Algebraic polygraphs modulo and linear rewriting

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Abstract – Convergent rewriting systems on algebraic structures give methods to prove coherence results and compute homological invariants of these structures. These methods are based on higher-dimensional extensions of the critical pair lemma that characterizes local confluence from confluence of critical pairs. The analysis of local confluence of rewriting systems on algebraic structures, such as groups or linear algebras, is complicated because of the underlying algebraic axioms, and local confluence properties require additional termination conditions. This article introduces the structure of algebraic polygraph modulo that formalizes the interaction between the rules of the rewriting system and the inherent algebraic axioms, and we show a critical pair lemma for algebraic polygraphs. We deduce from this result a critical pair lemma for rewriting systems on algebraic structures specified by rewriting systems convergent modulo associativity and commutativity axioms. As an illustration, we explicit our constructions on linear rewriting systems.

Keywords – Term rewriting modulo, algebraic polygraphs, linear rewriting.

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1. Introduction

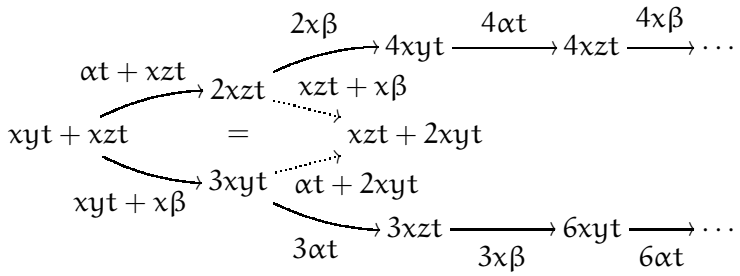
Completion procedures. The critical-pair completion (CPC) is an approach developed in the mid sixties that combines completion procedures and the notion of critical pair [4, 23]. It originates from theorem proving [22], polynomial ideal theory [3], and the word problem [13, 20]. This approach has found many applications to solve algorithmic problems, see [4] for an historical account. In the mid eighties CPC has found original and deep applications in algebra in order to solve coherence problems for monoids [8, 25], or to compute homological invariants of associative algebras, [1], and monoids, [14, 24]. More recently, higher-dimensional extensions of the critical-pair completion approach were used for the computation of cofibrant replacements of algebraic and categorical structures [6, 7, 16]. All these constructions are actually higher-dimensional extensions of the critical-pair completion approach, where the obstructions in each dimension are formulated in terms of critical branchings. While generators and rules are in dimension 1 and 2 respectively, the critical branchings describe 3-dimensional cocycles, and the 4-dimensional cocycles are described by critical 3-branchings, that is the overlapping of a rule on a critical branching. This generalizes in higher-dimensions, where for $n \geqslant 4$, the n-dimensional cocycles are described by overlappings of a rule on a critical $(n-1)$-branching. These constructions based on critical-pair completion are known for monoids, small categories, and algebras over a field. However, the extension of these methods to a wide range of algebraic structures is made difficult because of the interaction between the rewriting rules and the inherent axioms of the algebraic structure. For this reason, the higher-dimensional extensions of the critical-pair completion approach for a wide range of algebraic structures, including groups, Lie rings, is still an open problem.

Critical branching lemma. One of the main tools to reach confluence in critical-pair completion procedures for algebraic rewriting systems is the *critical pair lemma*, or *critical branching lemma*, by Knuth-Bendix, [13], and Nivat, [20]. Nivat showed that the local confluence of a string rewriting system is decidable, whether it is terminating or not. The proof of this result is based on classification of the local branchings into *orthogonal* branchings, that involve two rules that do not overlap, and *overlapping* branchings. A *critical branching* is a minimal overlapping application of two rules on the same redex. When the orthogonal branchings are confluent, if all critical branchings are confluent, then local confluence holds. Thus, the main argument to achieve critical branching lemma is to prove that orthogonal and overlapping branchings are confluent. For string and term rewriting systems, orthogonal branchings are always confluent, and confluence of critical branchings implies confluence of overlapping branchings. The situation is more complicated for rewriting systems on a linear structure.

The well known approaches of rewriting in the linear context consist in orienting the rules with respect to an ambiant monomial order, and critical branching lemma is well known in this context. However, some algebras do not admit any higher-dimensional finite convergent presentation on a fixed set of generators with respect to a monomial order, [6]. Due to algebraic perspectives, an approach of linear rewriting where the orientation of rules does not depend of a monomial order was introduced in [6]. However, in that setting there are two conditions to guarantee a critical branching lemma, namely termination and positivity of reductions. A positive reduction for a linear rewriting system, as defined in [6], is the application of a reduction rule on a monomial that does not appear in the polynomial context. For instance, consider the linear rewriting system on an associative algebra over a field $\mathbb{K}$ given in [6] defined by the following two rules

$$\alpha : xy \to xz, \qquad \beta : zt \to 2yt.$$

It has no critical branching, but it has the following non-confluent additive branching:



The dotted arrows correspond to non positive reductions. This example illustrates that the lack of termination is an obstruction to confluence of orthogonal branchings. Indeed, the critical branching lemma for linear 2-dimensional polygraphs states that a terminating left-monomial linear polygraph is locally confluent if and only if all its critical branchings are confluent, [6, Theorem 4.2.1]

Rewriting modulo. Rewriting modulo appears naturally in algebraic rewriting when studied reductions are defined modulo the axioms of an ambiant algebraic or categorical structure, eg. rewriting in commutative, groupoidal, linear, pivotal, weak structures. Furthermore, rewriting modulo facilitates the analysis of confluence. In particular, rewriting modulo a set of relations makes the property of confluence easier to prove. Indeed, the family of critical branchings that should be considered in the analysis of confluence is reduced, and the non-orientation of a part of the relations allows more flexibility when reaching confluence.

The most naive approach of rewriting modulo is to consider the rewriting system ${}_{P}R_{P}$ consisting in rewriting on congruence classes modulo the axioms P. This approach works for some equational theories, such as associative and commutative theories. However, it appears inefficient in general for the analysis of confluence. Indeed, the reducibility of an equivalence class needs to explore all the class, hence it requires all equivalence classes to be finite. Another approach of rewriting modulo has been considered by Huet in [9], where rewriting sequences involve only oriented rules and no equivalence steps, and the confluence property is formulated modulo equivalence. However, for algebraic rewriting systems such rewriting modulo is too restrictive for computations, see [12]. Peterson and Stickel introduced in [21] an extension of Knuth-Bendix's completion procedure, [13], to reach confluence of a rewriting system modulo an equational theory, for which a finite, complete unification algorithm is known. They applied their procedure to rewriting systems modulo axioms of associativity and commutativity, in order to rewrite in free commutative groups, commutative unitary rings, and distributive lattices. Jouannaud and Kirchner enlarged this approach in [11] with the definition of rewriting properties for any rewriting system modulo S such that $R \subseteq S \subseteq {}_{P}R_{P}$. They also proved a critical branching lemma and developed a completion procedure for rewriting systems modulo ${}_{P}R$, whose one-step reductions consist in application of a rule in R using P-matching. Their completion procedure is based on a finite P-unification algorithm. Bachmair and Dershowitz in [2] developed a generalization of Jouannaud-Kirchner's completion procedure using inference rules. Several other approaches have also been studied for term rewriting systems modulo to deal with various equational theories, see [18, 27].

Algebraic polygraphs. In this article, we introduce a categorical model for rewriting in algebraic structures which formalizes the interaction between the rules of the rewriting system and the inherent axioms of the algebraic structure. In Section 2, we recall the notion of cartesian 2-dimensional polygraph

introduced in [17], corresponding to rewriting systems that present a Lawvere algebraic theory. A cartesian 2-polygraph defines a categorical interpretation of term rewriting systems. It is defined by an equational signature (P_0, P_1) and a cellular extension of the free algebraic theory $P_1^{\times}$ on (P_0, P_1). One defines in Section 3 the structure of *algebraic polygraph* as a data made of a cartesian 2-polygraph P and a set Q of generating ground 1-cells and a cellular extension R on the ground 1-cells.

An algebraic critical branching lemma. In this work we introduce an algebraic setting for the formulation of the critical branching lemma. We define the structure of algebraic polygraph modulo which formalizes the interaction between the rules of the rewriting system and the inherent axioms of the algebraic structure. Then we prove a formulation of the Newman lemma modulo, [19], for quasi-terminating algebraic polygraphs modulo:

> **Theorem 4.1.4.** *Let $\mathcal{P}$ be a quasi-terminating algebraic polygraph modulo, and σ be a positive strategy on $\mathcal{P}$. If $\mathcal{P}$ is locally σ-confluent modulo, then it is σ-confluent modulo.*

We show a critical branching lemma for algebraic polygraphs modulo. We deduce from this result a critical branching lemma for rewriting systems on algebraic structures whose axioms are specified by term rewriting systems satisfying appropriate convergence relations modulo associativity and commutativity. Finally, we explicit our results in linear rewriting, and explain why termination is a necessary condition to characterize local confluence in that case.

> **Theorem 5.1.2.** *Let $\mathcal{P} = (P, Q, R, S)$ be an algebraic polygraph modulo with a positive strategy σ such that ${}_pR_P$ is quasi-terminating and positively σ-confluent. An algebraic rewriting system on $\mathcal{P}$ is locally confluent if and only if its critical branchings are confluent.*

Organisation of the article. In Section 2, we recall the categorical structure of cartesian polygraph introduced in [17], and we define progressive examples of cartesian 2-polygraphs presenting various algebraic theories, until the main cartesian 2-polygraph presenting the theory of modules over a commutative ring. In Section 3.1, we introduce the notion of algebraic polygraph modulo, and their rewriting properties. We define the notion of positive reduction with respect to an algebraic polygraph allowing to define algebraic rewriting systems, whose rules are quotients of the positive reductions. In Section 4, we present confluence property with respect to a positive strategy following [5] and we prove the Newman confluence lemma and the critical branching lemma in this setting. Finally, last section states the algebraic critical branching lemma, that we apply to string rewriting and linear rewriting.

2. Preliminaries on algebraic theories

In section we recall the notion of algebraic theory from [15] and of cartesian polygraph introduced in [17].

2.1. Cartesian polygraphs and theories

2.1.1. Signature and terms. A *signature* is defined by a set P_0 of *sorts* and a 1-polygraph, *i.e.* a directed graph,

$$P_0^* \underset{\partial_0^+}{\overset{\partial_0^-}{\leftleftarrows}} P_1$$

on the free monoid P_0^* over P_0. Elements of P_1 are called *operations*. For an operation α in P_1, its source $\partial_0^-(\alpha)$ is called its *arity* and its target $\partial_0^+(\alpha)$ its *coarity*. For sorts $s_1, \dots, s_k$, we denote $\underline{s} = s_1 \dots s_k$ their product in the free monoid P_0^*. We denote $|\underline{s}| = k$ the *length* of $\underline{s}$ and the sort s_i in $\underline{s}$ will be denoted by $\underline{s}_i$.

Recall from [15] that an *(multityped Lawvere algebraic) theory* for a given set of sorts P_0 is a category with finite products $\mathbb{T}$ together with a map ι from P_0 and with values in its set of 0-cells $\mathbb{T}_0$, and such that every 0-cell in $\mathbb{T}_0$ is isomorphic to a finite product of 0-cells in $\iota(P_0)$. We denote by $P_1^\times$ the free theory generated by a signature (P_0, P_1) whose products on 0-cells of $P_1^\times$ are induced by products of sorts in P_0^*, and the 1-cells of $P_1^\times$ are *terms* over P_1 defined by induction as follows:

i) the canonical projections $x_i^{\underline{s}} : \underline{s} \to \underline{s}_i$, for $1 \leqslant i \leqslant |\underline{s}|$ are terms, called *variables*,

ii) for any terms $f : \underline{s} \to r$ and $f' : \underline{s} \to r'$ in $P_1^\times$, there exists a unique 1-cell $\langle f, f' \rangle : \underline{s} \to rr'$, called *pairing* of terms f, f', such that $x_1^{rr'}\langle f, f' \rangle = f$ and $x_2^{rr'}\langle f, f' \rangle = f'$,

iii) for every operation $\varphi : \underline{r} \to s$ in P_1, $\underline{s}$ in S_0^* and terms $f_i : \underline{s} \to \underline{r}_i$ in $P_1^\times$ for $1 \leqslant i \leqslant |\underline{r}|$, there is a term $\varphi\langle f_1, \dots, f_{|\underline{r}|} \rangle : \underline{s} \to s$.

We define the *size* of a term f as the minimal number, denoted by $|f|$, of operations used to its definition.

For any 0-cells $\underline{s}, \underline{s}'$ in $P_1^\times$, we denote by $1_{\underline{s}}$ the identity 1-cell on a 0-cell $\underline{s}$, we denote by $\varepsilon_{\underline{s}}$ the *eraser* 1-cell defined as the unique 1-cell from $\underline{s}$ to the terminal 0-cell 0, and we denote by $\delta_{\underline{s}} = \langle 1_{\underline{s}}, 1_{\underline{s}} \rangle : \underline{s} \to \underline{s} \times \underline{s}$ the *duplicator* 1-cell. We denote respectively by $x_{\underline{s}}^{\underline{s}\underline{s}'} : \underline{s}\underline{s}' \to \underline{s}$ (resp. $x_{\underline{s}'}^{\underline{s}\underline{s}'} : \underline{s}\underline{s}' \to \underline{s}'$) the canonical projections. Finally, we denote by $\tau_{\underline{s},\underline{s}'} : \underline{s}\underline{s}' \to \underline{s}'\underline{s}$ the *exchange* 1-cell defined by $\tau_{\underline{s},\underline{s}'} = \langle x_{\underline{s}'}^{\underline{s}\underline{s}'}, x_{\underline{s}}^{\underline{s}\underline{s}'} \rangle$.

2.1.2. Two-dimensional cartesian polygraph. A *cartesian 2-polygraph* is a data (P_0, P_1, P_2) made of

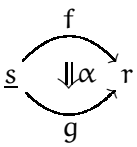
i) a signature (P_0, P_1),

ii) a cellular extension of the free theory $P_1^\times$, that is a set P_2 equipped with two maps

$$P_1^\times \underset{\partial_1^+}{\overset{\partial_1^-}{\leftleftarrows}} P_2$$

satisfying the following *globular conditions* $\partial_0^\mu \circ \partial_1^- = \partial_0^\mu \circ \partial_1^+$, for $\mu \in \{-, +\}$.

An element α of P_2 is called a *rule* with *source* $\partial_-(\alpha)$ and target $\partial_+(\alpha)$ that we denote respectively by α_- and α_+ so that such a rule is denoted by $\alpha : \alpha_- \Rightarrow \alpha_+$. The globular conditions impose that such a rule $f \Rightarrow g$ relates terms of same arity $\underline{s}$ and same coarity r, and it will be pictured as follows:



2.1.3. Two-dimensional theories. Recall that a 2-*dimensional theory*, or 2-*theory* for a given set of sorts P_0 is a 2-category with the additional following cartesian structure:

i) it has a terminal 0-cell, that is for every 0-cell $\underline{s}$ there exists a unique 1-cell $e_{\underline{s}} : \underline{s} \to 1$, called *eraser*, and the identity 2-cell is the unique endo-2-cell on an eraser,

ii) it has products, that is for all 0-cells $\underline{r}, \underline{r}'$ there is a product 0-cell $\underline{rr}'$ and 1-cells $x_{\underline{r}}^{\underline{rr}'} : \underline{rr}' \to \underline{r}$ and $x_{\underline{r}'}^{\underline{rr}'} \to \underline{r}'$ satisfying the two following conditions:

- for any 1-cells $f_1 : \underline{s} \to \underline{r}$ and $f_2 : \underline{s} \to \underline{r}'$, there exists a unique pairing 1-cell $\langle f_1, f_2\rangle : \underline{s} \to \underline{rr'}$, such that $x_{\underline{r}}^{\underline{rr}'}\langle f_1, f_2\rangle = f_1$, and $x_{\underline{r}'}^{\underline{rr}'}\langle f_1, f_2\rangle = f_2$,
- for any 2-cells $\alpha_1 : f_1 \Rightarrow f_1'$, $\alpha_2 : f_2 \Rightarrow f_2'$, there exists a unique 2-cell $\langle \alpha_1, \alpha_2\rangle : \langle f_1, f_2\rangle \Rightarrow \langle f_1', f_2'\rangle$.

We refer the reader to [17] for a detailed construction.

2.1.4. Free 2-theories. We denote by $P_2^\times$ the free 2-theory generated by a cartesian 2-polygraph (P_0, P_1, P_2). We briefly recall its construction and refer the reader to [17] for details. The underlying 1-category of $P_2^\times$ is the free theory $P_1^\times$ generated by the signature (P_0, P_1). Its 2-cells are defined inductively as follows:

i) for any 2-cell $\alpha : u \Rightarrow v$ in P_2 and 1-cell w in $P_1^\times$, there is a 2-cell $\alpha w : u \star_0 w \Rightarrow v \star_0 w$ in $P_2^\times$,

ii) for any 2-cells α, β in $P_2^\times$, there is a 2-cell $\langle \alpha, \beta\rangle : \langle \alpha_-, \beta_-\rangle \Rightarrow \langle \alpha_+, \beta_+\rangle$ in $P_2^\times$,

iii) for any 2-cell α in $P_2^\times$, there are 2-cells in $P_2^\times$ of the form $A[\alpha] : A[\alpha_-] \Rightarrow A[\alpha_+]$ where $A[\square]$ denotes an *algebraic context* of the form:

$$A[\square] := f\langle id_{f_1}, \dots, \square_i, \dots, id_{f_k}\rangle : \underline{s} \to r,$$

where $f_1, \dots, f_k : \underline{s} \to \underline{r_i}$ and $f : \underline{r} \to r$ are 1-cells of $P_1^\times$, and $\square_i$ is the i-th element of the pairing.

iv) these 2-cells are submitted to the following exchange relations

$$f\langle f_1, ..., f_i, ..., \beta, ..., f_k\rangle \star_1 f\langle f_1, \alpha, ..., f_j, ..., f_k\rangle = f\langle f_1, ..., \alpha, ..., f_j, ..., f_k\rangle \star_1 f\langle f_1, ..., f_i, ..., \beta, ..., f_k\rangle$$

where $f_i : \underline{s} \to \underline{r_i}$ and $f : \underline{r} \to r$ are 1-cells in $P_1^\times$, α and β are 2-cells in P_2. We will denote by $\langle f_1, ..., \alpha, ..., \beta, ..., f_k\rangle$ the 2-cell defined above.

v) The $\star_1$-composition of 2-cells in P_2 is given by sequential composition.

The source and target maps $\partial_1^\pm$ extend to $P_2^\times$ and we denote a_- and a_+ for $\partial_1^-(a)$ and $\partial_1^+(a)$.

2.1.5. Ground terms. Let (P_0, P_1, P_2) be a cartesian 2-polygraph. A *ground term* in the free theory $P_1^\times$ is a term with source $\mathbf{0}$. A 2-cell a in the free theory $P_2^\times$ is called *ground* when a_- is a ground term. Finally, an algebraic context $A[\square] = f\langle f_1, \dots, \square_i, \dots f_{|\underline{r}|}\rangle$ is called *ground* when the f_i are ground terms.

2.1.6. Free $(2,1)$-theory. A *free* $(2,1)$*-theory* is a theory $\mathbb{T}$ whose any 2-cell is invertible with respect the $\star_1$-composition. That is, any 2-cell α of $\mathbb{T}_2$ has an inverse $\alpha^- : \alpha_+ \Rightarrow \alpha_-$ satisfying the relations $\alpha \star_1 \alpha^- = 1_{\alpha_-}$ and $\alpha^- \star_1 \alpha = 1_{\alpha_+}$.

We denote by $P_2^\top$ the free $(2,1)$-theory generated by a cartesian 2-polygraph (P_0, P_1, P_2). The 2-cells of the $(2,1)$-theory $P_2^\top$ corresponds to elements of the equivalence relation generated by P_2.

2.1.7. Rewriting properties of cartesian polygraphs. Let P be a cartesian 2-polygraph. The algebraic contexts of the cartesian 2-polygraph P can be composed, and we will denote by $AA'[\Box] := A[A'[\Box]]$. In the same way, one defines a *multi-context* (of arity 2) as

$$B[\Box_i, \Box_j] := f\langle \mathrm{id}_{f_1}, \dots, \Box_i, \dots, \Box_j, \dots, \mathrm{id}_{f_k}\rangle,$$

where the $f_k : \underline{s} \to \underline{r_k}$ and $f : \underline{r} \to r$ are 1-cells in $P_1^\times(X)$, and $\Box_i$ (resp. $\Box_j$) has to be filled by a 1-cell $g_i : \underline{s} \to \underline{r_i}$ (resp. $g_j : \underline{\mathbf{s}} \to \underline{\mathbf{r}_j}$).

A 2-cell of the form $A[\alpha w]$ where A is an algebraic context, w is a 1-cell in $P_1^\times$ and α is a rule in P_2 is called a *rewriting step of* P. A *rewriting path* is a non-identity 2-cell of $P_2^\times$. Such a 2-cell can be decomposed as a $\star_1$-composition of rewriting steps:

$$\alpha = A_1[\alpha_1] \star_1 A_2[\alpha_2] \star_1 \dots A_k[\alpha_k],$$

The *length* of a 2-cell α in $P_1^\times$, denoted by $\ell(f)$, is the minimal number of rewriting steps in any $\star_1$-decomposition of α. In particular, a rewriting step is a 2-cell of length 1.

2.1.8. Notations. For the sake of readability, we will denote terms and rewriting rules of cartesian polygraphs as in term rewriting theory, [26]. The canonical projections $x_i^{\underline{s}} : \underline{s} \to \underline{s_i}$, for $1 \leqslant i \leqslant |\underline{s}|$ are identified to "*variables*" $x_1, \dots, x_{|\underline{s}|}$. And a 1-cell $f : \underline{s} \to r$ is denoted by $f(x_1, \dots, x_{|\underline{s}|})$, and a rule $\alpha : f \Rightarrow g$ with $f, g : \underline{s} \to r$ will be denoted by

$$\alpha_{x_1,\dots,x_{|\underline{s}|}} : f(x_1, \dots, x_{|\underline{s}|}) \Rightarrow g(x_1, \dots, x_{|\underline{s}|}).$$

2.2. Algebraic examples

2.2.1. Associative and commutative magmas. Denote by MAG the cartesian 2-polygraph whose signature has a unique sort denoted by 1 and an unique generating 1-cell $\mu : 2 \to 1$ and an empty set of generating 2-cells. Denote by Ass the cartesian 2-polygraph such that $\mathrm{Ass}_1 = \mathrm{MAG}_1$ and with an unique generating 2-cell:

$$A^\mu_{x,y,z} \ : \ \mu(\mu(x,y),z) \Rightarrow \mu(x,\mu(y,z)) \tag{2.2.2}$$

Denote by AC^μ (or simply AC when there is no ambiguity) the cartesian 2-polygraph such that $\mathrm{AC}_1 = \mathrm{MAG}_1$, and $\mathrm{AC}_2 = \mathrm{Ass}_2 \cup \{C\}$ with

$$C^\mu \ : \ \mu(x,y) \Rightarrow \mu(y,x) \tag{2.2.3}$$

that correspond to the rule $C^\mu : \mu\tau \Rightarrow \mu$, where τ is the exchanging operator defined in Section 2.1.1. Note that the cartesian 2-polygraph AC is not terminating, and that the rule C can not be oriented in a terminating way. As a consequence, in the sequel when P_2 is defined by a set of relations together with relations corresponding to commutativity and associativity axioms for some operation μ, we will chose to work modulo the polygraphs AC^μ.

2.2.4. Monoids. We define the cartesian polygraph Mon whose signature has a unique sort 1, $\mathrm{Mon}_1 = \mathrm{Ass}_1 \cup \{e : 0 \to 1\}$, and $\mathrm{Mon}_2 = \mathrm{Mag}_2 \cup \{E_l^\mu, E_r^\mu\}$ with

$$E_l^\mu \ : \ \mu(e, x) \Rightarrow x \qquad E_r^\mu \ : \ \mu(x, e) \Rightarrow x. \tag{2.2.5}$$

Then the theory $\overline{\mathrm{P}}$ is the theory of monoids that we will denote by $\mathbb{M}$. We also define the cartesian polygraph CMon by $\mathrm{CMon}_i = \mathrm{Mon}_i$ for $0 \leqslant i \leqslant 1$ and $\mathrm{CMon}_2 = \mathrm{Mon}_2 \cup \{C^\mu\}$ where C^μ is the commutativity 2-cell defined in (2.2.3).

2.2.6. Groups. We define the cartesian polygraph Grp whose signature has a unique sort 1, $\mathrm{Grp}_1 = \mathrm{Mon}_1 \cup \{\iota : 1 \to 1\}$, and $\mathrm{Grp}_2 = \mathrm{Mon}_2 \cup \{I_l^\mu, I_r^\mu\}$ with

$$I_l^{\mu,\iota} \ : \ \mu(\iota(x), x) \Rightarrow e \qquad I_r^{\mu,\iota} \ : \ \mu(x, \iota(x)) \Rightarrow e \tag{2.2.7}$$

Note that following [10], the following set of generating 2-cells gives a cartesian polygrah that is Tietze equivalent to Grp (that is it also presents the theory $\overline{\mathrm{Grp}}$) and convergent modulo the cartesian polygraph Ass:

$$G_1^{\mu,\iota} \ : \ \iota(e) \Rightarrow e \quad G_2^{\mu,\iota} \ : \ \iota(\iota(x)) \Rightarrow x \quad G_3^{\mu,\iota} \ : \ \iota(\mu(x, y)) \Rightarrow \mu(\iota(y), \iota(x)) \tag{2.2.8}$$

$$G_4^{\mu,\iota} \ : \ \mu(x, \mu(\iota(x), y)) \Rightarrow y \quad G_5^{\mu,\iota} \ : \ \mu(\iota(x), \mu(x, y)) \Rightarrow y \tag{2.2.9}$$

2.2.10. Abelian groups. Consider the cartesian polygraph Ab whose signature has a unique sort 1, $\mathrm{Ab}_1 = \mathrm{Grp}_1$ and $\mathrm{Ab}_2 = \mathrm{Grp}_2 \cup \{C\}$ where C is the commutativity generating 2-cell defined in (2.2.3).

2.2.11. Rings. Consider the cartesian polygraph Ring whose signature has a unique sort 1, $\mathrm{Ring}_1 = \mathrm{Ab}_1 \coprod \mathrm{Mon}_1$ with the following notations:

$$\mathrm{Ab}_1 \ = \{+ : \mathbf{2} \to \mathbf{1}, \ 0 : \mathbf{0} \to \mathbf{1}, \ - : \mathbf{1} \to \mathbf{1}\}, \qquad \mathrm{Mon}_1 \ = \{\cdot : \mathbf{2} \to \mathbf{1}, \ 1 : \mathbf{0} \to \mathbf{1}\},$$

and $\mathrm{Ring}_2 = \mathrm{Ab}_2 \cup \mathrm{Mon}_2 \cup \{D_l, D_r\}$, where

$$D_l : x \cdot (y + z) \Rightarrow x \cdot y + x \cdot z \qquad D_r : (y + z) \cdot x \Rightarrow y \cdot x + z \cdot x \tag{2.2.12}$$

The cartesian 2-polygraph CRing (commutative rings) is the cartesian 2-polygraph whose signature has a unique sort 1, $\mathrm{CRing} = \mathrm{Ring}_1$ with the same notations as above, and $\mathrm{CRing}_2 = \mathrm{Ring}_2 \cup \{C^\cdot\}$ where $C^\cdot$ is the commutativity generating 2-cell

$$C^\cdot \ : \ \cdot(x, y) \Rightarrow \cdot(y, x) \tag{2.2.13}$$

Following [21, Example 12.2], the following set of generating 2-cells gives a cartesian polygraph that is Tietze equivalent to CRing, and is convergent modulo AC:

$$E_r^+, \ I_r^{+,-}, \ G_1^{+,-}, \ G_2^{+,-}, \ G_3^{+,-}, \ D_r, \ R_1 \ : \ x \cdot 0 \Rightarrow 0, \ R_2 \ : \ x \cdot (-y) \Rightarrow -(x \cdot y), \ E_r^\cdot \tag{2.2.14}$$

2.2.15. Modules over a commutative ring. The cartesian 2-polygraph Mod with $\mathrm{Mod}_0 = \{\mathbf{m}, \mathbf{r}\}$, and $\mathrm{Mod}_1 = \mathrm{CRing}_1 \cup \mathrm{Ab}_1 \cup \{\eta : \mathbf{rm} \to \mathbf{m}\}$ with the following notations

i) $\mathrm{CRing}_0 = \{\mathbf{r}\}$, $\mathrm{CRing}_1 = \{+ : \mathbf{rr} \to \mathbf{r},\ 0 : \mathbf{0} \to \mathbf{r},\ - : \mathbf{r} \to \mathbf{r},\ \cdot : \mathbf{rr} \to \mathbf{r},\ 1 : \mathbf{0} \to \mathbf{r}\}$;

ii) $\mathrm{Ab}_0 = \{\mathbf{m}\}$, $\mathrm{Ab}_1 = \{\oplus : \mathbf{mm} \to \mathbf{m}, 0^{\oplus} : \mathbf{0} \to \mathbf{m},\ \iota : \mathbf{m} \to \mathbf{m}\}$;

iii) If there is no possible confusion, we will denote $\eta(\lambda, x) = \lambda.x$ for λ and x of type $\mathbf{r}$ and $\mathbf{m}$ respectively.

and $\mathrm{Mod}_2 = \mathrm{CRing}_2 \cup \mathrm{Ab}_2 \cup \{M_1, M_2, M_3, M_4\}$ with

$$M_1 : \lambda.(\mu.x) \Rightarrow (\lambda \cdot \mu).x \qquad M_2 : 1.x \Rightarrow x \tag{2.2.16}$$

$$M_3 : \lambda.(x \oplus y) \Rightarrow (\lambda.x) \oplus (\lambda.y) \qquad M_4 : \lambda.x \oplus \mu.x \Rightarrow (\lambda + \mu).x \tag{2.2.17}$$

Following [10], the 2-cells in (2.2.14) together with the following set of 2-cells

$$M_1, \quad M_2, \quad M_3, \quad M_4, \quad N_1 : x \oplus 0^{\oplus} \Rightarrow x, \quad N_2 : x \oplus (\lambda.x) \Rightarrow (1 + \lambda).x, \tag{2.2.18}$$

$$N_3 : x \oplus x \Rightarrow (1 + 1).x, \quad N_4 : x.0^{\oplus} \Rightarrow 0^{\oplus}, \quad N_5 : 0.x \Rightarrow 0^{\oplus}, \quad N_6 : \iota(x) \Rightarrow (-1).x \tag{2.2.19}$$

gives a convergent presentation of the theory of modules over a commutative ring modulo $\mathrm{AC}^{\cdot} \coprod \mathrm{AC}^{+}$, which contains all the associativity and commutativity relations for the operations $\cdot$ and $+$. This presentation can be summarized with the following set of generating 2-cells:

$x + 0 \Rightarrow x$	(ring_1)	$x + (-x) \Rightarrow 0$	(ring_2)
$-0 \Rightarrow 0$	(ring_3)	$-(-x) \Rightarrow x$	(ring_4)
$-(x + y) \Rightarrow (-x) + (-y)$	(ring_5)	$x \cdot (y + z) \Rightarrow x \cdot y + x \cdot z$	(ring_6)
$x \cdot 0 \Rightarrow 0$	(ring_7)	$x \cdot (-y) \Rightarrow -(x \cdot y)$	(ring_8)
$1 \cdot x \Rightarrow x$	(ring_9)	$a \oplus 0^{\oplus} \Rightarrow a$	(mod_1)
$x.(y.a) \Rightarrow (x \cdot y).a$	(mod_2)	$1.a \Rightarrow a$	(mod_3)
$x.a \oplus y.a \Rightarrow (x + y).a$	(mod_4)	$x.(a \oplus b) \Rightarrow (x.a) \oplus (y.b)$	(mod_5)
$a \oplus (r.a) \Rightarrow (1 + r).a$	(mod_6)	$a \oplus a \Rightarrow (1 + 1).a$	(mod_7)
$x.0^{\oplus} \Rightarrow 0^{\oplus}$	(mod_8)	$0.a \Rightarrow 0^{\oplus}$	(mod_9)
$I(a) \Rightarrow (-1).a$	(mod_{10})		

Let us denote by Mod_2' the set containing the 2-cells (2.2.14), (2.2.18) and (2.2.19), and denote by Mod^c the cartesian 2-polygraph $(\mathrm{Mod}_0, \mathrm{Mod}_1, \mathrm{Mod}_2' \cup \mathrm{AC}^{\cdot} \cup \mathrm{AC}^{+})$. It also presents the theory of modules over a commutative ring.

3. Algebraic polygraphs modulo

In this section we introduce the notion of algebraic polygraph as a cellular extension on closed terms. In Subsection 3.2, we introduce the notion of algebraic polygraph modulo and refer the reader to [5] for a categorical interpretation of the constructions given in this section.

3.1. Algebraic polygraphs

3.1.1. Constants. Let (P_0, P_1) be a signature, and Q be a set of *generating* 1*-cell (called constants)* with source $\mathbf{0}$ and target a sort in P_0. We denote by $P_1\langle Q\rangle$ the set of ground terms of the free theory $(P_1 \cup Q)^\times$.

3.1.2. Algebraic polygraph. An *algebraic polygraph* is a data (P, Q, R) where,

i) P is a cartesian 2-polygraph,

ii) Q is a family of set of generating constants $(Q_s)_{s\in P_0}$,

iii) R is a cellular extension of the set of ground terms $P_1\langle Q\rangle$.

Note that the cellular extension R is indexed by the sorts of P_0, that is it defines a family $(F_s, R_s)_{s\in P_0}$ of 1-polygraphs, where $F_s = P_1\langle Q\rangle_s$.

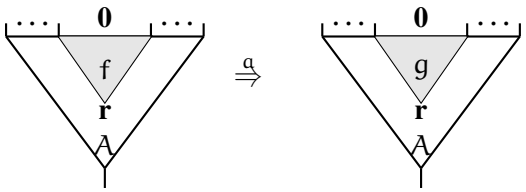
3.1.3. Example. Let Mon_2 be the cartesian 2-polygraph defined in (2.2.4). One defines an algebraic polygraph by setting:

$$Q = \{s, t : \mathbf{0} \to \mathbf{1}\}, \qquad R = \{\ \alpha\ :\ (s \cdot t) \cdot s \Rightarrow t \cdot (s \cdot t)\ \}. \tag{3.1.4}$$

3.1.5. Rewriting in algebraic prolygraphs. Let $\mathcal{P} = (P, Q, R)$ be an algebraic polygraph, and let $\alpha : f \Rightarrow g$ be a ground 2-cell in R. A R-*rewriting step* is a ground 2-cell in the free 2-theory $R^\times$ on $(P_1 \cup Q, R)$ of the form

$$A[\alpha] : A[f] \Rightarrow A[g],$$

where $A[\square]$ is a ground context. It can be depicted by the following diagram:

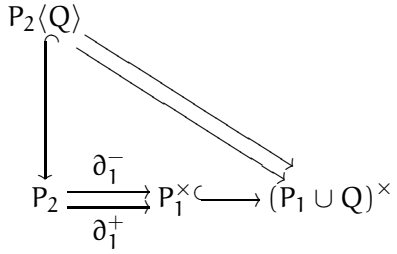


A R-*rewriting path* is a finite or infinite sequence $\underline{a} = a_1 \star_1 a_2 \star_1 \ldots \star_1 a_k \star_1 \ldots$ of R-rewriting steps a_i. The *length* of 2-cell $\underline{a}$ in $R^\times$, denoted by $\ell(a)$, is the minimal number of R-rewriting steps needed to write $\underline{a}$ as a composition as above

3.1.6. Example. Consider the rule α defined in (3.1.4). And the algebraic contexte $A[\square] = (s \cdot \square) \cdot t$, we have the rewriting step

$$A[\alpha] : (s \cdot ((s \cdot t) \cdot s)) \cdot t \Rightarrow (s \cdot (t \cdot (s \cdot t)) \cdot t.$$

3.1.7. Algebraic polygraph of axioms. The cellular extension P_2 defined on $P_1^\times$ extends to a cellular extension on the free 1-theory $(P_1 \cup Q)^\times$ denoted by $\widehat{P}_2$, whose source and target maps are defined in such a way that the following diagram commutes



and denote by $P_2\langle Q\rangle$ (resp $P_2\langle Q\rangle^\top$) the set of ground 2-cells in $\widehat{P}_2^\times$ (resp. $\widehat{P}_2^\top$). The set $P_2\langle Q\rangle$ thus contains the groundified 2-cells of P_2. The data $(P, Q, P_2\langle Q\rangle)$ defines an algebraic polygraph, that we call the *algebraic polygraph of axioms*. We say that two terms f and g in $P_1\langle Q\rangle$ are *algebraically equivalent* with respect to P, denoted by $f \equiv_{P_2} g$, if there exists a ground 2-cell in $P_2\langle Q\rangle^\top$ from f to g.

We will denote by $\overline{P\langle Q\rangle}$ the quotient of the full sub-category $P_1\langle Q\rangle$ of $P_1 \cup Q^\times$ by the congruence generated by the 2-cells in $P_2\langle Q\rangle$. Namely, two terms f and g that are related by a 2-cell in $P_2\langle Q\rangle^\top$ are identified in the quotient.

Note that the algebraic polygraph $(P, Q, P_2\langle Q\rangle)$ shares the rewriting properties of the cartesian 2-polygraph P. In particular, if P is terminating (resp. quasi-terminating, confluent, confluent modulo P'), then $(P, Q, P_2\langle Q\rangle)$ is terminating (resp. quasi-terminating, confluent, confluent modulo $(P', Q, P_2'\langle Q\rangle)$).

3.1.8. Example. In the example of the algebraic polygraph defined in (3.1.4), the set $P_2\langle Q\rangle$ is defined by the associativity relations on ground terms on the constants s and t. For instance, $P_2\langle Q\rangle$ contains the following ground 2-cell:

$$A_{s,t,s} : (s \cdot t) \cdot s \Rightarrow s \cdot (t \cdot s).$$

3.1.9. Positivity. Denote $\pi : P_1\langle Q\rangle \to \overline{P\langle Q\rangle}$ the canonical projection, and let $\sigma : \overline{P\langle Q\rangle} \to \mathsf{Set}$ be a map such that for any $\overline{f} \in \overline{P\langle Q\rangle}$, $\sigma(\overline{f})$ is a chosen non-empty subset of $\pi^{-1}(\overline{f})$. Such a map is called a *positive strategy* with respect to (P, Q). A rewriting step a in $R^\times$ is called σ-*positive* if a_- belongs to $\sigma(\overline{a_-})$. A rewriting path $a_1 \star_1 \ldots \star_1 a_k$ in $R^\times$ is called σ-*positive* if any of its rewriting steps is positive.

3.1.10. Strategies to define positivity. We introduce positivity strategies that depend on the inherent cartesian 2-polygraph P. Suppose that P is such that $P_2 = P_2' \cup P_2''$, with P_2' confluent modulo P_2''. For every 1-cell $\overline{f}$ in $\overline{P\langle Q\rangle}$, we set $\sigma(\overline{f}) = \mathrm{NF}(f, P_2' \bmod P_2'')$, where $f \in \pi^{-1}(\overline{f})$, the set of normal forms of f for P_2' modulo P_2''. Note that this is well-defined following [9, Lemma 2.6], since if $f, f' \in \pi^{-1}(\overline{f})$, then $\mathrm{NF}(f, P_2' \bmod P''_2) \equiv_{P''_2} \mathrm{NF}(f', P_2' \bmod P''_2)$.

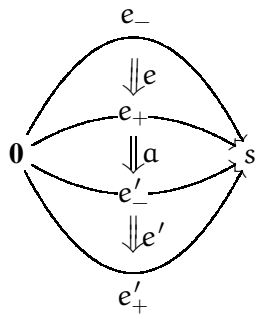
In many algebraic situations, we will set $\mathrm{Ass} \subseteq P_2''$. In particular, in the case of SRS, P_2' will be empty and $P_2'' = \mathrm{Ass}$. In that case, any term in $P_1\langle Q\rangle$ is a normal form for the empty polygraph modulo Ass, and thus the positive strategy consists in taking all the fiber. In the case of LRS, P_2'' will be AC, the algebraic polygraph corresponding to associativity and commutativity relations of the operations, and P_2' will be the convergent presentation of RMod modulo AC given in Section 2.2.15.

3.2. Algebraic polygraphs modulo

3.2.1. Algebraic polygraph modulo. Given an algebraic polygraph $\mathcal{P} = (P, Q, R)$ and a positive strategy σ on $\mathcal{P}$, one denotes by ${}_{P}R_{P}$ the cellular extension

$$P_1\langle Q\rangle \Longleftarrow {}_{P}R_{P}$$

made of triple (e, α, e'), where e and e' are ground 2-cells in $P_2\langle Q\rangle^{\top}$ and α is a R-rewriting step. Such a triple will be denoted by $e \star \alpha \star e'$, called a ${}_{P}R_{P}$-*rule*, and pictured by



We refer the reader to [5] for a detailed construction of the cellular extension ${}_{P}R_{P}$. Such a rule is called σ-positive if α is a σ-positive R-rewriting step. An *algebraic polygraph modulo* is a data (P, Q, R, S) made of

i) an algebraic polygraph (P, Q, R),

ii) a cellular extension S of $P_1\langle Q\rangle$ such that $R \subseteq S \subseteq {}_{P}R_{P}$.

Note that the data (P, Q, S) defines an algebraic polygraph modulo.

3.2.2. Example. Let us consider the algebraic polygraph (P, Q, R) defined in (3.1.4), then the following composition gives a rewriting step in ${}_{P}R_{P}$:

$$(s\cdot(s\cdot(t\cdot s)))\cdot t \equiv_{P_2} (s\cdot((s\cdot t)\cdot s))\cdot t \overset{A[\alpha]}{\Rightarrow} (s\cdot(t\cdot(s\cdot t))\cdot t \equiv_{P_2} ((s\cdot t)\cdot(s\cdot t))\cdot t.$$

3.2.3. Termination properties. An algebraic polygraph $\mathcal{P} = (P, Q, R)$ is called

i) *terminating* if there is no infinite rewriting sequence for $\mathcal{P}$, that is there is no sequence $(f_n)_{n\in\mathbb{N}}$ of 1-cells of $P_1\langle Q\rangle$ such that for each $n \in \mathbb{N}$, there is a rewriting step $f_n \to f_{n+1}$,

ii) *quasi-terminating* if for each sequence $(f_n)_{n\in\mathbb{N}}$ of 1-cells of $P_1\langle Q\rangle$ such that for each $n \in \mathbb{N}$, there is a rewriting step $f_n \to f_{n+1}$, the sequence $(f_n)_{n\in\mathbb{N}}$ contains an infinite number of occurrences of same 1-cell,

iii) *algebraically terminating* if for each sequence $(f_n)_{n\in\mathbb{N}}$ of 1-cells of $P_1\langle Q\rangle$ such that for each $n \in \mathbb{N}$, there is a rewriting step $f_n \to f_{n+1}$, the sequence $(f_n)_{n\in\mathbb{N}}$ contains an infinite number of occurrences of same 1-cell in context, that is, there exist $k, l \in \mathbb{N}$, such that $f_{k+l} = A[f_k]$ where A is a possibly empty ground context of $\mathcal{P}$,

iv) *exponentiation free* if there is no rewriting path with source a 1-cell f of $P_1\langle Q\rangle$ and target $C[f]$, where A is a nontrivial ground context of $\mathcal{P}$.

Any quasi-terminating polygraph is algebraically terminating. But the converse implication is false in general, indeed the rewriting system $a \to a \cdot a$ is algebraically terminating, but not quasi-terminating. In fact, it is not exponentiation free either. One proves that both properties algebraically terminating and exponentiation free implies the quasi-terminating property.

An algebraic polygraph modulo (P, Q, R, S) is called *terminating* (resp. *quasi-terminating*) if the algebraic polygraph (P, Q, S) is terminating (resp. quasi-terminating). Note that an algebraic polygraph is a special case of algebraic polygraph modulo when $S = R$. In the sequel we will consider only polygraphs modulo.

3.2.4. Quasi-normal forms. Let $\mathcal{P} = (P, Q, R, S)$ be an algebraic polygraph modulo. A 1-cell f of $P_1\langle Q\rangle$ is called a *quasi-irreducible* if for any S-rewriting step $f \to g$, there exists a S-rewriting sequence from g to f. A *quasi-normal form (with respect to $\mathcal{P}$)* of a 1-cell f in $P_1\langle Q\rangle$ is a quasi-irreducible 1-cell $\widetilde{f}$ of $P_1\langle Q\rangle$ such that there exists a S-rewriting sequence from f to $\widetilde{f}$.

When the algebraic polygraph modulo $\mathcal{P}$ is quasi-terminating, any 1-cell f of $P_1\langle Q\rangle$ admits at least a quasi-normal. Such a quasi-normal is neither S-irreducible nor unique in general. A *quasi-normal form strategy* is a map

$$s : P_1\langle Q\rangle \to P_1\langle Q\rangle$$

sending a 1-cell f on a chosen quasi-normal $\widetilde{f}$. We define a map

$$d : P_1\langle Q\rangle \to \mathbb{N}$$

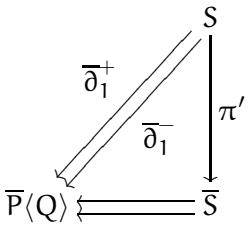
sending a 1-cell f to

$$d(f) = \min\{\, l(\underline{a}) \mid \underline{a} \text{ is a } {}_PR_P\text{-rewriting path from } f \text{ to } \widetilde{f}\,\},$$

counting the distance from f to $\widetilde{f}$.

3.3. Algebraic rewriting systems

3.3.1. Algebraic rewriting system. Note that the cellular extension S defined on $P_1\langle Q\rangle$ extends to a cellular extension of $\overline{P}\langle Q\rangle$, with source and target maps defined respectively by $\overline{\partial_1^-} := \pi \circ \partial_1^-$ and $\overline{\partial_1^+} := \pi \circ \partial_1^+$. An *algebraic rewriting system* on an algebraic polygraph modulo (P, Q, R, S) with a positive strategy σ is a cellular extension $\overline{S}$ of $\overline{P}\langle Q\rangle$ defined in such a way that the following diagram commutes



where the map π' assigns to a S-rule $e \star a \star e'$ an element $\overline{a}$ in $\overline{S}$ with source $\overline{a_-}$ and target $\overline{a_+}$. Explicitly,

$$\overline{S} = \{\overline{a} : \overline{a_-} \Rightarrow \overline{a_+} \mid e \star a \star e' \in S\}.$$

Note that $\overline{S} = \overline{R}$ for any $R \subseteq S \subseteq {}_PR_P$. Let us consider the subset $\overline{S}^\sigma$ of $\overline{S}$ defined by $\overline{S}^\sigma = \{\overline{a} : \overline{a_-} \Rightarrow \overline{a_+} \mid a \text{ is a } \sigma\text{-positive S-rule}\}$.

A $\overline{S}$-*rewriting step* (resp. a $\overline{S}^\sigma$-rewriting step) is the quotient of a S-rewriting step (resp. σ-positive rewriting step) by the canonical projection π, that is a 2-cell of the form $\overline{C}[\overline{a}] : \overline{C}[\overline{a_-}] \Rightarrow \overline{C}[\overline{a_+}]$, where C is a ground context of $P_1\langle Q\rangle$ and $C[a]$ is a S-rewriting step (resp. σ-positive S-rewriting step). A S-*rewriting path* is a sequence of $\overline{S}$-rewriting steps.

3.3.2. Example: string rewriting systems. A SRS can be deduced as a quotient algebraic polygraph as follows. We consider an algebraic polygraph (Mon, Q, R, S), where Mon is the cartesian polygraph defined in 2.2.4. The set of constants Q is the set of generating 1-cells of the SRS, and R corresponds to fibrations of rules of the SRS on the fibers modulo associativity.

For instance, consider the algebraic polygraph defined in (3.1.4). Then by quotient, we obtain the string rewriting system

$$\langle s, t \mid sts \Rightarrow tst \rangle$$

that presents the monoid B_3^+ of braids on 3 strands.

3.3.3. Example: linear rewriting systems. A linear rewriting system (LRS) is an algebraic rewriting system on an algebraic polygraph modulo (P, Q, R, S) such that $\text{Mod}^c \subseteq P$, where Mod^c is the cartesian 2-polygraph presenting the theory of modules over a commutative ring defined in Section 2.2.15.

4. Confluence in algebraic polygraphs modulo

In this section we present confluence properties of algebraic polygraphs modulo with fixed positive strategies.

4.1. Confluence modulo with respect to a positive strategy

4.1.1. Branchings in algebraic polygraphs modulo. Let $\mathcal{P} = (P, Q, R, S)$ be an algebraic polygraph modulo and σ a positive strategy on $\mathcal{P}$. A σ-*branching* of (P, Q, R, S) is a triple (a, e, b) where f and g are σ-positive 2-cells of $S^\times$ and e is a 1-cell of $P_2\langle Q\rangle^\top$ such that $e_- = a_-$ and $e_+ = b_-$. Such a σ-branching is depicted as follows

$$\begin{array}{ccc} u & \xrightarrow{a} & u' \\ {\scriptstyle e}\downarrow & & \\ v & \xrightarrow[b]{} & v' \end{array} .$$

Note that the 2-cells are represented by simple arrows in confluence diagrams for better readability in the diagrams in the sequel. The 2-cell b (resp. a) can be an identity 2-cell of $S^\times$, and in that case the σ-branching is of the form (a, e) (resp. (e, b)). The source of such a σ-branching is the pair (f, f) where $f = a_- = e_-$ (resp. $f = b_- = e_+$). The 2-cell e in $P_2\langle Q\rangle^\top$ can also be trivial, and in that

case the σ-branching modulo is a regular σ-branching (a, b). We denote by (u, u) its source, where $u = a_- = b_-$.

Such a σ-branching is σ-*confluent modulo* if there exist σ-positive 2-cells a' and b' in $S^\times$ and a 2-cell e' of $P_2\langle Q\rangle^\top$ as follows:

$$\begin{array}{ccccc} f & \xrightarrow{a} & f' & \overset{a'}{\dashrightarrow} & h \\ {\scriptstyle e}\downarrow & & & & \downarrow{\scriptstyle e'} \\ g & \xrightarrow[b]{} & g' & \underset{b'}{\dashrightarrow} & h' \end{array}$$

We say that the triple (a', e', b') is *a* σ-*confluence modulo* of the σ-branching modulo (a, e, b), and that the pair of terms (f, g) is the *source* of the σ-branching (a, e, b). Such a σ-branching is *local* if a is a rewriting step of S, b is $\ell(e) + \ell(b) = 1$. Namely, it is either of the form (a, e) or (a, b).

We say that the algebraic polygraph modulo (P, Q, R, S) is *confluent modulo* (resp. *locally confluent modulo*) if any σ-branching modulo (resp. local σ-branching modulo) is confluent modulo.

4.1.2. Remark. As noted in [2, Section 2], the algebraic polygraph R is the polygraph for which is the most difficult to reach σ-confluence modulo. Indeed, if R is confluent modulo P, then any algebraic polygraph modulo (P, Q, R, S) is confluent modulo P. For this reason, in many situation we relax by proving σ-confluence of ${}_PR$ or ${}_PR_P$ modulo P. They also noticed that when ${}_PR_P$ is terminating, R_P is confluent modulo P if and only if ${}_PR_P$ is confluent modulo P, and in that case R_P defines the same set of normal forms than ${}_PR_P$. As a consequence, we will either prove σ-confluence of R_P and ${}_PR_P$ in the sequel, leading to the same quotient algebraic rewriting system. Note finally that when ${}_PR \subseteq S \subseteq {}_PR_P$, any local σ-branching modulo of the form (a, e) is trivially confluent modulo P via the σ-confluence modulo $(1_{a_-}, e^- \star_1 a, 1_{a_+})$.

4.1.3. Double induction on the distance to the quasi-normal form. Consider the distance map $d : P_1\langle Q\rangle \to \mathbb{N}$ defined in Section 3.2.4. We extend this distance on 1-cells of $P_1\langle Q\rangle$ to a distance on σ-branchings modulo (a, e, b) by defining

$$d(a, e, b) := d(a_-) + d(a_+).$$

We then define a well-founded order $\prec$ on the set of σ-branchings of S modulo P by:

$$(a, e, b) \prec (a', e', b') \text{ if } d(a, e, b) < d(a', e', b').$$

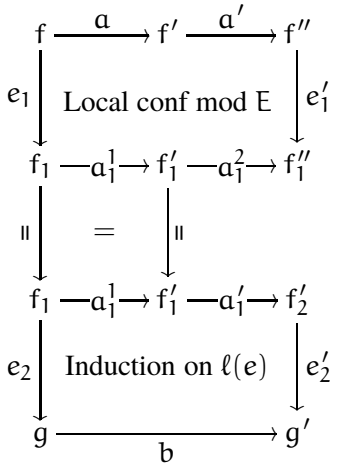
The confluence proofs in the sequel will be made using induction on this order. Note that this corresponds to a process of induction on sources of σ-branchings modulo, that is pairs of 1-cells in $P_1\langle Q\rangle$, with respect to distance of the quasi-normal form with respect to ${}_PR_P$. This follows Huet's double induction principle in the terminating setting, based on induction on an auxiliary rewriting system constructed on pairs of terms.

In this way, one proves for quasi-terminating algebraic polygraphs modulo the Newman lemma, following Huet's proof in the terminating setting.

4.1.4. Theorem (Newman lemma modulo for algebraic polygraphs modulo). *Let $\mathcal{P}$ be a quasi-terminating algebraic polygraph modulo, and σ be a positive strategy on $\mathcal{P}$. If $\mathcal{P}$ is locally* σ*-confluent modulo, then it is* σ*-confluent modulo.*

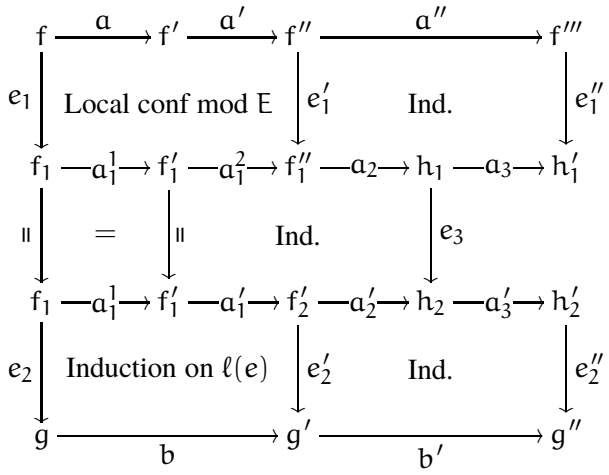
Proof. Assume that $\mathcal{P}$ is locally σ-confluent modulo P. We prove this result by induction on the well-founded order $\prec$ on σ-branchings modulo defined in 4.1.3. Let us pick a σ-branching modulo (a, e, b) of $\mathcal{P}$, and assume that for any σ-branching modulo (a', e', b') such that $(a, e, b) \prec (a', e', b')$, the σ-branching modulo (a', e', b') is confluent modulo P. Let us prove this results in two steps.

Step 1: We prove that any σ-branching modulo (a, e), where a is a S-rewriting step and e is a 2-cell in $P_2\langle Q\rangle^\top$ is σ-confluent modulo. Let us denote by u the 1-cell a_-, so that (a, e) is a σ-branching modulo of source (f, f). If f is S-irreducible, then a is an identity 2-cell, and the σ-branching is trivially confluent. We then suppose that a is not an identity, and proceed by induction on $\ell(e) \geqslant 1$. If $\ell(e) = 1$, (a, e) is a local σ-branching of S modulo P and it is confluent modulo by local σ-confluence. Now, let us assume that for $k \geqslant 1$, any σ-branching (a'', e'') of S modulo P such that $\ell(e'') = k$ is confluent modulo E, and let us consider a σ-branching (a, e) of S modulo P such that $\ell(e) = k + 1$. We choose a decomposition $e = e_1 \star e_2$ with e_1 of length 1. Using local σ-confluence on the σ-branching (a, e_1), there exists a σ-confluence modulo (a', e'_1, a_1) modulo P of this σ-branching. Then, we choose a decomposition $a_1 = a_1^1 \star a_1^2$ with a_1^1 of length 1. By induction hypothesis on the σ-branching modulo (a_1^1, e_2), there exists a σ-confluence modulo (a'_1, e'_2, b) of this σ-branching, as in the following diagram:



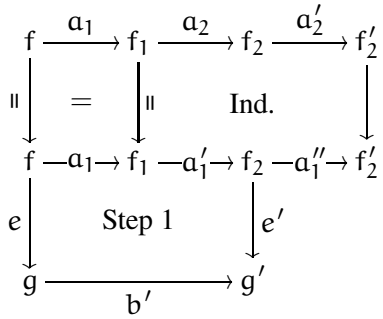
Now, since $f \equiv f_1 \overset{a_1^1}{\Rightarrow} f'_1$, we have $d(f'_1) = d(f) - 1$, and thus $(a_1^2, a'_1) \prec (a, e)$ so that we can use induction on the σ-branching (a_1^2, a'_1) of S modulo P of source (f'_1, f'_1) to prove that there exists a σ-confluence modulo (a_2, e_3, a'_2) of this σ-branching. By a similar argument, we have $d(f''_1) < d(f)$ and $d(f'_2) < d(f)$ and we can apply induction on the σ-branchings modulo $(a_2, (e'_1)^-)$ and (a'_2, e'_2) respectively. Therefore, there exist 2-cells a'',a_3, a'_3 and b' in $S^\times$ and 2-cells e''_1 and e''_2 in $P_2\langle Q\rangle^\top$ as in

the following diagram:



We can then use once again induction on the σ-branching (a_3, e_3, a_3') of S modulo P of source (h_1, h_2), and so on. Since the order $\prec$ is well-founded, this process terminates in finitely many steps until we reach the quasi-normal form $\widetilde{f}$. This yields the σ-confluence of the σ-branching (a, e).

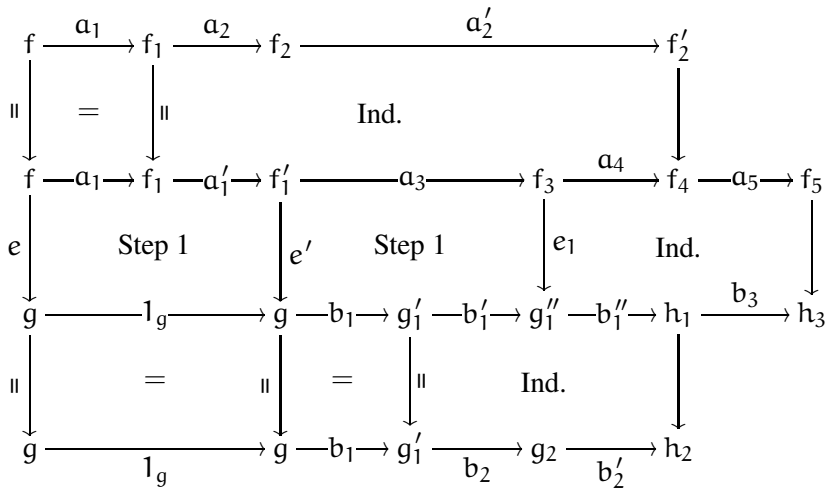
Step 2: Now, let us prove that any σ-branching modulo (a, e, b) is confluent modulo P. Let us choose such a σ-branching and denote by (f, g) its source. We assume that any σ-branching (a', e', b') of S modulo P with $(a', e', b') \prec (a, e, b)$ is confluent modulo P. We follow the proof scheme used by Huet in [9, Lemma 2.7]. Let us denote by $n := \ell(a)$ and $m := \ell(b)$. We assume without loss of generality that $n > 0$ and we fix a decomposition $a = a_1 \star_1 a_2$ with a_1 of length 1. If $m = 0$, by Step 1 on the σ-branching (a_1, e) of S modulo P, there exists a σ-confluence modulo (a_1', e', b') of this σ-branching. Then, since $d(f_1) = d(f) - 1$, we have $(a_2, a_1') \prec (a, e)$ and we can apply double induction on the σ-branching modulo (a_2, a_1'), as pictured in the following diagram:



We finish the proof of this case with a similar argument as in Step 1, using repeated inductions that terminate after a finite number of steps because the order $\prec$ is well-founded.

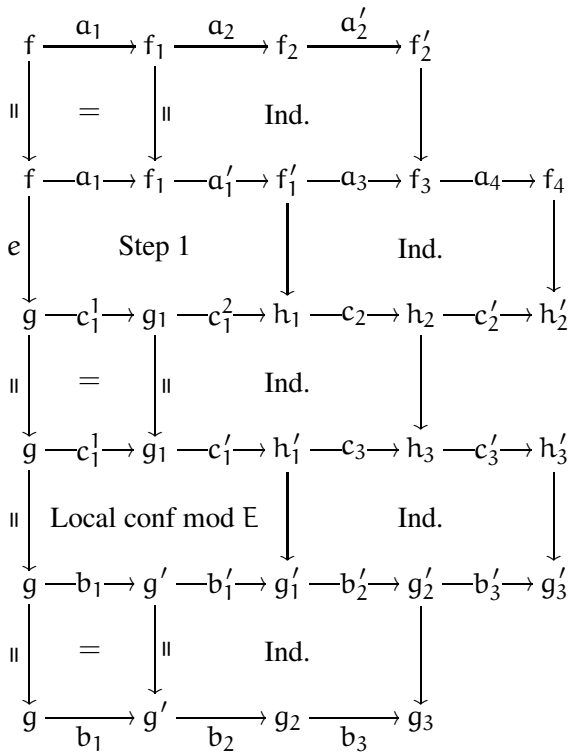
Now, assume that $m > 0$ and fix a decomposition $b = b_1 \star_1 b_2$ of b with b_1 of length 1. Using Step 1 on the σ-branching modulo (a_1, e), there exists a σ-confluence modulo (a_1', e_1, c_1) of this σ-branching. We distinguish two cases whether c_1 is trivial or not.

If c_1 is trivial, the σ-confluence of the σ-branching modulo (a, e, b) is given by the following diagram



where the σ-branchings modulo (a_1, e) and (b_1, e') are confluent modulo P by Step 1, and induction applies on the σ-branchings $(a_2, a'_1 \star_1 a_3)$, (b'_1, b_2) and (a_4, e_1, b''_1) since $d(f_1) < d(f)$, $d(f_1) < d(g)$, $d(g'_1) < d(g)$, $d(g'_1) < d(f)$ and $d(f_3) < d(f)$ respectively. We then reach a σ-confluence modulo of the σ-branching modulo (a, e, b) similarly.

If c_1 is not trivial, let us fix a decomposition $c_1 = c^1_1 \star_1 c^2_1$ with c^1_1 of length 1. The σ-confluence of the σ-branching modulo (a, e, b) is given by the following diagram:



where the σ-branching modulo (a_1, e) is confluent modulo by Step 1, the σ-branching modulo (c_1^1, b_1) is σ-confluent by local σ-confluence modulo, and one checks that induction applies on the σ-branchings (a_2, a_1'), (c_1^2, c_1'), (b_1', b_2), (a_3, c_2) and (c_3, b_2'). Similarly, we can repeat inductions to reach a σ-confluence modulo of (a, e, b). □

4.2. Critical σ-branchings modulo

4.2.1. Classification of local σ-branchings. The local σ-branchings modulo of $\mathcal{P}$ can be classified in the following families:

i) *trivial* σ-branchings of the form

$$\begin{array}{ccc} A[a_-] & \xrightarrow{A[a]} & A[a_+] \\ \| \downarrow & & \\ A[a_-] & \xrightarrow[A[a]]{} & A[a_+] \end{array}$$

for some ground context context A and σ-positive S-rewriting step a.

ii) *inclusion independant* σ-branchings modulo of the form

$$\begin{array}{ccc} A[a_-] & \xrightarrow{A[a]} & A[a_+] \\ \| \downarrow & & \\ A[A'[b_-]] & \xrightarrow[A[A'[b]]]{} & A[A'[b_+]] \end{array}$$

for some ground contexts context A and A', and σ-positive S-rewriting steps a and b.

iii) *orthogonal* σ-branchings modulo of the form

$$\begin{array}{ccc} B[a_-, b_-] & \xrightarrow{B[a, b_-]} & B[a_+, b_-] \\ \| \downarrow & & \\ B[a_-, b_-] & \xrightarrow[B[a_-, b]]{} & B[a_-, b_+] \end{array}$$

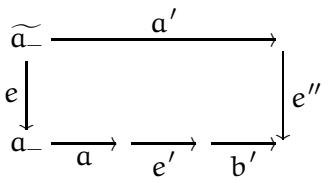
$$\begin{array}{ccc} B[a_-, e_-] & \xrightarrow{B[a, e_-]} & B[a_+, e_-] \\ B[a_-, e] \downarrow & & \\ B[a_-, e_+] & & \end{array} \qquad \begin{array}{ccc} B[e'_-, b_-] & \xrightarrow{B'[e'_-, b]} & B'[e'_-, b_+] \\ B'[e', b_-] \downarrow & & \\ B'[e'_+, b_-] & & \end{array}$$

for some ground multi-contexts B and B' of arity 2, S-rewriting steps a,b and c of S, and 2-cells e and e' in $P_2\langle Q\rangle^\top$.

iv) *non orthogonal* σ-branchings are the remaining local σ-branchings, that is nor inclusion independant nor orthogonal.

4.2.2. Critical σ-branchings. We define an order relation on σ-branchings modulo of an algebraic polygraph modulo (P, Q, R, S) by setting $(a, e, b) \sqsubseteq (a', e', b')$ if there exists a ground context A of $P_1\langle Q\rangle$ such that $a' = A[a]$, $e' = A[e]$ and $b' = A[b]$. A *critical σ-branching modulo* is a local σ-branching modulo P which is non trivial, non orthogonal and minimal for the order relation $\sqsubseteq$.

4.2.3. Positively confluence. An algebraic polygraph modulo (P, Q, R, S) with a positive strategy σ is called *positively σ-confluent* if, for any S-rewriting step f, there exists a representing $\widetilde{a_-} \in \sigma(a_-)$ of a_- and two σ-positive Sreductions a' and b' of length at most 1 as in the following diagram



4.2.4. Proposition (Terminating critical branching theorem modulo). *Let (P, Q, R, S) be a quasi-terminating and positively σ-confluent algebraic polygraph modulo with a positive strategy σ. Then it is locally σ-confluent modulo if and only if the two following properties hold:*

a_0) *any critical σ-branching modulo (a, b), where a and b are S-rewriting steps, is σ-confluent modulo.*

b_0) *any critical σ-branching modulo (a, e), where a is an S-rewriting step and e is a 2-cell in $P_2\langle Q\rangle^\top$ of length 1, is σ-confluent modulo.*

Proof. The left to right implication is trivial. Let us prove the converse. Suppose that condition **a_0)** holds and prove condition **a)**. The proof of the other implication is similar. We prove this by examine all the possible cases of local σ-branchings modulo given in Section **??**. Local aspherical σ-branchings are always σ-confluent modulo. Let us consider a local orthogonal σ-branching modulo of the form

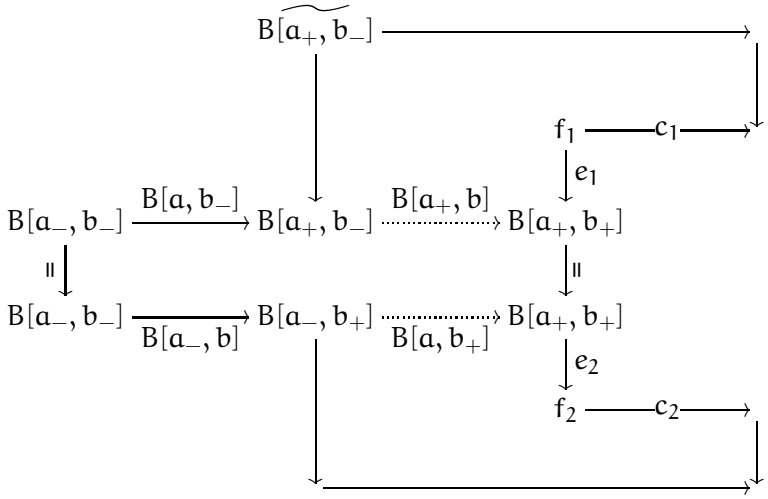
$$\begin{array}{ccc} B[a_-, b_-] & \xrightarrow{B[a, b_-]} & B[a_+, b_-] \\ \| \downarrow & & \\ B[a_-, b_-] & \xrightarrow[B[a_-, b]]{} & B[a_-, b_+] \end{array}$$

where $B[a, b_-]$ and $B[a_-, b]$ are σ-positive S-reductions. There are natural 2-cells in $S^\times$ that give a σ-confluence modulo of this diagram:

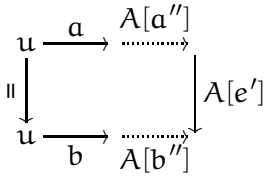
$$\begin{array}{ccccc} B[a_-, b_-] & \xrightarrow{B[a, b_-]} & B[a_+, b_-] & \overset{B[a_+, b]}{\cdots\cdots\rightarrow} & B[a_+, b_+] \\ \| \downarrow & & & & \downarrow \| \\ B[a_-, b_-] & \xrightarrow[B[a_-, b]]{} & B[a_-, b_+] & \overset{B[a, b_+]}{\cdots\cdots\rightarrow} & B[a_+, b_+] \end{array}$$

However, it may happen that these reductions are not σ-positive. Without loss of generality, let us assume that they are both not σ-positive. By positive σ-confluence assumption, there exists a representative

$\widetilde{B[a_+, b_-]}$ (resp. $\widetilde{B[a_-, b_+]}$) of $B[a_+, b_-]$ (resp. $B[a_-, b_+]$) in $P_1\langle Q\rangle$, σ-positive S-rewriting sequences h_1 and h_2, and 2-cells e_1,e_2 in $P_2\langle Q\rangle^\top$ as in the following diagram:



Then, we have $d(f_1) < d(B[a_-, b_-])$ and $d(f_2) < d(B[a_-, b_-])$ so that we can use induction of the σ-branching modulo $(c_1, e_1 \star e_2, c_2)$ of source (f_1, f_2). As a consequence, there exists a σ-confluence modulo (c_1', e, c_2') of this σ-branching modulo, and we then construct a σ-confluence modulo of $(B[a, b_-], B[a_-, b])$ by successive applications of induction as in the proof of Theorem 4.1.4. This process terminates since ${}_pR_P$ is quasi-terminating, and thus the order $\prec$ on σ-branchings modulo defined in Section 4.1.3 is well-founded. Let us now consider an overlapping σ-branching modulo of the form (a, b) where a and b are σ-positive S-rewriting steps. By definition, there exists a ground context A of $P_1\langle Q\rangle$ and a critical σ-branching modulo (a', b') such that $(a, b) = (A[a'], A[b'])$. Following condition $\mathbf{a_0}$), the critical σ-branching (a', b') is σ-confluent modulo, and there exists a σ-confluence modulo (a'', e', b'') of this σ-branching. However, the reductions $A[a'']$ and $A[b'']$ that would give a confluence modulo of (a, b) are not necessarily σ-positive:



However, using positive σ-confluence of S, we are able to construct a σ-confluence modulo of the σ-branching modulo (a, b) as in the previous case. □

4.2.5. Full positive strategy. When all reductions are positive, that is when $\sigma(\overline{f}) = \pi^{-1}(\overline{f})$ for any 1-cell $\overline{f}$, we say that σ is a *full positive strategy*. In that case, the quasi-termination assumption in Proposition 4.2.4 is not needed, since the natural confluences represented by dotted arrows are σ-positive. Moreover, the positive σ-confluence is always satisfied, by considering $a' = a$ and $b' = 1_{t_1(a)}$.

5. Algebraic critical branching lemma

By taking the quotient of the S-rewriting paths in Proposition 4.2.4, in this section we obtain an algebraic critical branching lemma, that we apply to string rewriting systems and linear rewriting systems.

5.1. Algebraic critical branchings

5.1.1. Critical branchings of algebraic polygraph. Let $\mathcal{P} = (P, Q, R, S)$ be an algebraic polygraph modulo with a positive strategy σ and let $\mathcal{A}$ be an algebraic rewriting system on $\mathcal{P}$. The *critical branchings* of $\mathcal{A}$ are the projections of the critical σ-branchings modulo of $\mathcal{P}$ of the form $\mathbf{a}_0$), that is pairs $(\overline{a}, \overline{b})$ of $\overline{S}^{\sigma}$-rewriting steps such that there is a σ-branching modulo in $\mathcal{P}$ with source $(\widetilde{a_-}, \widetilde{b_-})$. As a consequence of Proposition 4.2.4, we deduce the following result.

5.1.2. Theorem. *Let $\mathcal{P} = (P, Q, R, S)$ be an algebraic polygraph modulo with a positive strategy σ such that ${}_PR_P$ is quasi-terminating and positively σ-confluent. An algebraic rewriting system on $\mathcal{P}$ is locally confluent if and only if its critical branchings are confluent.*

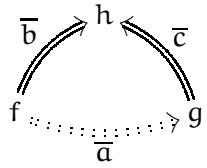
As an immediate consequence, we deduce the following usual critical branching lemma.

5.1.3. Corollary. *Let $\mathcal{P}$ be an algebraic polygraph modulo with a full positive strategy. Any algebraic rewriting system on $\mathcal{P}$ is locally confluent if and only if all its critical branchings are confluent.*

5.2. Examples

5.2.1. Critical branching lemma for string rewriting systems. When Mon is the cartesian 2-polygraph presenting the theory $\mathbb{M}$ of monoids given in (2.2.4), Theorem 5.1.2 corresponds to critical branching lemma for string rewriting systems as proved by Nivat, [20]. In that case, the choice of positive strategy σ making all the 2-cells in $S^{\times}$ be σ-positive implies that we do not need the additional quasi-termination and positive σ-confluence property, as explained in Remark 4.2.5.

5.2.2. Critical branching lemma for linear rewriting systems. Suppose that P contains the cartesian 2-polygraph Mod^c presenting the theory of modules over a commutative ring defined in Section 2.2.15. If P_2'' is the 2-polygraph $AC^+ \cup AC^{\cdot}$, and P_2' is Mod^c, then Theorem 5.1.2 corresponds to the critical branching lemma for linear rewriting systems proved in [6, Theorem 4.3.2]. Indeed, given an algebraic polygraph modulo (P, Q, R, S) with the σ-strategy of normal forms modulo AC defined in 3.1.10, the positivity confluence of S with respect to σ implies the factorization property of [6, Lemma 3.1.3], stating that any rewriting step $\overline{a}$ of $\overline{S}$ can be decomposed as $\overline{a} = \overline{b} \star \overline{c}^{-1}$ where $\overline{b}$ and $\overline{c}$ are either rewriting steps of $\overline{S}^{\sigma}$ or identities, as pictured in the following diagram:



Note that if $\overline{\alpha}$ is already a rewriting step of $\overline{S}^{\sigma}$, this factorization is trivial. When $\overline{\alpha}$ is in $\overline{S}$ but not in $\overline{S}^{\sigma}$, that is $\overline{\alpha}$ is a quotient of a non-σ-positive S-rewriting sequence, it states that $\overline{\alpha}$ can be factorized using positive reductions.

Note that in that case, ${}_{P}R_{P}$ can never be terminating: indeed, because of the linear context, for any R-rule $\alpha : f \Rightarrow g$, we have a ${}_{P}R_{P}$-rewriting step given by

$$g \equiv_{P} -f + (g + f) \xRightarrow{-\alpha + (g + f)} -g + (g + f) \equiv_{P} f \tag{5.2.3}$$

However, the quasi-termination assumption of ${}_{P}R_{P}$ is equivalent to the termination assumption of $\overline{S}^{\sigma}$ given in [6, Theorem 4.3.2]. Indeed, by definition an infinite rewriting path in $\overline{S}^{\sigma}$ comes from an infinite ${}_{P}R_{P}$-rewriting path that is not created by a cycle of the form (5.2.3), since the rule $-\alpha + (g + f)$ above is not σ-positive.

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