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► **To cite this version:**

M. A. Bertolim, C. Bonatti, M. P. Mello, G. M. Vago. Minimal number of periodic orbits for nonsingular Morse-Smale flows in odd dimension. 2023. hal-02946281v2

HAL Id: hal-02946281

<https://hal.science/hal-02946281v2>

Preprint submitted on 19 Oct 2023

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Minimal number of periodic orbits for non-singular Morse-Smale flows in odd dimension

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Thursday 19th October, 2023

Abstract

We consider the couples (M, Φ) where M is an odd-dimensional compact manifold with boundary, endowed with a non-singular Morse-Smale flow Φ , satisfying some *given* homological boundary information. We compute, in terms of such given homological information, a number p_{min} such that *any* non-singular Morse-Smale flow Φ on *any* manifold M satisfying the given abstract homological data must have *at least* p_{min} closed periodic orbits.

Moreover, we can provide, for any initial homological data, a non-singular Morse-Smale model (M_0, Φ_0) for which p_{min} is attained. Note that in the general case of a couple (M, Φ) satisfying the given homological information, such a number p_{min} is a lower bound.

The algorithm underlying this computation is based on optimization theory in network flows and transport systems.

Keywords: Morse-Conley theory, non-singular Morse-Smale flows, periodic orbits.

AMS Mathematics Subject Classification: 37B30, 37C27, 37D15.

*Supported by GDRI-RFBM (Réseau Franco-Brésilien en Mathématiques - Rede Franco-Brasileira de Matemática)

1 Introduction

In the setting of non-singular Morse-Smale flows on compact n -dimensional manifolds, a natural question is to understand how the topology of the manifold M forces the number and indices of the hyperbolic closed periodic orbits of any possible non-singular Morse-Smale flow over M . Of course it is not reasonable to try to answer precisely this question in its whole generality. However, it is possible to answer the same question in a wider context and still get some interesting dynamical information, which is optimal in the new more relaxed setting.

Inspired by the Morse flows theory, we focus on compact manifolds with boundary and discuss what is the periodic orbits information carried by the Betti numbers of the boundary, and how to algorithmically compute it. It is known, in Morse theory, that any Morse flow on any manifold satisfying some homological boundary information must have at least h_{min} singularities of some indices, that such a lower bound can be uniformly computed in terms of the Betti numbers of the boundary components, regardless of the topology of the underlying manifold, and that there exist a manifold and a flow for which this lower bound is optimal (see for instance [7] and [4]). We take here the analogous general approach for periodic orbits of a non-singular Morse-Smale flows.

More precisely, our main result consists of an algorithm computing a lower bound p_{min} of the number of periodic orbits of any non-singular Morse-Smale flow on any manifold with boundary satisfying some homological boundary conditions. Such a lower bound p_{min} can be uniformly computed in terms of the Betti numbers of the boundary components, regardless of the topology of the underlying manifold, and there exists a manifold and a non-singular Morse-Smale flow for which this lower bound is optimal. However, despite the analogy of the statement, some relevant differences appears both in the techniques and in the nature of the result, especially in large dimension and for large initial homological information. See Subsection 1.2.3.

Before discussing the interest and the limits of our main theorem, let us state it precisely.

We shall work with some partial abstract homological information, without any reference to a specific manifold or flow. We shall refer to this information as *abstract homological data*. This information consists of

- an odd integer n , $n \geq 3$;
- two positive integers e^+ and e^- ;
- $\frac{n-1}{2}$ integers, denoted by the expressions $B_j^+ - B_j^-$, for $j = 1, \dots, \frac{n-1}{2}$.

For any odd n , let us denote by M any n -dimensional compact connected oriented manifold with boundary ∂M consisting of $(e^+ + e^-)$ connected components, endowed with a non-singular Morse-Smale flow Φ transversally entering M through e^+ boundary components N_i^+ , $i = 1, \dots, e^+$, and transversally exiting through the remaining e^- boundary components N_i^- , $i = 1, \dots, e^-$. For all $j = 1, \dots, \frac{n-1}{2}$, if $\beta_j(N)$ represents the j -th Betti number of N ,

then we denote by $B_j^+ - B_j^-$ the sum:

$$B_j^+ - B_j^- = \sum_{k=1}^{e^+} \beta_j(N_k^+) - \sum_{k=1}^{e^-} \beta_j(N_k^-)$$

Within this notation, we say that a manifold M and a non-singular Morse-Smale flow Φ on M as in our context satisfy the *abstract homological data*

$$\left\{ n, e^+, e^-, \{B_j^+ - B_j^-\}_{j=1}^{\frac{n-1}{2}} \right\}.$$

Let us emphasize here that the flow Φ is non-singular Morse-Smale, that is, all of its recurrent sets are *closed periodic orbits* and lie in the interior of M . In the sequel, we shall simply call them periodic orbits because all the flows we shall consider are non-singular Morse-Smale.

We can prove the following theorem.

Theorem 1 *Let us be given the following abstract homological data $\left\{ n, e^+, e^-, \{B_j^+ - B_j^-\}_{j=1}^{\frac{n-1}{2}} \right\}$ satisfying (n odd and) $e^+ - e^- - \sum_{j=1}^{\frac{n-1}{2}} \{B_j^+ - B_j^-\} = 0$.*

Then the following conclusions hold true.

1. *Section 6 provides an explicit algorithm computing a number p_{min} associated with the given homological data, such that any non-singular Morse-Smale flow on any manifold satisfying these given abstract homological data must have at least p_{min} closed periodic orbits.*
2. *There exists a manifold M and a non-singular Morse-Smale flow Φ with exactly p_{min} periodic orbits satisfying the given abstract homological data.*

1.1 On the assumptions of Theorem 1

The assumption n odd will be discussed later (see Subsection 1.2.5).

The assumption $e^+ - e^- - \sum_{j=1}^{\frac{n-1}{2}} \{B_j^+ - B_j^-\} = 0$ means that the Euler characteristic of any manifold admitting a non-singular Morse-Smale flow is necessarily zero. It is known that this is required for having non-singular Morse-Smale flows. Hence this assumption on the abstract homological data is natural because of their interpretation.

Moreover, this condition is crucial in one of the main ingredients of the proof: the use of attaching handles vs. attaching round handles. On the one hand, in Morse theory, where each singularity of index j corresponds to a handle of index j , we know the effect of attaching one handle of index j of the Betti numbers of the boundary. This tells us that the list $\underline{h} = (h_0, \dots, h_n)$ of the number of singularities h_j of index j compatible with the initial homological data are of the form ((see [4]) $\underline{h} = \underline{h}_{min} + \underline{h}_{consecutive} + \underline{h}_{dual}$ where

the (integer) vector \underline{h}_{min} contains the minimal number of singularities that any Morse flow on any manifold compatible with the data must have and we have a finite number of choices for it which are explicit ;

the (integer) vector $\underline{h}_{consecutive}$ is a sum of vectors of type $\{(a_0, \dots, a_n) \mid \exists j, 0 \leq j < n, a_j = a_{j+1} = 1 \text{ and } a_i = 0 \text{ for all } i \neq j, i \neq j + 1\}$;

the (integer) vector \underline{h}_{dual} is a sum of vectors of type $\{(a_0, \dots, a_n) \mid \exists j, 0 < j < n, a_j = a_{n-j} = 1 \text{ and } a_i = 0 \text{ for all } i \neq j, i \neq n - j\}$;

On the other hand, in non-singular Morse-Smale theory ([2]), where each orbit of index j corresponds to a round handle of index j , replacing it by two handles of indices j and $j + 1$ (see [16]) implies that among all compatible vectors \underline{h} as above, there must be some \underline{h}' of the form $\underline{h}' = \underline{h}'_{consecutive}$. Well, the assumption on the Euler characteristic, ensures that each vector \underline{h}_{min} can be completed with some $\underline{h}_{consecutive} + \underline{h}_{dual}$ in order to obtain a vector \underline{h}' of the form $\underline{h}' = \underline{h}'_{consecutive}$ which is also compatible with the given homological data.

1.2 On the conclusions of Theorem 1

1.2.1 The topology behind Item 1

Item 1 of the above theorem starts from the fact that some loose information on the homology of any compatible underlying manifold M together with the knowledge of the entry and exit boundary of the underlying flow Φ allow us to guarantee the existence of a lower bound for the number of the periodic orbits of Φ . However, if we wish to work with a restricted class of manifolds admitting non-singular Morse-Smale flows, such a bound may not be sharp.

Consider for instance the data $\{n = 3, e^+ = e^- = 1, B_1^+ - B_1^- = 0\}$. For these fixed data one has $p_{min} = 0$. The manifold $M = \mathbb{T}^2 \times I$ endowed with the trivial flow Φ_M entering $\mathbb{T}^2 \times \{1\}$ and exiting $\mathbb{T}^2 \times \{0\}$ is an example of manifold and flow satisfying the homological data for which the computed p_{min} coincides with its minimal number of periodic orbits, denoted by $A_{min}(M, \partial)$. If we consider M' obtained by attaching a round handle to M as in Fig 1 below, we can see that the boundary of M' is also made of two copies of \mathbb{T}^2 .

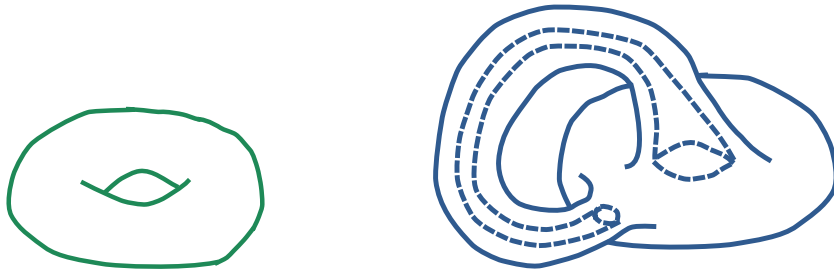


Figure 1: Exit and entry boundary of the nontrivial manifold M'

The respect of the homological data implies that the non-singular Morse-Smale flows we shall consider must enter through one torus component and exit through the other. The computation of the Conley index of the manifold and the way we constructed it show that $A_{min}(M', \partial) = 1$ which is strictly greater than p_{min} .

The situation here is analogous to the one discussed in [7] where the minimal number of singularities of Morse flows on manifolds satisfying some abstract homological data is

discussed. In our present context we can also consider that the manifolds and flows for which p_{min} is sharp (that is, $A_{min}(M, \partial) = p_{min}$) are in some sense the “simplest”, topologically speaking, among those admitting non-singular Morse-Smale flows and satisfying the given abstract homological information.

Let us also stress the fact that the abstract homological information only deals with the difference of the Betti numbers of the entry and exit boundary. For instance, in dimension 3, the couple $(N^+, N^-) = (\mathbb{S}^2 \sqcup \mathbb{S}^2, \mathbb{S}^2 \sqcup \mathbb{S}^2)$, the couple $(N^+, N^-) = (\mathbb{S}^2 \sqcup \mathbb{T}^2, \mathbb{S}^2 \sqcup \mathbb{T}^2)$ and the couple $(N^+, N^-) = (\mathbb{S}^2 \sqcup \mathbb{T}^2 \# \mathbb{T}^2, \mathbb{T}^2 \sqcup \mathbb{T}^2)$ all correspond to the same abstract homological data $\{n = 3, e^+ = e^- = 2, B_1^+ - B_1^- = 0\}$

1.2.2 On the algorithm of Item 1

The description of the algorithm of *Item 1* is very technical. Here are the guidelines. The conditions ensuring the existence of a solution starting from the abstract data are expressed in terms of a semi-algebraic system with integer coefficients. We are looking for the positive solutions of this system whose sum is minimal. We are therefore dealing with an optimization problem. The initial pairing problem is hence reformulated as a minimum cost flow (MCF) problem (see Theorem 12). Then the interpretation of the specific MCF problem as a transportation problem yields the wanted algorithm.

Let us emphasize that, especially in large dimension¹ and when the difference of the Betti numbers is also large and arbitrarily distributed, we cannot expect to compute the number p_{min} by any naive strategy. Let us take for instance the homological data

$$\left\{ n = 17, e^+ = 7, e^- = 6, B_1^+ - B_1^- = 3, B_2^+ - B_2^- = -1, B_3^+ - B_3^- = -2, B_4^+ - B_4^- = -7, \right. \\ \left. B_5^+ - B_5^- = -6, B_6^+ - B_6^- = 8, B_7^+ - B_7^- = -4, \frac{B_8^+ - B_8^-}{2} = -10 \right\}$$

Our algorithm gives us (among other information) the solution

- $p_{min} = 32$;
- the sequence $\underline{h}' = [h'_1, \dots, h'_{16}] = [5, 5, 1, 1, 2, 2, 0, 0, 10, 10, 0, 6, 6, 2, 8, 6]$, which can be translated into the sequence $[R_1, \dots, R_{15}] = [5, 0, 1, 0, 2, 0, 0, 0, 10, 0, 0, 6, 0, 2, 6]$ where R_j would correspond to the number of round handles (or equivalently, periodic orbits) of index j ;
- the fact that \underline{h}' has been obtained by completing $\underline{h}_{min} = [5, 0, 0, 0, 1, 2, 0, 0, 10, 10, 0, 6, 6, 2, 4, 6]$ by $\underline{h}_{consecutive} + \underline{h}_{dual} = [0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] + [0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 0]$

Now, trying to complete \underline{h}_{min} in any other naive way (by adding couples of handles of consecutive indices from left to right, or from right to left, or by adding couples of handles of dual indices from the extremities of \underline{h}_{min}) gives strict upper bounds of p_{min} and makes one understand that a more sophisticated and complex strategy is unavoidable.

This is a main difference with the Morse setting. There, the value h_{min} can be computed by hand directly from the homological data. Moreover, all the possible sequences of the

¹For $n = 3$, the computation is straightforward, see Section 5.

indices of the Morse singularities associated with h_{min} can be listed explicitly. Here, in contrast, the computation of p_{min} is intrinsically more laborious and our algorithm gives us just one way of realizing it as a sequence of round handles.

Let us underline that a Python program associated with this algorithm is freely available on the page <https://github.com/MargaridaMello/OddMinimumPairingProblem/> At the end of Section 6 it is explained how to use it and how to extract the needed information.

1.2.3 Unexpected indices of periodic orbits

As a counterpart of the complexity of the algorithm discussed above, we discover that, in contrast with the Morse setting, here the indices of the appearing periodic orbits cannot be always foreseen. For instance, for the homological data

$$\{n = 15, e^+ = 1, e^- = 1, B_1^+ - B_1^- = B_6^+ - B_6^- = 1, B_j^+ - B_j^- = 0 \forall j = 1 \dots 7, j \neq 1, 6\}$$

we have ($h_{min} = 2$ and) $p_{min} = 3$. The algorithm outputs an abstract realization of p_{min} with an orbit of index 3, an orbit of index 5 and an orbit of index 12 appearing from the combinatorics of the algorithm. The index 5 orbit is of course related to the fact that the 6-th (and the 8-th) Betti number vary; the index 12 orbit is related to the variation of the 1st (and 13-th) Betti number of the boundary. The index 3 orbit is somehow hidden in the data even though the difference $B_j^+ - B_j^-$ of the Betti numbers of indices 2, 3 and 4, as well as 10, 11 and 12, are zero). For the sake of completeness, let us mention that there are three other solutions (each with three orbits of indices respectively (1, 3, 5), (5, 10, 12) and (8, 10, 12) for which the analogous observations hold. Note that in this case only three of the four possible \underline{h}_{min} can be completed to a \underline{h}'_{min} combinatorially realizing $p_{min} = 3$ and one can be completed in two different ways².

1.2.4 On the realizations

Item 2 is constructive and explicit once we have run the algorithm of Item 1. In fact, as a result, such an algorithm gives not only the value of p_{min} but also a list of abstract round handles $R_1, \dots, R_{p_{min}}$, of given index and type, associated with p_{min} . The type contains the information of the effect of the corresponding round handle on the Betti number of the boundary. Using the realizations described in [12] one can easily conclude: for any R_k given by the algorithm, let (U_k, ϕ_k) be the isolating neighborhood of the corresponding index and type built as in [12]. A connected sum along the boundary of these U_k , $k = 1, \dots, p_{min}$ is a manifold M endowed with a non-singular Morse-Smale flow Φ (conjugate to ϕ_k on each U_k) such that (M, Φ) satisfies the given homological data. By construction, Φ has p_{min} periodic orbits.

1.2.5 Perspectives

The proof of this result opens the way to the even dimension. Even though the guidelines for finding an algorithm are comparable, the combinatorics in the even setting is richer because of

² \underline{h}_{min} such that $h_1 = 1, h_6 = 1$ yields the (1, 3, 5) triple; \underline{h}_{min} such that $h_6 = 1, h_{13} = 1$ yield the (3, 5, 12) and the (5, 10, 12) triples; \underline{h}_{min} such that $h_8 = 1, h_{13} = 1$ yields the (8, 10, 12) triple and \underline{h}_{min} such that $h_1 = 1, h_8 = 1$ cannot be completed in order to realize $p_{min} = 3$.

the existence of “invariant” handles³ and must be treated independently. Moreover, handles of invariant type are difficult to realize in the non-singular Morse-Smale context, so that the realization of the general abstract solution by a non-singular Morse-Smale model remains an open question.

1.2.6 Application to Lyapunov graphs

We shall conclude by an application of the reasoning to the problem of the continuation of Lyapunov graphs into Lyapunov graphs of Smale type (Lyapunov graphs associated with non-singular Morse-Smale flows). More precisely, in works [9], [10], [8] the authors showed that a generalized Lyapunov graph can be continued to a Lyapunov graph of Morse type if and only if some inequalities, called the Poincaré-Hopf inequalities are satisfied. In the present setting where we are concerned by non-singular Morse-Smale flows instead of Morse flows, we can prove

Theorem 2 *A (generalized) vertex of a Lyapunov graph associated with the homological $\{n, e^+, e^-, \{B_j^+ - B_j^-\}_{j=1}^{\frac{n-1}{2}}\}$ can be continued to a Lyapunov graph of Smale type if and only if a set of explicit inequalities (Poincaré-Hopf inequalities (1), (2), (5), (8) and (11) + coupling inequalities Lemma 4 (or Lemma 3)) are satisfied by such data.*

This theorem holds true in any dimension and constitutes the first step to tackle our main question in even dimension.

The paper is organized as follows. Section 2 contains background material. Section 3 is a first combinatorial translation of our problem into combinatorial terms. In Section 4 is given the explicit formulation and context of Theorem 2. Section 5 is devoted to the three-dimensional case. Section 6 explains the proof of our main theorem. Numerical examples are also treated there. Section 7 provides partial results concerning the even dimensional case.

2 Poincaré-Hopf inequalities

In this section we present an adaptation of the Poincaré-Hopf inequalities described in [9] for non-singular Morse-Smale flows on isolating blocks case.

A set $S \subset M$ is an *invariant set* of a flow ϕ_t if $\phi_t(S) = S$ for all $t \in \mathbb{R}$. A compact set $N \subset M$ is an *isolating neighborhood* if $\text{inv}(N, \phi) = \{x \in N : \phi_t(x) \subset N, \forall t \in \mathbb{R}\} \subset \text{int } N$. A compact set N is an *isolating block* if $N^- = \{x \in N : \phi_{[0,t)}(x) \not\subset N, \forall t > 0\}$ is closed and $\text{inv}(N, \phi) \subset \text{int } N$. An invariant set S is called an *isolated invariant set* if it is a maximal invariant set in some isolating neighborhood N , that is, $S = \text{inv}(N, \phi)$.

A component R of the *chain recurrent set*⁴, \mathcal{R} , of the flow ϕ_t , is an example of an invariant set. We will work under the hypothesis that \mathcal{R} is the finite union of isolated invariant sets

³Handles whose attaching produces no effect on the Betti numbers of the boundary.

⁴A point $x \in M$ is chain recurrent if given $\varepsilon > 0$ there exists an ε -chain from x to itself, i.e., there exists points $x = x_1, x_2, \dots, x_{n-1}, x_n = x$ and $t(i) \geq 1$ such that $d(\phi_{t(i)}(x_i), x_{i+1}) < \varepsilon \quad \forall 1 \leq i < n$. A set of such points will be denoted by \mathcal{R} and is called a *chain recurrent set*.

R_i . If f is a Lyapunov function⁵ associated with a flow and $c = f(R)$ then for $\varepsilon > 0$, the component of $f^{-1}[c - \varepsilon, c + \varepsilon]$ that contains R is an isolating neighborhood for R . Take $(N, N^-) = (f^{-1}[c - \varepsilon, c + \varepsilon], f^{-1}(c - \varepsilon))$ as an index pair for R . The Conley index is defined as the homotopy type of N/N^- . Its homology is denoted by $CH_*(S)$ and its rank denoted by $h_* = \text{rank } CH_*(S)$. For more details see [13].

Let N be any compact manifold of dimension n such that $\partial N = \partial N^+ \cup \partial N^-$, with ∂N^+ and ∂N^- non-empty where $\partial N^+(\partial N^-)$ is the disjoint union of $e^+(e^-)$ components of ∂N , and denote it by $\partial N^\pm = \bigcup_{i=1}^{e^\pm} N_i^\pm$. Also, consider the sum of the Betti numbers, $\beta_j(N_i^\pm)$, of these components, i.e. $B_j^\pm = \sum_{i=1}^{e^\pm} \beta_j(N_i^\pm)$ where $j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$.

The Poincaré-Hopf inequalities for an isolated invariant set S in an isolating block N with entering set for the flow N^+ and exiting set for the flow N^- , are obtained by analysis of long exact sequences of (N, N^+) and (N, N^-) . This analysis can be found in [9] in a more detailed exposition.

Note that (N, N^-) is an index pair for S and (N, N^+) is an index pair for the isolated invariant set of the reverse flow, S' .

Consider the long exact sequences for the pairs (N, N^-) and (N, N^+) , denoted by LES- and LES+, respectively:

$$0 \rightarrow H_n(N^-) \xrightarrow{i_n} H_n(N) \xrightarrow{p_n} H_n(N, N^-) \xrightarrow{\partial_n} H_{n-1}(N^-) \xrightarrow{i_{n-1}} H_{n-1}(N) \xrightarrow{p_{n-1}} H_{n-1}(N, N^-) \xrightarrow{\partial_{n-1}} \\ \dots \xrightarrow{\partial_2} H_1(N^-) \xrightarrow{i_1} H_1(N) \xrightarrow{p_1} H_1(N, N^-) \xrightarrow{\partial_1} H_0(N^-) \xrightarrow{i_0} H_0(N) \xrightarrow{p_0} H_0(N, N^-) \rightarrow 0$$

$$0 \rightarrow H_n(N^+) \xrightarrow{i'_n} H_n(N) \xrightarrow{p'_n} H_n(N, N^+) \xrightarrow{\partial'_n} H_{n-1}(N^+) \xrightarrow{i'_{n-1}} H_{n-1}(N) \xrightarrow{p'_{n-1}} H_{n-1}(N, N^+) \xrightarrow{\partial'_{n-1}} \\ \dots \xrightarrow{\partial'_2} H_1(N^+) \xrightarrow{i'_1} H_1(N) \xrightarrow{p'_1} H_1(N, N^+) \xrightarrow{\partial'_1} H_0(N^+) \xrightarrow{i'_0} H_0(N) \xrightarrow{p'_0} H_0(N, N^+) \rightarrow 0$$

Throughout the analysis, the Conley duality condition on the indices is assumed. That is, the isolated invariant sets S and S' have the property that $\text{rank } H_j(N, N^-) = h_j$ and $\text{rank } H_j(N, N^+) = \bar{h}_j = h_{n-j}$. Denote $\text{rank } H_0(N^-) = e^-$, $\text{rank } H_0(N^+) = e^+$ and $\text{rank } (H_j(N^\pm)) = B_j^\pm$.

By simultaneously analyzing the following pairs of maps

$$\{[(p_i, \partial'_i), (p'_i, \partial_i)], \dots, [(p_2, \partial'_2), (p'_2, \partial_2)]\}$$

and analyzing p_1 and p'_1 we obtain the Poincaré-Hopf inequalities in all its generality, where h_j is the dimension of the homology Conley index and $B_j^- = \sum_{i=1}^{e^-} (\beta_j^-)_i$, $B_j^+ = \sum_{i=1}^{e^+} (\beta_j^+)_i$, where $j \in \{1, \dots, n-2\}$. If n is odd, $n = 2i + 1, i \geq 1$, then (1)–(5) need to be satisfied.

⁵Given a continuous flow $\phi_t : M \rightarrow M$, on a closed n -manifold M , results of Conley [13] imply the existence of a continuous Lyapunov function $f : M \rightarrow \mathbb{R}$ associated with the flow with the property that it strictly decreases along the orbits outside the chain recurrent set \mathcal{R} , that is, if $x \notin R$ then $f(\phi_t(x)) < f(\phi_s(x))$ for $t > s$ and is constant on the chain recurrent components R of \mathcal{R} .

$$\begin{aligned}
h_j \geq & \sum_{k=1}^{j-1} (-1)^{k+j} (B_k^+ - B_k^-) + \sum_{k=0}^{j-1} (-1)^{k+j} (h_{n-k} - h_k) \\
& + (-1)^{j+1} (e^- - e^+), \quad j = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor
\end{aligned} \tag{1}$$

$$\begin{aligned}
h_{n-j} \geq & \sum_{k=1}^{j-1} (-1)^{k+j+1} (B_k^+ - B_k^-) + \sum_{k=0}^{j-1} (-1)^{k+j+1} (h_{n-k} - h_k) \\
& + (-1)^j (e^- - e^+), \quad j = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor
\end{aligned} \tag{2}$$

$$h_1 \geq h_0 - 1 + e^- \tag{3}$$

$$h_{n-1} \geq h_n - 1 + e^+ \tag{4}$$

$$\sum_{k=1}^{i-1} (-1)^k (B_k^+ - B_k^-) + (-1)^i \frac{B_i^+ - B_i^-}{2} - \sum_{k=0}^n (-1)^k h_k - (e^- - e^+) = 0 \tag{5}$$

Non-singular Morse-Smale flows on a smooth n -dimensional manifold M were considered together with a handle decomposition associated with a Lyapunov function in [14]. Thus, after the attachment of a handle corresponding to a singularity (or a round handle corresponding to a periodic orbit) one can consider the effect on the new regular level set. The authors completely describe how the Betti numbers of the level set vary after attaching a (round) handle when the homology coefficients are taken in $\frac{\mathbb{Z}}{2\mathbb{Z}}$. These results were generalized in [9] by considering continuous flows associated with Lyapunov functions on n -dimensional manifolds. More specifically, a flow in the isolating block N of an isolated invariant set S with possibly complicated dynamical behavior was considered. The effect on the Betti numbers of the regular level sets corresponding to the incoming N^+ and outgoing N^- boundaries of the flow in N were determined in terms of the homology indices of S . The classification of singularities presented in [14] was generalized in [6] when the homology coefficients are in \mathbb{Z} , \mathbb{Q} , \mathbb{R} , or $\frac{\mathbb{Z}}{p\mathbb{Z}}$ with p prime. In other words, as for the boundary, attaching a handle of index j ($j = 1 \dots n - 1$) to N^- can produce one of the following effects if n is odd:

- (H1) the j -th Betti number of the boundary N^+ is increased by 1 (or by 2, if $n = 2j + 1$), and the handle will be said of type j -d (d standing for disconnecting);
- (H2) the $(j-1)$ -th Betti number of the boundary N^+ is decreased by 1 (or by 2, if $n = 2j+1$), and the handle will be said of type $(j-1)$ -c (c standing for connecting);

Using these two effects, the Poincaré-Hopf inequalities described above were used in [9] in order to ensure that given an isolating neighborhood containing a singularity with possibly complicated dynamical behavior, this neighborhood can be replaced by a neighborhood related to a Morse flow, i.e., where the singularities of the flow are points. Being more precise, the Poincaré-Hopf inequalities ensure that given an isolating neighborhood N , with incoming N^+ and outgoing N^- boundaries of the flow in N , containing a singularity whose dimensions of the homology indices are (h_0, \dots, h_n) , this neighborhood can be replaced by

a neighborhood where the singularities of the flow are points, that is, $h_j = 1$, where j is the dimension of the unstable manifold of this isolated singularity for all $j = 0 \dots n$.

Consider the example as in Figure 3. This example is illustrating by means of a graph for facilitating the understanding. First, observe that the two handle effects described above, (H1) et (H2), can be viewed in terms of graphs in the following way. A handle containing a singularity of index ℓ corresponds to a vertex on the graph L_N labeled with $h_\ell = 1$, which can produce the two possible algebraic effects:

(G1) $h_\ell = 1$ is ℓ -d if it has the algebraic effect of increasing the corresponding β_ℓ label on the incoming edge.

(G2) $h_\ell = 1$ is $(\ell - 1)$ -c if it has the algebraic effect of decreasing the corresponding $\beta_{\ell-1}$ label on the incoming edge.

See the corresponding graphs in Figure 2.

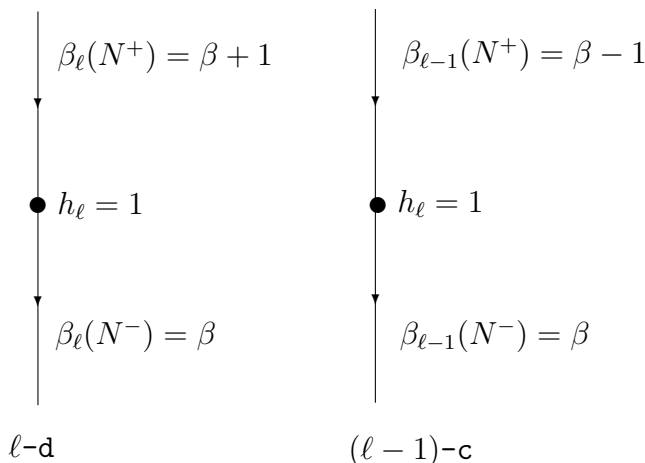


Figure 2: Two possible algebraic effects in odd dimension.

Coming back to Figure 3, we start with a neighborhood in dimension 5 containing a singularity whose the dimension of the homology indices are given by $(h_0 = 0, h_1 = 2, h_2 = 2, h_3 = 1, h_4 = 1, h_5 = 0)$, the components of the incoming and outgoing boundary components, N^+ and N^- , are labeled with the Betti numbers $(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$ as illustrated in Figure 3. Note that since $(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$ satisfy the Poincaré duality and since β_0 and β_4 represent the number of boundary components, we have put only (β_1, β_2) in Figure 3.

The procedure exemplified in Figure 3 is called *a continuation of an abstract Lyapunov graph to an abstract Lyapunov graph of Morse type* in [9], where the left side of Figure 3 represents the abstract Lyapunov graph while the right side represents the abstract Lyapunov graph of Morse type, in which any vertex v is labeled by $h_j = 1$ if it corresponds to a singularity of index j . The graphs here are used only to better illustrate the example, but what is important to note is that, since the Poincaré-Hopf inequalities (1)–(5) are satisfied,

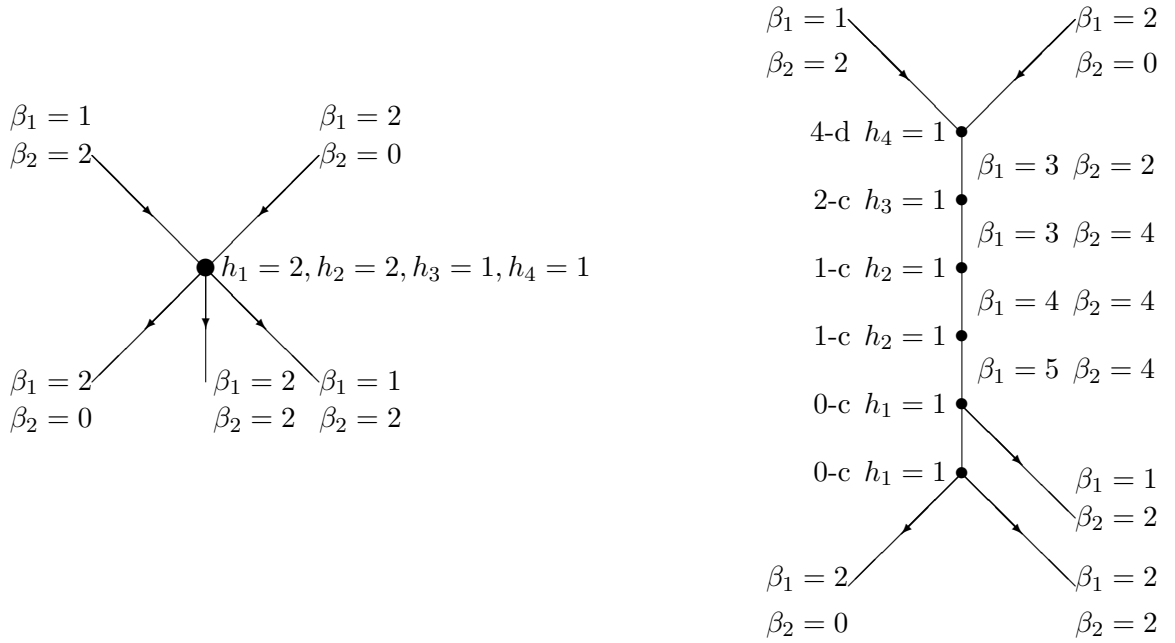


Figure 3: A continuation of an abstract Lyapunov graph to an abstract Lyapunov graph of Morse type.

it is possible to replace the initial neighborhood by another connected neighborhood with the same data, but with a simplified dynamical behavior.

The Poincaré-Hopf inequalities for non-singular Morse-Smale flows on isolating blocks are an adaptation of the Poincaré-Hopf inequalities above described. They differ only in inequalities (3) and (4). These adaptation should be done in order to ensure the continuation of an abstract Lyapunov graph L_N to an abstract Lyapunov graph of Smale type, in which any vertex v is labeled by $A_j = 1$ if it corresponds to a periodic orbit of index j .

The difference in inequalities (3) and (4) comes from the fact that in the case of the Poincaré-Hopf inequalities one treats h_0 by imposing, as necessary and sufficient condition, that $h_1 \geq h_0 + e^- - 1$, i.e., inequality (3). This inequality adjusts the problem of connectivity, that is, it ensures that the continued graph – as well as the corresponding isolating neighborhood – are connected. By this inequality we guarantee the possibility of having $h_0 + e^- - 1$ singularities of index 1 of type 0-c, h_1^c , which means connecting. Hence,

$$h_1^c \geq h_0 + e^- - 1. \quad (6)$$

It is important to observe that the singularities h_1 of type 0-c are responsible for connecting the outgoing boundary components. For example, if we have three outgoing boundary components, representing by $h_0 = 3$ we need two h_1 of type 0-c for producing a connected neighborhood. In order to better understand, consider the example in three-dimension presented in Figure 4. In this example we have a vertex v_k which is labeled by $h_0 = 1$ and $h_1 = 1$. Since these data satisfy inequality (3), we can replace the vertex as in the right side of Figure 4: by two vertices v_{k_1} and v_{k_2} respectively labeled with $h_0 = 1$ and $h_1 = 1$. It is important to note that the left and the right side of Figure 4 have the same number of incoming and outgoing edges and moreover they are both connected.

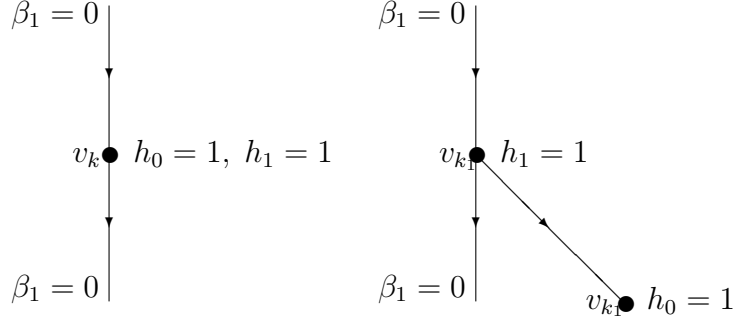


Figure 4: Connectivity of the outgoing edges.

In the case of the Poincaré-Hopf inequalities for non-singular Morse-Smale flows, inequality (3) should be modified, because the presence of $h_0 \neq 0$ implies the existence of periodic orbits of index 0. Each one is obtained by joining a singularity of index 0 with a singularity of index 1 of type 1-d, h_1^d (observe that all singularities h_1 of type 0-c, h_1^c , were already used for solving the connectivity problem as explained above). Hence, we have that

$$h_1^d \geq h_0. \quad (7)$$

For this reason, since $h_1 = h_1^c + h_1^d$ inequalities (6) and (7) imply that we have the following inequality replacing (3)

$$h_1 \geq 2h_0 - 1 + e^- \quad (8)$$

Each of the remaining singularities of index 1 is to be coupled with a singularity of index 2, thus creating a periodic orbit of index 1. Therefore, the left side of inequalities (1) remains the same for all $j \neq 1$.

Observe that the graph presented in Figure 4 cannot be replaced by a connected graph containing a periodic orbit of index 0: if it were the case we would use $h_1 = 1$ and $h_0 = 1$ to have a vertex labeled with $A_0 = 1$, with one incoming edge and no outgoing edges, and we would be in the situation of the right side of Figure 5, that is, a non connected graph. It is also important to observe that inequality (8) is not satisfied.

Figure 6 gives an example in dimension three where inequality (8) is satisfied and hence we can replace the initial vertex to a connected one containing only periodic orbits. It is important to note that the left and the right side of Figure 6 have the same number of incoming and outgoing edges and moreover they are both connected.

Analogously, the necessary and sufficient condition when one treats h_n is $h_{n-1} \geq h_n + e^+ - 1$, i.e. inequality (4), which assures the existence of $h_n + e^+ - 1$ singularities of type $(n-1)$ -d, which means disconnecting. Hence,

$$h_{n-1}^d \geq h_n + e^+ - 1. \quad (9)$$

In the case of the Poincaré-Hopf inequalities for non-singular Morse-Smale flows, the presence of $h_n \neq 0$ implies the existence of periodic orbits of index $n-1$. Each one is obtained by

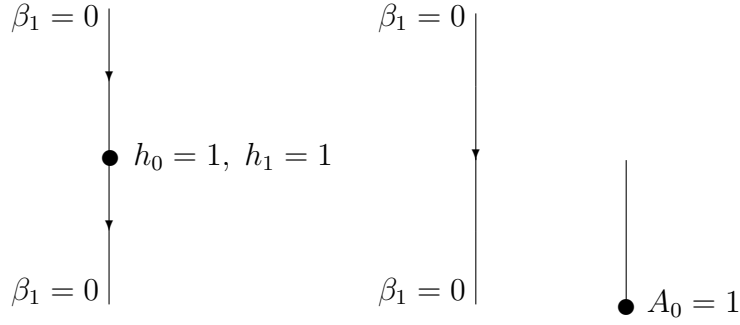


Figure 5: Impossibility of continuation to a connected graph containing a periodic orbit of index 0.

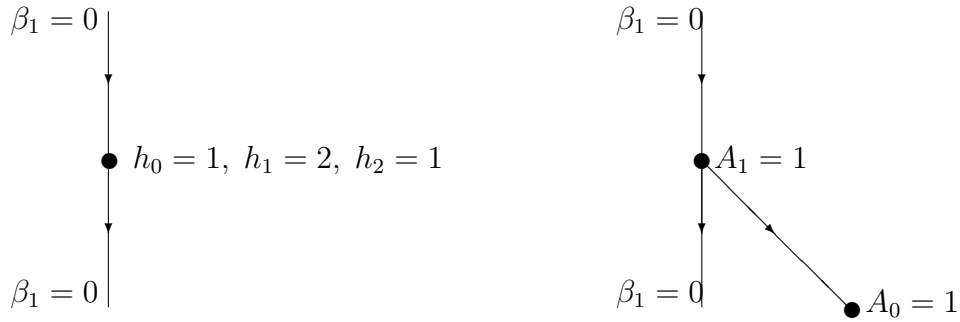


Figure 6: Continuation to a connected graph containing only periodic orbits.

joining a singularity of index n with a singularity of index $n - 1$ of type $(n - 1)$ -c, h_{n-1}^c . Hence, we have that

$$h_{n-1}^c \geq h_n. \quad (10)$$

For this reason, since $h_{n-1} = h_{n-1}^c + h_{n-1}^d$ inequalities (9) and (10) imply that we have the following inequality replacing (4)

$$h_{n-1} \geq 2h_n - 1 + e^+ \quad (11)$$

Each of the remaining singularities of index $n - 1$ is to be coupled with a singularity of index $n - 2$, thus creating a periodic orbit of index $n - 2$. Therefore, the left side of inequalities (1) remains the same for all $j \neq n - 1$.

In [9] it is shown that:

Proposition 1 *The system (110)–(115) has nonnegative integral solutions $(h, h_1^c, h_1^d, \dots, h_{n-1}^c, h_{n-1}^d)$ if and only if the Poincaré-Hopf inequalities (1)–(5) are satisfied.*

$$\left\{ \begin{array}{l} h_1^c = e^- - 1, \quad (12) \\ h_{n-1}^d = e^+ - 1, \quad (13) \\ h_j = h_j^c + h_j^d, \quad j = 1, \dots, n-1 \text{ and } j \neq \left\lfloor \frac{n}{2} \right\rfloor, \quad (14) \\ h_j^d - h_{j+1}^c - h_{n-j}^c + h_{n-j-1}^d = B_j^+ - B_j^-, \quad j = 1, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor, \quad (15) \\ h_i^d - h_{i+1}^c = \frac{B_i^+ - B_i^-}{2}, \quad \text{if } n = 2i + 1. \quad (16) \end{array} \right.$$

Observe that, as described previously, the inequalities (1)-(5) presented in [9] coincides with the Poincaré-Hopf inequalities for non-singular Morse-Smale flows except to the inequalities:

$$\begin{aligned} h_1 &\geq 2h_0 - 1 + e^- \\ h_{n-1} &\geq 2h_n - 1 + e^+ \end{aligned}$$

which are replaced by:

$$\begin{aligned} h_1 &\geq h_0 - 1 + e^- \\ h_{n-1} &\geq h_n - 1 + e^+ \end{aligned}$$

But, under the hypothesis conditions imposed in the labels, as described previously, these two groups of conditions are equivalent. Therefore, Proposition 1 remains true for the Poincaré-Hopf inequalities for non-singular Morse-Smale flows, i.e.:

Proposition 2 *The system (110)–(115) has nonnegative integral solutions $(h_1^c, h_1^d, \dots, h_{n-1}^c, h_{n-1}^d)$ if and only if the Poincaré-Hopf inequalities for non-singular Morse-Smale flows (1), (2), (5), (8) and (11) are satisfied.*

3 Periodic orbits and coupling inequalities

3.1 Periodic orbits

Recall that in [17] Franks proved that a hyperbolic periodic orbit of index j , can be viewed as the joining of two hyperbolic singularities p and q of adjacent indices j and $j+1$ respectively. Given a nondegenerate singularity of index j , one can associate with it the dimensions of the Conley homology indices, $h_j = 1$ and $h_k = 0$ for all $k \neq j$. Let A_j be the number of periodic orbits of index j and h_j be the number of singularities of index j .

In this dimension (n odd), given the data $(A_0, A_1, \dots, A_{n-1})$ we have $(h_0, h_1, \dots, h_{n-1}, h_n)$ associated with it and vice versa. If one starts the procedure with data consisting of singularities $(h_0, h_1, \dots, h_{n-1}, h_n)$ one can do a coupling procedure in order to construct our periodic orbits set $(A_0, A_1, \dots, A_{n-1})$. The number h_1 together with h_0 will construct A_0 but together with h_2 will construct A_1 . Then the number of periodic orbits of index 0 depends on the number of h_0 and the number of periodic orbits of index 1 depends on the number of h_1 ,

although some part of h_1 was already used for creating the periodic orbits of index 0. And the procedure continues by considering h_2 , and so on. Therefore, coupling from the left to right produces

$$\left\{ \begin{array}{l} A_0 = h_0 \\ A_1 = h_1 - h_0 \\ A_2 = h_2 - (h_1 - h_0) \\ \vdots \\ A_k = h_k - h_{k-1} + \dots \pm h_0 \\ \vdots \\ A_{n-1} = h_{n-1} - h_{n-2} + \dots \pm h_0 \end{array} \right. \quad \text{or equivalently} \quad \left\{ \begin{array}{l} A_0 = h_0 \\ A_1 = h_1 - A_0 \\ A_2 = h_2 - A_1 \\ \vdots \\ A_k = h_k - A_{k-1} \\ \vdots \\ A_{n-1} = h_{n-1} - A_{n-2} \end{array} \right. \quad (17)$$

Coupling from the right to the left produces

$$\left\{ \begin{array}{l} A_{n-1} = h_n \\ A_{n-2} = h_{n-1} - h_n \\ A_{n-3} = h_{n-2} - (h_{n-1} - h_n) \\ \vdots \\ A_{n-k} = h_{n-1-k} - h_{n-1-(k-1)} + \dots \pm h_n \\ \vdots \\ A_0 = h_1 - h_2 + \dots \pm h_n \end{array} \right. \quad \text{or equivalently} \quad \left\{ \begin{array}{l} A_{n-1} = h_n \\ A_{n-2} = h_{n-1} - A_{n-1} \\ A_{n-3} = h_{n-2} - A_{n-2} \\ \vdots \\ A_{n-k} = h_{n-1-k} - A_{n-k-1} \\ \vdots \\ A_0 = h_1 - A_1 \end{array} \right. \quad (18)$$

We can also do the coupling procedure symmetrically. First observe that $h_k = \text{rank}(H_k(N, N^-)) = \text{rank}(H^{n-k}(N, N^+)) = h_{n-k}$ (using Poincaré-Lefschetz duality). For details see [3] (Corollary 2.1) and [19]. The number h_1 together with h_0 will construct A_0 but together with h_2 will construct A_1 . Then the number of periodic orbits of index 0 depend on the number of h_0 and the number of periodic orbits of index 1 depend on the number of h_1 , but some part of h_1 was already used for creating the periodic orbits of index 0. And the procedure continues by considering h_2 , and so on. By duality of the indices, the number h_{n-1} together with h_n will construct A_{n-1} but together with h_{n-2} will construct A_{n-2} . Then the number of periodic orbits of index $n-1$ depends on the number of h_n and the number of periodic orbits of index $n-2$ depends on the number of h_{n-1} , but some part of h_{n-1} was already used for creating the periodic orbits of index $n-1$. And the procedure continues by considering h_{n-3} , and so on. Then we can define

$$\left\{ \begin{array}{l} A_0 = h_0 \\ A_1 = h_1 - h_0 \\ A_2 = h_2 - (h_1 - h_0) \\ \vdots \\ A_k = h_k - A_{k-1} \\ \vdots \\ A_i = h_i - A_{i-1} \end{array} \right. = \left\{ \begin{array}{l} A_{n-1} = h_n \\ A_{n-2} = h_{n-1} - h_n \\ A_{n-3} = h_{n-2} - (h_{n-1} - h_n) \\ \vdots \\ A_{n-1-k} = h_{n-k} - A_{n-k} \\ \vdots \\ A_{n-i-1} = h_{n-i} - A_{n-i} \end{array} \right. \quad (19)$$

3.2 Coupling inequalities

Lemma 3 A collection (h_0, \dots, h_n) can be decomposed in an union $\mathcal{A}_j = \frac{\sum_{j=0}^n h_j}{2}$ of disjoint couples of the form $(h_\ell, h_{\ell+1})$ with $\ell \in \{0, \dots, n\}$ if and only if

$$(-1)^k \sum_{j=0}^k (-1)^j h_j \geq 0, \quad k = 1, \dots, n-1, \quad (20)$$

$$\sum_{j=0}^n (-1)^j h_j = 0. \quad (21)$$

Proof: Trivial. In order to couple all singularities of index 0 with a singularity of index 1 it is necessary and sufficient that $h_1 \geq h_0$. In order to couple all singularities of index 1 with a singularity of index 2 it is necessary and sufficient that $h_2 \geq h_1 - h_0$ and so on until the coupling of the singularities of index $n-2$ with singularities of index $n-1$. Finally, the number of singularities of index $n-1$ not yet coupled and the number of singularities of index n must coincide for the coupling is complete. \square

Conversely, the coupling conditions can also be expressed by the system of the following lemma, whose proof is analogous to that of Lemma 3, the connection being made this time with the middle index(indices).

Lemma 4 A collection (h_0, \dots, h_n) can be decomposed in an union $\mathcal{A}_j = \frac{\sum_{j=0}^n h_j}{2}$ of disjoint pairs of the form $(h_\ell, h_{\ell+1})$ with $\ell \in \{0, \dots, n\}$ if and only if

$$(-1)^k \sum_{j=0}^k (-1)^j h_j \geq 0, \quad k = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \quad (22)$$

$$(-1)^k \sum_{j=0}^k (-1)^j h_{n-j} \geq 0, \quad k = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \quad (23)$$

$$\sum_{j=0}^n (-1)^j h_j = 0. \quad (24)$$

We observe that the last equation of each of the two previous systems is nothing other than the condition $\chi = \sum_{i=0}^n (-1)^i h_i = 0$, where χ denotes the Euler characteristic. It is known that this condition is a necessary condition for the realization of a flow non-singular of Morse-Smale type.

4 Continuation of a Lyapunov graph to a Lyapunov graph of Smale type

Notation:

1. \mathfrak{h}_{\min} is the minimal number of singularities satisfying the Poincaré-Hopf inequalities (1) – (5) (or equivalently, the system (110)–(115)). This minimal number was presented in [7].
2. $\mathfrak{h}_{\min p}$ is the minimal number of singularities satisfying the Poincaré-Hopf inequalities (1) – (5) and Lemma 3 (or Lemma 4).
3. \mathcal{A}_{\min} is the minimal number of periodic orbits, i.e., $\mathcal{A}_{\min} = \frac{\mathfrak{h}_{\min p}}{2}$.

Theorem 5 *A (generalized) vertex of a Lyapunov graph associated with the homological data $e^+ = B_0^+$, $e^- = B_0^-$ and $\{(B_j^+ - B_j^-)\}_{j=0}^2$ can be continued to a Lyapunov graph of Smale type if and only if its label (h_0, \dots, h_n) satisfies the Poincaré-Hopf inequalities of Smale type (1), (2), (5), (8), (11) and the coupling inequalities of Lemma 3. Equivalently, this same vertex can be continued to a Lyapunov graph of Smale type if and only if its label (h_0, \dots, h_n) satisfies the Poincaré-Hopf inequalities of Smale type (1), (2), (5), (8), (11) and the coupling inequalities of Lemma 4 (or Lemma 3).*

5 Dimension 3 case

5.1 Lyapunov graphs of Smale type in dimension 3

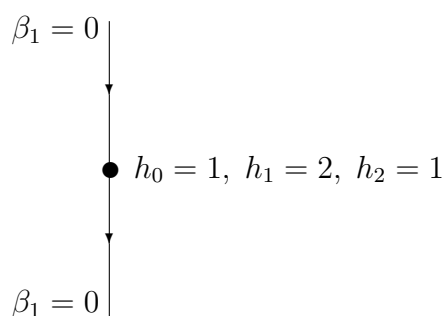


Figure 7: A Lyapunov graph in dimension 3

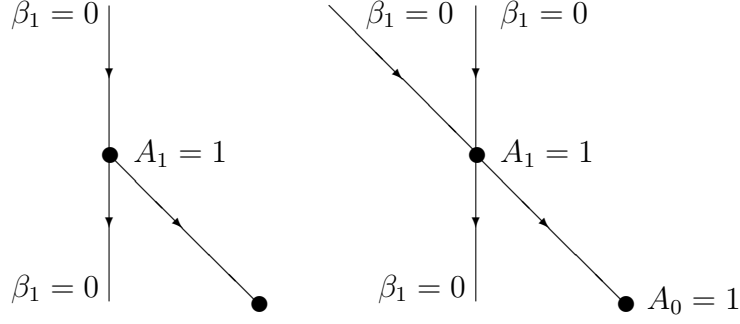


Figure 8: Explosion of a Lyapunov graph of Smale type in dimension 3

5.2 Combinatorial results

Proposition 3 *A (generalized) vertex of a Lyapunov graph associated with the homological data $e^+ = B_0^+$, $e^- = B_0^-$ and $\{(B_j^+ - B_j^-)\}_{j=0}^2$ can be continued to a Lyapunov graph of Smale type if and only if its labels (h_0, h_1, h_2, h_3) satisfies the following inequalities :*

$$\left\{ \begin{array}{l} h_0 \geq 0 \\ h_1 \geq 2h_0 + B_0^- - 1 \\ h_2 \geq 2h_3 + B_0^+ - 1 \\ h_3 \geq 0 \\ h_0 - h_1 + h_2 - h_3 = (B_0^+ - B_0^-) - \frac{(B_1^+ - B_1^-)}{2} \\ h_0 - h_1 + h_2 - h_3 = 0 \end{array} \right.$$

Moreover, by considering A_j the number of periodic orbits of index j , we have

$$\left\{ \begin{array}{l} A_0 = h_0 \\ A_1 = h_1 - h_0 (= h_2 - h_3) \\ A_2 = h_3 \end{array} \right.$$

Proof: A (generalized) vertex of a Lyapunov graph can be continued to a Lyapunov graph of Smale type if and only if it can be continued in a Lyapunov graph of Smale type with singularities which can be paired two by two according to the criterion of consecutive indices. This is equivalent to makes the label (h_0, h_1, h_2, h_3) of the vertex verify the Poincaré-Hopf inequalities for the continuation in a Lyapunov graph of the Smale type and the coupling inequalities. From the discussion of 2, Poincaré Hopf inequalities (PHS) for the continuation to a Lyapunov graph of Smale-type in dimension 3 are :

$$\left\{ \begin{array}{l} h_0 \geq 0 \\ h_1 \geq 2h_0 + B_0^- - 1 \\ h_2 \geq 2h_3 + B_0^+ - 1 \\ h_3 \geq 0 \\ h_0 - h_1 + h_2 - h_3 = (B_0^+ - B_0^-) - \frac{(B_1^+ - B_1^-)}{2} \end{array} \right. \quad (\text{PHS})$$

and, in dimension 3, the coupling conditions are reduced to the equation

$$h_0 - h_1 + h_2 - h_3 = 0$$

since the inequalities $h_1 \geq h_0$ and $h_2 \geq h_3$ are already guaranteed by the previous inequalities (PHS). \square

Corollary 6 *Let v be a vertex of a Lyapunov graph associated with the homological data $e^+ = B_0^+$, $e^- = B_0^-$ and $\{(B_j^+ - B_j^-)\}_{j=0}^2$ which can be continued to a Lyapunov graph of Smale type. Then its continuation contains at least $B_0^+ + B_0^- - 2$ periodic orbits thus distributed*

$$\begin{cases} A_0^{\min} = 0 \\ A_1^{\min} = \max(B_0^+, B_0^-) - 1 \\ A_2^{\min} = 0 \end{cases}$$

Among the A_1 periodic orbits of index 1, there will be

$$\begin{aligned} |B_0^+ - B_0^-| & \text{ of type } (0 - c, 2 - d) ; \\ |B_0^+ - B_0^-| & \text{ of type } (1 - d, 2 - d) \quad \text{si } B_0^+ \geq B_0^- , \\ & \text{ of type } (0 - c, 1 - c) \quad \text{if } B_0^+ < B_0^- . \end{aligned}$$

Moreover, this minimal decomposition corresponds to the labeling of v given by

$$h_1 = h_2 = \max(B_0^+, B_0^-) - 1$$

Proof: \square

Remark: In this case $\mathfrak{h}_{\min p} = \mathfrak{h}_{\min}$ and $\mathcal{A}_{\min} = \frac{\mathfrak{h}_{\min p}}{2} = \frac{\mathfrak{h}_{\min}}{2}$

Corollary 7 *Let v be a vertex of a Lyapunov graph associated with a slice of a manifold $(M, \partial^- M)$ of dimension 3, whose homological boundary data are e^+ , e^- and $\{(B_j^+ - B_j^-)\}_{j=0}^2$. Suppose that v is labeled by the ranks of the homological Conley index of $(M, \partial^- M)$ and that it can be continued in a Lyapunov graph of the Smale type. Then the periodic orbits of the continuation of v are in number of $\text{rk } H_1(M, \partial^- M) = \text{rk } H_2(M, \partial^- M)$ and distributed as*

$$\begin{cases} A_0 = 0 \\ A_1 = \text{rk } H_1(M, \partial^- M) \\ A_2 = 0 \end{cases}$$

Proof: \square

The following formula is a necessary condition for the existence of a Smale flow on a slice of a manifold $(M, \partial^- M)$ of odd dimension [cf. [2]] :

$$0 = \chi(M) - \chi(\partial^- M) = \chi(M, \partial^- M) = \sum_{j=0}^n \text{rk } H_j(M, \partial^- M)$$

Applied to the case of the dimension 3, since we only study connected slices, this gives :

$$\text{rk } H_1(M, \partial^- M) = \text{rk } H_2(M, \partial^- M)$$

5.3 Realization of Lyapunov graphs of Smale type in dimension 3

5.3.1 Attachment of round handles of index 0 (and 2)

Gluing a round handle of index 0 simply means :

- for the manifold creating one more connected component which is a solid torus $D^2 \times S^1$;
- for the boundary adding one more connected component which is a 2-dimensional torus, that is $N^+ = N^- \sqcup T^2$.

Taking the dual point of view tells us that the attachment of a round handle of index 2 makes a torus boundary component disappear, by filling it with a solid torus.

5.3.2 Attachment of round handles of index 1

We present here four elementary attachments of round handles of index 1 which realize each possible homological variations of the boundary (relative to the Betti numbers).

1. *Attachment of type $(0 - c, 2 - d)$*

As one removes and one adds at the same time a boundary component, it is necessary to start from a boundary N^- having at least two connected components. We can therefore restrict ourselves by considering the two components that we want modify and assume $N^- = N_1 \sqcup N_2$. After gluing (see Figure 9 and thinking $N_j = N_j \sharp S^2$ pour $j = 1, 2$), the boundary changes to $N^+ = (N_1 \sharp N_2) \sqcup S^2$. In particular, the Betti numbers of the edge remain unchanged.



Figure 9: Invariant (through β_0)

Concerning the quotients before and after the attachment, we have that $(N^- \times I)/(N^- \times \{0\})$ has the same homotopy type of a point while $((N^- \times I) \cup_{Fig9} R_1)/(N^- \times \{0\})$ has the same homotopy type of a pinched torus, so $h_1 = h_2 = 1$.

2. *Attachment of type $(1 - d, 1 - c)$* As we remove and add at the same time two generators of the first homology group from the boundary, it is necessary to start from an edge N^- having a non-trivial H_1 . We can therefore assume $N^- = N_1 \sharp T^2$. After gluing (see Figure 2), the boundary remains of the form $N^+ = N_1 \sharp T^2$. In particular, the Betti numbers of the boundary remain unchanged.

Concerning the quotients before and after the attachment, we have that $(N^- \times I)/(N^- \times \{0\})$ has the same type of homotopy of a point while $((N^- \times I) \cup_{Fig2} R_1)/(N^- \times \{0\})$ has the same homotopy type of $S^2 \vee S^1$, donc $h_1 = h_2 = 1$.



Figure 10: Invariant (through β_1)

3. *Attachment of type $(1 - d, 2 - d)$* Since we add a connected component and two generators of the first homology group from the boundary, we assume any N^- . After gluing (see Figure 3 and thinking $N^- \# S^2$), the boundary becomes $N^+ = (N^- \# T^2) \sqcup S^2$. In particular, the boundary β_0 was increased by 1 and by 2 the β_1 .



Figure 11: Increasing β_0 and β_1

Concerning the quotients before and after the attachment, we have that $(N^- \times I)/(N^- \times \{0\})$ has the same homotopy type of a point while $((N^- \times I) \cup_{Fig3} R_1)/(N^- \times \{0\})$ has the same homotopy type of a pinched torus, so $h_1 = h_2 = 1$.

4. *Attachment of type $(0 - c, 1 - c)$*

Since we remove two generators from the first homology group of the edge and also a connected component, it is necessary to start from a boundary N^- having at least two connected components and a non-trivial H_1 . We can therefore assume $N^- = N_1 \sqcup (N_2 \# T^2)$. After gluing (see Figure 4 and thinking $N_1 = N_1 \# S^2$), the boundary changes to $N^+ = N_1 \# N_2$. In particular, we have reduced by 1 the β_0 of the boundary and by 2 the β_1 .

Concerning the quotients before and after the attachment, we have that $(N^- \times I)/(N^- \times \{0\})$ has the same type of homotopy of a point while $((N^- \times I) \cup_{Fig4} R_1)/(N^- \times \{0\})$ has the same homotopy type of $S^2 \vee S^1$, so $h_1 = h_2 = 1$.

[To calculate the ranks of the index of Conley from $((N^- \times I) \cup_{Fig4} R_1)/(N^- \times \{0\})$, instead of studying this same space, we can also observe that the quotient space Y relative to the same neighborhood endowed with the opposite flow and defined by



Figure 12: Decreasing β_0 and β_1

$((N^- \times I) \cup_{\text{Fig 4}} R_1)/N^+$ has the same type of homotopy of S^3 prived of a ball and a full torus, i.e. the same homotopy type of $\mathbf{R}^3 \setminus S^1$, therefore, by McCord duality, $h_1 = \text{rk } H_2(Y) = 1$ and $h_2 = \text{rk } H_1(Y) = 1$.]

6 Proof of results

6.1 Background

As described in Section 2, if n is odd, then (1)–(5) need to be satisfied.

In [9], it was proved that if an abstract Lyapunov semi-graph satisfies the Poincaré-Hopf inequalities, then it can be continued to a Lyapunov semi-graph of Morse type. This means that any vertex of the initial abstract Lyapunov semi-graph \mathcal{L} can be replaced by a Lyapunov semi-graph of Morse type L_M , satisfying the same Betti numbers on the e^+ and e^- incoming and outgoing (dangling) edges, and such that the k -th ranks of the Conley homology indices in \mathcal{L} are equal to the number of singularities of index k in L_M . The continuation results are presented by means of a constructive algorithm which produces a linear system hereby called an h^{cd} -system. More precisely, in [9] Proposition 1 was proved.

In the sequel we adopt the following notation: $\mathcal{B}_j = B_j^+ - B_j^-$, for $j = 1, \dots, \lfloor (n-2)/2 \rfloor$, and, if n is odd, $\mathcal{B}_j = (B_j^+ - B_j^-)/2$ for $j = (n-1)/2$, $\mathcal{B}_0^+ = e^+ - 1$ and $\mathcal{B}_0^- = e^- - 1$.

6.2 Minimal number of singularities for pairing: $\mathfrak{h}_{\min p}$

In this section we adress a natural development of the problem addressed in [5]. In that work we presented a new proof for and extended results presented in [7] regarding the Ogasa number for ordered continuations of an abstract Lyapunov semi-graph \mathcal{L} .

Definition 1 *Given positive integers e^+ and e^- , and n integers $\{\mathcal{B}_j\}_{j=1}^{n-2}$ such that $\mathcal{B}_j = \mathcal{B}_{n-1-j}$ for $j = 1, \dots, n-2$, and $\mathcal{B}_i \equiv 0 \pmod{2}$ if $n = 2i + 1$, we say that an n -dimensional manifold M with boundary $\partial M = N^+ \cup N^-$ such that $N^+ \cap N^- = \emptyset$ satisfies the given (homological) boundary conditions if e^+ is the number of connected components of N^+ , e^- is the number of connected components of N^- and $\mathcal{B}_j = B_j^+ - B_j^-$ is the difference of the j -th Betti numbers of the boundary components, that is, $\mathcal{B}_j = \text{rank}(H_j(N^+)) - \text{rank}(H_j(N^-))$.*

It is shown in [7, 5] that the loose information about the boundary suffices to determine the abstract minimal number of singularities \mathfrak{h}_{\min} as well as their indices and types

(connecting and disconnecting). In other words, \mathfrak{h}_{\min} is the optimal value of the integer linear problem below. In [5], explicit analytical optimal solutions are given to the following problem, see Theorem 8 from [5] reproduced below for completeness.

$$\begin{aligned} & \text{Minimize } \sum_{j=1}^{n-1} h_j \\ & \text{subject to } (110)\text{--}(115) \\ & \quad h_j^c, h_j^d, h_j, \beta \in \mathbb{Z}_+, \quad j = 1, \dots, n-1. \end{aligned} \tag{25}$$

Recall that the positive part (resp., negative part) of a number x is $[x]^+ = x^+ = \max\{x, 0\}$ (resp., $[x]^- = x^- = \max\{-x, 0\}$). It follows that $|x| = x^+ + x^-$ and $x = x^+ - x^-$. We will use both notations, preferring the shorter one when possible, with the warning not to mistake the meaning of B_j^+ , B_j^- , \mathcal{B}_0^- , \mathcal{B}_0^+ , e^+ and e^- , defined independently. This inconsistency is acceptable in view of the standardization and simplification of notation that it will make possible.

Theorem 8 *Let e^+ and e^- be positive integers. Let $\{\mathcal{B}_j\}_{j=1}^{n-2}$ be integers such that $\mathcal{B}_j = \mathcal{B}_{n-1-j}$ for all $j = 1, \dots, n-2$. If $n = 2i + 1$, let $\mathcal{B}_i = 0 \pmod{2}$. Then any flow on any n -dimensional manifold M satisfying the given homological boundary conditions must have at least \mathfrak{h}_{\min} singularities, where*

$$\mathfrak{h}_{\min} = \mathcal{B}_0^- + \mathcal{B}_0^+ + \sum_{j=1}^i |\mathcal{B}_j|, \quad \text{if } n = 2i + 1.$$

Let h_j^d denote the number of singularities of index j and type j -d, and let h_j^c denote the number of singularities of index j and type $(j-1)$ -c. Then any nonnegative and integral h^{cd} satisfying

$$\begin{aligned} h_1^c &= \mathcal{B}_0^-, \\ h_{n-1}^d &= \mathcal{B}_0^+, \\ h_j^d + h_{n-j-1}^d &= \mathcal{B}_j^+, & j = 1, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor, \\ h_{j+1}^c + h_{n-j}^c &= \mathcal{B}_j^-, & j = 1, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor, \\ h_i^d &= \mathcal{B}_i^+, & \text{if } n = 2i + 1, \\ h_{i+1}^c &= \mathcal{B}_i^-, & \text{if } n = 2i + 1, \end{aligned}$$

constitutes a possible distribution of the \mathfrak{h}_{\min} singularities.

Suppose $n = 2i + 1$. Let $h_0 = h_n = 0$. In this work we are interested in the nonnegative integral solutions of (110)–(115) that allow for the pairing of the h 's. In Lemma 4, it was

shown that a necessary condition for this was the satisfaction of the following system of inequalities:

$$(-1)^k \sum_{j=1}^k (-1)^j h_j \geq 0, \quad k = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \quad (26)$$

$$(-1)^k \sum_{j=1}^k (-1)^j h_{n-j} \geq 0, \quad k = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \quad (27)$$

$$\sum_{j=1}^{n-1} (-1)^j h_j = 0. \quad (28)$$

In addition, we are interested in the solutions that allow pairings while using the minimum possible number of singularities. That is, we want to find, if they exist, optimal solutions to the integer linear problem

$$\begin{aligned} & \text{Minimize } \sum_{j=1}^{n-1} h_j \\ & \text{subject to } (110)\text{--}(115), (26)\text{--}(28) \\ & \quad h_j^c, h_j^d, h_j \in \mathbb{Z}_+, \quad j = 1, \dots, n-1. \end{aligned} \quad (29)$$

Denote by $\mathfrak{h}_{\min p}$ the optimal value of (29), if it exists. Then

$$\mathfrak{h}_{\min p} \geq \mathfrak{h}_{\min},$$

since (29) is obtained from (25) by adding restrictions.

6.3 Minimum Pairing Problem

We consider the case where n is the odd number $2i + 1$, where i is the middle dimension.

It will be shown in section 6.3.3 that the integer linear problem (29) is feasible if and only if the data $\mathcal{B}_0^+, \mathcal{B}_0^-, \mathcal{B}_j$, for $j = 1, \dots, (n-1)/2$ satisfy

$$\mathcal{B}_0^+ - \mathcal{B}_0^- + \sum_{j=1}^{(n-1)/2} (-1)^j \mathcal{B}_j = 0. \quad (30)$$

As a preliminary step, we show that (30) is a necessary condition for feasibility.

Lemma 9 *Given nonnegative integral \mathcal{B}_0^+ , \mathcal{B}_0^- , and integral \mathcal{B}_j , for $j = 1, \dots, (n-1)/2$, the system of inequalities (110)–(115), (28) has a nonnegative integral solution (h, h^{cd}) only if (30) is satisfied.*

Proof: It was shown in [9] that the system (110)–(115) may be interpreted as the system of equations defining a network flow problem. In this problem, the constants associated with the nodes are $\mathcal{B}_0^+, -\mathcal{B}_0^-, (-1)^j \mathcal{B}_j$, for $j = 1, \dots, (n-1)/2$, and $(-1)^j h_j$, for $j = 1, \dots, n-1$.

It is well known that the necessary condition for such a problem to have a solution is to have the sum of the values associated with the nodes be equal to zero:

$$-\mathcal{B}_0^- + \mathcal{B}_0^+ + \sum_{j=1}^{(n-1)/2} (-1)^j \mathcal{B}_j + \sum_{j=1}^{n-1} (-1)^j h_j = 0. \quad (31)$$

Therefore, a necessary condition for (28) and (31) to be simultaneously satisfied is that (30) holds. \square

Since all entries of the integer problem (29) are integral and the objective function of this problem has a trivial lower bound of zero, if feasible, (29) has an optimal value.

The instance $i = 1$ and $n = 2i + 1 = 3$ leads to the straightforward integer linear problem below.

$$\begin{array}{llll} \text{Minimize} & & h_1 + h_2 & \\ \text{s.t.} & h_1^c & & = \mathcal{B}_0^- \\ & h_1^d & & = \mathcal{B}_0^+ \\ & h_1^c + h_1^d & - h_1 & = 0 \\ & & h_2^c + h_2^d & - h_2 = 0 \\ & & h_1^d - h_2^c & = \mathcal{B}_1 \\ & h_1^c, h_1^d, h_2^c, h_2^d, h_1, h_2 & & \geq 0. \end{array}$$

This is a consequence of the fact that, for $n = 3$, the constraints (26) and (27) reduce to $h_1 \geq 0$ and $h_2 \geq 0$, constraints that were already present in (25). This problem has the general solution

$$\begin{aligned} h_1^c &= \mathcal{B}_0^- \\ h_1^d &= \mathcal{B}_0^+ + \alpha \\ h_1 &= \mathcal{B}_0^- + \mathcal{B}_0^+ + \alpha \\ h_2^c &= \mathcal{B}_1^- + \alpha \\ h_2^d &= \mathcal{B}_1^+ \\ h_2 &= \mathcal{B}_1^- + \mathcal{B}_1^+ + \alpha, \end{aligned}$$

where $\alpha \in \mathbb{Z}_+$. The value of the general solution is $\mathcal{B}_0^- + \mathcal{B}_0^+ + \alpha + \mathcal{B}_1^- + \mathcal{B}_1^+ + \alpha = 2\alpha + \mathcal{B}_0^- + \mathcal{B}_0^+ + |\mathcal{B}_1|$. Hence its optimal value, obtained when $\alpha = 0$, is $\mathfrak{h}_{\min} = \mathcal{B}_0^- + \mathcal{B}_0^+ + |\mathcal{B}_1| = \mathfrak{h}_{\min}$. Thus for the instance $n = 3$, the pairing requirement doesn't eliminate the optimal solution to the minimum singularities problem.

In many discussions, the special case $n = 3$ needs special treatment, since some equations and variables are not present. Since it is a trivial case from the optimization problem point of view, we will simplify the presentation by assuming from now on that $i \geq 2$ when analyzing the odd case.

Henceforth we will assume that (30) holds. This implies constraint (28) is redundant and may be dropped from (29).

6.3.1 1-arc and 2-arc flows equivalent network

The h^{cd} -system of equations (110)–(115) may be shown, by multiplying (an appropriate) half of the equations by -1 , to correspond to the flow conservation equations in an appropriate

network, see [9]. Figure 13 depicts the network corresponding to dimension $n = 7$, or $i = 3$. The underlying directed graph upon which the network is defined was studied in [9, 8, 10, 11]. It has $3i - 1$ nodes and $4i$ arcs. Ignoring the direction of arcs, we have a graph that contains $4 + (i - 1)$ biconnected components. The first four are simply the four arcs with their adjacent nodes at the left of the graph (assuming the planar embedding of Figure 13) and the $i - 1$ remaining ones are the diamond shaped subgraphs that follow on the right, joined at their sides by \mathcal{B} nodes. Arcs in each lozenge are directed from top to bottom. One may imagine the first four arcs on the left as a lozenge that was split along its left node. Attached to each node there is a supply/demand value, whereas the number associated with an arc is interpreted as flow along the arc. Flow conservation equations are associated with each node and state that flow into it minus flow out of it must equal its supply (if the result is negative), its demand (if positive) or zero, that is, the flow into a node must balanced the flow out of it. In the last case the node is called a transshipment node.

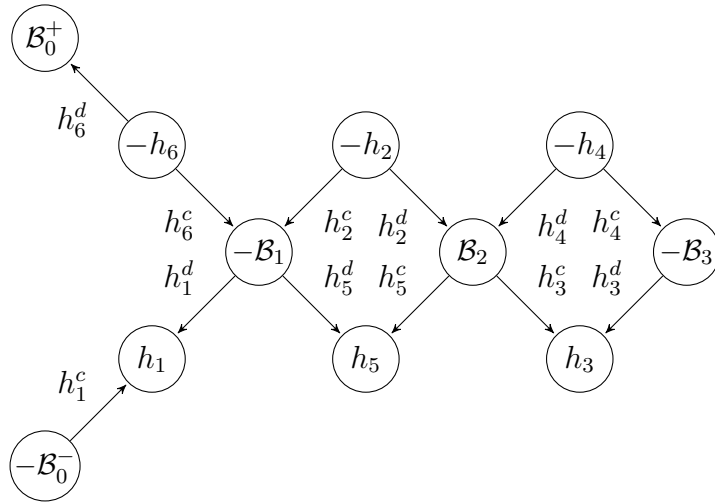


Figure 13: Network corresponding to h^{cd} -system, for $n = 7$.

Let (h, h^{cd}) be a nonnegative integral solution of (110)–(115). Then h^{cd} is a flow in a network whose nodes have supply/demands given by h and \mathcal{B}_0^- , \mathcal{B}_0^+ , \mathcal{B}_1 , \dots , \mathcal{B}_i . The flow into minus flow out of equations express the flow across a network in terms of flows along its arcs. Alternatively, one could express the network flow as flows along paths and cycles. The two are related by flow decomposition theorems. To this network we will apply the flow decomposition theorem given in [1, p. 80], a more general version of the one in [15, p. 8]. Since it is of interest in the present work, we add an integrality assumption to the statement of the theorem. That this results in the integrality of the elements in the decomposition follows directly from the constructive proof given in [1].

Theorem 10 (Flow Decomposition Theorem) *Let G be a network with ℓ nodes and m arcs and integral supplies and demands. Every integral path and cycle flow has a unique representation as nonnegative integral arc flows. Conversely, every nonnegative integral arc flow can be represented as an integral path and cycle flow (though not necessarily uniquely) with the following properties:*

- (a) Every directed path with positive flow connects a supply node to demand node.
- (b) At most $\ell + m$ paths and cycles have nonzero flow; out of these, at most m cycles have nonzero flow.

The networks considered herein contain no cycles, so the decomposition will involve only path flows. Furthermore, the paths either contain one or two arcs. Of course each arc is a 1-arc path. The arcs in the remaining paths fall into one of the possibilities: (1) h_j^d and h_{j-1}^c , for $j = 1, \dots, n-2$, (2) h_j^c and h_{n-j}^d , for $j = 2, \dots, i$, (3) h_j^d and h_{n-j}^c , for $j = 1, \dots, i-1$. Although in the general case the number of paths may potentially be exponential in the number of nodes, in this case the 2-arc paths number $2n - 5 = 4i - 4$. So the total number of paths is less than double the number of arcs. The notation adopted for the 2-arc paths is given in Table 1 below. The 1-arc paths are denoted by j^c and j^d , for $j = 1, \dots, n-1$, depending on whether they carry flow along arc h_j^c or h_j^d (as usual, there is an abuse of notation, the same symbol representing both the arc/path and the value of flow along it). Notice that the decomposition of flow also implies a decomposition of supplies and demands.

arcs in 2-arc path	notation	range
h_j^d, h_{j+1}^c	\mathfrak{d}_j	$j = 1, \dots, n-2$
h_j^c, h_{n-j}^d	\mathfrak{l}_j	$j = 2, \dots, i$
h_j^d, h_{n-j}^c	\mathfrak{r}_j	$j = 1, \dots, i-1$

Table 1: Notation for 2-arc flows.

In order to understand the relationship between arc flows and path flows, it is useful to imagine the flow along an arc, say h_2^d , either “vanishing” into node \mathcal{B}_2 or “passing through” node \mathcal{B}_2 to disappear either at node h_3 or h_{n-2} . The part of the flow corresponding to the first case is the 1-arc path flow along path 2^d , whereas the second part may be split between \mathfrak{r}_2 and \mathfrak{d}_2 . Applying this interpretation is easy to obtain the following equations to express the original h^{cd} variables in terms 1-arc and 2-arc flows:

$$h_1^c = 1^c \tag{32}$$

$$h_j^c = j^c + \mathfrak{l}_j + \mathfrak{d}_{j-1}, \quad j = 2, \dots, i \tag{33}$$

$$h_{i+1}^c = (i+1)^c + \mathfrak{d}_i \tag{34}$$

$$h_j^c = j^c + \mathfrak{r}_{n-j} + \mathfrak{d}_{j-1}, \quad j = i+2, \dots, n-1 \tag{35}$$

$$h_j^d = j^d + \mathfrak{r}_j + \mathfrak{d}_j, \quad j = 1, \dots, i-1 \tag{36}$$

$$h_i^d = i^d + \mathfrak{d}_i \tag{37}$$

$$h_j^d = j^d + \mathfrak{l}_{n-j} + \mathfrak{d}_j, \quad j = i+1, \dots, n-2 \tag{38}$$

$$h_{n-1}^d = (n-1)^d \tag{39}$$

When arc flows are replaced with path flows, all paths in the resulting equivalent network are of length one. In the dimension $n = 7$, the equivalent network is depicted in Figure 14. The 1-arc flows travel between an h node and a \mathcal{B} node. The 2-arc flows between h nodes.

The flow conservation equations for the general case $n = 2i + 1$ are

$$\begin{aligned}
1^c + 1^d + \mathfrak{r}_1 + \mathfrak{d}_1 &= h_1 \\
j^c + j^d + \mathfrak{r}_j + \mathfrak{l}_j + \mathfrak{d}_{j-1} + \mathfrak{d}_j &= h_j, \quad j = 2, \dots, i-1 \\
i^c + i^d + \mathfrak{l}_i + \mathfrak{d}_i + \mathfrak{d}_{i-1} &= h_i, \\
(i+1)^c + (i+1)^d + \mathfrak{l}_i + \mathfrak{d}_i + \mathfrak{d}_{i+1} &= h_{i+1}, \\
j^c + j^d + \mathfrak{r}_{n-j} + \mathfrak{l}_{n-j} + \mathfrak{d}_{j-1} + \mathfrak{d}_j &= h_j, \quad j = i+2, \dots, n-2 \\
(n-1)^c + (n-1)^d + \mathfrak{r}_1 + \mathfrak{d}_{n-2} &= h_{n-1} \\
1^c &= \mathcal{B}_0^- \\
(n-1)^d &= \mathcal{B}_0^+ \\
(n-j-1)^d + j^d &= \mathcal{B}_j^+, \quad j = 1, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor \\
(j+1)^c + (n-j)^c &= \mathcal{B}_j^-, \quad j = 1, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor \\
i^d &= \mathcal{B}_i^+ \\
(i+1)^c &= \mathcal{B}_i^-
\end{aligned}$$

Clearly, as predicted in Theorem 10, the relation between 1-arc and 2-arc flows and h^{cd} is not 1-to-1. Nevertheless, for any h^{cd} there is a set of 1-arc and 2-arc flows that constitute a decomposition thereof that is feasible for the new network and for any set of 1-arc and 2-arc flows satisfying the flow conservation equations in the new network, expressions (32)–(39) will result in a feasible h^{cd} for the original network.

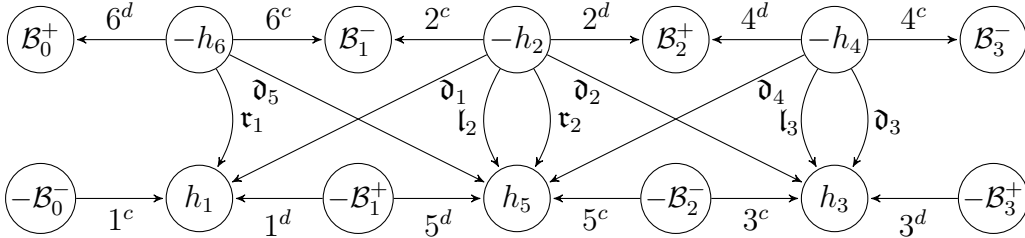


Figure 14: Network with 1-arc and 2-arc flows, case $n = 7$.

Arcs \mathfrak{l}_j and \mathfrak{r}_j in the equivalent network, for $j = 2, \dots, i-2$, are parallel arcs, that is, they share the same tail (node h_{2j} if j is even, node h_{n-2j} otherwise) and head (node h_{n-2j} if j is even, node h_{2j} otherwise). Likewise, \mathfrak{l}_i and \mathfrak{d}_i correspond to parallel arcs. Since arc flows have the same (zero) cost in problem (29), the same will be true for 1-arc and 2-arc flows. Therefore, we may, without affecting the cost of the solution, concentrate all flow along a pair of parallel arcs in one of said arcs, without affecting feasibility or cost of solution. This means we may, without loss of generality, eliminate from the equivalent network one arc from each pair. This further reduces the number of extra arcs added with the new formulation to $3i - 3$. We choose to eliminate \mathfrak{l}_j , for $j = 2, \dots, i$. The new flow conservation equations,

minus the multiplication by -1 , are:

$$1^c + 1^d + \mathbf{r}_1 + \mathbf{d}_1 = h_1 \quad (40)$$

$$j^c + j^d + \mathbf{r}_j + \mathbf{d}_{j-1} + \mathbf{d}_j = h_j, \quad j = 2, \dots, i-1 \quad (41)$$

$$i^c + i^d + \mathbf{d}_i + \mathbf{d}_{i-1} = h_i, \quad (42)$$

$$(i+1)^c + (i+1)^d + \mathbf{d}_i + \mathbf{d}_{i+1} = h_{i+1}, \quad (43)$$

$$j^c + j^d + \mathbf{r}_{n-j} + \mathbf{d}_{j-1} + \mathbf{d}_j = h_j, \quad j = i+2, \dots, n-2 \quad (44)$$

$$(n-1)^c + (n-1)^d + \mathbf{r}_1 + \mathbf{d}_{n-2} = h_{n-1} \quad (45)$$

$$1^c = \mathcal{B}_0^- \quad (46)$$

$$(n-1)^d = \mathcal{B}_0^+ \quad (47)$$

$$(n-j-1)^d + j^d = \mathcal{B}_j^+, \quad j = 1, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor \quad (48)$$

$$(j+1)^c + (n-j)^c = \mathcal{B}_j^-, \quad j = 1, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor \quad (49)$$

$$i^d = \mathcal{B}_i^+ \quad (50)$$

$$(i+1)^c = \mathcal{B}_i^- \quad (51)$$

6.3.2 Eliminating h and introducing alternate sum variables s

In order to eliminate h from the problem altogether, we must rewrite the objective function and the constraints involving the alternate sum inequalities. From Lemma 9, equation (28) may be dropped, since the data \mathcal{B}_0^+ , \mathcal{B}_0^- , \mathcal{B}_j , for $j = 1, \dots, \lfloor (n-1)/2 \rfloor$ are assumed to satisfy (30).

We may use (40)–(45) to remove h from the problem. This will of course require a rewriting of the objective function and of the constraints requiring the nonnegativity of the alternate sums (26)–(27). The objective function of (29) is transformed as follows.

$$\begin{aligned} \sum_{j=1}^{n-1} h_j &= \sum_{j=1}^{n-1} (j^c + j^d) + \sum_{j=2}^i (\mathbf{d}_{j-1}) + \mathbf{d}_i + \sum_{j=i+2}^{n-1} (\mathbf{r}_{n-j} + \mathbf{d}_{j-1}) \\ &\quad + \sum_{j=1}^{i-1} (\mathbf{r}_j + \mathbf{d}_j) + \mathbf{d}_i + \sum_{j=i+1}^{n-2} \mathbf{d}_j \\ &= \sum_{j=1}^{n-1} (j^c + j^d) + 2 \left[\sum_{j=1}^{i-1} \mathbf{r}_j + \sum_{j=1}^{n-2} \mathbf{d}_j \right] \\ &= \mathcal{B}_0^- + \mathcal{B}_0^+ + \sum_{j=1}^i |\mathcal{B}_j| + 2 \left[\sum_{j=1}^{i-1} \mathbf{r}_j + \sum_{j=1}^{n-2} \mathbf{d}_j \right] \\ &= \mathfrak{h}_{\min} + 2 \left[\sum_{j=1}^{i-1} \mathbf{r}_j + \sum_{j=1}^{n-2} \mathbf{d}_j \right], \end{aligned} \quad (52)$$

where the next-to-last equality is obtained by summing (46)–(51) and the last from Theorem 8.

Instead of working with full-blown definitions of the descending and ascending alternate sums

$$s_k = (-1)^k \sum_{j=1}^k (-1)^j h_j, \quad \text{for } k = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \quad (53)$$

and

$$s_{n-k} = (-1)^{n-k} \sum_{j=1}^k (-1)^j h_{n-j}, \quad \text{for } k = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \quad (54)$$

it is preferable to adopt the following equivalent recursive definition. The descending alternate sums are expressed as follows.

$$\begin{aligned} s_1 &= h_1 \\ &= 1^c + 1^d + \mathfrak{r}_1 + \mathfrak{d}_1 \end{aligned} \quad (55)$$

$$\begin{aligned} s_j &= h_j - s_{j-1} \\ &= j^c + j^d + \mathfrak{r}_j + \mathfrak{d}_j + \mathfrak{d}_{j-1} - s_{j-1}, \quad \text{for } j = 2, \dots, i-1 \end{aligned} \quad (56)$$

$$\begin{aligned} s_i &= h_i - s_{i-1} \\ &= i^c + i^d + \mathfrak{d}_i + \mathfrak{d}_{i-1} - s_{i-1}. \end{aligned} \quad (57)$$

The ascending alternate sums may be defined analogously:

$$\begin{aligned} s_{n-1} &= h_{n-1} \\ &= (n-1)^c + (n-1)^d + \mathfrak{r}_1 + \mathfrak{d}_{n-2} \end{aligned} \quad (58)$$

$$\begin{aligned} s_{n-j} &= h_{n-j} - s_{n-j+1} \\ &= (n-j)^c + (n-j)^d + \mathfrak{r}_j + \mathfrak{d}_{n-j-1} + \mathfrak{d}_{n-j} - s_{n-j+1}, \quad \text{for } j = 2, \dots, i-1 \end{aligned} \quad (59)$$

$$\begin{aligned} s_{i+1} &= h_{i+1} - s_{i+2} \\ &= (i+1)^c + (i+1)^d + \mathfrak{d}_i + \mathfrak{d}_{i+1} - s_{i+2}. \end{aligned} \quad (60)$$

The inequalities (26) require the nonnegativity of the descending alternate partial sums s_1, s_2, \dots, s_i , whereas (27) plays the same role for the ascending ones, $s_{n-1}, s_{n-2}, \dots, s_{i+1}$.

The symmetry present in the odd n case leads to the following lemma.

Lemma 11 *Given $\mathcal{B}_0^+, \mathcal{B}_0^-, \mathcal{B}_j$, $j = 1, \dots, i$, such that (30) holds, any solution (h, h^{cd}) of (110)–(115) will satisfy*

$$s_i = s_{i+1}. \quad (61)$$

Proof: The “middle” alternate sums given by (53) and (54) are

$$s_i = (-1)^i \sum_{j=1}^i (-1)^j h_j$$

and

$$s_{i+1} = (-1)^i \sum_{j=1}^i (-1)^j h_{n-j}.$$

Therefore,

$$s_{i+1} - s_i = (-1)^i \sum_{j=1}^i (-1)^j h_{n-j} - (-1)^i \sum_{j=1}^i (-1)^j h_j \quad (62)$$

$$= (-1)^{i+1} \sum_{j=1}^i (-1)^j [h_j - h_{n-j}] \quad (63)$$

$$= (-1)^{i+1} \sum_{j=1}^i (-1)^j [h_j^c + h_j^d - h_{n-j}^c - h_{n-j}^d] \quad (64)$$

$$= (-1)^{i+1} \left\{ -h_1^c + h_{n-1}^d + (-1)^i (h_i^d - h_{i+1}^c) \right. \quad (65)$$

$$\left. + \sum_{j=1}^{i-1} (-1)^j [-h_{j+1}^c + h_j^d - h_{n-j}^c + h_{n-j-1}^d] \right\} \quad (66)$$

$$= (-1)^{i+1} \left\{ -\mathcal{B}_0^- + \mathcal{B}_0^+ + (-1)^i \mathcal{B}_i + \sum_{j=1}^{i-1} (-1)^j \mathcal{B}_j \right\} \quad (67)$$

$$= 0, \quad (68)$$

where the last equality follows from (30). \square

Lemma 11 allows us to rewrite (60) as

$$s_i = (i+1)^c + (i+1)^d + \mathfrak{d}_i + \mathfrak{d}_{i+1} - s_{i+2}. \quad (60')$$

6.3.3 Formulation as a minimum cost flow (MCF) problem

With the replacement of arc flow by path flows, the elimination of h and the introduction of the alternate sum variables, problem (29) is converted into the following equivalent integer linear problem.

$$\begin{aligned} & \text{Minimize} \quad \left[\sum_{j=1}^{n-2} \mathfrak{d}_j + \sum_{j=1}^{i-1} \mathfrak{r}_j \right] \\ & \text{subject to} \quad (46)-(51), (55)-(60') \\ & \quad \quad \quad j^c, j_j^d, \mathfrak{r}_j, \mathfrak{d}_j, s_j \in \mathbb{Z}_+, \quad \text{for all valid } j. \end{aligned} \quad (69)$$

The above problem is equivalent to (29) in the sense that $\mathfrak{h}_{\min p}$ is equal to \mathfrak{h}_{\min} plus twice the optimal value of the above integer linear problem. The optimal value of (69) is denoted by ρ . Thus,

$$\mathfrak{h}_{\min p} = \mathfrak{h}_{\min} + 2\rho. \quad (70)$$

Theorem 12 *The minimum pairing problem is equivalent to an uncapacitated minimum cost flow problem.*

Proof: We show that the matrix of coefficients of the linear constraints (46)–(51), (55)–(60') may be transformed into a node-arc incidence matrix of a digraph by multiplying

appropriate subset of half the equations by -1 . In order to explicit the selection of rows, we must choose how equations (55)–(60′) are written to begin with, since there are variables in both sides of equal sign in these expressions. We fix the following choice:

$$-s_1 + 1^c + 1^d + \mathbf{r}_1 + \mathbf{d}_1 = 0 \quad (71)$$

$$-s_j + j^c + j^d + \mathbf{r}_j + \mathbf{d}_j + \mathbf{d}_{j-1} - s_{j-1} = 0, \text{ for } j = 2, \dots, i-1 \quad (72)$$

$$-s_i + i^c + i^d + \mathbf{d}_i + \mathbf{d}_{i-1} - s_{i-1} = 0 \quad (73)$$

$$-s_{n-1} + (n-1)^c + (n-1)^d + \mathbf{r}_1 + \mathbf{d}_{n-2} = 0 \quad (74)$$

$$-s_{n-j} + (n-j)^c + (n-j)^d + \mathbf{r}_j + \mathbf{d}_{n-j-1} + \mathbf{d}_{n-j} - s_{n-j+1} = 0, \text{ for } j = 2, \dots, i-1 \quad (75)$$

$$-s_i + (i+1)^c + (i+1)^d + \mathbf{d}_i + \mathbf{d}_{i+1} - s_{i+2} = 0 \quad (76)$$

This implies that the entries in the coefficient matrix of the constraints (46)–(51), (71)–(76) are either 0 or 1 for the columns corresponding to the 1-arc flows and 0, ± 1 for the 2-arc flow columns. We will examine each variable at a time. We will also assign labels to the equations to help drawing the network, since each equation will be the flow conservation equation in the associated node.

The selected equations for the sign change are (46); every other equation of (48), starting with the first one; every other equation of (49), starting with the second one; equation (50) if i is odd; equation (51) if i is even; every other equation of (71)–(73), starting with the second one; every other equation in (74)–(76), starting with the first one. We have to show that the matrix obtained changing the signs of these rows has exactly two entries per column, one 1 and the other -1 .

The label assignments for the equations in (46)–(51) is as follows. Equation (46) is assigned the label f_0 , (47) is assigned the label g_0 , the j th equation in (48) receives the labels f_j if j is odd and g_j otherwise, the j th equation in (49) receives the label g_j if j is odd and f_j otherwise. Equation (50) is assigned the label g_i if i is even, f_i otherwise. Equation (51) is assigned the label f_i if i is even, g_i otherwise. This means nodes f_0, f_1, \dots, f_i , associated with the equations in (46)–(51) that were multiplied by -1 , will be supply nodes, whereas g_0, g_1, \dots, g_i are associated with demand nodes.

The nodes associated with equations (71)–(76) will be transshipment nodes, since the constant in the right-hand-side of the flow conservation equation is zero. The j th equation in (71)–(73) is assigned the label t_j if j is odd and u_j if j is even. So equation (73) is labeled u_i if i is even and t_i otherwise. Similarly, the j th equation in (74)–(76) receives the label u_j if j is odd and t_j otherwise, ending with t_i if i is even and u_i otherwise. Notice that the labels u have been assigned to equations selected for the sign change, whereas the labels t were assigned to the unchanged equations.

Notice that each 1-arc flow variables appear exactly once in the set of equations (46)–(51) and once in the set of equations (55)–(60′). We just have to check that these entries will have the appropriate signs after the multiplication.

Consider the 1-arc flow variable 1^c and the associated column. This variable is present in the first equation of (46), labeled f_0 , one of the selected ones. So its coefficient is -1 in the corresponding row. It also appears in the first equation of (71)–(73), labeled t_1 . Since this is not a selected equation, its coefficient in this equation is 1. So this column satisfies the condition. The tail of the arc carrying flow 1^c is f_0 and its head is t_1 . A similar argument applies to $(n-1)^d$. The corresponding arc has tail u_1 and head g_0 .

Now consider the variable j^c , where $j \in \{2, \dots, i\}$. This variable is present with coefficient 1 in the $(j - 1)$ th equation in (49). This equation will be selected only if $j - 1$ is even, or j is odd. Thus if j is odd the variable is in an equation labeled f_{j-1} , and if j is even, in an equation labeled g_{j-1} . It is also present in the j th equation of (71)–(73), with label t_j if j is odd and u_j otherwise. Thus its coefficient will be -1 on this row in the modified matrix only if this equation is selected, which happens only if j is even. So if j is odd, the arc carrying flow j^c has tail f_{j-1} and head t_j . Otherwise, the arc has tail u_j and head g_{j-1} . In either case, the nonzero entries in the column of the modified matrix corresponding to this variable will be 1 and -1 .

Next consider j^d , for $j \in \{1, \dots, i - 1\}$. It has coefficient 1 in the j th equation of (48), so it will have coefficient -1 on the corresponding row in the modified matrix only if j is odd, with label f_j , otherwise the coefficient remains 1 in the equation labeled g_j . It is also present with coefficient 1 in the j th equation of (71)–(73), so its coefficient in this row of the modified matrix is -1 only if j is even. The equation has label t_j if j is odd and u_j otherwise. The arc with flow j^d thus has tail f_j and head t_j if j is odd and tail u_j and head g_j otherwise. Therefore, the corresponding column in the modified matrix will have nonzero entries 1 and -1 .

Next consider i^d . The corresponding column has coefficient 1 in row corresponding to equation (50). So the coefficient in this column and row will be -1 in the modified matrix only if i is odd, receiving label f_i in this case and g_i otherwise. The second equation containing this variable is the last, or i th, equation of (71)–(73). Thus the coefficient of this variable in the modified matrix will be -1 in this row only if i is even. The label in case i is odd (resp., even) is t_i (resp., u_i). The corresponding arc has tail f_i and head t_i if i is odd, and tail u_i and head g_i , otherwise. So again the corresponding column contains 1 and -1 in the modified matrix.

The analyses of columns corresponding to variables $(n - j)^c$ and $(n - j)^d$, for $j = 1, \dots, i$ are analogous.

The variables corresponding to 2-arc flows are not present in equations (46)–(51), only in (71)–(76). The variable \mathfrak{r}_j is present with coefficient 1 in the j th equation in (71)–(73), so it will have coefficient -1 on this in the modified matrix only if j is even. The associated label is t_j if j is odd and u_j otherwise. It is also present with coefficient 1 in the j th equation of (74)–(76), and this will change to -1 in the modified matrix only if j is odd. In this group of equations, the label is u_j if j is odd and t_j otherwise. Thus, in either case, the arc with flow \mathfrak{r}_j has tail u_j and head t_j and the corresponding column in the modified will have the appropriate configuration.

Now consider variable \mathfrak{d}_j , for $j \in \{1, \dots, i - 1\}$, It is present in two consecutive equations in (71)–(73) and so will have entries of opposite signs on these rows in the modified matrix. The tail (resp., head) of the arc carrying \mathfrak{d}_j is u_{j+1} (resp., t_j) if j is odd, and u_j and t_{j+1} otherwise. The variable \mathfrak{d}_{n-j} , for $j \in \{2, \dots, i - 1\}$ is present in two consecutive equations ($j - 1$ th and j th) in (74)–(76), so the corresponding column in the modified matrix has the appropriate pattern. The tail and head of the corresponding arc are u_{j-1} and t_j if j is even, and u_j and t_{j-1} otherwise. Finally, consider variable \mathfrak{d}_i . It is present in the last and i th equation of (71)–(73), and its coefficient will change to -1 in this row only if i is even. The second equation containing this variable is the last and i th equation of (74)–(76), so it will change to -1 in the modified matrix only if i is odd. So the corresponding arc will have tail

u_i and head t_i . Consequently, the new column in the modified matrix satisfies the desired conditions.

The last set of variables are the s_j 's. Each s_j , for $j = 1, \dots, n - 1$ is present in precisely two equations of (71)–(73), with coefficient -1 in both. For $j \neq i$, s_j is present in two consecutive equations either in (71)–(73) or in (74)–(76), so the corresponding column will have the appropriate pattern in the modified matrix. In both cases, the equations involved are the j th and $(j + 1)$ th. In the first case, since the even equations have had the sign changed, the arc corresponding to s_j has tail t_j and head u_{j+1} if j is odd, and tail t_{j+1} and head u_j otherwise. In the second case, since the odd numbered equations have had their sign changed, arc s_{n-j} has tail t_{j+1} and head u_j , if j is odd, and tail t_j and head u_{j+1} otherwise. The exception is s_i . It is present in the last and i th equation of (71)–(73) and in the last and i th equation of (74)–(76), as was the case for \mathfrak{d}_i . Thus the corresponding column in the modified matrix will have precisely two nonzero elements, 1 and -1 . Arc s_i 's tail and head are t_i and u_i . \square

Figure 15 depicts the network corresponding to the minimum cost flow (MCF) problem of the minimum pairing problem when $n = 7$.

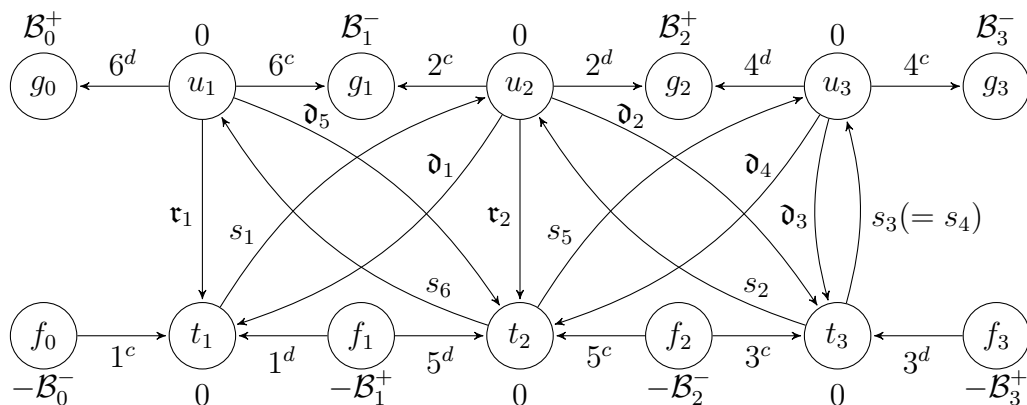


Figure 15: Minimum cost network flow for minimum pairing problem, case $n = 7$.

Theorem 12 allows us to tap into the abundant literature and results available for the MCF problem in uncapacitated networks (i.e., flow along arcs is unbounded above). Integrality constraints, for instance, may be dropped, since it is guaranteed that the linear relaxation of the problem will have an integral optimal solution if the data of the problem are integral, see, for instance, Theorem 9.10 of [1, p. 318]. A host of efficient algorithms for its solution is available, e.g. [18, 20]. If the numerical solution to a specific instance is desired, there are commercial and open source codes implemented for this class of problem. We proceed to explore the special characteristics of this problem to obtain analytical upper and lower bounds for the optimal value of (69).

The MCF network described by equations (46)–(51), (71)–(76), after the appropriate sign changes, has $i + 1$ supply nodes (nodes f_0, \dots, f_i , depicted at the bottom of Figure 15) and $i + 1$ demand nodes (nodes g_0, \dots, g_i , depicted at the top). The constant associated with each node is shown either above or below it. So the flow produced at the bottom needs to be carried to the top. The bottom to top arcs are the arcs associated with the alternate sum

variables. The 2-arc flows \mathfrak{d} and \mathfrak{r} are associated with the top to bottom arcs. The feasibility of the MCF problem is straightforward to show.

Lemma 13 *The MCF problem given by equations (46)–(51), (71)–(76), is feasible.*

Proof: The proof is constructive. Equations (46), (47), (50) and (51) uniquely determine flows 1^c , $(n-1)^d$, i^d and $(i+1)^c$. Arbitrarily choose remaining 1-arc flows so as to satisfy (48)–(49), as follows

$$j^d = \mathcal{B}_j^+ \quad \text{and} \quad (j+1)^c = \mathcal{B}_j^-, \quad \text{for } j = 1, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor,$$

and set remaining 1-arc flows to zero. This satisfies flow conservation constraints at all the demand and supply nodes of the MCF network.

Let $\mathfrak{r} = 0$ and $\mathfrak{d} = 0$. Use the flow conservation equations for the transshipment nodes (71)–(76) to calculate s . Flow s_1 is given by (71) and then s_2, s_3, \dots, s_i may be calculated, in this order, using (72)–(73). Similarly, flow s_{n-1} is given by (74) and s_{n-2}, \dots, s_{i+2} may be calculated, in this order, using (75). Equation (76) is redundant, from Lemma 11.

$$\begin{aligned} s_1 &= 1^c + 1^d \\ &= \mathcal{B}_0^- + \mathcal{B}_1^+ \end{aligned} \tag{77}$$

$$\begin{aligned} s_2 &= -s_1 + 2^c + 2^d \\ &= -\mathcal{B}_0^- - \mathcal{B}_1^+ + \mathcal{B}_1^- + \mathcal{B}_2^+ \end{aligned} \tag{78}$$

\vdots

$$s_j = \sum_{k=1}^j (j-1)^{k+j} (\mathcal{B}_{k-1}^- + \mathcal{B}_k^+), \quad \text{for } j = 2, \dots, i-1 \tag{79}$$

$$\begin{aligned} s_i &= -\sum_{k=1}^{i-1} (-1)^{k+i-1} (\mathcal{B}_{k-1}^- + \mathcal{B}_k^+) + \mathcal{B}_{i-1}^- + \mathcal{B}_i^+ \\ &= \sum_{k=1}^i (-1)^{k+i} (\mathcal{B}_{k-1}^- + \mathcal{B}_k^+) \end{aligned} \tag{80}$$

$$\begin{aligned} s_{n-1} &= (n-1)^c + (n-1)^d \\ &= \mathcal{B}_0^+ \end{aligned} \tag{81}$$

$$\begin{aligned} s_{n-2} &= -s_{n-1} + (n-2)^c + (n-2)^d \\ &= -\mathcal{B}_0^+ \end{aligned}$$

\vdots

$$\begin{aligned} s_{n-j} &= -s_{n-j+1} + (n-j)^c + (n-j)^d \\ &= (-1)^{j-1} \mathcal{B}_0^+, \quad \text{for } j = 1, \dots, i-1. \end{aligned} \tag{82}$$

This solution satisfies all flow conservation constraints. If s thus calculated is nonnegative, we have obtained a feasible solution for the MCF and the lemma is proven. Suppose not.

That is, there is some $j \in \{1, \dots, i\}$ (resp., $j \in \{i+2, \dots, n-1\}$) such that $s_j < 0$. Arc \mathfrak{d}_j (resp., arc \mathfrak{d}_{j-1}) forms a directed cycle with s_j , since the corresponding variables are present in the same pair of equations in (71)–(76) and they carry opposite signs in these equations. So if we send the flow s_j^- along this cycle the new solution will have $s_j = 0$ and $\mathfrak{d}_j = s_j^-$ (resp., $\mathfrak{d}_{j-1} = s_j^-$). Repeat this procedure for every negative component of s . Since the addition of flow along a directed cycle doesn't affect flow conservation constraints, the final nonnegative solution also satisfies (46)–(51) and is therefore feasible. \square

Since the MCF problem is a minimization problem, any feasible solution thereof gives an upper bound $\bar{\rho}$ for the optimal value ρ . For the choice adopted in the proof of Lemma 13, the cost is simply the sum of the 2-arc flows \mathfrak{d} . Using the formulas (77)–(82), we have

$$\begin{aligned} \bar{\rho}_1 &= \sum_{j=1}^i s_j^- + \sum_{j=1}^{i+1} s_{n-j}^- \\ &= \sum_{j=1}^i \left[\sum_{k=1}^j (-1)^{k+j} (\mathcal{B}_{k-1}^- + \mathcal{B}_k^+) \right]^- + \sum_{j=1}^{i+1} [(-1)^{j-1} \mathcal{B}_0^+]^+ \\ &= \sum_{j=1}^i \left[\sum_{k=1}^j (-1)^{k+j} (\mathcal{B}_{k-1}^- + \mathcal{B}_k^+) \right]^- + \left\lfloor \frac{i-1}{2} \right\rfloor \mathcal{B}_0^+. \end{aligned} \quad (83)$$

In this solution, all flow in the right-hand-side of (48)–(49) was concentrated in the 1-arc flow variable with smallest index. If we do the opposite, that is, let

$$(n-j-1)^d = \mathcal{B}_j^+ \quad \text{and} \quad (n-j)^c = \mathcal{B}_j^-, \quad \text{for } j = 1, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor,$$

and repeat the procedure, we obtain the following alternative upper bound for ρ :

$$\bar{\rho}_2 = \sum_{j=1}^{i-1} \left[\sum_{k=1}^j (-1)^{k+j} (\mathcal{B}_{k-1}^+ + \mathcal{B}_k^-) \right]^- + \left\lfloor \frac{i}{2} \right\rfloor \mathcal{B}_0^-. \quad (84)$$

Since both $\bar{\rho}_1$ and $\bar{\rho}_2$ are possible upper bounds for ρ , we may obtain a tighter one with

$$\bar{\rho} = \min\{\bar{\rho}_1, \bar{\rho}_2\}. \quad (85)$$

6.3.4 From an MCF problem to a transportation problem

Theoretically, one may solve an uncapacitated nonnegative costs MCF problem by calculating minimum cost paths (where the cost of a path is equal to sum of costs of arcs in path) from each supply node to each demand node and then solving a transportation problem. The set of supply and demand nodes of transportation problem are the same as in the MCF problem, but there are only arcs from supply nodes to demand nodes, and the cost of the arc from supply node ℓ to demand node j is equal to the cost of the minimum cost path from ℓ to j in the MCF problem. For general networks this approach is not advisable, since the number of paths may be prohibitive. But the special structure of our MCF network make

this not only feasible, but also attractive. As usual, when talking about minimum cost paths, we interpret costs as lengths. Notice that the arcs in the minimum pairing MCF problem have length either zero (all the original 1-arc flows and the alternate sum variables) or one (the 2-arc flows).

To facilitate our dealings in this transition from MCF network to transportation network, we will name the supply of node f_j as $-\theta_j$ and the demand of node g_j as δ_j . Table 2 gives the correspondence between the original and new notation.

$\theta_0 = \mathcal{B}_0^-$
$\theta_j = [(-1)^{j+1}\mathcal{B}_j]^+$, for $j = 1, \dots, i$
$\delta_0 = \mathcal{B}_0^+$
$\delta_j = [(-1)^j\mathcal{B}_j]^+$, for $j = 1, \dots, i$

Table 2: Relationship between θ , δ and \mathcal{B} .

The structure of the MCF network is such that each supply node has outgoing arcs to either one or two of the transshipment nodes with t labels, whereas each demand has incoming arcs from either one or two of the u -labeled transshipment nodes. Thus it behooves us to determine the minimum cost paths from the t -labeled nodes to the u -labeled nodes first. Supply and demand nodes, as well as the arcs incident thereto may be ignored for this task. In other words, we will first focus on the *transshipment subnetwork* of the MCF network, the subnetwork constituted by the transshipment nodes and arcs incident thereto. In the case $n = 7$, the subproblem is to compute the minimum cost paths in the network shown in Figure 16.

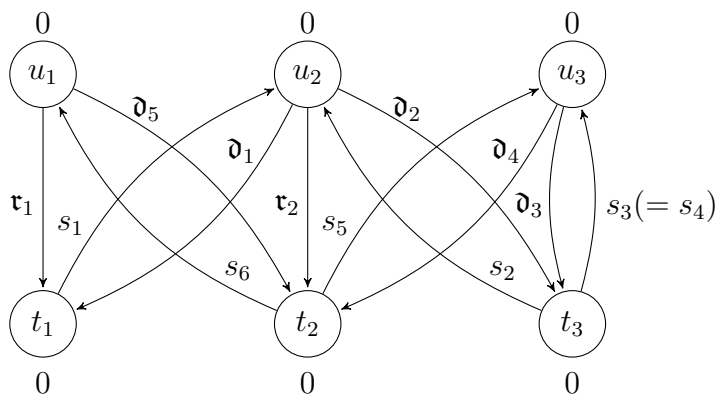


Figure 16: Transshipment subnetwork of the MCF network, case $n = 7$.

Minimum cost paths in the transshipment subnetwork for low values of n can be found by inspection. Since the subnetwork contains no parallel arcs, paths are uniquely determined by the sequence of nodes therein, so they will be described thus. Table 3 gives the correspondence between arcs in the MCF network and flow variables. Table 4 gives the minimum path distances and the respective minimum paths for $n = 7$.

In order to establish the minimum paths and distances for generic n , we will define two

(t_j, u_{j+1})	$\left\{ \begin{array}{l} s_j, \\ s_{n-j}, \end{array} \right.$ if j is odd otherwise	, for $j = 1, \dots, i - 1$
(t_j, u_{j-1})	$\left\{ \begin{array}{l} s_{j-1}, \\ s_{n-j+1}, \end{array} \right.$ if j is odd otherwise	, for $j = 2, \dots, i$
(t_i, u_i)	$s_i (= s_{i+1})$	
(u_j, t_j)	τ_j	, for $j = 1, \dots, i - 1$
(u_i, t_i)	\mathfrak{d}_i	
(u_j, t_{j+1})	$\left\{ \begin{array}{l} \mathfrak{d}_j, \\ \mathfrak{d}_{n-j-1}, \end{array} \right.$ if j is even otherwise	, for $j = 1, \dots, i - 1$
(u_j, t_{j-1})	$\left\{ \begin{array}{l} \mathfrak{d}_{j-1}, \\ \mathfrak{d}_{n-j}, \end{array} \right.$ if j is even otherwise	, for $j = 1, \dots, i - 1$
(f_j, t_j)	$\left\{ \begin{array}{l} j^d, \\ (n-j)^c, \end{array} \right.$ if j is odd otherwise	, for $j = 1, \dots, i$
(f_j, t_{j+1})	$\left\{ \begin{array}{l} (j+1)^c, \\ (n-j-1)^d, \end{array} \right.$ if j is even otherwise	, for $j = 0, \dots, i - 1$
(u_j, g_j)	$\left\{ \begin{array}{l} j^d, \\ (n-j)^c, \end{array} \right.$ if j is even otherwise	, for $j = 1, \dots, i$
(u_j, g_{j-1})	$\left\{ \begin{array}{l} j^c, \\ (n-j)^d, \end{array} \right.$ if j is even otherwise	, for $j = 1, \dots, i$

Table 3: Correspondence between arcs in MCF network and flow variables.

	u_1	u_2	u_3
t_1	1 t_1, u_2, t_2, u_1	0 t_1, u_2, t_2, u_1	1 t_1, u_2, t_3, u_3
t_2	0 t_2, u_1	1 t_2, u_1, t_1, u_2 t_2, u_3, t_3, u_2	0 t_2, u_3
t_3	1 t_3, u_2, t_2, u_1 t_3, u_3, t_2, u_1	0 t_1, u_2, t_2, u_1	0 t_3, u_3

Table 4: Minimum $t_\ell \rightsquigarrow u_j$ paths and distances for the case $n = 7$.

auxiliary functions γ and γ^0 , with \mathbb{Z}_+ as domain and image.

$$\gamma(k) = \begin{cases} 1, & \text{if } k = 0 \\ \lfloor \frac{k}{2} \rfloor, & \text{if } k \geq 1 \end{cases} \quad (86)$$

$$\gamma^0(k) = \left\lfloor \frac{k}{2} \right\rfloor, \quad \text{for } k \geq 0. \quad (87)$$

Their graphs are shown in Figure 17. Notice that, whereas γ^0 is monotone increasing, γ has a minimum at 1.

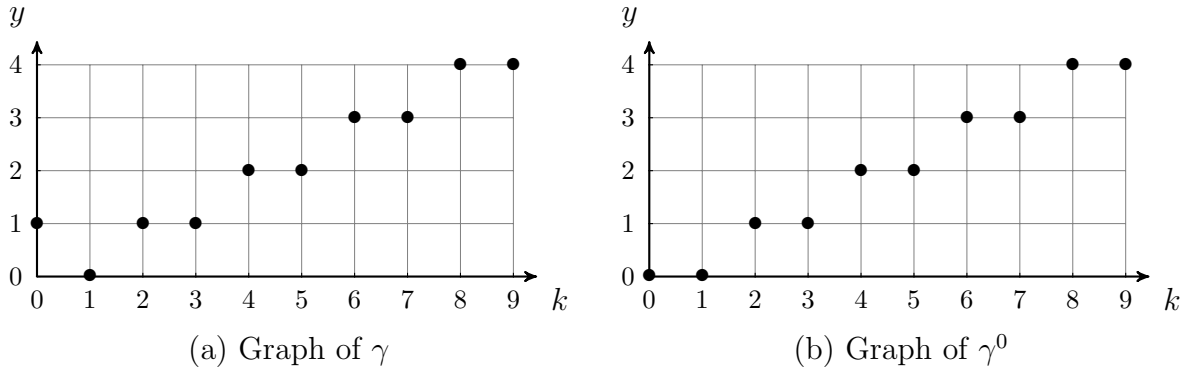


Figure 17: Graphs of γ and γ^0 .

$t_j \rightsquigarrow u_j$	$\left\{ \begin{array}{l} t_j, u_{j-1}, t_{j-1}, u_j \\ t_j, u_{j+1}, t_{j+1}, u_j \end{array} \right\}, \quad \text{if } 1 < j < i$
$t_1 \rightsquigarrow u_1$	t_1, u_2, t_2, u_1
$t_i \rightsquigarrow u_i$	t_i, u_i
$t_\ell \rightsquigarrow u_j$	$(t_{\ell+\alpha 2k}, u_{\ell+\alpha(2k+1)})_{k=0}^{\lfloor \frac{\ell-j}{2} \rfloor}, \quad \text{if } j-\ell \text{ is odd, where } \alpha = \text{sgn}(j-\ell)$
$t_\ell \rightsquigarrow u_j$	$\left\{ t_\ell \rightsquigarrow u_{\ell+\alpha k}, t_{\ell+\alpha k}, t_{\ell+\alpha k} \rightsquigarrow u_j \mid k \text{ odd, } 1 \leq k < j-\ell \right\}, \quad \text{if } j-\ell \text{ is even}$

Table 5: Shortest paths in transshipment subnetwork.

Lemma 14 *Let D be the $i \times i$ matrix whose entry $d_{\ell j}$ contains the minimum distance from t_ℓ to u_j in the transshipment subnetwork when $n = 2i + 1$. Then*

$$d_{\ell j} = \begin{cases} \gamma(|\ell - j|), & \text{if } \ell < i \\ \gamma^0(|\ell - j|), & \text{if } \ell = i, \end{cases} \quad (88)$$

and the shortest paths are given in Table 5.

Proof: From the construction of the MCF network in the proof of Theorem 12, it follows that there are no arcs between transshipment nodes, so the undirected version of the subnetwork is bipartite with the t -labeled nodes in one subset of the partition, called the t -subset, and the u -labeled nodes are in the other subset, called the u -subset. Therefore, paths from a t -labeled node to a u -labeled node must contain an odd number of arcs, as they must start

from one subset of nodes and end in the other, and the nodes in the path must alternate between the two subsets. Arcs outgoing from t -labeled nodes have cost zero, arcs incoming have cost 1. Furthermore, node t_j , for $1 < j < i$, has two outgoing arcs, to u_{j-1} and u_{j+1} , and three incoming arcs, from u_{j-1} , u_j and u_{j+1} . Node t_1 has only one outgoing arc, to u_2 , and two incoming arcs, from u_1 and u_2 . Node t_i has two outgoing arcs, to u_i and u_{i-1} and two incoming arcs, from the same pair of nodes.

First consider the shortest path from node t_j to u_j , for $j < i$. Since the transshipment subnetwork has no arc from t_j to u_j , a shortest path will have at least 3 arcs, with at least one arc in the path going from the u -subset to the t -subset. Thus any path will have length at least one. Since the paths $t_j, u_{j-1}, t_{j-1}, u_j$ and $t_j, u_{j+1}, t_{j+1}, u_j$ have length one and cover the possibilities of outgoing arcs from t_j , they are the shortest path from t_j to u_j . Therefore $d_{jj} = 1$, a value consistent with (88) and the paths in Table 5.

Next we look for the shortest distance from t_ℓ to u_j , for $|\ell - j| \geq 1$. If a path from t_ℓ to u_j has $m + 1$ nodes, then it must have m arcs and it has length $\lfloor m/2 \rfloor$, since m is odd and every other arc in the path, starting from the second one, goes from the u -subset to the t -subset. Thus finding a shortest path with respect to the lengths of the arcs is equivalent to finding a shortest path with respect to the number of arcs. Suppose $t_\ell = t_{\ell_0}, u_{\ell_1}, t_{\ell_2}, u_{\ell_3}, \dots, t_{\ell_{m-1}}, u_{\ell_m} = u_j$ is a path from t_ℓ to u_j in the transshipment subnetwork. Since $|\ell_r - \ell_{r-1}| \leq 1$, for $r = 1, \dots, m$, we have

$$m \geq \sum_{r=1}^m |\ell_r - \ell_{r-1}| \geq \begin{cases} \sum_{r=1}^m (\ell_r - \ell_{r-1}) = \ell_m - \ell_0 = j - \ell, \\ \sum_{r=1}^m (\ell_{r-1} - \ell_r) = \ell_0 - \ell_m = \ell - j, \end{cases}$$

which implies

$$m \geq |\ell - j|.$$

The inequality above implies that any path $t_\ell \rightsquigarrow u_j$ has at least $|\ell - j|$ arcs. Of course, since paths must have an odd number of arcs, the lower bound is $|\ell - j|$ if $|\ell - j|$ is odd and $|\ell - j| + 1$ otherwise. A path that achieves this lower bound must be a shortest path. Let $\alpha = \text{sgn}(j - \ell) = |j - \ell|/(j - \ell)$ (recall that $|j - \ell| \geq 1$). If $|\ell - j|$ is odd, the shortest path will be $(t_{\ell+\alpha 2k}, u_{\ell+\alpha(2k+1)})_{k=0}^{\lfloor |\ell-j|/2 \rfloor}$, with length $\lfloor |\ell - j|/2 \rfloor$. If $|\ell - j|$ is even, then the number of arcs in a shortest path must be $m = |\ell - j| + 1$, if at all achievable. A way to attain this lower bound is to choose odd k such that $1 \leq k < |j - \ell|$. Then the node $u_{\ell+\alpha k}$ is between $u_{\ell+\alpha}$ and u_j , in the sense of subindex value. Since $|\ell - j|$ is even and k is odd, it follows that $|j - (\ell + \alpha k)| = |j - \ell| - k$ is also odd. Consider the path constituted by the shortest path $t_\ell \rightsquigarrow u_{\ell+\alpha k}$, followed by $t_{\ell+\alpha k}$, followed by the shortest path $t_{\ell+\alpha k} \rightsquigarrow u_j$. The first and last parts of this path fall into the first case studied, since the absolute value of the difference of the indices of the extremes of the path is an odd number. So the number of arcs of this path is $k + 1 + |j - \ell| - k = |j - \ell| + 1$ and the length is $\lfloor (|j - \ell| + 1)/2 \rfloor = \lfloor |j - \ell|/2 \rfloor$, which makes it a shortest path from t_ℓ to u_j . Since there are $|j - \ell| - 2$ choices for k , there are $|j - \ell| - 2$ shortest paths in this case. Thus in both cases, the findings were compatible with (88) and Table 5. \square

Table 6 gives the minimum distances $d_{\ell j}$ for $n = 19$. The nature of the minimum distances formulas in (88) is reflected in the symmetry of this distance matrix. Notice that, for $\ell \neq j$, the minimum distance $d_{\ell j} = \gamma(|\ell - j|) = \gamma^0(|\ell - j|)$ depends on $|\ell - j|$ via the function γ . The only exception is d_{ii} , but this entry lies on the diagonal of the distance matrix. Hence, $d_{\ell j} = d_{j\ell}$, $d_{\ell, \ell+k} = d_{\ell, \ell-k}$, for valid values of ℓ , j and k . Therefore, we not only have that $D = D^T$, but also that the entries on each row and column are symmetric with respect to the diagonal element, increasing in value as they get farther from the diagonal entry.

Min distances $t_\ell \rightsquigarrow u_j$									
	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
t_1	1		1	1	2	2	3	3	4
t_2		1		1	1	2	2	3	3
t_3	1		1		1	1	2	2	3
t_4	1	1		1		1	1	2	2
t_5	2	1	1		1		1	1	2
t_6	2	2	1	1		1		1	1
t_7	3	2	2	1	1		1		1
t_8	3	3	2	2	1	1		1	
t_9	4	3	3	2	2	1	1		

Table 6: Minimum distances in the instance $n = 19$.

The shortest distances from f_ℓ to g_j are now straightforward to obtain, since from f_ℓ there are at most two choices of t -labeled nodes to go to, and at most two choices of u -labeled nodes to reach g_j . All arcs incident from (resp., incident to) supply nodes (resp., demand nodes) have distance zero. The diagrams and corresponding expression for $c_{\ell j}$, the minimum distance from f_ℓ to g_j , for the various possible choices are depicted below. Some nodes may

be absent from the diagram shown, for some values of j .

$$c_{0j} = \begin{cases} d_{11}, & \text{if } j = 0 \\ \min\{d_{1j}, d_{1,j+1}\}, & \text{if } 1 \leq j < i \\ d_{1i}, & \text{if } j = i \end{cases} \quad (89)$$

$$c_{lj} = \begin{cases} \min\{d_{l1}, d_{l+1,1}\}, & \text{if } j = 0 \\ \min\{d_{lj}, d_{l,j+1}, d_{l+1,j}, d_{l+1,j+1}\}, & \text{if } 1 \leq j < i \\ \min\{d_{li}, d_{l+1,i}\}, & \text{if } j = i \end{cases} \quad (90)$$

$$c_{ij} = \begin{cases} d_{i1}, & \text{if } j = 0 \\ \min\{d_{ij}, d_{i,j+1}\}, & \text{if } 1 \leq j < i \\ d_{ii}, & \text{if } j = i. \end{cases} \quad (91)$$

(92)

The shortest distances in the MCF network will closely resemble the ones in the transshipment subnetwork. The following function $\tilde{\gamma} : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ will be convenient for expressing these distances:

$$\tilde{\gamma}(k) = \begin{cases} 0, & \text{if } k = 0, \\ \gamma^0(k-1), & \text{if } k \geq 1. \end{cases} \quad (93)$$

We will show in Lemma 15 that the shortest source/sink paths in the MCF network are the ones in Table 7. These paths involve subpaths in the transshipment subnetwork, which are detailed in Table 5.

Lemma 15 *Let C be the $(i+1) \times (i+1)$ matrix, with row and column index set $\mathcal{I} = \{0, 1, \dots, i\}$, whose entry c_{lj} contains the minimum distance from f_l to g_j in the MCF network when $n = 2i + 1$. Then*

$$c_{lj} = \begin{cases} 1, & \text{if } l = j = 0, \\ \tilde{\gamma}(|l-j|), & \text{otherwise.} \end{cases} \quad (94)$$

Proof: From (89),

$$c_{00} = d_{11} = 1,$$

$f_0 \rightsquigarrow g_0$	$f_0, t_1 \rightsquigarrow u_1, g_0$
$f_0 \rightsquigarrow g_1$	$f_0, t_1 \rightsquigarrow u_2, g_1$
$f_0 \rightsquigarrow g_j$	$f_0, t_1 \rightsquigarrow u_j, g_j, \text{ for } j = 2, \dots, i$
$f_\ell \rightsquigarrow g_\ell$	$\left\{ \begin{array}{l} f_\ell, t_\ell \rightsquigarrow u_{\ell+1}, g_\ell \\ f_\ell, t_{\ell+1} \rightsquigarrow u_\ell, g_\ell \end{array} \right\}, \text{ for } \ell = 1, \dots, i-1$
$f_1 \rightsquigarrow g_0$	$f_1, t_2 \rightsquigarrow u_1, g_0$
$f_\ell \rightsquigarrow g_0$	$f_\ell, t_\ell \rightsquigarrow u_1, g_0, \text{ if } \ell = 2, \dots, i-1$
$f_\ell \rightsquigarrow g_{\ell+1}$	$\left\{ \begin{array}{l} f_\ell, t_\ell \rightsquigarrow u_{\ell+1}, g_{\ell+1} \\ f_\ell, t_{\ell+1} \rightsquigarrow u_{\ell+2}, g_{\ell+1} \end{array} \right\}, \text{ if } \ell = 1, \dots, i-1$
$f_\ell \rightsquigarrow g_{\ell-1}$	$\left\{ \begin{array}{l} f_\ell, t_{\ell+1} \rightsquigarrow u_\ell, g_{\ell-1} \\ f_\ell, t_\ell \rightsquigarrow u_{\ell-1}, g_{\ell-1} \end{array} \right\}, \text{ if } \ell = 2, \dots, i-1$
$f_\ell \rightsquigarrow g_{\ell+2}$	$f_\ell, t_{\ell+1} \rightsquigarrow u_{\ell+2}, g_{\ell+2}, \text{ if } \ell = 1, \dots, i-2$
$f_\ell \rightsquigarrow g_{\ell-2}$	$f_\ell, t_\ell \rightsquigarrow u_{\ell-1}, g_{\ell-2}, \text{ if } \ell = 2, \dots, i-1$
$f_\ell \rightsquigarrow g_j$	$\left\{ \begin{array}{l} f_\ell, t_{\ell+1} \rightsquigarrow u_j, g_j, \text{ if } j > \ell \\ f_\ell, t_\ell \rightsquigarrow u_{j+1}, g_j, \text{ otherwise} \end{array} \right\}, \text{ if } 1 \leq \ell < i, \quad 1 \leq j \leq i, \quad \ell - j \geq 3$
$f_i \rightsquigarrow g_i$	$f_i, t_i \rightsquigarrow u_i, g_i$
$f_i \rightsquigarrow g_j$	$\left\{ \begin{array}{l} f_i, t_i \rightsquigarrow u_j, g_j \\ f_i, t_i \rightsquigarrow u_{j+1}, g_j \end{array} \right\} \text{ if } i - j \text{ is odd, } j \leq i - 1$ $f_i, t_i \rightsquigarrow u_{j+1}, g_j, \text{ if } i - j \text{ is even, } j \leq i - 1$

Table 7: Shortest paths in MCF network.

achieved using the path $t_1 \rightsquigarrow u_1$, and so agrees with (94) and Table 7. Assume $1 \leq j < i$. Then, from (89),

$$\begin{aligned}
c_{0j} &= \min\{d_{1j}, d_{1,j+1}\} \\
&= \min\{\gamma(j-1), \gamma(j)\} \\
&= \begin{cases} \gamma(1) = \gamma^0(0), & \text{if } j = 1 \\ \gamma(j-1) = \gamma^0(j-1), & \text{if } 2 \leq j < i. \end{cases} \\
&= \tilde{\gamma}(j),
\end{aligned}$$

consistent with (94). When $j = 1$, then minimum is achieved at $\gamma(1) = d_{12}$ corresponding to path $t_1 \rightsquigarrow u_2$. When $2 \leq j < i$, the minimum corresponds to d_{1j} , or path $t_1 \rightsquigarrow u_j$. As a rule, the indices of the distance at which the minimum is achieved indicate the best subpath in the transshipment subnetwork. Both cases agree with Table 7. Lastly, again from (89), we have

$$c_{0i} = d_{1i} = \gamma(i-1) = \gamma^0(i-1) = \tilde{\gamma}(i),$$

as prescribed by (94), where the second inequality uses the assumption $i \geq 2$. Minimum is achieved at d_{1i} , also in accordance with Table 7.

Now assume $1 \leq \ell < i$. Let $j = \ell$. Then, from (90), we have

$$\begin{aligned}
c_{\ell\ell} &= \min\{d_{\ell\ell}, d_{\ell,\ell+1}, d_{\ell+1,\ell}, d_{\ell+1,\ell+1}\} \\
&= \min\{\gamma(1), \gamma(0)\} \\
&= \gamma(1) \\
&= \gamma^0(0) \\
&= \tilde{\gamma}(0),
\end{aligned}$$

as desired. The minimum is achieved at $d_{\ell,\ell+1}$ and $d_{\ell+1,\ell}$. Now let $j = 0$. Then

$$\begin{aligned}
c_{\ell 0} &= \min\{d_{\ell 1}, d_{\ell+1,1}\} \\
&= \min\{\gamma(\ell - 1), \gamma(\ell)\} \\
&= \begin{cases} \gamma(\ell) = \gamma(1) = 0, & \text{if } \ell = 1 \\ \gamma(\ell - 1), & \text{if } \ell \geq 2 \end{cases} \\
&= \begin{cases} \gamma^0(0) = \gamma^0(\ell - 1), & \text{if } \ell = 1 \\ \gamma^0(\ell - 1), & \text{if } \ell \geq 2 \end{cases} \\
&= \tilde{\gamma}(\ell).
\end{aligned}$$

When $\ell = 1$ the minimum is achieved at $d_{\ell+1,\ell} = d_{21}$, and when $2 \leq \ell < i$, at $d_{\ell 1}$.

Still letting $1 \leq \ell < i$, assume $1 \leq k = |\ell - j|$, $0 \neq j < i$. Then,

$$\begin{aligned}
c_{\ell j} &= \min\{d_{\ell j}, d_{\ell,j+1}, d_{\ell+1,j}, d_{\ell+1,j+1}\} \\
&= \min\{\gamma(k), \gamma(k+1), \gamma(k-1)\}.
\end{aligned}$$

If $k \leq 2$, then $1 \in \{k-1, k, k+1\}$, which implies $\min\{\gamma(k), \gamma(k+1), \gamma(k-1)\} = \gamma(1) = 0$. If $k = 1$ and $j = \ell + 1$, the minimum is achieved at $d_{\ell,\ell+1}$ and $d_{\ell+1,\ell+2}$. If $k = 1$ and $j = \ell - 1$, the minimum is achieved at $d_{\ell,\ell-1}$ and $d_{\ell+1,\ell}$. If $k = 2$ and $j = \ell + 2$, the minimum is $d_{\ell+1,\ell+2}$. If $k = 2$ and $j = \ell - 2$, the minimum is $d_{\ell,\ell-1}$. If $k > 2$, then $\min\{\gamma(k), \gamma(k+1), \gamma(k-1)\} = \gamma(k-1)$. When $j > \ell$, this minimum is achieved at $d_{\ell+1,j}$, and, when $j < \ell$, at $d_{\ell,j+1}$. The f_ℓ to g_j distance may be expressed as follows

$$\begin{aligned}
c_{\ell j} &= \begin{cases} \gamma^0(0) = \gamma^0(1), & \text{if } 1 \leq k = |\ell - j| \leq 2 \\ \gamma(k-1) = \gamma^0(k-1), & \text{if } 2 \leq k = |\ell - j| \end{cases} \\
&= \gamma^0(k-1) \\
&= \tilde{\gamma}(|\ell - j|)
\end{aligned}$$

Now let $1 \leq \ell < i$ and $j = i$. Then,

$$\begin{aligned}
c_{\ell i} &= \min\{d_{\ell i}, d_{\ell+1,i}\} \\
&= \min\{\gamma(i - \ell), \gamma(i - \ell - 1)\} \\
&= \gamma(i - \ell - 1) \\
&= \gamma^0(i - \ell - 1) \\
&= \tilde{\gamma}(|\ell - i|),
\end{aligned}$$

achieved at $d_{\ell+1,i}$, in accordance with (94) and Table 7.

Finally, let $\ell = i$. If $j = 0$, using (91) and (88),

$$c_{i0} = d_{i1} = \gamma^0(i-1) = \tilde{\gamma}(i).$$

Let $1 \leq j < i$. Then,

$$\begin{aligned} c_{ij} &= \min\{d_{ij}, d_{i,j+1}\} \\ &= \min\{\gamma^0(i-j), \gamma^0(i-j-1)\} \\ &= \gamma^0(i-j-1) \\ &= \tilde{\gamma}(|j-i|), \end{aligned}$$

achieved at $d_{i,j+1}$ if $i-j$ is even, and at both d_{ij} and $d_{i,j+1}$ if $i-j$ is odd, since $\gamma^0(2k) = \gamma^0(2k+1)$. The last possibility is $j = i$. Using (91),

$$c_{ii} = d_{ii} = \gamma^0(0) = 0 = \tilde{\gamma}(0),$$

as desired, which proves the lemma. \square

Figure 18 contains the minimum distance tables for $n = 5$ ($i = 2$), $n = 7$ ($i = 3$), $n = 9$ ($i = 4$) and $n = 19$ ($i = 9$), showing only nonzero values. It is possible to confirm that the table for smaller values of n are principal submatrices of the ones for higher values, in accordance with the formulas for the minimum distances given in (94). We now have a transportation problem that is equivalent to the MCF problem, namely

$$\begin{aligned} \text{Minimize} \quad & \sum_{j=0}^i \sum_{\ell=0}^i c_{\ell j} x_{\ell j} \\ \text{s. t.} \quad & \sum_{j=0}^i x_{\ell j} = \theta_{\ell}, \quad \text{for } \ell = 0, \dots, i \\ & \sum_{\ell=0}^i x_{\ell j} = \delta_j, \quad \text{for } j = 0, \dots, i \\ & x \geq 0. \end{aligned} \tag{95}$$

The variable $x_{\ell j}$ represents the flow from source f_{ℓ} to sink g_j . The entry (ℓ, j) of matrix C contains the cost of $x_{\ell j}$. The solution x may also be organized into a matrix. The supply satisfaction constraints of (95) translate to equating the sum of flows on the ℓ th row to the ℓ th supply θ_{ℓ} (first set of equations), and the demand satisfaction constraints requires the sum of flows along the j th column to be equal to the demand associated with that column, δ_j (second set of constraints). The pattern of distances will allow us to construct lower bounds for ρ , the optimal value of the MCF and of the transportation problem.

6.3.5 Lower bounds on ρ

We will construct two types of lower bounds on ρ : *column lower bounds* and *row lower bounds*. The rules are the same, only the roles are exchanged, columns for rows, demands for supplies.

	g_0	g_1	g_2
f_0	1		
f_1			
f_2			

(a) $n = 5$

	g_0	g_1	g_2	g_3
f_0	1			1
f_1				
f_2				
f_3	1			

(b) $n = 7$

	g_0	g_1	g_2	g_3	g_4
f_0	1			1	1
f_1					1
f_2					
f_3	1				
f_4	1	1			

(c) $n = 9$

	g_0	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9
f_0	1			1	1	2	2	3	3	4
f_1					1	1	2	2	3	3
f_2						1	1	2	2	3
f_3	1						1	1	2	2
f_4	1	1						1	1	2
f_5	2	1	1						1	1
f_6	2	2	1	1						1
f_7	3	2	2	1	1					
f_8	3	3	2	2	1	1				
f_9	4	3	3	2	2	1	1			

(d) $n = 19$

Figure 18: Minimum source/sink distances in MCF network.

The rationale for building a column lower bound is to calculate the minimum cost of satisfying the demand required at that column, using all resources (the supplies) in the most economical way for this column, without regard for the other requirements. The last is what makes this a lower bound, since we are considering only the most advantageous situation possible, irrespective of how this choice might impact others.

Let $\mathcal{I} = \{0, \dots, i\}$. Take a column $j \in \mathcal{I}$. The flows arriving at node g_j travel along arcs of costs $0, 1, \dots, \max\{c_{0j}, c_{ij}\} = \max\{\tilde{\gamma}(j), \tilde{\gamma}(i-j)\} = \max\{\lfloor (j-1)/2 \rfloor, \lfloor (i-j-1)/2 \rfloor\} = c_{\max}(j)$. Let

$$\Theta_k^j = \sum_{\ell \in \mathcal{I}: |\ell-j| \leq 2k+2} \theta_\ell$$

be the supply available to node g_j at cost at most k , for $k = 1, \dots, c_{\max}(j)$. Then the part of the demand at node g_j that must be satisfied at cost greater than or equal to $k+1$ is

$$[\delta_j - \Theta_k^j]^+.$$

Therefore the part of the demand of node g_j that must be satisfied at cost k in this most favorable scenario is

$$\left[\delta_j - \underbrace{[\delta_j - \Theta_k^j]^+}_{\text{demand that must be supplied at cost } \geq k+1} - \underbrace{\Theta_{k-1}^j}_{\text{available supply at cost } \leq k-1} \right]^+.$$

Therefore, summing the contributions of all columns, we obtain the following column lower bound for ρ :

$$\rho_1 = \sum_{j=0}^i \sum_{k=1}^{c_{\max}(j)} k \left[\delta_j - [\delta_j - \Theta_k^j]^+ - \Theta_{k-1}^j \right]^+. \quad (96)$$

The construction of the row lower bound is analogous, but instead of supply available at cost at most k , we need the concept of at cost at most k . For each supply node f_ℓ we define

$$\Delta_k^\ell = \sum_{j \in \mathcal{I}: |\ell-j| \leq 2k+2} \delta_j.$$

The expression of the row lower bound is then

$$\rho_2 = \sum_{\ell=0}^i \sum_{k=1}^{c_{\max}(\ell)} k \left[\theta_\ell - [\theta_\ell - \Delta_k^\ell]^+ - \Delta_{k-1}^\ell \right]^+. \quad (97)$$

Taking the maximum of the bounds we obtain the better lower bound

$$\underline{\rho} = \max\{\underline{\rho}_1, \underline{\rho}_2\}. \quad (98)$$

n	ρ^*
5	$[\mathcal{B}_0^- - (\mathcal{B}_1^- + \mathcal{B}_2^+)]^+ = [\mathcal{B}_0^+ - (\mathcal{B}_1^+ + \mathcal{B}_2^-)]^+$
7	$[\mathcal{B}_0^- - (\mathcal{B}_1^- + \mathcal{B}_2^+)]^+ + [\mathcal{B}_3^+ - ([-\mathcal{B}_0^- + \mathcal{B}_1^- + \mathcal{B}_2^+]^+ + \mathcal{B}_3^-)]^+$ $= [\mathcal{B}_0^+ - (\mathcal{B}_1^+ + \mathcal{B}_2^-)]^+ + [\mathcal{B}_3^- - ([-\mathcal{B}_0^+ + \mathcal{B}_1^+ + \mathcal{B}_2^-]^+ + \mathcal{B}_3^+)]^+$

Table 8: Analytical expressions for ρ^* , the optimal value of (69), for $n = 5$ and 7.

6.3.6 Analytical solutions for $n = 5$ and 7

In this section we show that the formulas for the optimal value of ρ^* of (69) given in Table 8 for $n = 5$ and $n = 7$ are correct. This will follow from considerations about the special transportation problem considered here. In each case, we give two alternative expressions for ρ^* . Notice that it is enough to prove the formula for $n = 7$. Case $n = 5$ may be transformed into case $n = 7$ by letting $\mathcal{B}_3 = 0$.

First we establish the equality of the alternate expressions furnished for ρ^* in the case $n = 7$. Equation (30) implies that

$$\mathcal{B}_0^- + \mathcal{B}_1^+ + \mathcal{B}_2^- + \mathcal{B}_3^+ = \mathcal{B}_0^+ + \mathcal{B}_1^- + \mathcal{B}_2^+ + \mathcal{B}_3^-. \quad (99)$$

If $\mathcal{B}_3 = 0$, (99) implies

$$\mathcal{B}_0^- - \mathcal{B}_1^- - \mathcal{B}_2^+ = \mathcal{B}_0^+ - \mathcal{B}_1^+ - \mathcal{B}_2^-,$$

and thus

$$\mathcal{B}_0^- - \mathcal{B}_1^- - \mathcal{B}_2^+ = \mathcal{B}_0^+ - \mathcal{B}_1^+ - \mathcal{B}_2^- \quad \Rightarrow \quad [\mathcal{B}_0^- - \mathcal{B}_1^- - \mathcal{B}_2^+]^+ = [\mathcal{B}_0^+ - \mathcal{B}_1^+ - \mathcal{B}_2^-]^+.$$

Furthermore,

$$[\mathcal{B}_3^+ - ([-\mathcal{B}_0^- + \mathcal{B}_1^- + \mathcal{B}_2^+]^+ + \mathcal{B}_3^-)]^+ = [-(-[\mathcal{B}_0^- + \mathcal{B}_1^- + \mathcal{B}_2^+]^+)]^+ = 0 = [-(-[\mathcal{B}_0^+ + \mathcal{B}_1^+ + \mathcal{B}_2^-]^+)]^+,$$

and the alternate expressions for the optimal value of ρ^* in Table 8, for each n , are equal.

If $\mathcal{B}_3 < 0$, the first expression for ρ^* when $n = 7$ simplifies as follows.

$$\begin{aligned} [\mathcal{B}_0^- - (\mathcal{B}_1^- + \mathcal{B}_2^+)]^+ + [\mathcal{B}_3^+ - ([-\mathcal{B}_0^- + \mathcal{B}_1^- + \mathcal{B}_2^+]^+ + \mathcal{B}_3^-)]^+ &= [\mathcal{B}_0^- - (\mathcal{B}_1^- + \mathcal{B}_2^+)]^+ \\ &= [\mathcal{B}_0^+ - (\mathcal{B}_1^+ + \mathcal{B}_2^-) + \mathcal{B}_3^-]^+ \end{aligned}$$

The second expression can be rewritten thus.

$$\begin{aligned} &[\mathcal{B}_0^+ - (\mathcal{B}_1^+ + \mathcal{B}_2^-)]^+ + [\mathcal{B}_3^- - ([-\mathcal{B}_0^+ + \mathcal{B}_1^+ + \mathcal{B}_2^-]^+ + \mathcal{B}_3^+)]^+ = \\ &= [\mathcal{B}_0^+ - (\mathcal{B}_1^+ + \mathcal{B}_2^-)]^+ + [\mathcal{B}_3^- - [-\mathcal{B}_0^+ + \mathcal{B}_1^+ + \mathcal{B}_2^-]^+]^+ \\ &= \begin{cases} [\mathcal{B}_0^+ - (\mathcal{B}_1^+ + \mathcal{B}_2^-)]^+ + [\mathcal{B}_3^-]^+, & \text{if } \mathcal{B}_0^+ - (\mathcal{B}_1^+ + \mathcal{B}_2^-) \geq 0 \\ [\mathcal{B}_3^- - ([-\mathcal{B}_0^+ + \mathcal{B}_1^+ + \mathcal{B}_2^-]^+)]^+, & \text{otherwise.} \end{cases} \\ &= [\mathcal{B}_0^+ - (\mathcal{B}_1^+ + \mathcal{B}_2^-) + \mathcal{B}_3^-]^+, \end{aligned}$$

therefore both expressions coincide. The case $\mathcal{B}_3 > 0$ is analogous.

Now we show that the expressions in Table 8 provide lower bounds for ρ^* . To do that, we build the dual of (95) for the case $n = 7$. The ℓ th supply satisfaction equation is associated with the dual variable u_ℓ , the j th demand equation with v_j . The dual is the linear optimization problem given below. It is known that the value of any feasible dual solution provides a lower bound for the primal optimal value. The dual optimal value coincides with the primal optimal value if both problems are feasible.

$$\begin{aligned} \text{Minimize } z(u, v) &= \sum_{\ell=0}^3 \theta_\ell u_\ell + \sum_{j=0}^3 \delta_j v_j \\ \text{s.t. } u_\ell + v_j &\leq \begin{cases} 1, & \text{if } (\ell, j) = (0, 0), (0, 3) \text{ or } (3, 0) \\ 0, & \text{otherwise} \end{cases} \\ u, v &\geq 0. \end{aligned} \tag{100}$$

The bound will be obtained with three dual feasible solutions. Solution $(u^0, v^0) = 0$ is trivially feasible. The feasibility of (u^1, v^1) and (u^2, v^2) may be verified in the matrices in Table 9, along with their values. For the first two matrices, the (ℓ, j) entry contains the sum $u_\ell + v_j$ for the (u, v) vector given along the right and bottom sides of the matrix. The entry on the top left corner of the (ℓ, j) cell contains the right-hand-side of the corresponding restraint in (100), when different from zero. The third matrix lists the original notation for supplies and demands, to facilitate checking the dual solution values.

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padding: 2px;">\mathcal{B}_2^+</td><td style="border: 1px solid black; padding: 2px;">\mathcal{B}_3^-</td><td style="border: 1px solid black; padding: 2px;"></td></tr> </table>		g_0	g_1	g_2	g_3		f_0	1			1	\mathcal{B}_0^-	f_1					\mathcal{B}_1^+	f_2					\mathcal{B}_2^-	f_3	1				\mathcal{B}_3^+		\mathcal{B}_0^+	\mathcal{B}_1^-	\mathcal{B}_2^+	\mathcal{B}_3^-	
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Table 9: Dual feasible solutions and original notation for supplies and demands.

If $\mathcal{B}_3 \geq 0$ and $\mathcal{B}_0^- \geq \mathcal{B}_1^- + \mathcal{B}_2^+$, then

$$\begin{aligned} &[\mathcal{B}_0^- - (\mathcal{B}_1^- + \mathcal{B}_2^+)]^+ + [\mathcal{B}_3^+ - ([-\mathcal{B}_0^- + \mathcal{B}_1^- + \mathcal{B}_2^+]^+ + \mathcal{B}_3^-)]^+ = \\ &= \mathcal{B}_0^- - (\mathcal{B}_1^- + \mathcal{B}_2^+) + [\mathcal{B}_3^+]^+ \\ &= \mathcal{B}_0^- - \mathcal{B}_1^- - \mathcal{B}_2^+ + \mathcal{B}_3^+ \\ &= z(u^1, v^1). \end{aligned}$$

If $\mathcal{B}_3 \geq 0$ and $\mathcal{B}_1^- + \mathcal{B}_2^+ - \mathcal{B}_3^+ \leq \mathcal{B}_0^- < \mathcal{B}_1^- + \mathcal{B}_2^+$, then

$$\begin{aligned} &[\mathcal{B}_0^- - (\mathcal{B}_1^- + \mathcal{B}_2^+)]^+ + [\mathcal{B}_3^+ - ([-\mathcal{B}_0^- + \mathcal{B}_1^- + \mathcal{B}_2^+]^+ + \mathcal{B}_3^-)]^+ = \\ &= [\mathcal{B}_3^+ - (-\mathcal{B}_0^- + \mathcal{B}_1^- + \mathcal{B}_2^+)]^+ \\ &= \mathcal{B}_0^- - \mathcal{B}_1^- - \mathcal{B}_2^+ + \mathcal{B}_3^+ \\ &= z(u^1, v^1). \end{aligned}$$

If $\mathcal{B}_3 \geq 0$ and $\mathcal{B}_0^- < \mathcal{B}_1^- + \mathcal{B}_2^+ - \mathcal{B}_3^+$, then

$$\begin{aligned} & [\mathcal{B}_0^- - (\mathcal{B}_1^- + \mathcal{B}_2^+)]^+ + [\mathcal{B}_3^+ - ([-\mathcal{B}_0^- + \mathcal{B}_1^- + \mathcal{B}_2^+]^+ + \mathcal{B}_3^-)]^+ = \\ & = [\mathcal{B}_3^+ - (-\mathcal{B}_0^- + \mathcal{B}_1^- + \mathcal{B}_2^+)]^+ \\ & = 0 \\ & = z(u^0, v^0). \end{aligned}$$

If $\mathcal{B}_3 < 0$ and $\mathcal{B}_0^- \geq \mathcal{B}_1^- + \mathcal{B}_2^+$, then

$$\begin{aligned} & [\mathcal{B}_0^- - (\mathcal{B}_1^- + \mathcal{B}_2^+)]^+ + [\mathcal{B}_3^+ - ([-\mathcal{B}_0^- + \mathcal{B}_1^- + \mathcal{B}_2^+]^+ + \mathcal{B}_3^-)]^+ = \\ & = [\mathcal{B}_0^- - (\mathcal{B}_1^- + \mathcal{B}_2^+)]^+ \\ & = \mathcal{B}_0^- - \mathcal{B}_1^- - \mathcal{B}_2^+ \\ & = z(u^2, v^2). \end{aligned}$$

If $\mathcal{B}_3 < 0$ and $\mathcal{B}_0^- < \mathcal{B}_1^- + \mathcal{B}_2^+$, then

$$[\mathcal{B}_0^- - (\mathcal{B}_1^- + \mathcal{B}_2^+)]^+ + [\mathcal{B}_3^+ - ([-\mathcal{B}_0^- + \mathcal{B}_1^- + \mathcal{B}_2^+]^+ + \mathcal{B}_3^-)]^+ = 0 = z(u^0, v^0).$$

Therefore $[\mathcal{B}_0^- - (\mathcal{B}_1^- + \mathcal{B}_2^+)]^+ + [\mathcal{B}_3^+ - ([-\mathcal{B}_0^- + \mathcal{B}_1^- + \mathcal{B}_2^+]^+ + \mathcal{B}_3^-)]^+$ is a lower bound for ρ^* . We show that it is indeed the actual optimal value by constructing a primal feasible solution that achieves this bound.

We will use the θ, δ notation. In this notation, the bound is

$$\rho^* = [\theta_0 - (\delta_1 + \delta_2)]^+ + [\theta_3 - ([-\theta_0 + \delta_1 + \delta_2]^+ + \delta_3)]^+.$$

Since we can't have both $\theta_3 = \mathcal{B}_3^+$ and $\delta_3 = \mathcal{B}_3^-$ positive, we will consider the case $\theta_3 \geq 0$, so that $\delta_3 = 0$. To solve the remaining case, $\delta_3 \geq 0$, simply transpose the problem, that is, switch supplies and demands, and reduce to the first case.

Let $x_{\ell j}$ be the flow from f_ℓ to g_j . Let

$$x_{00} = [\theta_0 - \delta_1 - \delta_2]^+ \tag{101}$$

and

$$x_{30} = [\theta_3 - ([-\theta_0 + \delta_1 + \delta_2]^+ + \delta_3)]^+ = [\theta_3 - [-\theta_0 + \delta_1 + \delta_2]^+]^+. \tag{102}$$

Notice that $\delta_3 = 0$ implies that $x_{\ell 3} = 0$, for all ℓ . Since x_{03} is the only variable, besides x_{00} and x_{30} , with unit cost, if we can assign values to the remaining entries of x so as to obtain a feasible vector, this feasible vector will have cost ρ^* , and therefore be an optimal solution.

First we have to show that the assignments are valid, that is, x_{00} and x_{30} are nonnegative integers (trivial) and don't introduce an obstacle to the completion of the solution. So they must satisfy

$$x_{00} \leq \theta_0 \tag{103}$$

$$x_{30} \leq \theta_3 \tag{104}$$

$$x_{00} + x_{30} \leq \delta_0. \tag{105}$$

Inequalities (103) and (104) are direct consequences of (101) and (102). To show (105), we consider two possibilities.

The first possibility is $\theta_0 - \delta_1 - \delta_2 \geq 0$. Then

$$x_{00} + x_{30} = \theta_0 - \delta_1 - \delta_2 + [\theta_3 - 0]^+ = \delta_0 - \theta_1 - \theta_2 - \theta_3 + \theta_3 \leq \delta_0,$$

where the last equality follows from (99).

The second possibility is $\theta_0 - \delta_1 - \delta_2 < 0$. Then

$$x_{00} + x_{30} = 0 + [\theta_3 - (-\theta_0 + \delta_1 + \delta_2)]^+ = [\delta_0 - \theta_1 - \theta_2]^+ \leq \delta_0.$$

Thus we conclude (105) is satisfied.

Now we proceed to complete the solution. Let

$$x_{01} = \min\{\delta_1, \theta_0 - x_{00}\} \quad \text{and} \quad x_{02} = \min\{\delta_2, \theta_0 - x_{01} - x_{00}\}.$$

Clearly, x_{01} and x_{02} are nonnegative integers. We claim that $\theta_0 = x_{00} + x_{01} + x_{02}$. It is easy to see that $\delta_1 + \delta_2 \geq \theta_0 - [\theta_0 - \delta_1 - \delta_2]^+ = \theta_0 - x_{00}$. If $\delta_1 \geq \theta_0 - x_{00}$, then $x_{01} = \theta_0 - x_{00}$, $x_{02} = 0$ and the claim is true in this case. If $\delta_1 < \theta_0 - x_{00}$, then $x_{01} = \delta_1$ and since $\delta_2 \geq \theta_0 - x_{00} - \delta_1 = \theta_0 - x_{00} - x_{01}$, we conclude that $x_{02} = \theta_0 - x_{01} - x_{00}$, which confirms the claim. With these assignments, the remaining demands at g_1 and g_2 are $\delta'_1 = \delta - x_{01}$ and $\delta'_2 = \delta_2 - x_{02}$.

Now we consider the constraint

$$x_{30} + x_{31} + x_{32} = [\theta_3 - [-\theta_0 + \delta_1 + \delta_2]^+] + x_{31} + x_{32} = \theta_3. \quad (106)$$

We claim that

$$x_{30} + \delta'_1 + \delta'_2 \geq \theta_3, \quad (107)$$

so that we may assign values to x_{31} and x_{32} , as done above to x_{01} and x_{02} , that will satisfy (106). Using the expressions for x_{30} , δ'_1 and δ'_2 , we have

$$\begin{aligned} x_{30} + \delta'_1 + \delta'_2 &= x_{30} + \delta_1 - x_{01} + \delta_2 - x_{02} \\ &= \delta_1 + \delta_2 + x_{30} - (\theta_0 - x_{00}). \end{aligned}$$

Using the above identity, the inequality (107) may be rewritten as

$$\begin{aligned} \delta_1 + \delta_2 &\geq \theta_3 - x_{30} + \theta_0 - x_{00} \\ &= \theta_0 + \theta_3 - [\theta_0 - \delta_1 - \delta_2]^+ - [\theta_3 - [-\theta_0 + \delta_1 + \delta_2]^+]^+. \end{aligned}$$

If $\theta_0 \geq \delta_1 + \delta_2$, the right-hand-side of the inequality above reduces to

$$\theta_3 - [\theta_3 - 0]^+ + \theta_0 - \theta_0 + \delta_1 + \delta_2 = \delta_1 + \delta_2,$$

so the inequality is satisfied. Otherwise, the right-hand-side is

$$\theta_0 + \theta_3 - [\theta_0 + \theta_3 - (\delta_1 + \delta_2)]^+,$$

so the inequality (107) is equivalent to the inequality

$$[\theta_0 + \theta_3 - (\delta_1 + \delta_2)]^+ \geq \theta_0 + \theta_3 - (\delta_1 + \delta_2),$$

which is clearly true.

So if we let

$$x_{31} = \min\{\delta'_1, \theta_3 - x_{30}\} \quad \text{and} \quad x_{32} = \min\{\delta'_2, \theta_3 - x_{30} - x_{31}\},$$

the partial solution is nonnegative integral and satisfies the conservation of flow equations for nodes f_0 and f_3 . Basically the flows assigned to the arcs incident out of f_0 and f_3 are nonnegative integrals that dispatch the supplies available at those nodes. There might be untapped supplies at f_1 and f_2 , and unsatisfied demand $\delta''_0 = \delta_0 - (x_{00} + x_{30})$, $\delta''_1 = \delta_1 - x_{01} - x_{31}$ and $\delta''_2 = \delta_2 - x_{02} - x_{32}$ at nodes g_1 and g_2 (recall $\delta_3 = 0$). Since

$$\begin{aligned} \delta''_0 + \delta''_1 + \delta''_2 &= \delta_0 + \delta_1 + \delta_2 - (x_{00} + x_{01} + x_{02} + x_{30} + x_{31} + x_{32}) \\ &= \delta_0 + \delta_1 + \delta_2 - \theta_0 - \theta_3 \\ &= \theta_1 + \theta_2. \end{aligned}$$

Thus the remaining variables and constraints constitute a transportation problem where all supply nodes are connected to all demand nodes and all flow costs are zero. This problem is trivially feasible and any solution to this problem combined with the values already assigned constitutes a feasible solution to the original transportation problem with cost ρ^* . Therefore ρ^* is an optimal solution to the transportation problem.

6.3.7 Numerical examples

A computer code was built to construct and solve the odd dimensional minimum pairing problem (29). It was written in Python 3.7.6 and accepts as inputs the parameter i (the instance to be solved is of dimension $n = 2i + 1$) and, optionally, the data $(\mathcal{B}_0^-, \mathcal{B}_0^+, \mathcal{B}_1, \dots, \mathcal{B}_i)$. If the latter are not supplied, random values are assigned to these constants, satisfying (30). The optimal value of ρ and various bounds are calculated, as well as a solution. The output is supplied as formatted latex code and a sample is provided below. This program is freely available on the page <https://github.com/MargaridaMello/OddMinimumPairingProblem/>. The relevant file is `ValueAndSolutionTN.py`. Once this code is executed, we may generate and solve a problem of dimension $n = 5$ by entering the command `ValueSolutionMinPairingProblem(2)` at the console window. Alternatively, if we want to solve the specific instance with $n = 5$ and $(\mathcal{B}_0^-, \mathcal{B}_0^+, \mathcal{B}_1, \mathcal{B}_2) = (6, 6, 5, 5)$, the command would be `ValueSolutionMinPairingProblem(2, [6, 6, 5, 5])`.

$$\boxed{i = 2 \quad n = 5}$$

$$(\mathcal{B}_0^-, \mathcal{B}_0^+, \mathcal{B}) = (6, 6, 5, 5)$$

In this case we obtain the following answer :

	g_0	g_1	g_2	θ
f_0	¹ 1		5	6
f_1	5			5
f_2				
δ	6		5	

Optimal value 1

Column lower bound 0

Row lower bound 0

First upper bound 6

Second upper bound 7

j^{cd}	^{1^c} 6	^{2^d} 5	^{3^d} 5	^{4^d} 6	
τ					
\mathfrak{d}	²	1			
s	¹	6	⁴	6	
h^{cd}	h_1^c 6	h_2^d 6	h_3^c 1	h_3^d 5	h_4^d 6
h	h_1 6	h_2 6	h_3 6	h_4 6	

If we are only interested in the minimal number of periodic orbits, we just need to read the last line labelled by “ h ”. In our example, it is the line

$$\boxed{h \parallel h_1 6 \mid h_2 6 \mid h_3 6 \mid h_4 6}.$$

From it, we obtain the following information :

- the minimal⁶ number of periodic orbits, p_{min} , is just the sum of the elements of the line “ h ” divided by 2. Here $p_{min} = \frac{6 + 6 + 6 + 6}{2} = 12$.
- the indices of the periodic orbits associated with this line, since each orbit of index j combinatorially corresponds to a couple of consecutive (h_j, h_{j+1}) (see Section 3). Here, the line “ h ” is uniquely partitioned into 6 couples (h_1, h_2) and 6 couples (h_3, h_4) , so that six of the twelve periodic orbits are of index 1 and the other six are of index 3.

If we are interested in realizing these abstract data by attaching round handles, we need further information about the compatible types “ c ” and “ d ” of such round handles (see [12]). Here, we obtain this information by the penultimate line :

$$\boxed{h^{cd} \parallel h_1^c 6 \mid h_2^d 6 \mid h_3^c 1 \mid h_3^d 5 \mid h_4^d 6},$$

that is, a realization can be obtained by attaching to a suitable boundary six round handles of index 1 and type (c, d) , one round handle of index 3 and type (c, d) and five round handles of index 3 and type (d, d) according to the pattern given in [12].

The other information created by the algorithm is useful whenever one wishes to follow step by step the algorithm presented in this paper. In this case we refer the reader particularly to Subsection 6.3.3 for the first table and Subsection 6.3.1 for the second one.

⁶in the sense of Theorem 1.

Let us emphasizing that the output is given in LaTeX format.

Coming back to the example in dimension 17 developed in the introduction, after entering `ValueSolutionMinPairingProblem(8, [5,6,3,-1,-2,-7,-6,8,-4,-10])` the output is :

$$i = 8 \quad n = 17$$

	g_0	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	θ
f_0	¹ 4			¹	¹	²	² 1	³	³	5
f_1	2			1	¹	¹	²	²	³	3
f_2				1		¹	¹	²	²	1
f_3	¹						¹	¹	²	
f_4	¹	¹					6	1	¹	7
f_5	²	¹	¹						¹	
f_6	²	²	¹	¹						
f_7	³	²	²	¹	¹					
f_8	³	³	²	²	¹	¹	6	4		10
δ	6			2		6	8	4		

Optimal value 6

Column lower bound 0

Row lower bound 0

First upper bound 33

Second upper bound 30

j^{cd}	^{1c} 5	^{5c} 1	^{6d} 2	^{9c} 10	^{10c} 4	^{10d} 6	^{12c} 6	^{13c} 6	^{14c} 2	^{15c} 1	^{15d} 3	^{16d} 6			
τ	² 4														
∂	² 1	⁴ 1													
s	¹ 5	³ 1	⁵ 2	¹⁰ 10	¹³ 6	¹⁵ 2	¹⁶ 6								
h^{cd}	h_1^c 5	h_2^d 5	h_3^c 1	h_4^d 1	h_5^c 2	h_6^d 2	h_9^c 10	h_{10}^c 4	h_{10}^d 6	h_{12}^c 6	h_{13}^c 6	h_{14}^c 2	h_{15}^c 5	h_{15}^d 3	h_{16}^d 6
h	h_1 5	h_2 5	h_3 1	h_4 1	h_5 2	h_6 2	h_9 10	h_{10} 10	h_{12} 6	h_{13} 6	h_{14} 2	h_{15} 8	h_{16} 6		

Hence, for these data $p_{min} = 32$.

7 Future work

The proof of this result opens the way to the even dimension. Even though the guidelines for finding an algorithm are comparable, the combinatorics in the even setting is richer because of the existence of “invariant” handles⁷ and must be treated independently. Moreover, handles of invariant type are difficult to realize in the non-singular Morse-Smale context, so that the realization of the general abstract solution by a non-singular Morse-Smale model remains an open question.

Research is underway regarding the even case. Partial results have already been obtained. We list here some partial results.

We start by remarking that the Poincaré-Hopf inequalities described in Section 2 are the

⁷Handles whose attaching produces no effect on the Betti numbers of the boundary

same except to equality (5), which is replaced by :

$$n = 2i, i \text{ odd}, i \geq 3, h_i - \sum_{k=1}^{i-1} (-1)^k (B_k^+ - B_k^-) - \sum_{k=0}^{i-1} (-1)^k (h_{n-k} - h_k) + (e^- - e^+) \equiv 0 \pmod{2}. \quad (108)$$

If $n = 2i = 0 \pmod{4}$ besides the handle effects (H1) and (H2) described in Section 2, we have also to consider

(H3) if $n = 4k$ and $j = 2k$ all the Betti numbers are kept unchanged in N^+ , and the handle will be said of type β -i (i standing for invariant)

and of course we should also add a third algebraic effect to (G1) and (G2) concerning a graph:

(G3) In the case $n = 2i = 0 \pmod{4}$, a vertex labeled with $h_i = 1$ is β -i, if all β_ℓ label on the incoming edge are kept constant

in this case.

Once again we follow Franks's idea in [17] and we consider a hyperbolic periodic orbit of index j as the joining of two hyperbolic singularities p and q of adjacent indices j and $j + 1$ respectively. Given a nondegenerate singularity of index j , one can associate with it the dimensions of the Conley homology indices, $h_j = 1$ and $h_k = 0$ for all $k \neq j$. Let A_j be the number of periodic orbits of index j and h_j be the number of singularities of index j .

Also, as in the previous case, given data $(A_0, A_1, \dots, A_{n-1})$ we have associated with it $(h_0, h_1, \dots, h_{n-1}, h_n)$ and vice versa. The coupling procedure performed with singularity data $(h_0, h_1, \dots, h_{n-1}, h_n)$ in order to construct a periodic orbit set $(A_0, A_1, \dots, A_{n-1})$ is similar to the odd case except in the middle dimension. In this case, we have the presence of β in h_i which does not have a dual. Coupling symmetrically produces

$$\left\{ \begin{array}{l} A_0 = h_0 \\ A_1 = h_1 - h_0 \\ A_2 = h_2 - (h_1 - h_0) \\ \vdots \\ \vdots \\ A_k = h_k - A_{k-1} \\ \vdots \\ \vdots \\ A_{i-1} = h_{i-1} - A_{i-2} \\ \quad (= h_i - A_{n-i}) \end{array} \right. = \left\{ \begin{array}{l} A_{n-1} = h_n \\ A_{n-2} = h_{n-1} - h_n \\ A_{n-3} = h_{n-2} - (h_{n-1} - h_n) \\ \vdots \\ \vdots \\ A_{n-1-k} = h_{n-k} - A_{n-k} \\ \vdots \\ \vdots \\ A_{n-i} = h_{n-(i-1)} - A_{n-(i-1)} \\ \quad (= h_i - A_{i-1}) \end{array} \right. \quad (109)$$

In this dimension Proposition 1 remains true and it can be reformulated as follow :

Proposition 4 *The system (110)–(115) has nonnegative integral solutions $(h, h_1^c, h_1^d, \dots, h_{n-1}^c, h_{n-1}^d, \beta)$ if and only if the Poincaré-Hopf inequalities (1)–(4) and equality (108) are satisfied.*

$$\left\{ \begin{array}{l} h_1^c = e^- - 1, \quad (110) \\ h_{n-1}^d = e^+ - 1, \quad (111) \\ h_j = h_j^c + h_j^d, \quad j = 1, \dots, n-1 \text{ and } j \neq \left\lfloor \frac{n}{2} \right\rfloor, \quad (112) \\ h_{\lfloor \frac{n}{2} \rfloor} = h_{\lfloor \frac{n}{2} \rfloor}^c + h_{\lfloor \frac{n}{2} \rfloor}^d + \beta, \quad \text{with } \beta = 0 \text{ if } n \not\equiv 0 \pmod{4}, \quad (113) \\ h_j^d - h_{j+1}^c - h_{n-j}^c + h_{n-j-1}^d = B_j^+ - B_j^-, \quad j = 1, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor, \quad (114) \\ h_i^d - h_{i+1}^c = \frac{B_i^+ - B_i^-}{2}, \quad \text{if } n = 2i + 1. \quad (115) \end{array} \right.$$

Acknowledgement

The authors thank Ruy S. Gonçalves for invaluable help with Python programming.

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