



HAL
open science

The Cayley transform in complex, real and graded K-theory

Chris Bourne, Johannes Kellendonk, Adam Rennie

► **To cite this version:**

Chris Bourne, Johannes Kellendonk, Adam Rennie. The Cayley transform in complex, real and graded K-theory. *Int.J.Math.*, 2020, 31 (09), pp.2050074. 10.1142/S0129167X20500743 . hal-02946239

HAL Id: hal-02946239

<https://hal.science/hal-02946239v1>

Submitted on 8 Mar 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The Cayley transform in complex, real and graded K -theory

Chris Bourne^{†,♠}, Johannes Kellendonk[§], Adam Rennie^{‡ *}

[†]WPI-Advanced Institute for Materials Research (WPI-AIMR), Tohoku University,
Sendai, Japan

[♠]RIKEN iTHEMS, Wako, Saitama, Japan

[§]Institute Camille Jordan, Université Claude Bernard Lyon 1,
Villeurbanne, France

[‡]School of Mathematics and Applied Statistics, University of Wollongong,
Wollongong, Australia

May 22, 2020

Abstract

We use the Cayley transform to provide an explicit isomorphism at the level of cycles from van Daele K -theory to KK -theory for graded C^* -algebras with a real structure. Isomorphisms between KK -theory and complex or real K -theory for ungraded C^* -algebras are a special case of this map. In all cases our map is compatible with the computational techniques required in physical and geometrical applications, in particular index pairings and Kasparov products. We provide applications to real K -theory and topological phases of matter.

1 Introduction

This paper presents explicit isomorphisms between K -theory and (unbounded) KK -theory for possibly graded C^* -algebras with or without a real structure. We use a K -theory due to van Daele [12, 13] to accommodate graded real or complex algebras. The key ingredient of our construction is the Cayley transform on Hilbert C^* -modules, which exchanges unbounded self-adjoint regular operators with unitary operators.

The isomorphism $DK(A) \cong KKR(\mathcal{C}\ell_{1,0}, A)$ is already known from work by Roe [42] and extended by Kubota [28, Theorem 5.11]. Roe shows that $DK(A)$ is isomorphic to $KK(\mathcal{C}\ell_1, A)$ and, for algebras with a real structure τ_A , $DK(A^{\tau_A})$ is isomorphic to $KKO(\mathcal{C}\ell_1, A^{\tau_A})$, emulating the proof given in [4, Section 17.5] for standard K -theory. The resulting isomorphism is, however, not given at the level of cycles.

The isomorphism we present is tailored to the needs of the applications, especially as they involve index pairings, and their more sophisticated cousins, Kasparov products.

Our work relies, ultimately, on results of Wood [48], whose proof of Bott periodicity informed van Daele and others. Wood's proof is expressed in terms of odd operators with square -1 (a trivial difference to the picture of van Daele that uses odd self-adjoint unitaries), and Wood's methods were flexible enough to be used widely. Our use of the Cayley transform is steeped in long tradition, and in particular has been applied to graded K -theory by Trout [47]. Utilising

*email: chris.bourne@tohoku.ac.jp, kellendonk@math.univ-lyon1.fr, renniea@uow.edu.au

the Cayley transform to obtain a converse of the functional calculus, Trout obtains a homotopy theoretic definition of graded K -theory.

Our work is motivated by results of the second author (and more recently [1]), who showed that homotopy classes of gapped Hamiltonians (with prescribed symmetries) are classified directly in terms of van Daele K -theory, [1, 23]. The work in [1, 23] complements and extends links of complex and real K -theory to topological states of matter, [17, 27, 29, 46].

Our isomorphism identifies a van Daele element with a concrete unbounded Kasparov cycle, which can then be used to take pairings/products quite explicitly in terms of cycles. Such computations are compatible with the bulk-edge correspondence as in [7, 29]. We present some examples and show how our technique facilitates computations. The Appendix provides sufficient conditions to be able to compute such products explicitly on the level of cycles.

We first give a review of van Daele K -theory, KK -theory and Kasparov's stabilisation theorem in Section 2. We use refinements of van Daele K -theory due to Roe [41, 42], describe the relationship between these two approaches, and prove an excision isomorphism.

In Section 3, we review the Cayley transform for ungraded Hilbert C^* -modules, which we use to construct an explicit isomorphism between odd K -theory and (unbounded) KK -theory for trivially graded complex C^* -algebras. An isomorphism $KK(\mathbb{C}, A) \rightarrow K_0(A)$ is also given using the graph projection of the (unbounded) operator of the KK -cycle.

In Section 4, we introduce a Cayley transform on graded C^* -modules, where the key difference with the ungraded case is that the transform interchanges a *pair* of odd self-adjoint unitaries with an unbounded, odd, self-adjoint and regular operator anti-commuting with an odd self-adjoint unitary. We remark that some of our constructions are similar to the converse functional calculus used by Trout [47] to study graded K -theory.

The graded Cayley transform is used to prove our main result in Section 4.2, where an explicit isomorphism $DK(A) \rightarrow KKR(\mathcal{C}\ell_{1,0}, A)$ and inverse is constructed. While our map from van Daele K -theory to KK -theory and its inverse are explicit, the proof that we obtain isomorphisms is somewhat involved. This is because a generic countably generated C^* -module need not be full, and so Morita equivalence relates the compact endomorphisms on the C^* -module to an ideal of the coefficient algebra. To prove that our maps give isomorphisms, we demonstrate the compatibility of the Cayley transform with Morita equivalence and non-full C^* -modules.

Some applications of our isomorphism are considered in Sections 5 and 6. In Section 5.1, we consider the case that $A = B \otimes \mathcal{C}\ell_{r,s}$ for some trivially graded B with real structure τ_B and so the image of our Cayley isomorphism is $KO_{1+s-r}(B^{\tau_B})$ or $K_{1-s-r}(B)$ if we ignore the real structure. We show how in these special cases we recover unitary descriptions of real K -theory as studied in [6] and [23, Section 5.6]. In Section 5.2, we show how our Cayley isomorphism interacts with the boundary map in van Daele K -theory. We can use this result to explicitly write the boundary map in KK -theory, $KKR(\mathcal{C}\ell_{1,0}, A) \xrightarrow{\delta} KKR(\mathbb{C}, I)$ coming from the semi-split short exact sequence

$$0 \rightarrow I \rightarrow E \rightarrow A \rightarrow 0$$

such that each map respects the \mathbb{Z}_2 -grading and real structure.

Finally in Section 6 we use our Cayley map to write down Kasparov modules representing bulk and boundary invariants of topological insulators. We note that we do not specify our algebra of observables and work with a generic complex C^* -algebra, possibly graded and possibly with a real structure implementing the anti-linear symmetries of the system. Of particular note are

the boundary invariants, where the Kasparov modules we construct are explicitly linked to a lift of the Hamiltonian under the bulk-boundary short exact sequence. Such lifts are typically related to half-space Hamiltonians and edge spectra. Hence our work complements recent results [1, 44] that express the boundary K -theory class using the half-space Hamiltonian.

Acknowledgements We would like to thank Alex Kumjian, David Pask, and Aidan Sims for pointing out some graded issues that were a bit odd. We also thank Christopher Max for sharing the results of [1] with us, and Matthias Lesch for helpful conversations. Finally, Bram Mesland has once again provided timely and useful advice. This work was supported by World Premier International Research Center Initiative (WPI), MEXT, Japan. AR was partially supported by the BFS/TFS project Pure Mathematics in Norway and CB is supported by a JSPS Grant-in-Aid for Early-Career Scientists (No. 19K14548). All authors thank the Erwin Schrödinger Institute program Bivariant K -Theory in Geometry and Physics for hospitality and support during the production of this work.

2 Preliminaries

Our paper is concerned with operator K -theory, van Daele K -theory and KK -theory. The Cayley transform gives us a method to pass between these theories.

Conventions: We assume that all C^* -algebras we encounter have a countable approximate identity (σ -unital). Recall that a real C^* -algebra is a C^* -algebra over the field of real numbers. In contrast, a Real C^* -algebra, written with large \mathbb{R} , is a complex C^* -algebra A with a *real structure*, that is, an anti-linear multiplicative map $\tau_A : A \rightarrow A$ which is of order 2. Elsewhere, e.g. in [24], Real C^* -algebras are also called $C^{*,r}$ -algebras. The subalgebra of elements fixed under τ_A is a real C^* -algebra A^{τ_A} . In specific situations where the context is clear, we will omit the subscript and simply denote a real structure by τ . All C^* -algebras are \mathbb{Z}_2 -graded (possibly trivially graded) unless otherwise stated, and we use \mathbb{Z}_2 -graded tensor products $\hat{\otimes}$.

The real Clifford algebra $Cl_{p,q}$ is the algebra generated by p self-adjoint odd elements e_1, \dots, e_p with square 1 and q skew-adjoint odd elements f_1, \dots, f_q with square -1 which all pairwise anti-commute. We denote by $\mathbb{C}l_{p,q}$ the Real C^* -algebra generated by e_1, \dots, e_p and f_1, \dots, f_q . That is, its elements are complex linear combinations of products of these generators, equipped with the real structure τ such that $e_j^\tau = e_j$, $f_j^\tau = f_j$. It is immediate that $\mathbb{C}l_{p,q} \cong \mathbb{C}l_{p+q}$ as complex algebras, and $\mathbb{C}l_{p,q}^\tau = Cl_{p,q}$. We will make frequent use of the Pauli matrices, where to establish notation we recall

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will freely take advantage of the isomorphism $\mathbb{C}l_{1,1} \cong C^*(\sigma_1, -i\sigma_2)$ with real structure by entrywise complex conjugation.

2.1 Van Daele K -theory

Definition 2.1. Let A be a real or complex C^* -algebra. We say that A has a balanced \mathbb{Z}_2 -grading if A contains an odd self-adjoint unitary¹ (OSU). That is, there is an odd element e satisfying $e = e^* = e^{-1}$. In particular A is unital. If A has a real structure τ_A , we also require $e^{\tau_A} = e$.

¹Odd self-adjoint unitaries are called super-symmetries by Roe [41, 42].

Let $V(A) = \bigsqcup_k \pi_0(\text{OSU}(M_k(A)))$, the disjoint union of homotopy classes of OSUs in $M_k(A)$, the $k \times k$ matrices with entries in A , $k \geq 1$. Here the grading and real structure on A are extended to $M_k(A)$ entrywise. The set $V(A)$ is an abelian semigroup with direct sum as operation, $[x] + [y] = [x \oplus y]$. The Grothendieck group obtained from this semigroup will be denoted $GV(A)$. The semigroup homomorphism $d : V(A) \rightarrow \mathbb{N}$ taking the value k on $M_k(A)$ induces a group homomorphism $d : GV(A) \rightarrow \mathbb{Z}$.

Definition 2.2. If A is unital and has a balanced \mathbb{Z}_2 -grading, then the van Daele group of A is $DK(A) = \text{Ker}(d : GV(A) \rightarrow \mathbb{Z})$.

If A is unital but is not balanced, then we set $DK(A) = \text{Ker}(d : GV(A \hat{\otimes} \mathbb{C}l_{1,1}) \rightarrow \mathbb{Z})$. The complex and real case is given by ignoring the real structure or passing to the real subalgebra $A^{\mathfrak{r}A} \hat{\otimes} \mathbb{C}l_{1,1}$.

If A is not unital then we set $DK(A) = \text{Ker}(q_* : DK(A^\sim) \rightarrow DK(\mathbb{C}))$ where $q : A^\sim \rightarrow \mathbb{C}$ quotients the minimal unitisation A^\sim by the ideal A , replace \mathbb{C} by \mathbb{R} if A is real, see (2.1).

Elements of $DK(A)$ are formal differences of OSUs denoted by $[x] - [y]$.

We elaborate on the non-unital case. Since \mathbb{C} (or \mathbb{R}) is trivially graded, the relevant exact sequence needed to define q_* is

$$0 \rightarrow A \hat{\otimes} \mathbb{C}l_{1,1} \rightarrow A^\sim \hat{\otimes} \mathbb{C}l_{1,1} \xrightarrow{q \hat{\otimes} \text{Id}} \mathbb{C} \otimes \mathbb{C}l_{1,1} \rightarrow 0$$

so that $DK(A) = \{[x] - [y] \in DK(A^\sim \hat{\otimes} \mathbb{C}l_{1,1}) : (q \hat{\otimes} \text{Id})_*[x] = (q \hat{\otimes} 1)_*[y]\}$. Adapting [12, Proposition 3.7] to Roe's formulation, one can find representatives x' for $[x]$ and y' for $[y]$ such that $q \hat{\otimes} \text{Id}(x') = q \hat{\otimes} \text{Id}(y')$. Thus, for non-unital A

$$DK(A) = \{[x] - [y] : x, y \in \text{OSU}(M_n(A^\sim \hat{\otimes} \mathbb{C}l_{1,1})), x - y \in M_n(A \hat{\otimes} \mathbb{C}l_{1,1})\}. \quad (2.1)$$

This is Roe's version of van Daele K -theory [41, 42]. As already mentioned, Roe shows that van Daele's K -groups are isomorphic to KK -groups from which we infer that they share all the standard properties of K -theory, though often we can only exploit this easily for ungraded algebras.

If A is complex with a real structure \mathfrak{r}_A , then we sometimes denote the van Daele K -theory group by $DK(A, \mathfrak{r}_A)$ to emphasise this. Clearly $DK(A, \mathfrak{r}_A) \cong DK(A^{\mathfrak{r}A})$.

If A is balanced graded, one may ask if we could equivalently use $A \hat{\otimes} \mathbb{C}l_{1,1}$ to define $DK(A)$. The following lemma shows that such a choice leads to consistent definitions of van Daele K -theory.

Lemma 2.3 ([41]). *Let A be balanced graded. The map*

$$[x] - [y] \mapsto \left[\frac{x+y}{2} \hat{\otimes} 1_2 + \frac{x-y}{2} \hat{\otimes} \sigma_3 \right] - [1 \hat{\otimes} \sigma_1] = \left[\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right] - \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \quad (2.2)$$

furnishes an isomorphism between $DK(A)$ and $DK(A \hat{\otimes} \mathbb{C}l_{1,1})$.

Roe refers to van Daele [12] for the proof. For the convenience of the reader we reproduce it below after having introduced van Daele's picture. We also note that for balanced graded A the identification of $M_2(A)$ with entrywise grading with $A \hat{\otimes} \mathbb{C}l_{1,1}$ depends on a choice of OSU in A , cf. Equation (2.4).

2.1.1 Base-points, and van Daele's picture

Van Daele's original definition of the version of K -theory which is named after him [12, 13] requires a choice of base point. This is a choice of OSU $e \in A$ if A is balanced graded, or $e \in A \hat{\otimes} \mathcal{C}l_{1,1}$ if not. This OSU is then used to embed $M_k(A)$ into $M_{k+1}(A)$ via $x \mapsto x \oplus e$. The semigroup $V_e(A) = \bigcup_k \pi_0(\text{OSU}(M_k(A)))$ is such that the union is no longer disjoint as $\pi_0(\text{OSU}(M_k(A)))$ is identified with a subset of $\pi_0(\text{OSU}(M_{k+1}(A)))$ via the above embedding.

The semigroup $V_e(A)$ depends on e up to homotopy. It has a unit element, the class of e . Van Daele's version of the K -group is thus given by the Grothendieck group $DK_e(A) = GV_e(A)$, where we include the chosen base point in our notation. If we denote for a moment the corresponding homotopy classes in $V_e(A)$ by $[x]_e$ then $[x] \mapsto [x]_e$ induces a map $\alpha_e : GV(A) \rightarrow GV_e(A)$, $\alpha_e([x] - [y]) = [x]_e - [y]_e$ between the corresponding Grothendieck groups. Restricted to the kernel of d the map α_e is a group isomorphism. We therefore arrive at two (isomorphic) presentations of the van Daele K -theory group. *In the sense that it uses the Grothendieck completion of a semigroup, Roe's formulation exhibits van Daele K -theory as a relative theory. In contrast, van Daele's formulation expresses the elements as relative to a chosen base point.*

A particularly handy situation arises if the base point e is homotopic to $-e$ (along a homotopy of OSUs in A). Then $V_e(A)$ is a group with inverse given by $-[x]_e = [-exe]_e$. If A is balanced this can always be achieved: choose a base point $e \in A$. Then $M_2(A)$ contains the OSU $e \oplus -e$ which is homotopic to its negative via the path $\begin{pmatrix} e \cos(t) & e \sin(t) \\ e \sin(t) & -e \cos(t) \end{pmatrix}$ for $t \in [0, \pi]$. The map $\varphi_e : GV_e(A) \rightarrow GV_{e \oplus -e}(M_2(A))$ given by

$$\varphi_e([x]_e - [y]_e) = \left[\begin{pmatrix} x & 0 \\ 0 & -eye \end{pmatrix} \right]_{e \oplus -e} \quad (2.3)$$

is then an isomorphism of groups. If A is not balanced but unital, then we start with $A \hat{\otimes} \mathcal{C}l_{1,1}$ and choose $e = 1 \hat{\otimes} \sigma_1$ as base point. The van Daele K -group of A is thus isomorphic to $GV_{\sigma_1 \oplus -\sigma_1}(M_4(A))$ (as it is originally defined in [12]).

If A is balanced graded, then $M_2(A)$ (with component-wise extension of the grading) is isomorphic to $A \hat{\otimes} \mathcal{C}l_{1,1}$, though the isomorphism depends on the choice of base point e . Given such an OSU, the isomorphism is given by $\psi_e : A \hat{\otimes} \mathcal{C}l_{1,1} \rightarrow M_2(A)$

$$\psi_e(x \hat{\otimes} 1) = \begin{pmatrix} x & 0 \\ 0 & (-1)^{|x|} e x e \end{pmatrix}, \quad \psi_e(1 \hat{\otimes} \sigma_1) = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}, \quad \psi_e(1 \hat{\otimes} i \sigma_2) = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix}. \quad (2.4)$$

Note that ψ_e^{-1} maps $e \oplus -e$ to $e \hat{\otimes} 1$ which is homotopic to $1 \hat{\otimes} \sigma_1$ via $t \mapsto \cos(t) e \hat{\otimes} 1 + \sin(t) 1 \hat{\otimes} \sigma_1$. These facts imply that the isomorphism of Lemma 2.3 is given by the composition of four isomorphisms $\alpha_{1 \hat{\otimes} \sigma_1}^{-1} \circ \psi_e^{-1} \circ \varphi_e \circ \alpha_e$.

2.1.2 Excision for van Daele K -theory

Excision for DK can be deduced from excision for ordinary K -theory when the algebra is trivially graded, but seems not to have been addressed for graded algebras.

For a balanced graded algebra A with a (closed two-sided graded) ideal I we define the relative van Daele group

$$DK(A, A/I) := \{[x] - [y] : x, y \in \text{OSU}(M_n(A)), x - y \in M_n(I)\}.$$

Here $[\cdot]$ denotes homotopy classes in $\text{OSU}(M_n(A))$. If A is not balanced, but only unital, then we use again $A \hat{\otimes} \mathbb{C}l_{1,1}$ in place of A in the above definition of the relative group; the ideal is then $I \hat{\otimes} \mathbb{C}l_{1,1}$. Again this is reasonable as the map from Lemma 2.3 provides an isomorphism between $DK(A, A/I)$ and $DK(A \hat{\otimes} \mathbb{C}l_{1,1}, A/I \hat{\otimes} \mathbb{C}l_{1,1})$ in the case that A is balanced. Indeed, for any OSU $y \in M_n(A)$, $w = \frac{1}{\sqrt{2}}(1 - y \hat{\otimes} \sigma_1)$ is an even unitary which is homotopic to 1 and therefore $\frac{x+y}{2} \hat{\otimes} 1_2 + \frac{x-y}{2} \hat{\otimes} \sigma_3$ is homotopic to $w(\frac{x+y}{2} \hat{\otimes} 1_2 + \frac{x-y}{2} \hat{\otimes} \sigma_3)w^*$. Finally, a computation shows that $w(\frac{x+y}{2} \hat{\otimes} 1_2 + \frac{x-y}{2} \hat{\otimes} \sigma_3)w^* - 1 \hat{\otimes} \sigma_1 \in I \hat{\otimes} \mathbb{C}l_{1,1}$ provided $x - y \in I$.

Equation (2.1) can now be interpreted in the way that $DK(A) = DK(A^\sim, A^\sim/A)$ for a non-unital algebra A .

Proposition 2.4 (Excision for DK). *Let I be a (closed two-sided graded) ideal in the unital algebra A . Then $DK(I) \cong DK(A, A/I)$.*

Proof. Since A is unital it contains I^\sim and hence any element $[x] - [y] \in DK(I^\sim, I^\sim/I)$ can be understood as an element of $DK(A, A/I)$. This defines a map $DK(I) \rightarrow DK(A, A/I)$.

We first show this map is surjective. Since I^\sim need not be balanced we work with $I^\sim \hat{\otimes} \mathbb{C}l_{1,1}$ and consequently also with $A \hat{\otimes} \mathbb{C}l_{1,1}$. Recall that $DK(A, A/I)$ is thus generated by elements of the form $[\frac{x+y}{2} \hat{\otimes} 1_2 + \frac{x-y}{2} \hat{\otimes} \sigma_3] - [1 \hat{\otimes} \sigma_1]$ such that $x - y \in M_n(I)$. Then with $w = \frac{1}{\sqrt{2}}(1 - y \hat{\otimes} \sigma_1)$, $[\frac{x+y}{2} \hat{\otimes} 1_2 + \frac{x-y}{2} \hat{\otimes} \sigma_3] = [w(\frac{x+y}{2} \hat{\otimes} 1_2 + \frac{x-y}{2} \hat{\otimes} \sigma_3)w^*]$ and $w(\frac{x+y}{2} \hat{\otimes} 1_2 + \frac{x-y}{2} \hat{\otimes} \sigma_3)w^* - 1 \hat{\otimes} \sigma_1 \in I \hat{\otimes} \mathbb{C}l_{1,1}$. Thus $\frac{x+y}{2} \hat{\otimes} 1_2 + \frac{x-y}{2} \hat{\otimes} \sigma_3$ has a representative in $M_n(I^\sim) \hat{\otimes} \mathbb{C}l_{1,1}$.

For injectivity, we let $[\frac{x+y}{2} \hat{\otimes} 1_2 + \frac{x-y}{2} \hat{\otimes} \sigma_3] - [\frac{x'+y'}{2} \hat{\otimes} 1_2 + \frac{x'-y'}{2} \hat{\otimes} \sigma_3]$ be trivial in $DK(A, A/I)$. There are then (perhaps after stabilisation) paths $x(t)$ and $y(t)$ of OSUs in $M_n(A)$ such that $x(0) = x$, $y(0) = y$, $x(1) = x'$, $y(1) = y'$ and, for all $t \in [0, 1]$, $x(t) - y(t) \in M_n(I)$. We let $w(t) = \frac{1}{\sqrt{2}}(1 - y(t) \hat{\otimes} \sigma_1)$. Then $w(t)(\frac{x(t)+y(t)}{2} \hat{\otimes} 1_2 + \frac{x(t)-y(t)}{2} \hat{\otimes} \sigma_3)w(t)^*$ is a homotopy in $M_n(I^\sim) \hat{\otimes} \mathbb{C}l_{1,1}$ between two representatives of $[\frac{x+y}{2} \hat{\otimes} 1_2 + \frac{x-y}{2} \hat{\otimes} \sigma_3]$ and $[\frac{x'+y'}{2} \hat{\otimes} 1_2 + \frac{x'-y'}{2} \hat{\otimes} \sigma_3]$ in $M_n(I^\sim) \hat{\otimes} \mathbb{C}l_{1,1}$. \square

Let us also consider a description of van Daele K -theory using base points in the case that A is not unital. We say that A is *weakly balanced graded* if its multiplier algebra contains an OSU (the grading on A extends uniquely to the multiplier algebra of A). Importantly $A \hat{\otimes} \mathbb{C}l_{1,1}$ is always weakly balanced graded. Having fixed an OSU e in the multiplier algebra, we define $A^{\sim e}$ to be the subalgebra of the multiplier algebra generated by A and e . We use the notation $e_n = e^{\oplus n}$.

Lemma 2.5. *Let A be a non-unital and weakly balanced graded algebra with base point e in the multiplier algebra. Then*

$$DK_e(A) := \{[x] - [y] \in DK_e(A^{\sim e}) : x - (e_k \oplus -e_{n-k}), y - (e_k \oplus -e_{n-k}) \in M_n(A), \text{ some } n, k\} \quad (2.5)$$

is isomorphic to $DK(A)$ (Definition 2.2).

Proof. Since A is weakly balanced, A is an ideal in $A^{\sim e}$. Hence $DK(A) \cong DK(A^{\sim e}, A^{\sim e}/A)$, that is, $DK(A)$ is given by elements $[x] - [y]$, $x, y \in \text{OSU}(M_n(A^{\sim e}))$ with $x - y \in M_n(A)$. Hence α_e^{-1} induces an injective map from $DK_e(A)$ into $DK(A^{\sim e}, A^{\sim e}/A)$.

Let $q : A^{\sim e} \rightarrow A^{\sim e}/A \cong \mathbb{C}l_{1,0}$ be the natural projection with $\tilde{e} = q(e)$ the generator of $\mathbb{C}l_{1,0}$. The only OSUs of $\mathbb{C}l_{1,0}$ are \tilde{e} and $-\tilde{e}$ and they are not homotopic. Therefore for

a given OSU $x \in M_n(A^{\sim e})$, $q(x)$ is homotopic $\tilde{e}_k \oplus (-\tilde{e})_{n-k}$ for some k . Hence there is an even unitary $\tilde{w} \in M_n(\mathbb{C}\ell_{1,0})$, homotopic to 1 along a path of even unitaries, such that $\tilde{w}q(x)\tilde{w}^* = \tilde{e}_k \oplus (-\tilde{e})_{n-k}$. We lift \tilde{w} to a unitary $w \in A^{\sim e}$ (via $\tilde{e} \mapsto e$). Then wxw^* is homotopic to x along a path of OSUs. Now let y be an OSU in $M_n(A^{\sim e})$ such that $x - y \in M_n(A)$. Then $q(y)$ is homotopic to $\tilde{e}_k \oplus (-\tilde{e})_{n-k}$ with k the same as $q(x)$. Similarly, we can find an even unitary $v \in A^{\sim e}$, homotopic to 1, such that $q(vyv^*) = \tilde{e}_k \oplus (-\tilde{e})_{n-k}$. We thus have found $[wxw^*] - [vyv^*] \in DK_e(A)$ which is a preimage of $[x] - [y] \in DK(A^{\sim e}, A^{\sim e}/A)$. The excision isomorphism of Proposition 2.4 completes the proof. \square

If we study homotopy classes of OSUs where there is a canonical or simple choice of base point, then our picture simplifies. For example, if A is a unital and trivially graded algebra, then OSUs of $A \otimes \mathbb{C}\ell_{1,1}$ are of the form

$$U = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$$

where $u \in A$ is unitary. If we choose $e = 1 \otimes \sigma_1$ as base point then we see that the map $U \mapsto u$ identifies $V_e(A \otimes \mathbb{C}\ell_{1,1})$ with the homotopy classes of unitaries in $(A \otimes \mathcal{K})^{\sim}$. Hence, we recover the group $K_1(A)$ or $KO_1(A^{\tau A})$.

2.2 C^* -modules and K -theory

Given a countably generated right- A C^* -module X_A we denote by $(\cdot | \cdot)_A$ the A -valued inner-product, $\text{End}_A(X)$ the adjointable endomorphisms of X_A and $\text{End}_A^0(X) \subset \text{End}_A(X)$ the ideal of compact endomorphisms, see [32]. Any right- A C^* -module X_A naturally gives rise to an ideal $J_X = \overline{\text{span}(X|X)}_A$ (closure in the norm of A). The module X_A is called full if $J_X = A$. In this section we work with \mathbb{Z}_2 -graded C^* -modules over \mathbb{Z}_2 -graded algebras. The ungraded case is analogous but simpler. Endomorphism algebras will always have a \mathbb{Z}_2 -grading inherited from acting on a \mathbb{Z}_2 -graded module. We say that X_A is *balanced graded* if $\text{End}_A(X)$ admits an OSU.

Recall that the standard graded Hilbert module over A is given by $\hat{\mathcal{H}}_A := \hat{\mathcal{H}} \hat{\otimes} A$ where $\hat{\mathcal{H}}$ is the graded infinite dimensional separable Hilbert space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})^\circ \cong \ell^2(\mathbb{N}) \otimes \mathbb{C}^2$ with grading operator $1 \otimes \sigma_3$. There is a standard OSU on $\hat{\mathcal{H}}_A \cong (\ell^2(\mathbb{N}) \otimes \mathbb{C}^2 \hat{\otimes} A)_A$ given by

$$Z = \text{Id}_{\ell^2(\mathbb{N})} \otimes \sigma_1 \hat{\otimes} 1_{\text{Mult}(A)} \quad (2.6)$$

The Kasparov stabilisation theorem says that for any countably generated X_A , $(X \oplus \hat{\mathcal{H}})_A \cong \hat{\mathcal{H}}_A$ as graded Hilbert A -modules [21]. The compact endomorphisms on $\hat{\mathcal{H}}_A$ are $\text{End}_A^0(\hat{\mathcal{H}}_A) \cong \hat{\mathcal{K}} \hat{\otimes} A$ with $\hat{\mathcal{K}}$ being the graded algebra of compact operators on $\hat{\mathcal{H}}$. If A is balanced with an OSU e , then we can apply the isomorphism from Equation (2.4) to obtain $\hat{\mathcal{K}} \hat{\otimes} A \cong \mathcal{K} \otimes A$, where \mathcal{K} is the trivially graded compact operators in which we absorbed a copy of $\mathbb{C}\ell_{1,1}$.

Lemma 2.6 (Morita invariance for DK). *Let X_A be a countably generated C^* -module over the C^* -algebra A and define the ideal $J_X = \overline{\text{span}(X|X)}_A$. Then X is full as a module over J_X and there is an isomorphism*

$$\zeta_X : DK(\text{End}_A^0(X)) \xrightarrow{\cong} DK(J_X). \quad (2.7)$$

Proof. The algebra $\text{End}_A^0(X) \hat{\otimes} \mathbb{C}\ell_{1,1}$ is balanced graded, and by definition X is a full module over J_X . Recall that the group $DK(\text{End}_A^0(X))$ is made up of differences $[U] - [V]$ with U, V OSUs over $\text{End}_A^0(X) \hat{\otimes} \mathbb{C}\ell_{1,1}$ such that $U - V \in \text{End}_A^0(X) \hat{\otimes} \mathbb{C}\ell_{1,1}$.

Let K be an odd self-adjoint finite rank endomorphism with $U - V - K$ small enough in norm so that $V + K$ is invertible. This is possible since

$$U = V + U - V = V + K + (U - V - K).$$

Because the finite rank operators are stable under the holomorphic functional calculus [20, Lemma 6.3], we can take $\tilde{U} = \text{phase}(V + K)$ such that $\tilde{U} - V$ is finite rank. The path $[0, 1] \ni t \mapsto \text{phase}(V + K + t(U - V - K))$ gives a homotopy from \tilde{U} to U , and so every class $[U] - [V] \in DK(\text{End}_A^0(X))$ is represented by a class $[\tilde{U}] - [V]$ with $\tilde{U} - V$ finite rank. Any two such representatives are homotopic (in $DK(\text{End}_A^0(X))$) by construction. Thus the difference $\begin{pmatrix} \tilde{U} & 0 \\ 0 & V \end{pmatrix} - \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}$ is finite rank and $\begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}$ is homotopic to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Now let $W : (X \oplus \hat{\mathcal{H}})_{J_X \hat{\otimes} \mathbb{C}\ell_{1,1}} \rightarrow \hat{\mathcal{H}}_{J_X \hat{\otimes} \mathbb{C}\ell_{1,1}}$ be a stabilisation isomorphism, and $W_2 = W \oplus W$ two copies of W . Then for any OSU Z on $\hat{\mathcal{H}}_{J_X \hat{\otimes} \mathbb{C}\ell_{1,1}}$,

$$\left[\begin{pmatrix} \tilde{U} & 0 & 0 & 0 \\ 0 & Z & 0 & 0 \\ 0 & 0 & V & 0 \\ 0 & 0 & 0 & Z \end{pmatrix} \right] - \left[\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & Z & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z \end{pmatrix} \right] =: [\tilde{U} \oplus Z \oplus V \oplus Z] - [\hat{Z}]$$

defines a class in $DK(\text{End}_{J_X}^0(X \oplus \hat{\mathcal{H}}))$ and $(\text{Ad}_W)_* : DK(\text{End}_{J_X}^0(X \oplus \hat{\mathcal{H}})) \rightarrow DK(\text{End}_{J_X \hat{\otimes} \mathbb{C}\ell_{1,1}}^0(\hat{\mathcal{H}}))$.

Note that \hat{Z} is homotopic to $V \oplus Z \oplus V \oplus Z$ and the latter differs from $\tilde{U} \oplus Z \oplus V \oplus Z$ by a finite rank operator. Therefore $W_2 \hat{Z} W_2^*$ is homotopic to $W_2(V \oplus Z \oplus V \oplus Z)W_2^*$ and the latter differs from $W_2(\tilde{U} \oplus Z \oplus V \oplus Z)W_2^*$ by a finite rank operator, i.e. a matrix over $J_X \hat{\otimes} \mathbb{C}\ell_{1,1}$. It follows that

$$[W_2(\tilde{U} \oplus Z \oplus V \oplus Z)W_2^*] - [W_2 \hat{Z} W_2^*]$$

is a well-defined element in $DK(J_X)$. Thus we have a well-defined and clearly injective map

$$\begin{aligned} DK(\text{End}_A^0(X)) \ni [U] - [V] &\mapsto \zeta_X([U] - [V]) \\ &= [W_2(\tilde{U} \oplus Z \oplus V \oplus Z)W_2^*] - [W_2 \hat{Z} W_2^*] \in DK(M_2(J_X)). \end{aligned}$$

For surjectivity, suppose that $R, S \in M_n(J_X \hat{\otimes} \mathbb{C}\ell_{1,1})$ are OSUs with $R - S \in M_n(J_X \hat{\otimes} \mathbb{C}\ell_{1,1})$. We consider the corresponding class $[\frac{1}{2}(R + S) \otimes 1 + \frac{1}{2}(R - S) \otimes \sigma_3] - [1 \hat{\otimes} \sigma_1] = [R \oplus S] - [1 \hat{\otimes} \sigma_1]$ via the isomorphism (2.2). As operators on $\hat{\mathcal{H}}_{J_X \hat{\otimes} \mathbb{C}\ell_{1,1}}$, we note that $Z = \text{Id}_{\ell^2(\mathbb{N})} \otimes \sigma_1 \hat{\otimes} 1_{\text{Mult}(J_X) \hat{\otimes} \mathbb{C}\ell_{1,1}}$ and $\text{Id} \otimes 1 \hat{\otimes} \sigma_1$ anti-commute and so are homotopic by a path of OSUs. Therefore, using the stabilisation map $W : (X \oplus \hat{\mathcal{H}})_{J_X \hat{\otimes} \mathbb{C}\ell_{1,1}} \rightarrow \hat{\mathcal{H}}_{J_X \hat{\otimes} \mathbb{C}\ell_{1,1}}$, we can define a class $[W^*(R \oplus S)W|_{X_A}] - [W^*ZW|_{X_A}] \in DK(\text{End}_A^0(X))$, whose representative OSUs may be homotoped so that the difference is a finite rank operator.

We then apply ζ_X and obtain the class

$$\begin{aligned} &\left[W_2 \begin{pmatrix} W^*(R \oplus S)W|_{X_A} & 0 & 0 & 0 \\ 0 & Z & 0 & 0 \\ 0 & 0 & W^*ZW|_{X_A} & 0 \\ 0 & 0 & 0 & Z \end{pmatrix} W_2^* \right] - [W_2 \hat{Z} W_2^*] = [R \oplus S] - [1 \otimes \sigma_1] \\ &= [R] - [S], \end{aligned}$$

where the last equality following from an application of the isomorphism (2.2). \square

Corollary 2.7. *Let X_A be a balanced graded countably generated C^* -module over the C^* -algebra A and define the ideal $J_X = \overline{\text{span}(X|X)}_A$. There is an isomorphism (abusively still called ζ_X) $\zeta_X : DK(\text{End}_A(X), \text{End}_A(X)/\text{End}_A^0(X)) \xrightarrow{\cong} DK(J_X)$ given by applying excision and then*

$$\zeta_X([U] - [V]) = [W_2(\tilde{U} \oplus Z \oplus V \oplus Z)W_2^{-1}] - [W_2\hat{Z}W_2^*]. \quad (2.8)$$

Proof. The excision map $DK(\text{End}_A(X), \text{End}_A(X)/\text{End}_A^0(X)) \rightarrow DK(\text{End}_A^0(X))$ simply picks representatives of each class which lie in $\text{End}_A^0(X) \sim \hat{\otimes} \mathcal{C}\ell_{1,1}$, as shown in Proposition 2.4. We then apply the Morita isomorphism of Lemma 2.6. \square

Finally, we can take all spaces and unitaries to be Real, and carry through the same discussion without issue. We observe that Exel's elegant Fredholm index proof [16] of Equation (2.7) for complex K -theory would require the development of Fredholm theory in our setting, and so we have opted for this more direct route.

2.3 KK -theory with real structures

We now briefly review Real Kasparov theory or KKR -theory [22]. A complex Hilbert C^* -module X_B is a Real Hilbert C^* -module if there is an antilinear map $\mathfrak{r}_X : X_B \rightarrow X_B$, called the real structure, such that $(x^{\mathfrak{r}_X})^{\mathfrak{r}_X} = x$, $x^{\mathfrak{r}_X} \cdot b^{\mathfrak{r}_B} = (x \cdot b)^{\mathfrak{r}_X}$ and $(x_1^{\mathfrak{r}_X} \mid x_2^{\mathfrak{r}_X})_B = ((x_1 \mid x_2)_B)^{\mathfrak{r}_B}$. The real structure on the C^* -module induces a real structure \mathfrak{r} on $\text{End}_B(X)$ via $S^{\mathfrak{r}}x = (S(x^{\mathfrak{r}_X}))^{\mathfrak{r}_X}$. Representations of Real algebras $\pi : A \rightarrow \text{End}_B(X)$ should be compatible with this real structure, $\pi(a^{\mathfrak{r}_A}) = \pi(a)^{\mathfrak{r}}$.

We will often work with unbounded operators on C^* -modules, see [32, Chapter 9]. We recall that a densely defined closed self-adjoint operator $D : \text{Dom}(D) \subset X_B \rightarrow X_B$ is *regular* if the operator $1 + D^2 : \text{Dom}(D^2) \rightarrow X_B$ has dense range. We write that $D^{\mathfrak{r}} = D$ if $(\text{Dom}(D))^{\mathfrak{r}_X} \subset \text{Dom}(D)$ and $(Dx^{\mathfrak{r}_X})^{\mathfrak{r}_X} = Dx$ for all $x \in \text{Dom}(D)$. We also recall the graded commutator, where for endomorphisms S, T with homogenous parity $[S, T]_{\pm} = ST - (-1)^{|S||T|}TS$.

Definition 2.8. Let A and B be \mathbb{Z}_2 -graded Real C^* -algebras. A Real unbounded Kasparov module $(\mathcal{A}, \pi X_B, D)$ consists of

1. a Real and \mathbb{Z}_2 -graded C^* -module X_B ,
2. a Real and graded $*$ -homomorphism $\pi : A \rightarrow \text{End}_B(X)$,
3. an unbounded self-adjoint, regular and odd operator $D = D^{\mathfrak{r}}$ and a dense $*$ -subalgebra $\mathcal{A} \subset A$ such that for all $a \in \mathcal{A} \subset A$, $\pi(a)\text{Dom}(D) \subset \text{Dom}(D)$ and

$$[D, \pi(a)]_{\pm} \in \text{End}_B(X), \quad \pi(a)(1 + D^2)^{-1} \in \text{End}_B^0(X). \quad (2.9)$$

If both algebras and the module are trivially graded but the self-adjoint regular operator D still satisfies the conditions (2.9), then we have an odd Kasparov module.

We will often omit the representation $\pi : A \rightarrow \text{End}_B(X)$ if the context is clear. An unbounded Kasparov modules represent a class $[(A, X_B, D(1 + D^2)^{-1/2})] \in KKR(A, B)$ [2], though we remark that the group $KKR(A, B)$ depends on the choice of real structures for A and B . If A and B are ungraded and (\mathcal{A}, X_B, D) is an odd unbounded Kasparov module, then we can turn

it into a graded Kasparov module $(\mathcal{A} \hat{\otimes} \mathbb{C}\ell_{0,1}, X_B \hat{\otimes} \mathbb{C}^2, D \hat{\otimes} \sigma_1)$, where the generator of the left $\mathbb{C}\ell_{0,1}$ -action is represented by the matrix $-i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

If (\mathcal{A}, X_B, D) is a Real unbounded Kasparov module, then we can ignore the real structures and obtain a complex unbounded Kasparov module and class in $KK(A, B)$. If we restrict the Real C^* -module X_B to the elements fixed under τ_X , we obtain a real C^* -module $X_{B^{\tau_B}}$. Similarly, the Real left action of A becomes a real left action $\pi : A^{\tau_A} \rightarrow \text{End}_{B^{\tau_B}}(X^{\tau_X})$. We do not lose any information by restricting Real Kasparov modules to real C^* -modules and algebras. Similarly, real Kasparov modules can be complexified to obtain Real Kasparov modules and so $KKR(A, B) \cong KKO(A^{\tau_A}, B^{\tau_B})$. If the algebra B is trivially graded, we can also consider real K -theory, where $KKR(\mathbb{C}\ell_{r,s}, B) \cong KKO(Cl_{r,s}, B^{\tau_B}) \cong KO_{r-s}(B^{\tau_B})$.

Remark 2.9 (Normalisation of classes in $KKR(\mathbb{C}\ell_{1,0}, A)$). The KK group we focus on in this manuscript is $KKR(\mathbb{C}\ell_{1,0}, A)$. We review some basic simplifications of representatives of KK -classes, cf. [4, Section 17.4]. To emphasise the generator e of the Clifford representation, we will denote the Kasparov module $(\mathbb{C}\ell_{1,0}, X_A, T)$ (bounded or unbounded) by (e, X_A, T) .

We can assume without loss of generality that $e^2 = 1_X \in \text{End}_A(X)$. (If e^2 acts as a projection P , we can restrict the module to PX_A , replacing T by PTP and the remaining part of the Kasparov module will be degenerate.) This therefore allows us to assume that X_A is a balanced graded C^* -module. Lastly, we can guarantee that the generator e of the $\mathbb{C}\ell_{1,0}$ action on X_A anti-commutes (graded-commutes) with the operator T . If $Te + eT \neq 0$, then we can take the perturbation $\tilde{T} = \frac{1}{2}(T - eTe)$ which anti-commutes with e without changing the KKR -class. \diamond

3 A Cayley isomorphism of complex K -theory and KK -theory

In this section, we consider the more familiar complex ungraded K -theory and define an isomorphism to (odd) KK -theory using the Cayley transform. Our map provides an alternative approach to the well-known isomorphism which the reader can find in [4, Section 17.5]. To highlight its usefulness, we show that our map is well-suited for the constructive form of the Kasparov product which is based on unbounded KK -cycles. The more general case of graded algebras and real structures is more efficiently handled with van Daele K -theory and will be studied in Section 4.

3.1 The Cayley transform on ungraded Hilbert modules

Here we briefly recall and expand on some results from [32, Chapter 9, 10].

Proposition 3.1. *Let A be a C^* -algebra, X_A a Hilbert C^* -module and T a self-adjoint regular (possibly unbounded) and right- A -linear operator on X_A . Then*

$$\mathcal{C}(T) := (T + i)(T - i)^{-1} \in \text{End}_A(X)$$

is a unitary operator. If T has compact resolvent then $\mathcal{C}(T) - 1 \in \text{End}_A^0(X)$. Similarly, if $V \in \text{End}_A(X)$ is unitary, then

$$\mathcal{C}^{-1}(V) = i(V + 1)(V - 1)^{-1}$$

is a (possibly unbounded) self-adjoint regular operator with domain $(V - 1)X_A$. If $V - 1 \in \text{End}_A^0(X)$, then $\mathcal{C}^{-1}(V)$ has compact resolvent.

Proof. The proof of self-adjointness and regularity can be found in [32, Chapter 10]. If T has compact resolvent, then

$$(T + i)(T - i)^{-1} - 1 = 2i(T - i)^{-1} \in \text{End}_A^0(X).$$

Similarly, if $V - 1 \in \text{End}_A^0(X)$, then a short calculation yields

$$1 + \mathcal{C}^{-1}(V)^2 = 4((V - 1)(V^* - 1))^{-1},$$

whence $(1 + \mathcal{C}^{-1}(V)^2)^{-1} \in \text{End}_A^0(X)$ and so is compact as an endomorphism. Taking the square root remains inside the compact operators. \square

Remark 3.2. Note that if T is invertible, then -1 is not in the spectrum of $\mathcal{C}(T)$. If T is bounded, then 1 is not in the spectrum of $\mathcal{C}(T)$. \diamond

Despite the suggestive notation, \mathcal{C} and \mathcal{C}^{-1} are not complete inverses of each other. If $U \in \text{End}_A(X)$ is unitary, then $\mathcal{C} \circ \mathcal{C}^{-1}(U)$ is the restriction of U to the C^* -module $\overline{(U - 1)X_A}$, the closure of $\text{Dom}(\mathcal{C}^{-1}(U))$ in X_A , and may not recover all of X_A in general. However, we can recover the essential information of U at the level of K -theory classes.

Lemma 3.3. *Let A be a unital C^* -algebra and $U \in A$ unitary. Define the ideal $J_U = \overline{A(U - 1)A}$. Then U defines a class in $[U] \in K_1(J_U)$. With $\iota_U : J_U \hookrightarrow A$ the inclusion, $(\iota_U)_*([U]) = [U] \in K_1(A)$.*

Proof. That $[U] \in K_1(A)$ defines $[U] \in K_1(J_U)$ follows since $U|_{J_U} = 1 \bmod J_U$. Let $q : A \rightarrow A/J_U$ be the quotient map, and observe that $[q(U)] = [1] = 0 \in K_1(A/J_U)$. So $q(U|_{J_U})$ is stably homotopic to 1 , whence there exists $w \in M_{n+1}(A/J_U)$ a unitary in the connected component of the identity such that

$$w(q(U|_{J_U}) \oplus 1_n)w^{-1} = 1_{n+1}.$$

Now lift w to a unitary \tilde{w} over A in the connected component of the identity. Then

$$\tilde{w}(U|_{J_U} \oplus 1_n)\tilde{w}^{-1} = \tilde{w}(U \oplus 1_n)\tilde{w}^{-1} = 1_{n+1} \bmod J$$

and so

$$\iota_*([U]) = \iota_*([\tilde{w}(U \oplus 1_n)\tilde{w}^{-1}]) = \iota_*([\tilde{w}(U \oplus 1_n)\tilde{w}^{-1}]) = [\tilde{w}(U \oplus 1_n)\tilde{w}^{-1}] = [U] \in K_1(A). \quad \square$$

3.2 The Cayley transform and odd K -theory

Let A be a C^* -algebra and consider a unitary element $u \in M_N(A)$ (or $M_N(A^\sim)$ if A is non-unital). We consider the (inverse) Cayley transform $\mathcal{C}^{-1}(u)$ of u as an unbounded self-adjoint regular operator on a suitable A -module. Namely,

$$\text{Dom}(\mathcal{C}^{-1}(u)) = (u - 1)A^N, \quad \mathcal{C}^{-1}(u)v = i(u + 1)(u - 1)^{-1}v, \quad v \in (u - 1)A^N.$$

Let $\overline{(u - 1)A_A^N}$ be the closure of $\text{Dom}(\mathcal{C}^{-1}(u))$ in A^N .

Proposition 3.4. *The triple $(\mathbb{C}, \overline{(u - 1)A_A^N}, \mathcal{C}^{-1}(u))$ is an unbounded odd Kasparov module.*

Proof. Most of the result immediately follows from Proposition 3.1. We compute that $(1 + \mathcal{C}(u)^2)^{-1} = \frac{1}{4}(u - 1)(u^* - 1) \in M_N(A)$ (as opposed to $M_N(A^\sim)$) and so is compact as an endomorphism. As in the proof of Proposition 3.1, taking the square root remains inside the compact operators. \square

Theorem 3.5. *The map $K_1(A) \rightarrow KK^1(\mathbb{C}, A)$ defined by sending a unitary to the class of its Cayley transform,*

$$K_1(A) \ni [u] \mapsto [(\mathbb{C}, \overline{(u-1)A_A^N}, \mathcal{C}^{-1}(u))] \in KK^1(\mathbb{C}, A),$$

is well-defined and an isomorphism. The inverse is provided by the Cayley transform on densely defined self-adjoint and regular operators,

$$KK^1(\mathbb{C}, A) \ni [(\mathbb{C}, X_A, T)] \mapsto \iota_* \circ \zeta_X([\mathcal{C}(T)])$$

where $\zeta_X : K_1(\text{End}_A^0(X)) \rightarrow K_1(J_X)$ is the Morita isomorphism for $J_X = \overline{\text{span}(X|X)}_A$ and $\iota : J_X \rightarrow A$ is the inclusion.

Proof. Additivity is clear, and if $u = 1_A$ then the module X_A is the zero module, and $\mathcal{C}^{-1}(u) = 0$. Hence the resulting class is zero in $KK^1(\mathbb{C}, A)$.

Now suppose that $\{u_t\}_{t \in [0,1]}$ is a norm continuous path of unitaries over A^\sim . Define a right $A \otimes C([0, 1])$ -module by

$$\mathcal{X}_{A \otimes C([0,1])} = \{f : [0, 1] \rightarrow A^N : f(t) \in \overline{(u_t - 1)A_A^N} \text{ for all } t \in [0, 1]\}$$

Similarly we define

$$\text{Dom}(\mathcal{C}^{-1}(u_\bullet)) = \{f : [0, 1] \rightarrow A^N : f(t) \in \text{Dom}(\mathcal{C}^{-1}(u_t)) \text{ for all } t \in [0, 1]\}$$

and $(\mathcal{C}^{-1}(u_\bullet)f)(t) = \mathcal{C}^{-1}(u_t)f(t)$. Then because u_\bullet is a unitary over $A^\sim \otimes C([0, 1])$, the triple $(\mathbb{C}, \mathcal{X}_{A \otimes C([0,1])}, \mathcal{C}^{-1}(u_\bullet))$ is an odd Kasparov \mathbb{C} - $A \otimes C([0, 1])$ -module. Composing with the evaluation map on $C[0, 1]$, the Kasparov modules $(\mathbb{C}, X_A, \mathcal{C}^{-1}(u_0))$ and $(\mathbb{C}, X_A, \mathcal{C}^{-1}(u_1))$ define the same class in $KK^1(\mathbb{C}, A)$, and our map is well-defined and injective.

For surjectivity we will display the inverse map. Given an odd unbounded Kasparov \mathbb{C} - A -module (\mathbb{C}, X_A, T) with T self-adjoint and regular, we define $u = \mathcal{C}(T) = (T + i)(T - i)^{-1}$. Then $u - 1_X = 2i(T - i)^{-1}$ is compact by Proposition 3.1, so that $u \in (\text{End}_A^0(X))^\sim \subset (\mathcal{K} \otimes A)^\sim$. Provided we obtain a well-defined map, additivity is obvious.

Suppose that (\mathbb{C}, X_A, T) represents zero. Then (modulo degenerate Kasparov modules) the bounded transform $F_T = T(1 + T^2)^{-1/2}$ of T is operator homotopic to an invertible, since the only obstruction to triviality of the module is $1_X - F_T^2$. This means that we can suppose that T is invertible, and then we see that $(T + i)(T - i)^{-1}$ has an arc containing -1 in its resolvent set by Remark 3.2. Thus the unitary $(T + i)(T - i)^{-1}$ is homotopic to 1_X , and so represents zero in $K_1(A)$.

If (\mathbb{C}, X_A, T_t) is an operator homotopy, $t \in [0, 1]$, then we obtain a norm continuous path of unitaries $\mathcal{C}(T_t)$. This is done in the Real case in Lemma 4.14, see Equation (4.4), and the proof carries over. Hence the unitaries $\mathcal{C}(T_0)$ and $\mathcal{C}(T_1)$ define the same class in $K_1(\text{End}_A^0(X))$.

We define the inverse map to be

$$KK^1(\mathbb{C}, A) \ni [(\mathbb{C}, X_A, T)] \mapsto \iota_* \circ \zeta_X([\mathcal{C}(T)])$$

where $\zeta_X : K_1(\text{End}_A^0(X)) \rightarrow K_1(J_X)$ is the Morita isomorphism (defined analogously to Lemma 2.6), $J_X = \overline{\text{span}(X|X)}_A$ and $\iota : J_X \rightarrow A$ is the inclusion.

The inverse map is well-defined and so we now check that the two maps are indeed mutual inverses. For $u \in M_N(A)$ we find that our recipe gives $\mathcal{C}(\mathcal{C}^{-1}(u)) = u$ as an element of $\text{End}_A^0((u-1)A_A^N)^\sim$. Applying $\zeta_{\overline{(u-1)A^N}}$ to the class $[u] \in K_1(\text{End}_A^0((u-1)A^N))$ gives $[u]^J \in K_1(J_u)$ where $J_u = \overline{A^N(u-1)A^N}$. By Lemma 3.3, $\iota_*([u]^J) = [u] \in K_1(A)$.

In the other direction, given an odd Kasparov module (\mathbb{C}, X_A, T) , we have that $\mathcal{C}^{-1}(\mathcal{C}(T)) = T$ as operators on X . To prove that we obtain an isomorphism of groups, we consider the two homomorphisms $\mathcal{C}_A^{-1} : K_1(A) \rightarrow KK^1(\mathbb{C}, A)$ and $\mathcal{C}_{J_X}^{-1} : K_1(J_X) \rightarrow KK^1(\mathbb{C}, J_X)$ defined by the Cayley transform. These two homomorphisms are related by

$$\mathcal{C}_A^{-1} \circ \iota_* \circ \zeta_X([\mathcal{C}(T)]) = \iota_* \circ \mathcal{C}_{J_X}^{-1} \circ \zeta_X([\mathcal{C}(T)]) = \iota_* \circ \zeta_X^{KK} \circ \mathcal{C}_{\text{End}_A^0(X)}^{-1}([\mathcal{C}(T)]) \quad (3.1)$$

where $\mathcal{C}_{\text{End}_A^0(X)}^{-1} : K_1(\text{End}_A^0(X)) \rightarrow KK^1(\mathbb{C}, \text{End}_A^0(X))$ and $\zeta_X^{KK} = \cdot \hat{\otimes}_{\text{End}_A^0(X)} [(\text{End}_A^0(X), X_{J_X}, 0)]$ is the product with the Morita equivalence bimodule. The first equality essentially follows from Lemma 3.3, while the second comes from a direct calculation and simple homotopy. The details in the Real case are presented in Lemmas 4.12 and 4.13.

Applying (3.1) to the class $[(\mathbb{C}, X_A, T)] \in KK^1(\mathbb{C}, A)$, we have

$$\begin{aligned} \mathcal{C}_A^{-1} \circ \iota_* \circ \zeta_X([\mathcal{C}(T)]) &= \iota_* \circ \zeta_X^{KK} [(\mathbb{C}, \overline{(\mathbb{C}, (T-i)^{-1} \text{End}_A^0(X))_{\text{End}_A^0(X)}}), \mathcal{C}^{-1} \circ \mathcal{C}(T)] \\ &= \iota_* \circ \zeta_X^{KK} [(\mathbb{C}, \text{End}_A^0(X)_{\text{End}_A^0(X)}), T] \\ &= \iota_* [(\mathbb{C}, X_{J_X}, T)] = [(\mathbb{C}, X_A, T)], \end{aligned}$$

where we used that $\mathcal{C}(T) - 1 = 2i(T-i)^{-1}$ and the resolvent of T has dense range. \square

Corollary 3.6. *The generator of $KK^1(\mathbb{C}, C_0(\mathbb{R})) = \mathbb{Z}$ is represented by the unbounded Kasparov module*

$$\left(\mathbb{C}l_1, \begin{pmatrix} C_0(\mathbb{R}) \\ C_0(\mathbb{R}) \end{pmatrix}_{C_0(\mathbb{R})}, \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \right),$$

where the odd generator of $\mathbb{C}l_1$ acts by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Proof. The generator of $K_1(C_0(\mathbb{R}))$ is given by

$$u = e^{-2i \tan^{-1}(x) + i\pi} = -e^{-2i \tan^{-1}(x)}$$

since under the isomorphism $C_0((-1, 1)) \rightarrow C_0(\mathbb{R})$ given by $t \mapsto \tan(\pi t/2)$ the unitary u is mapped onto the generator of $K_1(C_0(-1, 1))$ described in [18, Example 4.8.7]. We then calculate the Cayley transform of u and find $\mathcal{C}^{-1}(u)$ is the operator of multiplication by x . Representing the odd Kasparov module as a class in $KK^1(\mathbb{C}, A)$ gives the desired result. \square

3.3 The graph projection, even K -theory and index pairings

Theorem 3.5 is a special case of a more general result explicitly relating unbounded Kasparov theory with van Daele K -theory studied in Section 4. Before going on to describe this relation

in general, we complete the picture for complex ungraded algebras by providing an isomorphism $KK(\mathbb{C}, A) \cong K_0(A)$ which is compatible with both the Cayley transform and the Kasparov product. The following example provides important motivation.

Example 3.7. The generator of $K_0(C_0(\mathbb{R}^2))$ is well-known to be the external product of the generator of $K_1(C_0(\mathbb{R}))$ with itself [18, Example 4.8.7]. It is also known to be the class $[p_B] - [1]$ where $[1]$ denotes the class of the trivial line bundle and

$$p_B(x, y) = \frac{1}{1 + x^2 + y^2} \begin{pmatrix} 1 & x - iy \\ x + iy & x^2 + y^2 \end{pmatrix}$$

is the Bott projector. Using the unbounded external Kasparov product (see [2]) we easily find that the external product of two copies of the generator of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$ from Corollary 3.6 to be represented by

$$\left(\mathbb{C}, \begin{pmatrix} C_0(\mathbb{R}^2) \\ C_0(\mathbb{R}^2) \end{pmatrix}_{C_0(\mathbb{R}^2)}, T = \begin{pmatrix} 0 & x - iy \\ x + iy & 0 \end{pmatrix} \right).$$

To relate these two representatives, we recall the general representation of K -theory classes below. \diamond

Let A be a complex C^* -algebra. Any class in $K_0(A)$ can be represented by a difference $[p] - [q]$ where for some $N \in \mathbb{N}$ and unitisation A_b we have projections $p, q \in M_N(A_b)$ and some $W \in M_N(A_b)$ such that $WpW^* - q \in M_N(A)$. Excision says that we can always take $A_b = A^\sim$, the minimal unitisation, and W can be taken to be a lift of a partial isometry over A^\sim/A .

More generally, Morita invariance of K -theory says that if $p, q \in \text{Mult}(A \otimes \mathcal{K})$ are projections in the multiplier algebra such that there exists $W \in \text{Mult}(A \otimes \mathcal{K})$ unitary with $WpW^* - q \in A \otimes \mathcal{K}$, then $[p] - [q] \in K_0(A \otimes \mathcal{K}) \cong K_0(A)$ and all classes are of this form.

Theorem 3.8. *If $T : \text{Dom}(T) \subset X_A \rightarrow X_A$ is an (unbounded regular) odd self-adjoint operator on the graded C^* -module X_A with compact resolvent, let $P_{T_+} \in \text{End}_A(X)$ be the graph projection of $T_+ : X_+ \rightarrow X_-$ and P_{X_-} the projection onto the negative part of X_A . Then*

$$[P_{T_+}] - [P_{X_-}]$$

defines a class in $K_0(A)$. The map

$$KK(\mathbb{C}, A) \ni [(\mathbb{C}, X_A, T)] \mapsto [P_{T_+}] - [P_{X_-}] \in K_0(\text{End}_A^0(X)) \cong K_0(A)$$

provides an inverse to the isomorphism $K_0(A) \rightarrow KK(\mathbb{C}, A)$

$$[p] - [q] \mapsto \left[\left(\mathbb{C}, p(A^\sim)_A^N \oplus q(A^\sim)_A^N, \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix} \right) \right], \quad WpW^* - q \in M_N(A). \quad (3.2)$$

Proof. That the groups $K_0(A)$ and $KK(\mathbb{C}, A)$ are isomorphic is [22, Theorem 3, Section 6] in the unital case and [22, Corollary 2, Section 6] in general. The remainder of the proof is a careful reiteration of [22, Corollary 2, Section 6]. By [2] every Kasparov class in $KK(\mathbb{C}, A)$ with A σ -unital can be represented by an unbounded Kasparov module (\mathbb{C}, X_A, T) , where unbounded means ‘not necessarily bounded’.

Given (\mathbb{C}, X_A, T) , the graph projection of T_+ is

$$\begin{aligned} P_{T_+} &= \begin{pmatrix} (1 + T_+^* T_+)^{-1} & (1 + T_+^* T_+)^{-1} T_+^* \\ T_+(1 + T_+^* T_+)^{-1} & T_+(1 + T_+^* T_+)^{-1} T_+^* \end{pmatrix} \\ &= \begin{pmatrix} (1 + T_+^* T_+)^{-1} & (1 + T_+^* T_+)^{-1} T_+^* \\ T_+(1 + T_+^* T_+)^{-1} & P_{X_-} - (1 + T_+ T_+^*)^{-1} \end{pmatrix}. \end{aligned} \quad (3.3)$$

Hence $P_{T_+} = P_{X_-}$ mod $\text{End}_A^0(X)$, and so $[P_{T_+}] - [P_{X_-}]$ defines a K -theory class for $\text{End}_A^0(X)$ and so a class in $K_0(A)$. In addition there is an isomorphism

$$v_+ = ((1 + T_+^* T_+)^{-1/2}, (1 + T_+^* T_+)^{-1/2} T_+^*) : \begin{pmatrix} X_+ \\ X_- \end{pmatrix} \rightarrow X_+$$

with $v_+ v_+^* = 1_{X_+}$ and $v_+^* v_+ = P_{T_+}$. Thus *provided that* $(1 + T_+^* T_+)^{-1/2} T_+^* \in \text{End}_A^0(X)$ *we have*

$$[P_{T_+}] - [P_{X_-}] = [P_{X_+}] - [P_{X_-}]. \quad (3.4)$$

To show that Equation (3.2) is an inverse to the graph projection map, we need to consider classes in $K_0(A \otimes \mathcal{K})$. The analogue of the map (3.2) for $K_0(A \otimes \mathcal{K}) \rightarrow KK(\mathbb{C}, A)$ can be written (for a separable Hilbert space \mathcal{H}) as

$$K_0(A \otimes \mathcal{K}) \ni [p] - [q] \mapsto \left[\left(\mathbb{C}, p(\mathcal{H} \otimes A^\sim)_A \oplus q(\mathcal{H} \otimes A^\sim)_A, \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix} \right) \right]. \quad (3.5)$$

We can consider the modules as being over A or A^\sim , as the difference lives in $K_0(A)$.

Now let (\mathbb{C}, X_A, T) be an even Kasparov module over A and map it to $[P_{T_+}] - [P_{X_-}] \in K_0(\text{End}_A^0(X)) \cong K_0(A)$. The formula (3.3) for the graph projection shows that we obtain a Kasparov module

$$\left(\mathbb{C}, \begin{pmatrix} P_{T_+} \begin{pmatrix} X_+ \\ X_- \end{pmatrix} \\ X_- \end{pmatrix}_A, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right).$$

One can now check directly that the map $[(\mathbb{C}, X_A, T)] \mapsto [P_{T_+}] - [P_{X_-}]$ is well-defined and provides an inverse to the isomorphism in Equation (3.5). \square

Remark 3.9. In the next section we will consider a more general approach which is compatible with real structures and gradings. We will compare the more general method with the graph projection approach in Remark 4.17. \diamond

Returning to Example 3.7 and the Kasparov module

$$\left(\mathbb{C}, \begin{pmatrix} C_0(\mathbb{R}^2) \\ C_0(\mathbb{R}^2) \end{pmatrix}_{C_0(\mathbb{R}^2)}, T = \begin{pmatrix} 0 & x - iy \\ x + iy & 0 \end{pmatrix} \right),$$

we obtain precisely $P_{x+iy} = p_B$ the Bott projection. Observe that with this representative of the class $[p_B] - [1]$, we can not take the formal difference of modules “[X_+] - [X_-]”, as $(x \pm iy)(1 + x^2 + y^2)^{-1/2}$ is not in the minimal unitisation of $C_0(\mathbb{R}^2)$. One way to think about this issue is to observe that both X_\pm are of the form $X_\pm = Y_\pm \otimes_{C_0(\mathbb{R})} C_0(\mathbb{R})$, with Y_\pm finite projective $C_0(\mathbb{R})^\sim$ modules. The modules Y_\pm can not both be trivial, as $x \pm iy$ would not provide an operator between them: see [40]. The graph projection of the “intertwining

operator” $x + iy$ provides the most direct way to access the K -theoretic difference of these two (seemingly trivial) modules.

The graph projection approach has been exploited many times before in noncommutative approaches to index theory, see for example [11, 15, 34, 35].

The Cayley transform can be conveniently used to express the index pairing between K_1 -theory and K^1 -homology, as this pairing becomes a Kasparov product when the transform is applied to the K_1 -class. We describe such pairings and products in the Appendix. Together with other known properties of the index pairing, we obtain the following result.

Proposition 3.10. *Let $(A, \mathcal{H}, \mathcal{D})$ be an odd spectral triple with $\overline{A\mathcal{H}} = \mathcal{H}$, and let $u \in M_N(A^\sim)$ be unitary with Cayley transform $\mathcal{C}^{-1}(u)$. Suppose that we have an approximate unit $v_n \in C^*((u-1), (u^*-1))$ such that $[\mathcal{D}, v_n](1-u^*) \rightarrow 0$ $*$ -strongly. Then with $\tilde{\mathcal{D}} = \mathcal{D}|_{(u-1)\text{Dom}(\mathcal{D})}$, the index pairing between $K_1(A)$ and the spectral triple is given by*

$$\begin{aligned} \langle [u], [\mathcal{D}] \rangle &= sf(\mathcal{D}, u\mathcal{D}u^*) = \text{Index}(PuP - (1-P)) \\ &= \text{Index}(\mathcal{C}^{-1}(u) + i\tilde{\mathcal{D}} : \overline{(u-1)\mathcal{H}^N} \rightarrow \overline{(u-1)\mathcal{H}^N}), \end{aligned}$$

where $sf(\mathcal{D}, u\mathcal{D}u^*)$ is the spectral flow, [36, 37, 10], and $P = \chi_{[0, \infty)}(\mathcal{D})$ is the non-negative spectral projection.

Proof. We know from [36, 37, 10, 20] (for instance) that the index pairing of $[u]$ and the class of $(A, \mathcal{H}, \mathcal{D})$ is given by

$$\langle [u], [\mathcal{D}] \rangle = sf(\mathcal{D}, u\mathcal{D}u^*) = \text{Index}(PuP - (1-P)).$$

Applying the Cayley transform to the unitary u , the index pairing becomes the Kasparov product of the class introduced in Proposition 3.4 with the KK -class defined by the spectral triple,

$$[(\mathbb{C}, \overline{(u-1)A_A^N}, \mathcal{C}^{-1}(u))] \otimes_A [(A, \mathcal{H}, \mathcal{D})]$$

followed by the standard isomorphism $KK(\mathbb{C}, \mathbb{C}) \ni [(\mathbb{C}, \mathcal{H}, T)] \mapsto \text{Index}(T_+) \in \mathbb{Z}$. Observing that $\overline{(u-1)A^N} \otimes_A \mathcal{H} = \overline{(u-1)\mathcal{H}^N}$, a representative of this product is given by

$$\left(\mathbb{C}, \overline{(1-u)\mathcal{H}^N} \otimes \mathbb{C}^2, \begin{pmatrix} 0 & \mathcal{C}^{-1}(u) - i\tilde{\mathcal{D}} \\ \mathcal{C}^{-1}(u) + i\tilde{\mathcal{D}} & 0 \end{pmatrix} \right),$$

see Theorem A.2. Taking the index,

$$\text{Index}(PuP) = sf(\mathcal{D}, u\mathcal{D}u^*) = \text{Index}(\mathcal{C}^{-1}(u) + i\tilde{\mathcal{D}} : \overline{(u-1)\mathcal{H}^N} \rightarrow \overline{(u-1)\mathcal{H}^N}). \quad \square$$

More general odd Kasparov products can be handled by Theorem A.2 in the appendix.

Example 3.11. Let $A = C(S^1)$ and take the usual spectral triple $\mu = (C^\infty(S^1), L^2(S^1), \frac{1}{i} \frac{d}{d\theta})$ for the circle. For $u = e^{-i\theta}$, $-\pi \leq \theta \leq \pi$, we can represent the Cayley transform on the domain of functions vanishing (fast enough) at $\theta = 0$ by $\mathcal{C}^{-1}(u) = \cot(\theta/2)$. Example A.3 checks that the product $[e^{-i\bullet}] \otimes_{C(S^1)} \mu$ is represented by the spectral triple

$$\left(\mathbb{C}, L^2(S^1) \otimes \mathbb{C}^2, \begin{pmatrix} 0 & -\frac{d}{d\theta} + \cot(\theta/2) \\ \frac{d}{d\theta} + \cot(\theta/2) & 0 \end{pmatrix} \right).$$

The equation $\frac{dy}{d\theta} \pm \cot(\theta/2)y = 0$ is satisfied for the function $y = C(\sin(\theta/2))^{\pm 2}$ with C constant, and while $(\sin(\theta/2))^2$ is square-summable, $(\sin(\theta/2))^{-2}$ is not. Hence $\text{Ker}(\frac{d}{d\theta} + \cot(\theta/2)) = \text{span}\{(\sin(\theta/2))^2\}$ and $\text{Index}(\frac{d}{d\theta} + \cot(\theta/2)) = 1$. We have therefore obtained the correct index, giving a check of signs. \diamond

4 A graded Cayley isomorphism between DK -theory and KK -theory

In this section we define a graded version of the Cayley transform on C^* -modules which allows us to define an explicit map between van Daele K -theory and KK -theory. As in the complex ungraded case, we show this map is an isomorphism and consider some applications of this result in Sections 5 and 6.

4.1 The Cayley transform on graded Hilbert modules

Here we extend results on the Cayley transform to *odd* operators on *graded* Hilbert C^* -modules. Throughout this section we assume that the Hilbert C^* -module X_A is balanced graded, i.e., $\text{End}_A(X)$ has at least one OSU.

Definition 4.1. Let X_A be a countably generated and balanced graded C^* -module with OSU $e \in \text{End}_A(X)$. Suppose T is an odd self-adjoint regular (possibly unbounded) right- A -linear operator that anti-commutes with e . Define

$$\mathcal{C}_e(T) := e(T + e)(T - e)^{-1}$$

as the graded Cayley transform of T relative to e .

We first note that $\mathcal{C}_e(T)$ is well-defined as $(T \pm e)^2 = 1 + T^2$, so $(T - e)^{-1}$ is bounded. Moreover the range of $(T - e)^{-1}$ is $\text{Dom}(T)$, and so \mathcal{C}_e is everywhere defined and bounded.

Lemma 4.2. *The operator $\mathcal{C}_e(T)$ is an odd self-adjoint unitary on X_A . Moreover, if $(1 + T^2)^{-1}$ is compact then $\mathcal{C}_e(T) - e$ is compact.*

Proof. Clearly $\mathcal{C}_e(T)$ is odd, and since $eT = -Te$ we have $T(T - e)^{-1} = (T + e)^{-1}T$ and $e(T - e)^{-1} = (-T - e)^{-1}e$. Then we see that $\mathcal{C}_e(T)$ is self-adjoint by computing

$$\begin{aligned} (e(T + e)(T - e)^{-1})^* &= (T - e)^{-1}(T + e)e = T(T + e)^{-1}e - e(T + e)^{-1}e \\ &= T(e(T + e))^{-1} - (e(T + e)e)^{-1} = (Te - 1)(-T + e)^{-1} = e(T + e)(T - e)^{-1}. \end{aligned}$$

Now we compute the square by

$$\begin{aligned} \mathcal{C}_e(T)^2 &= e(T + e)(T - e)^{-1}e(T + e)(T - e)^{-1} = e(T + e)e(-T - e)^{-1}(T + e)(T - e)^{-1} \\ &= -e(T + e)e(T - e)^{-1} = -e^2(-T + e)(T - e)^{-1} = 1. \end{aligned}$$

We have

$$\mathcal{C}_e(T) - e = e((T + e) - (T - e))(T - e)^{-1} = 2(T - e)^{-1}.$$

Since compact operators are closed under continuous functional calculus and an ideal, compactness of $(1 + T^2)^{-1}$ implies compactness of $|T - e|^{-1}$ which then implies compactness of $(T - e)^{-1}$. As a consequence, if $(1 + T^2)^{-1}$ is compact then $\mathcal{C}_e(T) - e$ is compact as well. \square

Remark 4.3. We can recover the ungraded Cayley transform as a special case of our graded map. Namely, if Y_A is an ungraded C^* -module, we consider $X_A = Y_A \otimes \mathbb{C}^2$ with the obvious

grading. Then for S a self-adjoint regular operator on Y_A , we consider $T = S \otimes \sigma_2$ and $e = 1 \otimes \sigma_1$. Using $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & b^{-1} \\ a^{-1} & 0 \end{pmatrix}$ we obtain

$$\mathfrak{C}_e(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -iS + 1 \\ +iS + 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & (iS - 1)^{-1} \\ (-iS - 1)^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathfrak{C}(S)^* \\ \mathfrak{C}(S) & 0 \end{pmatrix}.$$

If $(1 + S^2)^{-1} \in \text{End}_A^0(Y)$, then $\mathfrak{C}(S) - 1 \in \text{End}_A^0(Y)$ and $\mathfrak{C}_e(T) - 1 \otimes \sigma_1 \in \text{End}_A^0(X)$. \diamond

For the inverse Cayley transform, we again consider self-adjoint odd unitaries relative to a base point OSU e .

Definition 4.4. Let X_A be a balanced graded C^* -module with $U, e \in \text{End}_A(X)$ odd self-adjoint unitaries. Define

$$\mathfrak{C}_e^{-1}(U) := e(U + e)(U - e)^{-1}$$

with domain $(U - e)X_A$.

We let $\overline{(U - e)X_A}$ be the closure of $\text{Dom}(\mathfrak{C}_e^{-1}(U))$ in X_A where, by construction, $\mathfrak{C}_e^{-1}(U)$ is densely defined.

Lemma 4.5. *Let X_A be a countably generated and balanced graded C^* -module with OSU $e \in \text{End}_A(X)$. If $U \in \text{End}_A(X)$ is an odd self-adjoint unitary, the operator $\mathfrak{C}_e^{-1}(U)$ is an odd self-adjoint regular (possibly unbounded) operator on $\overline{(U - e)X_A}$ which anti-commutes with e . Moreover, if $U - e$ is compact, then $(1 + \mathfrak{C}_e^{-1}(U)^2)^{-1/2}$ is a compact operator on $\overline{(U - e)X_A}$.*

Proof. It is immediate that $\mathfrak{C}_e^{-1}(U)$ is odd. Since e and U are OSUs we have $e(U \pm e) = (e \pm U)U$. Hence $e(U - e)X_A = (e - U)UX_A = (U - e)X_A$ showing that the domain $(U - e)X_A$ is preserved by e . Furthermore, for any $\psi \in (U - e)X_A$,

$$e(U \pm e)^{-1}\psi = ((U \pm e)e^{-1})^{-1}\psi = (U(e \pm U))^{-1}\psi = (e \pm U)^{-1}U\psi.$$

Therefore e anti-commutes with $(U + e)(U - e)^{-1}$ and hence also with $\mathfrak{C}_e^{-1}(U)$ (on the domain) since

$$e\mathfrak{C}_e^{-1}(U) = e^2(U + e)(U - e)^{-1} = e(e + U)U(U - e)^{-1} = e(e + U)(e - U)^{-1}e = -\mathfrak{C}_e^{-1}(U)e.$$

To show that $\mathfrak{C}_e^{-1}(U)$ is self-adjoint and regular, we employ [32, Theorem 10.4]. Consider the operator $F = \frac{1}{2}\mathfrak{C}_e(U)(2 - Ue - eU)^{1/2} = \frac{1}{2}e(Ue + 1)(Ue - 1)^{-1}(2 - Ue - eU)^{1/2}$. This operator is self-adjoint by direct computation using the normality of Ue , has norm bounded above by 1, and also

$$F^2 = \frac{1}{4}(2 + eU + Ue) \quad \text{and so} \\ 1 - F^2 = \frac{1}{2} - \frac{1}{4}(eU + Ue) = \frac{1}{4}(2 - eU - Ue) = \frac{1}{4}(eU - 1)(Ue - 1)$$

which shows that $1 - F^2$ is positive. The operator $\mathfrak{C}_e^{-1}(U)$ is defined on the range of $(U - e)$, which by the unitarity of e is the same as the range of $(Ue - 1)$. The operator $(1 - F^2)^{1/2} = \frac{1}{2}|(Ue - 1)|$ then has dense range equal to $(U - e)X_A$.

We can now apply [32, Theorem 10.4] which implies that the operator $F(1 - F^2)^{-1/2}$ is a densely defined, regular self-adjoint operator on $\overline{(U - e)X_A}$. This operator is

$$\begin{aligned} F(1 - F^2)^{-1/2} &= \frac{1}{2}e(Ue + 1)(Ue - 1)^{-1}(2 - Ue - eU)^{1/2} \left(\frac{1}{4}(2 - Ue - eU) \right)^{-1/2} \\ &= e(Ue + 1)(Ue - 1)^{-1} = e(U + e)(U - e)^{-1} = \mathcal{C}_e^{-1}(U) \end{aligned}$$

and so we find that $\mathcal{C}_e^{-1}(U)$ is regular and self-adjoint.

Next we compute $1 + \mathcal{C}_e^{-1}(U)^2 = 1 + \mathcal{C}_e^{-1}(U)^* \mathcal{C}_e^{-1}(U)$, where

$$\begin{aligned} 1 + \mathcal{C}_e^{-1}(U)^* \mathcal{C}_e^{-1}(U) &= 1 + (U - e)^{-1}(U + e)(U + e)(U - e)^{-1} \\ &= 1 + (2 + eU + Ue)(2 - eU - Ue)^{-1} \\ &= 4(2 - eU - Ue)^{-1} = 4(U - e)^{-2}. \end{aligned} \tag{4.1}$$

Therefore $(1 + \mathcal{C}_e^{-1}(U)^2)^{-1/2} = \frac{1}{2}|U - e|$, which is compact if $U - e$ is compact. \square

Remark 4.6. We again show how to recover the ungraded case. Take Y_A ungraded and $X_A = Y_A \otimes \mathbb{C}^2$. Then odd self-adjoint unitaries U take the form $U = \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix}$ with $V \in \text{End}_A(Y)$ unitary. We then compute for $e = 1 \otimes \sigma_1$,

$$\begin{aligned} \mathcal{C}_e^{-1}(U) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & (V^* + 1) \\ (V + 1) & 0 \end{pmatrix} \begin{pmatrix} 0 & (V - 1)^{-1} \\ (V^* - 1)^{-1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & (V + 1)(V - 1)^{-1} \\ (V^* + 1)(V^* - 1)^{-1} & 0 \end{pmatrix} \\ &= \mathcal{C}^{-1}(V) \otimes \sigma_2 \end{aligned}$$

with $\mathcal{C}^{-1}(V)$ the ungraded Cayley transform of V . \diamond

Remark 4.7. If X_A is a balanced \mathbb{Z}_2 -graded module over the non-trivially \mathbb{Z}_2 -graded algebra A , then it is not necessarily the case that the even and odd halves $X_A = (X_+ \oplus X_-)_A$ are isomorphic with $X_A = (X_+ \oplus UX_+)_A$ and the isomorphism U providing an OSU $\tilde{U} = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$ in $\text{End}_A(X)$. The issue is that \tilde{U} need not be adjointable. \diamond

In a sense which we will make precise, the maps \mathcal{C}_e and \mathcal{C}_e^{-1} are mutual inverses.

Proposition 4.8. *Let X_A be a countably generated and balanced graded C^* -module with OSU $e \in \text{End}_A(X)$. If T is an odd self-adjoint regular operator which anti-commutes with e then $\mathcal{C}_e^{-1} \circ \mathcal{C}_e(T) = T$ on X_A . If $U \in \text{End}_A(X)$ is an OSU, then $\mathcal{C}_e \circ \mathcal{C}_e^{-1}(U)$ is the restriction of U to $\overline{(U - e)X_A}$.*

Proof. Let x be a right- A -linear operator on X_A and $Y_A \subset X_A$ be a submodule on which $(x - e)^{-1}$ is well-defined. Apart from their domains, both expressions $\mathcal{C}_e^{-1} \circ \mathcal{C}_e(x)$ and $\mathcal{C}_e \circ \mathcal{C}_e^{-1}(x)$ are equal to $e(e(x + e)(x - e)^{-1} + e)(e(x + e)(x - e)^{-1} - e)^{-1}$. As $(x + e)(x - e)^{-1} - 1 = 2e(x - e)^{-1}$ the above expression is well-defined on Y_A and given by

$$e(e(x + e)(x - e)^{-1} + e)(e(x + e)(x - e)^{-1} - e)^{-1} = x.$$

For the first statement we substitute $x = T$ and $Y_A = X_A$, which is possible as T anti-commutes with e . For the second statement we take $x = U$ and $Y_A = \overline{(U - e)X_A}$. \square

As in the ungraded case, for $U \in \text{End}_A(X)$ unitary, $\mathfrak{C}_e \circ \mathfrak{C}_e^{-1}(U)$ is the restriction of U to the C^* -module $\overline{(U - e)X_A}$, which need not recover all of X_A in general. However, we have a graded analogue of Lemma 3.3 describing the K -theory classes.

Lemma 4.9. *Let A be a balanced graded unital C^* -algebra and $U, V \in A$ odd self-adjoint unitaries. Define the ideal $J = \overline{A(U - V)A}$. Then U, V defines a class in $[U]^J - [V]^J \in DK(J)$, and with $\iota : J \hookrightarrow A$ the inclusion $\iota_*([U]^J - [V]^J) = [U] - [V] \in DK(A)$.*

Proof. We see that $U = V$ modulo J and so $q_*([U] - [V]) \in DK(A/J)$ will be trivial for $q : A \rightarrow A/J$ the quotient map. By [12, Proposition 2.3] there is an even unitary w over A/J in the connected component of the identity such that

$$q \left(\left(\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \oplus \sigma_1 \oplus \cdots \oplus \sigma_1 \right) \right) = w \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \sigma_1 \oplus \cdots \oplus \sigma_1 \right) w^{-1} \right).$$

Since w is in the connected component of the identity, it lifts to an even unitary \tilde{w} in A connected to the identity and such that

$$W := \tilde{w}^{-1} \left(\left(\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \oplus \sigma_1^{\oplus n} \right) \tilde{w} \right) \quad (4.2)$$

is equal to $\sigma_1^{\oplus(n+1)}$ modulo $J \hat{\otimes} \mathcal{C}\ell_{1,1}$. Hence the unitary W of (4.2) is a unitary over $J \sim \hat{\otimes} \mathcal{C}\ell_{1,1}$. Now we have

$$[W] - [\sigma_1^{\oplus(n+1)}] = \left[\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \right] - [\sigma_1] \in DK(A)$$

because \tilde{w} is an even unitary over A connected to the identity. Since W is a unitary over $J \sim \hat{\otimes} \mathcal{C}\ell_{1,1}$, we may define a class in $DK(J)$ by $[W]^J - [\sigma_1^{\oplus(n+1)}]^J$, where the J just indicates that we regard these as unitaries over $J \sim \hat{\otimes} \mathcal{C}\ell_{1,1}$. Applying the inclusion map

$$\iota_*([W]^J - [\sigma_1^{\oplus(n+1)}]^J) = [W] - [\sigma_1^{\oplus(n+1)}] = \left[\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \right] - [1 \otimes \sigma_1] = [U] - [V] \in DK(A)$$

where we have applied the isomorphism from (2.2) in the last equality. \square

Lastly, we note that the graded Cayley transforms do not involve any complex structure and therefore are valid also for operators on real Hilbert modules.

4.2 The isomorphism of DK and KK

Here we use our results on the graded Cayley transform to construct an explicit isomorphism between the van Daele K -group $DK(A)$ and KK -group $KK(\mathcal{C}\ell_{1,0}, A)$. To cover the complex and real case simultaneously, we work with KKR -theory and real structures.

Using [2] and Remark 2.9, we represent any class in $KKR(\mathcal{C}\ell_{1,0}, A)$ by an unbounded Kasparov module (e, X_A, T) , where X_A is a countably generated and balanced graded Real C^* -module, e^2 acts as the identity and e anti-commutes with T .

Let us first describe a map $\mathfrak{C} : DK(A) \rightarrow KKR(\mathcal{C}\ell_{1,0}, A)$.

Lemma 4.10. *Let A be a unital and balanced graded algebra with $V, W \in M_n(A)$ OSUs. The inverse Cayley transform induces a homomorphism $\mathfrak{C} : DK(A, \tau_A) \rightarrow KKR(\mathbb{C}\ell_{1,0}, A)$,*

$$\mathfrak{C}([V] - [W]) = [(W, \overline{(V - W)A_A^n}, \mathfrak{C}_W^{-1}(V))], \quad \mathfrak{C}_W^{-1}(V) = W(V + W)(V - W)^{-1}.$$

If A is non-unital and weakly balanced graded, and B is any balanced graded unital algebra containing A as a graded ideal, then we can use the Cayley transform to define a homomorphism $\mathfrak{C} : DK(B, B/A, \tau_A) \rightarrow KKR(\mathbb{C}\ell_{1,0}, A)$

$$\mathfrak{C}([V] - [W]) = [(W, \overline{(V - W)B_A^n}, \mathfrak{C}_W^{-1}(V))] = [(W, \overline{(V - W)A_A^n}, \mathfrak{C}_W^{-1}(V))].$$

If A is not balanced nor weakly balanced, let $\xi_{\mathbb{C}\ell_{1,1}} = \cdot \hat{\otimes} [(\mathbb{C}\ell_{1,1}, \mathbb{C}^2_{\mathbb{C}}, 0)]$ be the isomorphism given by the external Kasparov product with the class of the Morita equivalence $(\mathbb{C}\ell_{1,1}, \mathbb{C}^2_{\mathbb{C}}, 0)$. Then given OSUs $V, W \in M_n(A \hat{\otimes} \mathbb{C}\ell_{1,1})$, the Cayley transform defines a homomorphism $\mathfrak{C} : DK(A, \tau_A) \rightarrow KKR(\mathbb{C}\ell_{1,0}, A)$

$$\begin{aligned} \mathfrak{C}([V] - [W]) &= \xi_{\mathbb{C}\ell_{1,1}} [(W, \overline{(V - W)(A \hat{\otimes} \mathbb{C}\ell_{1,1})^n_{A \hat{\otimes} \mathbb{C}\ell_{1,1}}}, \mathfrak{C}_W^{-1}(V))] \\ &= [(W, \overline{(V - W)(A \otimes \mathbb{C}^2)_A^n}, \mathfrak{C}_W^{-1}(V))]. \end{aligned} \quad (4.3)$$

Observe that the map (4.3) encompasses all cases stated in the proposition.

Proof. We deal with the unital and balanced case, as the other cases are only notationally more complex.

As V and W are odd and Real, $\mathfrak{C}_W^{-1}(V)$ is also odd and $\mathfrak{C}_W^{-1}(V)^\tau = \mathfrak{C}_W^{-1}(V)$. We note that $W(V \pm W) = (W \pm V)V$, so $\overline{(W - V)WA_A^n} = \overline{(V - W)A_A^n}$. Thus the action of both the generator of $\mathbb{C}\ell_{1,0}$ and V preserve the A -module and the domain of $\mathfrak{C}_W^{-1}(V)$. Applying Lemma 4.5, $\mathfrak{C}_W^{-1}(V)$ is self-adjoint, regular, anti-commutes with the $\mathbb{C}\ell_{1,0}$ -action and $(1 + \mathfrak{C}_W^{-1}(V)^2)^{-1/2}$ is compact. Hence we obtain a Real (unbounded) Kasparov module so all that is left is to make sure that the map \mathfrak{C} is well-defined.

Suppose that we have a continuous path of odd self-adjoint unitaries $[0, 1] \ni t \mapsto V_t$, with $[V_t] - [W] \in \text{Ker } q_*$ and $V_t^\tau = V_t$ for all t . The continuity of V_t ensures that the pointwise C^* -module $\overline{(V_t - W)A_A^n}$ can be extended to a $A \otimes C([0, 1])$ -module, $\overline{(V_\bullet - W)A^n_{A \otimes C([0, 1])}}$, where the real structure on $A \otimes C([0, 1]) \cong C([0, 1], A)$ is such that $a^\tau(t) = (a(t))^\tau$. Recalling Equation (4.1), the bounded transform of $\mathfrak{C}_W^{-1}(V_t)$ is given by

$$F_t = \frac{1}{2}W(V_t + W)(V_t - W)^{-1}|V_t - W|$$

for all t . Once again the continuity of V_t ensures that $\{F_t\}_{t \in [0, 1]}$ is a well-defined and self-adjoint operator F_\bullet on $\overline{(V_\bullet - W)A^n_{A \otimes C([0, 1])}}$. Assembling this information and using the pointwise properties of F_t , we obtain a Kasparov module

$$(W, \overline{(V_\bullet - W)A^n_{A \otimes C([0, 1])}}, F_\bullet).$$

We therefore obtain a homotopy in KKR and, hence, \mathfrak{C} is well-defined. It is easily seen that \mathfrak{C} respects direct sums and so is a homomorphism. \square

Remark 4.11. Let us briefly note that if $WV + VW = 0$, so W and V are homotopic as OSUs, then the resulting Kasparov module $(W, \overline{(V - W)A_A^n}, \mathfrak{C}_W^{-1}(V))$ is degenerate. We first observe that if W and V anti-commute, $(W - V)^{-1} = \frac{1}{2}(W - V)$ and so

$$\mathfrak{C}_W^{-1}(V) = \frac{1}{2}W(W + V)(V - W) = \frac{1}{2}(V - WWV) = V.$$

Hence the Kasparov module simplifies to $(W, \overline{(V - W)A_A^n}, V)$ which is clearly degenerate. \diamond

To define a map from KK to DK , we need to know that our construction is compatible with Morita invariance. To work with explicit cycles, we also need to consider C^* -modules that are not full. The first issue arises because for a Kasparov module (e, X_A, T) , we most easily construct OSUs in $\text{End}_A(X)$ and need to get back to the coefficient algebra. This is the Morita invariance requirement.

If X is not full, then the Cayley transformation will only have range in $J_X = \overline{\text{span}(X|X)}_A$. Hence we also need to understand the dependence of the Cayley transformation on the inclusion $J_X \hookrightarrow A$. The next two lemmas address these points.

Lemma 4.12. *Let A be a unital and balanced graded algebra with $V, W \in M_n(A)$ OSUs, and let $J = \overline{A^n(V - W)A^n}$. Then we have classes $[V]^J - [W]^J \in DK(J)$ and $[V] - [W] \in DK(A)$ related by $\iota_*([V]^J - [W]^J) = [V] - [W]$ where $\iota : J \rightarrow A$ is the inclusion.*

Letting $\mathfrak{C}^A : DK(A, \mathfrak{r}_A) \rightarrow KKR(\mathbb{C}\ell_{1,0}, A)$ and $\mathfrak{C}^J : DK(A, A/J) \rightarrow KKR(\mathbb{C}\ell_{1,0}, J)$ be the homomorphisms of Lemma 4.10, we have

$$\mathfrak{C}^A \circ \iota_*([V]^J - [W]^J) = \mathfrak{C}^A([V] - [W]) = \iota_* \circ \mathfrak{C}^J([V]^J - [W]^J).$$

Proof. The first statements are proved in Lemma 4.9. The subsequent equalities are true by the construction of the Kasparov modules. \square

Lemma 4.13. *Let X_A be a balanced graded Real C^* -module with $J_X = \overline{\text{span}(X|X)}_A$. The maps $\mathfrak{C}^{J_X} : DK(J_X) \rightarrow KKR(\mathbb{C}\ell_{1,0}, J_X)$ and $\mathfrak{C}^{\text{End}_A^0(X)} : DK(\text{End}_A^0(X)) \rightarrow KKR(\mathbb{C}\ell_{1,0}, \text{End}_A^0(X))$ are such that*

$$\mathfrak{C}^{J_X} \circ \zeta_X = \zeta_X^{KK} \circ \mathfrak{C}^{\text{End}_A^0(X)}$$

where $\zeta_X : DK(\text{End}_A^0(X)) \xrightarrow{\cong} DK(J_X)$ is the isomorphism of Equation (2.7) and $\zeta_X^{KK} = \cdot \hat{\otimes}_{\text{End}_A^0(X)} [(\text{End}_A^0(X), X_{J_X}, 0)]$ the Morita isomorphism in KK .

Proof. Let $[U] - [V] \in DK(\text{End}_A^0(X))$, so that $U, V \in M_n(\text{End}_A^0(X) \hat{\otimes} \mathbb{C}\ell_{1,1})$ with $U - V \in M_n(\text{End}_A^0(X) \hat{\otimes} \mathbb{C}\ell_{1,1})$. Because we deal with matrices over $\text{End}_A^0(X) \hat{\otimes} \mathbb{C}\ell_{1,1}$, the Morita isomorphism of Lemma 2.6 gives that

$$\begin{aligned} \mathfrak{C}^{J_X} \circ \zeta_X([U] - [V]) &= \mathfrak{C}^{J_X}([W_{2n}(\tilde{U} \oplus Z^{\oplus n} \oplus V \oplus Z^{\oplus n})W_{2n}^*] - [W_{2n}\hat{Z}^{\oplus n}W_{2n}^*]) \\ &= \left[\left(\hat{Z}^{\oplus n}, \overline{\begin{pmatrix} \tilde{U} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & V & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} (X \oplus \hat{\mathcal{H}})_{J_X}^{\oplus 2n}, \mathfrak{C}_{\hat{Z}^{\oplus n}}^{-1}(\tilde{U} \oplus Z^{\oplus n} \oplus V \oplus Z^{\oplus n}) \right) \right], \end{aligned}$$

where we have removed W_{2n} using unitary invariance of KK -classes. Similarly, we have that

$$\begin{aligned}\zeta_X^{KK} \circ \mathfrak{C}^{\text{End}_A^0(X)}([U] - [V]) &= \zeta_X^{KK}([\overline{(V, (U - V) \text{End}_A^0(X)^n_{\text{End}_A^0(X)}, \mathfrak{C}_V^{-1}(U))}]]) \\ &= [(V, \overline{(U - V)X_{J_X}^n}, \mathfrak{C}_V^{-1}(U))].\end{aligned}$$

As in the proof of Lemma 4.10, the continuous homotopy from U to \tilde{U} gives a homotopy of Kasparov modules.

To complete the proof, we homotopy the Kasparov module representing $\mathfrak{C}^{J_X} \circ \zeta_X([U] - [V])$. First observe that

$$\begin{pmatrix} \sin(t)V & 0 & \cos(t)1_n & 0 \\ 0 & Z^{\oplus n} & 0 & 0 \\ \cos(t)1_n & 0 & \sin(t)V & 0 \\ 0 & 0 & 0 & Z^{\oplus n} \end{pmatrix}$$

is a homotopy of OSUs. This yields a homotopy of C^* -modules

$$\overline{\begin{pmatrix} \tilde{U} - \sin(t)V & 0 & -\cos(t)1_n & 0 \\ 0 & 0 & 0 & 0 \\ -\cos(t)1_n & 0 & V - \sin(t)V & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} (X \oplus \hat{\mathcal{H}})_{J_X}^{\oplus 2n}$$

to $\overline{(\tilde{U} - V)X_{J_X}^n} \oplus 0$. Simultaneously, we obtain a homotopy of operators from $\mathfrak{C}_{\hat{Z}^{\oplus n}}^{-1}(\tilde{U} \oplus Z^{\oplus n} \oplus V \oplus Z^{\oplus n})$ to $\mathfrak{C}_V^{-1}(\tilde{U}) \oplus 0$ compatible with the obvious path of domains and the (constant) left action of $\mathcal{C}\ell_{1,0}$. Thus

$$\begin{aligned}\mathfrak{C}^{J_X} \circ \zeta_X([U] - [V]) &= [(V, \overline{(\tilde{U} - V)X_{J_X}^n}, \mathfrak{C}_V^{-1}(\tilde{U}))] \\ &= \zeta_X^{KK} \circ \mathfrak{C}^{\text{End}_A^0(X)}([U] - [V]).\end{aligned} \quad \square$$

We now consider the map $KKR(\mathcal{C}\ell_{1,0}, A) \rightarrow DK(A)$.

Lemma 4.14. *Let (e, X_A, T) be an unbounded Real Kasparov module with $e^2 = 1_X$ and e anti-commuting with T . Let $J_X = \overline{\text{span}(X|X)}_A$, $\zeta_X : DK(\text{End}_A(X), \text{End}_A(X)/\text{End}_A^0(X)) \xrightarrow{\cong} DK(J_X)$ the isomorphism of Corollary 2.7 and $\iota : J_X \hookrightarrow A$ the inclusion. Then the Cayley transform defines a homomorphism $\mathfrak{A} : KKR(\mathcal{C}\ell_{1,0}, A) \rightarrow DK(A, \mathfrak{r}_A)$,*

$$KKR(\mathcal{C}\ell_{1,0}, A) \ni [(e, X_A, T)] \xrightarrow{\mathfrak{A}} \iota_* \circ \zeta_X([\mathfrak{C}_e(T)] - [e]) \in DK(A, \mathfrak{r}_A).$$

Proof. Lemma 4.2 tells us that $\mathfrak{C}_e(T) = e(T + e)(T - e)^{-1}$ is odd, self-adjoint, unitary, $\mathfrak{C}_e(T) - e \in \text{End}_A^0(X)$ and $\mathfrak{C}_e(T)^\tau = \mathfrak{C}_e(T)^\tau$. Hence we obtain a class $[\mathfrak{C}_e(T)] - [e] \in DK(\text{End}_A(X), \text{End}_A(X)/\text{End}_A^0(X))$. Thus $\iota_* \circ \zeta_X([\mathfrak{C}_e(T)] - [e])$ is a well-defined element in $DK(A, \mathfrak{r}_A)$ and we just need to check that the map respects the relevant equivalence relations. We use the equivalence relation on KKR generated by unitary equivalence, addition of degenerate Kasparov modules and operator homotopy [4, Section 17].

Any (bounded) degenerate Kasparov module (e, X_A, F) has F invertible and anticommutes with e . So suppose that the operator T of our unbounded Kasparov module (e, X_A, T) is invertible, self-adjoint and graded commutes with the $\mathcal{C}\ell_{1,0}$ -action. The phase of T then defines a degenerate bounded Kasparov module, whose class in KKR is zero. Consider the

homotopy $V(\lambda) = e(T + e\lambda)(T - e\lambda)^{-1}$ for $\lambda \in [0, 1]$. Using the normality of T , we compute that for $\lambda, \varrho \in [0, 1]$,

$$e(T + \lambda e)(T - \lambda e)^{-1} - e(T + \varrho e)(T - \varrho e)^{-1} = 2e(\lambda - \varrho)T(T - \varrho e)^{-1}(T - \lambda e)^{-1}.$$

Hence the map $\lambda \mapsto V(\lambda)$ is norm continuous as $T(T - \varrho e)^{-1}(T - \lambda e)^{-1}$ is uniformly bounded since T is invertible. The path is also invariant under the real structure as $T^{\mathfrak{r}} = T$ and $e^{\mathfrak{r}} = e$. We obtain a homotopy of OSUs such that

$$V(\lambda) - e \in \text{End}_A^0(X), \quad V(1) = e(T + e)(T - e)^{-1} \sim V(0) = e$$

and $[\mathcal{C}_e(T)] - [e] = 0$. Thus degenerate Kasparov classes map to zero.

Given an operator homotopy (e, X_A, F_t) of bounded Kasparov modules with $F_0 = T(1 + T^2)^{-1/2}$, we can define the class

$$(\mathbb{C}\ell_{1,0}, (X \otimes C([0, 1]))_{A \otimes C([0, 1])}, F_{\bullet})$$

as a bounded Kasparov module. As shown in [14, Proposition 2.8, Theorem 2.9], there is some self-adjoint regular T_{\bullet} such that $F_{T_{\bullet}}$ is F_{\bullet} , and we can moreover take T_0 to be operator homotopic to T . Averaging allows us to ensure that $T_t e + e T_t = 0$ and $T_t^{\mathfrak{r}} = T_t$ for all $t \in [0, 1]$. Then we have a homotopy of unbounded operators T_t such that $T_t(1 + T_t^2)^{-1/2}$ is operator norm continuous for all t . Then using $(T_t - e)^{-2} = (1 + T^2)^{-1}$ we compute

$$\begin{aligned} e(T_t + e)(T_t - e)^{-1} &= e(T_t + e)(1 + T_t^2)^{-1/2}(1 + T_t^2)^{1/2}(T_t - e)^{-1} \\ &= e(T_t + e)(1 + T_t^2)^{-1/2}(1 + T_t^2)^{1/2}(T_t - e)(T_t - e)^{-2} \\ &= e(T_t + e)(1 + T_t^2)^{-1/2}(T_t - e)(1 + T_t^2)^{-1/2} \end{aligned} \tag{4.4}$$

which is a product of norm continuous paths by assumption. So we obtain a homotopy of odd Real self-adjoint unitaries. Then the class $[\mathcal{C}_e(T_t)] - [e]$ is constant in the relative group $DK(\text{End}_A(X), \text{End}_A(X)/\text{End}_A^0(X))$ for all $t \in [0, 1]$ and so \mathfrak{A} is constant under operator homotopies.

The invariance of the map under unitary equivalence is a simple check. Finally, because group addition is induced by the direct sum, it follows that \mathfrak{A} is a homomorphism. \square

We combine Lemmas 4.10 and 4.14 to obtain our main result.

Theorem 4.15. *The homomorphisms $\mathfrak{A} : KKR(\mathbb{C}\ell_{1,0}, A) \rightarrow DK(A, \mathfrak{r}_A)$ and $\mathfrak{C} : DK(A, \mathfrak{r}_A) \rightarrow KKR(\mathbb{C}\ell_{1,0}, A)$ are mutually inverse isomorphisms.*

Proof. We do not assume that the unitisation A^{\sim} is balanced graded, so the homomorphism $\mathfrak{C} : DK(A, \mathfrak{r}_A) \rightarrow KKR(\mathbb{C}\ell_{1,0}, A)$ is defined as in (4.3).

We have already shown that $\mathfrak{A} : KKR(\mathbb{C}\ell_{1,0}, A) \rightarrow DK(A)$ and $\mathfrak{C}^A : DK(A) \rightarrow KKR(\mathbb{C}\ell_{1,0}, A)$ are well-defined. We just need to show they are mutual inverses.

We first consider $\mathfrak{C} \circ \mathfrak{A}$. Take an element $[(e, X_A, T)] \in KKR(\mathbb{C}\ell_{1,0}, A)$ with $e^2 = 1_X$ and e

anti-commuting with T . We set $J_X = \overline{\text{span}(X|X)_A}$ and compute using Lemmas 4.12 and 4.13,

$$\begin{aligned}
[(e, X_A, T)] &\xrightarrow{\mathfrak{A}} \iota_* \circ \zeta_X([\mathcal{C}_e(T)] - [e]) \\
&\xrightarrow{\mathfrak{C}} \mathfrak{C}^A \circ \iota_* \circ \zeta_X([\mathcal{C}_e(T)] - [e]) \\
&= \iota_* \circ \mathfrak{C}^{J_X} \circ \zeta_X([\mathcal{C}_e(T)] - [e]) \\
&= \iota_* \circ \zeta_X^{KK} \circ \mathfrak{C}^{\text{End}_A^0(X)}([\mathcal{C}_e(T)] - [e]) \\
&= \iota_* \circ \zeta_X^{KK}[(e, \overline{(T-e)^{-1} \text{End}_A^0(X)_{\text{End}_A^0(X)}}), \mathcal{C}_e^{-1} \circ \mathcal{C}_e(T)] \\
&= \iota_* \circ \zeta_X^{KK}[(e, \text{End}_A^0(X)_{\text{End}_A^0(X)}, T)] \\
&= \iota_*[(e, X_{J_X}, T)] \\
&= [(e, X_A, T)],
\end{aligned}$$

where we have used that $\mathcal{C}_e(T) - e = 2(T - e)^{-1}$, $(T - e)^{-1}$ has dense range, and $\mathcal{C}_e^{-1} \circ \mathcal{C}_e(T) = T$ by Proposition 4.8.

We now consider $\mathfrak{A} \circ \mathfrak{C}$. We do not assume A^\sim is balanced graded and so consider OSUs $U, V \in M_n(A^\sim \hat{\otimes} \mathcal{C}l_{1,1})$ with $U - V \in M_n(A \hat{\otimes} \mathcal{C}l_{1,1})$. Our Cayley map then gives

$$\mathfrak{C}([U] - [V]) = [(V, \overline{(U - V)(A \hat{\otimes} \mathcal{C}^2)_A^n}, \mathcal{C}_V^{-1}(U))].$$

We let $Y_A = \overline{(U - V)(A \hat{\otimes} \mathcal{C}^2)_A^n}$ and recall from Proposition 4.8 that $\mathcal{C}_V^{-1} \circ \mathcal{C}_V(U) = U|_Y$. We let $J_Y = \text{span}(Y|Y)_A$ and use Equation (2.7) and Lemma 4.12 to compute

$$\begin{aligned}
\mathfrak{A} \circ \mathfrak{C}([U] - [V]) &= \iota_* \circ \zeta_Y([U|_{Y_A}]^{\text{End}_A^0(Y)} - [V|_Y]^{\text{End}_A^0(Y)}) \\
&= \iota_*([W_2(\tilde{U}|_{Y_A} \oplus Z \oplus V|_{Y_A} \oplus Z)W_2^*]^{J_Y} - [W_2\hat{Z}W_2^*]^{J_Y}) \\
&= \iota_*([W_2(U|_{Y_A} \oplus Z \oplus V|_{Y_A} \oplus Z)W_2^*]^{J_Y} - [W_2\hat{Z}W_2^*]^{J_Y}) \\
&= [W_2(U \oplus Z \oplus V \oplus Z)W_2^*] - [W_2\hat{Z}W_2^*] \\
&= \left[\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \right] - \left[\begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \right] \in DK(A \hat{\otimes} \mathcal{C}l_{1,1}).
\end{aligned}$$

This completes the proof. If A is balanced graded, then we can apply (the inverse of) the isomorphism (2.2) to recover $[U] - [V] \in DK(A)$ explicitly. \square

For completeness, let us list a few immediate corollaries of our result.

- Corollary 4.16.**
1. Let A be a graded C^* -algebra. Then $KK(\mathcal{C}l_1, A) \cong DK(A)$.
 2. Let B be a real C^* -algebra, $B = A^{\text{tA}}$ for some Real C^* -algebra A . Then $KKO(\mathcal{C}l_{1,0}, B) \cong DK(B)$.
 3. Recall the complex graded K -theory groups $K_j^{\text{gr}}(A) := KK(\mathbb{C}, A \hat{\otimes} \mathcal{C}l_j)$ from [31]. Then $DK(A) \cong K_1^{\text{gr}}(A)$.
 4. Let $KR'(A) = [C_0(\mathbb{R}), A \hat{\otimes} \mathcal{K}]$ denote the group of \mathbb{Z}_2 -graded and Real asymptotic morphisms from [47], where $C_0(\mathbb{R})$ has grading α such that $\alpha(f)(x) = f(-x)$ and real structure by complex conjugation. Then $KR'(A \hat{\otimes} \mathcal{C}l_{0,1}) \cong DK(A)$.

Proof. The first two results come from either ignoring the real structure or passing to a real subalgebra. For the third statement, we use that

$$DK(A) \cong KK(\mathcal{C}\ell_1, A) \cong KK(\mathcal{C}\ell_2, A \hat{\otimes} \mathcal{C}\ell_1) \cong KK(\mathbb{C}, A \hat{\otimes} \mathcal{C}\ell_1) = K_1^{\text{gr}}(A),$$

where we take the external product by $1_{KK(\mathcal{C}\ell_1, \mathcal{C}\ell_1)}$ and use the Morita equivalence between $\mathcal{C}\ell_2$ and \mathbb{C} . Similarly, it is shown in [47, Theorem 4.7] that $KR'(A) \cong KKR(\mathbb{C}, A)$. Hence

$$KR'(A \hat{\otimes} \mathcal{C}\ell_{0,1}) \cong KKR(\mathbb{C}, A \hat{\otimes} \mathcal{C}\ell_{0,1}) \cong KKR(\mathcal{C}\ell_{1,0}, A) \cong DK(A). \quad \square$$

Remark 4.17. Let us briefly consider the map \mathfrak{A} applied to a complex Kasparov module (\mathbb{C}, X_A, T) with A trivially graded. We first inflate this Kasparov module to a class in $KK(\mathcal{C}\ell_1, A \otimes \mathcal{C}\ell_1)$ by taking the external product with the ring identity of $KK(\mathcal{C}\ell_1, \mathcal{C}\ell_1)$. Given the Kasparov module $(e, (X \hat{\otimes} \mathcal{C}\ell_1)_{A \hat{\otimes} \mathcal{C}\ell_1}, T \hat{\otimes} 1)$, we choose the ‘ordered basis’

$$(X \hat{\otimes} \mathcal{C}\ell_1)_{A \hat{\otimes} \mathcal{C}\ell_1} = [(X_+ \otimes \mathcal{C}\ell_{1,+}) \oplus (X_- \otimes \mathcal{C}\ell_{1,-}) \oplus (X_+ \otimes \mathcal{C}\ell_{1,-}) \oplus (X_- \otimes \mathcal{C}\ell_{1,+})]_{A \hat{\otimes} \mathcal{C}\ell_1}$$

and then compute

$$\mathfrak{C}_e(T) - e = 2(T - e)^{-1} = \begin{pmatrix} 0_2 & (\tilde{T} - \sigma_3)^{-1} \\ (\tilde{T} - \sigma_3)^{-1} & 0_2 \end{pmatrix}, \quad \tilde{T} = \begin{pmatrix} 0 & T_- \otimes 1 \\ T_+ \otimes 1 & 0 \end{pmatrix}$$

with $T_{\pm} : X_{\pm} \rightarrow X_{\mp}$. Further expanding and suppressing the tensor product notation

$$(\tilde{T} - \sigma_3)^{-1} = \begin{pmatrix} -(1 + T_- T_+)^{-1} & T_- (1 + T_+ T_-)^{-1} \\ T_+ (1 + T_- T_+)^{-1} & (1 + T_+ T_-)^{-1} \end{pmatrix} = P_{T_-} - P_{X_+}.$$

Hence, as an operator on $(X \hat{\otimes} \mathcal{C}\ell_1)_{A \hat{\otimes} \mathcal{C}\ell_1} \cong (X_{\pm}^{\oplus 2})_{A \hat{\otimes} \mathcal{C}\ell_1}$, $\mathfrak{C}_e(T) - e$ acts as $(P_{T_-} - P_{X_+}) \otimes \sigma_1$. Therefore our general Cayley map \mathfrak{A} is precisely the negative of the graph projection map we employed in Section 3.3. \diamond

5 Applications to real and complex K -theory

In this section we consider some special cases of Theorem 4.15 to study examples and problems coming from real and complex K -theory.

5.1 Unitary descriptions of K -theory

Given a complex and ungraded C^* -algebra A with real structure \mathfrak{r}_A , we know from [22, §5] that there are isomorphisms $KKR(\mathcal{C}\ell_{r,s}, A) \cong KO_{r-s}(A^{\mathfrak{r}_A})$, where this identification is shown via a (generalised) Clifford-module index, see [43, Section 2.2].

Alternatively, descriptions of KO -theory using Real C^* -algebras and unitaries have appeared in [6] and [23, Section 5.6]. In this section, we show in a few cases how these unitary descriptions of KO -theory are compatible with our Cayley isomorphism. We note that many of these descriptions will be of use to us for studying the bulk invariants of topological insulators in Section 6.

Example 5.1 (Trivially graded algebras and KO_1). Let A be trivially graded and (e, X_A, T) an unbounded Real Kasparov module representing an element in $KKR(\mathbb{C}\ell_{1,0}, A)$. As A is trivially graded, without loss of generality we can write $X_A \cong Y_A \oplus Y_A$ with Y_A an ungraded C^* -module. Because T anti-commutes with the generator $\mathbb{C}\ell_{1,0}$ generator, our Kasparov module reduces to the form

$$(e, Y_A \otimes \mathbb{C}^2, T = T_+ \otimes f), \quad e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with $T_+^* = -T_+$. The real structure on \mathbb{C}^2 is pointwise complex conjugation, which ensures that $e^{\mathfrak{r}} = e$, $f^{\mathfrak{r}} = f$ and $T_+^{\mathfrak{r}} = T_+$.

Because T_+ is skew-adjoint $(T_+ \pm 1)$ is invertible and we compute

$$\begin{aligned} \mathcal{C}_e(T) &= e(T + e)(T - e)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -T_+ + 1 \\ T_+ + 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -T_+ - 1 \\ T_+ - 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & (T_+ + 1)(T_+ - 1)^{-1} \\ (T_+ - 1)(T_+ + 1)^{-1} & 0 \end{pmatrix}. \end{aligned}$$

One finds that $U_{T_+} = (T_+ + 1)(T_+ - 1)^{-1}$ is unitary, $(U_{T_+})^{\mathfrak{r}} = U_{T_+}$ and

$$1 - (T_+ + 1)(T_+ - 1)^{-1} = -2(T_+ - 1)^{-1} \in \text{End}_A^0(Y)$$

as we have an unbounded Kasparov module. Thus for $J_Y = \overline{\text{span}(Y|Y)}_A$ we have a class $[U_{T_+}] \in K_1(\text{End}_A^0(Y)) \cong K_1(J_Y)$ where we use an ungraded version of the isomorphism of Equation (2.7) from Section 2.2. If we ignore real structures, then denoting $\iota : J_Y \hookrightarrow A$ and $\zeta_Y : K_1(\text{End}_A^0(Y)) \xrightarrow{\cong} K_1(J_Y)$, we obtain a map

$$KK(\mathbb{C}\ell_1, A) \ni [(\mathbb{C}\ell_1, Y_A \otimes \mathbb{C}^2, T_+ \otimes f)] \mapsto \iota_* \circ \zeta_Y [(T_+ + 1)(T_+ - 1)^{-1}] \in K_1(A)$$

which is a skew-adjoint analogue of (the inverse of) the isomorphism in Theorem 3.5. Similarly, passing to real subalgebras $\iota_* \circ \zeta_{Y^{\mathfrak{r}Y}} [(T_+ + 1)(T_+ - 1)^{-1}] \in KO_1(A^{\mathfrak{r}A})$.

Let us also consider the inverse map. If $U \in A^{\sim}$ is a unitary and $U^{\mathfrak{r}A} = U$, then $(U+1)(U-1)^{-1}$ is an unbounded skew-adjoint operator and

$$\left(\sigma_1, \overline{(U-1)A_A} \otimes \mathbb{C}^2, (U+1)(U-1)^{-1} \otimes f \right) \quad (5.1)$$

is an unbounded Kasparov module, where the real structure on $\overline{(U-1)A_A}$ comes from \mathfrak{r}_A and the real structure on \mathbb{C}^2 is pointwise complex conjugation. A direct check or Theorem 4.15 (combined with the equivalence between van Daele and operator K -theory for ungraded algebras) gives that the map $KKO(\mathbb{C}\ell_{1,0}, A^{\mathfrak{r}A}) \rightarrow KO_1(A^{\mathfrak{r}A})$ or $KK(\mathbb{C}\ell_1, A) \rightarrow K_1(A)$ is an isomorphism with the inverse given by the unbounded Kasparov module in Equation (5.1). \diamond

Example 5.2 (An isomorphism $KKR(\mathbb{C}\ell_{0,1}, A) \rightarrow KO_{-1}(A^{\mathfrak{r}A})$). Here we consider the Cayley map for elements in $KKR(\mathbb{C}\ell_{0,1}, A)$ that reduces to our original ungraded complex Cayley isomorphism from Theorem 3.5 if we ignore the real structure.

Let $(\mathbb{C}\ell_{0,1}, X_A, T)$ be a Real Kasparov module with A trivially graded and $f \in \mathbb{C}\ell_{0,1}$ the generator. Making analogous simplifications as Example 5.1, we write the Kasparov module as

$$\left(\mathbb{C}\ell_{0,1}, (Y \oplus Y)_A, T = \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} \right), \quad f \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (y_1, y_2)^{\mathfrak{r}Y} = (y_1^{\mathfrak{r}Y}, y_2^{\mathfrak{r}Y}),$$

which also implies that $S = S^*$ and $S^r = S$ on the ungraded C^* -module Y_A . As S is self-adjoint, unbounded and has compact resolvent, we can apply the ungraded Cayley transform

$$U_S = \mathcal{C}(S) = (S + i)(S - i)^{-1},$$

which by Proposition 3.1 is unitary and $U_S - 1 \in \text{End}_A^0(Y)$. Applying the real structure on $\text{End}_A(Y)$, $U_S^r = U_S^*$. Hence, we obtain a map from cycles in $KKR(\mathcal{C}\ell_{0,1}, A)$ to unitaries $u \in \text{End}_A^0(Y)^\sim$ such that $u^r = u^*$.

The group $KO_{-1}(A^{rA})$ can be characterised by equivalence classes of complex unitaries in $M_n(A^\sim)$ such that $u^r = u^*$ [23, Section 5.6]. We also compare our presentation of KO_{-1} to that of Boersema and Loring, who characterise $KO_{-1}(A, \rho)$ as equivalence classes of unitaries $u \in M_n(A^\sim)$ such that $u^\rho = u$ for ρ an *anti-multiplicative* involution [6]. We can recover this picture by defining $\rho = * \circ r$, so that $u^r = u^*$ implies that $u^\rho = u$. Hence, our Cayley map determines a class $\iota_* \circ \zeta_Y[U_S] \in KO_{-1}(A^r)$.

Now, suppose that A is a Real C^* -algebra and the real structure in $M_n(A)$ is applied entrywise. Given $u \in M_n(A^\sim)$ unitary and such that $u^{rA} = u^*$, by Proposition 3.1 there is a well-defined self-adjoint operator $\mathcal{C}^{-1}(u)$,

$$\text{Dom}(\mathcal{C}^{-1}(u)) = (u - 1)A^n, \quad \mathcal{C}^{-1}(u)v = i(u + 1)(u - 1)^{-1}v, \quad v \in \text{Dom}(\mathcal{C}^{-1}(u)).$$

Using the obvious real structure on the C^* -module $\overline{(u - 1)A_A^n}$, we check that

$$\mathcal{C}^{-1}(u)^r = -i(u^* + 1)(u^* - 1)^{-1} = -i(u^* + 1)u((u^* - 1)u)^{-1} = i(u + 1)(u - 1)^{-1} = \mathcal{C}^{-1}(u)$$

and so the argument in Proposition 3.4 extends to give that

$$\left(\mathcal{C}\ell_{0,1}, \overline{(u - 1)A_A^n} \otimes \mathbb{C}^2, \begin{pmatrix} 0 & \mathcal{C}^{-1}(u) \\ \mathcal{C}^{-1}(u) & 0 \end{pmatrix} \right), \quad f \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

is an unbounded KKR -cycle with real structure on \mathbb{C}^2 by complex conjugation. Following the proof of Theorem 3.5, we obtain that the maps

$$\begin{aligned} KO_{-1}(A^{rA}) \ni [u] &\mapsto \left[(\mathcal{C}\ell_{0,1}, \overline{(u - 1)A_A^n} \otimes \mathbb{C}^2, \mathcal{C}^{-1}(u) \otimes \sigma_1) \right] \in KKR(\mathcal{C}\ell_{0,1}, A) \\ KKR(\mathcal{C}\ell_{0,1}, A) \ni [(\mathcal{C}\ell_{0,1}, Y_A \otimes \mathbb{C}^2, S \otimes \sigma_1)] &\mapsto \iota_* \circ \zeta_Y[\mathcal{C}(S)] \in KO_{-1}(A^{rA}) \end{aligned}$$

are well-defined and mutual inverses. The main difference is that the identity element in $KO_{-1}(A^r)$ is given by the class of i times the unit of A , $[i1_A]$ and we need to ensure that any homotopy of unitaries respects the condition $v_t^r = v_t^*$.

Clearly if we ignore the real structure, then we recover our original ungraded Cayley map $K_1(A) \rightarrow KK^1(\mathbb{C}, A)$ from Theorem 3.5. \diamond

Example 5.3 (Unitary and projective descriptions of K_0). We consider $A \otimes \mathcal{C}\ell_{1,0}$ with A ungraded and Real. In this case, any odd self-adjoint unitary is of the form $x \otimes e$ with e the generator of $\mathcal{C}\ell_{1,0}$ and $x = x^* = x^{rA}$ and unitary. Suppose that we have two self-adjoint Real unitaries $x \in M_n(A^\sim)$, $y \in M_m(A^\sim)$ with $x - y \in M_N(A)$. Then we obtain an element $[x \otimes e] - [y \otimes e] \in DK(A \otimes \mathcal{C}\ell_{1,0})$.

Applying our Cayley map, we first note that, because $x - y$ is compact (over A),

$$\overline{(x \otimes e - y \otimes e)(A \otimes \mathcal{C}\ell_{1,0})^N}_{A \otimes \mathcal{C}\ell_{1,0}} \cong \frac{1}{2} \overline{(x \otimes 1 - y \otimes 1)(A \otimes \mathcal{C}\ell_{1,0})^N}_{A \otimes \mathcal{C}\ell_{1,0}}$$

Furthermore, on its domain, the Cayley transform $\mathcal{C}_{y \otimes e}^{-1}(x \otimes e)$ acts as the zero-map and so our map $DK(A \otimes \mathcal{C}l_{1,0}) \rightarrow KKR(\mathcal{C}l_{1,0}, A \otimes \mathcal{C}l_{1,0})$ reduces to

$$\begin{aligned} [x \otimes e] - [y \otimes e] &\mapsto [(\mathcal{C}l_1, \frac{1}{2}(x \otimes 1 - y \otimes 1)(A \otimes \mathcal{C}l_1)_{A \otimes \mathcal{C}l_1}^N, 0)] \\ &= [(\mathbb{C}, \frac{1}{2}(1-x)A_A^n \oplus \frac{1}{2}(1-y)A_A^m, 0)] \hat{\otimes}_{\mathbb{C}} [(\mathcal{C}l_1, \mathcal{C}l_{1\mathcal{C}l_1}, 0)] \\ &= [(\mathbb{C}, \frac{1}{2}(1-x)A_A^n \oplus \frac{1}{2}(1-y)A_A^m, 0)] \hat{\otimes}_{\mathbb{C}} 1_{KKR(\mathcal{C}l_{1,0}, \mathcal{C}l_{1,0})}. \end{aligned}$$

Hence, given projections $p, q \in M_N(A)$, with $p^{\mathfrak{r}A} = p$ and $q^{\mathfrak{r}A} = q$, we recover the usual map $KO_0(A^{\mathfrak{r}A}) \rightarrow KKR(\mathbb{C}, A)$ via the self-adjoint unitaries $x = 1 - 2p$, $y = 1 - 2q$ and our van Daele map. If we ignore the real structure, then our Cayley map recovers the isomorphism $K_0(A) \rightarrow KK(\mathbb{C}, A)$ from Section 3.3. \diamond

Example 5.4 ($KKR(\mathcal{C}l_{1,0}, M_2(A) \otimes \mathcal{C}l_{0,1}) \rightarrow KO_2(A^{\mathfrak{r}A})$). Suppose that we have the algebra $M_2(A) \otimes \mathcal{C}l_{0,1}$ with A unital, trivially graded and the real structure on $M_2(A)$ given entrywise by \mathfrak{r}_A . Any C^* -module $Y_{M_2(A)}$ can be decomposed into an *ungraded* sum $(X \oplus X)_{M_2(A)}$. We use the presentation $\mathcal{C}l_{0,1} \cong \mathbb{C} \oplus \mathbb{C}$ with grading by the flip automorphism and real structure $(\alpha, \beta)^{\mathfrak{r}0,1} = (\bar{\beta}, \bar{\alpha})$. Then if we take a class in $KKR(\mathcal{C}l_{1,0}, M_2(A) \otimes \mathcal{C}l_{0,1})$, we can write

$$\left(\mathcal{C}l_{1,0}, ((X \oplus X) \otimes \mathcal{C}l_{0,1})_{M_2(A) \otimes \mathcal{C}l_{0,1}}, T \right), \quad \mathcal{C}l_{1,0} = C^*(e), \quad e = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes (1, -1)$$

where using the real structure $((x_1, x_2) \otimes (\alpha, \beta))^{\mathfrak{r}} = (x_1^{\mathfrak{r}X}, x_2^{\mathfrak{r}X}) \otimes (\bar{\beta}, \bar{\alpha})$ we see that $e^{\mathfrak{r}} = (-1)^2 e = e$. Similarly the right-action of $\mathcal{C}l_{0,1}$ is given by multiplication by $1_2 \otimes (i, -i)$. The decomposition of the Kasparov module means that we can write T in the form $T = S \otimes (1, -1)$, where S is a self-adjoint unbounded operator on $X \oplus X$, $S\sigma_2 + \sigma_2 S = 0$ and $S^{\mathfrak{r}} = S$. We then compute that

$$\mathcal{C}_e(T) = e(T + e)(T - e)^{-1} = \sigma_2(S + \sigma_2)(S - \sigma_2)^{-1} \otimes (1, -1).$$

Letting $U_S = \sigma_2(S + \sigma_2)(S - \sigma_2)^{-1} \in \text{End}_{M_2(A)}^0(X \oplus X)^{\sim}$, we see that $U_S^{\mathfrak{r}} = -U_S$, $U_S^* = U_S$ and

$$\begin{aligned} U_S^2 &= \sigma_2(S + \sigma_2)(S - \sigma_2)^{-1} \sigma_2(S + \sigma_2)(S - \sigma_2)^{-1} \\ &= \sigma_2(S + \sigma_2)(S - \sigma_2)^{-1} (-S + \sigma_2) \sigma_2(S - \sigma_2)^{-1} \\ &= -\sigma_2(S + \sigma_2) \sigma_2(S - \sigma_2)^{-1} = -\sigma_2^2(-S + \sigma_2)(S - \sigma_2)^{-1} = 1. \end{aligned}$$

Hence, U_S is an (ungraded) self-adjoint and imaginary unitary.

Summarising our discussion, given a class in $KKR(\mathcal{C}l_{1,0}, M_2(A) \otimes \mathcal{C}l_{0,1})$, we can construct a unitary operator $V \in \text{End}_{M_2(A)}^0(X \oplus X)^{\sim}$ such that $V^* = V$ and $V^{\mathfrak{r}} = -V$. Applying the (ungraded) Morita invariance from Equation (2.7), we recover the unitary description of $KO_2(A^{\mathfrak{r}A})$ as homotopy classes of self-adjoint unitaries with $u^{\mathfrak{r}} = -u$ given in [6, 23]. Such self-adjoint and imaginary unitaries can be abstractly characterised as spectrally flattened Hamiltonians with a particle-hole symmetry. We will return to this point in Section 6. \diamond

Example 5.5 (KO_3 and KKR). Using the Künneth formula for real K -theory [5], we can express $KO_3(A^{\mathfrak{r}A}) \cong KO_{-1}(A^{\mathfrak{r}A} \otimes \mathbb{H})$, where \mathbb{H} is considered as a real ungraded C^* -algebra. In particular, we use the presentation $\mathbb{H} \cong M_2(\mathbb{C})^{\text{Ad-}i\sigma_2 \circ \mathfrak{c}}$ with \mathfrak{c} complex conjugation. We again note that this is an ungraded isomorphism (putting in a grading, the right hand side of the isomorphism becomes $Cl_{0,2}$).

To note this equivalence concretely, we use the description of KO_3 from [23, Section 5.6], which characterises $KO_3(A^{\text{r}A})$ as equivalence classes of unitaries $u \in M_n(A^\sim)$ such that $u^{\text{r}A} = -u^*$. Given such a $u \in A$ we consider the matrix $v = u \otimes \sigma_1 \in A \otimes M_2(\mathbb{C})$, where one can check that $v^{\text{Ad}_{-i\sigma_2} \circ \text{r}A} = v^*$ and as such we get a class $[u \otimes \sigma_1] \in KO_{-1}(M_2(A)^{\text{Ad}_{-i\sigma_2} \circ \text{r}A})$. Hence we can apply the map from Example 5.2 to get a Real Kasparov module

$$\left(\mathbb{C}l_{0,1} \overline{((u \otimes \sigma_1) - 1_2)M_2(A)}_{A \otimes M_2(\mathbb{C})} \otimes \mathbb{C}^2, \mathbb{C}^{-1}(u \otimes \sigma_1) \otimes \sigma_1 \right)$$

with $(a_1, a_2)^{\text{r}} = (a_1^{\text{Ad}_{-i\sigma_2} \circ \text{r}A}, a_2^{\text{Ad}_{-i\sigma_2} \circ \text{r}A})$ and the left $\mathbb{C}l_{0,1}$ -action generated by $1 \otimes (-i\sigma_2)$. Passing to real subalgebras and applying the Künneth formula, we obtain an element in $KKO(Cl_{0,1}, A^{\text{r}A} \otimes M_2(\mathbb{C})^{\text{Ad}_{-i\sigma_2} \circ \text{r}}) \cong KKO(Cl_{0,1}, A^{\text{r}A} \otimes Cl_{0,4})$. \diamond

5.2 Short exact sequences and boundary maps

Here we consider the compatibility of our Cayley isomorphism with the boundary map of van Daele K -theory and KK -theory. Suppose that

$$0 \rightarrow I \rightarrow E \xrightarrow{q} A \rightarrow 0 \quad (5.2)$$

is a short exact sequence of graded C^* -algebras with a completely positive linear splitting. If the algebras possess a real structure, then we also assume that these maps are equivariant with respect to this structure. By [45, Theorem 1.1] there are connecting homomorphisms $KKR(I, B) \xrightarrow{\delta} KKR(A, B \hat{\otimes} \mathbb{C}l_{1,0})$ and $KKR(B, A) \xrightarrow{\delta} KKR(B, I \hat{\otimes} \mathbb{C}l_{1,0})$. We will consider a special case of the latter of these boundary maps using van Daele K -theory and our Cayley isomorphism. We note that boundary maps in van Daele K -theory and their compatibility with Kasparov theory has already been extensively studied by Kubota [28, Section 5]. In particular, the boundary maps in van Daele K -theory inherit many properties from KK -theory such as naturality.

If the quotient algebra A is unital, we assume that it is balanced (it contains an OSU). If A is non-unital, we assume that $\text{Mult}(A)$ contains an OSU e and use the description of van Daele that includes a base point $DK_e(A) \cong DK(A^{\sim e}, A^{\sim e}/A) \cong DK(A)$ from Lemma 2.5, where

$$DK_e(A) = \{[x] - [y] \in GV_e(A^{\sim e}) : x - (e_k \oplus -e_{n-k}), y - (e_k \oplus -e_{n-k}) \in M_n(A), \text{ some } n, k\}$$

and $A^{\sim e} \subset \text{Mult}(A)$ the algebra generated by A and e . Let us recall the formula for the boundary map in van Daele K -theory.

Lemma 5.6 ([13], Proposition 3.4). *The boundary map of the short exact sequence (5.2), $\delta : DK(A) \rightarrow DK(I \hat{\otimes} \mathbb{C}l_{1,0})$, is given by*

$$\delta([x_1] - [x_2]) = [Y_1] - [Y_2], \quad Y_i = -\exp(\pi \tilde{x}_i \hat{\otimes} \rho)(1 \hat{\otimes} \rho), \quad (5.3)$$

where $\tilde{x}_i \in E$ is an odd self-adjoint lift of x_i and ρ is the odd generator of $\mathbb{C}l_{1,0}$. We may assume that $\|\tilde{x}_i\| = 1$.

If e is a choice of base point in A which lifts to an OSU in E (we may simply take the image of a base point in E) then one easily finds $\delta([x] - [e]) = [Y] - [1 \hat{\otimes} \rho]$ and so we may simplify the formulas (as does van Daele) by writing $\delta([x]) = [Y]$. We also remark that Equation (5.3)

is simpler than the formula given in [13, Proposition 3.4]. This is because we have chosen $1 \hat{\otimes} \rho$ as a constant basepoint and noting that

$$-(\sin(\pi \tilde{x} \hat{\otimes} 1) + (1 \hat{\otimes} \rho) \cos(\pi \tilde{x} \hat{\otimes} 1)) = -\exp(\pi \tilde{x} \hat{\otimes} \rho)(1 \hat{\otimes} \rho),$$

which can be shown by the Taylor series expansion.

Proposition 5.7. *Let $x \in M_n(A^{\sim e})$ be an OSU. Under the isomorphism of Theorem 4.15 and the identification $KKR(\mathbb{C}l_{1,0}, I \hat{\otimes} \mathbb{C}l_{1,0}) \cong KKR(\mathbb{C}, I)$, the class $\delta([x]) \in DK(I \hat{\otimes} \mathbb{C}l_{1,0})$ can be identified with the class of the unbounded Kasparov module*

$$\left(\mathbb{C}, \overline{\cos(\frac{\pi}{2} \tilde{x}) I_I^n}, \tan(\frac{\pi}{2} \tilde{x}) \right),$$

with $\tilde{x} \in M_n(E)$ an odd self-adjoint lift of x .

Proof. We use Lemma 5.6 and calculate the inverse Cayley transform, where

$$\begin{aligned} \mathcal{C}_{1 \hat{\otimes} \rho}^{-1}(Y) &= (1 \hat{\otimes} \rho) \left(-\exp(\pi \tilde{x} \hat{\otimes} \rho)(1 \hat{\otimes} \rho) + (1 \hat{\otimes} \rho) \right) \left(-\exp(\pi \tilde{x} \hat{\otimes} \rho)(1 \hat{\otimes} \rho) - (1 \hat{\otimes} \rho) \right)^{-1} \\ &= (1 \hat{\otimes} \rho) \left(-\exp(\pi \tilde{x} \hat{\otimes} \rho) + 1 \right) \left(-\exp(\pi \tilde{x} \hat{\otimes} \rho) - 1 \right)^{-1} \\ &= -(1 \hat{\otimes} \rho) \tanh(\frac{\pi}{2} \tilde{x} \hat{\otimes} \rho) \end{aligned}$$

with domain $\sinh(\frac{\pi}{2} \tilde{x} \hat{\otimes} \rho)(I^n \hat{\otimes} \mathbb{C}l_{1,0})$. Using that \tilde{x} and ρ are odd, $(\tilde{x} \hat{\otimes} \rho)^{2j+1} = (-1)^j \tilde{x}^{2j+1} \hat{\otimes} \rho$. Therefore,

$$-(1 \hat{\otimes} \rho)(-1)^j (\tilde{x} \hat{\otimes} \rho)^{2j+1} = -(1 \hat{\otimes} \rho)(-1)^j (-1)^j (\tilde{x}^{2j+1} \hat{\otimes} \rho) = \tilde{x}^{2j+1} \hat{\otimes} 1$$

and so by the Taylor series expansion, $-(1 \hat{\otimes} \rho) \tanh(\frac{\pi}{2} \tilde{x} \hat{\otimes} \rho) = \tan(\frac{\pi}{2} \tilde{x}) \hat{\otimes} 1$ on the domain $\cos(\frac{\pi}{2} \tilde{x}) I_I^n \hat{\otimes} \mathbb{C}l_1$. Hence our Kasparov module can be factorised

$$(\mathbb{C}l_{1,0}, \overline{\cos(\frac{\pi}{2} \tilde{x}) I_I^n} \hat{\otimes} \mathbb{C}l_{1,0 \mathbb{C}l_{1,0}}, \tan(\frac{\pi}{2} \tilde{x}) \hat{\otimes} 1) = (\mathbb{C}, \overline{\cos(\frac{\pi}{2} \tilde{x}) I_I^n}, \tan(\frac{\pi}{2} \tilde{x})) \hat{\otimes}_{\mathbb{C}} (\mathbb{C}l_{1,0}, \mathbb{C}l_{1,0 \mathbb{C}l_{1,0}}, 0),$$

and removing the element $1_{KKR(\mathbb{C}l_{1,0}, \mathbb{C}l_{1,0})}$ gives the identification $KKR(\mathbb{C}l_{1,0}, I \hat{\otimes} \mathbb{C}l_{1,0}) \cong KKR(\mathbb{C}, I)$. \square

Corollary 5.8. *Let $(\mathbb{C}l_{1,0}, X_A, T)$ be an (unbounded) Kasparov module such that the operator T anti-commutes with the left Clifford generator e . The image of this Kasparov module under the composition $KKR(\mathbb{C}l_{1,0}, A) \xrightarrow{\delta_{KK}} KKR(\mathbb{C}l_{1,0}, I \hat{\otimes} \mathbb{C}l_{1,0}) \xrightarrow{\cong} KKR(\mathbb{C}, I)$ can be represented by the Kasparov module*

$$\left(\mathbb{C}, \overline{\cos(\frac{\pi}{2} \tilde{\mathcal{C}}_e(T)) I_I}, \tan(\frac{\pi}{2} \tilde{\mathcal{C}}_e(T)) \right)$$

with $\tilde{\mathcal{C}}_e(T) \in E$ a lift of $\mathcal{C}_e(T)$.

Proof. By [28, Proposition 5.13], the diagram

$$\begin{array}{ccc} KKR(\mathbb{C}l_{1,0}, A) & \xrightarrow{\delta_{KK}} & KKR(\mathbb{C}l_{1,0}, I \hat{\otimes} \mathbb{C}l_{1,0}) \\ \simeq \downarrow & & \uparrow \simeq \\ DK(A) & \xrightarrow{\delta_{DK}} & DK(I \hat{\otimes} \mathbb{C}l_{1,0}) \end{array}$$

is commutative. The result then immediately follows from Proposition 5.7. \square

A representative of the boundary map considered in Corollary 5.8 is guaranteed to exist by [45, Theorem 1.1]. The advantage of the corollary is that it gives an explicit representative that is constructed from the given unbounded Kasparov module. To finish this section, we also give a simple representative of the boundary map in van Daele as a bounded Kasparov module.

Proposition 5.9. *Let $x \in M_n(A^{\sim e})$ be an OSU. The class $\delta([x]) \in DK(I \hat{\otimes} \mathbb{C} \ell_{1,0}) \cong KKR(\mathbb{C}, I)$ can be identified with the element*

$$[(\mathbb{C}, I_I^n, \tilde{x})] \in KKR(\mathbb{C}, I)$$

with $\tilde{x} \in M_n(E)$ an odd self-adjoint lift of x .

Proof. Applying the bounded transform to the Kasparov module from Proposition 5.7, we get the bounded operator $\sin(\frac{\pi}{2}\tilde{x}) \in \text{End}_I(I^n)$. We can then take a straight-line operator homotopy from $\sin(\frac{\pi}{2}\tilde{x})$ to \tilde{x} . \square

Proposition 5.9 implies that the non-triviality of the class $\delta([x])$ as an element of $KKR(\mathbb{C}, I)$ is entirely contained in the failure of the lift \tilde{x} to be invertible. For the case of x related to a bulk Hamiltonian, such a condition can be linked to the presence of topological boundary spectrum.

6 Applications to topological phases

Van Daele K -theory has recently been employed by the second author and others to provide a classification of topological phases of materials with respect to an algebra of observables A [23, 24, 1]. We now use our Cayley isomorphism to consider the corresponding class in KK -theory.

One reason for representing our bulk invariant as a Kasparov module is that we are then free to apply the full machinery of Kasparov theory to conduct further study on the invariants of interest. For example, if the algebra A is a crossed product or groupoid algebra typically studied in the C^* -algebraic approach to condensed matter theory [3], then we immediately obtain a bulk-boundary correspondence for pairings of our bulk invariant with a ‘Dirac element’ that extracts the strong numerical phase of the system [7, 8, 9, 25, 26, 29, 39].

6.1 Bulk invariants for topological insulators

For simplicity we will assume that A is unital, which is roughly equivalent to working under a tight-binding approximation. We first briefly review some physical terms.

Definition 6.1 (Abstract insulators and symmetries). We say a self-adjoint element $h \in A$ is an insulator if h has a spectral gap. Taking a constant shift if necessary, we assume that an insulator h is such that $0 \notin \sigma(h)$.

We say that an insulator h has a chiral symmetry if A is graded and h is an odd element under this grading.

Let A be a C^* -algebra with real structure τ_A .

1. An insulator h has a time-reversal symmetry (TRS) if $h^{\tau_A} = h$.

2. An insulator h has a particle-hole symmetry (PHS) if $h^{\tau A} = -h$.

Because we take insulators h to be self-adjoint invertible operators, the spectrally flattened operator $\bar{h} := h|h|^{-1}$ is a self-adjoint unitary. Therefore, if h has a chiral symmetry, then \bar{h} gives an element in $V(A)$. Provided we have another odd self-adjoint unitary for comparison, we obtain an element in $DK(A)$. We call this van Daele element the *bulk invariant* of the topological phase. If there is no chiral symmetry we take the tensor product $A \otimes \mathbb{C}\ell_1$ and consider $\bar{h} \otimes \rho$ instead, which is an OSU.

6.1.1 With chiral symmetry

We consider a chiral symmetry which is inner in the sense that the grading of A is given by Ad_Γ for some $\Gamma = \Gamma^* \in A$.

Example 6.2 (Chiral symmetry, no real structure). If we do not make any reference to real structures on the graded algebra A , then taking the projection $\Pi_+ = \frac{1}{2}(1 + \Gamma)$, A is isomorphic to $A_{++} \otimes \mathbb{C}\ell_2$ with $A_{++} = \Pi_+ A \Pi_+$ a trivially graded algebra [23, Proposition 3.5]. This isomorphism depends on a choice of OSU $e \in A$. Using this isomorphism, the operator of interest is $u_h = \Pi_+ e \bar{h} \Pi_+$, which is unitary and gives a class $[u_h] \in K_1(A_{++})$. Hence we can apply our complex ungraded Cayley map (Theorem 3.5) to obtain the KK^1 -class of the bulk invariant,

$$[(\mathbb{C}\ell_1, \overline{(u_h - 1)A_{++}A_{++}} \otimes \mathbb{C}^2, \mathcal{C}^{-1}(u_h) \otimes \sigma_1)], \quad u_h = \Pi_+ e \bar{h} \Pi_+, \quad \mathcal{C}^{-1}(u_h) = i(u_h + 1)(u_h - 1)^{-1}. \quad \diamond$$

The above expressions are for insulators with complex symmetries. We now consider symmetries involving a real structure like TRS or PHS.

Example 6.3 (Real grading). If we have an inner chiral symmetry with real grading operator $\Gamma = \Gamma^{\tau A}$ then $A_{++}^{\tau A} = A_{++}$, $A^{\tau A} \cong A_{++}^{\tau A} \otimes Cl_{1,1}$. If $(e\bar{h})^{\tau A} = e\bar{h}$, then $u_h^{\tau A} = u_h$ is a unitary in $A_{++}^{\tau A}$ and we are in the same situation as Example 5.1. Hence the Kasparov module of interest is

$$\left[(\mathbb{C}\ell_{1,0}, \overline{(u_h - 1)A_{++}A_{++}} \otimes \mathbb{C}^2, (u_h + 1)(u_h - 1)^{-1} \otimes f) \right] \in KKR(\mathbb{C}\ell_{1,0}, A_{++}) \cong KO_1(A_{++}^{\tau A}).$$

If $(e\bar{h})^{\tau A} = -e\bar{h}$, then iu_h is unitary and $(iu_h)^{\tau A} = iu_h \in A_{++}^{\tau A}$. Our Cayley map then gives the Kasparov module

$$(\mathbb{C}\ell_{1,0}, \overline{(u_h + i)A_{++}A_{++}} \otimes \mathbb{C}^2, (u_h - i)(u_h + i)^{-1} \otimes f). \quad \diamond$$

Example 6.4 (Imaginary grading). If the grading operator Γ is imaginary, $\Gamma^{\tau A} = -\Gamma$, then for $a_{++} \in A_{++}$, $a_{++}^{\tau A} \in A_{--} = \Pi_- A \Pi_-$ with $\Pi_- = \frac{1}{2}(1 - \Gamma)$. In this situation the real subalgebra $A^{\tau A} \cong A_{++}^{\text{Ad}_{e\text{or}A}} \otimes Cl_{2,0}$ [23, Theorem 3.10]. If $(e\bar{h})^{\tau A} = e\bar{h}$, we check that

$$(\Pi_+ e \bar{h} \Pi_+)^{\text{Ad}_{e\text{or}A}} = e \Pi_- e \bar{h} \Pi_- e = \Pi_+ \bar{h} e \Pi_+ = (\Pi_+ e \bar{h} \Pi_+)^*.$$

That is, $u_h^{\text{Ad}_{e\text{or}A}} = u_h^*$ and we are in the case of Example 5.2. Therefore, for $\mathcal{C}^{-1}(u_h) = i(u_h + 1)(u_h - 1)$,

$$\left(\mathbb{C}\ell_{0,1}, \overline{(u_h - 1)A_{++}A_{++}} \otimes \mathbb{C}^2, \begin{pmatrix} 0 & \mathcal{C}^{-1}(u_h) \\ \mathcal{C}^{-1}(u_h) & 0 \end{pmatrix} \right), \quad (v_1, v_2)^\tau = (v_1^{\text{Ad}_{e\text{or}A}}, v_2^{\text{Ad}_{e\text{or}A}})$$

is the Real Kasparov module of interest and determines a class in $KKR(\mathbb{C}\ell_{0,1}, A_{++}) \cong KO_{-1}(A_{++}^{\text{Ad}_e \circ \tau_A})$.

Suppose now that $(e\bar{h})^{\tau_A} = -e\bar{h}$. Then $u_h^{\text{Ad}_e \circ \tau_A} = -u_h^*$ and we are in the setting of Example 5.5. Hence we can consider the ungraded unitary

$$v_h = \begin{pmatrix} 0 & u_h \\ u_h & 0 \end{pmatrix} \in M_2(A), \quad v_h^{\text{Ad}_{-i\sigma_2} \circ \text{Ad}_e \circ \tau_A} = v_h^*,$$

so $v_h \in KO_{-1}(M_2(A_{++})^{\text{Ad}_{-i\sigma_2} \circ \text{Ad}_e \circ \tau_A}) \cong KO_{-1}(A_{++}^{\text{Ad}_e \circ \tau_A} \otimes \mathbb{H}) \cong KO_3(A_{++}^{\text{Ad}_e \circ \tau_A})$. Applying the Cayley transformation,

$$\left(\mathbb{C}\ell_{0,1}, \overline{(v_h - 1_2)M_2(A)}_{A \otimes M_2(\mathbb{C})} \hat{\otimes} \mathbb{C}^2, \mathcal{C}^{-1}(v_h) \otimes \sigma_1 \right)$$

is an unbounded KKR -cycle with real structure $\text{Ad}_{-i\sigma_2} \circ \text{Ad}_e \circ \tau_A$ applied pointwise on the direct sum. The unbounded cycle represents an element in $KKO(Cl_{0,1}, A_{++}^{\text{Ad}_e \circ \tau_A} \otimes Cl_{0,4})$. \diamond

6.1.2 Without chiral symmetry

If there is no chiral symmetry, then the relevant algebra is $A \otimes \mathbb{C}\ell_1$ (potentially with a real structure), where we have the odd self-adjoint unitary $\bar{h} \otimes \rho$.

Example 6.5 (No symmetry or TRS only). We consider the two OSUs $\bar{h} \otimes \rho$ and $1 \otimes \rho$ (where $1 \otimes \rho$ plays the role of a base point) and the element $[\bar{h} \otimes \rho] - [1 \otimes \rho] \in DK(A \otimes \mathbb{C}\ell_1)$ encodes the obstruction of a homotopy of \bar{h} to a trivial Hamiltonian. We are in the setting of Example 5.3, where there is a map

$$[\bar{h} \otimes \rho] - [1 \otimes \rho] \mapsto [(\mathbb{C}, \frac{1}{2}(\bar{h} - 1)A_A, 0)] \hat{\otimes}_{\mathbb{C}} 1_{KK(\mathbb{C}\ell_1, \mathbb{C}\ell_1)}.$$

More simply still, we recover the class of the Fermi projection $[\frac{1}{2}(\bar{h} - 1)] = [\chi_{(-\infty, 0]}(h)] \in K_0(A)$.

If in addition A has a real structure with $h^{\tau_A} = h$, then the same argument applies and we get the KKR -class $[(\mathbb{C}, \frac{1}{2}(\bar{h} - 1)A_A, 0)]$ or the class of the projection $[\frac{1}{2}(\bar{h} - 1)] \in KO_0(A^{\tau_A})$. Note that in many cases of interest, $A^{\tau_A} \cong A^{\tilde{\tau}_A} \otimes \mathbb{H}$ for some other real structure $\tilde{\tau}_A$. In such a situation, by the Künneth formula $[\frac{1}{2}(\bar{h} - 1)] \in KO_0(A^{\tilde{\tau}_A} \otimes \mathbb{H}) \cong KO_4(A^{\tilde{\tau}_A})$. \diamond

Example 6.6 (Particle-hole symmetric Hamiltonians). Let A be a trivially graded and complex C^* -algebra and suppose there is a real structure τ_A on A with $\bar{h}^{\tau_A} = -\bar{h}$ for some insulator $h \in A$. That is, h has a particle-hole symmetry.

The algebra A is ungraded so we consider the element $\bar{h} \otimes (1, -1) \in \mathbb{C}\ell_1$, where we use the real structure on $\mathbb{C}\ell_1 \cong \mathbb{C} \oplus \mathbb{C}$ given by $(\alpha, \beta)^{\tau_{0,1}} = (\bar{\beta}, \bar{\alpha})$. One then checks that $\bar{h} \otimes (1, -1)$ is self-adjoint, square one and

$$(\bar{h} \otimes (1, -1))^{\tau_A \otimes \tau_{0,1}} = \bar{h}^{\tau_A} \otimes (1, -1)^{\tau_{0,1}} = -\bar{h} \otimes (-1, 1) = \bar{h} \otimes (1, -1).$$

Let us consider the bulk phase of the odd self-adjoint unitary $\bar{h} \otimes (1, -1)$ relative to a fixed base point. Namely, suppose that A has a real skew-adjoint unitary J , $J^* = -J$, $J^2 = -1$ and $J^{\tau_A} = J$. Then $iJ \otimes (1, -1) \in A \otimes \mathbb{C}\ell_1$ is an odd self-adjoint unitary invariant under $\tau_A \otimes \tau_{0,1}$. We therefore obtain a class

$$[\bar{h} \otimes (1, -1)] - [iJ \otimes (1, -1)] \in DK(A \otimes \mathbb{C}\ell_1, \tau_A \otimes \tau_{0,1}).$$

Applying our Cayley map, we note that $\mathbb{C}_{iJ \otimes (1, -1)}^{-1}(\overline{h}) = iJ(\overline{h} + iJ)(\overline{h} - iJ)^{-1} \otimes (1, -1)$ and so our Kasparov module is

$$\left(\mathbb{C}l_{1,0}, \overline{(\overline{h} - iJ) \otimes (1, -1)}(A \otimes \mathbb{C}l_1)_{A \otimes \mathbb{C}l_{0,1}}, iJ(\overline{h} + iJ)(\overline{h} - iJ)^{-1} \otimes (1, -1) \right),$$

where the left Clifford action is multiplication by $iJ \otimes (1, -1)$. The class of this Kasparov module gives an element in $KKR(\mathbb{C}l_{1,0}, A \otimes \mathbb{C}l_{0,1}) \cong KO_2(A^{\text{rA}})$. \diamond

6.2 Boundary invariants for topological insulators

We now consider the boundary map of the bulk invariant from a (Real, graded) short exact sequence with positive linear splitting

$$0 \rightarrow I \rightarrow E \rightarrow A \rightarrow 0$$

and the corresponding image in DK -theory and KK -theory. Our work complements recent descriptions of the boundary K -theory class of topological phases via the Cayley transform by Schulz-Baldes and Toniolo [44]. Similarly, Alldridge, Max and Zirnbauer use the van Daele boundary map and Roe's isomorphism $DK(I \hat{\otimes} \mathbb{C}l_{1,0}) \rightarrow KKR(\mathbb{C}, I)$ to write down a bounded representative of the boundary invariant [1]. Our work has different motivations and constructions to [1] though there are clear similarities.

Let A be unital and $h \in A$ an insulator with spectral gap Δ at 0; we may suppose that $\Delta = (-\delta, \delta)$. We set $t_\Delta = \frac{2\pi}{|\Delta|} = \frac{\pi}{\delta}$, a characteristic time. Let \tilde{h} be a lift of h in E . Then

$$\tilde{a} = \frac{\tilde{h}}{\delta} P_\Delta(\tilde{h}) + P_{\geq \delta}(\tilde{h}) - P_{\leq -\delta}(\tilde{h}) \quad (6.1)$$

is a lift of the spectrally flattened \overline{h} .

As we will show, our explicit lift \tilde{a} combined with our general results about the boundary map from Section 5.2 will allow us to write down the boundary invariants of topological insulators explicitly in terms of the lift \tilde{h} .

6.2.1 Without chiral symmetry

If A is trivially graded then $x = \overline{h} \otimes e$ is an OSU of $A \otimes \mathbb{C}l_1$ where e is the square one generator of $\mathbb{C}l_1$. It follows that $\tilde{x} = \tilde{a} \otimes e$ is a lift of x . We recall Lemma 5.6, which gives the element $\delta(\overline{h} \otimes e) \in DK(I \otimes \mathbb{C}l_1 \hat{\otimes} \mathbb{C}l_1)$. Using the identification $I \otimes \mathbb{C}l_1 \hat{\otimes} \mathbb{C}l_1 \cong I \otimes \mathbb{C}l_2$, the class $\delta(\overline{h} \otimes e)$ is represented by the element $[Y] - [1 \otimes \rho]$ with ρ a generator of $\mathbb{C}l_2$ and

$$\begin{aligned} Y &= \exp(\pi \tilde{a} \otimes e \rho)(1 \otimes \rho) \\ &= -\exp(-i\pi(\tilde{a} \otimes \Gamma))(1 \otimes \rho) \\ &= (P_\Delta(\tilde{h})^\perp \otimes 1 - (P_\Delta(\tilde{h}) \otimes 1) \exp(-it_\Delta(\tilde{h} \otimes \Gamma)))(1 \otimes \rho), \end{aligned}$$

where $\Gamma = ie\rho$ is the grading operator on $I \otimes \mathbb{C}l_2$.

Example 6.7 (Boundary KK -class, No symmetries). Let us consider the Kasparov module representing the boundary invariant without reference to a real structure. By Proposition 5.7, the boundary class $\delta([x]) \in DK(I \otimes \mathbb{C}l_2)$ is represented by the unbounded Kasparov module

$$\left(\mathbb{C}, \overline{\cos(\frac{\pi}{2} \tilde{a} \otimes e)}(I \otimes \mathbb{C}l_1)_{I \otimes \mathbb{C}l_1}, \tan(\frac{\pi}{2} \tilde{a} \otimes e) \right).$$

We note that because $\overline{h} \in A$ and is not a matrix, we do not have to take a direct sum of the module I_I . We can simplify this Kasparov module by noting that $\overline{\cos(\frac{\pi}{2}\tilde{a} \otimes e)(I \otimes \mathbb{C}l_1)} \cong \overline{\cos(\frac{\pi}{2}\tilde{a})I} \otimes \mathbb{C}l_1$ and $\tan(\frac{\pi}{2}\tilde{a} \otimes e) = \tan(\frac{\pi}{2}\tilde{a}) \otimes e$. Recalling our definition of \tilde{a} , Equation (6.1), and writing $P_\Delta := P_\Delta(\tilde{h})$, we further reduce our boundary Kasparov module to

$$\left(\mathbb{C}, \overline{\cos(\frac{1}{2}t_\Delta \tilde{h})P_\Delta I_I} \otimes \mathbb{C}l_1, \tan(\frac{1}{2}t_\Delta \tilde{h}) \otimes e \right),$$

where we denote by $\mathbb{C}l_1$ the C^* -module $\mathbb{C}l_{1\mathbb{C}l_1}$.

If we consider bounded representatives of the KK -class, then by Proposition 5.9, the boundary invariant is represented by the Kasparov module

$$[(\mathbb{C}, I_I \otimes \mathbb{C}l_1, \tilde{a} \otimes e)] = [(\mathbb{C}, P_\Delta I_I \otimes \mathbb{C}l_1, \tilde{h} \otimes e)] \in KK(\mathbb{C}, I \otimes \mathbb{C}l_1).$$

In many cases of interest, the lift \tilde{h} is the restriction of h to a system with boundary and P_Δ the projection onto edge spectrum. Hence our boundary Kasparov module closely lines up with the physical intuition of a boundary topological invariant. \diamond

Example 6.8 (Boundary KK -class, TRS and PHS Hamiltonians). If h has a TRS, $h^{\text{rA}} = h$, then \tilde{a} is real and self-adjoint, so e must be real and self-adjoint. Hence $Y \in I^{\text{r}} \otimes Cl_{2,0}$ and we have the real Kasparov module

$$\left(\mathbb{C}, \overline{\cos(\frac{1}{2}t_\Delta \tilde{h})P_\Delta I_I} \otimes \mathbb{C}l_{1,0}, \tan(\frac{1}{2}t_\Delta \tilde{h}) \otimes e \right)$$

which gives a class in $KKR(\mathbb{C}, I \otimes \mathbb{C}l_{1,0}) \cong KO_{-1}(I^{\text{r}})$, where the last isomorphism is given by considering the ungraded Cayley transform $\mathcal{C}(\tan(\frac{1}{2}t_\Delta \tilde{h}))$. The bounded representative of this KKR -class is $(\mathbb{C}, P_\Delta I_I \otimes \mathbb{C}l_{1,0}, \tilde{h} \otimes e)$.

If h has a PHS, $h^{\text{rA}} = -h$, then \tilde{h} is imaginary and self-adjoint. Hence e must be imaginary and self-adjoint for Y to be real and self-adjoint. Therefore $Y \in I^{\text{r}} \otimes Cl_{1,1}$ and our boundary invariant is represented by the Kasparov module

$$\left(\mathbb{R}, \overline{\cos(\frac{1}{2}t_\Delta \tilde{h})P_\Delta I_{I^{\text{r}}}} \otimes Cl_{0,1}, \tan(\frac{1}{2}t_\Delta \tilde{h}) \otimes (1, -1) \right)$$

and corresponding class in $KKO(\mathbb{R}, I^{\text{r}} \otimes Cl_{0,1}) \cong KO_1(I^{\text{r}})$. The bounded representative is given by the Kasparov module $(\mathbb{R}, P_\Delta I_{I^{\text{r}}} \otimes Cl_{0,1}, \tilde{h} \otimes (1, -1))$. \diamond

6.2.2 With chiral symmetry

We first note a general result on graded algebras that will be of use to us. Suppose B is \mathbb{Z}_2 -graded and the grading is implemented by a self-adjoint unitary $\Gamma \in \text{Mult}(B)$. Then the map

$$\eta(b \hat{\otimes} \rho^k) := b \Gamma^k \otimes \rho^{k+|b|} \tag{6.2}$$

defines a graded isomorphism between the graded tensor product $B \hat{\otimes} \mathbb{C}l_1$ with grading Ad_Γ on B and the ungraded tensor product $B \otimes \mathbb{C}l_1$ with trivial grading on B .

If an insulator $h \in A$ has a chiral symmetry, then the lift \tilde{a} is an odd self-adjoint lift of \overline{h} . Therefore, by Lemma 5.6, the class of $[\overline{h}]$ under the boundary map in van Daele K -theory is represented by

$$Y = -\exp(\pi \tilde{a} \hat{\otimes} \rho)(1 \hat{\otimes} \rho) = \left(P_\Delta(\tilde{h})^\perp - P_\Delta(\tilde{h}) \exp(t_\Delta \tilde{h} \hat{\otimes} \rho) \right) (1 \hat{\otimes} \rho).$$

Example 6.9 (Chiral symmetry only). We can again apply Proposition 5.7 and obtain a representative of $\delta([\bar{h}])$ in $KK(\mathbb{C}, I)$ as the class of the Kasparov module

$$\left(\mathbb{C}, \overline{\cos(\frac{\pi}{2}\tilde{a})I_I}, \tan(\frac{\pi}{2}\tilde{a}) \right).$$

Because we have used the specific lift \tilde{a} , we write $P_\Delta = P_\Delta(\tilde{h})$ and simplify this Kasparov module to

$$\left(\mathbb{C}, \overline{\cos(\frac{1}{2}t_\Delta\tilde{h})P_\Delta I_I}, \tan(\frac{1}{2}t_\Delta\tilde{h}) \right).$$

Taking the bounded transform, we use Proposition 5.9 and obtain the boundary invariant

$$[(\mathbb{C}, I_I, \tilde{a})] = [(\mathbb{C}, P_\Delta I_I, \tilde{h})] \in KK(\mathbb{C}, I). \quad (6.3)$$

Suppose now that I is inner-graded, e.g. the grading is implemented by an inner chiral symmetry on the boundary. Then using the isomorphism from Equation (6.2), we know that $\delta([\bar{h}]) \in DK(I \hat{\otimes} \mathbb{C}\ell_1) \cong DK(I \otimes \mathbb{C}\ell_1) \cong K_0(I)$. Equation (6.3) gives a representative of this class, but not a canonical one as the Kasparov module in (6.3) uses the grading on I . \diamond

Example 6.10 (TRS with chiral symmetry). Let us now consider the boundary map of chiral symmetric Hamiltonians with a real structure. If h has a TRS, $h^{\varepsilon_A} = h$, then \tilde{h} is real and $Y \in I^\nu \otimes Cl_{1,0}$. Our Kasparov module of interest is

$$\left(\mathbb{C}, \overline{\cos(\frac{1}{2}t_\Delta\tilde{h})P_\Delta I_I}, \tan(\frac{1}{2}t_\Delta\tilde{h}) \right)$$

Taking the bounded transform, the boundary invariant is also represented by the Kasparov module

$$[(\mathbb{C}, P_\Delta I_I, \tilde{h})] \in KKR(\mathbb{C}, I).$$

If I is inner-graded by the element $\Gamma \in \text{Mult}(I)$ and is such that $\Gamma^\varepsilon = \Gamma$, then the map η from Equation (6.2) gives an isomorphism $I^\nu \hat{\otimes} Cl_{1,0}$ to $I^\nu \otimes Cl_{1,0}$, where I^ν has trivial grading on the right-hand side. Hence our boundary invariant can also be regarded as an element in $KO_0(I^\nu)$. \diamond

Example 6.11 (PHS with chiral symmetry). Suppose that h has a PHS so \tilde{h} and the lift $\tilde{a} \in E$ are imaginary. In order to apply the van Daele boundary map, we first need to construct a real OSU. We consider the algebra $A \hat{\otimes} \mathbb{C}\ell_{0,2}$, where one can check that $\bar{h} \hat{\otimes} i f_1 f_2$ is an odd Real self-adjoint unitary. Therefore, for e the self-adjoint odd generator in $\mathbb{C}\ell_{1,2}$ and $\omega = e_1 f_1 f_2 \in \mathbb{C}\ell_{1,2}$ the orientation element, Lemma 5.6 gives that

$$Y = -\exp(\pi\tilde{a} \hat{\otimes} i\omega)(1 \hat{\otimes} e) = \left(P_\Delta(\tilde{h})^\perp - P_\Delta(\tilde{h}) \exp(t_\Delta\tilde{h} \hat{\otimes} i\omega) \right) (1 \hat{\otimes} e)$$

represents the class $\delta([\bar{h} \hat{\otimes} i f_1 f_2]) \in DK(I \hat{\otimes} \mathbb{C}\ell_{1,2})$.

We can now apply Proposition 5.7 to obtain the unbounded Kasparov module

$$\left(\mathbb{C}, \overline{\cos(\frac{\pi}{2}\tilde{a} \hat{\otimes} i f_1 f_2) I \hat{\otimes} \mathbb{C}\ell_{0,2} I \hat{\otimes} \mathbb{C}\ell_{0,2}}, \tan(\frac{\pi}{2}\tilde{a} \hat{\otimes} i f_1 f_2) \right)$$

representing the boundary. We can simplify this Kasparov module to

$$\left(\mathbb{C}, \overline{\cos(\frac{\pi}{2}\tilde{a}) I_I \hat{\otimes} \mathbb{C}\ell_{0,2}}, \tan(\frac{\pi}{2}\tilde{a}) \hat{\otimes} i f_1 f_2 \right)$$

and using the explicit lift \tilde{a} from Equation (6.1), the boundary Kasparov module becomes

$$\left(\mathbb{C}, \overline{\cos(\frac{1}{2}t_\Delta \tilde{h}) P_\Delta I_I} \hat{\otimes} \mathbb{C}l_{0,2}, \tan(\frac{1}{2}t_\Delta \tilde{h}) \hat{\otimes} if_1 f_2 \right).$$

We also take the bounded transform, where by a straight-line homotopy, the boundary class is represented by the Kasparov module

$$\left[(\mathbb{C}, P_\Delta I_I \hat{\otimes} \mathbb{C}l_{0,2}, \tilde{h} \hat{\otimes} if_1 f_2) \right] \in KKR(\mathbb{C}, I \hat{\otimes} \mathbb{C}l_{0,2}).$$

If the grading on I is inner and the grading operator imaginary, $\Gamma^r = -\Gamma$, then the isomorphism η from Equation (6.2) is such that $I^{r_I} \hat{\otimes} Cl_{0,2} \cong I^{r_I} \otimes Cl_{0,2}$ with I^{r_I} trivially graded on the right-hand side. Therefore, for inner and imaginary chiral symmetries, the boundary invariant gives a class in $KO_2(I^{r_I})$. \diamond

A Kasparov products with the Cayley transform

In this appendix we outline how our Cayley map on K -theory is compatible with the constructive form of the Kasparov product. We will address the complex case. The real case can be adapted from the complex one, for while the algebraic details change, the analytic details are the same.

We will typically be interested in products of odd (ungraded) Kasparov modules (\mathcal{A}, X_B, D) , but note that our results also hold when $A = \overline{A}$ and B are \mathbb{Z}_2 -graded. In the graded case, the triple (\mathcal{A}, X_B, D) should be interpreted as the Kasparov module $(\mathcal{A} \hat{\otimes} \mathbb{C}l_1, X_B^{\oplus 2}, D \hat{\otimes} \sigma_1)$, where $D : \text{Dom}(D) \rightarrow X_B$ is even and the left Clifford action is generated by $-i\sigma_2$.

Given an unbounded Kasparov module (\mathcal{A}, X_B, D) and an even unitary $u \in \mathcal{A} \subset A$ we are interested in representatives of the product

$$[(\mathbb{C}, \overline{(u-1)A_A}, \mathcal{C}^{-1}(u))] \otimes_A [(\mathcal{A}, X_B, D)] \in KKR(\mathbb{C}, B), \quad \mathcal{C}^{-1}(u) = i(u+1)(u-1)^{-1}.$$

To construct a representative, we need to work on the module $\overline{(u-1)X_B} \cong \overline{(u-1)A} \otimes_A X_B$ (or rather two copies of this module) and consider the operators

$$\mathcal{C}^{-1}(u) \pm i\tilde{D} : (u-1)\text{Dom}(D) \subset \overline{(u-1)X_B} \rightarrow \overline{(u-1)X_B},$$

where we need to make sense of the restriction $\tilde{D} = D|_{(u-1)\text{Dom}(D)}$ in spite of the possible lack of complementability of $\overline{(u-1)X_B}$ in X_B .

Lemma A.1. *Let (\mathcal{A}, X_B, D) be an unbounded Kasparov module and $u \in \mathcal{A}^\sim$ unitary. Suppose that there exists an approximate unit $(v_n) \subset C^*((u-1), (u^*-1))$ such that for all n the commutator $[D, v_n]$ is defined and bounded, $v_n X_B \subset (u-1)X_B = (u^*-1)X_B$, and finally $[D, v_n](u^*-1) \rightarrow 0$ $*$ -strongly.*

Then with $\tilde{D} = s\text{-lim } v_n D|_{(u-1)X}$ we find that $(\mathcal{C}^{-1}(u) \pm i\tilde{D})^ = \mathcal{C}^{-1}(u) \mp i\tilde{D}$.*

Proof. We prove the lemma for $\mathcal{C}^{-1}(u) + i\tilde{D}$ since the other case is proved identically. Let $y \in \text{Dom}((\mathcal{C}^{-1}(u) + i\tilde{D})^*)$. That is, there exists $z \in \overline{(u-1)X_B}$ such that for all elements $x \in \text{Dom}(\mathcal{C}^{-1}(u) + i\tilde{D}) = (u-1)\text{Dom}(D)$ we have

$$((\mathcal{C}^{-1}(u) + i\tilde{D})x \mid y)_B = (x \mid z)_B.$$

Write $x = (u - 1)\xi$ with $\xi \in \text{Dom}(D)$. We observe that $v_n[\mathcal{D}, u] = [\mathcal{D}, v_n(u - 1)] - [\mathcal{D}, v_n](u - 1) \rightarrow [\mathcal{D}, u]$ strongly. Then

$$\begin{aligned}
((\mathcal{C}^{-1}(u) + i\tilde{D})x \mid y)_B &= ((\mathcal{C}^{-1}(u) + i\tilde{D})(u - 1)\xi \mid y)_B \\
&= (i(u + 1)\xi \mid y)_B + (i \lim v_n[D, u]\xi \mid y)_B + (\lim v_n(u - 1)iD\xi \mid y)_B \\
&= (i(u + 1)\xi \mid y)_B + (i[D, u]\xi \mid y)_B + ((u - 1)iD\xi \mid y)_B \\
&= (\xi \mid -i(1 + u^*)y)_B + (\xi \mid i[D, u^*]y)_B + (D\xi \mid -i(u^* - 1)y)_B \\
&= (\xi \mid (u^* - 1)z)_B.
\end{aligned}$$

Rearranging the last equality shows that

$$(D\xi \mid -i(u^* - 1)y)_B = (\xi \mid (u^* - 1)z)_B + (\xi \mid i(u^* + 1)y)_B - (\xi \mid i[D, u^*]y)_B \quad (\text{A.1})$$

and as this holds for all $\xi \in \text{Dom}(D)$, we see that $(u^* - 1)y \in \text{Dom}(D^*) = \text{Dom}(D)$. Moreover, since $z = (\mathcal{C}^{-1}(u) + i\tilde{D})^*y$ we learn that

$$-iD(u^* - 1)y = (u^* - 1)(\mathcal{C}^{-1}(u) + i\tilde{D})^*y + i(u^* + 1)y - i[D, u^*]y$$

or

$$(u^* - 1)(\mathcal{C}^{-1}(u) + i\tilde{D})^*y = -iD(u^* - 1)y + i[D, u^*]y - i(u^* + 1)y. \quad (\text{A.2})$$

If $y \in \text{Dom}(D)$ then

$$-iD(u^* - 1)y + i[D, u^*]y = -i(u^* - 1)Dy$$

and in this case we see from Equation (A.2) that $(u^* + 1)y$ is in $(u^* - 1)\text{Dom}(D) \subset (u^* - 1)X_B = \text{Dom}(\mathcal{C}^{-1}(u))$. Thus we can multiply through by $(u^* - 1)^{-1}$ and find that

$$(\mathcal{C}^{-1}(u) + i\tilde{D})^*y = -iDy + \mathcal{C}^{-1}(u)y.$$

Hence

$$\text{Dom}((\mathcal{C}^{-1}(u) + i\tilde{D})^*) \cap \text{Dom}(D) = \text{Dom}(\mathcal{C}^{-1}(u) - i\tilde{D}),$$

and we need only show that $y \in \text{Dom}(D)$. So let $(v_n) \subset C^*((u - 1), (u^* - 1))$ be as in the statement of the Lemma. Then we find that

$$\begin{aligned}
-iD(u^* - 1)y + i[D, u^*]y &= \lim_n v_n(-iD(u^* - 1)y + i[D, u^*]y) \\
&= \lim_n \left(-i[v_n, D](u^* - 1)y - iD(u^* - 1)v_ny + i[v_n, [D, u^*]]y + i[D, u^*]v_ny \right) \\
&= \lim_n \left(-i[v_n, D](u^* - 1)y - i(u^* - 1)Dv_ny + i[v_n, [D, u^*]]y \right) \\
&= \lim_n \left(-i[v_n, D](u^* - 1)y - i(u^* - 1)Dv_ny \right)
\end{aligned}$$

where the last equality follows since $[v_n, [D, u^*]] \rightarrow 0$ strongly. Now $v_ny \in (1 - u)X$ and $v_ny \in \text{Dom}(D)$ by Equation (A.1). Thus if $[v_n, D](u^* - 1) \rightarrow 0$ strongly we deduce that y is in the closure of $(u - 1)\text{Dom}(D)$ in the graph norm of D , and so in $\text{Dom}(D)$. \square

Theorem A.2. *Let (A, X_B, D) be an odd unbounded Kasparov module and $u \in \mathcal{A}^\sim$ an even unitary. Suppose $C^*((u - 1), (u^* - 1))$ has an approximate unit (v_n) as in Lemma A.1 and $\|[D, u]\| < 2$. Then*

$$\left(\mathbb{C}, \overline{(u - 1)X_B} \oplus \overline{(u - 1)X_B}, \mathcal{C}^{-1}(u) \hat{+} \tilde{D} \right), \quad \mathcal{C}^{-1}(u) \hat{+} \tilde{D} := \begin{pmatrix} 0 & \mathcal{C}^{-1}(u) - i\tilde{D} \\ \mathcal{C}^{-1}(u) + i\tilde{D} & 0 \end{pmatrix}$$

is an unbounded Kasparov module representing the Kasparov product of the class of the Cayley transform $[(\mathbb{C}, \overline{(u - 1)X_B}, \mathcal{C}^{-1}(u))] \in KK_1(\mathbb{C}, A)$ with $[(A, X_B, D)] \in KK^1(A, B)$.

Remark If $[D, u]$ is bounded we can ensure that $\|[D, u]\| < 2$ is satisfied by rescaling D . \diamond

Proof. We employ the main result of [30]. First, if we make the identification $\overline{(u-1)A} \otimes_A X_B \cong \overline{(u-1)X_B}$ by left multiplication, then the map

$$X_B \ni x \mapsto (u-1)aDx - D|_{(u-1)X}(u-1)ax = -[D, (u-1)a]x$$

is bounded. This proves that Kucerovsky's connection condition is satisfied [30].

Since $\text{Dom}(\mathcal{C}^{-1}(u)) \subset \text{Dom}(\mathcal{C}^{-1}(u) \pm i\tilde{D})$, the domain condition is satisfied, and we need only check Kucerovsky's positivity condition and that we have a Kasparov module.

The assumption that $[\mathcal{D}, v_n](u^* - 1) \rightarrow 0$ strongly and Lemma A.1 tells us that $\mathcal{C}^{-1}(u) \hat{+} \tilde{D}$ is self-adjoint. For regularity, let $\phi : B \rightarrow \mathbb{C}$ be a state, and form the Hilbert space $X \otimes_B L^2(B, \phi)$. The sequence $v_n \otimes 1$ satisfies the same domain mapping properties with respect to $(\mathcal{C}^{-1}(u) \hat{+} \tilde{D}) \otimes 1$ as v_n did for $\mathcal{C}^{-1}(u) \hat{+} \tilde{D}$, and so the above arguments show that $(\mathcal{C}^{-1}(u) \hat{+} \tilde{D}) \otimes 1$ is self-adjoint. As ϕ was an arbitrary state, the local global-principle [19, 38] implies the regularity of $\mathcal{C}^{-1}(u) \hat{+} \tilde{D}$.

To check the positivity condition we first compute the anti-commutator. So

$$\begin{aligned} & \begin{pmatrix} 0 & \mathcal{C}^{-1}(u) \\ \mathcal{C}^{-1}(u) & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathcal{C}^{-1}(u) - i\tilde{D} \\ \mathcal{C}^{-1}(u) + i\tilde{D} & 0 \end{pmatrix} \\ & + \begin{pmatrix} 0 & \mathcal{C}^{-1}(u) - i\tilde{D} \\ \mathcal{C}^{-1}(u) + i\tilde{D} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathcal{C}^{-1}(u) \\ \mathcal{C}^{-1}(u) & 0 \end{pmatrix} \\ & = 2 \begin{pmatrix} \mathcal{C}^{-1}(u)^2 & 0 \\ 0 & \mathcal{C}^{-1}(u)^2 \end{pmatrix} + \begin{pmatrix} i[\mathcal{C}^{-1}(u), D] & 0 \\ 0 & -i[\mathcal{C}^{-1}(u), D] \end{pmatrix} \\ & = 2 \begin{pmatrix} 4(u-1)^{-1}(u^* - 1)^{-1} - 1 & 0 \\ 0 & 4(u-1)^{-1}(u^* - 1)^{-1} - 1 \end{pmatrix} \\ & + \begin{pmatrix} 2(u-1)^{-1}[u, D](u^* - 1)^{-1} & 0 \\ 0 & -2(u-1)^{-1}[u, D](u^* - 1)^{-1} \end{pmatrix} \\ & = 2 \begin{pmatrix} (u-1)^{-1}(4 - [u, D]u^*)(u^* - 1)^{-1} - 1 & 0 \\ 0 & (u-1)^{-1}(4 + [u, D]u^*)(u^* - 1)^{-1} - 1 \end{pmatrix} \end{aligned}$$

Recasting this computation in terms of quadratic forms shows that the required positivity holds when $[D, u]u^* \leq 4$, which is satisfied since we assume $\|[D, u]\| < 2$.

Because the left-action is by the complex numbers, commutators of the left action with $\mathcal{C}^{-1}(u) \hat{+} \tilde{D}$ are trivially bounded. Thus all that remains is to check the compact resolvent condition. The self-adjointness of $\mathcal{C}^{-1}(u) \hat{+} \tilde{D}$ on $(u-1)\text{Dom}(\mathcal{D})$ tells us that

$$(i \pm \mathcal{C}^{-1}(u) \hat{+} \tilde{D})^{-1} \overline{(u-1)X_B}^{\oplus 2} = (u-1)\text{Dom}(\mathcal{D})^{\oplus 2} = (u-1)(i \pm \mathcal{D})^{-1} X_B^{\oplus 2} \hookrightarrow X_B^{\oplus 2}$$

where the last inclusion is compact since $u-1 \in \mathcal{A}$ and \mathcal{D} has locally compact resolvent. \square

Example A.3. For $z \in S^1$ we let $\rho(z) = 2 - z - \bar{z}$ and define $v_n = \rho(\rho + 1/n)^{-1}$. Elementary trigonometry shows that

$$\left[\frac{1}{i} \frac{d}{d\theta}, v_n \right] (1 - \bar{z}) = 2(1 - v_n) \left(\frac{\sin(\theta)}{1 - \cos(\theta) + 1/2n} - \cot(\theta/2) \right) e^{i\theta/2} \sin(\theta/2) + 2(1 - v_n) e^{i\theta/2} \cos(\theta/2)$$

which does indeed go to zero strongly on $L^2(S^1) = \overline{(z-1)L^2(S^1)}$. \diamond

References

- [1] A. Alldridge, C. Max and M. R. Zirnbauer. *Bulk-boundary correspondence for disordered free-fermion topological phases*. Comm. Math. Phys., online first (2019), <https://dx.doi.org/10.1007/s00220-019-03581-7>
- [2] S. Baaj and P. Julg. *Théorie bivariante de Kasparov et opérateurs non bornés dans les C^* -modules hilbertiens*. C. R. Acad. Sci. Paris Sér. I Math, **296** (1983), no. 21, 875–878.
- [3] J. Bellissard. *K-theory of C^* -algebras in solid state physics*. In Statistical mechanics and field theory: mathematical aspects (Groningen, 1985), volume 257 of Lecture Notes in Phys., Springer, Berlin (1986), 99–156.
- [4] B. Blackadar. *K-Theory for Operator Algebras*. Volume 5 of Mathematical Sciences Research Institute Publications, Cambridge Univ. Press, Cambridge (1998).
- [5] J. L. Boersema. *Real C^* -algebras, united K-theory and the Künneth formula*. K-theory, **26** (2002), 345–402.
- [6] J. L. Boersema and T. A. Loring. *K-Theory for real C^* -algebras via unitary elements with symmetries*. New York J. Math. **22** (2016), 1139–1220.
- [7] C. Bourne, J. Kellendonk and A. Rennie. *The K-theoretic bulk-edge correspondence for topological insulators*. Ann. Henri Poincaré, **18** (2017), no. 5, 1833–1866.
- [8] C. Bourne and B. Mesland. *Index theory and topological phases of aperiodic lattices*. Ann. Henri Poincaré, **20** (2019), no. 6, 1969–2038.
- [9] C. Bourne and A. Rennie. *Chern numbers, localisation and the bulk-edge correspondence for continuous models of topological phases*. Math. Phys. Anal. Geom., **21** (2018), no. 3, art. 16, 62 pages.
- [10] A. L. Carey and J. Phillips. *Spectral flow in Fredholm modules, eta invariants and the JLO cocycle*. K-Theory, **31** (2004), no. 2, 135–194.
- [11] A. Connes and H. Moscovici. *Cyclic cohomology, the Novikov conjecture and hyperbolic groups*. Topology, **29** (1990), no. 3, 345–388.
- [12] A. van Daele. *K-theory for graded Banach algebras I*. Quart. J. Math. Oxford Ser. (2), **39** (1988), no. 154, 185–199.
- [13] A. van Daele. *K-theory for graded Banach algebras II*. Pacific J. Math., **134** (1988), no. 2, 377–392.
- [14] K. van den Dungen and B. Mesland. *Homotopy equivalence in unbounded KK-theory*. Ann. K-Theory, to appear. arXiv:1907.04049 (2019).
- [15] E. van Erp. *The index of hypoelliptic operators on foliated manifolds*. J. Noncommut. Geom., **5** (2011), no. 1, 107–124.
- [16] R. Exel, *A Fredholm operator approach to Morita equivalence*. K-Theory, **7** (1993), 285–308.

- [17] D. S. Freed and G. W. Moore. *Twisted equivariant matter*. Ann. Henri Poincaré, **14** (2013), no. 8, 1927–2023.
- [18] N. Higson and J. Roe. *Analytic K-homology*. Oxford University Press, Oxford (2000).
- [19] J. KAAD and M. Lesch. *A local-global principle for regular operators on Hilbert modules*. J. Funct. Anal., **262** (2012), no. 10, 4540–4569.
- [20] J. KAAD, R. Nest and A. Rennie. *KK-theory and spectral flow in von Neumann algebras*. J. K-Theory, **10** (2012), no. 2, 241–277.
- [21] G. G. Kasparov. *Hilbert C^* -modules: theorems of Stinespring and Voiculescu*. J. Operator Theory, **4** (1980), no. 2, 133–150.
- [22] G. G. Kasparov. *The operator K-functor and extensions of C^* -algebras*. Math. USSR Izv., **16** (1981), 513–572.
- [23] J. Kellendonk. *On the C^* -algebraic approach to topological phases for insulators*. Ann. Henri Poincaré, **18** (2017), no. 7, 2251–2300.
- [24] J. Kellendonk. *Cyclic cohomology for graded $C^{*,r}$ -algebras and its pairings with van Daele K-theory*. Comm. Math. Phys., **368** (2019), no. 2, 467–518.
- [25] J. Kellendonk, T. Richter and H. Schulz-Baldes. *Edge current channels and Chern numbers in the integer quantum Hall effect*. Rev. Math. Phys., **14** (2002), no. 1, 87–119.
- [26] J. Kellendonk and H. Schulz-Baldes. *Boundary maps for C^* -crossed products with \mathbb{R} with an application to the quantum Hall effect*. Comm. Math. Phys., **249** (2004), no. 3, 611–637.
- [27] A. Kitaev. *Periodic table for topological insulators and superconductors*. In V. Lebedev & M. Feigel'man, editors, volume 1134 of American Institute of Physics Conference Series, pages 22–30 (2009).
- [28] Y. Kubota. *Notes on twisted equivariant K-theory for C^* -algebras*. Internat. J. Math., **27** (2016), no. 6, 1650058.
- [29] Y. Kubota. *Controlled topological phases and bulk-edge correspondence*. Comm. Math. Phys., **349** (2017), no. 2, 493–525.
- [30] D. Kucerovsky. *The KK-product of unbounded modules*. K-Theory, **11** (1997), 17–34.
- [31] A. Kumjian, D. Pask and A. Sims. *Graded C^* -algebras, graded K-theory, and twisted P-graph C^* -algebras*. J. Operator Theory, **80** (2018), no. 2, 295–348.
- [32] E. C. Lance. *Hilbert C^* -Modules: A Toolkit for Operator Algebraists*. Cambridge University Press, Cambridge (1995).
- [33] J. A. Mingo. *K-theory and multipliers of stable C^* -algebras*. Trans. Amer. Math. Soc., **299** (1987), no. 1, 397–411.
- [34] T. Natsume. *A new role of graph projections in index theory*. In Quantum and non-commutative analysis (Kyoto, 1992), volume 16 of Math. Phys. Stud., pages 285–290 (1993).

- [35] R. Nest and B. Tsygan. *Algebraic index theorem*. Comm. Math. Phys., **172** (1995), no. 2, 223–262.
- [36] J. Phillips. *Self-adjoint Fredholm operators and spectral flow*. Canad. Math. Bull., **39** (1996), 460–467.
- [37] J. Phillips. *Spectral flow in type I and type II factors—a new approach*. Fields Inst. Commun., **17** (1997), 137–153.
- [38] F. Pierrot. *Opérateurs réguliers dans les C^* -modules et structure des C^* -algèbres des groupes de Lie semisimple complexes simplement connexes*. J. Lie Theory, **16** (2006), 651–689.
- [39] E. Prodan and H. Schulz-Baldes. *Bulk and Boundary Invariants for Complex Topological Insulators: From K -Theory to Physics*. Springer, Cham (2016).
- [40] A. Rennie and A. Sims. *Non-commutative vector bundles for non-unital algebras*. SIGMA **13**, (2017), 041, 12 pages, arXiv:1612.03559.
- [41] J. Roe. *Kasparov products and dual algebras*. J. Funct. Anal., **155** (1998), 286–296.
- [42] J. Roe. *Paschke duality for real and graded C^* -algebras*. Q. J. Math., **55** (2004), no. 3, 325–331.
- [43] H. Schröder. *K -Theory for Real C^* -algebras and Applications*. Taylor & Francis, New York (1993).
- [44] H. Schulz-Baldes and D. Toniolo. *Dimensional reduction and scattering formulation for even topological invariants*. arXiv:1811.11781 (2018).
- [45] G. Skandalis. *Exact sequences for the Kasparov groups of graded algebras*. Canad. J. Math., **37** (1985), 193–216.
- [46] G. C. Thiang. *On the K -theoretic classification of topological phases of matter*. Ann. Henri Poincaré. **17** (2016), no. 4, 757–794.
- [47] J. Trout. *On graded K -theory, elliptic operators and the functional calculus*. Illinois J. Math., **44** (2000), no. 2, 294–309.
- [48] R. Wood. *Banach algebras and Bott periodicity*. Topology, **4** (1965/66), 371–389.