solution given the initial condition $(\overline{\gamma}_0,\overline{p}_0)$. Unfortunately this does not seem to be the case in the random setting.

Proof of Proposition 2.12. Set $\widetilde{b}_t(x) = -D_p\widetilde{H}_t(D\widetilde{u}_t(x), x)$. Then, in view of (2.21) and the definition of a solution $\widetilde{u}$ of (1.3), $\widetilde{b}_t(x) = -\widetilde{a}_t(x)D\widetilde{u}_t(x) - \widetilde{B}_t(x)$ is bounded and one-side Lipschitz, that is, there exists $C_0 > 0$ such that, for all $(x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T]$,

$$\|\widetilde{b}\|_\infty \le C_0 \ \text{ and } \ (\widetilde{b}_t(x) - \widetilde{b}_t(y)) \cdot (x - y) \ge -C_0|x - y|^2.$$

Then we can apply ω by ω the uniqueness result to the continuity equation given in Proposition A.1 in the appendix. □

2.6. **Existence of optimal paths of the stochastic control problem.** We now address the problem of the existence of optimal paths for the control representation of $\widetilde{u}$ established in Proposition 2.7 . For simplicity we assume again that $t = 0$ and recall that

$$\widetilde{u}_0(x) = \operatorname{essinf}_{\gamma\in\mathcal{A}_{0,x}} \mathbb{E}\left[\int_t^T \widetilde{L}_s(\dot\gamma_s, \gamma_s)ds + \widetilde{G}(\gamma_T)\right]. \tag{2.23}$$

The problem is that (2.23) is a non convex stochastic optimal control problem with a (a priori) non smooth value function, hence the existence of an optimal path is far from obvious.

Proposition 2.13. *Assume* (H1), (2.21) *and* (2.22). *Then,* $\mathbb{P}-$*a.s. and for a.e.* $x \in \mathbb{R}^d$ *there exists a unique minimizer* $\overline{\gamma}^x \in \mathcal{A}_{0,x}$ *of the stochastic optimal control problem* (2.23) *and this minimizer satisfies* $\dot{\overline{\gamma}}^x_t = -D_p\widetilde{H}_t(D\widetilde{u}_t(\overline{\gamma}_t), \overline{\gamma}_t)$.

Proof. Fix $\overline{m}_0 \in \mathcal{P}_2(\mathbb{R}^d)$ with smooth and positive density. According to Proposition 2.12, the random continuity equation

$$\partial_t\widetilde{m}_t - \operatorname{div}(\widetilde{m}_t D_p\widetilde{H}_t(D\widetilde{u}_t, x)) = 0 \ \text{ in } \ \mathbb{R}^d \times (0,\infty) \quad \widetilde{m}_0 = \overline{m}_0 \ \text{ in } \ \mathbb{R}^d$$

has a unique solution $\widetilde{m}$ with bounded density.

A simple adaptation of the Lagrangian approach introduced in Ambrosio [2] shows that, in view of the uniqueness of the solution to the continuity equation, there exists a $dx\times d\mathbb{P}-$a.e. unique Borel measurable map $\mathbb{R}^d \times \Omega \ni (x, \omega) \to \overline{\gamma}^{x,\omega} \in \Gamma = C([0, T]; \mathbb{R}^d)$, which is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, such that, $\mathbb{P}$-a.s. and for any $t \in [0, T]$, $\overline{\gamma}^{x,\omega}_0 = x$ and

$$\widetilde{m}_t(\cdot, \omega) = \int_{\mathbb{R}^d} \delta_{\overline{\gamma}^{x,\omega}_t}\overline{m}_0(x)dx.$$

We show next that the process $(\overline{\gamma}^x_t)_{t\in[0,T]}$ is optimal in the optimization problem (2.23) for $\overline{m}_0-$a.e. $x \in \mathbb{R}^d$. For this, we claim that

$$\int_{\mathbb{R}^d} \mathbb{E}\left[\int_0^T \widetilde{L}_t(\dot{\overline{\gamma}}^x_t, \overline{\gamma}^x_t)dt + \widetilde{G}(\overline{\gamma}^x_T)\right] m_0(x)dx = \int_{\mathbb{R}^d} \widetilde{u}_0(x)m_0(x)dx. \tag{2.24}$$

Assuming for the moment (2.24), we proceed with the proof of the optimality, and recall that, for any $x \in \mathbb{R}^d$,

$$\widetilde{u}_0(x) \le \mathbb{E}\left[\int_0^T \widetilde{L}_t(\dot{\overline{\gamma}}^x_t, \overline{\gamma}^x_t)dt + \widetilde{G}(\overline{\gamma}^x_T)\right].$$

Integrating the inequality above against $\overline{m}_0$, we infer by (2.24) that $\overline{\gamma}^x$ is optimal for $\overline{m}_0-$a.e. $x \in \mathbb{R}^d$. Since $\overline{m}_0 > 0$, this holds for a.e. $x \in \mathbb{R}^d$.

It remains to prove (2.24). For this let, $t_n = nT/N$ for $n \in \{0, \dots, N\}$ with $N \in \mathbb{N}$ and note that, in view of the equation satisfied by $\widetilde{u}$, we have, for a.e. x,

$$\widetilde{u}_{t_{n+1}}(x) - \widetilde{u}_{t_n}(x) = -\int_{t_n}^{t_{n+1}} \widetilde{H}_t(D\widetilde{u}_t(x), x)dt - (\widetilde{M}_{t_{n+1}}(x) - \widetilde{M}_{t_n}(x)). \tag{2.25}$$

Integrating (2.25) against $\widetilde{m}_{t_n}$, which is absolutely continuous with a bounded density, and summing over n gives

$$\sum_{n=0}^{N-1}\int_{\mathbb{R}^d}(\widetilde{u}_{t_{n+1}}(x)-\widetilde{u}_{t_n}(x))\widetilde{m}_{t_n}(x)dx$$
$$=-\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}^d}\widetilde{H}_t(D\widetilde{u}_t(x),x)\widetilde{m}_{t_n}(x)dxdt-\sum_{n=0}^{N-1}\int_{\mathbb{R}^d}(\widetilde{M}_{t_{n+1}}(x)-\widetilde{M}_{t_n}(x))\widetilde{m}_{t_n}(x)dx.$$

Reorganizing the left-hand side of the expression above taking into account the equation satisfied by $\widetilde{m}$ yields

$$\sum_{n=0}^{N-1}\int_{\mathbb{R}^d}(\widetilde{u}_{t_{n+1}}(x)-\widetilde{u}_{t_n}(x))\widetilde{m}_{t_n}(x)dx$$
$$=\int_{\mathbb{R}^d}\widetilde{u}_T(x)\widetilde{m}_{t_{N-1}}(x)dx-\int_{\mathbb{R}^d}\widetilde{u}_0(x)\widetilde{m}_0(x)dx-\sum_{n=1}^{N-1}\int_{\mathbb{R}^d}\widetilde{u}_{t_n}(x)(\widetilde{m}_{t_n}(x)-\widetilde{m}_{t_{n-1}}(x))dx$$
$$=\int_{\mathbb{R}^d}\widetilde{G}(x)\widetilde{m}_{t_{N-1}}(x)dx-\int_{\mathbb{R}^d}\widetilde{u}_0(x)\overline{m}_0(x)dx$$
$$+\sum_{n=1}^{N-1}\int_{t_{n-1}}^{t_n}\int_{\mathbb{R}^d}D\widetilde{u}_{t_n}(x)\cdot D_p\widetilde{H}_t(D\widetilde{u}_t(x),x)\widetilde{m}_t(x)dxdt.$$

We let $N\to+\infty$ and take expectation to find

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\widetilde{G}(x)\widetilde{m}_T(x)dx-\int_{\mathbb{R}^d}\widetilde{u}_0(x)\overline{m}_0(x)dx+\int_0^T\int_{\mathbb{R}^d}D\widetilde{u}_t(x)\cdot D_p\widetilde{H}_t(D\widetilde{u}_t(x),x)\widetilde{m}_t(x)\right]$$
$$=\mathbb{E}\left[-\int_0^T\int_{\mathbb{R}^d}\widetilde{H}_t(D\widetilde{u}(x)_t,x)\widetilde{m}_t(x)dxdt-\int_0^T\int_{\mathbb{R}^d}\widetilde{m}_t(x)d\widetilde{M}_t(x)dx\right].$$

Recalling that $\widetilde{M}$ is a martingale and that $p\cdot D_p\widetilde{H}_t(p,x)+\widetilde{H}_t(p,x)=\widetilde{L}_t(-D_p\widetilde{H}_t(D\widetilde{u}_t(x),x),x)$, we rearrange the last expression to get

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\widetilde{G}(x)\widetilde{m}_T(x)dx+\int_0^T\int_{\mathbb{R}^d}\widetilde{L}_t(-D_p\widetilde{H}_t(D\widetilde{u}_t(x),x),x)\widetilde{m}_t(x)\right]=\int_{\mathbb{R}^d}\widetilde{u}_0(x)\overline{m}_0(x)dx.$$

Finally, the facts that

$$\widetilde{m}_t=\int_{\mathbb{R}^d}\delta_{\overline{\gamma}_t^x}\overline{m}_0(x)dx\quad\text{and}\quad\dot{\overline{\gamma}}_t^x=-D_p\widetilde{H}_t(D\widetilde{u}_t(\overline{\gamma}_t^x),\overline{\gamma}_t^x),\overline{\gamma}_t^x)\ \ \mathbb{P}\otimes\overline{m}_0-\text{a.e.}(\omega,x)$$

imply that (2.24) holds. □

3. The stochastic MFG system

We investigate the stochastic MFG system (1.1). We begin recalling that, after the change of the unknowns in (1.5) we obtain, at least formally, (1.6) which we study here.

3.1. **The assumptions and the notion of solution.** To study (1.6) we assume that

$$\overline{m}_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \tag{MFG1}$$

$$\begin{cases} \widetilde{F} : \mathbb{R}^d \times [0,T] \times \mathcal{P}_1(\mathbb{R}^d) \times \Omega \to \mathbb{R} \text{ is such that, for any } m \in \mathcal{P}_1(\mathbb{R}^d), \\ \widetilde{F}_\cdot(\cdot, m) \in \mathcal{S}^r(C^2(\mathbb{R}^d)) \text{ for any } r \geq 1, \text{ and} \\ \widetilde{F}_t \text{ satisfies (1.11) uniformly in } t \in [0,T] \text{ and in } \omega \in \Omega, \end{cases} \tag{MFG2}$$

and

$$\begin{cases} \widetilde{G} : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \times \Omega \to \mathbb{R} \text{ is } \mathcal{F}_T-\text{measurable}, \ G \in C(\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d); \mathbb{R}), \\ \text{and (1.11) is satisfied uniformly in } \omega \in \Omega. \end{cases} \tag{MFG3}$$

For $\widetilde{H} : \mathbb{R}^d \times [0,T] \times \Omega \to \mathbb{R}$, we assume that

$$\widetilde{H}_t \text{ satisfies (1.10) uniformly in } t \in [0,T] \text{ and in } \omega \in \Omega. \tag{MFG4}$$

Moreover, if

$$\omega_R^N = \sup_{|s-t| \leq 1/N, y \in \mathbb{R}^d, m \in \mathcal{P}_1(\mathbb{R}^d), |p| \leq R} \Big[|\widetilde{H}_s(y,p) - \widetilde{H}_t(y,p)| + |\widetilde{F}_s(y,m) - \widetilde{F}_t(y,m)| \Big],$$

then, for any $R > 0$,

$$\mathop{\mathbb{E}}_{N \to \infty} [\omega_R^N] = 0. \tag{MFG5}$$

Finally, we assume that

$$\begin{cases} \widetilde{F}_t \text{ and } \widetilde{G} \text{ are strongly monotone uniformly in } t \in [0,T] \text{ and in } \omega \in \Omega, \\ \text{and } \widetilde{F}_t \text{ is strictly monotone for all } t \in [0,T] \text{ and } \mathbb{P}-\text{a.s. in } \omega. \end{cases} \tag{MFG6}$$

A classical example of a map $\widetilde{F}$ satisfying the above conditions, which goes back to [28, 29, 30, 31], is of the form

$$\widetilde{F}_t(x,m) = \widetilde{f}_t(\cdot, m * \rho(\cdot)) * \rho,$$

where

$$\begin{cases} \rho \text{ is a smooth, non negative and even kernel,} \\ \text{with Fourier transform } \widehat{\rho} \text{ vanishing almost nowhere,} \end{cases}$$

and
$\widetilde{f} : \mathbb{R}^d \times [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is $(\mathcal{F}_t)_{t \in [0,T]}$ adapted and, for any $R > 0$,

$$\begin{cases} \text{there exists } C_R > 0 \text{ such that} \\ \sup_{0 \leq s \leq R, \ t \in [0,T]} \Big[\|\widetilde{f}_t(\cdot, s)\|_\infty + \|D\widetilde{f}_t(\cdot, s)\|_\infty + \|D^2 \widetilde{f}_t(\cdot, s)\|_\infty \Big] \leq C_R, \end{cases}$$

and

$$\mathop{\mathbb{E}}_{N \to \infty} [\sup_{|t_1 - t_2| \leq 1/N, 0 \leq s \leq R} |\widetilde{f}_{t_1}(y,s) - \widetilde{f}_{t_2}(y,s)|] = 0,$$

and, finally,

$\widetilde{f}$ is strictly increasing and Lipschitz in the second variable, that is, there exists $\alpha \in (0,1)$ such that

$$\alpha \leq \frac{\partial \widetilde{f}_t}{\partial s}(x,s) \leq \alpha^{-1}.$$

It is immediate that $\widetilde{F}$ satisfies the regularity conditions in (MFG2) and (MFG5) and, moreover,

$$\begin{aligned}
&\int_{\mathbb{R}^d} (\widetilde{F}_t(x,m_1) - \widetilde{F}_t(x,m_2))(m_1 - m_2)(dx) \\
&\quad = \int_{\mathbb{R}^d} (\widetilde{f}_t(x, m_1 * \rho(x)) - \widetilde{f}_t(x, m_2 * \rho(x)))(m_1 * \rho(x) - m_2 * \rho(x))dx \\
&\quad \geq \alpha \int_{\mathbb{R}^d} (m_1 * \rho(x)) - m_2 * \rho(x)))^2 dx.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^d} (\widetilde{F}_t(x,m_1) - \widetilde{F}_t(x,m_2))^2 dx &\leq \|\rho\|_\infty^2 \int_{\mathbb{R}^d} (\widetilde{f}_t(x, m_1 * \rho(x)) - \widetilde{F}_t(x, m_2 * \rho(x)))^2 dx \\
&\leq \|\rho\|_\infty^2 \alpha^{-2} \int_{\mathbb{R}^d} (m_1 * \rho(x) - m_2 * \rho(x))^2 dx \\
&\leq \|\rho\|_\infty^2 \alpha^{-3} \int_{\mathbb{R}^d} (\widetilde{F}_t(x,m_1) - \widetilde{F}_t(x,m_2))(m_1 - m_2)(dx),
\end{aligned}$$

and $\widetilde{F}$ is strongly monotone.

The strict monotonicity follows from the observation that, if

$$\int_{\mathbb{R}^d} (\widetilde{F}_t(x,m_1) - \widetilde{F}_t(x,m_2))(m_1 - m_2)(dx) = 0,$$

then $(m_1 - m_2) * \rho = 0$, which implies that $(\widehat{m_1 - m_2})\widehat{\rho} = 0$. Since $\widehat{\rho}$ vanishes almost nowhere, it follows that $(\widehat{m_1 - m_2}) = 0$ and, hence, $m_1 = m_2$.

Remark 3.1. It follows from (MFG2) and (MFG6) that $\widetilde{F}$ is Hölder continuous in m with respect to the $\mathbf{d}_1-$ distance, that is, for all $m_1, m_2 \in \mathcal{P}_1(\mathbb{R}^d)$

$$\|\widetilde{F}_t(\cdot, m_1) - \widetilde{F}_t(\cdot, m_2)\|_\infty \leq C\mathbf{d}_1^{1/(d+2)}(m_1, m_2).$$

Indeed, in view of the interpolation inequality

$$\|f\|_\infty \leq C_d \|Df\|_\infty^{1/(d+2)} \|f\|_{L^2}^{2/(d+2)},$$

with C_d depending only on the dimension, we find

$$\begin{aligned}
&\|\widetilde{F}_t(\cdot,m_1)-\widetilde{F}_t(\cdot,m_2)\|_\infty^{d+2}\\
&\quad\le C_d(\|D\widetilde{F}_t(\cdot,m_1)\|_\infty+\|D\widetilde{F}_t(\cdot,m_2)\|_\infty)\int_{\mathbb{R}^d}(\widetilde{F}_t(x,m_1)-\widetilde{F}_t(x,m_2))^2dx\\
&\quad\le 2C_0C_d\alpha^{-1}\int_{\mathbb{R}^d}(\widetilde{F}_t(x,m_1)-\widetilde{F}_t(x,m_2))(m_1-m_2)(dx)\\
&\quad\le 2C_0C_d\alpha^{-1}(\|D\widetilde{F}_t(\cdot,m_1)\|_\infty+\|D\widetilde{F}_t(\cdot,m_2)\|_\infty)\mathbf{d}_1(m_1,m_2)\\
&\quad\le 4C_0^2C_d\alpha^{-1}\mathbf{d}_1(m_1,m_2).
\end{aligned}$$

We continue with the definition of a weak solution of (1.6).

Definition 3.2. The triplet $(\widetilde{u},\widetilde{m},\widetilde{M})$ is a solution of (1.6) if:
(i) $\widetilde{u}\in\mathcal{S}^r(W^{1,1}_{loc}(\mathbb{R}^d))$, $\widetilde{M}\in\mathcal{S}^r(L^1_{loc}(\mathbb{R}^d))$ and $\widetilde{m}\in\mathcal{S}^r(\mathcal{P}_1(\mathbb{R}^d))$ for any $r\ge 1$,
(ii) there exits $C>0$ such that, $\mathbb{P}-$a.s., for a.e. $t\in[0,T]$, all $z\in\mathbb{R}^d$ such that $|z|\le 1$, and in the sense of distributions,

$$\|\widetilde{m}\|_\infty+\|\widetilde{u}_t\|_{W^{1,\infty}(\mathbb{R}^d)}+\|\widetilde{M}_t\|_\infty+D^2\widetilde{u}_t\ z\cdot z\le C,$$

(iii) the process $(\widetilde{M}_t(x))$ is a $(\mathcal{F}_t)_{t\in[0,T]}$ continuous martingale for a.e. $x\in\mathbb{R}^d$.
(iv) for a.e $(x,t)\in\mathbb{R}^d\times[0,T]$ and $\mathbb{P}-$a.s. in ω,

$$\widetilde{u}_t(x)=\widetilde{G}(x,\widetilde{m}_T)-\int_t^T(\widetilde{H}_s(D\widetilde{u}_s(x),x)-\widetilde{F}_s(x,\widetilde{m}_s))ds-\widetilde{M}_T(x)+\widetilde{M}_t(x),$$

and
(v) in the sense of distributions and $\mathbb{P}-$a.s. in ω,

$$d_t\widetilde{m}_t=\mathrm{div}(\widetilde{m}_tD_p\widetilde{H}_t(D\widetilde{u}_t,x))\ \text{ in }\ \mathbb{R}^d\times(0,T)\quad\widetilde{m}_0=\overline{m}_0\ \text{ in }\ \mathbb{R}^d.$$

3.2. **The existence and uniqueness result.** The main result of the paper about the existence and uniqueness of a solution of (1.6) is stated next.

Theorem 3.3. *Assume* (MFG1), (MFG2), (MFG3), (MFG4), (MFG5) *and* (MFG6). *Then there exists a unique solution of* (1.6).

The proof consists of several steps. Similarly to the Hamilton-Jacobi case, the solution is constructed by discretizing the noise in time. Hence, the first step is to recall and refine known regularity results for deterministic MFG systems. Then we explain the construction of an approximate solution by time discretization and, finally, we pass to the limit to obtain the solution of (1.6).

3.3. **The deterministic MFG system.** We consider the deterministic MFG system

$$\begin{cases}
-\partial_t u+H(Du,x)=F(x,\mu_t)\ \text{ in }\ \mathbb{R}^d\times(0,T),\\
\partial_t\mu_t-\mathrm{div}(\mu_tD_pH(Du_t,x))=0\ \text{ in }\ \mathbb{R}^d\times(0,T),\\
m_0=\overline{m}_0\quad u(\cdot,T)=G(\cdot,m_T).
\end{cases}\tag{3.1}$$

A solution of (3.1) is a pair (u, m) such that u is a continuous, bounded and semiconcave in x uniformly in t viscosity solution of the HJ equation, while $m \in C([0,T], \mathcal{P}_2(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (0,T))$ is a solution of the continuity equation in the sense of distribution.

Next we state some general hypotheses, which imply the existence of a solution of (3.1).

We assume that

$$H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \ \text{ satisfies (1.10)}, \tag{3.2}$$

$$\begin{cases} F \in C(\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d); \mathbb{R}) \ \text{ is Lipschitz continuous and} \\ \text{semiconcave in } x \text{ uniformly in } m, \end{cases} \tag{3.3}$$

$$\begin{cases} \text{there exits } \alpha_F > 0 \text{ such that, for all } m_1, m_2 \in \mathcal{P}_1(\mathbb{R}^d), \\ \int_{\mathbb{R}^d} (F(x, m_1) - F(x, m_2))(m_1 - m_2)(dx) \geq \alpha_F \|F(\cdot, m_1) - F(\cdot, m_2)\|_\infty^{d+2}, \end{cases} \tag{3.4}$$

and

$$\begin{cases} G : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R} \text{ is Hölder continuous in } m \text{ uniformly in } x, \text{ bounded and} \\ \text{semiconcave in } x \text{ uniformly in } m, \text{ and there exists } \alpha_G > 0 \text{ such that} \\ \text{for all } m_1, m_2 \in \mathcal{P}_1(\mathbb{R}^d), \\ \int_{\mathbb{R}^d} (G(x, m_1) - G(x, m_2))(m_1 - m_2) dx \geq \alpha_G \|G(\cdot, m_1) - G(\cdot, m_2)\|_\infty^{d+2}. \end{cases} \tag{3.5}$$

We refer to Remark 3.1 about about the connection between (3.4) and the more standard strong monotonicity condition.

The following result can be derived from [30]; see also [11]. In the sequel, we give some details about the proof of the estimates that are needed for the proof of Theorem 3.3.

Lemma 3.4. *Assume* (3.2), (3.3), (3.4) *and* (3.5). *There exists $C_0 > 1$ such that, for any $\overline{m}_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, there exists a unique solution u of* (3.1) *such that*

$$\|u\|_\infty \leq \|G\|_\infty + C_0 T, \tag{3.6}$$

$$\begin{cases} u \text{ is semiconcave in } x \text{ uniformly in } m, \text{ that is, if, for some } C_1 > 0, \\ \text{all } z \in \mathbb{R}^d \text{ with } |z| \leq 1, \text{ all } m \in \mathcal{P}_2, \text{ and in the sense of distributions,} \\ \text{if } D^2 G(\cdot, m) z \cdot z - \lambda G(\cdot, m) \leq C_1, \text{ then for all } t \in [0,T], \\ \qquad D^2 u_t(\cdot) z \cdot z - \lambda u_t(\cdot) \leq C_1 + C_0 T, \end{cases} \tag{3.7}$$

and

$$\begin{cases} m \text{ is bounded in } \mathbb{R}^d \times (0,T) \text{ and has finite second moments, that is, there} \\ \text{exists } C > 0 \text{ depending on } \|Du\|_\infty \text{ and the semiconcavity} \\ \text{constant of } u \text{ such that, for all } t \in [0,T], \\ \qquad M_2(m_t) \leq M_2(\overline{m}_0) + CT \quad \text{and} \quad \|m_t\|_\infty \leq \|\overline{m}_0\|_\infty + CT. \end{cases} \tag{3.8}$$

We note the claim is that u remains bounded and uniformly semiconcave in x uniformly in time and in the initial measure $\overline{m}_0$.

In addition, u is also uniformly Lipschitz continuous in x. This is a consequence of the elementary fact that, if $v : \mathbb{R}^d \to \mathbb{R}$ is bounded by some M and semiconcave with constant K, v is Lipschitz continuous with a Lipschitz constant bounded by $2(MK)^{1/2}$.

Finally, the fact that the estimates on u and m grow only linearly in time T will be important in the construction of the next subsection and justifies the awkward formulation of the semiconcavity estimate.

The proof of Lemma 3.4. The existence and uniqueness of the solution (u, m) and the estimates on m can be found in [11, 30]. The bound and the semiconcavity estimates on u can be established as in the proof of Lemma 2.4. Here we only repeat some the formal argument for convenience, noting that everything can be justified using "viscous" regularizations.

Formally, it is immediate that

$$\begin{aligned}\frac{d}{dt}M_2(m_t) = \frac{d}{dt}\int_{\mathbb{R}^d}|x|^2 m_t(dx) &= -\int_{\mathbb{R}^d} 2x\cdot D_pH(Du_t(x),x)m_t(dx)\\ &\leq 4\|D_pH(Du_t)\|_\infty^2 + M_2(m_t),\end{aligned}$$

and the estimate on $M_2(m_t)$ follows by Gronwall's Lemma.

For the $L^\infty-$bound, we rewrite the continuity equation as

$$\partial_t m - D_pH(Du_t,x)\cdot Dm_t - m_t\mathrm{div}(D_pH(Du_t,x)) = 0,$$

where

$$\mathrm{div}(D_pH(Du_t(x),x)) = \mathrm{Tr}(D^2_{pp}H(Du_t,x)D^2u_t(x) + D^2_{px}H(Du_t(x),x)) \leq C,$$

in view of the Lipschitz and semiconcavity estimates of u. The bound follows using the maximum principle. □

Later in the paper it will be convenient to define the solution of (3.1) in a unique way for less regular initial measures. For this we use the following regularity result.

Proposition 3.5. *Assume* (3.2), (3.3), (3.4), *and* (3.5). *Then, if* $(u^1,m^1),(u^2,m^2)$ *are the solutions of the MFG system* (3.1) *with* $m_0^1, m_0^2 \in \mathcal{P}_2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, *then, there exist positive constants* C, C', *which depend on* d, α_G, α_F *and* $\|Du^i\|_\infty$, *such that*

$$\|u^1-u^2\|_\infty^{d+2} \leq C\int_{\mathbb{R}^d}(u^1(x,0)-u^2(x,0))(m_0^1-m_0^2)(dx) \leq C'\mathbf{d}_1(m_0^1,m_0^2).$$

It follows that the map $U : \mathbb{R}^d\times[0,T]\times(\mathcal{P}_2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)) \to \mathbb{R}$ given by

$$U_t(x,\overline{m}_0) = u_t(x),$$

where (u,m) is the solution of the MFG system (3.1) with initial condition $m(0) = \overline{m}_0$, has a unique extension on $\mathbb{R}^d\times[0,T]\times\mathcal{P}_1(\mathbb{R}^d)$.

Moreover, in view of Lemma 3.4 and Proposition 3.5, the extended map $U : \mathbb{R}^d\times[0,T]\times\mathcal{P}_1(\mathbb{R}^d)\to\mathbb{R}$ is Hölder continuous in m uniformly in x, Lipschitz continuous and semiconcave in x uniformly in m, and strongly monotone in the sense of (3.4).

We also note that the map $\overline{m}_0 \to m$, where (u,m) is the solution of the MFG system (3.1) with initial condition $m(0) = \overline{m}_0$, is continuous in $C([0,T];\mathcal{P}_1(\mathbb{R}^d))$. Indeed, Proposition 3.5 gives that the map $\overline{m}_0 \to u$ is continuous from $\mathcal{P}_2(\mathbb{R}^d)$ to $C(\mathbb{R}^d\times[0,T])$. In view of the semiconcavity estimate on u, this in turn yields the continuity of the map $\overline{m}_0 \to Du$ in $L^1_{loc}(\mathbb{R}^d\times[0,T])$. The claimed continuity of $\overline{m}_0 \to m$ follows combining the uniform in

time continuity of m in $\mathcal{P}_1(\mathbb{R}^d)$, which depends on $\|Du\|_\infty$, the $L^\infty-$estimate on m and the uniqueness of the solution of the associated continuity equation.

Proof of Proposition 3.5. Using a viscous approximation to justify it, the standard proof of uniqueness of the MFG system (3.1) yields

$$\int_0^T \int_{\mathbb{R}^d} (F(x,m_t^1) - F(x,m_t^2))(m_t^1 - m_t^2)(dx) \le -\left[\int_{\mathbb{R}^d} (u^1-u^2)(m^1-m^2)(dx)\right]_0^T.$$

It then follows from the strong monotonicity condition on F and and G that

$$\begin{aligned}
&\alpha_G \|G(\cdot,m_T^1) - G(\cdot,m_T^2)\|_\infty^{d+2} + \alpha_F \int_0^T \|F(\cdot,m_t^1) - F(\cdot,m_t^2))\|_\infty^{d+2} dt \\
&\le \int_{\mathbb{R}^d} (G(x,m_T^1) - G(x,m_T^2))(m_T^1 - m_T^2)(dx) + \int_0^T \int_{\mathbb{R}^d} (F(x,m_t^1) - F(x,m_t^2))(m_t^1 - m_t^2)(dx) \\
&\le \int_{\mathbb{R}^d} (u^1(x,0) - u^2(x,0))(m_0^1 - m_0^2)(dx) \le (\|Du^1\|_\infty + \|Du^2\|_\infty) \mathbf{d}_1(m_0^1, m_0^2).
\end{aligned}$$

Using the uniform Lipschitz estimates on u^i and the comparison principle in the equations for the u^i we find

$$\|u^1 - u^2\|_\infty \le \|G(\cdot,m_T^1) - G(\cdot,m_T^2)\|_\infty + \int_0^T \|F(\cdot,m_t^1) - F(\cdot,m_t^2))\|_\infty dt.$$

Hence, there exists $C>0$, which depends on d, α_G, α_F and $\|Du^i\|_\infty$, such that

$$\begin{aligned}
\|u^1 - u^2\|_\infty^{d+2} &\le C\Big(\|G(\cdot,m_T^1) - G(\cdot,m_T^2)\|_\infty^{d+2} + \int_0^T \|F(\cdot,m_t^1) - F(\cdot,m_t^2)\|_\infty^{d+2} dt\Big) \\
&\le C \int_{\mathbb{R}^d} (u^1(x,0) - u^2(x,0))(m_0^1 - m_0^2)(dx) \le C \mathbf{d}_1(m_0^1, m_0^2).
\end{aligned}$$

□

3.4. **The discretized stochastic MFG system.** We use the same discretization technique and the same notation as in the proof of Theorem 2.3. Let $N \in \mathbb{N}$ and set $t_n^N = Tn/N$,

$$\widetilde{H}_t^N(p,x) = \widetilde{H}_{t_n^N}(p,x) \quad \text{and} \quad \widetilde{F}_t^N(x,m) = \widetilde{F}_{t_n^N}(x,m) \quad \text{on} \quad [t_n^N, t_{n+1}^N),$$

and consider the filtration $(\mathcal{F}_t^N)_{t\in[0,T]}$ defined by

$$\mathcal{F}_t^N = \mathcal{F}_{t_n^N} \quad \text{for} \quad t \in [t_n^N, t_{n+1}^N).$$

The goal here is to build a triplet $(\widetilde{u}^N, \widetilde{M}^N, \widetilde{\mu}^N)$ such that

(i) $(\widetilde{u}^N, \widetilde{M}^N, \widetilde{\mu}^N)$ is adapted to the filtration $(\mathcal{F}_t^N)_{t\in[0,T]}$,

(ii) on each interval (t_n^N, t_{n+1}^N) with $n = 0, \dots, N-1$, $\widetilde{u}^N$ is a viscosity solutions of the backward HJ equation

$$\begin{cases} -\partial_t \widetilde{u}^N + \widetilde{H}_t^N(D\widetilde{u}^N, x) = \widetilde{F}_t^N(x, \widetilde{\mu}_t^N) \quad \text{on} \quad \mathbb{R}^d \times (t_n^N, t_{n+1}^N), \\ \widetilde{u}^N(\cdot, t_{n+1}^{N,-}) = \mathbb{E}\left[\widetilde{u}^N(\cdot, t_{n+1}^{N,+}) | \mathcal{F}_{t_n^N}\right] \quad \text{in} \quad \mathbb{R}^d, \end{cases}$$

(iii) $\Delta\widetilde{M}^N$ is defined by

$$\Delta\widetilde{M}_t^N(x) = \widetilde{u}^N(x, t_{n+1}^{N,-}) - \mathbb{E}\left[\widetilde{u}^N(x, t_{n+1}^{N,+}) | \mathcal{F}_{t_n}\right] \quad \text{on} \quad (t_n^N, t_{n+1}^N),$$

and

$$\widetilde{M}_t^N(x) = \sum_{t_n^N < t} \Delta \widetilde{M}_{t_n^N}^N(x).$$

(iv) $\widetilde{\mu}^N \in C([0,T]; \mathcal{P}_2(\mathbb{R}^d))$ is $\mathbb{P}-$a.s. a weak solution of

$$\begin{cases} \partial_t \widetilde{\mu}_t^N - \operatorname{div}(\widetilde{\mu}_t^N D_p \widetilde{H}_t^N(D\widetilde{u}_t^N, x)) = 0 \ \text{ in } \ \mathbb{R}^d \times (0,T), \\ \widetilde{\mu}_0^N = \overline{m}_0 \ \text{ in } \ \mathbb{R}^d. \end{cases}$$

This is the topic of the next lemma.

Lemma 3.6. *Assume* (3.2), (3.3), (3.4) *and* (3.5). *Then, there exists at least one solution* $(\widetilde{u}^N, \widetilde{M}^N, \widetilde{\mu}^N)$ *of the problem above and* $C > 0$ *such that,* $\mathbb{P}-$*a.s., a.e. in* $t \in [0,T]$, *all* $z \in \mathbb{R}^d$ *with* $|z| \le 1$, *and in the sense of distributions,*

(i) $\|\widetilde{u}^N\|_\infty + \|D\widetilde{u}^N\|_\infty + \|\widetilde{M}\|_\infty + \|\widetilde{\mu}^N\|_\infty + D^2\widetilde{u}_t^N z \cdot z \le C$,

(ii) for any $x \in \mathbb{R}^d$, $(\widetilde{u}_t^N(x))_{t\in[0,T]}$ *and* $(\widetilde{\mu}_t^N(x))_{t\in[0,T]}$ *are adapted to the filtration* $(\mathcal{F}_t^N)_{t\in[0,T]}$ *and, therefore, the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$, *and* $(\widetilde{M}_{t_n}^N(x))_{n=0,\dots,N}$ *is a martingale with respect to the discrete filtration* $(\mathcal{F}_{t_n})_{n=0,\dots,N}$.

Proof. We show first $(\widetilde{u}^N, \widetilde{M}^N, \widetilde{\mu}^N)$ is well-posed. For this, we define by backward induction a sequence of maps $\widetilde{U}^N : \mathbb{R}^d \times [t_n^N, t_{n+1}^N) \times \mathcal{P}_1(\mathbb{R}^d) \times \Omega \to \mathbb{R}$ such that, for each $t \in [t_n^N, t_{n+1}^N)$, $\widetilde{U}_t^N$ is $\mathcal{F}_{t_n^N}-$measurable, Hölder continuous in m uniformly in x, bounded and semiconcave in x uniformly in m and strongly monotone in the sense of (3.4).

We set $\widetilde{U}_N^N = \widetilde{G}$ and, given $\widetilde{U}_{t_{n+1}^N}^N$, we define $\widetilde{U}_t^N$ on $[t_n^N, t_{n+1}^N)$ as follows: for any $\overline{m}_0 \in \mathcal{P}_2 \cap L^\infty$ we solve the MFG system

$$\begin{cases} -\partial_t \widetilde{v}_t + \widetilde{H}_t^N(D\widetilde{v}_t, x) = \widetilde{F}^N(x, \widetilde{m}_t) \ \text{ in } \ \mathbb{R}^d \times (t_n^N, t_{n+1}^N), \\ \partial_t \widetilde{m}_t - \operatorname{div}(\widetilde{m}_t D_p \widetilde{H}_t^N(D\widetilde{v}_t, x)) = 0 \ \text{ in } \ \mathbb{R}^d \times (t_n^N, t_{n+1}^N), \\ \widetilde{m}_{t_n} = \overline{m}_0 \ \text{ nd } \ \widetilde{v}_{t_{n+1}^{N,-}} = \mathbb{E}\left[\widetilde{U}_{t_{n+1}^N}^N(\cdot, \widetilde{m}_{t_{n+1}^{N,+}}^N) \mid \mathcal{F}_{t_n^N}\right] \ \text{ in } \ \mathbb{R}^d. \end{cases}$$

We know from the discussion after Proposition 3.5 that, if we set $\widetilde{U}_t^N(x, \overline{m}_0) = \widetilde{v}_t(x)$, then $\widetilde{U}_t^N$ can be extended on $\mathbb{R}^d \times \mathcal{P}_1$ and satisfies the required regularity properties.

In what follows it will be convenient to set $\widetilde{\rho}_t^{N,n}(x, \overline{m}_0) = \widetilde{m}_t(x)$. We remark that $\widetilde{\rho}^{N,n}$ is $\mathcal{F}_{t_n}-$measurable and continuous in $\overline{m}_0$ in $\mathcal{P}_1(\mathbb{R}^d)$.

Given $\overline{m}_0 \in \mathcal{P}_2 \cap L^\infty$, we now build $(\widetilde{u}^N, \widetilde{\mu}^N, \widetilde{M}^N)$. We set, for $t \in [0, t_1^N]$,

$$(\widetilde{u}_t^N(x), \widetilde{\mu}_t^N(x)) = (\widetilde{U}_t^N(x, \overline{m}_0), \widetilde{\rho}_t^{N,0}(x, \overline{m}_0)),$$

and note that, in view of Lemma 3.4, $\widetilde{\mu}_t^N$ is bounded in L^∞ and has bounded finite second order moment both uniformly in N. Then, using a forward in time induction, we define, for $t \in [t_n^N, t_{n+1}^N)$,

$$(\widetilde{u}_t^N(x), \widetilde{\mu}_t^N(x)) = (\widetilde{U}_t^N(x, \widetilde{\mu}_{t_n^N}^N), \widetilde{\rho}_t^{N,n}(x, \widetilde{\mu}_{t_n^N}^N)),$$

and

$$\Delta \widetilde{M}_t^N(x) = \widetilde{u}^N(x, t_{n+1}^{N,-}) - \mathbb{E}\left[\widetilde{u}^N(x, t_{n+1}^{N,+}) | \mathcal{F}_{t_n^N}\right] \ \text{ on } \ (t_n^N, t_{n+1}^N),$$

and

$$\widetilde{M}_t^N(x) = \sum_{t_n^N < t} \Delta \widetilde{M}_{t_n^N}^N(x).$$

In view of the definition of $\widetilde{U}^N$ and $\rho^{N,n}$, the triplet $(\widetilde{u}^N, \widetilde{M}^N, \widetilde{\mu}^N)$ solves the required equations and is adapted to the discrete filtration $(\mathcal{F}_t^N)_{t\in[0,T]}$. The estimates on $\widetilde{u}^N$ and $\widetilde{\mu}^N$ follow from Lemma 3.4 applied on each time interval (t_n^N, t_{n+1}^N). We remark that this is the place we use that the estimates in Lemma 3.4 is grow linearly in time. The bound on $\widetilde{M}^N$ is obtained as in the proof of Lemma 2.4. □

3.5. **Passing to the limit.** The aim is to pass to the limit in the discrete MFG system. Using the strong monotonicity of $\widetilde{F}$ and $\widetilde{G}$, we obtain the following estimate.

Lemma 3.7. *Let $(\widetilde{u}^N, \widetilde{M}^N, \widetilde{\mu}^N)$ be defined as above. There exists a random variable ω^N such that $\mathop{\mathbb{E}}_{N\to\infty}\Big[\omega^N\Big] = 0$ and, for $K, N \in \mathbb{N}$ with $K \geq N$,*

$$\mathbb{E}\Big[\|\widetilde{G}(\cdot,\widetilde{\mu}_T^N) - \widetilde{G}(\cdot,\widetilde{\mu}_T^K)\|_\infty^{d+2} + \int_0^T \|\widetilde{F}_t(\cdot,\widetilde{\mu}_t^N) - \widetilde{F}_t(\cdot,\widetilde{\mu}_t^K)\|_\infty^{d+2} dt\Big] \leq C\mathbb{E}\Big[\omega^N\Big].$$

Proof. Using the fact that the pair $(\widetilde{u}^N, \widetilde{\mu}^N)$ is piecewise classical solution of the MFG system, with $\widetilde{\mu}^N$ continuous in time and adapted, we find, following the classical Lasry-Lions computation, that

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\Big[\int_{\mathbb{R}^d} (\widetilde{u}_t^N(x) - \widetilde{u}_t^K(x))(\widetilde{\mu}_t^N(x) - \widetilde{\mu}_t^K(x))dx\Big] \\
&= \mathbb{E}\Big[\int_{\mathbb{R}^d} (\widetilde{H}_t^N(D\widetilde{u}_t^N(x),x) - \widetilde{F}_t^N(x,\widetilde{\mu}_t^N) - \widetilde{H}_t^K(D\widetilde{u}_t^K(x),x) + \widetilde{F}_t^K(x,\widetilde{\mu}_t^K))(\widetilde{\mu}_t^N(x) - \widetilde{\mu}_t^K(x))dx \\
&+ \int_{\mathbb{R}^d} (\widetilde{u}_t^N(x) - \widetilde{u}_t^K(x))(\mathrm{div}(\widetilde{\mu}_t^N(x) D_p\widetilde{H}_t^N(D\widetilde{u}_t^N(x),x)) - \mathrm{div}(\widetilde{\mu}_t^K(x) D_p\widetilde{H}_t^K(D\widetilde{u}_t^K(x),x)))dx\Big] \\
&= -\mathbb{E}\Big[\int_{\mathbb{R}^d} (\widetilde{F}_t^N(x,\widetilde{\mu}_t^N) - \widetilde{F}_t^K(x,\widetilde{\mu}_t^K))(\widetilde{\mu}_t^N(x) - \widetilde{\mu}_t^K(x))dx\Big] \\
&- \mathbb{E}\Big[\int_{\mathbb{R}^d} \mu_t^N(x)\Big(\widetilde{H}_t^K(D\widetilde{u}_t^K(x),x) - \widetilde{H}_t^N(D\widetilde{u}_t^N(x),x) \\
&- (D\widetilde{u}_t^N(x) - D\widetilde{u}_t^K(x)) \cdot D_p\widetilde{H}_t^N(D\widetilde{u}_t^N(x),x)\Big)dx + \int_{\mathbb{R}^d} \mu_t^K(x)\Big(\widetilde{H}_t^N(D\widetilde{u}_t^N(x),x) \\
&- \widetilde{H}_t^K(D\widetilde{u}_t^K(x),x) - (D\widetilde{u}_t^K(x) - D\widetilde{u}_t^N(x)) \cdot D_p\widetilde{H}_t^K(D\widetilde{u}_t^K(x),x)\Big)dx\Big].
\end{aligned}$$

In order to use the strong monotonicity assumption on $\widetilde{F}$ and the convexity of $\widetilde{H}$, we replace the discretized maps $\widetilde{F}^N$ and $\widetilde{H}^N$ by the continuous ones and find

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\Big[\int_{\mathbb{R}^d} (u_t^N(x) - u_t^K(x))(\widetilde{\mu}_t^N(x) - \widetilde{\mu}_t^K(x))dx\Big] \\
&\leq -\mathbb{E}\Big[\int_{\mathbb{R}^d} (\widetilde{F}_t(x,\widetilde{\mu}_t^N) - \widetilde{F}_t(x,\widetilde{\mu}_t^K))(\widetilde{\mu}_t^N(x) - \widetilde{\mu}_t^K(x))dx\Big] + C\mathbb{E}\Big[\omega^N\Big],
\end{aligned}$$

where, with C being the uniform bound on $D\widetilde{u}^N$ and on $D\widetilde{u}^K$,

$$\omega^N = \sup_{x\in\mathbb{R}^d,|t-s|\le 1/N,|p|\le C,m\in\mathcal{P}_1} \Big[|\widetilde{F}_t(x,m)-\widetilde{F}_s(x,m)| + |\widetilde{H}_t(p,x)-\widetilde{H}_s(p,x)|\Big]. \tag{3.9}$$

Integrating in time inequality above and using the fact that $\mu_0^N = \mu_0^K$, we get

$$\mathbb{E}\Big[\int_{\mathbb{R}^d}(\widetilde{G}(x,\widetilde{\mu}_T^N)-\widetilde{G}(x,\widetilde{\mu}_T^K))(\widetilde{\mu}_T^N(x)-\widetilde{\mu}_T^K(x))dx\Big]$$

$$+\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^d}(\widetilde{F}_t(x,\widetilde{\mu}_t^N)-\widetilde{F}_t(x,\widetilde{\mu}_t^K))(\widetilde{\mu}_t^N(x)-\widetilde{\mu}_t^K(x))dxdt\Big] \le C\mathbb{E}\Big[\omega^N\Big].$$

Therefore, in view of the strong monotonicity of $\widetilde{F}$ and $\widetilde{G}$ in (3.4) and (3.5), we find

$$\mathbb{E}\Big[\int_{\mathbb{R}^d}(\widetilde{G}(x,\widetilde{\mu}_T^N)-\widetilde{G}(x,\widetilde{\mu}_T^K))^2dx + \int_0^T\int_{\mathbb{R}^d}(\widetilde{F}_t(x,\widetilde{\mu}_t^N)-\widetilde{F}_t(x,\widetilde{\mu}_t^K))^2dxdt\Big] \le C\mathbb{E}\Big[\omega^N\Big].$$

Finally, the uniform Lipschitz regularity of $\widetilde{F}$ and $\widetilde{G}$ in space and an elementary interpolation yield

$$\mathbb{E}\Big[\sup_x|\widetilde{G}(x,\widetilde{\mu}_T^N)-\widetilde{G}(x,\widetilde{\mu}_T^K)|^{d+2} + \int_0^T\sup_x|\widetilde{F}_t(x,\widetilde{\mu}_t^N)-\widetilde{F}_t(x,\widetilde{\mu}_t^K)|^{d+2}dt\Big] \le C\mathbb{E}\Big[\omega^N\Big].$$

□

Next we estimate the difference between $\widetilde{u}^N$ and $\widetilde{u}^K$.

Lemma 3.8. *The sequence $(\widetilde{u}^N)_{N\ge1}$ is a Cauchy with the respect to the family of seminorms*

$$\left(\sup_{t\in[0,T]}\mathbb{E}\left[\|\widetilde{u}_t\|^{d+1}_{L^\infty(B_R)}\right]\right)_{R>0}$$

Proof. Since the arguments are almost identical to the ones used in the proof of Theorem 2.3 and Proposition 2.5, here we only present a sketch.

Let $1<N<K$ and $\phi:\mathbb{R}\to\mathbb{R}^+$ smooth, Lipschitz continuous, convex and nonincreasing map, and set $w_t(x)=\phi(\widetilde{u}_t^N(x)-\widetilde{u}_t^K(x))$. Using induction and the convexity of ϕ to cancel the jump terms, which are martingales, for any $t\in[0,T)$ and $h\in(0,T-h)$, we find

$$\mathbb{E}\Big[w_{t+h}(x)-w_t(x)\Big]$$

$$\ge \mathbb{E}\Big[\int_t^{t+h}\phi'(\widetilde{u}_s^N(x)-\widetilde{u}_s^K(x))(\widetilde{H}_s^N(D\widetilde{u}^N(x)_s,x)-\widetilde{F}_s^N(x,\widetilde{\mu}_s^N)$$

$$-\widetilde{H}_s^K(D\widetilde{u}_s^K(x),x)+\widetilde{F}_s^N(x,\widetilde{\mu}_s^K))ds\Big] \ge \mathbb{E}\Big[\int_t^{t+h}(\mathbf{b}_s(x)\cdot Dw_s(x)+\zeta_s(x))ds\Big],$$

where

$$\zeta_s(x)=\phi'(\widetilde{u}_s^N(x)-\widetilde{u}_s^K(x))(\widetilde{H}_s^N(D\widetilde{u}_s^K(x),x)-\widetilde{H}_s^K(D\widetilde{u}_s^K(x),x)-\widetilde{F}_s^N(x,\widetilde{\mu}_s^K)+\widetilde{F}_s^N(x,\widetilde{\mu}_s^N))$$

and

$$\mathbf{b}_s(x)=\int_0^1 D_p\widetilde{H}_s^N((1-\lambda)D\widetilde{u}_s^N(x)+\lambda D\widetilde{u}_s^K(x))d\lambda.$$

For some $\alpha, \beta > 0$ to be chosen below, let

$$e_t = \mathbb{E}\left[\int_{B_{\alpha+\beta t}} w_t(y)dy\right].$$

As in the proof of Theorem 2.3 and for β large enough, but independent of N and K, we get

$$e_t \le C(e_T + \mathbb{E}\Big[\int_t^T \int_{B_{\alpha+\beta s}} \zeta_s(y)dyds\Big]).$$

Choosing, after approximation, $\phi(s) = (-s)_+$, we derive that, for all $t \in [0,T]$, ω^N is as in (3.9), and constants $C > 0$ and $C_{\alpha,\beta} = C(\alpha,\beta) > 0$,

$$\mathbb{E}\left[\int_{B_{\alpha+\beta t}} (-(\widetilde{u}_t^N(y) - \widetilde{u}_t^K(y)))_+ dy\right] \le C\mathbb{E}\left[\int_{B_{\alpha+\beta t}} (-(\widetilde{G}(y,\widetilde{\mu}_T^N) - \widetilde{G}(y,\widetilde{\mu}_T^K)))_+ dy\right]$$
$$+ C\mathbb{E}\Big[\int_t^T \int_{B_{\alpha+\beta s}} |\widetilde{F}_s(y,\widetilde{\mu}_T^N) - \widetilde{F}_s(y,\widetilde{\mu}_T^K)|dyds\Big] + C_{\alpha,\beta}\mathbb{E}[\omega^N].$$

Reversing the roles of u^N and u^K, we then obtain, for all $t \in [0,T]$,

$$\mathbb{E}\left[\int_{B_{\alpha+\beta t}} |\widetilde{u}_t^N(y) - \widetilde{u}_t^K(y)|dy\right] \le C\mathbb{E}\left[\int_{B_{\alpha+\beta t}} |\widetilde{G}(y,\widetilde{\mu}_T^N) - \widetilde{G}(y,\widetilde{\mu}_T^K)|dy\right]$$
$$+ C\mathbb{E}\Big[\int_t^T \int_{B_{\alpha+\beta s}} |\widetilde{F}_s(y,\widetilde{\mu}_T^N) - \widetilde{F}_s(y,\widetilde{\mu}_T^K)|dyds\Big] + C_{\alpha,\beta}\mathbb{E}[\omega^N].$$

It follows from Lemma 3.7, for some $\varepsilon_{\alpha,\beta}(N) \to 0$ as $N \to +\infty$,

$$\mathbb{E}\left[\int_{B_{\alpha+\beta t}} |\widetilde{u}_t^N(y) - \widetilde{u}_t^K(y)|dy\right] \le \varepsilon_{\alpha,\beta}(N).$$

The uniform Lipschitz estimate for the $\widetilde{u}^N$ gives the result. □

We have now established all the ingredients needed for the proof of the existence and uniqueness of solution of the stochastic MFG system.

Proof of Theorem 3.3. Lemma 3.8 and the properties of $\widetilde{F}$ and $\widetilde{G}$ yield that the sequences $(\widetilde{u}^N)_{N\in\mathbb{N}}$, $(\widetilde{F}(\cdot,\widetilde{\mu}_\cdot^N))_{N\in\mathbb{N}}$ and $\widetilde{G}(\cdot,\widetilde{\mu}_T^N)_{N\in\mathbb{N}}$ are Cauchy, with respective limits $\widetilde{u}$, $\widetilde{f}$ and $\widetilde{g}$.

It follows, as in the proof of Theorem 2.3, that the sequences $(D\widetilde{u}^N)_{N\in\mathbb{N}}$ and $(\widetilde{M}^N)_{N\in\mathbb{N}}$ also converge, as $N \to \infty$, to $D\widetilde{u}$ and $\widetilde{M}$ respectively, $\widetilde{M}$ is a continuous process, and, in addition, $(\widetilde{u}, \widetilde{M})$ solves

$$d\widetilde{u}_t = (\widetilde{H}_t(D\widetilde{u}_t, x) - \widetilde{f}_t(x))dt + d\widetilde{M}_t \text{ in } \mathbb{R}^d \times [0,T) \quad \widetilde{u}_T = \widetilde{g} \text{ on } \mathbb{R}^d.$$

Next we need to check that the sequence $(\widetilde{\mu}^N)_{N\in\mathbb{N}}$ has a limit $\widetilde{\mu}$ and that $\widetilde{f}_t(x) = \widetilde{F}_t(x,\widetilde{\mu}_t)$ and $\widetilde{g}(x) = \widetilde{G}(x,\widetilde{\mu}_T)$.

Fix ω for which $\widetilde{u}^N$ converges to $\widetilde{u}$ locally uniformly and $D\widetilde{u}^N$ converges to $D\widetilde{u}$ a.e.. In view of the bound on $(\widetilde{\mu}^N)_{N\in\mathbb{N}}$, the sequence $(\widetilde{\mu}^N)_{N\in\mathbb{N}}$ is relatively compact in $C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ and in L^∞ weak-*. So we can find a subsequence, which we denote in the same way, which

converges, in $C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ and in $L^\infty-$weak-*, to some $\widetilde{\mu}$, which is a bounded solution of the continuity equation

$$\partial_t\widetilde{\mu}_t = \mathrm{div}(\widetilde{\mu}_t D_p\widetilde{H}_t(D\widetilde{u}_t,x)) \text{ in } \mathbb{R}^d\times(0,TT \quad \widetilde{\mu}_0=\overline{m}_0 \text{ on}\mathbb{R}^d.$$

In addition, the continuity of $\widetilde{F}_t$ with respect to the measure argument implies that $(\widetilde{F}_t(x,\widetilde{\mu}_t^N))$ converges to $\widetilde{F}_t(x,\widetilde{\mu}_t)$ for any $(x,t)\in[0,T]\times\mathbb{R}^d$. Hence,

$$\widetilde{F}_t(x,\widetilde{\mu}_t)=\widetilde{f}_t(x). \tag{3.10}$$

Since (3.10) holds true along any converging subsequence, we infer from the strict monotonicity of $\widetilde{F}_t$ that the compact sequence $(\widetilde{\mu}^N)_{N\in\mathbb{N}}$ has a unique accumulation point in $C([0,T];\mathcal{P}_1(\mathbb{R}^d))$, and, thus it converges a.s. to an adapted and bounded process $\widetilde{\mu}$ satisfying the continuity equation.

It follows that the sequences $(\widetilde{F}_t^N(x,\widetilde{\mu}_t^N))_{N\in\mathbb{N}}$ and $(\widetilde{G}^N(x,\widetilde{\mu}_T^N))_{N\in\mathbb{N}}$ converge locally uniformly to $\widetilde{F}_t(x,\widetilde{\mu}_t)$ and $\widetilde{G}(x,\widetilde{\mu}_T)$ respectively. We can therefore conclude that the pair $(\widetilde{u},\widetilde{M},\widetilde{\mu})$ is a solution to the MFG system (1.6).

Since the proof of the uniqueness of solutions follows the standard argument, we only sketch it. Let $(\widetilde{u}^1,\widetilde{M}^1,\widetilde{m}^1)$ and $(\widetilde{u}^2,\widetilde{M}^2,\widetilde{m}^2)$ be two solutions of (1.6). We show that

$$\begin{aligned}\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^d}(\widetilde{F}_t(x,\widetilde{m}_t^1)-\widetilde{F}_t(x,\widetilde{m}_t^2))(\widetilde{m}_t^1-\widetilde{m}_t^2)(dx)dt \\ +\int_{R^d}(\widetilde{G}(x,\widetilde{m}_T^1)-\widetilde{G}(x,\widetilde{m}_T^2))(\widetilde{m}_T^1-\widetilde{m}_T^2)(dx)\Big]\le 0.\end{aligned} \tag{3.11}$$

For this, for $n=0,\dots,N$, let $t_n^N=nT/N$, and note that, in view of the equation satisfied by $\widetilde{u}=\widetilde{u}^1-\widetilde{u}^2$, we have, letting $\widetilde{M}=\widetilde{M}^1-\widetilde{M}^2$,

$$\begin{aligned}\widetilde{u}_{t_{n+1}^N}(x)-\widetilde{u}_{t_n^N}(x) = &-\int_{t_n^N}^{t_{n+1}^N}(\widetilde{H}_t(D\widetilde{u}^1,x)-\widetilde{F}_t(x,\widetilde{m}^1(t))-\widetilde{H}_t(D\widetilde{u}^2,x)+\widetilde{F}_t(x,\widetilde{m}^2(t))dt\\ &-(\widetilde{M}_{t_{n+1}^N}(x)-\widetilde{M}_{t_n^N}(x)).\end{aligned}$$

Let $\widetilde{m}=\widetilde{m}^1-\widetilde{m}^2$. Integrating the equality above against $\widetilde{m}_{t_n^N}$ and summing over n gives

$$\begin{aligned}&\sum_{n=0}^{N-1}\int_{\mathbb{R}^d}(\widetilde{u}_{t_{n+1}^N}(x)-\widetilde{u}_{t_n^N}(x))\widetilde{m}_{t_n^N}(x)dx\\ &\quad=-\sum_{n=0}^{N-1}\int_{t_n^N}^{t_{n+1}^N}\int_{\mathbb{R}^d}(\widetilde{H}_t(D\widetilde{u}^1(x),x)-\widetilde{F}_t(x,\widetilde{m}_t^1)-\widetilde{H}_t(D\widetilde{u}_t^2(x),x)+\widetilde{F}_t(x,\widetilde{m}_t^2))\widetilde{m}_{t_n^N}(x)dxdt\\ &\qquad-\sum_{n=0}^{N-1}\int_{\mathbb{R}^d}(\widetilde{M}_{t_{n+1}^N}(x)-\widetilde{M}_{t_n^N}(x))\widetilde{m}_{t_n^N}(x)dx.\end{aligned}$$

After reorganizing the left-hand side above by taking into account the equation satisfied by $\widetilde{m}$ yields

$$\begin{aligned}
&\sum_{n=0}^{N-1}\int_{\mathbb{R}^d}(\widetilde{u}_{t_{n+1}^N}(x)-\widetilde{u}_{t_n^N}(x))\widetilde{m}_{t_n^N}(x)dx\\
&=\int_{\mathbb{R}^d}\widetilde{u}_T(x)\widetilde{m}_{t_{N-1}^N}(x)dx-\int_{\mathbb{R}^d}\widetilde{u}_0(x)\widetilde{m}_0(x)dx-\sum_{n=1}^{N-1}\int_{\mathbb{R}^d}\widetilde{u}_{t_n^N}(x)(\widetilde{m}_{t_n^N}(x)-\widetilde{m}_{t_{n-1}^N}(x))dx\\
&=\int_{\mathbb{R}^d}(\widetilde{G}(x,\widetilde{m}_T^1)-\widetilde{G}(x,\widetilde{m}_T^2))(\widetilde{m}^1_{t_{N-1}^N}(x)-\widetilde{m}^2_{t_{N-1}^N}(x))dx\\
&\quad+\sum_{n=1}^{N-1}\int_{t_{n-1}^N}^{t_n^N}\int_{\mathbb{R}^d}D\widetilde{u}_{t_n^N}(x)\cdot(D_p\widetilde{H}_t(D\widetilde{u}_t^1(x),x)\widetilde{m}_t^1(x)-D_p\widetilde{H}_t(D\widetilde{u}_t^2(x),x)\widetilde{m}_t^2(x))dxdt.
\end{aligned}$$

We let $N\to+\infty$ and find, after taking expectation,

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\mathbb{R}^d}(\widetilde{G}(x,\widetilde{m}_t^1)-\widetilde{G}(x,\widetilde{m}_T^2))(\widetilde{m}_T^1(x)-\widetilde{m}_T^2(x))dx\\
&\qquad+\int_0^T\int_{\mathbb{R}^d}D\widetilde{u}_t(x)\cdot(D_p\widetilde{H}_t(D\widetilde{u}_t^1(x),x)\widetilde{m}_t^1(x)-D_p\widetilde{H}_t(D\widetilde{u}_t^2(x),x)\widetilde{m}_t^2(x))dxdt\Big]\\
&=-\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^d}(\widetilde{H}_t(D\widetilde{u}_t^1(x),x)-\widetilde{F}_t(x,\widetilde{m}_t^1)-\widetilde{H}_t(D\widetilde{u}_t^2(x),x)+\widetilde{F}_t(x,\widetilde{m}_t^2))\widetilde{m}_t(x)dxdt\Big].
\end{aligned}$$

We can now rearrange the expression in the usual way, and taking into account the convexity of $\widetilde{H}_t=\widetilde{H}_t(p,x)$ in p, to conclude that (3.11) holds.

Using the strict monotonicity assumption on $\widetilde{F}$ we infer that, $\mathbb{P}-$a.s. and for a.e. $(x,t)\in[0,T]\times\mathbb{R}^d$, $\widetilde{m}_t^1=\widetilde{m}_t^2$. Thus $\widetilde{u}^1$ and $\widetilde{u}^2$ solve the same HJ equation. It follows from Proposition (2.5) that $\widetilde{u}^1=\widetilde{u}^2$.

The equality $\widetilde{M}^1=\widetilde{M}^2$ follows from the equation satisfied by the $\widetilde{u}^i$. □

3.6. **Application to $N-$player differential games.** We consider here a game with N players and show that, if N is large enough, the optimal controls given by the solution of the stochastic MFG system (1.6) provide an approximate Nash equilibrium for the game.

We begin with the notation, terminology and the general setting. In what follows, $N\in\mathbb{N}$, $\overline{m}_0\in\mathcal{P}_2(\mathbb{R}^d)$ with an $L^\infty-$density, $(Z^i)_{i\in\mathbb{N}}$ is a sequence of independent random initial conditions on $\mathbb{R}^d$ with law $\overline{m}_0$, and W is a Brownian motion independent of the $(Z^i)_{i\in\mathbb{N}}$.

The state X^{α^i} of the $i-$th player satisfies, for $i=1,\dots,N$, the stochastic differential equation

$$dX_t^{\alpha^i}=\alpha_t^i dt+\sqrt{2\beta}dW_t\ \text{ in }\ [0,T]\qquad X_0^{\alpha^i}=Z^i,\tag{3.12}$$

with α^i an admissible control of player i, that is, an $\mathbb{R}^d-$valued measurable process adapted to the filtration generated by $(W_s)_{s\le t}$ and the $(Z^j)_{j\in\mathbb{N}}$, and such that $\mathbb{E}[\int_0^T|\alpha_t^i|^2dt]<+\infty$. Note that the noise W is the same for all the players.

The cost of player i, associated to the admissible control α^i and given the admissible controls of the $(\alpha^j)_{j\neq i}$ of the other players, is

$$J^{N,i}(\alpha^i,(\alpha^j_{j\neq i}))=\mathbb{E}\left[\int_0^T(L(\alpha_t^i,X_t^{\alpha^i})+F(X_t^{\alpha^i},m^{N,i}_{\mathbf{X}_t}))dt+g(X_T^{\alpha^i},m^{N,i}_{\mathbf{X}_T^N})\right],$$

where $\mathbf{X}_t = (X_t^{\alpha^1}, \dots, X_t^{\alpha^N})$ with X^{α^j} a solution of (3.12) and

$$m_{\mathbf{X}_t}^{N,i} = \frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \delta_{X_t^{\alpha^j}}.$$

Given $\varepsilon > 0$, we say that the family $(\overline{\alpha}^i)_{i=1,\dots,N}$ of admissible controls is an $\epsilon-$ Nash equilibrium of the game, if, for any $i = 1, \dots, N$ and for any admissible control α^i of the player i,

$$J^{N,i}(\overline{\alpha}^i, (\overline{\alpha}^j)_{j\neq i}) \leq J^{N,i}(\alpha^i, (\overline{\alpha}^j)_{j\neq i}) + \varepsilon.$$

The associated Hamiltonian of is

$$H(p,x) = \sup_{\alpha\in\mathbb{R}^d} [-p\cdot\alpha - L(\alpha, x)].$$

We assume that

$$\begin{cases} F, G : \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R} \text{ are globally Lipschitz continuous and, for some } C_0 > 0, \\ \qquad \sup_{m\in\mathcal{P}_1} \|F(\cdot, m)\|_{C^2} + \|G(\cdot, m)\|_{C^2} \leq C_0, \end{cases} \tag{3.13}$$

F and G are strongly monotone with constant α and F is strictly monotone (3.14)

and the Hamiltonian H is of the form

$$\begin{cases} H(p,x) = \frac{a(x)}{2}|p|^2 + B(x)\cdot p \\ \text{with } a \in C^2(\mathbb{R}^d),\ B \in C^2(\mathbb{R}^d;\mathbb{R}^d) \text{ and } C_0^{-1} \leq a(x) \leq C_0. \end{cases} \tag{3.15}$$

It is then immediate that

$$L(\alpha, x) = \frac{1}{2a(x)}|\alpha + B(x)|^2.$$

As in the earlier parts of the paper, we set

$$\begin{cases} \widetilde{H}_t(x,p) = H(p, x + \sqrt{2\beta}W_t), \\ \widetilde{F}_t(x,m) = F(x + \sqrt{2\beta}W_t, (id + \sqrt{2\beta}W_t)\sharp m_t) \ \text{ and} \\ \widetilde{G}(x) = G(x - \sqrt{2\beta}W_T, (id + \sqrt{2\beta}W_T)\sharp m_T). \end{cases} \tag{3.16}$$

In view of the conditions above, $\widetilde{H}$, $\widetilde{F}$, $\widetilde{G}$ and $\overline{m}_0$ satisfy (MFG1), (MFG2), (MFG3), (MFG4), (MFG5), and (MFG6).

We denote by $(\widetilde{u}, \widetilde{M}, \widetilde{m})$ the solution of (1.6), and recall that Proposition 2.13 implies, for a.e. $x \in \mathbb{R}^d$, the existence of a family $(\overline{\gamma}_t^x)_{t\in[0,T]}$ of adapted processes which minimize

$$\widetilde{u}_0(x) = \inf_{\gamma_0 = x} \mathbb{E}\left[\int_t^T (\widetilde{L}_s(\dot\gamma_s, \gamma_s) + \widetilde{F}_s(\gamma_s, \widetilde{m}_s))ds + \widetilde{G}(\gamma_T, \widetilde{m}_T)\right], \tag{3.17}$$

where

$$\widetilde{L}_t(p,\alpha,x,\omega) = \frac{1}{2\widetilde{a}_t(x)}|p + \widetilde{B}_t(x)|^2 \ \text{ and } \ \widetilde{a}_t(x) = a(x + \sqrt{2\beta}W_t),\ \widetilde{B}_t(x) = B(x + \sqrt{2\beta}W_t).$$

Set $\overline{\alpha}_t^x = \dot{\overline{\gamma}}_t^x = -D_p\widetilde{H}_t(D\widetilde{u}_t(\overline{\gamma}_t^x), \overline{\gamma}_t^x)$.

Proposition 3.9. *Assume* (MFG1), (3.13), (3.14) *and* (3.15). *Then, for any $\varepsilon > 0$, there exists $N_0 = N_0(\varepsilon) \in \mathbb{N}$ such that, for any $N \geq N_0$, the family of random controls $(\overline{\alpha}^{Z^i})_{i=1,\dots,N}$ is an $\epsilon-$Nash equilibrium of the game.*

Proof. To simplify the notation, in what follows we set $\overline{X}^i = X^{\overline{\alpha}^{Z^i}}$ and note that

$$\overline{X}^i_t = Z^i + \int_0^t \overline{\alpha}^{Z^i}_s ds + \sqrt{2\beta} W_t = \overline{\gamma}^{Z^i}_t + \sqrt{2\beta} W_t.$$

We check that the conditional law of $(\overline{X}^i_t)_{t\in[0,T]}$ given $(W_t)_{t\in[0,T]}$ is $m_t = (Id + \sqrt{2\beta}W_t)\sharp \widetilde{m}_t$. Indeed, in view of Proposition II-2.7 of [14] and since $\overline{\gamma}^{Z^i}$ solves the ODE

$$\dot{\overline{\gamma}}^{Z^i}_t = -D_p \widetilde{H}_t(D\widetilde{u}_t(\overline{\gamma}^{Z^i}_t), \overline{\gamma}^{Z^i}_t),$$

the conditional law $\widetilde{\mu}_t$ of $(\overline{\gamma}^{Z^i}_t)$ given W is a solution in the sense of distributions of the continuity equation

$$\partial_t \widetilde{\mu}_t - \operatorname{div}(\widetilde{\mu}_t D_p \widetilde{H}_t(D\widetilde{u}_t(x), x)) = 0 \ \text{ in } \ \mathbb{R}^d \times (0,T) \qquad \widetilde{\mu}_0 = \overline{m}_0.$$

It follows from Proposition 2.12 that this solution is unique. Therefore, the conditional law $\widetilde{\mu}_t$ of $(\overline{\gamma}^{Z^i}_t)_{t\in[0,T]}$ given W is $\widetilde{m}_t$. Since $\overline{X}^i_t = \overline{\gamma}^{Z^i}_t + \sqrt{2\beta}W_t$, this implies that the conditional law of $\overline{X}^i_t$ given W is m_t.

Since the Z^i's and W are independent, it is clear that the $\overline{X}^i$'s are conditionally independent and have the same law m given W. It then follows from the Glivenko-Cantelli law of large numbers that, $\mathbb{P}-$a.s.,

$$\lim_{N\to+\infty} \mathbb{E}\left[\mathbf{d}_2(m^{N,i}_{\overline{\mathbf{X}}_t}, m_t) \mid W\right] = 0.$$

In view of the Lipschitz continuity and boundedness of F and G, the limit above implies that, for any (x,t),

$$\lim_{N\to+\infty} \mathbb{E}\left[|F(x, m^{N,i}_{\overline{\mathbf{X}}_t}) - F(x,m_t)| + |G(x, m^{N,i}_{\overline{\mathbf{X}}_T}) - G(x,m_T)|\right] = 0.$$

As the integrant is uniformly continuous in x uniformly in t and N and has a uniform modulus in t, which in expectation is uniform in x and N, we deduce that, for any $R>0$ and $\varepsilon>0$, there exists $N_R \in \mathbb{N}$ such that, if $N \geq N_R$,

$$\mathbb{E}\left[\sup_{x\in B_R, t\in[0,T]} |F(x, m^{N,i}_{\overline{\mathbf{X}}_t}) - F(x,m_t)| + |G(x, m^{N,i}_{\overline{\mathbf{X}}_T}) - G(x,m_T)|\right] \leq \frac{\varepsilon}{4(T+1)}. \tag{3.18}$$

Note that, since $D\widetilde{u}$ is bounded, there exists $M>0$ such that $\overline{\alpha}^{Z^i}_t = -D_p\widetilde{H}_t(D\widetilde{u}_t(\overline{\gamma}^{Z^i}_t, t), \overline{\gamma}^{Z^i}_t)$ is bounded by M and, thus, we have

$$\begin{cases} J^{N,i}(\overline{\alpha}^{Z^i}, (\overline{\alpha}^{Z^j})_{j\neq i}) = \mathbb{E}\Big[\int_0^T \frac{1}{2a_t(\overline{X}^i_t)} |\overline{\alpha}^{Z^i}_t + B(\overline{X}^i_t)|^2 + F(\overline{X}^i_t, m^{N,i}_{\overline{\mathbf{X}}_t}) dt \\ \qquad\qquad + G(\overline{X}^i_T, m^{N,i}_{\overline{\mathbf{X}}_T})\Big] \leq T(C_0M^2 + C_0\|B\|^2_\infty + \|F\|_\infty) + \|G\|_\infty. \end{cases} \tag{3.19}$$

Let α^i be an admissible control for player i. To prove the claim, it is necessary to estimate $J^{N,i}(\alpha^i, (\overline{\alpha}^{Z^j})_{j\neq i})$ in terms $\mathbb{E}[\int_0^T |\alpha^i_s|^2 ds]$.

In what follows, we introduce the constant

$$A = \max\left[4TC_0(C_0M^2 + 2C_0\|B\|^2_\infty + 2\|F\|_\infty) + 8C_0\|G\|_\infty \ , \ \mathbb{E}[\int_0^T |\overline{\alpha}^{Z^i}_s|^2 ds]\right],$$

which is independent of N and i, since the law of $\overline{\alpha}^{Z^i}_s$ does not depend on i.

If $\mathbb{E}[\int_0^T |\alpha_s^i|^2 ds] \geq A$, then, in view of (3.19) and the choice of A, we find

$$\begin{aligned} J^{N,i}(\alpha^i, (\overline{\alpha}^{Z^j})_{j\neq i}) &= \mathbb{E}\left[\int_0^T \frac{1}{2a_t(X_t^{\alpha^i})}|\alpha_t^i + B(X_t^{\alpha^i})|^2 + F(X_t^{\alpha^i}, m_{\overline{\mathbf{X}}_t}^{N,i})dt + G(X_T^{\alpha^i}, m_{\overline{\mathbf{X}}_T}^{N,i})\right] \\ &\geq \frac{1}{4C_0}\mathbb{E}\left[\int_0^T |\alpha_t^i|^2 dt\right] - TC_0\|B\|_\infty^2 - T\|F\|_\infty - \|G\|_\infty \\ &\geq A/(4C_0) - TC_0\|B\|_\infty^2 - T\|F\|_\infty - \|G\|_\infty \geq J^{N,i}(\overline{\alpha}^{Z^i}, (\overline{\alpha}^{Z^j})_{j\neq i}). \end{aligned}$$

If $\mathbb{E}[\int_0^T |\alpha_s^i|^2 ds] \leq A$, an estimate which is satisfied by $\overline{\alpha}^{Z^i}$, then, for any $R > 0$, we obtain

$$\begin{cases} \mathbb{P}\left[\sup_{t\in[0,T]} |X_t^{\alpha^i}| \geq R\right] \\ \leq \mathbb{P}\left[|Z^i| \geq R/3\right] + \mathbb{P}\left[\int_0^T |\alpha_t^i| dt \geq R/3\right] + \mathbb{P}\left[\sqrt{2\beta}\sup_{t\in[0,T]} |W_t| \geq R/3\right] \\ \leq 9R^{-2}\left(\mathbb{E}[|Z^i|^2] + T\mathbb{E}\left[\int_0^T |\alpha_t^i|^2 dt\right] + 2\beta\mathbb{E}\left[\sup_{t\in[0,T]} |W_t|^2\right]\right) \leq CR^{-2}(1+A). \end{cases} \tag{3.20}$$

We fix R large enough to be chosen below and $N \geq N_R$ as in (3.18). Then

$$\begin{aligned} J^{N,i}(\alpha^i, (\overline{\alpha}^{Z^j})_{j\neq i}) &= \mathbb{E}\left[\int_0^T \frac{1}{2a(X_t^{\alpha^i})}|\alpha_t^i + B(X_t^{\alpha^i})|^2 + F(X_t^{\alpha^i}, m_{\overline{\mathbf{X}}_t}^{N,i})dt + G(X_T^{\alpha^i}, m_{\overline{\mathbf{X}}_T}^{N,i})\right] \\ &\geq \mathbb{E}\left[\int_0^T \frac{1}{2a(X_t^{\alpha^i})}|\alpha_t^i + B(X_t^{\alpha^i})|^2 + F(X_t^{\alpha^i}, m_t)dt + G(X_T^{\alpha^i}, m_T)\right] \\ &\quad - \mathbb{E}\left[T\sup_t |F(X_t^{\alpha^i}, m_{\overline{\mathbf{X}}_t}^{N,i}) - F(X_t^{\alpha^i}, m_t)| + |G(X_T^{\alpha^i}, m_{\overline{\mathbf{X}}_T}^{N,i}) - G(X_T^{\alpha^i}, m_T)|\right], \end{aligned}$$

where, in view of (3.18) and (3.20) and R suffciently large,

$$\begin{aligned} &\mathbb{E}\left[T\sup_{t\in[0,T]} |F(X_t^{\alpha^i}, m_{\overline{\mathbf{X}}_t}^{N,i}) - F(X_t^{\alpha^i}, m_t)| + |G(X_T^{\alpha^i}, m_{\overline{\mathbf{X}}_T}^{N,i}) - G(X_T^{\alpha^i}, m_T)|\right] \\ &\quad \leq (T+1)\mathbb{E}\left[\sup_{x\in B_R, t\in[0,T]} |F(x, m_{\overline{\mathbf{X}}_t}^{N,i}) - F(x, m_t)| + |G(x, m_{\overline{\mathbf{X}}_T}^{N,i}) - G(x, m_T)|\right] \\ &\qquad + 2(T+1)(\|F\|_\infty + \|G\|_\infty)\mathbb{P}\Big[\sup_{t\in[0,T]} |X_t^\alpha| \geq R\Big] \leq \frac{\varepsilon}{4} + \frac{C}{R^2}(1+A) < \frac{\varepsilon}{2}. \end{aligned}$$

It follows that

$$J^{N,i}(\alpha^i, (\overline{\alpha}^{Z^j})_{j\neq i}) \geq \mathbb{E}\left[\int_0^T \frac{1}{2a(X_t^{\alpha^i})}|\alpha_t^i + B(X_t^{\alpha^i})|^2 + F(X_t^{\alpha^i}, m_t)dt + G(X_T^{\alpha^i}, m_T)\right] - \frac{\varepsilon}{2}.$$

Therefore, setting $\gamma_t^{\alpha^i} := X_t^{\alpha^i} - \sqrt{2\beta}W_t$, we get

$$J^{N,i}(\alpha^i, (\overline{\alpha}^{Z^j})_{j\neq i}) \geq \mathbb{E}\left[\int_0^T \widetilde{L}(\alpha_t^i, \gamma_t^{\alpha^i}) + \widetilde{F}(\gamma_t^{\alpha^i}, \widetilde{m}_t)dt + \widetilde{G}(\gamma_T^{\alpha^i}, \widetilde{m}_T)\right] - \frac{\varepsilon}{2}.$$

The same argument, with an estimate from above instead of an estimate from below, shows that

$$J^{N,i}(\overline{\alpha}^{Z^i},(\overline{\alpha}^{Z^j})_{j\neq i}) \leq \mathbb{E}\left[\int_0^T \widetilde{L}(\overline{\alpha}_t^{Z^i},\overline{\gamma}_t^{Z^i}) + \widetilde{F}(\overline{\gamma}_t^{Z^i},\widetilde{m}_t)dt + \widetilde{G}(\overline{\gamma}_T^{Z^i},\widetilde{m}_T)\right] + \frac{\varepsilon}{2}.$$

Since, in view of the optimality of $\overline{\alpha}^{Z^i}$ in (3.17), we also have

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \widetilde{L}(\alpha_t^i,\gamma_t^{\alpha^i}) + \widetilde{F}(\gamma_t^{\alpha^i},\widetilde{m}_t)dt + \widetilde{G}(\gamma_T^{\alpha^i},\widetilde{m}_T)\right]\\ &\qquad \geq \mathbb{E}\left[\int_0^T \widetilde{L}(\overline{\alpha}_t^{Z^i},\overline{\gamma}_t^{Z^i}) + \widetilde{F}(\overline{\gamma}_t^{Z^i},\widetilde{m}_t)dt + \widetilde{G}(\overline{\gamma}_T^{Z^i},\widetilde{m}_T)\right],\end{aligned}$$

combining the three last inequalities we conclude that $(\overline{\alpha}^{Z^i})_{i=1,\dots,N}$ is an $\epsilon-$Nash equilibrium. □

Appendix A. A uniqueness result for a continuity equation

We study here the uniqueness of distributional solutions of the forward continuity equation

$$\partial_t m + \operatorname{div}(mb) = 0 \ \text{ in } \ \mathbb{R}^d\times(0,T) \quad m(0)=\overline{m}_0 \ \text{ in } \ \mathbb{R}^d, \tag{A.1}$$

where $\overline{m}_0$ is a Borel probability measure on $\mathbb{R}^d$ with a bounded density and $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ is a bounded, half-Lipschitz from below Borel vector field, that is, there is a constant C_0 such that, for all $x,y\in\mathbb{R}^d$ and $t\in[0,T]$,

$$|b_t(x)|\leq C_0 \ \text{ and } \ (b(t,x)-b(t,y))\cdot(x-y)\geq -C_0|x-y|^2. \tag{A.2}$$

The existence and uniqueness of distributional solution of (A.1) is closely related to the existence and uniqueness of solutions of the ODE

$$\dot{x}_t = b_t(x_t) \ \text{ in } \ [0,t_0], \quad x(t_0)=x_0. \tag{A.3}$$

The following result is a variant of a theorem in [7]:

Proposition A.1. *Assume* (A.2) *and let* $\overline{m}_0\in\mathcal{P}_1(\mathbb{R}^d)$ *be absolutely continuous with bounded density. Then there exists a unique bounded and absolutely continuous with respect to the Lebesgue measure solution of* (A.1).

The proof requires several steps and is based on the Filippov regularization b^F of b; see Filippov [19]. Recall that b^F is an upper semicontinuous set-valued map with convex compact values. It follows from (A.2), that, if $\mathcal{N}=\{(x,t):\{b_t(x)\}\neq b_t^F(x)\}$, then

$$\mathcal{L}^{d+1}(\mathcal{N})=0. \tag{A.4}$$

It is known that, in view of (A.2), for any $(x_0,t_0)\in\mathbb{R}^d\times(0,T]$, there exists a unique absolutely continuous solution $X(x_0,t_0,\cdot)$ of the backward differential inclusion

$$\dot{x}_t\in b_t^F(x_t) \ \text{ in } \ [0,t_0], \quad x_{t_0}=x_0,$$

which is referred to as the Filippov solution of the ODE, and, for all $x_0,x_1\in\mathbb{R}^d$ and $t\in[0,t_0]$,

$$|X(x_0,t_0,t)-X(x_1,t_0,t)|\leq e^{C_0T}|x_0-x_1|. \tag{A.5}$$

Note that any smooth approximation b^ε of b obtained by, for example, a convolution with a nonnegative, smooth kernel, satisfies (A.2) with the same constant C_0.

It then follows that the classical backward flow $X^\varepsilon(x_0,t_0,\cdot)$ of

$$\dot{x}^\varepsilon_t = b^\varepsilon_t(x^\varepsilon_t) \ \text{ in } \ [0,t_0], \quad x^\varepsilon_{t_0} = x,$$

satisfies (A.5) with a uniform constant, hence, it converges locally uniformly to X.

Lemma A.2. *Assume* (A.2). *Then, for any $0 \le t \le t_0 \le T$, the map $x \to X(x,t_0,t)$ is surjective.*

Proof. Arguing by contradiction, we assume that, for some $t \in [0,t_0]$, there exists $y \in \mathbb{R}^d \backslash X(\mathbb{R}^d,t_0,t)$. The finite speed of propagation property and the continuity the flow yield some $\delta > 0$ such that $B_\delta(y) \cap X(\mathbb{R}^d,t_0,t) = \emptyset$.

Let $\overline{m}_0$ be a probability measure with a smooth density supported in $B_\delta(y)$ and b^ε a smooth approximation of b satisfying (A.2) with a constant independent of ε. Note that, for any $0 \le t \le t_0 \le T$, the map $x \to X^\varepsilon(x,t_0,t)$ is smooth and one-to-one.

We consider the classical solution m^ε to the continuity equation

$$\partial_t m^\varepsilon + \mathrm{div}(m^\varepsilon b^\varepsilon) = 0 \ \text{ in } \ \mathbb{R}^d \times (t,T) \quad m^\varepsilon(t) = \overline{m}_0.$$

Since $\mathrm{div}(b^\varepsilon) \ge -C_0$, we infer from the maximum principle that

$$\|m^\varepsilon\|_\infty \le \|\overline{m}_0\|_\infty e^{C_0 T}.$$

Passing (up to a subsequence) to the $\varepsilon \to 0$ limit, we obtain $m \in L^\infty(\mathbb{R}^d \times (0,T))$ solving

$$\partial_t m + \mathrm{div}(mb) = 0 \ \text{ in } \ \mathbb{R}^d \times (t,T) \quad m(t) = \overline{m}_0 \ \text{ in } \ \mathbb{R}^d, \tag{A.6}$$

in the sense of distributions.

Next, we use Ambrosio's superposition principle [2] which provides a connection between solutions of the continuity equation (A.6) and the ODE (A.3) in the form of a Borel measure η on $\Gamma = C([t,t_0],\mathbb{R}^d)$ which is concentrated on solutions of (A.3) and is such that $m(s) = e_s \sharp \eta$ for $s \in [t,T]$.

We claim that, for $\eta-$a.e. $\gamma \in \Gamma$, γ is a Filippov solution of (A.3). Indeed, it follows from (A.4) that, since m is absolutely continuous with respect to the Lebesgue measure,

$$0 = \int_t^T \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{N}}(x,s) m(x,s) dx ds = \int_\Gamma \int_t^T \mathbf{1}_{\mathcal{N}}(\gamma(s),s) ds \eta(d\gamma).$$

Hence, for $\eta-$a.e. $\gamma \in \Gamma$, $(\gamma_s,s) \notin \mathcal{N}$ for a.e. $s \in [t,T]$ and, therefore, for a.e. $s \in [t,T]$,

$$\dot{\gamma}_s = b_t(\gamma_s) \in b^F_s(\gamma_s).$$

Thus, for $\eta-$a.e. $\gamma \in \Gamma$, γ is a Filippov solution of (A.3), and, in view of the uniqueness of the backward solution, $\gamma(s) = X(\gamma(t_0),t_0,s)$. Since $\gamma(t)$ belongs to the support of $\overline{m}_0$ which is contained in $B_\delta(y)$, this leads to a contradiction. □

Lemma A.3. *Assume* (A.2). *Then, for any $s,t \in [0,T]$ with $s < t$, there exists a set $E_{s,t}$ of full $\mathcal{L}^d-$measure on which $X^{-1}(\cdot,t,s)$ is a singleton.*

Proof. The main step of the proof is the fact that, for any $x \in \mathbb{R}^d$, the set $X^{-1}(\{x\},s,t)$, which, in view of Lemma A.2, is nonempty, is connected.

Since $X^{-1}(\{x\},s,t)$ is compact, it suffices to show that, if O_1 and O_2 are two open subsets of $\mathbb{R}^d$ such that $X^{-1}(\{x\},s,t) \subset O_1 \cup O_2$ and $\overline{O}_1 \cap \overline{O}_2 = \emptyset$, then $X^{-1}(\{x\},s,t)$ is contained either in O_1 or in O_2.

Let O_1 and O_2 be as above. The upper-semicontinuity of $X^{-1}(\cdot, s, t)$, which is a consequence of the stability of the flow, yields some $r > 0$ such that $X^{-1}(B_r(x), s, t) \subset O_1 \cup O_2$.

Let b^ε and X^ε be respectively a smooth approximation of b and the associated group of solution. Then $(X^\varepsilon)^{-1}(B_r(x), s, t) = X^\varepsilon(B_r(x), t, s)$ is connected and, for ε small enough, is contained in $O_1 \cup O_2$. Thus, it is contained either in O_1 or in O_2. Without loss of generality, we can assume that there exists $\varepsilon_n \to 0$ such that $(X^{\varepsilon_n})^{-1}(B_r(x), s, t) \subset O_1$.

Passing to the limit up to this subsequence, we infer that

$$X^{-1}(\{x\}, s, t) \subset \bigcap_{r>0} \operatorname{limsup}(X^{\varepsilon_n})^{-1}(B_r(x), s, t) \subset \overline{O_1}.$$

Since $\overline{O}_1 \cap \overline{O}_2 = \emptyset$ and $X^{-1}(\{x\}, s, t) \subset O_1 \cup O_2$, this implies that $X^{-1}(\{x\}, s, t)$ is contained in O_1, and, hence, $X^{-1}(\{x\}, s, t)$ is connected.

The fact that $x \to X(x, s, t)$ is Lipschitz continuous and the area formula imply that $X^{-1}(\{x\}, s, t)$ is at most countable for a.e. $x \in \mathbb{R}^d$. Then the fact that $X^{-1}(\{x\}, s, t)$ is nonempty and connected implies that, as soon it is countable, $X^{-1}(\{x\}, s, t)$ must be a singleton for a.e. $x \in \mathbb{R}^d$. □

The next Lemma is about the existence and uniqueness of a forward solution of the Fillipov ODE.

Lemma A.4. *Assume* (A.2). *Then, there exists $E \subset R^d$ of full $\mathcal{L}^d-$measure such that, for any $x \in E$, there exists a unique forward maximal absolutely continuous solution of $\dot{x}_t \in b_t^F(x_t)$ on $[0, T]$ with $x_0 = x$.*

Proof. Let $(t_n)_{n\in N}$ be a countable and dense set of times in $[0, T]$ with $t_0 = T$ and $E = \bigcap_n E_{0,t_n}$, where $E_{s,t}$ is given in Lemma A.3. Note that E had a full $\mathcal{L}^d-$measure in $\mathbb{R}^d$. Then, for any $x \in E$, there exists a unique $y \in \mathbb{R}^d$ such that $X(y, T, 0) = x$.

We claim that $t \to X(y, T, t)$ is the claimed unique forward maximal solution. Indeed, by definition, it is a maximal solution. Assume that $x : [0, t^*) \to \mathbb{R}^d$ is another maximal solution defined on an interval $[0, t^*)$ with $t^* \in (0, T]$ and $x_0 = x$. Note that, the backward uniqueness of the flow, implies that $x_s = X(x_t, t, s)$ for any $0 \le s < t < t^*$.

Let $t_n \in (0, t^*)$. Then $t \to X(y, T, t)$ is a solution on $[0, t_n]$ starting from $X(y, T, t_n)$. Therefore x_{t_n} and $X(y, T, t_n)$ belong to $X^{-1}(\{x\}, t_n, 0)$, which is a singleton by the definition of E_{0,t_n}. It follows that $x_{t_n} = X(y, T, t_n)$. So $x_t = X(y, T, t)$ on $[0, t^*)$, and, hence, the uniqueness. □

We are now in a position to complete the proof of the existence and uniqueness of bounded and absolutely continuous distributional solutions of the continuity equation.

Proof of Proposition A.1. Since the existence of a bounded solution of the continuity equation can be achieved by standard approximation, we concentrate only on the uniqueness.

Let m be an absolutely continuous solution to the continuity equation with initial condition $\overline{m}_0$. The Ambrosio superposition Theorem [2] yields a measure η on Γ such that $m(t) = e_t \sharp \eta$, where $e_t(\gamma) = \gamma_t$, and for $\eta-$a.e. $\gamma \in \Gamma$, γ is an absolutely continuous solution to the ODE $\dot{\gamma}_t = b_t(\gamma_t)$.

Arguing as in the proof of Lemma A.2, we find that, that $\eta-$a.e. $\gamma \in \Gamma$ is a Filippov solution of the ODE.

We now disintegrate η with respect to $\overline{m}_0$ into $\eta(d\gamma) = \int_{\mathbb{R}^d} \eta_x(d\gamma)\overline{m}_0(x)dx$ in such a way that, for $\overline{m}_0-$a.e. $x \in \mathbb{R}^d$ and η_x-a.e. $\gamma \in \Gamma$, $\gamma_0 = x$. Since, by Lemma A.4, the Filippov solution to the ODE is unique for a.e. $x \in \mathbb{R}^d$, we obtain that, for $\overline{m}_0-$a.e. $x \in \mathbb{R}^d$, η_x is a Dirac mass. The uniqueness of the bounded and absolutely continuous solution of the continuity equation then easily follows from [2]. □

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