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Heavy range of the randomly biased walk on Galton-Watson trees in the slow movement regime

Xinxin Chen*

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Abstract

We consider the randomly biased random walk on trees in the slow movement regime as in [HS16], whose potential is given by a branching random walk in the boundary case. We study the heavy range up to the n -th return to the root, i.e., the number of edges visited more than k_n times. For $k_n = n^\theta$ with $\theta \in (0, 1)$, we obtain the convergence in probability of the rescaled heavy range, which improves one result of [AD20].

MSC: 60K37, 60J80, 60G50

Keywords: randomly biased random walk, branching random walk, Seneta-Heyde norming.

1 Introduction

Let \mathbb{T} be a supercritical Galton-Watson tree rooted at ρ . And to any vertex $x \in \mathbb{T} \setminus \{\rho\}$, we assign a random bias $A_x \geq 0$. For any vertex $x \in \mathbb{T}$, denote its parent by x^* and denote its children by x^1, x^2, \dots, x^{N_x} where N_x denotes the number of its children which could be 0 if there is none. Now for given $\mathcal{E} := \{\mathbb{T}, (A_x)_{x \in \mathbb{T} \setminus \{\rho\}}\}$, let $(X_n)_{n \geq 0}$ be a nearest-neighbour random walk on \mathbb{T} , started from $X_0 = \rho$, with the biased transition probabilities: for any $x, y \in \mathbb{T}$,

$$p^{\mathcal{E}}(x, y) = \begin{cases} \frac{A_{x^j}}{1 + \sum_{i=1}^{N_x} A_{x^i}}, & \text{if } y = x^j \text{ for some } j \in \{1, \dots, N_x\} \\ \frac{1}{1 + \sum_{i=1}^{N_x} A_{x^i}}; & \text{if } y = x^*. \end{cases} \quad (1.1)$$

For convenience, to the root ρ , we add artificially a vertex ρ^* to be its parent and let (1.1) holds also for $x = \rho$ with $p^{\mathcal{E}}(\rho^*, \rho) = 1$. Obviously, this is a random walk in random environment. In particular, when A_x equals some constant $\lambda > 0$ for any x , this is known as λ -biased random walk on Galton-Watson tree, which was introduced and deeply studied by Lyons [Lyo90, Lyo92] and Lyons, Pemantle and Peres [LPP95, LPP96].

In our setting, we assume that $\{A_{x^1}, \dots, A_{x^{N_x}}\}$, $x \in \mathbb{T}$ are i.i.d. copies of the point process $\mathcal{A} = \{A_1, \dots, A_N\}$ where $N \in \mathbb{N}$ represents the offspring of the Galton-Watson tree \mathbb{T} . Let \mathbf{P} denote the

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probability measure of the environment \mathcal{E} . Given the environment \mathcal{E} , denote the quenched probability by $\mathbb{P}^{\mathcal{E}}$. Then $\mathbb{P}(\cdot) := \int \mathbb{P}^{\mathcal{E}}(\cdot) \mathbf{P}(d\mathcal{E})$ denotes the annealed probability. We always assume $\mathbf{E}[N] > 1$ so that \mathbb{T} is supercritical, i.e. \mathbb{T} survives with positive probability. Let $\mathbf{P}^*(\cdot) = \mathbf{P}(\cdot | \mathbb{T} \text{ survives})$ and $\mathbb{P}^*(\cdot) = \mathbb{P}^*(\cdot | \mathbb{T} \text{ survives})$.

In this setting, we could describe the environment \mathcal{E} by a branching random walk. For any $x \in \mathbb{T}$, let $|x|$ be its generation, i.e., the graph distance between the root ρ and x . For any $x, y \in \mathbb{T}$, we write $x \leq y$ if x is an ancestor of y and $x < y$ if $x \leq y$ and $x \neq y$. Let $\llbracket \rho, x \rrbracket$ be the ancestral line of x , that is, the set of vertices on the unique shortest path from ρ to x . In this ancestral line, for any $0 \leq i \leq |x|$, let x_i be the ancestor of x in the i -th generation; in particular, $x_0 = \rho$ and $x_{|x|} = x$. Then, define

$$V(x) := - \sum_{i=1}^{|x|} \log A_{x_i}, \forall x \in \mathbb{T} \setminus \{\rho\},$$

with $V(\rho) := 0$. Usually, $(V(x), x \in \mathbb{T})$ is viewed as the potential of the random walk. Immediately, we see that $(V(x), x \in \mathbb{T})$ is a branching random walk whose law is governed by that of $\mathcal{L} := \{V(x), |x| = 1\}$. Note that \mathcal{A} is distributed as $\{e^{-V(x)}, |x| = 1\}$.

From now on, we write the environment by this branching random walk, i.e., $\mathcal{E} = (V(x), x \in \mathbb{T})$. Then, the transition probabilities of the random walk $(X_n)_{n \geq 0}$ can be written as follows

$$\begin{cases} \mathbb{P}^{\mathcal{E}}(X_{n+1} = x^* | X_n = x) = \frac{e^{-V(x)}}{e^{-V(x)} + \sum_{y: y^* = x} e^{-V(y)}} \\ \mathbb{P}^{\mathcal{E}}(X_{n+1} = y | X_n = x) = \frac{e^{-V(y)} \mathbf{1}_{\{y^* = x\}}}{e^{-V(x)} + \sum_{z: z^* = x} e^{-V(z)}}. \end{cases} \quad (1.2)$$

Throughout the paper, we assume that the branching random walk is in the boundary case, that is,

$$\mathbf{E} \left[\sum_{|x|=1} e^{-V(x)} \right] = 1, \quad \mathbf{E} \left[\sum_{|x|=1} V(x) e^{-V(x)} \right] = 0. \quad (1.3)$$

We also assume the following integrability condition: there exists certain $\delta_0 > 0$ such that

$$\mathbf{E} \left[\sum_{|x|=1} e^{-(1+\delta_0)V(x)} \right] + \mathbf{E} \left[\sum_{|x|=1} e^{\delta_0 V(x)} \right] < \infty. \quad (1.4)$$

In addition, we assume that

$$\mathbf{E}[N^2] + \mathbf{E} \left[\left(\sum_{|u|=1} (1 + V_+(u))^2 e^{-V(u)} \right)^2 \right] < \infty, \quad (1.5)$$

where $V_+(u) := \max\{V(u), 0\}$. Immediately, one sees that $\sigma^2 := \mathbf{E}[\sum_{|u|=1} V(u)^2 e^{-V(u)}] \in (0, \infty)$. We take $\sigma = \sqrt{\sigma^2}$.

The criteria of recurrence/transience for random walks on trees is established by Lyons and Pemantle [LP92], which shows that the walk $(X_n)_{n \geq 0}$ is recurrent under (1.3). Further, Faraud [Far11]

proved that the walk is null recurrent under (1.3) and (1.4). Hu and Shi studied the walk under these assumptions, and showed in [HS07] that if \mathbb{T} is regular tree, then a.s., asymptotically, $\max_{0 \leq i \leq n} |X_i| = \Theta((\log n)^3)$. So the walk is called in a regime of **slow movement**. Later, under (1.3) and (1.4), Faraud, Hu and Shi proved in [FHS12], on the survival of \mathbb{T} , a.s. ,

$$\lim_{n \rightarrow \infty} \frac{\max_{0 \leq i \leq n} |X_i|}{(\log n)^3} = Cst. \quad (1.6)$$

Further, Hu and Shi obtained in [HS16] that $\frac{|X_n|}{(\log n)^2}$ converges weakly. The spread and the range of this walk have been studied in [AD14] and [AC18]. In this paper, we study the heavy range of the walk in this slow regime.

Define the edge local time for the edge (x^*, x) as follows

$$\bar{L}_x(n) := \sum_{k=0}^n \mathbf{1}_{\{X_{k-1}=x^*, X_k=x\}}, \forall n \geq 1.$$

Let $\tau_0 := 0$ and

$$\tau_n := \inf\{k > \tau_{n-1} : X_{k-1} = \rho^*, X_k = \rho\}, \forall n \geq 1.$$

Then $\bar{L}_\rho(\tau_n) = n$. It can be seen from [HS15] that $\max_{x \in \mathbb{T}} \bar{L}_x(\tau_n)$ is of order n in probability. For any $\theta \in (0, 1)$, define

$$R^{\geq n^\theta}(\tau_n) := \sum_{x \in \mathbb{T}} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta\}}.$$

We are interested in this so-called heavy range, which was first considered by Andreoletti and Diel [AD20]. They show that in any recurrent case, under \mathbb{P}^* , in probability, $R^{\geq n^\theta}(\tau_n) = n^{\zeta_\theta + o(1)}$ where $\zeta_\theta > 0$ is a constant depending on the regimes and on θ . In the sub-diffusive and diffusive regimes, our upcoming paper with de Raphélis [CdR] will prove the convergence in law of $\frac{R^{\geq n^\theta}(\tau_n)}{n^{\zeta_\theta}}$ under the annealed and quenched probability. In the slow movement regime, it is given in [AD20] that $\zeta_\theta = 1 - \theta$. We obtain the convergence in probability of $\frac{R^{\geq n^\theta}(\tau_n)}{n^{1-\theta}}$ under \mathbb{P}^* in this paper.

Let us state the main result as follows.

Theorem 1.1. *For any $\theta \in (0, 1)$, the following convergence in probability holds:*

$$\frac{R^{\geq n^\theta}(\tau_n)}{n^{1-\theta}} \xrightarrow[n \rightarrow \infty]{\text{in } \mathbb{P}^*} \Lambda(\theta) D_\infty, \quad (1.7)$$

where $D_\infty > 0$ is the \mathbb{P}^* -a.s. limit of the derivative martingale $(D_n := \sum_{|x|=n} V(x)e^{-V(x)})_{n \geq 0}$ and $\Lambda(\theta)$ is a positive real number whose value is given in (1.12) later.

Under (1.3), D_n is a martingale with respect to the natural filtration $\{\mathcal{F}_n; n \geq 0\}$ with $\mathcal{F}_n := \sigma(V(u); |u| \leq n)$. Under (1.4), it converges a.s. towards some non-degenerate limit according to Theorem of [BK04]. Moreover, $\mathbf{P}(D_\infty > 0) = \mathbf{P}(\mathbb{T} \text{ survives})$ under (1.4).

1.1 Sketch of proofs and organisation of the paper

Write $\bar{L}_x^{(n)}$ for $\bar{L}_x(\tau_n)$. In addition, up to the n -th return to ρ^* , define the number of excursions visiting x by

$$E_x^{(n)} := \sum_{k=1}^n \mathbf{1}_{\{\exists j \in (\tau_{k-1}, \tau_k], X_j = x\}}, \forall n \geq 1.$$

Then define

$$R^{\geq n^\theta}(\tau_n, j) := \sum_{x \in \mathbb{T}} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = j\}}, \forall 1 \leq j \leq n.$$

It is clear that

$$R^{\geq n^\theta}(\tau_n) = \sum_{j=1}^n R^{\geq n^\theta}(\tau_n, j) = \sum_{j=2}^n R^{\geq n^\theta}(\tau_n, j) + R^{\geq n^\theta}(\tau_n, 1).$$

It is known in [AD20] that under \mathbb{P}^* ,

$$\frac{\log R^{\geq n^\theta}(\tau_n)}{\log n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}^*} 1 - \theta.$$

We are going to treat $\sum_{j=2}^n R^{\geq n^\theta}(\tau_n, j)$ and $R^{\geq n^\theta}(\tau_n, 1)$ separately and show the convergences in probability of

$$\frac{1}{n^{1-\theta}} \sum_{j=2}^n R^{\geq n^\theta}(\tau_n, j) \text{ and } \frac{1}{n^{1-\theta}} R^{\geq n^\theta}(\tau_n, 1),$$

under the annealed probability \mathbb{P}^* . In fact, we have the following results.

Proposition 1.2. *For any $\theta \in (0, 1)$, the following convergence in probability holds:*

$$\frac{1}{n^{1-\theta}} \sum_{j=2}^n R^{\geq n^\theta}(\tau_n, j) \xrightarrow[n \rightarrow \infty]{\mathbb{P}^*} \Lambda_0(\theta) D_\infty, \quad (1.8)$$

where

$$\Lambda_0(\theta) := \frac{\sqrt{2}}{\sqrt{\pi} \sigma^2} \int_0^\infty \mathcal{C}_0\left(\frac{\theta}{\sqrt{u}}, \frac{1-\theta}{\sqrt{u}}\right) \frac{du}{u} \quad (1.9)$$

with \mathcal{C}_0 defined in (A.28).

Proposition 1.3. *For any $\theta \in (0, 1)$, the following convergence in probability holds:*

$$\frac{1}{n^{1-\theta}} R^{\geq n^\theta}(\tau_n, 1) \xrightarrow[n \rightarrow \infty]{\mathbb{P}^*} \Lambda_1(\theta) D_\infty, \quad (1.10)$$

where

$$\Lambda_1(\theta) := c_{\mathcal{R}} \int_0^\infty \mathcal{G}\left(\frac{1}{\sqrt{s}}, \frac{\theta}{\sqrt{s}}\right) \frac{ds}{s}, \quad (1.11)$$

with $\mathcal{G}(a, b)$ defined in (A.30) and $c_{\mathcal{R}}$ defined in (A.4).

Theorem 1.1 follows directly from Propositions 1.2 and 1.3 with

$$\Lambda(\theta) = \Lambda_0(\theta) + \Lambda_1(\theta). \quad (1.12)$$

Here the finiteness of $\Lambda_0(\theta)$ can be checked immediately as $\Lambda_0(\theta) \leq \frac{\sqrt{2}}{\sqrt{\pi\sigma^2}} \int_0^\infty \varphi\left(\frac{1-\theta}{\sigma\sqrt{u}}\right) \frac{du}{\sigma u} < \infty$. For $\Lambda_1(\theta)$, by change of variables $r = s(1-u)$ and $t = su$, one sees that

$$\begin{aligned} \Lambda_1(\theta) &= \mathbf{c}_{\mathcal{R}} \frac{\mathbf{c}_-}{\sigma} \int_0^\infty \int_0^1 \mathcal{C}_{\frac{1}{\sqrt{su}}, \frac{1}{\sqrt{su}}} \mathcal{C}_0\left(\frac{1-\theta}{\sqrt{s(1-u)}}, \frac{\theta}{\sqrt{s(1-u)}}\right) \frac{du}{(1-u)u} \frac{ds}{s} \\ &= \mathbf{c}_{\mathcal{R}} \frac{\mathbf{c}_-}{\sigma} \int_0^\infty \mathcal{C}_0\left(\frac{1-\theta}{\sqrt{r}}, \frac{\theta}{\sqrt{r}}\right) \frac{dr}{r} \int_0^\infty \mathcal{C}_{\frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}} \frac{dt}{t} = \frac{\mathbf{c}_{\mathcal{R}} \mathbf{c}_- \sqrt{\pi\sigma}}{\sqrt{2}} \Lambda_0(1-\theta) \int_0^\infty \mathcal{C}_{\frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}} \frac{dt}{t}, \end{aligned}$$

where the finiteness of $\int_0^\infty \frac{\mathcal{C}_{t^{-1/2}, t^{-1/2}}}{t} dt$ has been verified in Lemma A.1 of [AC18].

Let us do some basic calculations here. For any $x \in \mathbb{T} \cup \{\rho^*\}$, let T_x be the first hitting time at x :

$$T_x := \inf\{k \geq 0 : X_k = x\}.$$

Then, it is known that

$$a_x := \mathbb{P}_\rho^\mathcal{E}(T_x < T_{\rho^*}) = \frac{1}{\sum_{y \in \llbracket \rho, x \rrbracket} e^{V(y)}} = \frac{e^{-V(x)}}{H_x}, \quad (1.13)$$

$$b_x := \mathbb{P}_{x^*}^\mathcal{E}(T_x < T_{\rho^*}) = 1 - \frac{1}{H_x}, \quad (1.14)$$

$$\text{where } H_x := \sum_{y \in \llbracket \rho, x \rrbracket} e^{V(y) - V(x)}.$$

As a consequence, for any fixed $x \in \mathbb{T}$,

$$\mathbb{P}_\rho^\mathcal{E}(\bar{L}_x(\tau_1) = 0) = 1 - a_x \text{ and } \mathbb{P}_\rho^\mathcal{E}(\bar{L}_x(\tau_1) \geq k) = a_x b_x^{k-1}, \forall k \geq 1.$$

Then by Markov property, under $\mathbb{P}_\rho^\mathcal{E}$, $(\bar{L}_x(\tau_{n+1}) - \bar{L}_x(\tau_n))_{n \geq 1}$ are i.i.d. random variables distributed as $\bar{L}_x(\tau_1)$. Moreover, $E_x^{(n)}$ is a Binomial random variable with parameters n and a_x . Let

$$\bar{V}(x) := \max_{y \in \llbracket \rho, x \rrbracket} V(y) \text{ and } \underline{V}(x) := \min_{y \in \llbracket \rho, x \rrbracket} V(y), \forall x \in \mathbb{T}.$$

Then $a_x \leq e^{-\bar{V}(x)}$, $H_x \leq e^{\bar{V}(x) - V(x)}$.

To get $\sum_{j=2}^n R^{\geq n^\theta}(\tau_n, j)$, we need to consider the individuals $x \in \mathbb{T}$ such that $\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}$. As $a_x \leq e^{-\bar{V}(x)}$, the individuals with $\bar{V}(x) \gtrsim \log n$ would be visited in at most one excursion with high probability. We thus take $\{x \in \mathbb{T} : \bar{V}(x) \lesssim \log n\}$. On the other hand, $\mathbb{E}^\mathcal{E}[\bar{L}_x(\tau_n)] = ne^{-V(x)}$. So, the event $\{V(x) \lesssim (1-\theta) \log n\}$ involves also. Actually, we could verify that with high probability under \mathbb{P}^* ,

$$\sum_{j=2}^n R^{\geq n^\theta}(\tau_n, j) \approx \sum_{x \in \mathbb{T}} \mathbf{1}_{\{\bar{V}(x) \lesssim \log n, V(x) \lesssim (1-\theta) \log n\}}.$$

The asymptotic of the latter will be treated in Lemma 3.2.

To get $R^{\geq n^\theta}(\tau_n, 1)$, we are going to compare it with its quenched expectation. We see that

$$\mathbb{E}^{\mathcal{E}}[R^{\geq n^\theta}(\tau_n, 1)] = \sum_{x \in \mathbb{T}} na_x(1 - a_x)^{n-1} b_x^{n^\theta - 1} \approx n^{1-\theta} \sum_{x \in \mathbb{T}} e^{-V(x)} \left(\frac{n^\theta}{H_x} e^{-\frac{n^\theta}{H_x}} \right)$$

as we only need to count the individuals with $\{\bar{V}(x) \gtrsim \log n\}$ so that they are visited only by one excursion with high probability. Here we take $\{\bar{V}(x) - V(x) \approx \theta \log n\}$ so that $H_x = \Theta(n^\theta)$ as H_x and $e^{\bar{V}(x) - V(x)}$ are comparable. In additional, it is known in [HS16] that up to τ_n , with high probability, the walker has not visited the stopping line $\{x \in \mathbb{T} : \max_{\rho \leq y < x} H_y < \gamma_n \leq H_x\}$ with $\gamma_n = \frac{n}{(\log n)^\gamma}$ for any $\gamma > 0$. By bounding the quenched variance of $R^{\geq n^\theta}(\tau_n, 1)$, we could verify that with high probability,

$$R^{\geq n^\theta}(\tau_n, 1) \approx n^{1-\theta} \sum_{x \in \mathbb{T}} e^{-V(x)} \left(\frac{n^\theta}{H_x} e^{-\frac{n^\theta}{H_x}} \right) \mathbf{1}_{\{\bar{V}(x) \gtrsim \log n, \bar{V}(x) - V(x) \approx \theta \log n, \max_{\rho \leq y \leq x} H_y < r_n\}}.$$

The asymptotic of the latter will be given in Lemma 4.5.

The rest of the paper is organised as follows. In section 2, we state some basic facts on the branching random walk and the biased random walk. In section 3, we study $\sum_{j=2}^n R^{\geq n^\theta}(\tau_n, j)$ and prove Proposition 1.2 by choosing the suitable environment. In section 4, we prove Proposition 1.3. Next, Section 5 is devoted to proving the generalised Seneta-Heyde norming results: Lemmas 4.5 and 3.2, by applying the new method introduced by [BM19]. In Section 6, we complete the proofs of the technical lemmas.

In this paper, we use $(c_i)_{i \geq 0}$ and $(C_i)_{i \geq 0}$ for positive constants which may change from line to line. And we write $f(n) \sim g(n)$ when $\frac{f(n)}{g(n)} \rightarrow 1$ as $n \rightarrow \infty$.

2 Preliminary results

In this section, we state some facts and lemmas which will be used later.

2.1 Many-to-One Lemma

Recall that \mathbf{P} is the law of the branching random walk $(V(u), u \in \mathbb{T})$ started from $V(\rho) = 0$. Let $\mathbf{P}_a((V(u), u \in \mathbb{T}) \in \cdot) = \mathbf{P}((a + V(u), u \in \mathbb{T}) \in \cdot)$ for any $a \in \mathbb{R}$. Let E_a be the corresponding expectation. Then the following lemma holds because of (1.3).

Lemma 2.1 (Many-to-One). *For any $n \geq 1$, $a \in \mathbb{R}$ and any measurable function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$, we have*

$$\mathbf{E}_a \left[\sum_{|u|=n} e^{-V(u)} f(V(u_1), \dots, V(u_n)) \right] = e^{-a} \mathbf{E} [f(S_1 + a, \dots, S_n + a)], \quad (2.1)$$

where $(S_n)_{n \geq 0}$ is one dimensional centred random walk with i.i.d. increments and $S_0 = 0$.

Moreover, by (1.4), $\mathbf{E}[S_1^2] = \sigma^2 \in (0, \infty)$ and

$$\mathbf{E}[e^{-\delta_0 S_1}] + \mathbf{E}[e^{(1+\delta_0)S_1}] < \infty. \quad (2.2)$$

For any $n \geq 0$, let $\bar{S}_n := \max_{0 \leq k \leq n} S_k$ and $\underline{S}_n := \min_{0 \leq k \leq n} S_k$. More estimates and rescaling results on the random walk $(S_n)_{n \in \mathbb{N}}$ can be found in Appendix A.2.

2.2 Lyons' change of measure

Define the additive martingale with respect to the natural filtration $\{\mathcal{F}_n\}_{n \geq 0}$ by

$$W_n := \sum_{|u|=1} e^{-V(u)}, \forall n \geq 0.$$

Under (1.3), this is a non-negative martingale and it converges \mathbf{P} -a.s. to zero according to [Lyo97]. By Kolmogorov extension theorem, for any \mathbb{R} , we can define a probability measure \mathbf{Q}_a on $\mathcal{F}_\infty := \bigvee_{n \geq 0} \mathcal{F}_n$ such that

$$\frac{d\mathbf{Q}_a}{d\mathbf{P}_a} \Big|_{\mathcal{F}_n} := e^a \sum_{|u|=n} e^{-V(u)}, \forall n \geq 0.$$

Let $\mathbf{E}_{\mathbf{Q}_a}$ denote the corresponding expectation and write \mathbf{Q} for \mathbf{Q}_0 .

Let us introduce a probability measure $\widehat{\mathbf{Q}}_a$ on the space of marked branching random walks so that its marginal distribution is exactly \mathbf{Q}_a . Recall that the reproduction law of the branching random walk $(V(x), x \in \mathbb{T})$ is given by the point process $\mathcal{L} = \{V(x), |x| = 1\}$. Let $\widehat{\mathcal{L}}$ be the point process having Radon-Nykodim derivative $\sum_{z \in \mathcal{L}} e^{-z}$ with respect to the law of \mathcal{L} . We start with w_0 the root, located at $V(w_0) = 0$. At time 1, it dies and reproduces a random number of individuals whose displacements with respect to $V(w_0)$, viewed as a point process, are distributed as $\widehat{\mathcal{L}}$. All children of w_0 form the first generation, among which we choose x to be w_1 with probability proportional to $e^{-V(x)}$. Then recursively, at time $n + 1$, the individuals of the n -th generation die and reproduce independently their children according to the law of \mathcal{L} , except w_n which gives birth to its children according to $\widehat{\mathcal{L}}$. The w_{n+1} is selected among the children of w_n with probability proportional to $e^{-V(u)}$ for each child u of w_n . This construction gives us a branching random walk with a marked ray $(w_n)_{n \geq 0}$, which is called the spine. The law of this marked branching random walk $(V(x), x \in \mathbb{T}; (w_n)_{n \geq 0})$ is denoted by $\widehat{\mathbf{Q}}_0$. Again, $\widehat{\mathbf{Q}}_a$ denotes the law of $(a + V(x), x \in \mathbb{T}; (w_n)_{n \geq 0})$ under $\widehat{\mathbf{Q}}_0$. We use $\mathbf{E}_{\widehat{\mathbf{Q}}_a}$ to represent the corresponding expectation and use $\widehat{\mathbf{Q}}$ instead of $\widehat{\mathbf{Q}}_0$ for brevity.

It is known that the marginal law of $\widehat{\mathbf{Q}}_a$ on the branching random walk is the same as \mathbf{Q}_a defined above. We also state the following proposition from [Lyo97], which gives some properties of $\widehat{\mathbf{Q}}_a$.

Proposition 2.2. (i)

$$\widehat{\mathbf{Q}}_a((V(w_0), \dots, V(w_n)) \in \cdot) = \mathbf{P}((a + S_0, \dots, a + S_n) \in \cdot).$$

(ii) For any $|u| = n$,

$$\widehat{\mathbf{Q}}_a(w_n = u | \mathcal{F}_n) = \frac{e^{-V(u)}}{W_n}.$$

For the marked branching random walk $(V(x), x \in \mathbb{T}; (w_n)_{n \geq 0})$, let $\Omega(w_j) = \{u \in \mathbb{T} : u^* = w_{j-1}, u \neq w_j\}$ be the collection of brothers of w_j for any $j \geq 1$. Let \mathcal{G} be the sigma-field containing all information along the spine, that is,

$$\mathcal{G} := \sigma\{(w_k, V(w_k))_{k \geq 0}, (u, V(u))_{u \in \cup_{k \geq 0} \Omega(w_k)}\}.$$

2.3 Quenched probability for edge local times

Recall that under $\mathbb{P}_\rho^\mathcal{E}$, $(\bar{L}_x(\tau_{n+1}) - \bar{L}_x(\tau_n))_{n \geq 1}$ are i.i.d. random variables distributed as $\bar{L}_x(\tau_1)$. Observe also that $E_x^{(n)} = \sum_{k=1}^n \mathbf{1}_{\{\bar{L}_x(\tau_k) - \bar{L}_x(\tau_{k-1}) \geq 1\}}$. Let us state the following lemma on the joint distribution of $(\bar{L}_x(\tau_n), E_x^{(n)})$ under the quenched probability.

Lemma 2.3. *Let $a, b \in (0, 1)$. Suppose that $(\zeta_i)_{i \geq 1}$ are i.i.d. random variables taking values in \mathbb{N} such that*

$$\mathbf{P}(\zeta_1 = 0) = 1 - a, \text{ and } \mathbf{P}(\zeta_1 \geq k) = ab^{k-1}, \forall k \geq 1.$$

1. *If $n^\theta(1-b) \geq (1+\eta)na$ with some $\eta > 0$, then there exists $c_\eta > 0$ such that for any $n \geq 1$,*

$$\mathbf{P}\left(\sum_{i=1}^n \zeta_i \geq n^\theta\right) \leq 2nae^{-c_\eta n^\theta(1-b)}, \quad (2.3)$$

and

$$\mathbf{P}\left(\sum_{i=1}^n \zeta_i \geq n^\theta; \sum_{i=1}^n \mathbf{1}_{\{\zeta_i \geq 1\}} \geq 2\right) \leq 2(na)^2 e^{-c_\eta n^\theta(1-b)}. \quad (2.4)$$

2. *For $A > 0, 0 < \lambda < 1$ and for any $n \geq 1$,*

$$\mathbf{P}\left(\sum_{i=1}^n \zeta_i \leq A\right) \leq e^{-\lambda(\frac{na}{1+\lambda} - (1-b)A)}. \quad (2.5)$$

The proof of Lemma 2.3 will be postponed in Appendix A.1.

3 Proof of Proposition 1.2

In this section, we study $\sum_{j=2}^n R^{\geq n^\theta}(\tau_n, j)$ and prove Proposition 1.2.

First, it is proved in [FHS12] that $\max_{1 \leq i \leq \tau_n} |X_i| = O((\log n)^3)$, \mathbb{P}^* -a.s. So,

$$\sum_{j \geq 2} R^{\geq n^\theta}(\tau_n, j) = \sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}} + o_n(1), \mathbb{P}^*\text{-a.s.},$$

with some large and fixed constant $c_0 > 0$. On the other hand, it is known that \mathbf{P}^* -a.s.,

$$0 \geq \inf_{u \in \mathbb{T}} V(u) > -\infty.$$

So, we only need to consider $\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}} \mathbf{1}_{\{V(x) \geq -\alpha\}}$ for any fixed $\alpha > 0$. Now for any $a, b \in \mathbb{R}$, let

$$A_n(a, b) := \{x \in \mathbb{T} : \bar{V}(x) - V(x) \leq \theta \log n + a, V(x) \leq (1 - \theta) \log n + b\}, \forall n \geq 1,$$

and

$$A_n^+(a, b) := \{x \in \mathbb{T} : \bar{V}(x) \leq \log n + a, V(x) \leq (1 - \theta) \log n + b\}, \forall n \geq 1.$$

Then, we stress that for any $\alpha > 0, b > 0, a_n = a \log \log n$ with $a > 3$,

$$\begin{aligned} \sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{V(x) \geq -\alpha\}} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}} + o_{\mathbf{P}}(n^{1-\theta}) &\leq \sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{V(x) \geq -\alpha\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}} \\ &\leq \sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{V(x) \geq -\alpha\}} \mathbf{1}_{\{x \in A_n^+(a_n, b)\}} + o_{\mathbf{P}}(n^{1-\theta}), \end{aligned}$$

because of the following lemma.

Lemma 3.1. *Let $b > 0, \alpha > 0$. For $a_n = a \log \log n$ with $a > 3$, we have*

$$\frac{1}{n^{1-\theta}} \sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) < n^\theta \text{ or } E_x^{(n)} \leq 1\}} \xrightarrow[n \rightarrow \infty]{in \mathbf{P}} 0, \quad (3.1)$$

and

$$\frac{1}{n^{1-\theta}} \sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{V(x) \geq -\alpha\}} \mathbf{1}_{\{x \notin A_n^+(a_n, b)\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}} \xrightarrow[n \rightarrow \infty]{in \mathbf{P}} 0. \quad (3.2)$$

It remains to study $\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}}$ and $\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{x \in A_n^+(a_n, b)\}}$, which is done in the next lemma.

Lemma 3.2. *Let $b > 0$. For $a_n = o(\log n)$, we have*

$$\frac{\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}}}{n^{1-\theta}} \xrightarrow[n \rightarrow \infty]{in \mathbf{P}^*} D_\infty \Lambda_0(\theta) e^{-b}, \quad (3.3)$$

and

$$\frac{\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{x \in A_n^+(a_n, b)\}}}{n^{1-\theta}} \xrightarrow[n \rightarrow \infty]{in \mathbf{P}^*} D_\infty \Lambda_0(\theta) e^b. \quad (3.4)$$

The proof of Lemma 3.1 will be given later in Section 6, and the proof of Lemma 3.2 will be in Section 5. Now we are ready to prove Proposition 1.2.

Proof of Proposition 1.2. Recall that $D_\infty > 0$, \mathbb{P}^* -a.s. We only need to show that for any $\delta \in (0, 1)$, as $n \rightarrow \infty$,

$$\mathbb{P}^* \left(\frac{\sum_{j \geq 2} R^{\geq n^\theta}(\tau_n, j)}{n^{1-\theta}} \geq (1 + \delta)\Lambda_0(\theta)D_\infty \text{ or } \leq (1 - \delta)\Lambda_0(\theta)D_\infty \right) \rightarrow 0.$$

Observe that for any $\alpha > 0$ and $\beta \in (0, 1)$,

$$\begin{aligned} & \mathbb{P}^* \left(\frac{\sum_{j \geq 2} R^{\geq n^\theta}(\tau_n, j)}{n^{1-\theta}} \geq (1 + \delta)\Lambda_0(\theta)D_\infty \text{ or } \leq (1 - \delta)\Lambda_0(\theta)D_\infty \right) \\ & \leq \mathbf{P}(\inf_{x \in \mathbb{T}} V(x) < -\alpha) + \mathbf{P}^* \left(\max_{1 \leq i \leq \tau_n} |X_i| > c_0(\log n)^3 \right) + \mathbf{P}^*(D_\infty < \beta) \\ & \quad + \mathbf{P}^* \left(\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{V(x) \geq -\alpha\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}} \geq (1 + \delta)n^{1-\theta}\Lambda_0(\theta)D_\infty \text{ or } \leq (1 - \delta)n^{1-\theta}\Lambda_0(\theta)D_\infty; D_\infty \geq \beta \right) \end{aligned}$$

It is known (see [Aid13]) that for any $\alpha > 0$, $\mathbf{P}(\inf_{x \in \mathbb{T}} V(x) < -\alpha) \leq e^{-\alpha}$. Note also that $\mathbf{P}^*(D_\infty < \beta) = o_\beta(1)$ as $\beta \downarrow 0$. Therefore,

$$\begin{aligned} & \mathbb{P}^* \left(\frac{\sum_{j \geq 2} R^{\geq n^\theta}(\tau_n, j)}{n^{1-\theta}} \geq (1 + \delta)\Lambda D_\infty \text{ or } \leq (1 - \delta)\Lambda D_\infty \right) \tag{3.5} \\ & \leq c_2 e^{-\alpha} + o_n(1) + o_\beta(1) + \mathbf{P}^* \left(\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{V(x) \geq -\alpha\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}} \geq (1 + \delta)n^{1-\theta}\Lambda_0(\theta)D_\infty; D_\infty \geq \beta \right) \\ & \quad + \mathbf{P}^* \left(\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{V(x) \geq -\alpha\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}} \leq (1 - \delta)n^{1-\theta}\Lambda_0(\theta)D_\infty; D_\infty \geq \beta, \inf_{x \in \mathbb{T}} V(x) \geq -\alpha \right). \end{aligned}$$

On the one hand, for any $b > 0$ and $a_n = a \log \log n$ with $a > 3$, one has

$$\begin{aligned} & \mathbb{P}^* \left(\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{V(x) \geq -\alpha\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}} \geq (1 + \delta)n^{1-\theta}\Lambda_0(\theta)D_\infty; D_\infty > \beta \right) \\ & \leq \mathbb{P}^* \left(\frac{1}{n^{1-\theta}} \sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{x \notin A_n^+(a_n, b)\}} \mathbf{1}_{\{V(x) \geq -\alpha\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}} \geq \frac{\delta\Lambda_0(\theta)\beta}{2} \right) \\ & \quad + \mathbb{P}^* \left(\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{V(x) \geq -\alpha\}} \mathbf{1}_{\{x \in A_n^+(a_n, b)\}} \mathbf{1}_{\{V(x) \geq -\alpha\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}} \geq (1 + \delta/2)n^{1-\theta}\Lambda_0(\theta)D_\infty \right) \\ & \leq o_n(1) + \mathbb{P}^* \left(\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{x \in A_n^+(a_n, b)\}} \geq (1 + \delta/2)n^{1-\theta}\Lambda_0(\theta)D_\infty \right), \end{aligned}$$

where the last line follows from (3.2). For the second term on the righthand side, taking $b > 0$ small so

that $e^b < 1 + \delta/2$ and using (3.4) yields that

$$\mathbb{P}^* \left(\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{x \in A_n^+(a_n, b)\}} \geq (1 + \delta/2)n^{1-\theta} \Lambda_0(\theta) D_\infty \right) \rightarrow 0,$$

as n goes to infinity. On the other hand, observe that

$$\begin{aligned} & \mathbb{P}^* \left(\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{\bar{V}(x) \geq -\alpha\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}} \leq (1 - \delta)n^{1-\theta} \Lambda_0(\theta) D_\infty; D_\infty \geq \beta, \inf_{x \in \mathbb{T}} V(x) \geq -\alpha \right) \\ & \leq \mathbb{P}^* \left(\frac{1}{n^{1-\theta}} \sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) < n^\theta \text{ or } E_x^{(n)} \leq 1\}} \geq \frac{\delta \Lambda_0(\theta) \beta}{2} \right) \\ & \quad + \mathbb{P}^* \left(\sum_{|x| \leq c_0(\log n)^3} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}} \leq (1 - \delta/2) \Lambda_0(\theta) D_\infty \right) \\ & = o_n(1) \end{aligned}$$

by (3.1) and (3.3) with $b > 0$ small enough so that $e^{-b} > 1 - \delta/2$. Going back to (3.5), one sees that

$$\mathbb{P}^* \left(\frac{\sum_{j \geq 2} R_n^{\geq n^\theta}(\tau_n, j)}{n^{1-\theta}} \geq (1 + \delta) \Lambda_0(\theta) D_\infty \text{ or } \leq (1 - \delta) \Lambda_0(\theta) D_\infty \right) \leq c_2 e^{-\alpha} + o_\beta(1) + o_n(1).$$

Letting $n \rightarrow \infty$ then $\alpha \uparrow \infty$ and $\beta \downarrow 0$ concludes (1.8). \square

4 Proof of Proposition 1.3

This section is devoted to proving Proposition 1.3. Similarly as above, we have \mathbb{P}^* -a.s.,

$$R_n^\theta(\tau_n, 1) = \sum_{\ell=1}^{c_0(\log n)^3} \sum_{|x|=\ell} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)}=1\}} + o_n(1).$$

For $a_n = a \log \log n$ with $a > 3$, set

$$\mathfrak{B}_n^\pm := \{x \in \mathbb{T} : \bar{V}(x) \geq \log n \pm a_n\}, \text{ and } \mathfrak{D}_n := \{x \in \mathbb{T} : \bar{V}(x) - V(x) \in [\theta \log n - a_n, \theta \log n + a_n]\}.$$

We first show that with high probability, $R_n^\theta(\tau_n, 1) \approx \sum_{\ell=1}^{c_0(\log n)^3} \sum_{|x|=\ell} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)}=1\}} \mathbf{1}_{\{z \in \mathfrak{B}_n^-\}} \mathbf{1}_{\{z \in \mathfrak{D}_n\}}$.

This comes from the following lemma whose proof is stated in Section 6.

Lemma 4.1. *As $n \uparrow \infty$, we have*

$$\mathbb{E} \left[\sum_{\ell=1}^{c_0(\log n)^3} \sum_{|x|=\ell} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)}=1\}} \mathbf{1}_{\{\bar{V}(x) < \log n - a_n\}} \right] = o(n^{1-\theta}), \quad (4.1)$$

$$\mathbb{E} \left[\sum_{\ell=1}^{c_0(\log n)^3} \sum_{|x|=\ell} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)}=1\}} \mathbf{1}_{\{x \notin \mathfrak{D}_n\}} \right] = o(n^{1-\theta}). \quad (4.2)$$

Here we introduce the stopping line

$$\mathcal{L}_r := \{x \in \mathbb{T} : \max_{y < x} H_y < r \leq H_x\}, \forall r > 1.$$

It is known that in [HS16] that

$$\mathbb{P}(\exists k \leq \tau_n : X_k \in \mathcal{L}_n) \rightarrow 0.$$

This means that $\mathbb{P}(\{X_k, k \leq \tau_n\} \subset \{x \in \mathbb{T} : x < \mathcal{L}_n\} \cup \{\rho^*\}) \rightarrow 1$. For any $r > 1$, define

$$\mathfrak{L}_r := \{x \in \mathbb{T} : \max_{y < x} H_y < r\}.$$

So, we only need to study $\sum_{\ell=1}^{c_0(\log n)^3} \sum_{|x|=\ell} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)}=1\}} \mathbf{1}_{\{z \in \mathfrak{B}_n^-\}} \mathbf{1}_{\{z \in \mathfrak{D}_n\}} \mathbf{1}_{\{z \in \mathfrak{L}_n\}}$. In fact, only the generations of order $(\log n)^2$ should be counted and \mathfrak{B}_n^- can be replaced by \mathfrak{B}_n^+ , in view of the following lemma.

Lemma 4.2. *As $\varepsilon \downarrow 0$, we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{1-\theta}} \mathbb{E} \left[\sum_{\ell=1}^{\varepsilon(\log n)^2} \sum_{|x|=\ell} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)}=1\}} \mathbf{1}_{\{x \in \mathfrak{B}_n^-\}} \mathbf{1}_{\{x \in \mathfrak{D}_n, \underline{V}(x) \geq -\alpha\}} \right] = o_\varepsilon(1), \quad (4.3)$$

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{1-\theta}} \mathbb{E} \left[\sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{|x|=\ell} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)}=1\}} \mathbf{1}_{\{x \in \mathfrak{B}_n^-\}} \mathbf{1}_{\{x \in \mathfrak{D}_n, \underline{V}(x) \geq -\alpha\}} \mathbf{1}_{\{x \in \mathfrak{L}_n\}} \right] = o_\varepsilon(1). \quad (4.4)$$

For any $\varepsilon \in (0, 1)$,

$$\frac{1}{n^{1-\theta}} \mathbb{E} \left[\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{|x|=\ell} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)}=1\}} \mathbf{1}_{\{\bar{V}(x) \in [\log n - a_n, \log n + a_n], \underline{V}(x) \geq -\alpha, x \in \mathfrak{D}_n\}} \right] = o_n(1) \quad (4.5)$$

Instead of \mathfrak{L}_n , we are going to use \mathfrak{L}_{r_n} with $r_n = \frac{n}{(\log n)^\gamma}$ to control the quenched variance of $\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{|x|=\ell} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)}=1\}} \mathbf{1}_{\{z \in \mathfrak{B}_n^+\}} \mathbf{1}_{\{z \in \mathfrak{D}_n\}} \mathbf{1}_{\{z \in \mathfrak{L}_{r_n}\}}$.

Lemma 4.3. *For any $\varepsilon \in (0, 1)$ fixed, $\alpha > 0$ and for $\gamma_n = \frac{n}{(\log n)^\gamma}$ with fixed $\gamma > 0$, we have*

$$\mathbb{E} \left[\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{|z|=\ell} \mathbf{1}_{\{\bar{L}_z(\tau_n) \geq n^\theta, E_z^{(n)}=1\}} \mathbf{1}_{\{z \in \mathfrak{B}_n^+\}} \mathbf{1}_{\{z \in \mathfrak{D}_n\}} \mathbf{1}_{\{\underline{V}(z) \geq -\alpha, \gamma_n \leq \max_{x \leq z} H_x < n\}} \right] = o(n^{1-\theta}). \quad (4.6)$$

Let $\mathfrak{D}_n^K := \{x \in \mathbb{T} : \bar{V}(x) - V(x) \in [\theta \log n - K, \theta \log n + K]\}$ with large constant $K \geq 1$. Then, as $K \rightarrow \infty$,

$$\mathbb{E} \left[\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{|z|=\ell} \mathbf{1}_{\{\bar{L}_z(\tau_n) \geq n^\theta, E_z^{(n)}=1\}} \mathbf{1}_{\{z \in \mathfrak{B}_n^+\}} \mathbf{1}_{\{z \in \mathfrak{D}_n \setminus \mathfrak{D}_n^K\}} \mathbf{1}_{\{\underline{V}(z) \geq -\alpha, \max_{x \leq z} H_x < n\}} \right] = o_K(1) n^{1-\theta}. \quad (4.7)$$

Let

$$\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha) := \sum_{|x|=\ell} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1\}} \mathbf{1}_{\{x \in \mathfrak{B}_n^+\}} \mathbf{1}_{\{x \in \mathfrak{D}_n\}} \mathbf{1}_{\{x \in \mathfrak{L}_{\gamma_n}\}} \mathbf{1}_{\{V(x) \geq -\alpha\}}.$$

It immediately follows that

$$\mathbb{E}^\mathcal{E} [\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)] = \sum_{|x|=\ell} \mathbb{P}^\mathcal{E}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1) \mathbf{1}_{\{x \in \mathfrak{B}_n^+\}} \mathbf{1}_{\{x \in \mathfrak{D}_n\}} \mathbf{1}_{\{x \in \mathfrak{L}_{\gamma_n}\}} \mathbf{1}_{\{V(x) \geq -\alpha\}},$$

where $\mathbb{P}^\mathcal{E}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1) = (1 + o_n(1))n^{1-\theta}e^{-V(x)}\mathbf{f}(\frac{n^\theta}{H_x})$ with $\mathbf{f}(u) = ue^{-u}$.

Let $\text{Var}^\mathcal{E}$ denote the quenched variance. We state the following estimate.

Lemma 4.4. *Let $0 < A < B < \infty$. For $\ell \in [A(\log n)^2, B(\log n)^2] \cap \mathbb{N}$, one has*

$$\mathbb{E}[\text{Var}^\mathcal{E}(\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha))] \leq c_1 \frac{n^{2-2\theta}}{(\log n)^{a \wedge \gamma - 4}}. \quad (4.8)$$

All these previous lemmas will be proved in Section 6. The following lemma states the asymptotic behaviour of the quenched expectation $\mathbb{E}^\mathcal{E} [\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)]$.

Lemma 4.5. *For any $0 < A < B < \infty$ and $a + \gamma > 6$, one has*

$$\sum_{\ell=A(\log n)^2}^{B(\log n)^2} \sum_{|x|=\ell} e^{-V(x)} \mathbf{f}(\frac{n^\theta}{H_x}) \mathbf{1}_{\{x \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}\}} \xrightarrow[n \rightarrow \infty]{\mathbf{P}^*} D_\infty \times \int_A^B \mathcal{G}(\frac{1}{\sqrt{u}}, \frac{\theta}{\sqrt{u}}) \frac{du}{u}.$$

In fact, because of (4.7), we only need to prove that

$$\sum_{\ell=A(\log n)^2}^{B(\log n)^2} \sum_{|x|=\ell} e^{-V(x)} \mathbf{f}(\frac{n^\theta}{H_x}) \mathbf{1}_{\{x \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n^k \cap \mathfrak{L}_{\gamma_n}\}} \xrightarrow[n \rightarrow \infty]{\mathbf{P}^*} C_0(A, B, K) D_\infty, \quad (4.9)$$

where $C_0(A, B, K) \in (0, \infty)$ and $\lim_{K \rightarrow \infty} C_0(A, B, K) = \int_A^B \mathcal{G}(\frac{1}{\sqrt{u}}, \frac{1}{\sqrt{u}}, \frac{\theta}{\sqrt{u}}) \frac{du}{u}$. The proof of (4.9) is postponed in Section 5.

Let us prove Proposition 1.3 by use of these lemmas.

Proof of Proposition 1.3. Note that for any $\delta > 0$ and $\beta > 0$,

$$\begin{aligned} & \mathbb{P}^* \left(\left| \frac{R_{n^\theta}(\tau_n, 1)}{n^{1-\theta}} - \Lambda_1(\theta) D_\infty \right| \geq \delta D_\infty \right) \\ & \leq \mathbb{P}^* \left(\inf_{x \in \mathbb{T}} V(x) < -\alpha \right) + \mathbb{P}^* \left(\max_{1 \leq i \leq \tau_n} |X_i| > c_0(\log n)^3 \right) + \mathbb{P}^*(D_\infty < \beta) + \mathbb{P}^*(\exists k \leq \tau_n, X_k \in \mathcal{L}_n) \\ & \quad + \mathbb{P}^* \left(\left| \frac{\sum_{\ell=1}^{c_0(\log n)^3} \sum_{|x|=\ell} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1\}} \mathbf{1}_{\{x \in \mathfrak{L}_n\}} \mathbf{1}_{\{V(x) \geq -\alpha\}}}{n^{1-\theta}} - \Lambda_1(\theta) D_\infty \right| \geq \delta D_\infty, D_\infty \geq \beta \right). \end{aligned}$$

Here $\mathbb{P}^*(\exists k \leq \tau_n, X_k \in \mathcal{L}_n) = o_n(1)$ according to [HS16]. By Lemmas 4.1, 4.2 and 4.3, one has

$$\begin{aligned} \mathbb{P}^* \left(\left| \frac{R_{n^\theta}(\tau_n, 1)}{n^{1-\theta}} - \Lambda_1(\theta)D_\infty \right| \geq \delta D_\infty \right) &\leq c_2 e^{-\alpha} + o_n(1) + o_\beta(1) \\ &+ \mathbb{P}^* \left(\left| \frac{\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)}{n^{1-\theta}} - \Lambda_1(\theta)D_\infty \right| \geq \frac{\delta}{2} D_\infty, D_\infty \geq \beta \right). \end{aligned} \quad (4.10)$$

Here, one sees that

$$\begin{aligned} &\mathbb{P}^* \left(\left| \frac{\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)}{n^{1-\theta}} - \Lambda_1(\theta)D_\infty \right| \geq \frac{\delta}{2} D_\infty, D_\infty \geq \beta \right) \\ &\leq \mathbb{P}^* \left(\left| \frac{\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha) - \mathbb{E}^\mathcal{E}[\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)]}{n^{1-\theta}} \right| \geq \delta\beta/4 \right) \\ &+ \mathbb{P}^* \left(\left| \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbb{E}^\mathcal{E}[\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)] - \Lambda_1(\theta)D_\infty \right| \geq \frac{\delta}{4} D_\infty, D_\infty \geq \beta \right) \end{aligned}$$

By Chebyshev's inequality and then Cauchy-Schwartz inequality,

$$\begin{aligned} &\mathbb{P} \left(\left| \frac{\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha) - \mathbb{E}^\mathcal{E}[\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)]}{n^{1-\theta}} \right| \geq \delta\beta/4 \right) \\ &\leq \frac{16}{(\delta\beta)^2 n^{2-2\theta}} \mathbb{E} \left[\left(\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha) - \mathbb{E}^\mathcal{E}[\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)] \right)^2 \right] \\ &\leq \frac{16}{(\delta\beta)^2 n^{2-2\theta}} \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} 1 \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbb{E} \left[\text{Var}^\mathcal{E}(\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)) \right], \end{aligned}$$

which is $o_n(1)$ by Lemma 4.4 as long as $a \wedge \gamma > 8$. On the other hand,

$$\begin{aligned} &\mathbb{P}^* \left(\left| \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbb{E}^\mathcal{E}[\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)] - \Lambda_1(\theta)D_\infty \right| \geq \frac{\delta}{4} D_\infty, D_\infty \geq \beta \right) \\ &\leq \mathbb{P}^* \left(\left| \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{|x|=\ell} (1 + o_n(1)) e^{-V(x)} \mathbf{f}\left(\frac{n^\theta}{H_x}\right) \mathbf{1}_{\{x \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}\}} - \Lambda_1(\theta)D_\infty \right| \geq \frac{\delta}{4} D_\infty, D_\infty \geq \beta \right) \\ &+ \mathbb{P}^*(\inf V(u) < -\alpha). \end{aligned}$$

We thus deduce from Lemma 4.5 that

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}^* \left(\left| \frac{\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)}{n^{1-\theta}} - \Lambda_1(\theta)D_\infty \right| \geq \frac{\delta}{2} D_\infty, D_\infty \geq \beta \right) \leq c_2 e^{-\alpha}.$$

Going back to (4.10) and letting $\alpha \rightarrow \infty$ and $\beta \downarrow 0$, we therefore conclude that for any $\delta > 0$.

$$\limsup_{n \rightarrow \infty} \mathbb{P}^* \left(\left| \frac{R_{n^\theta}(\tau_n, 1)}{n^{1-\theta}} - \Lambda_1(\theta) D_\infty \right| \geq \delta D_\infty \right) = 0.$$

□

5 Generalised Seneta-Heyde scaling: proof of Lemmas 3.2 and 4.5

In this section, we study the following sum: for any $0 < A < B < \infty$,

$$\chi_i(A, B, r) := \sum_{Ar^2 \leq m \leq Br^2} \sum_{|z|=m} e^{-V(z)} F_i(z, r), \text{ for } i = 1, 2, 3; \quad (5.1)$$

where

$$F_1(z, r) := e^{V(z) - (1-\theta)r - b} \mathbf{1}_{\{\bar{V}(z) - V(z) \leq \theta r + t_r, V(z) \leq (1-\theta)r + b\}} \quad (5.2)$$

$$F_2(z, r) := e^{V(z) - (1-\theta)r - b} \mathbf{1}_{\{\bar{V}(z) \leq r + t_r, V(z) \leq (1-\theta)r + b\}} \quad (5.3)$$

$$F_3(z, r) := \mathbf{f} \left(\frac{e^{\theta r}}{H_z} \right) \mathbf{1}_{\{\bar{V}(z) \geq r + t_r, \max_{y \leq z} (\bar{V}(y) - V(y)) \leq r + s_r, \bar{V}(z) - V(z) \in [\theta r - K, \theta r + K]\}} \quad (5.4)$$

with $t_r = o(r)$, $s_r = o(r)$, $K > 0$ and $b \in \mathbb{R}$ such that $s_r + 6 \log r < t_r$. We are going to show that as $r \rightarrow \infty$.

$$\chi_i(A, B, r) \xrightarrow{\text{in } \mathbb{P}^*} \mathbf{C}_i(A, B) D_\infty, \text{ for } i = 1, 2, 3, \quad (5.5)$$

where $\mathbf{C}_i(A, B)$ are positive constants which will be determined later.

One can see immediately that Lemma 3.2 is mainly based on the convergences of χ_1 and χ_2 and that Lemma 4.5 is based on the convergence of χ_3 with $r = \log n$. To complete the proof of Lemma 3.2, as $F_1 \leq F_2$, we still need to check the following estimate.

Lemma 5.1. *For any $\alpha > 0$, as $\varepsilon \downarrow 0$, we have*

$$\limsup_{n \rightarrow \infty} \mathbf{E} \left[\sum_{m=1}^{\varepsilon(\log n)^2} \sum_{|z|=m} e^{-V(z)} F_2(z, \log n) \mathbf{1}_{\{\underline{V}(z) \geq -\alpha\}} \right] = o_\varepsilon(1); \quad (5.6)$$

$$\limsup_{n \rightarrow \infty} \mathbf{E} \left[\sum_{(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{|z|=m} e^{-V(z)} F_2(z, \log n) \mathbf{1}_{\{\underline{V}(z) \geq -\alpha\}} \right] = o_\varepsilon(1). \quad (5.7)$$

To conclude Lemma 4.5, in other words, to get (4.9), we need to compare $\{z \in \mathcal{L}_{\gamma n}\} = \{\max_{y \leq z} H_y \leq \frac{n}{(\log n)^\gamma}\}$ with $\{\max_{y \leq z} (\bar{V}(y) - V(y)) \leq r + s_r\}$. In fact, note that $e^{\bar{V}(y) - V(y)} \leq H_y \leq |z| e^{\bar{V}(y) - V(y)}$. Thus for $|z| \leq B(\log n)^2$ with $n \gg 1$,

$$\mathbf{1}_{\{\max_{y \leq z} (\bar{V}(y) - V(y)) \leq \log n - (\gamma+3) \log \log n\}} \leq \mathbf{1}_{\{z \in \mathcal{L}_{\gamma n}\}} \leq \mathbf{1}_{\{\max_{y \leq z} (\bar{V}(y) - V(y)) \leq \log n - \gamma \log \log n\}}$$

Note also that in Lemma 4.5, $t_r = a \log \log n$ with $a + \gamma > 6$. Therefore, we can deduce Lemma 4.5 from (5.5) for $i = 3$.

In the following, we prove (5.5) and check Lemma 5.1 in Section 6. Our proof of (5.5) mainly follows the idea of [BM19].

Outline of proof of (5.5). It is known that for any $\varepsilon \in (0, 1)$, there exists $k_0 \geq 1$ such that

$$\mathbf{P} \left(\inf_{n \geq k_0} \inf_{|z|=n} V(z) \geq 0 \right) \geq 1 - \varepsilon, \quad (5.8)$$

with the convention that $\inf \emptyset = \infty$. For any r such that $Ar^2 \geq 2k_0$, let

$$\tilde{\chi}_i(A, B, r, k_0) := \sum_{Ar^2 \leq m \leq Br^2} \sum_{|z|=m} \tilde{F}_i(z, r, k_0)$$

where $\tilde{F}_i(z, r, k_0) := F_i(z, r) \mathbf{1}_{\{\min_{z_0 \leq y \leq z} V(y) \geq 0\}}$ with $z_0 := z_{k_0}$. It then follows from (5.8) that for any $\varepsilon > 0$ and $i = 1, 2, 3$, there exists $k_0 \geq 1$ such that for any $k \geq k_0$,

$$\mathbf{P}(\forall r \geq 1, \chi_i(A, B, r) \neq \tilde{\chi}_i(A, B, r, k)) \leq 2\varepsilon. \quad (5.9)$$

So, according to [BM19], it suffices to show that for any $\lambda > 0$ and $i = 1, 2, 3$, a.s.,

$$\lim_{k_0 \rightarrow \infty} \limsup_{r \rightarrow \infty} \mathbf{E}[e^{-\lambda \tilde{\chi}_i(A, B, r, k_0)} | \mathcal{F}_{k_0}] = \lim_{k_0 \rightarrow \infty} \liminf_{r \rightarrow \infty} \mathbf{E}[e^{-\lambda \tilde{\chi}_i(A, B, r, k_0)} | \mathcal{F}_{k_0}] = \exp\{-\lambda \mathbf{C}_i(A, B) D_\infty\}. \quad (5.10)$$

By Lemma B.1 of [BM19] and a Cantor diagonal extraction argument, this yields the convergence in probability of $\tilde{\chi}_i(A, B, r, k_0(r))$ towards $\mathbf{C}_i(A, B) D_\infty$. Then (5.5) follows immediately from (5.9).

Let us check (5.10). Observe that by Jensen's inequality,

$$\begin{aligned} \mathbf{E}[e^{-\lambda \tilde{\chi}_i(A, B, r, k_0)} | \mathcal{F}_{k_0}] &= \prod_{|u|=k_0} \mathbf{E} \left[\exp\left\{-\lambda \sum_{Ar^2 \leq m \leq Br^2} \sum_{|z|=m} \mathbf{1}_{\{z_0=u\}} e^{-V(z)} \tilde{F}_i(z, r)\right\} \middle| \mathcal{F}_{k_0} \right] \\ &\geq \exp \left\{ -\lambda \sum_{|u|=k_0} \sum_{Ar^2 \leq m \leq Br^2} \mathbf{E} \left[\sum_{|z|=m} \mathbf{1}_{\{z_0=u\}} e^{-V(z)} \tilde{F}_i(z, r) \right] \middle| \mathcal{F}_{k_0} \right\} \\ &\geq \exp \left\{ -\lambda \sum_{|u|=k_0} (1 + o_r(1)) \mathbf{E}_{V(u)}[\hat{\chi}_i] \mathbf{1}_{\{\max_{|u|=k_0} \bar{V}(u) \leq r^{1/3}, \min_{|u|=k_0} \underline{V}(u) \geq -r^{1/3}\}} \right\}, \end{aligned} \quad (5.11)$$

where $\hat{\chi}_i = \hat{\chi}_i(A, B, r, k_0) := \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} \sum_{|z|=m} e^{-V(z)} \hat{F}_i(z, r, k_0)$ with

$$\begin{aligned} \hat{F}_1(z, r, k_0) &:= e^{V(z) - (1-\theta)r - b} \mathbf{1}_{\{\underline{V}(z) \geq 0, \bar{V}(z) - V(z) \leq \theta r + t_r, V(z) \leq (1-\theta)r + b\}}; \\ \hat{F}_2(z, r, k_0) &:= e^{V(z) - (1-\theta)r - b} \mathbf{1}_{\{\underline{V}(z) \geq 0, \bar{V}(z) \leq r + t_r, V(z) \leq (1-\theta)r + b\}}; \\ \hat{F}_3(z, r, k_0) &:= f \left(\frac{e^{\theta r}}{H_z} \right) \mathbf{1}_{\{\underline{V}(z) \geq 0, \bar{V}(z) \geq r + t_r, \max_{y \leq z} (\bar{V}(y) - V(y)) \leq r + s_r, \bar{V}(z) - V(z) \in [\theta r - K, \theta r + K]\}}. \end{aligned}$$

Let us explain a little the last inequality in (5.11). Note that if $\{\max_{|u|=k_0} \bar{V}(u) \leq r^{1/3}\}$, one has $\bar{V}(z) = \max_{z_0 \leq y \leq z} V(y)$. Thus, for $i = 1, 2$,

$$\mathbf{E} \left[\sum_{|z|=m} \mathbf{1}_{\{z_0=u\}} e^{-V(z)} \tilde{F}_i(z, r) \mid \mathcal{F}_{k_0} \right] = \mathbf{E}_{V(u)} \left[\sum_{|z|=m-k_0} e^{-V(z)} \hat{F}_i(z, r, k_0) \right].$$

For $i = 3$, one can see that given $\{\max_{|u|=k_0} \bar{V}(u) \leq r^{1/3}, \min_{|u|=k_0} \underline{V}(u) \geq -r^{1/3}\}$ and $\{\bar{V}(z) \geq r + t_r, \bar{V}(z) - V(z) \in [\theta r + d, \theta r + d + h]\}$, we have moreover $\{\max_{y \leq z} (\bar{V}(y) - V(y)) \leq r + s_r\} = \{\max_{z_0 \leq y \leq z} (\bar{V}(y) - V(y)) \leq r + s_r\}$ and $f(\frac{e^{\theta r}}{H_z}) = (1 + o_r(1))f(\frac{e^{\theta r}}{\sum_{z_0 \leq y \leq z} e^{V(y)-V(z)}})$ as

$$\frac{|H_z - \sum_{z_0 \leq y \leq z} e^{V(y)-V(z)}|}{H_z} \leq k_0 e^{r^{1/3} - r - t_r} = o_r(1).$$

This leads to

$$\mathbf{E} \left[\sum_{|z|=m} \mathbf{1}_{\{z_0=u\}} e^{-V(z)} \tilde{F}_3(z, r) \mid \mathcal{F}_{k_0} \right] = (1 + o_r(1)) \mathbf{E}_{V(u)} \left[\sum_{|z|=m-k_0} e^{-V(z)} \hat{F}_3(z, r, k_0) \right].$$

We next turn to the upper bound of $\mathbf{E}[e^{-\lambda \tilde{\chi}_i(A, B, r, k_0)} \mid \mathcal{F}_{k_0}]$. For any $\delta \in (0, 1)$, let $\lambda_\delta := \lambda e^{-\lambda \delta}$ and

$$\hat{\chi}_i^{(\delta)} := \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} \sum_{|z|=m} e^{-V(z)} \hat{F}_i(z, r, k_0) \mathbf{1}_{\{\sum_{|z|=m} e^{-V(z)} \hat{F}_i(z, r, k_0) \leq \frac{\delta}{Br^2}\}}$$

As a consequence of the fact $e^{-\lambda t} \leq 1 - \lambda_\delta t$ for any $t \in [0, \delta]$,

$$\begin{aligned} & \mathbf{E}[e^{-\lambda \tilde{\chi}_i(A, B, r, k_0)} \mid \mathcal{F}_{k_0}] \\ & \leq \prod_{|u|=k_0} \mathbf{E} \left[\exp\left\{-\lambda \sum_{Ar^2 \leq m \leq Br^2} \sum_{|z|=m} \mathbf{1}_{\{z_0=u\}} e^{-V(z)} \tilde{F}_i(z, r) \mathbf{1}_{\{\sum_{|z|=m} \mathbf{1}_{\{z_0=u\}} e^{-V(z)} \tilde{F}_i(z, r) \leq \frac{\delta}{Br^2}\}}\right\} \mid \mathcal{F}_{k_0} \right] \\ & \leq \prod_{|u|=k_0} \left(1 - \lambda_\delta \mathbf{E} \left[\sum_{Ar^2 \leq m \leq Br^2} \sum_{|z|=m} \mathbf{1}_{\{z_0=u\}} e^{-V(z)} \tilde{F}_i(z, r) \mathbf{1}_{\{\sum_{|z|=m} \mathbf{1}_{\{z_0=u\}} e^{-V(z)} \tilde{F}_i(z, r) \leq \frac{\delta}{Br^2}\}} \mid \mathcal{F}_{k_0} \right] \right) \\ & \leq \exp \left\{ -\lambda_\delta \sum_{|u|=k_0} \mathbf{E} \left[\sum_{Ar^2 \leq m \leq Br^2} \sum_{|z|=m} \mathbf{1}_{\{z_0=u\}} e^{-V(z)} \tilde{F}_i(z, r) \mathbf{1}_{\{\sum_{|z|=m} \mathbf{1}_{\{z_0=u\}} e^{-V(z)} \tilde{F}_i(z, r) \leq \frac{\delta}{Br^2}\}} \mid \mathcal{F}_{k_0} \right] \right\} \end{aligned} \quad (5.12)$$

which as explained above, for r large enough, is bounded by

$$\exp \left\{ -\lambda_\delta \sum_{|u|=k_0} (1 + o_r(1)) \mathbf{E}_{V(u)}[\hat{\chi}_i^{(\delta/2)}] \right\} + \mathbf{1}_{\{\max_{|u|=k_0} \bar{V}(u) > r^{1/3}\}} + \mathbf{1}_{\{\min_{|u|=k_0} \underline{V}(u) < -r^{1/3}\}}.$$

For (5.11) and (5.12), letting $r \rightarrow \infty$ brings out that

$$\begin{aligned} \liminf_{r \rightarrow \infty} \exp \left\{ -\lambda \sum_{|u|=k_0} (1 + o_r(1)) \mathbf{E}_{V(u)}[\tilde{\chi}_i] \right\} & \leq \liminf_{r \rightarrow \infty} \mathbf{E}[e^{-\lambda \tilde{\chi}_i(A, B, r, k_0)} \mid \mathcal{F}_{k_0}] \\ & \leq \limsup_{r \rightarrow \infty} \mathbf{E}[e^{-\lambda \tilde{\chi}_i(A, B, r, k_0)} \mid \mathcal{F}_{k_0}] \leq \limsup_{r \rightarrow \infty} \exp \left\{ -\lambda_\delta \sum_{|u|=k_0} (1 + o_r(1)) \mathbf{E}_{V(u)}[\hat{\chi}_i^{(\delta/2)}] \right\} \end{aligned} \quad (5.13)$$

Next, we claim that for any $x \geq 0$, $\lim_{r \rightarrow \infty} \mathbf{E}_x[\hat{\chi}_i] = C_i(A, B)\mathcal{R}(x)e^{-x}$ with $C_i(A, B)$ a positive constant and $\mathcal{R}(\cdot)$ is the renewal function defined in (A.3). Moreover, we stress that for $\delta > 0$ and $x \gg 1$,

$$\limsup_{r \rightarrow \infty} \mathbf{E}_x[\hat{\chi}_i - \hat{\chi}_i^{(\delta)}] = o_x(1)\mathcal{R}(x)e^{-x}.$$

These are stated in the following lemma.

Lemma 5.2. *For any $x \geq 0$, $\delta > 0$, as $r \rightarrow \infty$,*

$$\lim_{r \rightarrow \infty} \mathbf{E}_x \left[\sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} \sum_{|z|=m} e^{-V(z)} \hat{F}_i(z, r, k_0) \right] = C_i(A, B)\mathcal{R}(x)e^{-x}, \quad (5.14)$$

$$\limsup_{r \rightarrow \infty} \mathbf{E}_x \left[\sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} \sum_{|z|=m} e^{-V(z)} \hat{F}_i(z, r, k_0) \mathbf{1}_{\{\sum_{|z|=m} e^{-V(z)} \hat{F}_i(z, r, k_0) > \frac{\delta}{Br^2}\}} \right] = o_x(1)\mathcal{R}(x)e^{-x}, \quad (5.15)$$

where

$$C_1(A, B) = C_2(A, B) = \frac{\mathbf{c}_+}{\sigma} \int_A^B \mathcal{C}_0\left(\frac{\theta}{\sqrt{u}}, \frac{1-\theta}{\sqrt{u}}\right) \frac{du}{u},$$

and

$$C_3(A, B) = \int_{-K}^K \mathbf{E}\left[\mathbf{f}\left(\frac{e^{-s}}{\mathcal{H}_\infty + \mathcal{H}_\infty^{(-)} - 1}\right)\right] ds \int_A^B \mathcal{G}\left(\frac{1}{\sqrt{u}}, \frac{\theta}{\sqrt{u}}\right) \frac{du}{u}.$$

By (A.4), $\mathcal{R}(u) \sim \mathbf{c}_\mathcal{R}u$ as $u \rightarrow \infty$. Recall also that the derivative martingale $D_{k_0} = \sum_{|u|=k_0} V(u)e^{-V(u)}$ converges a.s. to some non-negative limit D_∞ . As a result, we obtain

$$\lim_{k_0 \rightarrow \infty} \lim_{r \rightarrow \infty} \mathbf{E}[e^{-\lambda \tilde{\chi}_i(A, B, r, k_0)} | \mathcal{F}_{k_0}] = \exp\{-\lambda \mathbf{c}_\mathcal{R} C_i(A, B) D_\infty\}.$$

By Lemma B.1 of [BM19] and a Cantor diagonal extraction argument, this yields convergence in probability of $\tilde{\chi}_i(A, B, r, k_0(r))$ towards $\mathbf{c}_\mathcal{R} C_i(A, B) D_\infty$. In view of (5.9), we obtain the convergence in probability of $\chi_i(A, B, r)$ towards $\mathbf{C}_i(A, B) D_\infty$ under \mathbf{P} (hence under \mathbf{P}^*) with $\mathbf{C}_i(A, B) = \mathbf{c}_\mathcal{R} C_i(A, B)$. Note that $\int_{\mathbb{R}} \mathbf{E}\left[\mathbf{f}\left(\frac{e^{-s}}{\mathcal{H}_\infty + \mathcal{H}_\infty^{(-)} - 1}\right)\right] ds = 1$. So Lemma 4.5 holds and finally Proposition 1.3 holds with

$$\Lambda_1(\theta) = \mathbf{c}_\mathcal{R} \int_0^\infty \mathcal{G}\left(\frac{1}{\sqrt{u}}, \frac{\theta}{\sqrt{u}}\right) \frac{du}{u}.$$

And Proposition 1.2 holds with

$$\Lambda_0(\theta) = \mathbf{c}_\mathcal{R} \frac{\mathbf{c}_+}{\sigma} \int_0^\infty \mathcal{C}_0\left(\frac{\theta}{\sqrt{u}}, \frac{1-\theta}{\sqrt{u}}\right) \frac{du}{u} = \frac{\sqrt{2}}{\sqrt{\pi}\sigma^2} \int_0^\infty \mathcal{C}_0\left(\frac{\theta}{\sqrt{u}}, \frac{1-\theta}{\sqrt{u}}\right) \frac{du}{u},$$

because of (A.5).

In order to conclude (5.5), we only need to prove Lemma 5.2 mainly for $i = 2, 3$.

Proof of Lemma 5.2. Proof of (5.14). By Many-to-one lemma, we have

$$\mathbf{E}_x[\hat{\chi}_2] = e^{-x} \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} \mathbf{E}_x \left[e^{S_m - (1-\theta)r - b}; \underline{S}_m \geq 0, \bar{S}_m \leq r + t_r, S_m \leq (1-\theta)r + b \right]$$

By (A.31), as $r \rightarrow \infty$,

$$\begin{aligned} \mathbf{E}_x[\hat{\chi}_2] &= \mathcal{R}(x) e^{-x} \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} \frac{1 + o_r(1)}{m} \mathcal{C}_0\left(\frac{r}{\sqrt{m}}, \frac{(1-\theta)r}{\sqrt{m}}\right) \\ &= \mathcal{R}(x) e^{-x} (1 + o_r(1)) \int_{A - \frac{k_0}{r^2}}^{B - \frac{k_0}{r^2}} \frac{\mathbf{c}_+}{\sigma} \mathcal{C}_0\left(\frac{\theta}{\sqrt{u}}, \frac{1-\theta}{\sqrt{u}}\right) \frac{du}{u} \end{aligned}$$

which converges to $\mathcal{R}(x) e^{-x} \frac{\mathbf{c}_+}{\sigma} \int_A^B \mathcal{C}_0\left(\frac{\theta}{\sqrt{u}}, \frac{1-\theta}{\sqrt{u}}\right) \frac{du}{u}$. By (A.32) instead of (A.31), we get (5.14) for $i = 1$. Moreover, we get that

$$C_1(A, B) = C_2(A, B) = \int_A^B \frac{\mathbf{c}_+}{\sigma} \mathcal{C}_0\left(\frac{\theta}{\sqrt{u}}, \frac{1-\theta}{\sqrt{u}}\right) \frac{du}{u}.$$

For $i = 3$, by (A.29), as $r \rightarrow \infty$,

$$\begin{aligned} \mathbf{E}_x[\hat{\chi}_3] &= e^{-x} \mathcal{R}(x) \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} \int_{-K}^K \mathbf{E} \left[\mathbf{f}\left(\frac{e^{-s}}{\mathcal{H}_\infty + \mathcal{H}_\infty^{(-)}} - 1\right) \right] ds \frac{1 + o_r(1)}{m} \mathcal{G}\left(\frac{r}{\sqrt{m}}, \frac{\theta r}{\sqrt{m}}\right) \\ &\rightarrow \mathcal{R}(x) e^{-x} \int_{-K}^K \mathbf{E} \left[\mathbf{f}\left(\frac{e^{-s}}{\mathcal{H}_\infty + \mathcal{H}_\infty^{(-)}}\right) \right] ds \int_A^B \mathcal{G}\left(\frac{1}{\sqrt{u}}, \frac{\theta}{\sqrt{u}}\right) \frac{du}{u} \end{aligned}$$

Proof of (5.15). First, by Markov inequality,

$$\begin{aligned} \mathbf{E}_x[\hat{\chi}_i - \hat{\chi}_i^{(\delta)}] &= \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} \mathbf{E}_x \left[\sum_{|z|=m} e^{-V(z)} \hat{F}_i(z, r, k_0) \mathbf{1}_{\left\{ \sum_{|z|=m} e^{-V(z)} \hat{F}_i(z, r, k_0) > \frac{\delta}{Br^2} \right\}} \right] \\ &\leq \sum_{m=Ar^2 - k_0}^{Br^2 - k_0} \mathbf{E}_x \left[\sum_{|z|=m} e^{-V(z)} \hat{F}_i(z, r, k_0) \left(\frac{Br^2 \sum_{|z|=m} e^{-V(z)} \hat{F}_i(z, r, k_0)}{\delta} \wedge 1 \right) \right]. \end{aligned}$$

Note that $\hat{F}_1 \leq \hat{F}_2$. So, we only need to treat it for $i = 2, 3$. By Lyons' change of measure and Proposition 2.2, we then get that

$$\begin{aligned} &\mathbf{E}_x[\hat{\chi}_i - \hat{\chi}_i^{(\delta)}] \\ &\leq \sum_{m=Ar^2 - k_0}^{Br^2 - k_0} e^{-x} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\hat{F}_i(w_m, r, k_0) \left(\left[\frac{Br^2}{\delta} \sum_{j=1}^m \sum_{u \in \Omega(w_j)} \sum_{|z|=m, z \geq u} e^{-V(z)} \hat{F}_i(z, r, k_0) + e^{-V(w_m)} \hat{F}_i(w_m, r, k_0) \right] \wedge 1 \right) \right] \\ &\leq UB_1(A, B, r, i) + UB_2(A, B, r, i), \end{aligned} \tag{5.16}$$

where

$$\begin{aligned} UB_1(A, B, r, i) &:= \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} e^{-x} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\hat{F}_i(w_m, r, k_0) \left(\frac{Br^2}{\delta} e^{-V(w_m)} \hat{F}_i(w_m, r, k_0) \wedge 1 \right) \right], \\ UB_2(A, B, r, i) &:= \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} e^{-x} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\hat{F}_i(w_m, r, k_0) \left(\left[\frac{Br^2}{\delta} \sum_{j=1}^m \sum_{u \in \Omega(w_j)} \sum_{|z|=m, z \geq u} e^{-V(z)} \hat{F}_i(z, r, k_0) \right] \wedge 1 \right) \right]. \end{aligned}$$

Observe that for $i = 3$, by Proposition 2.2 and (A.1),

$$\begin{aligned} UB_1(A, B, r, 3) &\leq \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} \frac{Br^2 e^{-x}}{\delta} \mathbf{E}_x \left[e^{-S_m} \mathbf{1}_{\{\underline{S}_m \geq 0, \bar{S}_m \geq r + t_r, \bar{S}_m - S_m \in [\theta r - K, \theta r + K]\}} \right] \\ &\leq \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} \frac{Br^2 e^{-(1-\theta)r - t_r + K}}{\delta \sqrt{m}} c_3 (1+x) e^{-x} = o_r(1) \mathcal{R}(x) e^{-x}. \end{aligned}$$

Note also that as $\widehat{F}_2 \leq 1$, by (A.2),

$$\begin{aligned} UB_1(A, B, r, 2) &\leq \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} e^{-x} \frac{Br^2}{\delta} \mathbf{E}_{\widehat{Q}_x} \left[\widehat{F}_2(w_k, r, k_0) e^{-V(w_m)} \right] \\ &\leq \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} e^{-x} \frac{Br^2 e^{-(1-\theta)r - b}}{\delta} \mathbf{P}_x(\underline{S}_m \geq 0, S_m \leq (1-\theta)r + b) \\ &\leq \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} e^{-x} \frac{Br^2 e^{-(1-\theta)r - b}}{\delta} \frac{c_4(1+x)(1+r)^2}{m^{3/2}} = o_r(1) \mathcal{R}(x) e^{-x}. \end{aligned}$$

Recall that $\mathcal{G} = \sigma\{(w_k, V(w_k))_{k \geq 0}, (u, V(u))_{u \in \cup_{k \geq 0} \Omega(w_k)}\}$. So,

$$\begin{aligned} UB_2(A, B, r, i) &\leq \\ &\sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} e^{-x} \mathbf{E}_{\widehat{Q}_x} \left[\widehat{F}_i(w_m, r, k_0) \left(\left(\frac{Br^2}{\delta} \sum_{j=1}^m \sum_{u \in \Omega(w_j)} \mathbf{E}_{\widehat{Q}_x} \left[\sum_{|z|=m, z \geq u} e^{-V(z)} \widehat{F}_i(z, r, k_0) \Big| \mathcal{G} \right] \right) \wedge 1 \right) \right], \end{aligned} \quad (5.17)$$

where for $i = 2$ and $u \in \Omega(w_j)$, by branching property at u and then by (A.2),

$$\begin{aligned} &\mathbf{E}_{\widehat{Q}_x} \left[\sum_{|z|=m, z \geq u} e^{-V(z)} \widehat{F}_2(z, r, k_0) \Big| \mathcal{G} \right] \\ &\leq e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \mathbf{E}_{V(u)} [e^{S_{m-j} - (1-\theta)r - b}; \underline{S}_{m-j} \geq 0, S_{m-j} \leq (1-\theta)r - b] \\ &\leq e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \frac{c_5(1+V(u))(1+r)}{(m-j+1)^{3/2}} \mathbf{1}_{\{j < m/2\}} + e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \mathbf{1}_{\{j \geq m/2\}}, \end{aligned} \quad (5.18)$$

and for $i = 3$ and $u \in \Omega(w_j)$,

$$\begin{aligned} &\mathbf{E}_{\widehat{Q}_x} \left[\sum_{|z|=m, z \geq u} e^{-V(z)} \widehat{F}_3(z, r, k_0) \Big| \mathcal{G} \right] \\ &= \mathbf{E}_{\widehat{Q}_x} \left[\sum_{|z|=m, z \geq u} e^{-V(z)} \widehat{F}_3(z, r, k_0) \mathbf{1}_{\{\bar{V}(z) = \bar{V}(u)\}} \Big| \mathcal{G} \right] + \mathbf{E}_{\widehat{Q}_x} \left[\sum_{|z|=m, z \geq u} e^{-V(z)} \widehat{F}_3(z, r, k_0) \mathbf{1}_{\{\bar{V}(z) > \bar{V}(u)\}} \Big| \mathcal{G} \right] \\ &\leq e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0, \bar{V}(u) - V(u) \leq r + s_r, \bar{V}(u) \geq r + t_r\}} \mathbf{P}_{V(u)}(\underline{S}_{m-j} \geq 0, x - S_{m-j} \in [\theta r - K, \theta r + K]) \Big|_{x = \bar{V}(u)} \\ &+ e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \mathbf{P}_{V(u)}(\underline{S}_{m-j} \geq 0, \bar{S}_{m-j} - S_{m-j} \in [\theta r - K, \theta r + K], \max_{k \leq m-j} (\bar{S}_k - S_k) \leq r + s_r, \bar{S}_{m-j} \geq r + t_r), \end{aligned}$$

where by (A.25) for $j < m/2$ and $V(u) \leq r/2$, one has

$$\begin{aligned} & \mathbf{P}_{V(u)}(\underline{S}_{m-j} \geq 0, \bar{S}_{m-j} - S_{m-j} \in [\theta r - K, \theta r + K], \max_{k \leq m-j} (\bar{S}_k - S_k) \leq r + s_r, \bar{S}_{m-j} \geq r + t_r) \\ & \leq \mathbf{1}_{\{j \geq m/2\}} + \mathbf{1}_{\{j < m/2, V(u) \geq r/2\}} + c_6(1 + V(u)) \frac{(1 + K^2)(1 + r)}{(m - j)^{3/2}} \mathbf{1}_{\{j < m/2, V(u) \leq r/2\}}. \end{aligned} \quad (5.19)$$

Moreover, by (A.2), one sees that

$$\begin{aligned} & \mathbf{E}_{\hat{Q}_x} \left[\sum_{|z|=m, z \geq u} e^{-V(z)} \hat{F}_3(z, r, k_0) | \mathcal{G} \right] \\ & \leq e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0, \bar{V}(u) - V(u) \leq r + s_r, \bar{V}(u) \geq r + t_r\}} \left[\frac{c_7(1 + V(u))(\bar{V}(u) - \theta r + K)(1 + 2K)}{(m - j)^{3/2}} \wedge 1 \right] \\ & \quad + e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \mathbf{1}_{\{j \geq m/2\}} + e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \mathbf{1}_{\{j < m/2, V(u) \geq r/2\}} \\ & \quad + c_6(1 + V(u)) e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0, V(u) \leq r/2, j < m/2\}} \frac{(1 + K^2)(1 + r)}{(m - j)^{3/2}} \\ & \leq c_8(1 + V(u))^2 e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \frac{(1 + K^2)(1 + r)}{(m - j)^{3/2}} \mathbf{1}_{\{j < m/2\}} + 2e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \mathbf{1}_{\{j \geq m/2\}} \\ & \quad + e^{-r/4} e^{-V(u)/2} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \mathbf{1}_{\{j < m/2\}}. \end{aligned} \quad (5.21)$$

Plugging (5.18) or (5.20) to (5.17) yields that

$$UB_2(A, B, r, 2) \leq UB_2^<(A, B, r, 2) + UB_2^>(A, B, r, 2) \quad (5.22)$$

where

$$\begin{aligned} UB_2^<(A, B, r, 2) & := \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} e^{-x} \frac{Br^3}{\delta m^{3/2}} \sum_{j=1}^{m/2} \mathbf{E}_{\hat{Q}_x} \left[\hat{F}_2(w_m, r, k_0) \sum_{u \in \Omega(w_j)} (1 + V(u)) e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \right] \\ UB_2^>(A, B, r, 2) & := \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} e^{-x} \mathbf{E}_{\hat{Q}_x} \left[\hat{F}_2(w_m, r, k_0) \left(\sum_{j=m/2}^m \sum_{u \in \Omega(w_j)} \frac{Br^2}{\delta} e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \right) \wedge 1 \right] \end{aligned}$$

and that

$$UB_2(A, B, r, 3) \leq UB_2^{(1)}(A, B, r, 3) + UB_2^{(2)}(A, B, r, 3) + UB_2^{(3)}(A, B, r, 3), \quad (5.23)$$

where

$$\begin{aligned} UB_2^{(1)}(A, B, r, 3) & := \sum_{m=Ar^2 - k_0}^{Br^2 - k_0} e^{-x} \frac{2B(1 + K^2)r^3}{\delta m^{3/2}} \mathbf{E}_{\hat{Q}_x} \left[\hat{F}_3(w_m, r, k_0) \left(\sum_{j=1}^{m/2} \sum_{u \in \Omega(w_j)} (1 + V(u))^2 e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \right) \right] \\ UB_2^{(2)}(A, B, r, 3) & := \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} e^{-x} \frac{Br^2}{\delta} e^{-r/4} \mathbf{E}_{\hat{Q}_x} \left[\hat{F}_3(w_m, r, k_0) \left(\sum_{j=1}^{m/2} \sum_{u \in \Omega(w_j)} e^{-V(u)/2} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \right) \right] \\ UB_2^{(3)}(A, B, r, 3) & := \sum_{Ar^2 - k_0 \leq m \leq Br^2 - k_0} e^{-x} \mathbf{E}_{\hat{Q}_x} \left[\hat{F}_3(w_m, r, k_0) \left(\sum_{j=m/2}^m \sum_{u \in \Omega(w_j)} 2 \frac{Br^2}{\delta} e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \right) \wedge 1 \right]. \end{aligned}$$

In the rest part, we will check that all these terms are $o_x(1)\mathcal{R}(x)e^{-x}$ for $r \rightarrow \infty$ and then $x \gg 1$.

We will first treat $UB_2^<(A, B, r, 2)$, $UB_2^{(1)}(A, B, r, 3)$ and $UB_2^{(2)}(A, B, r, 3)$ in the similar way. For any $u \in \mathbb{T}$, let $\Delta V(u) = V(u) - V(u^*)$ be its displacement. Write $\Delta_+ V(u)$ for $\Delta V(u) \vee 0$. Then,

$$\sum_{u \in \Omega(w_j)} (1 + V(u))e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \leq \sum_{u \in \Omega(w_j)} (1 + V(u))^2 e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \leq e^{-V(w_{j-1})/2} \mathbf{1}_{\{\underline{V}(w_{j-1}) \geq 0\}} V_j^+,$$

with $V_j^+ := \sum_{u \in \Omega(w_j)} e^{-\Delta V(u)/2}$. Consequently,

$$\begin{aligned} & UB_2^<(A, B, r, 2) \\ & \leq \sum_{m=Ar^2-k_0}^{Br^2-k_0} e^{-x} \frac{Br^3}{\delta m^{3/2}} \sum_{j=1}^{m/2} \mathbf{E}_{\hat{\mathcal{Q}}_x} \left[e^{V(w_m) - (1-\theta)r - b} \mathbf{1}_{\{\underline{V}(w_m) \geq 0, \bar{V}(w_m) \leq r + t_r, V(w_m) \leq (1-\theta)r + b\}} e^{-V(w_{j-1})/2} V_j^+ \right], \end{aligned}$$

which by Markov property at time j and then by (A.23), is bounded by

$$\begin{aligned} & \sum_{m=Ar^2-k_0}^{Br^2-k_0} e^{-x} \frac{Br^3}{\delta m^{3/2}} \sum_{j=1}^{m/2} \mathbf{E}_{\hat{\mathcal{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_j) \geq 0\}} e^{-V(w_{j-1})/2} V_j^+ \mathbf{E}_{V(w_j)} [e^{S_{m-j} - (1-\theta)r + b} \mathbf{1}_{\{S_{m-j} \geq 0, S_{m-j} \leq (1-\theta)r - b\}}] \right] \\ & \leq \sum_{m=Ar^2-k_0}^{Br^2-k_0} e^{-x} \frac{Br^3}{\delta m^{3/2}} \sum_{j=1}^{m/2} \mathbf{E}_{\hat{\mathcal{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_j) \geq 0\}} e^{-V(w_{j-1})/2} V_j^+ (1 + V(w_j)) \right] \frac{c_9 r}{(m-j)^{3/2}}. \end{aligned}$$

Here $(1 + V(w_j)) \mathbf{1}_{\{\underline{V}(w_j) \geq 0\}} \leq (1 + V(w_{j-1})) \mathbf{1}_{\{\underline{V}(w_{j-1}) \geq 0\}} (1 + \Delta_+ V(w_j))$ and then Markov property at time $j-1$ implies that

$$\begin{aligned} & \mathbf{E}_{\hat{\mathcal{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_j) \geq 0\}} e^{-V(w_{j-1})/2} V_j^+ (1 + V(w_j)) \right] \\ & \leq \mathbf{E}_{\hat{\mathcal{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_{j-1}) \geq 0\}} (1 + V(w_{j-1})) e^{-V(w_{j-1})/2} \right] \mathbf{E}_{\hat{\mathcal{Q}}} [V_1^+ (1 + V_+(w_1))], \end{aligned}$$

where by Proposition 2.2,

$$\mathbf{E}_{\hat{\mathcal{Q}}} [V_1^+ (1 + V_+(w_1))] = \mathbf{E} \left[\sum_{|u|=1} (1 + V_+(u)) e^{-V(u)} \sum_{|v|=1, v \neq u} e^{-V(v)/2} \right].$$

By Cauchy-Schwartz inequality and (1.5),

$$\begin{aligned} \mathbf{E}_{\hat{\mathcal{Q}}} [V_1^+ (1 + V_+(w_1))]^2 & \leq \mathbf{E} \left[\left(\sum_{|u|=1} (1 + V_+(u)) e^{-V(u)} \right)^2 \right] \mathbf{E} \left[\left(\sum_{|u|=1} e^{-V(u)/2} \right)^2 \right] \\ & \leq \mathbf{E} \left[\left(\sum_{|u|=1} (1 + V_+(u)) e^{-V(u)} \right)^2 \right] \mathbf{E} \left[N \sum_{|u|=1} e^{-V(u)} \right] \\ & \leq \mathbf{E} \left[\left(\sum_{|u|=1} (1 + V_+(u)) e^{-V(u)} \right)^2 \right] \mathbf{E}[N^2] \mathbf{E} \left[\left(\sum_{|u|=1} e^{-V(u)} \right)^2 \right] < \infty. \end{aligned}$$

Similarly, we also have $\mathbf{E}_{\hat{\mathbf{Q}}}[V_1^+(1 + V_+(w_1))^2] < \infty$. It follows that

$$\begin{aligned} & UB_2^{\leq}(A, B, r, 2) \\ & \leq \sum_{m=Ar^2-k_0}^{Br^2-k_0} e^{-x} \frac{Br^3}{\delta m^{3/2}} \sum_{j=1}^{m/2} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_{j-1}) \geq 0\}} (1 + V(w_{j-1})) e^{-V(w_{j-1})/2} \right] \frac{c_9 r}{(m-j)^{3/2}} \\ & \leq \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \frac{c_{10} r^4}{\delta m^3} \sum_{j=1}^{m/2} \mathbf{E}_x \left[\mathbf{1}_{\{\underline{S}_{j-1} \geq 0\}} e^{-S_{j-1}/4} \right] \leq e^{-x} \mathbf{E}_x \left[\sum_{j \geq 0} e^{-S_j/4} \mathbf{1}_{\{\underline{S}_j \geq 0\}} \right], \end{aligned}$$

which by (A.12) shows that $UB_2^{\leq}(A, B, r, 2) = o_x(1)\mathcal{R}(x)e^{-x}$. For $UB_2^{(1)}(A, B, r, 3)$, as $\mathbf{f}(t) \leq 1$, we have

$$\begin{aligned} & \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\hat{F}_3(w_m, r, k_0) \left(\sum_{j=1}^{m/2} \sum_{u \in \Omega(w_j)} (1 + V(u))^2 e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \right) \right] \tag{5.24} \\ & \leq \sum_{j=1}^{m/2} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_m) \geq 0, \bar{V}(w_m) \geq r+t_r, \max_{k \leq m} (\bar{V}(k) - V(k)) \leq r+s_r, \bar{V}(w_m) - V(w_m) \in [\theta r - K, \theta r + K]\}} e^{-V(w_{j-1})/2} V_j^+ \right] \\ & = \sum_{j=1}^{m/2} \sum_{i=1}^m \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_m) \geq 0, \tau_m^V = i, \bar{V}(w_m) \geq r+t_r, \max_{k \leq m} (\bar{V}(k) - V(k)) \leq r+s_r, \bar{V}(w_m) - V(w_m) \in [\theta r - K, \theta r + K]\}} e^{-V(w_{j-1})/2} V_j^+ \right] \end{aligned}$$

where $\tau_m^V := \inf\{i \leq m : V(w_i) = \bar{V}(w_m)\}$.

On the one hand, if $\tau_m^V \geq j+1$, by Markov property at time j , one sees that

$$\begin{aligned} & \sum_{j=1}^{m/2} \sum_{i=j+1}^m \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_m) \geq 0, \tau_m^V = i, \bar{V}(w_m) \geq r+t_r, \max_{k \leq m} (\bar{V}(k) - V(k)) \leq r+s_r, \bar{V}(w_m) - V(w_m) \in [\theta r - K, \theta r + K]\}} e^{-V(w_{j-1})/2} V_j^+ \right] \\ & \leq \sum_{j=1}^{m/2} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_j) \geq 0\}} e^{-V(w_{j-1})/2} V_j^+ \mathbf{E}_{V(w_j)} \left[\mathbf{1}_{\{\underline{S}_{m-j} \geq 0, \bar{S}_{m-j} \geq r+t_r, \max_{k \leq m-j} (\bar{S}_k - S_k) \leq r+s_r, \bar{S}_{m-j} - S_{m-j} \in [\theta r - K, \theta r + K]\}} \right] \right], \end{aligned}$$

which by (A.25) and (A.12) is bounded by

$$\begin{aligned} & c_{11} \sum_{j=1}^{m/2} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_j) \geq 0\}} e^{-V(w_{j-1})/2} V_j^+ (1 + V(w_j)) \frac{(1 + K^2)(1 + r)}{(m-j)^{3/2}} \right] \\ & \leq \frac{c_{11}(1 + K^2)(1 + r)}{m^{3/2}} \mathbf{E}_x \left[\sum_{j \geq 1} (1 + S_{j-1}) e^{-S_{j-1}/2} \mathbf{1}_{\{\underline{S}_{j-1} \geq 0\}} \right] \mathbf{E}_{\hat{\mathbf{Q}}} [(1 + V_+(w_1) V_1^+)] \\ & = \frac{c_{11}(1 + K^2)(1 + r)}{m^{3/2}} o_x(1)\mathcal{R}(x). \end{aligned}$$

On the other hand, if $\tau_m^V \leq j$, again by Markov property at time j ,

$$\begin{aligned} & \sum_{j=1}^{m/2} \sum_{i=1}^j \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_m) \geq 0, \tau_m^V = i, \bar{V}(w_m) \geq r+t_r, \max_{k \leq m} (\bar{V}(k) - V(k)) \leq r+s_r, \bar{V}(w_m) - V(w_m) \in [\theta r - K, \theta r + K]\}} e^{-V(w_{j-1})/2} V_j^+ \right] \\ & \leq \sum_{j=1}^{m/2} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_j) \geq 0\}} e^{-V(w_{j-1})/2} V_j^+ \mathbf{P}_{V(w_j)}(\underline{S}_{m-j} \geq 0, x - S_{m-j} \in [\theta r - K, \theta r + K]) \Big|_{x=\bar{V}(w_j)} \right] \end{aligned}$$

where by (A.2),

$$\mathbf{P}_{V(w_j)}(S_{m-j} \geq 0, x - S_{m-j} \in [\theta r - K, \theta r + K])|_{x=\bar{V}(w_j)} \leq \frac{c_{12}(1+K^2)(1+V(w_j))(1+\bar{V}(w_j) - \theta r)}{(m-j)^{3/2}}$$

which is bounded by $\frac{c_{13}(1+K^2)(1+V(w_j))^2(1+r)}{(m-j)^{3/2}}$ because $\bar{V}(w_j) - V(w_j) \leq r + s_r$. Again by Markov property at time $j-1$ and (A.12), we get that

$$\begin{aligned} & \sum_{j=1}^{m/2} \sum_{i=1}^j \mathbf{E}_{\tilde{\mathbf{Q}}_x} \left[\mathbf{1}_{\{V(w_m) \geq 0, \tau_m^V = i, \bar{V}(w_m) \geq r + t_r, \max_{k \leq m} (\bar{V}(k) - V(k)) \leq r + s_r, \bar{V}(w_m) - V(w_m) \in [\theta r - K, \theta r + K]\}} e^{-V(w_{j-1})/2} V_j^+ \right] \\ & \leq \frac{c_{13}(1+K^2)(1+r)}{m^{3/2}} \mathbf{E}_x \left[\sum_{j \geq 1} (1+S_{j-1})^2 e^{-S_{j-1}/2} \mathbf{1}_{\{\bar{S}_{j-1} \geq 0\}} \right] \mathbf{E}_{\tilde{\mathbf{Q}}} [V_1^+ (1+V_+(w_1))^2] \\ & = \frac{c_{13}(1+K^2)(1+r)}{m^{3/2}} o_x(1) \mathcal{R}(x). \end{aligned}$$

Combining these inequalities and going back to (5.24), we have

$$\mathbf{E}_{\tilde{\mathbf{Q}}_x} \left[\hat{F}_3(w_m, r, k_0) \left(\sum_{j=1}^{m/2} \sum_{u \in \Omega(w_j)} (1+V(u))^2 e^{-V(u)} \mathbf{1}_{\{V(u) \geq 0\}} \right) \right] \leq \frac{c_{14}(1+K^2)(1+r)}{m^{3/2}} o_x(1) \mathcal{R}(x).$$

This implies that

$$UB_2^{(1)}(A, B, r, 3) \leq \sum_{m=Ar^2-k_0}^{Br^2-k_0} e^{-x} \frac{2B(1+K^2)r^3}{\delta m^{3/2}} \frac{c_{14}(1+K^2)(1+r)}{m^{3/2}} o_x(1) \mathcal{R}(x) = o_x(1) \mathcal{R}(x) e^{-x}. \quad (5.25)$$

Note that $\sum_{u \in \Omega(w_j)} e^{-V(u)/2} \leq e^{-V(w_{j-1})/2} V_j^+$. So similarly as above,

$$UB_2^{(2)}(A, B, r, 3) = o_x(1) \mathcal{R}(x) e^{-x}. \quad (5.26)$$

Let us turn to bound $UB_2^{(3)}(A, B, r, 3)$ in (5.23). Let $\underline{V}(w_{[j,m]}) := \min_{j \leq k \leq m} V(w_k)$ and $S_{[j,m]} := \min_{j \leq k \leq m} S_k$. Observe that

$$\begin{aligned} & UB_2^{(3)}(A, B, r, 3) = \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \mathbf{E}_{\tilde{\mathbf{Q}}_x} \left[\hat{F}_3(w_m, r, k_0) \left(\sum_{j=m/2}^m \sum_{u \in \Omega(w_j)} 2 \frac{Br^2}{\delta} e^{-V(u)} \mathbf{1}_{\{V(u) \geq 0\}} \right) \wedge 1 \right] \\ & \leq \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \mathbf{E}_{\tilde{\mathbf{Q}}_x} \left[\hat{F}_3(w_m, r, k_0) \mathbf{1}_{\{\underline{V}(w_{[m/2-1,m]}) \leq 6 \log r\}} \right] \\ & \quad + \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \frac{2Br^2}{\delta} \sum_{j=m/2}^m \mathbf{E}_{\tilde{\mathbf{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_n) \geq 0, \underline{V}(w_{[m/2-1,m]}) \geq 6 \log r\}} e^{-V(w_{j-1})} \sum_{u \in \Omega(w_j)} e^{-\Delta V(u)} \right]. \quad (5.27) \end{aligned}$$

On the one hand, by Proposition 2.2,

$$\begin{aligned}
& \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\hat{F}_3(z, r, k_0) \mathbf{1}_{\{V(w_{[m/2-1, m]}) \leq 6 \log r\}} \right] \\
& \leq \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{V(w_n) \geq 0, \bar{V}(w_n) \geq r + t_r, \max_{k \leq n} (\bar{V}(w_k) - V(w_k)) \leq r + s_r, \bar{V}(w_n) - V(w_n) \in [\theta r - K, \theta r + K]\}} \mathbf{1}_{\{V(w_{[m/2-1, m]}) \leq 6 \log r\}} \right] \\
& \leq \sum_{j=m/2-1}^{m-1} \mathbf{P}_x(\underline{S}_m \geq 0, \bar{S}_m \geq r + t_r, \underline{S}_{[m/2-1, m]} = S_j \leq 6 \log r, \bar{S}_m - S_m \in [\theta r - K, \theta r + K], \max_{k \leq m} (\bar{S}_k - S_k) \leq r + s_r)
\end{aligned}$$

Recall that $t_r > s_r + 6 \log r$. So $\bar{S}_m > \bar{S}_j$. By Markov property at time j , this leads to the following inequality:

$$\begin{aligned}
& \sum_{j=m/2-1}^{m-1} \mathbf{P}_x(\underline{S}_m \geq 0, \bar{S}_m \geq r + t_r, \underline{S}_{[m/2-1, m]} = S_j \leq 6 \log r, \bar{S}_m - S_m \in [\theta r - K, \theta r + K], \max_{k \leq m} (\bar{S}_k - S_k) \leq r + s_r) \\
& \leq \sum_{j=m/2-1}^{m-1} \mathbf{P}_x(\underline{S}_j \geq 0, S_j \leq 6 \log r) \mathbf{P}(\underline{S}_{m-j} \geq 0, \max_{k \leq m-j} (\bar{S}_k - S_k) \leq r + s_r, \bar{S}_{m-j} - S_{m-j} \in [\theta r - K, \theta r + K]) \\
& \leq \frac{c_{15}(1+x)(6 \log r)^2}{m^{3/2}} \sum_{j=m/2-1}^{m-1} \mathbf{P}(\underline{S}_{m-j} \geq 0, \max_{k \leq m-j} (\bar{S}_k - S_k) \leq r + s_r, \bar{S}_{m-j} - S_{m-j} \in [\theta r - K, \theta r + K])
\end{aligned}$$

where the last inequality comes from (A.2). Then by (A.26), one gets that

$$\mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\hat{F}_3(z, r, k_0) \mathbf{1}_{\{V(w_{[m/2-1, m]}) \leq 6 \log r\}} \right] \leq \frac{c_{16}(1+K)(1+x)(6 \log r)^2}{m^{3/2}}$$

which ensures that $\sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\hat{F}_3(z, r, k_0) \mathbf{1}_{\{V(w_{[m/2-1, m]}) \leq 6 \log r\}} \right] = o_r(1) \mathcal{R}(x) e^{-x}$.

On the other hand, by Markov property at time j ,

$$\begin{aligned}
& \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \frac{2Br^2}{\delta} \sum_{j=m/2}^m \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{V(w_n) \geq 0, V(w_{[m/2-1, m]}) \geq 6 \log r\}} e^{-V(w_{j-1})} \sum_{u \in \Omega(w_j)} e^{-\Delta V(u)} \right] \\
& \leq \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \frac{2Br^2}{\delta} \sum_{j=m/2}^m \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{V(w_{j-1}) \geq 0\}} e^{-V(w_{j-1})/3} \right] \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\sum_{u \in \Omega(w_1)} e^{-V(u)} \right]
\end{aligned}$$

where by Proposition 2.2 and (A.12),

$$\sum_{j=m/2}^m \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{V(w_{j-1}) \geq 0\}} e^{-V(w_{j-1})/3} \right] = \sum_{j=m/2}^m \mathbf{E}_x \left[\mathbf{1}_{\{\underline{S}_{j-1} \geq 0\}} e^{-S_{j-1}/3} \right] \leq \mathbf{E}_x \left[\sum_{j \geq 0} e^{-S_j/4} \right] = o_x(1) \mathcal{R}(x).$$

Moreover by Proposition 2.2 and (1.5),

$$\mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\sum_{u \in \Omega(w_1)} e^{-V(u)} \right] \leq \mathbf{E} \left[\left(\sum_{|u|=1} e^{-V(u)} \right)^2 \right] < \infty.$$

We thus deduce that

$$\sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \frac{2Br^2}{\delta} \sum_{j=m/2}^m \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_n) \geq 0, \underline{V}(w_{[m/2-1, m]}) \geq 6 \log r\}} e^{-V(w_{j-1})} \sum_{u \in \Omega(w_j)} e^{-\Delta V(u)} \right] = o_x(1) \mathcal{R}(x) e^{-x}. \quad (5.28)$$

Going back to (5.27), we obtain that $UB_2^{(3)}(A, B, r, 3) = o_x(1) \mathcal{R}(x) e^{-x}$.

It remains to bound $UB_2^{\geq}(A, B, r, 2)$ in (5.22). Similarly as above, observe that

$$\begin{aligned} UB_2^{\geq}(A, B, r, 2) &= \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\hat{F}_2(w_k, r, k_0) \left(\sum_{j=m/2}^m \sum_{u \in \Omega(w_j)} \frac{Br^2}{\delta} e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq 0\}} \right) \wedge 1 \right] \\ &\leq \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\hat{F}_2(w_k, r, k_0) \mathbf{1}_{\{\underline{V}(w_{[m/2-1, m]}) \leq 6 \log r\}} \right] \\ &\quad + \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \sum_{j=m/2}^m \frac{Br^2}{\delta} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\mathbf{1}_{\{\underline{V}(w_n) \geq 0, \underline{V}(w_{[m/2-1, m]}) \geq 6 \log r\}} e^{-V(w_{j-1})} \sum_{u \in \Omega(w_j)} e^{-\Delta V(u)} \right] \\ &= \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\hat{F}_2(w_k, r, k_0) \mathbf{1}_{\{\underline{V}(w_{[m/2-1, m]}) \leq 6 \log r\}} \right] + o_x(1) \mathcal{R}(x) e^{-x}, \end{aligned}$$

where the last line comes from (5.28).

For the first term on the right hand side, by Proposition 2.2,

$$\begin{aligned} &\sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\hat{F}_2(w_k, r, k_0) \mathbf{1}_{\{\underline{V}(w_{[m/2-1, m]}) \leq 6 \log r\}} \right] \\ &\leq \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \sum_{j=m/2-1}^m \mathbf{E}_x \left[e^{S_m - (1-\theta)r - b} \mathbf{1}_{\{\underline{S}_m \geq 0, S_m \leq (1-\theta)r + b, S_j = \underline{S}_{[m/2-1, m]} \leq 6 \log r\}} \right] \end{aligned}$$

which by Markov property at time j , is bounded by

$$\begin{aligned} &\sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \sum_{j=m/2-1}^m \mathbf{E}_x \left[\mathbf{1}_{\{\underline{S}_j \geq 0, S_j \leq 6 \log r\}} \mathbf{E} \left[e^{S_{m-j} - [(1-\theta)r + b - v]} \mathbf{1}_{\{\underline{S}_{m-j} \geq 0, S_{m-j} \leq (1-\theta)r + b - v\}} \mid v = S_j \right] \right] \\ &\leq \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \sum_{k=0}^{6 \log r} \sum_{j=m/2-1}^m \mathbf{P}_x(\underline{S}_j \geq 0, S_j \in [k, k+1]) e \mathbf{E} \left[e^{S_{m-j} - [(1-\theta)r + b - k]} \mathbf{1}_{\{\underline{S}_{m-j} \geq 0, S_{m-j} \leq (1-\theta)r + b - k\}} \right]. \end{aligned}$$

Then by (A.2) and by (A.24), we have

$$\begin{aligned} &\sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{-x} \mathbf{E}_{\hat{\mathbf{Q}}_x} \left[\hat{F}_2(w_k, r, k_0) \mathbf{1}_{\{\underline{V}(w_{[m/2-1, m]}) \leq 6 \log r\}} \right] \\ &\leq \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{1-x} \sum_{k=0}^{6 \log r} \frac{c_{17}(1+x)(2+k)}{m^{3/2}} \sum_{j=m/2-1}^m \mathbf{E} \left[e^{S_{m-j} - [(1-\theta)r + b - k]} \mathbf{1}_{\{\underline{S}_{m-j} \geq 0, S_{m-j} \leq (1-\theta)r + b - k\}} \right] \\ &\leq c \sum_{Ar^2-k_0 \leq m \leq Br^2-k_0} e^{1-x} \frac{c_{18}(1+x)(6 \log r)^2}{m^{3/2}} = o_r(1) \mathcal{R}(x) e^{-x}. \end{aligned}$$

We hence completes the proof of (5.15). \square

6 Proof of Lemmas 3.1, 5.1, 4.1, 4.2, 4.3 and 4.4

Proof of Lemma 3.1. It suffices to show that

$$\mathbb{E}_{(6.1)} := \mathbb{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) < n^\theta \text{ or } E_x^{(n)} \leq 1\}} \right] = o(n^{1-\theta}). \quad (6.1)$$

$$\mathbb{E}_{(6.2)} := \mathbb{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{x \notin A_n^+(a_n, b)\}} \mathbf{1}_{\{V(x) \geq -a\}} \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2\}} \right] = o(n^{1-\theta}). \quad (6.2)$$

Proof of (6.1). Observe that

$$\begin{aligned} \mathbb{E}_{(6.1)} &= \mathbb{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}} \left(\mathbf{1}_{\{\bar{L}_x(\tau_n) < n^\theta\}} + \mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \leq 1\}} \right) \right] \\ &= \mathbb{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}} \left\{ \mathbb{P}^{\mathcal{E}} \left(\bar{L}_x(\tau_n) < n^\theta \right) + \mathbb{P}^{\mathcal{E}} \left(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1 \right) \right\} \right] \end{aligned}$$

So (6.1) follows the following convergences:

$$\mathbb{E}_{(6.3)} := \mathbb{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}} \mathbb{P}^{\mathcal{E}} \left(\bar{L}_x(\tau_n) < n^\theta \right) \right] = o(n^{1-\theta}); \quad (6.3)$$

$$\mathbb{E}_{(6.4)} := \mathbb{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}} \mathbb{P}^{\mathcal{E}} \left(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1 \right) \right] = o(n^{1-\theta}). \quad (6.4)$$

Note that for $x \in A_n(-a_n, -b)$, we have $n^\theta(1 - b_x) = \frac{n^\theta}{H_x} \leq e^{-b} \frac{ne^{-V(x)}}{H_x} = e^{-b} n a_x$. So, by (2.5) with $\lambda = b$,

$$\mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) < n^\theta) \leq e^{-\lambda \left(\frac{na_x}{1+\lambda} - b_x n^\theta \right)} \leq e^{-c_{19}(b) \frac{n^\theta}{H_x}}, \quad (6.5)$$

with $c_{19}(b) := b \left(\frac{e^b}{1+b} - 1 \right) > 0$. Moreover, we see that for $x \in A_n(-a_n, -b)$ with $|x| \leq c_0(\log n)^3$,

$$H_x \leq |x| e^{\bar{V}(x) - V(x)} \leq c_0(\log n)^3 n^\theta e^{-a_n}. \quad (6.6)$$

Plugging it into (6.5) yields that for $a_n = a \log \log n$ with $a > 3$,

$$\mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) < n^\theta) \leq e^{-c_{20}(\log n)^{a-3}}.$$

This implies that

$$\mathbb{E}_{(6.3)} \leq e^{-c_{20}(\log n)^{a-3}} \mathbb{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}} \right]$$

Then by Many-to-One Lemma, one sees that

$$\begin{aligned} \mathbb{E}_{(6.3)} &\leq e^{-c_2(\log n)^{a-3}} \sum_{k=1}^{c_0(\log n)^3} \mathbf{E} \left[e^{S_k} \mathbf{1}_{\{\bar{S}_k - S_k \leq \theta \log n - a_n, S_k \leq (1-\theta) \log n - b\}} \right] \\ &\leq e^{-c_{20}(\log n)^{a-3}} n^{1-\theta} c_0(\log n)^3 = o(n^{1-\theta}), \end{aligned}$$

which shows (6.3). On the other hand, for $n \geq 2$ and $x \in A_n(-a_n, -b)$, we could get that

$$\mathbb{P}^{\mathcal{E}} \left(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1 \right) = na_x b_x^{n^\theta - 1} (1 - a_x)^{n-1} \leq n^{1-\theta} e^{-V(x)} e^{-c_{21} \frac{n^\theta}{H_x}}.$$

In fact, for $c_{21} = 1 - \frac{1}{2^\theta}$,

$$na_x b_x^{n^\theta - 1} (1 - a_x)^{n-1} = n^{1-\theta} e^{-V(x)} \left(1 - \frac{1}{H_x}\right)^{n^\theta - 1} \frac{n^\theta}{H_x} \left(1 - \frac{e^{-V(x)}}{H_x}\right)^{n-1} \leq n^{1-\theta} e^{-V(x)} e^{-c_{21} \frac{n^\theta}{H_x}} \frac{n^\theta}{H_x} e^{-\frac{n}{2} e^{-V(x)}/H_x}$$

where $ne^{-V(x)} \geq e^b n^\theta \geq n^\theta$ for $x \in A_n(-a_n, -b)$. So $\frac{n^\theta}{H_x} e^{-\frac{n}{2} e^{-V(x)}/H_x} \leq \frac{n^\theta}{H_x} e^{-\frac{n}{2H_x}} \leq \sup_{t \geq 0} t e^{-t/2} < 1$. Therefore, $\mathbb{P}^{\mathcal{E}} \left(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1 \right) \leq n^{1-\theta} e^{-V(x)} e^{-c_{21} \frac{n^\theta}{H_x}}$. Further, for $x \in A_n(-a_n, -b)$, by (6.6), we get that

$$\mathbb{P}^{\mathcal{E}} \left(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1 \right) \leq n^{1-\theta} e^{-V(x)} e^{-c_{22}(\log n)^{a-3}}.$$

Consequently,

$$\mathbb{E}_{(6.4)} \leq n^{1-\theta} e^{-c_{22}(\log n)^{a-3}} \mathbf{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{x \in A_n(-a_n, -b)\}} e^{-V(x)} \right],$$

which, by the Many-to-One Lemma, leads to

$$\begin{aligned} \mathbb{E}_{(6.4)} &\leq n^{1-\theta} e^{-c_{22}(\log n)^{a-3}} \sum_{k=1}^{c_0(\log n)^3} \mathbf{P}(\bar{S}_k - S_k \leq \theta \log n - a_n, S_k \leq (1-\theta) \log n - b) \\ &\leq n^{1-\theta} e^{-c_{22}(\log n)^{a-3}} c_0(\log n)^3 = o(n^{1-\theta}), \end{aligned}$$

which concludes (6.4).

Proof of (6.2). Observe that

$$\begin{aligned} \mathbb{E}_{(6.2)} &\leq \mathbf{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{\bar{V}(x) > \log n + a_n\}} \mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2) \right] \\ &\quad + \mathbf{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{\underline{V}(x) \geq -\alpha, \bar{V}(x) \leq \log n + a_n, V(x) > (1-\theta) \log n + b\}} \mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2) \right]. \end{aligned}$$

So, to get (6.2), we only need to show that

$$\mathbb{E}_{(6.7)} := \mathbf{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{\bar{V}(x) > \log n + a_n\}} \mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2) \right] = o(n^{1-\theta}); \quad (6.7)$$

$$\mathbb{E}_{(6.8)} := \mathbf{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{\underline{V}(x) \geq -a, \bar{V}(x) \leq \log n + a_n, V(x) > (1-\theta) \log n + b\}} \mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2) \right] = o(n^{1-\theta}). \quad (6.8)$$

Let us begin with (6.7). For $x \in \mathbb{T}$ such that $\bar{V}(x) > \log n + a_n$ with $a_n = a \log \log n$, we could show that

$$\mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2) \leq n^{1-\theta} (\log n)^{-a} e^{-V(x)}.$$

In fact, by Markov inequality, one sees that

$$\begin{aligned} \mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2) &\leq \frac{\mathbb{E}^{\mathcal{E}}[\bar{L}_x(\tau_n); E_x^{(n)} \geq 2]}{n^\theta} = \frac{H_x}{n^\theta} \mathbb{E}^{\mathcal{E}}[E_x^{(n)}; E_x^{(n)} \geq 2] \\ &= \frac{H_x}{n^\theta} [\mathbb{E}^{\mathcal{E}}[E_x^{(n)}] - \mathbb{P}^{\mathcal{E}}(E_x^{(n)} = 1)] = \frac{H_x}{n^\theta} n a_x (1 - (1 - a_x)^{n-1}), \end{aligned}$$

where $\frac{H_x}{n^\theta} n a_x = n^{1-\theta} e^{-V(x)}$ and $1 - (1 - a_x)^{n-1} \leq (n-1)a_x$. So,

$$\mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2) \leq n^{1-\theta} e^{-V(x)} (n-1)a_x.$$

Note that as $\bar{V}(x) > \log n + a_n$ with $a_n = a \log \log n$, we have $(n-1)a_x \leq n e^{-\bar{V}(x)} \leq (\log n)^{-a}$. So, $\mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2) \leq n^{1-\theta} (\log n)^{-a} e^{-V(x)}$. It hence follows that for $a > 3$,

$$\begin{aligned} \mathbb{E}_{(6.7)} &\leq n^{1-\theta} (\log n)^{-a} \mathbf{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} e^{-V(x)} \right] \\ &= c_0 (\log n)^{3-a} n^{1-\theta} = o(n^{1-\theta}), \end{aligned}$$

where the last line comes from Many-to-One Lemma. This proves (6.7).

It remains to prove (6.8). Observe that

$$\begin{aligned} \mathbb{E}_{(6.8)} &\leq \mathbf{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{\bar{V}(x) \leq \log n - a_n, V(x) > (1-\theta) \log n + b\}} \mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta) \right] \\ &\quad + \mathbf{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{\underline{V}(x) \geq -a, \bar{V}(x) \in [\log n - a_n, \log n + a_n], V(x) > (1-\theta) \log n + b\}} \mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2) \right]. \end{aligned}$$

Therefore, we only need to check that

$$\mathbf{E}_{(6.9)} := \mathbf{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{\bar{V}(x) \leq \log n - a_n, V(x) > (1-\theta) \log n + b\}} \mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta) \right] \quad (6.9)$$

$$\mathbf{E}_{(6.10)} := \mathbf{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{V(x) \geq -\alpha, \bar{V}(x) \in [\log n - a_n, \log n + a_n], V(x) > (1-\theta) \log n + b\}} \mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2) \right] \quad (6.10)$$

For $\mathbf{E}_{(6.9)}$, as $n^\theta(1-b_x) = \frac{n^\theta}{H_x} \geq e^b n a_x$ if $V(x) > (1-\theta) \log n + b$, by (2.3) with $\eta = b$,

$$\mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta) \leq 2n a_x e^{-c_\eta n^\theta b_x} = 2n^{1-\theta} e^{-V(x)} \frac{n^\theta}{H_x} e^{-c_\eta \frac{n^\theta}{H_x}} \leq c_{23} n^{1-\theta} e^{-V(x)} e^{-\frac{c_\eta}{2} \frac{n^\theta}{H_x}},$$

where we use the fact that $t e^{-c_\eta t} \leq c_{24} e^{-c_\eta t/2}$ for any $t > 0$ and $c_{24} := \sup_{t \geq 0} t e^{-c_\eta t/2}$. In addition, given $\{\bar{V}(x) \leq \log n - a_n, V(x) > (1-\theta) \log n + b\}$, we get $H_x \leq |x| e^{\bar{V}(x) - V(x)} \leq c_0(\log n)^{3-a} n^\theta$. Thus,

$$\mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta) \leq c_{23} n^{1-\theta} e^{-V(x)} e^{-c_{25}(\log n)^{a-3}}.$$

This combined with Many-to-One Lemma implies that

$$\begin{aligned} \mathbf{E}_{(6.9)} &\leq c_{23} n^{1-\theta} e^{-c_{25}(\log n)^{a-3}} \mathbf{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} e^{-V(x)} \right] \\ &= c_{23} c_0 (\log n)^3 e^{-c_{25}(\log n)^{a-3}} n^{1-\theta} = o(n^{1-\theta}), \end{aligned}$$

which shows (6.9).

For $\mathbf{E}_{(6.10)}$, again, as $n^\theta(1-b_x) = \frac{n^\theta}{H_x} \geq e^b n a_x$, by (2.4) with $\eta = b$, one has

$$\mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} \geq 2) \leq 2(n a_x)^2 e^{-c_\eta n^\theta(1-b_x)} = 2n^{2(1-\theta)} e^{-2V(x)} \left(\frac{n^\theta}{H_x}\right)^2 e^{-c_\eta \frac{n^\theta}{H_x}},$$

which is less than $c_{26} n^{2(1-\theta)} e^{-2V(x)}$ since $\left(\frac{n^\theta}{H_x}\right)^2 e^{-c_\eta \frac{n^\theta}{H_x}} \leq \sup_{t \geq 0} t^2 e^{-c_\eta t} < \infty$. As a result,

$$\begin{aligned} \mathbf{E}_{(6.10)} &\leq c_{26} n^{1-\theta} \mathbf{E} \left[\sum_{k=1}^{c_0(\log n)^3} \sum_{|x|=k} \mathbf{1}_{\{V(x) \geq -\alpha, \bar{V}(x) \in [\log n - a_n, \log n + a_n], V(x) > (1-\theta) \log n + b\}} e^{-V(x)} e^{(1-\theta) \log n - V(x)} \right] \\ &= c_{26} n^{1-\theta} \sum_{k=1}^{c_0(\log n)^3} \mathbf{E} \left[e^{(1-\theta) \log n - S_k}; \underline{S}_k \geq -\alpha, \bar{S}_k \in [\log n - a_n, \log n + a_n], S_k > (1-\theta) \log n + b \right], \end{aligned}$$

where the last equality follows from Many-to-One Lemma. So it suffices to show that

$$\mathbf{E}_{(6.11)} := \sum_{k=1}^{c_0(\log n)^3} \mathbf{E} \left[e^{(1-\theta) \log n - S_k} \mathbf{1}_{\{S_k \geq -\alpha, \bar{S}_k \in [\log n - a_n, \log n + a_n], S_k \geq (1-\theta) \log n\}} \right] = o_n(1). \quad (6.11)$$

Apparently, $e^{(1-\theta)\log n - S_k} \leq e^{-a_n}$ if $S_k > (1-\theta)\log n + a_n$. As a result, for $a_n = a \log \log n$ with $a > 3$,

$$\mathbf{E}_{(6.11)} \leq o_n(1) + \sum_{k=1}^{c_0(\log n)^3} \mathbf{E}_{(6.12)}(k),$$

where

$$\mathbf{E}_{(6.12)}(k) := \mathbf{E} \left[e^{(1-\theta)\log n - S_k} \mathbf{1}_{\{\underline{S}_k \geq -\alpha, \bar{S}_k \in [\log n - a_n, \log n + a_n], S_k \in [(1-\theta)\log n, (1-\theta)\log n + a_n]\}} \right]. \quad (6.12)$$

We only need to show that $\sum_{k=1}^{c_0(\log n)^3} \mathbf{E}_{(6.12)}(k) = o_n(1)$. For $1 \leq k \leq \varepsilon(\log n)^2$ with $\varepsilon \in (0, 1)$ small, by (A.19),

$$\begin{aligned} \sum_{k=1}^{\varepsilon(\log n)^2} \mathbf{E}_{(6.12)}(k) &\leq \sum_{r=(1-\theta)\log n}^{(1-\theta)\log n + a_n} e^{(1-\theta)\log n - r} \sum_{k=1}^{\varepsilon(\log n)^2} \mathbf{P}(\underline{S}_k \geq -\alpha, S_k \in [r, r+1]) \\ &\leq c_{27}(1+\alpha)\varepsilon = o_\varepsilon(1). \end{aligned}$$

For $k \geq (\log n)^2/\varepsilon$, by (A.2), one has

$$\begin{aligned} \sum_{k=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \mathbf{E}_{(6.12)}(k) &\leq \sum_{r=(1-\theta)\log n}^{(1-\theta)\log n + a_n} e^{(1-\theta)\log n - r} \sum_{k=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \mathbf{P}(\underline{S}_k \geq -\alpha, S_k \in [r, r+1]) \\ &\leq c_{28}(1+\alpha)^2\sqrt{\varepsilon} = o_\varepsilon(1). \end{aligned}$$

It remains to check that $\limsup_{n \rightarrow \infty} \sum_{k=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbf{E}_{(6.12)}(k) = o_\varepsilon(1)$. By considering the first time that $(S_i)_{0 \leq i \leq k}$ hits \bar{S}_k , we get that

$$\begin{aligned} &\sum_{k=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbf{E}_{(6.12)}(k) \\ &= \sum_{k=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=1}^k \mathbf{E} \left[e^{(1-\theta)\log n - S_k} \mathbf{1}_{\{\underline{S}_k \geq -\alpha, \bar{S}_{j-1} < S_j = \bar{S}_k \in [\log n - a_n, \log n + a_n], S_k \in [(1-\theta)\log n, (1-\theta)\log n + a_n]\}} \right] \\ &\leq \sum_{k=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{s=-a_n}^{a_n} \sum_{t=0}^{a_n} e^{-t} \sum_{j=1}^k \mathbf{E} \left[\mathbf{1}_{\{\underline{S}_k \geq -\alpha, \bar{S}_{j-1} < S_j = \bar{S}_k \in [\log n + s, \log n + s + 1], S_k \in [(1-\theta)\log n + t, (1-\theta)\log n + t + 1]\}} \right] \end{aligned}$$

By Markov property at time j , one sees that

$$\begin{aligned} &\mathbf{E} \left[\mathbf{1}_{\{\underline{S}_k \geq -\alpha, \bar{S}_{j-1} < S_j = \bar{S}_k \in [\log n + s, \log n + s + 1], S_k \in [(1-\theta)\log n + t, (1-\theta)\log n + t + 1]\}} \right] \\ &\leq \mathbf{P}(\underline{S}_j \geq -\alpha, \bar{S}_j = S_j \in [\log n + s, \log n + s + 1]) \mathbf{P}(\bar{S}_{k-j} \leq 0, S_{k-j} + \theta \log n \in [t - s - 1, t - s + 1]). \end{aligned}$$

So,

$$\sum_{k=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbf{E}_{(6.12)}(k) \leq \sum_{k=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=1}^{k-1} \mathbf{P}_{(6.14)}(j, k), \quad (6.13)$$

with

$$\begin{aligned}
& \mathbf{P}_{(6.14)}(j, k) \\
&= \sum_{s=-a_n}^{a_n} \sum_{t=0}^{a_n} e^{-t} \mathbf{P}(\underline{S}_j \geq -\alpha, \bar{S}_j = S_j \in [\log n + s, \log n + s + 1]) \mathbf{P}(\bar{S}_{k-j} \leq 0, S_{k-j} + \theta \log n \in [t - s - 1, t - s + 1])
\end{aligned} \tag{6.14}$$

Observe that $\sum_{j=1}^{k-1} \mathbf{P}_{(6.14)}(j, k) \leq \sum_{j=1}^{\varepsilon^2(\log n)^2} + \sum_{j=(\varepsilon \log n)^2}^{k-(\varepsilon \log n)^2} + \sum_{j=k-(\varepsilon \log n)^2}^{k-1} \mathbf{P}_{(6.14)}(j, k)$. We bound the three sums separately. First, by (A.7) and (A.2) for $j \leq (\varepsilon \log n)^2 \leq \varepsilon k$, we have

$$\begin{aligned}
& \sum_{j=1}^{\varepsilon^2(\log n)^2} \mathbf{P}_{(6.14)}(j, k) \\
&\leq \sum_{j=1}^{\varepsilon^2(\log n)^2} \mathbf{P}(\underline{S}_j \geq -\alpha, \bar{S}_j = S_j \geq \log n - a_n) \sup_{0 \leq t \leq a_n, |s| \leq a_n} \mathbf{P}(\bar{S}_{k-j} \leq 0, S_{k-j} + \theta \log n \in [t - s - 1, t - s + 1]) \\
&\leq c_{29} \frac{\log n}{k^{3/2}} \sum_{j=1}^{\varepsilon^2(\log n)^2} \frac{1 + \alpha}{j^{1/2} \log n} \leq c_{30} \frac{(1 + \alpha) \varepsilon \log n}{k^{3/2}}
\end{aligned}$$

For $(\varepsilon \log n)^2 \leq j \leq k - (\varepsilon \log n)^2$, by (A.2) and (A.18), one sees that

$$\begin{aligned}
\sum_{j=(\varepsilon \log n)^2}^{k-(\varepsilon \log n)^2} \mathbf{P}_{(6.14)}(j, k) &\leq 2a_n \sum_{j=(\varepsilon \log n)^2}^{k-(\varepsilon \log n)^2} \sup_{0 \leq t \leq a_n, |s| \leq a_n} \mathbf{P}(\underline{S}_j \geq -\alpha, \bar{S}_j = S_j - \log n \in [s, s + 1]) \frac{(1 + \log n)}{(k - j)^{3/2}} \\
&\leq 2a_n \sum_{j=(\varepsilon \log n)^2}^{k-(\varepsilon \log n)^2} \frac{c_{31} (1 + \alpha)^4 (\log n)^4}{j^3 (k - j)^{3/2}} \leq \frac{c_{32} a_n (1 + \alpha)^4}{\varepsilon^4 k^{3/2}} + c_{32} (1 + \alpha)^4 \frac{(\log n)^3 a_n}{\varepsilon k^3},
\end{aligned}$$

As $k \geq \varepsilon (\log n)^2$, we get that $\sum_{j=(\varepsilon \log n)^2}^{k-(\varepsilon \log n)^2} \mathbf{P}_{(6.14)}(j, k) \leq c_{33} (1 + \alpha)^4 \frac{a_n}{\varepsilon^4 k^{3/2}}$.

For $k - (\varepsilon \log n)^2 \leq j < k$, by (A.18) and (A.19), one sees that

$$\begin{aligned}
& \sum_{j=k-(\varepsilon \log n)^2}^{k-1} \mathbf{P}_{(6.14)}(j, k) \\
&\leq \frac{c_{34} a_n (1 + \alpha)^4 (\log n)^3}{k^3} \sup_{0 \leq t \leq a_n, |s| \leq a_n} \sum_{j=1}^{(\varepsilon \log n)^2} \mathbf{P}(\bar{S}_j \leq 0, -S_j \in [\theta \log n + s - t - 1, \theta \log n + s - t + 1]) \\
&\leq \frac{c_{35} a_n (1 + \alpha)^4 (\log n)^3}{k^3} \varepsilon^2.
\end{aligned}$$

As a consequence,

$$\sum_{j=1}^{k-1} \mathbf{P}_{(6.14)}(j, k) \leq c_{30} \frac{(1 + \alpha) \varepsilon \log n}{k^{3/2}} + c_{33} (1 + \alpha)^4 \frac{a_n}{\varepsilon^4 k^{3/2}} + \frac{c_{35} a_n (1 + \alpha)^4 (\log n)^3}{k^3} \varepsilon^2.$$

Plugging it into (6.13) yields that

$$\sum_{k=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbf{E}_{(6.12)}(k) = o_n(1) + o_\varepsilon(1),$$

which completes the proof of (6.11). We thus conclude (6.10) and (6.2). \square

Proof of Lemma 5.1. Let

$$\mathbf{E}_{(5.6)} := \mathbf{E} \left[\sum_{m=1}^{\varepsilon(\log n)^2} \sum_{|z|=m} e^{-V(z)} F_2(z, \log n) \mathbf{1}_{\{V(z) \geq -\alpha\}} \right],$$

$$\mathbf{E}_{(5.7)} := \mathbf{E} \left[\sum_{m=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{|z|=m} e^{-V(z)} F_2(z, \log n) \mathbf{1}_{\{V(z) \geq -\alpha\}} \right].$$

Let us bound $\mathbf{E}_{(5.6)}$ first. By Many-to-One Lemma,

$$\begin{aligned} E_{(5.6)} &= \sum_{k=1}^{\varepsilon(\log n)^2} \mathbf{E} \left[e^{S_k - (1-\theta) \log n - b} \mathbf{1}_{\{\underline{S}_k \geq -\alpha, \bar{S}_k \leq \log n + a_n, S_k \leq (1-\theta) \log n + b\}} \right] \\ &\leq \sum_{k=1}^{\varepsilon(\log n)^2} e^{-\frac{1-\theta}{2} \log n - b} + \sum_{k=1}^{\varepsilon(\log n)^2} \mathbf{E} \left[e^{S_k - (1-\theta) \log n - b} \mathbf{1}_{\{\underline{S}_k \geq -\alpha, \bar{S}_k \leq \log n + a_n, S_k \in [\frac{1-\theta}{2} \log n, (1-\theta) \log n + b]\}} \right] \\ &\leq o_n(1) + \sum_{t=\frac{1-\theta}{2} \log n}^{(1-\theta) \log n + b} e^{t - (1-\theta) \log n - b} \sum_{k=1}^{\varepsilon(\log n)^2} \mathbf{P}(\underline{S}_k \geq -\alpha, S_k \in [t, t+1]). \end{aligned}$$

We then deduce from (A.19) that $\mathbf{E}_{(5.6)} = o_n(1) + o_\varepsilon(1)$. This suffices to conclude (5.6).

On the other hand, by Many-to-One Lemma,

$$\begin{aligned} \mathbf{E}_{(5.7)} &= \sum_{k=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \mathbf{E} \left[e^{S_k - (1-\theta) \log n - b} \mathbf{1}_{\{\underline{S}_k \geq -\alpha, \bar{S}_k \leq \log n + a_n, S_k \leq (1-\theta) \log n + b\}} \right] \\ &\leq \sum_{k=(\log n)^2/\varepsilon}^{c_0(\log n)^3} e^{-\frac{1-\theta}{2} \log n - b} + \sum_{k=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{t=\frac{1-\theta}{2} \log n}^{(1-\theta) \log n + b} e^{t - (1-\theta) \log n - b} \mathbf{P}(\underline{S}_k \geq -\alpha, S_k \in [t, t+1]). \end{aligned}$$

By use of (A.2), we obtain that

$$\mathbf{E}_{(5.7)} \leq o_n(1) + \sum_{k=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \frac{c_{36}(1+\alpha)(1+(1-\theta)\log n + b + \alpha)}{k^{3/2}} = o_n(1) + o_\varepsilon(1).$$

This ends the proof of Lemma 5.1. \square

Proof of Lemma 4.1. In fact, as $\mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1) = na_x(1-a_x)^{n-1}b_x^{n^\theta-1}$, we are going to show that

$$\mathbf{E}_{(6.15)} := \sum_{\ell=1}^{c_0(\log n)^3} \mathbf{E} \left[\sum_{|x|=\ell} na_x(1-a_x)^{n-1}b_x^{n^\theta-1} \mathbf{1}_{\{\bar{V}(x) \leq \log n - a_n\}} \right] = o(n^{1-\theta}); \quad (6.15)$$

$$\mathbf{E}_{(6.16)} := \sum_{\ell=1}^{c_0(\log n)^3} \mathbf{E} \left[\sum_{|x|=\ell} na_x(1-a_x)^{n-1}b_x^{n^\theta-1} \mathbf{1}_{\{x \notin \mathcal{D}_n\}} \right] = o(n^{1-\theta}). \quad (6.16)$$

First, observe that if $\bar{V}(x) \leq \log n - a_n$ with $a_n = a \log \log n$ and $|x| \leq c_0(\log n)^3$, then

$$na_x(1-a_x)^{n-1}b_x^{n^\theta-1} \leq n^{1-\theta} e^{-V(x)} \frac{n^\theta}{H_x} e^{-\frac{n^\theta}{2H_x}} e^{-(n-1)a_x} \leq n^{1-\theta} e^{-V(x)} e^{-c_{37}(\log n)^{a-3}}$$

as $a_x \geq \frac{1}{|x|e^{\bar{V}(x)}}$. This follows that

$$\mathbf{E}_{(6.15)} \leq n^{1-\theta} e^{-c_{37}(\log n)^{a-3}} \sum_{\ell=1}^{c_0(\log n)^3} \mathbf{E} \left[\sum_{|x|=\ell} e^{-V(x)} \right]$$

which by Many-to-One lemma, is bounded by $n^{1-\theta} c_0(\log n)^3 e^{-c_{37}(\log n)^{a-3}} = o(n^{1-\theta})$. This proves (6.15).

Next, let

$$\mathbf{E}_{(6.17)} := \sum_{\ell=1}^{c_0(\log n)^3} \mathbf{E} \left[\sum_{|x|=\ell} na_x(1-a_x)^{n-1}b_x^{n^\theta-1} \mathbf{1}_{\{\bar{V}(x) - V(x) > \theta \log n + a_n\}} \right] \quad (6.17)$$

$$\mathbf{E}_{(6.18)} := \sum_{\ell=1}^{c_0(\log n)^3} \mathbf{E} \left[\sum_{|x|=\ell} na_x(1-a_x)^{n-1}b_x^{n^\theta-1} \mathbf{1}_{\{\bar{V}(x) - V(x) < \theta \log n - a_n\}} \right]. \quad (6.18)$$

It is immediate that

$$\mathbf{E}_{(6.16)} = \mathbf{E}_{(6.17)} + \mathbf{E}_{(6.18)}. \quad (6.19)$$

So, we only need to check that $\mathbf{E}_{(6.17)} = o(n^{1-\theta})$ and $\mathbf{E}_{(6.18)} = o(n^{1-\theta})$.

On the one hand, note that $|x|e^{\bar{V}(x)-V(x)} \geq H_x \geq e^{\bar{V}(x)-V(x)}$. If $\bar{V}(x) - V(x) \geq \theta \log n + a_n$, then $\frac{n^\theta}{H_x} \leq (\log n)^{-a}$ and

$$na_x(1-a_x)^{n-1}b_x^{n^\theta-1} \leq n^{1-\theta} e^{-V(x)} \frac{n^\theta}{H_x} \leq n^{1-\theta} e^{-V(x)} (\log n)^{-a}.$$

This brings out that

$$\mathbf{E}_{(6.17)} \leq n^{1-\theta} (\log n)^{-a} \sum_{\ell=1}^{c_0(\log n)^3} \mathbf{E} \left[\sum_{|x|=\ell} e^{-V(x)} \right] = o(n^{1-\theta}). \quad (6.20)$$

On the other hand, if $\bar{V}(x) - V(x) \leq \theta \log n - a_n$ and $|x| \leq c_0(\log n)^3$, one has $\frac{n^\theta}{H_x} \geq \frac{1}{c_0} (\log n)^{a-3}$ and

$$na_x(1-a_x)^{n-1}b_x^{n^\theta-1} \leq n^{1-\theta} e^{-V(x)} e^{-\frac{n^\theta}{2H_x}} \leq n^{1-\theta} e^{-V(x)} e^{-c_{38}(\log n)^{a-3}}.$$

As a consequence,

$$\mathbf{E}_{(6.18)} \leq n^{1-\theta} e^{-c(\log n)^{a-3}} \sum_{\ell=1}^{c_0(\log n)^3} \mathbf{E} \left[\sum_{|x|=\ell} e^{-V(x)} \right] = o(n^{1-\theta}). \quad (6.21)$$

We then deduce (6.16) from (6.19), (6.20) and (6.21). \square

Proof of Lemma 4.2. Again, as $\mathbb{P}^{\mathcal{E}}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1) = na_x(1-a_x)^{n-1}b_x^{n^\theta-1}$, we are going to show that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{1-\theta}} \sum_{\ell=1}^{\varepsilon(\log n)^2} \mathbf{E} \left[\sum_{|x|=\ell} na_x(1-a_x)^{n-1}b_x^{n^\theta-1} \mathbf{1}_{\{x \in \mathfrak{B}_n^- \cap \mathfrak{D}_n, \underline{V}(x) \geq -\alpha\}} \right] = o_\varepsilon(1); \quad (6.22)$$

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{1-\theta}} \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \mathbf{E} \left[\sum_{|x|=\ell} na_x(1-a_x)^{n-1}b_x^{n^\theta-1} \mathbf{1}_{\{x \in \mathfrak{B}_n^- \cap \mathfrak{D}_n \cap \mathfrak{L}_n, \underline{V}(x) \geq -\alpha\}} \right] = o_\varepsilon(1); \quad (6.23)$$

$$\frac{1}{n^{1-\theta}} \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbf{E} \left[\sum_{|x|=\ell} na_x(1-a_x)^{n-1}b_x^{n^\theta-1} \mathbf{1}_{\{x \in \mathfrak{D}_n\}} \mathbf{1}_{\{\bar{V}(x) \in [\log n - a_n, \log n + a_n], \underline{V}(x) \geq -\alpha\}} \right] = o_n(1). \quad (6.24)$$

Proof of (6.22). Let

$$\mathbf{E}_{(6.22)}(\ell) := \mathbf{E} \left[\sum_{|x|=\ell} na_x(1-a_x)^{n-1}b_x^{n^\theta-1} \mathbf{1}_{\{x \in \mathfrak{B}_n^- \cap \mathfrak{D}_n, \underline{V}(x) \geq -\alpha\}} \right].$$

To get (6.22), we need to bound $\sum_{\ell=1}^{(\log n)^{1+\delta}} \mathbf{E}_{(6.22)}(\ell)$ and $\sum_{\ell=(\log n)^{1+\delta}}^{\varepsilon(\log n)^2} \mathbf{E}_{(6.22)}(\ell)$ with $\delta \in (0, 1)$.

As $na_x(1-a_x)^{n-1}b_x^{n^\theta-1} \leq n^{1-\theta}e^{-V(x)}$, we see that by Many-to-One Lemma,

$$\begin{aligned} \mathbf{E}_{(6.22)} &\leq n^{1-\theta} \mathbf{E} \left[\sum_{|x|=\ell} e^{-V(x)} \mathbf{1}_{\{\underline{V}(x) \geq -\alpha, \bar{V}(x) \geq \log n - a_n, \bar{V}(x) - V(x) \in [\theta \log n - a_n, \theta \log n + a_n]\}} \right] \\ &= n^{1-\theta} \mathbf{P}(\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \bar{S}_\ell - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n]). \end{aligned}$$

For $1 \leq \ell \leq (\log n)^{1+\delta}$ with $\delta \in (0, 1/3)$, one sees that by (A.17),

$$\begin{aligned} \sum_{\ell=1}^{(\log n)^{1+\delta}} \mathbf{E}_{(6.22)}(\ell) &\leq n^{1-\theta} \sum_{\ell=1}^{(\log n)^{1+\delta}} \mathbf{P}(\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \bar{S}_\ell - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n]) \\ &\leq n^{1-\theta} \sum_{\ell=1}^{(\log n)^{1+\delta}} \mathbf{P}(S_\ell \geq (1-\theta) \log n - 2a_n) \\ &\leq n^{1-\theta} e^{-c_{39}(\log n)^{1-\delta}} = o(n^{1-\theta}). \end{aligned}$$

For $\ell \geq (\log n)^{1+\delta}$, by considering the first time hitting \bar{S}_ℓ , one sees that

$$\begin{aligned} &\frac{1}{n^{1-\theta}} \sum_{\ell=(\log n)^{1+\delta}}^{\varepsilon(\log n)^2} \mathbf{E}_{(6.22)}(\ell) \\ &\leq \sum_{\ell=(\log n)^{1+\delta}}^{\varepsilon(\log n)^2} \sum_{j=1}^{\ell-1} \mathbf{P}(\underline{S}_\ell \geq -\alpha, \bar{S}_{j-1} < S_j = \bar{S}_\ell, S_j \geq \log n - a_n, S_j - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n]), \end{aligned}$$

which by Markov property at time j , is less than

$$\begin{aligned}
& \sum_{\ell=(\log n)^{1+\delta}}^{\varepsilon(\log n)^2} \sum_{j=1}^{\ell-1} \mathbf{P}(S_j \geq -\alpha, \bar{S}_j = S_j \geq \log n - a_n) \mathbf{P}(\bar{S}_{\ell-j} \leq 0, -S_{\ell-j} \in [\theta \log n - a_n, \theta \log n + a_n]) \\
& \leq \sum_{\ell=(\log n)^{1+\delta}}^{\varepsilon(\log n)^2} \left(\sum_{j=1}^{(\log n)^{1+\delta}/2} \mathbf{P}(S_j \geq \log n - a_n) + \sum_{j=\ell-(\log n)^{1+\delta}/2}^{\ell} \mathbf{P}(-S_{\ell-j} \in [\theta \log n - a_n, \theta \log n + a_n]) \right) \\
& \quad + \sum_{j=(\log n)^{1+\delta}/2}^{\ell-(\log n)^{1+\delta}/2} \mathbf{P}(S_j \geq -\alpha, \bar{S}_j = S_j \geq \log n - a_n) \mathbf{P}(\bar{S}_{\ell-j} \leq 0, -S_{\ell-j} \in [\theta \log n - a_n, \theta \log n + a_n])
\end{aligned}$$

Again by (A.17), one has

$$\sum_{j=1}^{(\log n)^{1+\delta}/2} \mathbf{P}(S_j \geq \log n - a_n) + \sum_{\ell-(\log n)^{1+\delta}/2}^{\ell} \mathbf{P}(-S_{\ell-j} \in [\theta \log n - a_n, \theta \log n + a_n]) \leq e^{-c_{40}(\log n)^{1-\delta}}.$$

On the other hand, by (A.16) and (A.14), one has

$$\begin{aligned}
& \sum_{j=(\log n)^{1+\delta}/2}^{\ell-(\log n)^{1+\delta}/2} \mathbf{P}(S_j \geq -\alpha, \bar{S}_j = S_j \geq \log n - a_n) \mathbf{P}(\bar{S}_{\ell-j} \leq 0, -S_{\ell-j} \in [\theta \log n - a_n, \theta \log n + a_n]) \\
& \leq \sum_{j=(\log n)^{1+\delta}/2}^{\ell-(\log n)^{1+\delta}/2} \frac{c_{41}(1+\alpha)}{\sqrt{j} \log n} e^{-c_{42} \frac{(\log n)^2}{j}} \frac{1}{\ell-j} e^{-c_{43} \frac{(\log n)^2}{\ell-j}} \\
& \leq \frac{2}{\ell} e^{-c_{43} \frac{(\log n)^2}{\ell}} \sum_{j=(\log n)^{1+\delta}/2}^{\ell/2} \frac{c_{41}(1+\alpha)}{\sqrt{j} \log n} e^{-c_{42} \frac{(\log n)^2}{j}} + \frac{c_{41}(1+\alpha)}{\log n \sqrt{\ell/2}} e^{-c_{42} \frac{(\log n)^2}{\ell}} \sum_{j=\ell/2}^{\ell} \frac{1}{\ell-j} e^{-c_{43} \frac{(\log n)^2}{\ell-j}} \\
& \leq \frac{c_{44}(1+\alpha)}{\ell} e^{-c_{45} \frac{(\log n)^2}{\ell}}.
\end{aligned}$$

As a consequence,

$$\frac{1}{n^{1-\theta}} \sum_{\ell=(\log n)^{1+\delta}}^{\varepsilon(\log n)^2} \mathbf{E}_{(6.22)}(\ell) \leq \varepsilon(\log n)^2 e^{-c_{40}(\log n)^{1-\delta}} + \sum_{\ell=(\log n)^{1+\delta}}^{\varepsilon(\log n)^2} \frac{c_{44}(1+\alpha)}{\ell} e^{-c_{45} \frac{(\log n)^2}{\ell}},$$

which converges to 0 as $n \rightarrow \infty$ then $\varepsilon \downarrow 0$.

Proof of (6.23). Let

$$\mathbf{E}_{(6.23)} := \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \mathbf{E} \left[\sum_{|x|=\ell} na_x (1-a_x)^{n-1} b_x^{n^\theta-1} \mathbf{1}_{\{x \in \mathfrak{B}_n^- \cap \mathfrak{D}_n \cap \mathfrak{L}_n, V(x) \geq -\alpha\}} \right]$$

Note that $na_x (1-a_x)^{n-1} b_x^{n^\theta-1} \leq c_{46} n^{1-\theta} e^{-V(x)} \left(\frac{n^\theta}{H_x} \wedge \frac{H_x}{n^\theta} \right)$ as $xe^{-x/2} \leq c_{46}(x \wedge \frac{1}{x})$. Therefore, by Many-

to-One Lemma,

$$\begin{aligned} \mathbf{E}_{(6.23)} &\leq c_{46} n^{1-\theta} \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \mathbf{E} \left[\sum_{|x|=\ell} e^{-V(x)} \left(\frac{n^\theta}{H_x} \wedge \frac{H_x}{n^\theta} \right) \mathbf{1}_{\{x \in \mathfrak{B}_n^- \cap \mathfrak{D}_n \cap \mathfrak{L}_n, \mathcal{V}(x) \geq -\alpha\}} \right] \\ &= c_{46} n^{1-\theta} \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \mathbf{E} \left[\left(\frac{n^\theta}{H_\ell^S} \wedge \frac{H_\ell^S}{n^\theta} \right) \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n]\}} \right] \end{aligned}$$

where $H_\ell^S := \sum_{k=0}^{\ell} e^{S_k - S_\ell}$. Observe that

$$\begin{aligned} &\mathbf{E} \left[\left(\frac{n^\theta}{H_\ell^S} \wedge \frac{H_\ell^S}{n^\theta} \right) \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n]\}} \right] \\ &\leq \sum_{x=-a_n}^{-1} \mathbf{E} \left[\frac{H_\ell^S}{n^\theta} \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] \\ &\quad + \sum_{x=0}^{a_n-1} \mathbf{E} \left[\frac{n^\theta}{H_\ell^S} \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] \end{aligned}$$

Note that $H_\ell^S \geq e^{\bar{S}_\ell - S_\ell}$. It follows that

$$\begin{aligned} &\sum_{x=0}^{a_n-1} \mathbf{E} \left[\frac{n^\theta}{H_\ell^S} \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] \\ &\leq \sum_{x=0}^{a_n-1} \mathbf{E} \left[n^\theta e^{S_\ell - \bar{S}_\ell} \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] \\ &\leq \sum_{x=0}^{a_n-1} e^{-x} \mathbf{E} \left[\mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right]. \end{aligned}$$

On the other hand, as $H_\ell^S = \sum_{k=0}^{\ell} e^{S_k - \bar{S}_\ell} e^{\bar{S}_\ell - S_\ell}$,

$$\begin{aligned} &\sum_{x=-a_n}^{-1} \mathbf{E} \left[\frac{H_\ell^S}{n^\theta} \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] \\ &\leq \sum_{x=-a_n}^{-1} e^x \mathbf{E} \left[\sum_{k=0}^{\ell} e^{S_k - \bar{S}_\ell} \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] \end{aligned}$$

So, it suffices to prove that uniformly for $x \in [-a_n, a_n]$,

$$\sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \mathbf{E} \left[\mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] = o_{n,\varepsilon}(1) \quad (6.25)$$

$$\sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \mathbf{E} \left[\sum_{k=0}^{\ell} e^{S_k - \bar{S}_\ell} \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] = o_{n,\varepsilon}(1). \quad (6.26)$$

First, we consider (6.25) and let $\mathbf{E}_{(6.25)}(\ell) := \mathbf{E} \left[\mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right]$.
By considering the first time hitting \bar{S}_ℓ , we have

$$\sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \mathbf{E}_{(6.25)}(\ell) = \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=1}^{\ell-1} \mathbf{E}_{(6.27)}(j, \ell), \quad (6.27)$$

where

$$\mathbf{E}_{(6.27)}(j, \ell) := \mathbf{E} \left[\mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell = S_j > \bar{S}_{j-1}, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right].$$

By Markov property at time j , It is immediate that

$$\begin{aligned} \mathbf{E}_{(6.27)}(j, \ell) &\leq \mathbf{P} \left(\underline{S}_j \geq -\alpha, \max_{k \leq j} (\bar{S}_k - S_k) \leq \log n, \bar{S}_j = S_j \geq \log n - a_n \right) \\ &\quad \times \mathbf{P} \left(\bar{S}_{\ell-j} \leq 0, \underline{S}_{\ell-j} \geq -\log n, -S_{\ell-j} \in [\log n + x, \log n + x + 1] \right). \end{aligned} \quad (6.28)$$

Observe that for $j \leq (\log n)^{1+\delta}$ or $j \geq \ell - (\log n)^{1+\delta}$ with $\delta \in (0, 1)$,

$$\begin{aligned} &\sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \left[\sum_{j=1}^{(\log n)^{1+\delta}} \mathbf{E}_{(6.27)}(j, \ell) + \sum_{j=\ell-(\log n)^{1+\delta}}^{\ell-1} \mathbf{E}_{(6.27)}(j, \ell) \right] \\ &\leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \left[\sum_{j=1}^{(\log n)^{1+\delta}} \mathbf{P}(S_j \geq \log n - a_n) + \sum_{j=\ell-(\log n)^{1+\delta}}^{\ell-1} \mathbf{P}(-S_{\ell-j} \geq \theta \log n - a_n) \right]. \end{aligned}$$

By (A.17), we get that

$$\sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \left[\sum_{j=1}^{(\log n)^{1+\delta}} \mathbf{E}_{(6.27)}(j, \ell) + \sum_{j=\ell-(\log n)^{1+\delta}}^{\ell-1} \mathbf{E}_{(6.27)}(j, \ell) \right] \leq c_0(\log n)^3 e^{-c_{47}(\log n)^{1-\delta}} = o_n(1). \quad (6.29)$$

For $(\log n)^{1+\delta} \leq j \leq (\log n)^2$, by (6.28), (A.16) and (A.2), one has

$$\begin{aligned} &\sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=(\log n)^{1+\delta}}^{(\log n)^2} \mathbf{E}_{(6.27)}(j, \ell) \\ &\leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=(\log n)^{1+\delta}}^{(\log n)^2} \mathbf{P}(\underline{S}_j \geq -\alpha, \bar{S}_j = S_j \geq \log n - a_n) \mathbf{P}(\bar{S}_{\ell-j} \leq 0, -S_{\ell-j} - \theta \log n \in [x, x + 1]) \\ &\leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=(\log n)^{1+\delta}}^{(\log n)^2} c_{48} \frac{1 + \alpha}{\sqrt{j} \log n} e^{-c_{49} \frac{(\log n)^2}{j}} \frac{\theta \log n + a_n}{(\ell - j)^{3/2}} \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} c_{50} (1 + \alpha) \frac{\log n}{\ell^{3/2}}, \end{aligned}$$

which is $o_\varepsilon(1)$ as $\varepsilon \downarrow 0$. For $(\log n)^2 \leq j \leq \ell - (\log n)^2$, by (6.28), (A.11) and (A.2), one has

$$\begin{aligned} & \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=(\log n)^2}^{\ell-(\log n)^2} \mathbf{E}_{(6.27)}(j, \ell) \\ & \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=(\log n)^2}^{\ell-(\log n)^2} \mathbf{P}(\underline{S}_j \geq -\alpha, \max_{k \leq j} (\bar{S}_k - S_k) \leq \log n, \bar{S}_j = S_j) \mathbf{P}(\bar{S}_{\ell-j} \leq 0, -S_{\ell-j} - \theta \log n \in [x, x+1]) \\ & \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=(\log n)^2}^{\ell-(\log n)^2} c_{51} \frac{1+\alpha}{j} e^{-c_{52} \frac{j}{(\log n)^2}} \frac{\theta \log n + a_n}{(\ell-j)^{3/2}} \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} c_{53} (1+\alpha) \frac{\log n}{\ell^{3/2}}, \end{aligned}$$

which is also $o_\varepsilon(1)$. For $\ell - (\log n)^2 \leq j \leq \ell - (\log n)^{1+\delta}$, by (6.28), (A.11) and (A.14), one has

$$\begin{aligned} & \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=\ell-(\log n)^2}^{\ell-(\log n)^{1+\delta}} \mathbf{E}_{(6.27)}(j, \ell) \\ & \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=\ell-(\log n)^2}^{\ell-(\log n)^{1+\delta}} \mathbf{P}(\underline{S}_j \geq -\alpha, \max_{k \leq j} (\bar{S}_k - S_k) \leq \log n, \bar{S}_j = S_j) \mathbf{P}(\bar{S}_{\ell-j} \leq 0, -S_{\ell-j} - \theta \log n \in [x, x+1]) \\ & \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=\ell-(\log n)^2}^{\ell-(\log n)^{1+\delta}} c_{54} \frac{1+\alpha}{j} e^{-c_{55} \frac{j}{(\log n)^2}} \frac{1}{\ell-j} e^{-c_{56} \frac{(\log n)^2}{\ell-j}} \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} c_{57} \frac{1+\alpha}{\ell} e^{-c_{58} \frac{\ell}{(\log n)^2}} \end{aligned}$$

which is $o_\varepsilon(1)$. We hence end up with

$$\sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \mathbf{E}_{(6.25)}(\ell) = o_n(1) + o_\varepsilon(1),$$

which shows (6.25).

Let us turn to check (6.26). Let

$$\mathbf{E}_{(6.26)}(\ell) := \sum_{k=0}^{\ell} \mathbf{E} \left[e^{S_k - \bar{S}_\ell} \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right],$$

and

$$\mathbf{E}_{(6.30)}(j, k, \ell) := \mathbf{E} \left[e^{S_k - S_j} \mathbf{1}_{\{\bar{S}_\ell = S_j > \bar{S}_{j-1}\}} \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n - a_n, \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1]\}} \right]. \quad (6.30)$$

By considering the first time hitting \bar{S}_ℓ , one sees that

$$\begin{aligned} \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \mathbf{E}_{(6.26)}(\ell) &= \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{k=0}^{\ell} \sum_{j=1}^{\ell-1} \mathbf{E}_{(6.30)}(j, k, \ell) \\ &= \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=1}^{\ell-1} \sum_{k=0}^{j-1} \mathbf{E}_{(6.30)}(j, k, \ell) + \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=1}^{\ell-1} \sum_{k=j}^{\ell} \mathbf{E}_{(6.30)}(j, k, \ell). \end{aligned} \quad (6.31)$$

For $k \geq j$, by Markov property at time j ,

$$\mathbf{E}_{(6.30)}(j, k, \ell) \leq \mathbf{P}(\underline{S}_j \geq -\alpha, S_j = \bar{S}_j \geq \log n - a_n, \max_{1 \leq i \leq j} (\bar{S}_i - S_i) \leq \log n) \times \mathbf{E} \left[e^{S_{k-j}} \mathbf{1}_{\{\bar{S}_{\ell-j} \leq 0; -S_{\ell-j} \in [\theta \log n + x, \theta \log n + x + 1]\}} \right]. \quad (6.32)$$

Similarly as above, we use different inequalities for different j to bound the second sum on the right hand side of (6.31).

1. For $j \leq (\log n)^{1+\delta}$ with $\delta \in (0, 1)$, by (6.32), (A.21) and (A.17), one sees that

$$\begin{aligned} & \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=1}^{(\log n)^{1+\delta}} \sum_{k=j}^{\ell} \mathbf{E}_{(6.30)}(j, k, \ell) \\ & \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=1}^{(\log n)^{1+\delta}} \mathbf{P}(S_j \geq \log n - a_n) \mathbf{E} \left[\sum_{k=0}^{\ell-j} e^{S_k} \mathbf{1}_{\{\bar{S}_{\ell-j} \leq 0; -S_{\ell-j} \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] \\ & \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=1}^{(\log n)^{1+\delta}} \mathbf{P}(S_j \geq \log n - a_n) c_{59} \frac{\theta \log n + a_n}{(\ell/2)^{3/2}} \leq c_{60} \sqrt{\varepsilon} e^{-c_{61}(\log n)^{1-\delta}}, \end{aligned}$$

which is $o_n(1)$.

2. For $(\log n)^{1+\delta} \leq j \leq (\log n)^2$, by (6.32), (A.16) and (A.21), one sees that

$$\begin{aligned} & \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=(\log n)^{1+\delta}}^{(\log n)^2} \sum_{k=j}^{\ell} \mathbf{E}_{(6.30)}(j, k, \ell) \\ & \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=(\log n)^{1+\delta}}^{(\log n)^2} \mathbf{P}(\underline{S}_j \geq -\alpha, S_j = \bar{S}_j \geq \log n - a_n) \mathbf{E} \left[\sum_{k=0}^{\ell-j} e^{S_k} \mathbf{1}_{\{\bar{S}_{\ell-j} \leq 0; -S_{\ell-j} \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] \\ & \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=(\log n)^{1+\delta}}^{(\log n)^2} c_{62} \frac{1 + \alpha}{\sqrt{j} \log n} e^{-c_{63} \frac{(\log n)^2}{j}} \frac{\theta \log n + a_n}{(\ell - j)^{3/2}} \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} c_{64} (1 + \alpha) \frac{\log n}{\ell^{3/2}}, \end{aligned}$$

which is $o_\varepsilon(1)$.

3. For $(\log n)^2 \leq j \leq \ell - (\log n)^2$, by (6.32), (A.11) and (A.21), one sees that

$$\begin{aligned} & \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=(\log n)^2}^{\ell - (\log n)^2} \sum_{k=j}^{\ell} \mathbf{E}_{(6.30)}(j, k, \ell) \\ & \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=(\log n)^2}^{\ell - (\log n)^2} \mathbf{P}(S_j \geq -\alpha, S_j = \bar{S}_j, \max_{1 \leq i \leq j} (\bar{S}_i - S_i) \leq \log n) \mathbf{E} \left[\sum_{k=0}^{\ell-j} e^{S_k} \mathbf{1}_{\{\bar{S}_{\ell-j} \leq 0; -S_{\ell-j} \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] \\ & \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=(\log n)^2}^{\ell - (\log n)^2} c_{65} \frac{1 + \alpha}{j} e^{-c_{66} \frac{j}{(\log n)^2}} \frac{\theta \log n + a_n}{(\ell - j)^{3/2}} \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} c_{67} (1 + \alpha) \frac{\log n}{\ell^{3/2}}, \end{aligned}$$

which is $o_\varepsilon(1)$.

4. For $\ell - (\log n)^2 \leq j \leq \ell$, by (6.32), (A.11) and (A.20), one sees that

$$\begin{aligned}
& \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=\ell-(\log n)^2}^{\ell-1} \sum_{k=j}^{\ell} \mathbf{E}_{(6.30)}(j, k, \ell) \\
& \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{\ell-j=1}^{(\log n)^2} \mathbf{P}(\underline{S}_j \geq -\alpha, S_j = \bar{S}_j, \max_{1 \leq i \leq j} (\bar{S}_i - S_i) \leq \log n) \mathbf{E} \left[\sum_{k=0}^{\ell-j} e^{S_k} \mathbf{1}_{\{\bar{S}_{\ell-j} \leq 0; -S_{\ell-j} \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] \\
& \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=\ell-(\log n)^2}^{\ell-1} c_{68} \frac{1+\alpha}{\ell} e^{-c_{69} \frac{j}{(\log n)^2}} \mathbf{E} \left[\sum_{k=0}^{\ell-j} e^{S_k} \mathbf{1}_{\{\bar{S}_{\ell-j} \leq 0; -S_{\ell-j} \in [\theta \log n + x, \theta \log n + x + 1]\}} \right] \\
& \leq \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} c_{70} \frac{1+\alpha}{\ell} e^{-c_{71} \frac{\ell}{(\log n)^2}},
\end{aligned}$$

which is $o_\varepsilon(1)$.

Combining all these terms, we get that

$$\sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=1}^{\ell-1} \sum_{k=j}^{\ell} \mathbf{E}_{(6.30)}(j, k, \ell) = o_n(1) + o_\varepsilon(1). \quad (6.33)$$

Next, let us bound $\sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=1}^{\ell-1} \sum_{k=0}^{j-1} \mathbf{E}_{(6.30)}(j, k, \ell)$. For $k < j$, Markov property at time j implies that

$$\begin{aligned}
\mathbf{E}_{(6.30)}(j, k, \ell) & \leq \mathbf{E} \left[e^{S_k - S_j} \mathbf{1}_{\{\underline{S}_j \geq -\alpha, S_j > \bar{S}_{j-1}, \bar{S}_j \geq \log n - a_n, \max_{0 \leq i \leq j} (\bar{S}_i - S_i) \leq \log n\}} \right] \\
& \quad \times \mathbf{P}(\bar{S}_{\ell-j} \leq 0, -S_{\ell-j} - \theta \log n \in [x, x + 1]).
\end{aligned}$$

If $\bar{S}_k \leq \frac{1}{2} \log n$, then $\mathbf{E}_{(6.30)}(j, k, \ell) \leq e^{-\frac{1}{2} \log n + a_n}$. Therefore,

$$\begin{aligned}
\mathbf{E}_{(6.30)}(j, k, \ell) & \leq e^{-\frac{1}{2} \log n + a_n} + \mathbf{E} \left[e^{S_k - S_j} \mathbf{1}_{\{\underline{S}_j \geq -\alpha, \bar{S}_k \geq \frac{\log n}{2}, S_j > \bar{S}_{j-1}, \bar{S}_j \geq \log n - a_n, \max_{0 \leq i \leq j} (\bar{S}_i - S_i) \leq \log n\}} \right] \\
& \quad \times \mathbf{P}(\bar{S}_{\ell-j} \leq 0, -S_{\ell-j} - \theta \log n \in [x, x + 1]). \quad (6.34)
\end{aligned}$$

By Markov property at time k and by the fact that $(S_j - S_{j-i})_{0 \leq i \leq j}$ is distributed as $(S_i)_{0 \leq i \leq j}$,

$$\begin{aligned}
& \mathbf{E} \left[e^{S_k - S_j} \mathbf{1}_{\{\bar{S}_k \geq \frac{1}{2} \log n, \underline{S}_j \geq -\alpha, S_j > \bar{S}_{j-1}, \bar{S}_j \geq \log n - a_n, \max_{0 \leq i \leq j} (\bar{S}_i - S_i) \leq \log n\}} \right] \\
& \leq \mathbf{E} \left[\mathbf{1}_{\{\underline{S}_k \geq -\alpha, \bar{S}_k \geq \frac{1}{2} \log n, \max_{i \leq k} (\bar{S}_i - S_i) \leq \log n\}} \mathbf{E} [e^{-S_{j-k}} \mathbf{1}_{\{S_{j-k} = \bar{S}_{j-k} \geq x_0\}}] \Big|_{x_0 = \bar{S}_k - S_k} \right] \\
& = \mathbf{E} \left[\mathbf{1}_{\{\underline{S}_k \geq -\alpha, \bar{S}_k \geq \frac{1}{2} \log n, \max_{i \leq k} (\bar{S}_i - S_i) \leq \log n\}} \mathbf{E} [e^{-S_{j-k}} \mathbf{1}_{\{S_{j-k} \geq 0, S_{j-k} \geq x_0\}}] \Big|_{x_0 = \bar{S}_k - S_k} \right]
\end{aligned}$$

which by (A.22) is less than $\frac{c_{72}}{(j-k)^{3/2}} \mathbf{E} \left[e^{(S_k - \bar{S}_k)/2} \mathbf{1}_{\{\underline{S}_k \geq -\alpha, \bar{S}_k \geq \frac{1}{2} \log n, \max_{i \leq k} (\bar{S}_i - S_i) \leq \log n\}} \right]$. As a result,

$$\sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=1}^{\ell-1} \sum_{k=0}^{j-1} \mathbf{E}_{(6.30)}(j, k, \ell) \leq o_n(1) + \sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=1}^{\ell-1} \sum_{k=1}^{j-1} \frac{c_{72}}{(j-k)^{3/2}} \mathbf{E}_{(6.35)}(j, k, \ell), \quad (6.35)$$

where

$$\mathbf{E}_{(6.35)}(j, k, \ell) := \mathbf{E} \left[e^{(S_k - \bar{S}_k)/2} \mathbf{1}_{\{\underline{S}_K \geq -\alpha, \bar{S}_k \geq \frac{1}{2} \log n, \max_{i \leq k} (\bar{S}_i - S_i) \leq \log n\}} \right] \mathbf{P}(\bar{S}_{\ell-j} \leq 0, -S_{\ell-j} - \theta \log n \in [x, x+1]).$$

For $j \leq \ell/2$, by (A.2), we have

$$\begin{aligned} & \sum_{j=1}^{\ell/2} \sum_{k=1}^{j-1} \frac{c_{72}}{(j-k)^{3/2}} \mathbf{E}_{(6.35)}(j, k, \ell) \leq \sum_{j=1}^{\ell/2} \sum_{k=1}^{j-1} \frac{c_{73}}{(j-k)^{3/2}} \mathbf{E} \left[e^{(S_k - \bar{S}_k)/2} \mathbf{1}_{\{\underline{S}_K \geq -\alpha, \bar{S}_k \geq \frac{1}{2} \log n, \max_{i \leq k} (\bar{S}_i - S_i) \leq \log n\}} \right] \frac{\log n}{(\ell/2)^{3/2}} \\ & \leq \frac{\log n}{(\ell/2)^{3/2}} \sum_{k=1}^{\ell/2-1} \mathbf{E} \left[e^{(S_k - \bar{S}_k)/2} \mathbf{1}_{\{\underline{S}_K \geq -\alpha, \bar{S}_k \geq \frac{1}{2} \log n, \max_{i \leq k} (\bar{S}_i - S_i) \leq \log n\}} \right] \sum_{j=k+1}^{\ell/2} \frac{c_{73}}{(j-k)^{3/2}} \\ & \leq c_{74} \frac{\log n}{\ell^{3/2}} \left[\sum_{k=1}^{(\log n)^2} + \sum_{k=(\log n)^2}^{\ell/2} \mathbf{E} \left[e^{(S_k - \bar{S}_k)/2} \mathbf{1}_{\{\underline{S}_K \geq -\alpha, \bar{S}_k \geq \frac{1}{2} \log n, \max_{i \leq k} (\bar{S}_i - S_i) \leq \log n\}} \right] \right] \end{aligned}$$

By (A.10) for $k \leq (\log n)^2$ and by (A.9) for $k \geq (\log n)^2$, we see that

$$\begin{aligned} & \sum_{j=1}^{\ell/2} \sum_{k=1}^{j-1} \frac{c_{72}}{(j-k)^{3/2}} \mathbf{E}_{(6.35)}(j, k, \ell) \\ & \leq c_{75} \frac{\log n}{\ell^{3/2}} \left[\sum_{k=1}^{(\log n)^2} \frac{1+\alpha}{\sqrt{k} \log n} + \sum_{k=(\log n)^2}^{\ell/2} \left(\frac{(1+\alpha) \log k}{k^{3/2}} + \frac{(1+\alpha)}{k} e^{-c_{76} \frac{k}{(\log n)^2}} \right) \right] \leq c_{77} (1+\alpha) \frac{\log n}{\ell^{3/2}}. \end{aligned}$$

This yields that

$$\sum_{\ell=(\log n)^2/\varepsilon}^{c_0(\log n)^3} \sum_{j=1}^{\ell/2} \sum_{k=1}^{j-1} \frac{c_{72}}{(j-k)^{3/2}} \mathbf{E}_{(6.35)}(j, k, \ell) = o_\varepsilon(1). \quad (6.36)$$

For $\ell/2 \leq j \leq \ell - (\log n)^2$, by (A.2), we have

$$\begin{aligned} & \sum_{j=\ell/2}^{\ell - (\log n)^2} \sum_{k=1}^{j-1} \frac{c_{72}}{(j-k)^{3/2}} \mathbf{E}_{(6.35)}(j, k, \ell) \\ & \leq \sum_{j=\ell/2}^{\ell - (\log n)^2} \sum_{k=1}^{j-1} \frac{c_{78}}{(j-k)^{3/2}} \mathbf{E} \left[e^{(S_k - \bar{S}_k)/2} \mathbf{1}_{\{\underline{S}_K \geq -\alpha, \bar{S}_k \geq \frac{1}{2} \log n, \max_{i \leq k} (\bar{S}_i - S_i) \leq \log n\}} \right] \frac{\log n}{(\ell-j)^{3/2}} \end{aligned}$$

By (A.10) for $k \leq (\log n)^2$ and by (A.9) for $k \geq (\log n)^2$, we see that

$$\begin{aligned} & \sum_{j=\ell/2}^{\ell - (\log n)^2} \sum_{k=1}^{j-1} \frac{c_{72}}{(j-k)^{3/2}} \mathbf{E}_{(6.35)}(j, k, \ell) \\ & \leq \sum_{j=\ell/2}^{\ell - (\log n)^2} \left[\sum_{k=1}^{(\log n)^2} \frac{c_{79}}{(j-k)^{3/2}} \frac{1+\alpha}{\sqrt{k} \log n} + \sum_{k=(\log n)^2}^{j-1} \frac{c_{80}}{(j-k)^{3/2}} (1+\alpha) \left(\frac{\log k}{k^{3/2}} + \frac{e^{-c_{81} \frac{k}{(\log n)^2}}}{k} \right) \right] \frac{\log n}{(\ell-j)^{3/2}} \\ & \leq c_{82} (1+\alpha) \left(\frac{\log \ell}{\ell^{3/2}} + \frac{e^{-c_{83} \frac{\ell}{(\log n)^2}}}{\ell} \right). \end{aligned}$$

This leads to

$$\sum_{\ell=(\log n)^{2/\varepsilon}}^{c_0(\log n)^3} \sum_{j=\ell/2}^{\ell-(\log n)^2} \sum_{k=1}^{j-1} \frac{c}{(j-k)^{3/2}} \mathbf{E}_{(6.35)}(j, k, \ell) = o_\varepsilon(1). \quad (6.37)$$

For $\ell - (\log n)^2 \leq j \leq \ell - 1$, by (A.10) for $k \leq (\log n)^2$ and by (A.9) for $k \geq (\log n)^2$, one sees that

$$\begin{aligned} & \sum_{j=\ell-(\log n)^2}^{\ell-1} \sum_{k=1}^{j-1} \frac{c_{72}}{(j-k)^{3/2}} \mathbf{E}_{(6.35)}(j, k, \ell) \\ & \leq \sum_{j=\ell-(\log n)^2}^{\ell-1} \left[\sum_{k=1}^{(\log n)^2} \frac{c_{84}}{(j-k)^{3/2}} \frac{1+\alpha}{\sqrt{k} \log n} + \sum_{k=(\log n)^2}^{j-1} \frac{c_{85}}{(j-k)^{3/2}} (1+\alpha) \left(\frac{\log k}{k^{3/2}} + \frac{e^{-c_{86} \frac{k}{(\log n)^2}}}{k} \right) \right] \\ & \quad \times \mathbf{P}(\bar{S}_{\ell-j} \leq 0, -S_{\ell-j} - \theta \log n \in [x, x+1]) \\ & \leq c_{87}(1+\alpha) \left(\frac{\log \ell}{\ell^{3/2}} + \frac{e^{-c_{88} \frac{\ell}{(\log n)^2}}}{\ell} \right) \sum_{j=1}^{(\log n)^2} \mathbf{P}(\bar{S}_j \leq 0, -S_j - \theta \log n \in [x, x+1]) \end{aligned}$$

which by (A.19) is less than

$$c_{89}(1+\alpha) \left(\frac{\log \ell}{\ell^{3/2}} + \frac{e^{-c_{88} \frac{\ell}{(\log n)^2}}}{\ell} \right).$$

Consequently,

$$\sum_{\ell=(\log n)^{2/\varepsilon}}^{c_0(\log n)^3} \sum_{j=\ell-(\log n)^2}^{\ell-1} \sum_{k=1}^{j-1} \frac{c_{72}}{(j-k)^{3/2}} \mathbf{E}_{(6.35)}(j, k, \ell) = o_n(1) + o_\varepsilon(1). \quad (6.38)$$

In view of (6.36), (6.37) and (6.38), we end up with

$$\sum_{\ell=(\log n)^{2/\varepsilon}}^{c_0(\log n)^3} \sum_{j=1}^{\ell-1} \sum_{k=1}^{j-1} \frac{c_{72}}{(j-k)^{3/2}} \mathbf{E}_{(6.35)}(j, k, \ell) = o_n(1) + o_\varepsilon(1).$$

This, combined with (6.35), (6.33) and (6.31), gives (6.26). We thus conclude (6.23).

Proof of (6.24). Let

$$\mathbf{E}_{(6.24)}(\ell) := \mathbf{E} \left[\sum_{|x|=\ell} n a_x (1-a_x)^{n-1} b_x^{n^\theta-1} \mathbf{1}_{\{x \in \mathfrak{D}_n\}} \mathbf{1}_{\{\bar{V}(x) \in [\log n - a_n, \log n + a_n], \underline{V}(x) \geq -\alpha\}} \right].$$

We are going to show that

$$\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^{2/\varepsilon}} \mathbf{E}_{(6.24)}(\ell) = o(n^{1-\theta}). \quad (6.39)$$

Recall that $n a_x (1-a_x)^{n-1} b_x^{n^\theta-1} \leq n^{1-\theta} e^{-V(x)}$. It then follows from Many-to-One Lemma that

$$\begin{aligned} & \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^{2/\varepsilon}} \mathbf{E}_{(6.24)}(\ell) \leq n^{1-\theta} \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^{2/\varepsilon}} \mathbf{E} \left[\sum_{|x|=\ell} e^{-V(x)} \mathbf{1}_{\{x \in \mathfrak{D}_n\}} \mathbf{1}_{\{\bar{V}(x) \in [\log n - a_n, \log n + a_n], \underline{V}(x) \geq -\alpha\}} \right] \\ & = n^{1-\theta} \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^{2/\varepsilon}} \mathbf{P}(\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \in [\log n - a_n, \log n + a_n], \bar{S}_\ell - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n]). \end{aligned}$$

which by considering the first time hitting \bar{S}_ℓ is equal to

$$n^{1-\theta} \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=1}^{\ell-1} \mathbf{P}(\underline{S}_\ell \geq -\alpha, \bar{S}_{j-1} < S_j = \bar{S}_\ell \in [\log n - a_n, \log n + a_n], S_j - S_\ell - \theta \log n \in [-a_n, a_n]).$$

So by Markov property at time j , we get that

$$\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbf{E}_{(6.24)}(\ell) \leq n^{1-\theta} \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=1}^{\ell-1} \mathbf{P}_{(6.40)}(j, \ell), \quad (6.40)$$

where

$$\mathbf{P}_{(6.40)}(j, \ell) := \mathbf{P}(\underline{S}_j \geq -\alpha, \bar{S}_j = S_j \in [\log n - a_n, \log n + a_n]) \mathbf{P}(\bar{S}_{\ell-j} \leq 0, -S_{\ell-j} - \theta \log n \in [-a_n, a_n]).$$

We will divide the sum on the right hand side of (6.40) into four parts: $\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=1}^{(\log n)^{1+\delta}}$ with $\delta \in (1/2, 1)$, $\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=(\log n)^{1+\delta}}^{(\log n)^2}$, $\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=(\log n)^2}^{\ell - (\log n)^2}$ and $\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=\ell - (\log n)^2}^{\ell-1}$ and bound them separately.

1. For $1 \leq j \leq (\log n)^{1+\delta}$ with $\delta \in (0, 1)$, by (A.2) and (A.17), one sees that

$$\begin{aligned} & \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=1}^{(\log n)^{1+\delta}} \mathbf{P}_{(6.40)}(j, \ell) \\ & \leq \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=1}^{(\log n)^{1+\delta}} \mathbf{P}(S_j \geq \log n - a_n) c_{90} \frac{a_n \log n}{(\ell - j)^{3/2}} \leq \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} c_{90} \frac{a_n \log n}{\ell^{3/2}} e^{-c_{91}(\log n)^{1+\delta}} = o_n(1). \end{aligned}$$

2. For $(\log n)^{1+\delta} \leq j \leq (\log n)^2$ with $\delta \in (1/2, 1)$, by (A.18) and (A.2), one gets that

$$\begin{aligned} & \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=(\log n)^{1+\delta}}^{(\log n)^2} \mathbf{P}_{(6.40)}(j, \ell) \\ & \leq \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=(\log n)^{1+\delta}}^{(\log n)^2} c_{92} (1 + \alpha)^4 \frac{a_n (\log n)^3}{j^3} \frac{a_n \log n}{(\ell - j)^{3/2}} \leq \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} c_{92} (1 + \alpha)^4 \frac{a_n^2 (\log n)^{2(1-\delta)}}{\ell^{3/2}} = o_n(1). \end{aligned}$$

3. For $(\log n)^2 \leq j \leq \ell - (\log n)^2$, by (A.8) and (A.2), one sees that

$$\begin{aligned} & \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=(\log n)^2}^{\ell - (\log n)^2} \mathbf{P}_{(6.40)}(j, \ell) \\ & \leq \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=(\log n)^2}^{\ell - (\log n)^2} \mathbf{P}(\underline{S}_j \geq -\alpha, S_j \in [\log n - a_n, \log n + a_n]) \mathbf{P}(\bar{S}_{\ell-j} \leq 0, -S_{\ell-j} - \theta \log n \in [-a_n, a_n]) \\ & \leq \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=(\log n)^2}^{\ell - (\log n)^2} c_{93} \frac{(1 + \alpha) a_n}{j^{3/2}} \frac{a_n \log n}{(\ell - j)^{3/2}} \leq \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} c_{94} (1 + \alpha) \frac{a_n^2}{\ell^{3/2}} = o_n(1). \end{aligned}$$

4. For $\ell - (\log n)^2 \leq j \leq \ell - 1$, by (A.8) and (A.19), one gets that

$$\begin{aligned}
& \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=\ell-(\log n)^2}^{\ell-1} \mathbf{P}_{(6.40)}(j, \ell) \\
& \leq \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} c_{95} \frac{(1+\alpha)a_n}{\ell^{3/2}} \sum_{r=\theta \log n - a_n}^{\theta \log n + a_n} \sum_{j=1}^{(\log n)^2} \mathbf{P}(\bar{S}_j \leq 0, -S_j \in [r, r+1]) \\
& \leq \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} c_{96} (1+\alpha) \frac{a_n^2}{\ell^{3/2}} = o_n(1).
\end{aligned}$$

Going back to (6.40), we deduce that $\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbf{E}_{(6.24)}(\ell)$. This completes the proof of (6.24). \square

Proof of Lemma 4.3. Proof of (4.6). Recall that for $x \in \mathfrak{B}^+ n \cap \mathfrak{D}_n \cap \mathfrak{L}_n$, one has $\mathbb{P}^\mathcal{E}(\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1) = na_x(1-a_x)^{n-1}b_x^{n^\theta-1} = (1+o_n(1))n^{1-\theta}e^{-V(x)}f(\frac{n^\theta}{H_x})$ with $\mathbf{f}(t) = te^{-t}$. It then follows that

$$\begin{aligned}
& \frac{1}{n^{1-\theta}} \mathbb{E} \left[\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{|z|=\ell} \mathbf{1}_{\{\bar{L}_z(\tau_n) \geq n^\theta, E_z^{(n)} = 1\}} \mathbf{1}_{\{z \in \mathfrak{B}_n^+\}} \mathbf{1}_{\{z \in \mathfrak{D}_n\}} \mathbf{1}_{\{\underline{V}(z) \geq -\alpha, \gamma_n \leq \max_{x \leq z} H_z < n\}} \right] \\
& = (1+o_n(1)) \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbb{E} \left[\sum_{|z|=\ell} e^{-V(z)} \mathbf{f}\left(\frac{n^\theta}{H_z}\right) \mathbf{1}_{\{z \in \mathfrak{B}_n^+\}} \mathbf{1}_{\{z \in \mathfrak{D}_n\}} \mathbf{1}_{\{\underline{V}(z) \geq -\alpha, \gamma_n \leq \max_{x \leq z} H_z < n\}} \right] \\
& = (1+o_n(1)) \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbb{E} \left[\mathbf{f}\left(\frac{n^\theta}{H_\ell^S}\right) \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n + a_n, \bar{S}_\ell - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n], \gamma_n \leq \max_{k \leq \ell} H_k^S \leq n\}} \right] \quad (6.41)
\end{aligned}$$

which is less than

$$c \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbb{E} \left[\mathbf{f}\left(\frac{n^\theta}{H_\ell^S}\right) \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n + a_n, \bar{S}_\ell - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n], \log n - r \log \log n - \log \ell \leq \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n\}} \right],$$

as $e^{\bar{S}_k - S_k} \leq H_k^S \leq ke^{\bar{S}_k - S_k}$. To conclude, we only need to show that for any $a_n = o(\log n)$,

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \sum_{\ell=a(\log n)^2}^{A(\log n)^2} \mathbb{E} \left[\mathbf{f}\left(\frac{n^\theta}{H_\ell^S}\right) \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n + a_n, \bar{S}_\ell - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n], \max_{k \leq \ell} (\bar{S}_k - S_k) \leq \log n + a_n\}} \right] \\
& = \mathcal{R}(\alpha) \int_a^A \mathcal{G}\left(\frac{1}{\sqrt{u}}, \frac{1}{\sqrt{u}}, \frac{\theta}{\sqrt{u}}\right) \frac{du}{u} \quad (6.42)
\end{aligned}$$

which follows immediately from (A.29) and (4.7). By comparing the convergences for $a_n = 0$ and $a_n = -(r+3) \log \log n$, we obtain what we want.

Proof of (4.7). Similarly as (6.41), we get that

$$\begin{aligned} & \frac{1}{n^{1-\theta}} \mathbb{E} \left[\sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{|z|=\ell} \mathbf{1}_{\{\bar{L}_z(\tau_n) \geq n^\theta, E_z^{(n)}=1\}} \mathbf{1}_{\{z \in \mathfrak{B}_n^+\}} \mathbf{1}_{\{z \in \mathfrak{D}_n \setminus \mathfrak{D}_n^K\}} \mathbf{1}_{\{\underline{V}(z) \geq -\alpha, \max_{x \leq z} H_x < n\}} \right] \\ &= (1 + o_n(1)) \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbb{E} \left[\mathbf{f}\left(\frac{n^\theta}{H_\ell^S}\right) \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n + a_n, \bar{S}_\ell - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n] \setminus [\theta \log n - K, \theta \log n + K], \max_{k \leq \ell} H_k^S \leq n\}} \right]. \end{aligned}$$

Similarly as in the proof of (6.23), one has

$$\begin{aligned} & \mathbb{E} \left[\mathbf{f}\left(\frac{n^\theta}{H_\ell^S}\right) \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n + a_n, \bar{S}_\ell - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n] \setminus [\theta \log n - K, \theta \log n + K], \max_{k \leq \ell} H_k^S \leq n\}} \right] \\ & \leq \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=1}^{\ell-1} \mathbb{E} \left[\left(\frac{n^\theta}{H_\ell^S} \wedge \frac{H_\ell^S}{n^\theta}\right) \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \tau_\ell(\bar{S})=j, \bar{S}_\ell \geq \log n + a_n, \bar{S}_\ell - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n] \setminus [\theta \log n - K, \theta \log n + K], \max_{k \leq \ell} (\bar{S}_k - S_k) \leq n\}} \right] \\ & \leq \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{x=-a_n}^{-K} e^x \sum_{j=1}^{\ell-1} \mathbb{E} \left[\sum_{k=0}^{\ell} e^{S_k - \bar{S}_\ell} \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \tau_\ell(\bar{S})=j, \bar{S}_\ell \geq \log n + a_n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1], \max_{k \leq \ell} (\bar{S}_k - S_k) \leq n\}} \right] \\ & \quad + \sum_{\ell=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{x=K}^{a_n} e^{-x} \sum_{j=1}^{\ell-1} \mathbb{E} \left[\mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \tau_\ell(\bar{S})=j, \bar{S}_\ell \geq \log n + a_n, \bar{S}_\ell - S_\ell \in [\theta \log n + x, \theta \log n + x + 1], \max_{k \leq \ell} (\bar{S}_k - S_k) \leq n\}} \right] \end{aligned}$$

Using the same arguments as for (6.25) and (6.26), one sees that

$$\mathbb{E} \left[\mathbf{f}\left(\frac{n^\theta}{H_\ell^S}\right) \mathbf{1}_{\{\underline{S}_\ell \geq -\alpha, \bar{S}_\ell \geq \log n + a_n, \bar{S}_\ell - S_\ell \in [\theta \log n - a_n, \theta \log n + a_n] \setminus [\theta \log n - K, \theta \log n + K], \max_{k \leq \ell} H_k^S \leq n\}} \right] \leq c_{97}(1 + \alpha)e^{-K},$$

which is $o_K(1)$ as $K \rightarrow \infty$. \square

Proof of Lemma 4.4. Let us consider the quenched variance of $\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)$ which is

$$\begin{aligned} & \text{Var}^\mathcal{E}(\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)) = \mathbb{E}^\mathcal{E} \left[(\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha) - \mathbb{E}^\mathcal{E}[\Xi_n(\ell, \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \alpha)])^2 \right] \\ &= \sum_{|x|=\ell} na_x(1-a_x)^{n-1} b_x^{n^\theta-1} [1 - na_x(1-a_x)^{n-1} b_x^{n^\theta-1}] \mathbf{1}_{\{x \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \underline{V}(x) \geq -\alpha\}} + \Sigma \text{Var} \end{aligned} \quad (6.43)$$

where

$$\begin{aligned} \Sigma \text{Var} &:= \sum_{|x|=|z|=\ell, x \neq z} \mathbf{1}_{\{x, z \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \underline{V}(x) \geq -\alpha, \underline{V}(z) \geq -\alpha\}} \\ & \quad \times [\mathbb{E}^\mathcal{E}(\mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)}=1\}}) \mathbf{1}_{\{\bar{L}_z(\tau_n) \geq n^\theta, E_z^{(n)}=1\}}) - n^2 a_x a_z (1-a_x)^{n-1} b_x^{n^\theta-1} (1-a_z)^{n-1} b_z^{n^\theta-1}]. \end{aligned} \quad (6.44)$$

On the one hand, for the first term on the right hand side of (6.43), as $\ell = \Theta((\log n)^2)$,

$$\begin{aligned} & \sum_{|x|=\ell} na_x(1-a_x)^{n-1} b_x^{n^\theta-1} [1 - na_x(1-a_x)^{n-1} b_x^{n^\theta-1}] \mathbf{1}_{\{x \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \underline{V}(x) \geq -\alpha\}} \\ & \leq \sum_{|x|=\ell} na_x(1-a_x)^{n-1} b_x^{n^\theta-1} \mathbf{1}_{\{x \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \underline{V}(x) \geq -\alpha\}} \end{aligned}$$

whose expectation under \mathbf{E} is $\Theta\left(\frac{n^{1-\theta}}{\ell}\right)$ according to (A.29) and (6.41). For $x \neq z$, one sees that $\{E_x^{(n)} = E_z^{(n)} = 1\}$ means that either x and z are visited in two different excursions or they are both visited in the same excursion. Let $a_{x,z} := \mathbb{P}_\rho^\mathcal{E}(T_x \wedge T_z < T_{\rho^*})$. Then,

$$\begin{aligned} & \mathbb{E}^\mathcal{E}(\mathbf{1}_{\{\bar{L}_x(\tau_n) \geq n^\theta, E_x^{(n)} = 1\}} \mathbf{1}_{\{\bar{L}_z(\tau_n) \geq n^\theta, E_z^{(n)} = 1\}}) \\ &= n(n-1)a_x a_z (1 - a_{x,z})^{n-2} (b_x b_z)^{n^\theta - 1} + n(1 - a_{x,z})^{n-1} \mathbb{P}^\mathcal{E}(\bar{L}_x(\tau_1) \geq n^\theta, \bar{L}_z(\tau_1) \geq n^\theta) \\ &\leq n^2 a_x a_z (1 - a_{x,z})^{n-2} (b_x b_z)^{n^\theta - 1} + n(1 - a_{x,z})^{n-1} \mathbb{E}^\mathcal{E} \left[\frac{\bar{L}_x(\tau_1) \bar{L}_z(\tau_1)}{n^{2\theta}} \right] \end{aligned}$$

Let $u = x \wedge z$ be the latest common ancestor of x and z . Say that u_x is the child of u such that $u_x \leq x$ and u_z is the child of u such that $u_z \leq z$. Then

$$\begin{aligned} \mathbb{E}^\mathcal{E} [\bar{L}_x(\tau_1) \bar{L}_z(\tau_1)] &= \mathbb{E}^\mathcal{E} [\bar{L}_{u_x}(\tau_1) \bar{L}_{u_z}(\tau_1)] e^{-V(x) - V(z) + V(u_x) + V(u_z)} \\ &= \mathbb{E}^\mathcal{E} [\bar{L}_u(\tau_1) (\bar{L}_u(\tau_1) + 1)] e^{-V(x) - V(z) + 2V(u)} \\ &= 2H_u e^{-V(u)} \times e^{-V(x) - V(z) + 2V(u)} \end{aligned}$$

It follows that

$$\begin{aligned} \Sigma_{\text{Var}} &\leq \sum_{x \neq z, |z|=|x|=\ell} n^2 a_x a_z (b_x b_z)^{n^\theta - 1} [(1 - a_{x,z})^{n-2} - (1 - a_x)^{n-1} (1 - a_z)^{n-1}] \mathbf{1}_{\{x,z \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \underline{V}(x) \geq -\alpha, \underline{V}(z) \geq -\alpha\}} \\ &\quad + \sum_{k=0}^{\ell-1} \sum_{|u|=k} \sum_{|x|=|z|=\ell, x \wedge z = u} 2n^{1-2\theta} H_u e^{-V(u)} \times e^{-V(x) - V(z) + 2V(u)} \mathbf{1}_{\{x,z \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \underline{V}(x) \geq -\alpha, \underline{V}(z) \geq -\alpha\}} \end{aligned}$$

By Lemma 4.2 of [AC18], $(1 - a_{x,z})^{n-2} - (1 - a_x)^{n-1} (1 - a_z)^{n-1} \leq n a_z + n a_x$. Moreover, $a_x \leq e^{-\bar{V}(x)} \leq e^{-\log n - a_n}$ for $x \in \mathfrak{B}_n^+$. Consequently,

$$\begin{aligned} & \sum_{x \neq z, |z|=|x|=\ell} n^2 a_x a_z (b_x b_z)^{n^\theta - 1} [(1 - a_{x,z})^{n-2} - (1 - a_x)^{n-1} (1 - a_z)^{n-1}] \mathbf{1}_{\{x,z \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \underline{V}(x) \geq -\alpha, \underline{V}(z) \geq -\alpha\}} \\ &\leq n^{2-2\theta} \sum_{k=0}^{\ell-1} \sum_{|u|=k} \sum_{|x|=|z|=\ell, x \wedge z = u} e^{-V(x) - V(z)} [n a_z + n a_x] \mathbf{1}_{\{x,z \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \underline{V}(x) \geq -\alpha, \underline{V}(z) \geq -\alpha\}} \\ &\leq 2e^{-a_n} n^{2-2\theta} \sum_{k=0}^{\ell-1} \sum_{|u|=k} \sum_{|x|=|z|=\ell, x \wedge z = u} e^{-V(x) - V(z)} \mathbf{1}_{\{x,z \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \underline{V}(x) \geq -\alpha, \underline{V}(z) \geq -\alpha\}} \end{aligned}$$

As $x, z \in \mathfrak{L}_{\gamma_n}$, $H_u \leq \gamma_n$. So,

$$\begin{aligned} \Sigma_{\text{Var}} &\leq \frac{2n^{2-2\theta}}{(\log n)^a} \sum_{k=0}^{\ell-1} \sum_{|u|=k} \sum_{|x|=|z|=\ell, x \wedge z = u} e^{-V(x) - V(z)} \mathbf{1}_{\{x,z \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \underline{V}(x) \geq -\alpha, \underline{V}(z) \geq -\alpha\}} \\ &\quad + 2 \frac{n^{2-2\theta}}{(\log n)^\gamma} \sum_{k=0}^{\ell-1} \sum_{|u|=k} \sum_{|x|=|z|=\ell, x \wedge z = u} e^{-V(x) - V(z) + V(u)} \mathbf{1}_{\{x,z \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \underline{V}(x) \geq -\alpha, \underline{V}(z) \geq -\alpha\}} \\ &\leq \left(\frac{2n^{2-2\theta}}{(\log n)^a} e^\alpha + \frac{2n^{2-2\theta}}{(\log n)^\gamma} \right) \sum_{k=0}^{\ell-1} \sum_{|u|=k} \sum_{|x|=|z|=\ell, x \wedge z = u} e^{-V(x) - V(z) + V(u)} \mathbf{1}_{\{x,z \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{L}_{\gamma_n}, \underline{V}(x) \geq -\alpha, \underline{V}(z) \geq -\alpha\}} \end{aligned}$$

since $V(u) \geq \underline{V}(x) \geq -\alpha$. Observe that

$$\begin{aligned}
& \mathbf{E} \left[\sum_{k=0}^{\ell-1} \sum_{|u|=k} \sum_{|x|=|z|=\ell, x \wedge z = u} e^{-V(x)-V(z)+V(u)} \mathbf{1}_{\{x, z \in \mathfrak{B}_n^+ \cap \mathfrak{D}_n \cap \mathfrak{S}_{\gamma_n}, \underline{V}(x) \geq -\alpha, \underline{V}(z) \geq -\alpha\}} \right] \\
& \leq \mathbf{E} \left[\sum_{k=0}^{\ell-1} \sum_{|u|=k} \sum_{\substack{u_z^* = u_x^* = u \\ u_z \neq u_x}} e^{-V(u_x) - \Delta V(u_z)} \sum_{z > u_z, |z| = \ell} e^{-[V(z) - V(u_z)]} \mathbf{1}_{\{\underline{V}(z) \geq -\alpha\}} \sum_{x > u_x, |x| = \ell} e^{-[V(x) - V(u_x)]} \mathbf{1}_{\{\underline{V}(x) \geq -\alpha\}} \right] \\
& = \mathbf{E} \left[\sum_{k=0}^{\ell-1} \sum_{|u|=k} \sum_{\substack{u_z^* = u_x^* = u \\ u_z \neq u_x}} e^{-V(u_x) - \Delta V(u_z)} \mathbf{1}_{\{\underline{V}(u_x) \wedge \underline{V}(u_z) \geq -\alpha\}} \mathbf{P}_{V(u_z)}(\underline{S}_{\ell-1-k} \geq -\alpha) \mathbf{P}_{V(u_x)}(\underline{S}_{\ell-1-k} \geq -\alpha) \right],
\end{aligned}$$

where the last equality follows from Many-to-One Lemma. By (A.1), we deduce that

$$\begin{aligned}
& \mathbf{E} \left[\sum_{k=0}^{\ell-1} \sum_{|u|=k} \sum_{\substack{u_z^* = u_x^* = u \\ u_z \neq u_x}} e^{-V(u_x) - \Delta V(u_z)} \mathbf{1}_{\{\underline{V}(u_x) \wedge \underline{V}(u_z) \geq -\alpha\}} \mathbf{P}_{V(u_z)}(\underline{S}_{\ell-1-k} \geq -\alpha) \mathbf{P}_{V(u_x)}(\underline{S}_{\ell-1-k} \geq -\alpha) \right] \\
& \leq c_{98} \mathbf{E} \left[\sum_{k=0}^{\ell-1} \sum_{|u|=k} (1 + \alpha + V(u))^2 e^{-V(u)} \mathbf{1}_{\{\underline{V}(u) \geq -\alpha\}} \sum_{\substack{u_z^* = u_x^* = u \\ u_z \neq u_x}} e^{-\Delta V(u_x) - \Delta V(u_z)} \frac{(1 + \Delta_+ V(u_z))(1 + \Delta_+ V(u_x))}{\ell - k} \right] \\
& \leq \sum_{k=0}^{\ell-1} \frac{c_{99}}{\ell - k} \mathbf{E}[(1 + \alpha + S_k)^2; \underline{S}_k \geq -\alpha] \leq \sum_{k=0}^{\ell-1} c_{100} \frac{k + (1 + \alpha)^2}{\ell - k} \leq c_{101} \ell^2.
\end{aligned}$$

We therefore end up with

$$\Sigma_{\mathbb{V}ar} \leq \frac{c_{102} n^{2-2\theta}}{(\log n)^{a \wedge r}} \ell^2 \leq \frac{c_{103} n^{2-2\theta}}{(\log n)^{a \wedge r - 4}}.$$

which suffices to conclude Lemma 4.4. \square

A Appendix

A.1 Proof of Lemma 2.3

Proof. We first prove (2.5). Observe that as $b \in (0, 1)$, for any $\lambda > 0$, by Markov inequality,

$$\begin{aligned}
\mathbf{P} \left(\sum_{i=1}^n \zeta_i \leq A \right) &= \mathbf{P} \left(e^{-\lambda(1-b) \sum_{i=1}^n \zeta_i} \geq e^{-\lambda(1-b)A} \right) \\
&\leq e^{\lambda(1-b)A} \mathbf{E} \left[e^{-\lambda(1-b)\zeta_1} \right]^n
\end{aligned}$$

where $\mathbf{E}[e^{-\lambda(1-b)\zeta_1}] = 1 - \frac{a(e^{\lambda(1-b)} - 1)}{e^{\lambda(1-b)} - b}$. We have $1 - x \leq e^{-x}$ for any $x \in [0, 1]$. It follows that

$$\begin{aligned} \mathbf{P}\left(\sum_{i=1}^n \zeta_i \leq A\right) &\leq \exp\left\{\lambda(1-b)A - na \frac{(e^{\lambda(1-b)} - 1)}{e^{\lambda(1-b)} - b}\right\} \\ &= \exp\left\{\lambda(1-b)A - na \frac{(e^{\lambda(1-b)} - 1)}{(e^{\lambda(1-b)} - 1) + (1-b)}\right\} \end{aligned}$$

Since $0 < 1 - b \leq \frac{e^{\lambda(1-b)} - 1}{\lambda}$, one gets $\frac{(e^{\lambda(1-b)} - 1)}{(e^{\lambda(1-b)} - 1) + (1-b)} \geq \frac{\lambda}{\lambda + 1}$ and then

$$\mathbf{P}\left(\sum_{i=1}^n \zeta_i \leq A\right) \leq e^{-\lambda\left(\frac{na}{\lambda+1} - (1-b)A\right)}, \forall n \geq 1.$$

Let us turn to check (2.3) and (2.4). We only prove (2.3), (2.4) follows from similar arguments. Note that for any $s \in [1, \frac{1}{b})$, Markov inequality implies that

$$\begin{aligned} \mathbf{P}\left(\sum_{i=1}^n \zeta_i \geq n^\theta\right) &\leq s^{-n^\theta} \mathbf{E}\left[s^{\sum_{i=1}^n \zeta_i}; \sum_{i=1}^n \mathbf{1}_{\{\zeta_i \geq 1\}} \geq 1\right] = \frac{\mathbf{E}[s^{\zeta_1}]^{n^\theta} - \mathbf{P}(\sum_{i=1}^n \mathbf{1}_{\{\zeta_i \geq 1\}} = 0)}{s^{n^\theta}} \\ &= \frac{1}{s^{n^\theta}} \left[\left(1 - a + \frac{a(1-b)s}{1-bs}\right)^{n^\theta} - (1-a)^{n^\theta} \right] \\ &\leq \frac{1}{s^{n^\theta}} \frac{na(1-b)s}{1-bs} \left(1 - a + \frac{a(1-b)s}{1-bs}\right)^{n^\theta - 1}, \end{aligned}$$

since $(1 - a + x)^n - (1 - a)^n \leq nx(1 - a + x)^{n-1}$ for any $x > 0$. Now take $s = \frac{1+\delta b}{(1+\delta)b}$ with some $\delta > 0$. Apparently, $s \in [1, \frac{1}{b})$ and for any $\eta \in (0, 1)$, there exists $M_\eta > 1$ such that $\log(1 + \frac{1-b}{(1+\delta)b}) \geq (1 - \eta/3) \frac{1-b}{(1+\delta)b}$ as long as $\delta b \geq M_\eta$. Consequently, for $\delta \geq M_\eta/b > 0$,

$$\begin{aligned} \mathbf{P}\left(\sum_{i=1}^n \zeta_i \geq n^\theta\right) &\leq na \frac{1+\delta b}{\delta b} \left(1 + \frac{1-b}{(1+\delta)b}\right)^{-n^\theta} \left(1 + \frac{a}{\delta b}\right)^{n^\theta - 1} \\ &\leq 2(na) e^{-(1-\eta/3) \frac{n^\theta(1-b)}{(1+\delta)b} + n \frac{a}{\delta b}}. \end{aligned}$$

Now we take $\eta \in (0, 1)$ such that $n^\theta(1-b) > na(1+\eta)$ and $\delta = \max\{\frac{M_\eta}{b}, \frac{2}{\eta-\eta^2}\}$ so that

$$\frac{na}{\delta b} \leq \frac{n^\theta(1-b)}{(1+\eta)\delta b} = \frac{n^\theta(1-b)}{(1+\delta)b} \frac{1+\delta}{\delta(1+\eta)} \leq (1-\eta/2) \frac{n^\theta(1-b)}{(1+\delta)b}$$

This yields that

$$\mathbf{P}\left(\sum_{i=1}^n \zeta_i \geq n^\theta\right) \leq 2(na) e^{-\frac{\eta}{6(1+\delta)b} n^\theta(1-b)},$$

where $(1+\delta)b \leq 1 + M_\eta + \frac{2}{\eta-\eta^2}$. We hence conclude (2.3) with $c_\eta = \frac{\eta}{6(1+M_\eta + \frac{2}{\eta-\eta^2})} \in (0, \infty)$. \square

A.2 Results on one-dimensional random walks

We state some facts and inequalities on centred random walk $(S_n)_{n \geq 0}$ introduced in the Many-to-One Lemma. The proofs are postponed in Section A.3.

Let $\zeta_n := S_n - S_{n-1}$ for any $n \geq 1$. Note that $\mathbf{E}[\zeta_1] = 0$, $\sigma^2 = \mathbf{E}[\zeta_1^2] < \infty$. Moreover, by (1.4),

$$\mathbf{E}[e^{-\delta_0 \zeta_1} + e^{(1+\delta_0)\zeta_1}] < \infty.$$

We start with some well known inequalities (see [AC18] for instance). Recall that $\underline{S}_n = \min_{0 \leq k \leq n} S_k$ and $\bar{S}_n = \max_{0 \leq k \leq n} S_k$. Note that the inequalities in the following hold also for the random walk $(-S_n)_{n \geq 0}$. For any $\alpha \geq 0$ and $n \geq 1$, we have

$$\mathbf{P}(\underline{S}_n \geq -\alpha) \leq \frac{C_4(1+\alpha)}{\sqrt{n}} \text{ and } \mathbf{P}(\bar{S}_n \leq \alpha) \leq \frac{C_4(1+\alpha)}{\sqrt{n}}. \quad (\text{A.1})$$

For any $\alpha \geq 0$, $b \geq a \geq -\alpha$ and for any $n \geq 1$,

$$\mathbf{P}(\underline{S}_n \geq -\alpha, S_n \in [a, b]) = \mathbf{P}_\alpha(S_n \geq 0, S_n \in [\alpha + a, \alpha + b]) \leq \frac{C_5(1+\alpha)(1+b+\alpha)(1+b-a)}{n^{3/2}}. \quad (\text{A.2})$$

We define the renewal function associated with the strict descending ladder process as follows:

$$\mathcal{R}(u) := \sum_{k=0}^{\infty} \mathbf{P}(S_k < \underline{S}_{k-1}, S_k \geq -u), \forall u \geq 0. \quad (\text{A.3})$$

It is known from Renewal theorem that

$$\frac{1}{u} \mathcal{R}(u) \rightarrow \mathbf{c}_R \text{ as } u \rightarrow \infty. \quad (\text{A.4})$$

Moreover there exist $0 < C_6 < C_7 < \infty$ such that for any $u \geq 0$,

$$C_6(1+u) \leq \mathcal{R}(u) \leq C_7(1+u).$$

Recall that there exists some positive constant \mathbf{c}_+ such that $\mathbf{P}(\underline{S}_n \geq 0) \sim \frac{\mathbf{c}_+}{\sqrt{n}}$ as $n \rightarrow \infty$. According to Lemma 2.1 of [AS14],

$$\mathbf{c}_R \mathbf{c}_+ = \sqrt{\frac{2}{\pi \sigma^2}}. \quad (\text{A.5})$$

Fact A.1. 1. For any $u, \alpha \geq 0$ and for any $n \geq 1$,

$$\mathbf{P}_u(\underline{S}_n \geq -\alpha, S_n = \bar{S}_n) \leq \frac{C_8(1+\alpha+u)}{n}. \quad (\text{A.6})$$

2. For any $n \geq 1$ and $A > 0$, $\alpha \geq 0$,

$$\mathbf{P}(\underline{S}_n \geq -\alpha, S_n = \bar{S}_n \geq A) \leq \frac{C_9(1+\alpha)}{A\sqrt{n}}. \quad (\text{A.7})$$

3. For any $B > 0$ fixed, there exists $c(B) > 0$ such that for any $n \geq 1$ and $-B\sqrt{n} \leq -\alpha \leq 0 < a < b \leq B\sqrt{n}$,

$$\mathbf{P}(S_n \geq -\alpha, S_n \in [a, b]) \leq \frac{c(B)(1+\alpha)(b-a)}{n^{3/2}}. \quad (\text{A.8})$$

4. For $A > 0$ sufficiently large and any $\lambda > 0, \alpha \geq 0$ and $n \geq 1$,

$$\mathbf{E}[e^{\lambda(S_n - \bar{S}_n)}; \max_{1 \leq k \leq n} (\bar{S}_k - S_k) \leq A, \underline{S}_n \geq -\alpha] \leq C_{10}(1+\alpha) \left[\frac{\log n}{n^{3/2}} + \frac{1}{n} e^{-C_{11}n/A^2} \right]. \quad (\text{A.9})$$

5. For any $A \geq 1, \lambda > 0, \alpha \geq 0$ and $n \geq 1$,

$$\mathbf{E}[e^{\lambda(S_n - \bar{S}_n)}; \bar{S}_n \geq A, \underline{S}_n \geq -\alpha] \leq \frac{C_{12}(1+\alpha)}{A\sqrt{n}}. \quad (\text{A.10})$$

6. For $\alpha \geq 0$ and $A \geq 1$ sufficiently large,

$$\mathbf{P}(\underline{S}_n \geq -\alpha, \bar{S}_n = S_n, \max_{1 \leq k \leq n} (\bar{S}_k - S_k) \leq A) \leq C_{13} \frac{1+\alpha}{n} e^{-C_{14} \frac{n}{A^2}}. \quad (\text{A.11})$$

7. As $x \rightarrow \infty$,

$$\mathbf{E}_x \left[\sum_{n=0}^{\infty} e^{-S_n/4} \mathbf{1}_{\{\underline{S}_n \geq 0\}} \right] = o_x(1) \mathcal{R}(x). \quad (\text{A.12})$$

According to [Afa93], conditioned on $\{\underline{S}_n \geq 0\}$, the rescaled path $(\frac{S_{[nt]}}{\sqrt{n}}; 0 \leq t \leq 1)$ and $\sum_{i=0}^n e^{-S_i}$ converge jointly in law to a Brownian meander $(m_t, t \in [0, 1])$ and a positive random variable \mathcal{H}_∞ which is independent of the Brownian meander. One can refer to [AC18] for more details. Let us state (A.12) of [AC18] here.

Fact A.2. Let $\alpha \geq 0, a, b > 0$ fixed and $a_n = o(\sqrt{n}), b_n = o(\sqrt{n})$. For any uniformly continuous and bounded function $g : [1, \infty) \rightarrow \mathbb{R}_+$, we have

$$\lim_{n \rightarrow \infty} n \mathbf{E} \left[g \left(\sum_{j=1}^n e^{S_j - S_n} \right) \mathbf{1}_{\{\underline{S}_n \geq -\alpha, S_n > \bar{S}_{n-1}, \max_{1 \leq i \leq n} (\bar{S}_i - S_i) \leq a\sqrt{n} + a_n, S_n \geq b\sqrt{n} + b_n\}} \right] = C_{a,b} \mathcal{R}(\alpha) \mathbf{E}[g(\mathcal{H}_\infty)]. \quad (\text{A.13})$$

where \mathcal{R} is the renewal function and $C_{a,b}$ is defined in (3.20) of [AC18].

The previous two Facts can be found in [AC18]. The following lemmas state some inequalities that will be proved in Appendix A.3.

Lemma A.3. Let $\alpha \geq 0$. There exists $\varepsilon_0 \in (0, 1)$ such that for m sufficiently large and for any $1 \leq r \leq \varepsilon_0 m$, we have

$$\mathbf{P}(\underline{S}_m \geq -\alpha, S_m \in [r, r+1]) \leq C_{15} \frac{1+\alpha}{m} e^{-C_{16} \frac{r^2}{m}}, \quad (\text{A.14})$$

and

$$\mathbf{P}(\underline{S}_m \geq -\alpha, S_m \geq r) \leq C_{17} \frac{1+\alpha}{r} e^{-C_{18} \frac{r^2}{m}}. \quad (\text{A.15})$$

Moreover,

$$\mathbf{P}(\underline{S}_m \geq -\alpha, \bar{S}_m = S_m \geq r) \leq C_{19} \frac{1+\alpha}{\sqrt{mr}} e^{-C_{20} \frac{r^2}{m}}. \quad (\text{A.16})$$

Lemma A.4. 1. For $\delta \in [0, 1)$ and $A \geq 1$ sufficiently large,

$$\sum_{k=1}^{A^{1+\delta}} \mathbf{P}(S_k \geq A) \leq e^{-C_{21}A^{1-\delta}}. \quad (\text{A.17})$$

2. Let $\alpha \geq 0$, for any $n \geq 1$ and $r \geq 0$,

$$\mathbf{P}(\underline{S}_n \geq -\alpha, \bar{S}_n = S_n \in [r, r+1]) \leq C_{22}(1+\alpha)^4 \frac{(1+r)^3}{n^3} \quad (\text{A.18})$$

3. Let $\eta > 0, \alpha \geq 0$. For r sufficiently large, one has

$$\sum_{1 \leq k \leq \eta r^2} \mathbf{P}(\underline{S}_k \geq -\alpha, S_k \in [r, r+1]) \leq C_{23}(1+\alpha)\eta. \quad (\text{A.19})$$

Moreover,

$$\sum_{k=1}^{\eta r^2} \mathbf{E} \left[\sum_{i=0}^k e^{-S_i}; \underline{S}_k \geq 0, S_k \in [r, r+1] \right] \leq C_{24}\eta. \quad (\text{A.20})$$

4. For any $x \geq 0$ and $n \geq 1$,

$$\mathbf{E} \left[\sum_{k=0}^n e^{S_k}; \bar{S}_n \leq 0, S_n \in [-x-1, -x] \right] \leq C_{25} \frac{1+x}{n^{3/2}}. \quad (\text{A.21})$$

5. For any $A \geq 0, \alpha \geq 0$ and $n \geq 1$,

$$\mathbf{E}_\alpha [e^{-S_n}; \underline{S}_n \geq 0, S_n \geq A] \leq \frac{C_{26}(1+\alpha)}{n^{3/2}} e^{-A/2}. \quad (\text{A.22})$$

6. For any $\alpha, A > 0$ and $n \geq 1$,

$$\mathbf{E}_\alpha [e^{S_n-A}; \underline{S}_n \geq 0, S_n \leq A] \leq C_{27} \frac{(1+\alpha)(1+A)}{n^{3/2}}. \quad (\text{A.23})$$

7. There exists $c \in \mathbb{R}_+^*$ such that for any $A > 0$,

$$\sum_{n \geq 0} \mathbf{E} [e^{S_n-A}; \underline{S}_n \geq 0, S_n \leq A] < C_{28}. \quad (\text{A.24})$$

8. For $\alpha \geq 0, a, b, c > 0, K \geq 1, n \leq Ar^2$ with $A > 0$,

$$\mathbf{P}(\underline{S}_n \geq -\alpha, \bar{S}_n \geq ar, \max_{k \leq n} (\bar{S}_k - S_k) \leq br, \bar{S}_n - S_n \in [cr - K, cr + K]) \leq C_{29}(1+\alpha) \frac{(1+K^2)r}{n^{3/2}} \quad (\text{A.25})$$

9. For $a, b, \eta > 0$ and $r \gg 1$ sufficiently large,

$$\sum_{k=1}^{\eta r^2} \mathbf{P}(\underline{S}_k \geq 0, \bar{S}_k \geq ar, \bar{S}_k - S_k \in [br, br+1]) \leq C_{30}(a, b)\eta^{3/2}. \quad (\text{A.26})$$

The following lemma focus on asymptotic results that we need.

Lemma A.5. *Let $\alpha \geq 0$. Then the following convergences hold.*

1. *For any continuous and bounded function $g : [0, \infty) \rightarrow \mathbb{R}_+$, the following convergence holds uniformly for x, y in any compact set of $(0, \infty)$ and for $z = o(\sqrt{n})$, $h > 0$,*

$$\mathbf{E} \left[g \left(\sum_{i=1}^n e^{-S_i} \right) \mathbf{1}_{\{\underline{S}_n \geq -\alpha, \bar{S}_n \leq x\sqrt{n}, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h]\}} \right] = \frac{\mathbf{c}_+ h \mathcal{R}(\alpha)}{\sigma n} \mathbf{E}_\alpha [g(e^\alpha \mathcal{H}_\infty - 1)] \mathcal{C}_0(x-y, y) + \frac{o_n(1)}{n}, \quad (\text{A.27})$$

where

$$\mathcal{C}_0(a, b) = \varphi\left(\frac{b}{\sigma}\right) \mathbf{P}(\bar{R}_1 - R_1 \leq \frac{a}{\sigma} | R_1 = \frac{b}{\sigma}), \quad (\text{A.28})$$

and $\mathbf{c}_+ = \lim_{n \rightarrow \infty} \sqrt{n} \mathbf{P}(\underline{S}_n \geq 0)$.

2. *Let $a, b > 0$ be fixed constants. For $F(x, y) = \frac{x}{y} e^{-x/y}$ with $x \in \mathbb{R}$ and $y \geq 1$ and for $a_n = o(\sqrt{n})$, $a'_n = o(\sqrt{n})$ and fixed $K > 0$, we have*

$$\begin{aligned} \lim_{n \rightarrow \infty} n \mathbf{E} \left[F(e^{b\sqrt{n} - (\bar{S}_n - S_n)}, \sum_{i=0}^n e^{S_i - \bar{S}_n}) \mathbf{1}_{\{\underline{S}_n \geq -\alpha, \bar{S}_n \geq a\sqrt{n} + a_n, \max_{0 \leq k \leq n} (\bar{S}_k - S_k) \leq a\sqrt{n} + a'_n, \bar{S}_n - S_n \in [b\sqrt{n} - K, b\sqrt{n} + K]\}} \right] \\ = \mathcal{G}(a, b) \mathcal{R}(\alpha) \int_{-K}^K \mathbf{E}[(F(e^{-s}, \mathcal{H}_\infty + \mathcal{H}_\infty^{(-)} - 1)] ds, \quad (\text{A.29}) \end{aligned}$$

where

$$\mathcal{G}(a, b) := \int_0^1 C_{\frac{a}{\sqrt{u}}, \frac{a}{\sqrt{u}}} \frac{\mathbf{c}_-}{\sigma} \mathcal{C}_0\left(\frac{a-b}{\sqrt{1-u}}, \frac{b}{\sqrt{1-u}}\right) \mathbf{1}_{\{a > b\}} \frac{du}{u(1-u)} \quad (\text{A.30})$$

with $C_{a,b}$ defined in (3.20) of [AC18], $\mathbf{c}_- := \lim_{n \rightarrow \infty} \sqrt{n} \mathbf{P}(\bar{S}_n \leq 0)$ and $\mathcal{H}_\infty^{(-)} := \sum_{k=0}^{\infty} e^{-\zeta_k^{(-)}}$ with $(\zeta_k^{(-)})_{k \geq 0}$ the Markov chain obtained from the reflected walk $-S$. Moreover, this convergences holds uniformly for a, b in any compact set of $(0, \infty)$.

The following result is a direct consequence of (A.27).

Corollary A.6. *Let $\alpha \geq 0$ and $a, b > 0$. For $a_n = o(\sqrt{n})$ and $b_n = o(\sqrt{n})$, the following convergence holds.*

$$\lim_{n \rightarrow \infty} n \mathbf{E} \left[e^{S_n - b\sqrt{n} - b_n}; \underline{S}_n \geq -\alpha, \bar{S}_n - S_n \leq a\sqrt{n} + a_n, S_n \leq b\sqrt{n} + b_n \right] = \frac{\mathbf{c}_+ \mathcal{R}(\alpha)}{\sigma} \mathcal{C}_0(a, b); \quad (\text{A.31})$$

$$\lim_{n \rightarrow \infty} n \mathbf{E} \left[e^{S_n - b\sqrt{n} - b_n}; \underline{S}_n \geq -\alpha, \bar{S}_n \leq (a+b)\sqrt{n} + a_n, S_n \leq b\sqrt{n} + b_n \right] = \frac{\mathbf{c}_+ \mathcal{R}(\alpha)}{\sigma} \mathcal{C}_0(a, b). \quad (\text{A.32})$$

where $\mathcal{C}_0(a, b) = \varphi\left(\frac{b}{\sigma}\right) \mathbf{P}(\bar{R}_1 - R_1 \leq \frac{a}{\sigma} | R_1 = \frac{b}{\sigma})$ as in (1.9).

A.3 Proofs of (A.14) - (A.29)

Proof of (A.14). This is given in Lemma B6 of [AD14] when $\alpha = 0$ and the increments are bounded. Let us prove the general case.

For $1 \leq r \leq A\sqrt{m}$ with $A > 10$ fixed, by (A.2), it is clear that

$$\mathbf{P}(\underline{S}_m \geq -\alpha, S_m \in [r, r+1]) \leq \frac{C_5(1+\alpha)(1+r+\alpha)}{m^{3/2}} \leq C_{31} \frac{1+\alpha}{m} e^{-C_{32}r^2/m}.$$

It suffices to show (A.14) for $A\sqrt{m} \leq r \leq \varepsilon_0 m$. For any $x \in \mathbb{R}$, let

$$T_x^+ := \inf\{k \geq 0 : S_k \geq x\}, \text{ and } T_x^- := \inf\{k \geq 0 : S_k < x\}.$$

Then it is known that for any $0 \leq x \leq y$,

$$\mathbf{P}_x(T_y^+ < T_0^-) \leq C_{33} \frac{x+1}{y+1}.$$

Recall that the increments of S are $\xi_k, k \geq 0$ which have finite exponential moments. Therefore, one has

$$\begin{aligned} \mathbf{P}(\underline{S}_m \geq -\alpha, S_m \in [r, r+1]) &\leq \mathbf{P}(\max_{k \leq m} \xi_k \geq r/2) + \mathbf{P}(\underline{S}_m \geq -\alpha, S_m \in [r, r+1], S_{T_{\sqrt{m}}^+} \leq \sqrt{m} + \frac{r}{2}) \\ &\leq C_{34} m e^{-\delta_0 r/2} + \sum_{j=1}^m \mathbf{P}_\alpha(\underline{S}_m \geq 0, S_m \in [r+\alpha, r+\alpha+1], T_{\sqrt{m}+\alpha}^+ = j, S_j \in [\sqrt{m} + \alpha, r/2 + \sqrt{m} + \alpha]). \end{aligned}$$

By Markov property at $T_{\sqrt{m}}^+$,

$$\begin{aligned} &\sum_{j=1}^m \mathbf{P}_\alpha(\underline{S}_m \geq 0, S_m \in [r+\alpha, r+\alpha+1], T_{\sqrt{m}+\alpha}^+ = j, S_j \in [\sqrt{m} + \alpha, r/2 + \sqrt{m} + \alpha]) \\ &\leq \sum_{j=1}^{m-1} \mathbf{P}_\alpha(S_j \geq 0, T_{\sqrt{m}+\alpha}^+ = j) \max_{\sqrt{m}+\alpha \leq x \leq \sqrt{m}+\alpha+r/2} \mathbf{P}(S_{m-j} \in [r+\alpha-x, r+\alpha-x+1]) \\ &\leq \mathbf{P}_\alpha(T_{\sqrt{m}+\alpha}^+ < T_0^-) \max_{1 \leq j \leq m} \max_{r/3 \leq x \leq r} \mathbf{P}(S_j \in [x, x+1]) \leq C_{35} \frac{\alpha+1}{\sqrt{m} + \alpha + 1} \max_{1 \leq j \leq m} \max_{r/3 \leq x \leq r} \mathbf{P}(S_j \in [x, x+1]). \end{aligned}$$

On the one hand, for $j \geq Kr$ with $K \geq 1$ fixed and $r \gg 1$, by Chernoff's bound,

$$\max_{1 \leq j < Kr} \max_{r/3 \leq x \leq r} \mathbf{P}(S_j \in [x, x+1]) \leq \max_{1 \leq j < Kr} \mathbf{P}(S_j \geq r/3) \leq e^{-C_{36}r}.$$

On the other hand, for $Kr \leq j \leq m$, we use the following change of measure

$$\mathbf{P}^{(t)}((S_1, \dots, S_j) \in \cdot) = E[e^{tS_j - j\phi_S(t)}; (S_1, \cdot, S_j) \in \cdot]$$

with $\phi_S(t) := \log \mathbf{E}[e^{t\xi_1}]$. The probability $\mathbf{P}^{(t)}$ is well defined when $\phi_S(t) < \infty$. The corresponding expectation is denoted by $E^{(t)}$. It hence follows that for $t \in (-\delta_0/2, \delta_0/2)$,

$$\begin{aligned} \mathbf{P}(S_j \in [x, x+1]) &= \mathbf{E}^{(t)}[e^{-tS_j + j\phi_S(t)}; S_j \in [x, x+1]] \\ &\leq e^{-tx + j\phi_S(t)} \mathbf{P}^{(t)}(S_j \in [x, x+1]) \leq e^{-tr/3 + C_{37}jt^2} \mathbf{P}^{(t)}(S_j \in [x, x+1]), \end{aligned}$$

as $\phi_S(t) \leq C_{37}t^2$ for $|t| \leq \delta_0/2$. Let us take $t = t_j = \frac{r}{6C_{37}j}$ so that $e^{-tr/3+C_{37}jt^2} \leq e^{-\frac{r^2}{36C_{37}j}}$. Moreover, as under $\mathbf{P}^{(t)}$, (S_k) is a random walk with i.i.d. increments and $\mathbf{E}^{(t)}[e^{sS_1}] < \infty$ for $s \in (0, \delta_0/2)$, Berry-Esseen theorem shows that there exists C such that for $Kr \leq j \leq m$,

$$\mathbf{P}^{(t)}(S_j \in [x, x+1]) \leq \frac{C}{\sqrt{j}}.$$

As a result,

$$\max_{Kr \leq j \leq m} \max_{r/3 \leq x \leq r} \mathbf{P}(S_j \in [x, x+1]) \leq \max_{Kr \leq j \leq m} \frac{C}{\sqrt{j}} e^{-\frac{r^2}{36C_{37}j}} \leq \frac{C}{\sqrt{m}} e^{-\frac{r^2}{36C_{37}m}},$$

as long as $r \geq A\sqrt{m}$ with $A \geq \sqrt{18C_{37}}$. We thus end up with

$$\mathbf{P}(\underline{S}_m \geq -\alpha, S_m \in [r, r+1]) \leq C_{34}me^{-\delta_0 r/2} + C_{35} \frac{\alpha+1}{\sqrt{m} + \alpha + 1} \left(e^{-C_{36}r} \vee \frac{C}{\sqrt{m}} e^{-\frac{r^2}{36C_{37}m}} \right)$$

which suffices to obtain (A.14). □

Proof of (A.15). Observe that by (A.14) and Chernoff's bound,

$$\begin{aligned} \mathbf{P}(\underline{S}_m \geq -\alpha, S_m \geq r) &\leq \sum_{t=r}^{\varepsilon_0 m} \mathbf{P}(\underline{S}_m \geq -\alpha, S_m \in [t, t+1]) + \mathbf{P}(S_m \geq \varepsilon_0 m) \\ &\leq \sum_{t=r}^{\varepsilon_0 m} C_{15} \frac{1+\alpha}{m} e^{-C_{16} \frac{t^2}{m}} + e^{-C_{38}m} \\ &\leq C_{17} \frac{1+\alpha}{r} e^{-C_{18} r^2/m}. \end{aligned}$$

□

Proof of (A.16). Note that $(S_m - S_{m-i})_{0 \leq i \leq m/2}$ is an independent copy of $(S_i)_{0 \leq i \leq m}$. So, by (A.15) and (A.1),

$$\begin{aligned} &\mathbf{P}(\underline{S}_m \geq -\alpha, S_m = \bar{S}_m \geq r) \\ &\leq \mathbf{P}(\underline{S}_{m/2} \geq -\alpha, S_{m/2} \geq r/2) \mathbf{P}(\underline{S}_{m/2} \geq 0) + \mathbf{P}(\underline{S}_{m/2} \geq -\alpha) \mathbf{P}(\underline{S}_{m/2} \geq 0, S_{m/2} \geq r/2) \\ &\leq C_{19} \frac{1+\alpha}{\sqrt{mr}} e^{-C_{20} r^2/m}. \end{aligned}$$

□

Proof of (A.17). Because of (2.2), for $\lambda \in (0, 1 + \delta_0)$ and $k \geq 1$,

$$\mathbf{P}(S_k \geq A) \leq e^{-\lambda A} \mathbf{E}[e^{\lambda S_k}] = e^{-\lambda A + k\phi_S(\lambda)},$$

where $\phi_S(\lambda) = \log \mathbf{E}[e^{\lambda S_1}]$. Note that $\phi'_S(0) = \mathbf{E}[S_1] = 0$ and $\phi_S(\lambda) \leq C_{37}\lambda^2$ for $\lambda \in (0, \delta_0/2)$ small. By taking $\lambda = \frac{1}{2C_{37}A^\delta}$ with A sufficiently large, we have

$$\begin{aligned} \sum_{1 \leq k \leq A^{1+\delta}} \mathbf{P}(S_k \geq A) &\leq \sum_{1 \leq k \leq A^{1+\delta}} e^{-\lambda A + k\phi_S(\lambda)} \\ &\leq \sum_{1 \leq k \leq A^{1+\delta}} e^{-\lambda A + C_{37}k\lambda^2} \leq A^{1+\delta} e^{-\frac{A^{1-\delta}}{4C_{37}}} \end{aligned}$$

which suffices to conclude (A.17) for $\delta \in (0, 1)$. In particular, for $\delta = 0$, we can take $C_{37} > 1/\delta_0$ so that (A.17) holds. \square

Proof of (A.18). Observe that by Markov property at time $n/2$,

$$\begin{aligned} \mathbf{P}(\underline{S}_n \geq -\alpha, \bar{S}_n = S_n \in [r, r+1]) &\leq \mathbf{E}[\mathbf{1}_{\{\underline{S}_{n/2} \geq -\alpha, S_{n/2} \leq r+1\}} \mathbf{P}(\bar{S}_{n/2} = S_{n/2} \in [r-x, r-x+1])|_{x=S_{n/2}}] \\ &= \mathbf{E}[\mathbf{1}_{\{\underline{S}_{n/2} \geq -\alpha, S_{n/2} \leq r+1\}} \mathbf{P}(\underline{S}_{n/2} \geq 0, S_{n/2} \in [r-x, r-x+1])|_{x=S_{n/2}}] \end{aligned}$$

which by (A.2), is bounded by

$$C_{39} \mathbf{E}[\mathbf{1}_{\{\underline{S}_{n/2} \geq -\alpha, S_{n/2} \leq r+1\}} \frac{(2+r-S_{n/2})}{n^{3/2}}] \leq C_{39} \frac{(2+r+\alpha)}{n^{3/2}} \mathbf{P}(\underline{S}_{n/2} \geq -\alpha, S_{n/2} \leq r+1)$$

which by (A.2) implies that

$$\mathbf{P}(\underline{S}_n \geq -\alpha, \bar{S}_n = S_n \in [r, r+1]) \leq C_{40} \frac{(1+\alpha)(2+r+\alpha)^3}{n^3} \leq C_{40}(1+\alpha)^4 \frac{(1+r)^3}{n^3}.$$

This completes the proof of (A.18). \square

Proof of (A.19). By use of (A.17) and (A.14), we see that for $r \geq \eta^{-2}$ sufficiently large,

$$\begin{aligned} \sum_{1 \leq k \leq \eta r^2} \mathbf{P}(\underline{S}_k \geq -\alpha, S_k \in [r, r+1]) &\leq \sum_{k=1}^{r^{3/2}} \mathbf{P}(S_k \geq r) + \sum_{k=r^{3/2}}^{\eta r^2} \mathbf{P}(\underline{S}_k \geq -\alpha, S_k \in [r, r+1]) \\ &\leq e^{-C_{21}r^{1/2}} + \sum_{k=r^{3/2}}^{\eta r^2} C_{15} \frac{1+\alpha}{k} e^{-C_{16} \frac{r^2}{k}} \leq C_{41}(1+\alpha)\eta \end{aligned}$$

as $\sum_{k=r^{3/2}}^{\eta r^2} \frac{1}{k} e^{-C_{16} \frac{r^2}{k}} \leq \int_{r^{3/2}}^{\eta r^2+1} \frac{2}{x} e^{-C_{16}r^2/x} dx \leq \int_{\frac{1}{2\eta}}^{\sqrt{r}} \frac{1}{t} e^{-C_{16}t} dt \leq \frac{2\eta}{C_{16}}$. \square

Proof of (A.20). It is immediate that

$$\begin{aligned} \sum_{1 \leq k \leq \eta r^2} \mathbf{E} \left[\sum_{i=0}^k e^{-S_i}; \underline{S}_k \geq 0, S_k \in [r, r+1] \right] &\leq \sum_{1 \leq k \leq \eta r^2} k \mathbf{P}(\underline{S}_k \geq 0, S_k \in [r, r+1]) \\ &\leq r^{3/2} \sum_{k=1}^{r^{3/2}} \mathbf{P}(S_k \geq r) + \sum_{k=r^{3/2}}^{\eta r^2} k \mathbf{P}(\underline{S}_k \geq -\alpha, S_k \in [r, r+1]) \end{aligned}$$

which by (A.17) and (A.14), is bounded by $C_{42}\eta$. \square

Proof of (A.21). In fact, we only need to check that

$$\sum_{k=1}^{n-1} \mathbf{E}[e^{-S_k}; \underline{S}_n \geq 0, S_n \in [x, x+1]] \leq C_{43} \frac{1+x}{n^{3/2}}.$$

By Markov property time at time k and then by (A.2), one sees that

$$\begin{aligned} \sum_{k=1}^{n-1} \mathbf{E}[e^{-S_k}; \underline{S}_n \geq 0, S_n \in [x, x+1]] &= \sum_{k=1}^{n-1} \mathbf{E}[e^{-S_k} \mathbf{1}_{\{S_k \geq 0\}} \mathbf{P}_{S_k}(\underline{S}_{n-k} \geq 0, S_{n-k} \in [x, x+1])] \\ &\leq \sum_{k=1}^{n-1} \frac{C_5(2+x)}{(n-k)^{3/2}} \mathbf{E}[(1+S_k)e^{-S_k} \mathbf{1}_{\{S_k \geq 0\}}] \\ &\leq \sum_{k=1}^{n-1} \frac{C_{44}(1+x)}{(n-k)^{3/2}} \sum_{t=0}^{\infty} \frac{(1+t)^2 e^{-t}}{k^{3/2}} \leq C_{45}(1+x) \sum_{k=1}^{n-1} \frac{1}{k^{3/2}(n-k)^{3/2}}, \end{aligned}$$

which is less than $C_{46}(1+x)n^{-3/2}$. \square

Proof of (A.22). Observe that by (A.2),

$$\begin{aligned} \mathbf{E}_\alpha[e^{-S_n}; \underline{S}_n \geq 0, S_n \geq A] &\leq \sum_{t=A}^{\infty} e^{-t} \mathbf{P}_\alpha(\underline{S}_n \geq 0, S_n \in [t, t+1]) \\ &\leq \sum_{t=A}^{\infty} e^{-t} \frac{C_5(1+\alpha)(2+t)}{n^{3/2}} \leq C_{47} \frac{1+\alpha}{n^{3/2}} e^{-A/2}. \end{aligned}$$

\square

Proof of (A.23). Note that by (A.2),

$$\begin{aligned} \mathbf{E}_\alpha[e^{S_n-A}; \underline{S}_n \geq 0, S_n \leq A] &\leq \sum_{t=0}^A e^{t+1-A} \mathbf{P}_\alpha(\underline{S}_n \geq 0, S_n \in [t, t+1]) \\ &\leq C_5 \frac{1+\alpha}{n^{3/2}} \sum_{t=0}^A (2+t) e^{t+1-A} \leq C_{48} \frac{(1+\alpha)(1+A)}{n^{3/2}}. \end{aligned}$$

\square

Proof of (A.24). In fact, by setting $\tau^- := \inf\{k \geq 0 : S_k < 0\}$ and $\mathcal{R}^-(dx)$ the renewal measure associated with the weak ascending ladder process of $(S_n)_{n \geq 0}$, we have

$$\begin{aligned} \sum_{n \geq 0} \mathbf{E}[e^{S_n-A}; \underline{S}_n \geq 0, S_n \leq A] &= \mathbf{E} \left[\sum_{n=0}^{\tau^- - 1} e^{S_n-A} \mathbf{1}_{\{S_n \leq A\}} \right] \\ &= \int_0^A e^{x-A} \mathcal{R}^-(dx) \leq C_{49}, \end{aligned}$$

because there exists a constant $\bar{c}_{\mathcal{R}} > 0$ such that for any $h > 0$, $\mathcal{R}^-([x, x+h]) \sim \bar{c}_{\mathcal{R}} h$ as $x \rightarrow \infty$. \square

Proof of (A.25). Let

$$\mathbf{P}_{(A.25)} := \mathbf{P}(\underline{S}_n \geq -\alpha, \bar{S}_n \geq ar, \max_{k \leq n} (\bar{S}_k - S_k) \leq br, \bar{S}_n - S_n \in [cr - K, cr + K]).$$

By considering the first time hitting \bar{S}_n and by Markov property,

$$\begin{aligned} \mathbf{P}_{(A.25)} &= \sum_{j=1}^{n-1} \mathbf{P}(\underline{S}_n \geq -\alpha, S_j = \bar{S}_n \geq ar, S_j - S_n \in [cr - K, cr + K]) \\ &\leq \sum_{j=1}^{n-1} \mathbf{P}(\underline{S}_j \geq -\alpha, S_j = \bar{S}_j \geq ar) \mathbf{P}(\bar{S}_{n-j} \leq 0, -S_{n-j} \in [cr - K, cr + K]) \end{aligned}$$

which by (A.7) and by (A.2) for $(-S_n)_{n \geq 0}$ is bounded by

$$\sum_{j=1}^{n-1} \frac{C_{50}(1+\alpha)}{\sqrt{j}ar} \frac{(1+cr+K)(1+2K)}{(n-j)^{3/2}}.$$

which is bounded by $\frac{C_{29}(1+\alpha)(1+K)^2(1+r)}{n^{3/2}}$ as $n \leq Ar^2$. □

Proof of (A.26). By considering the first time hitting \bar{S}_n and by Markov property, we have

$$\begin{aligned} &\sum_{k=1}^{\eta r^2} \mathbf{P}(\underline{S}_k \geq 0, \bar{S}_k \geq ar, \bar{S}_k - S_k \in [br, br + 1]) \\ &\leq \sum_{k=1}^{\eta r^2} \sum_{j=1}^{k-1} \mathbf{P}(\underline{S}_k \geq 0, S_j = \bar{S}_j \geq ar) \mathbf{P}(\bar{S}_{k-j} \leq 0, -S_{k-j} \in [br, br + 1]) \\ &\leq \sum_{j=1}^{\eta r^2} \mathbf{P}(\underline{S}_k \geq 0, S_j = \bar{S}_j \geq ar) \sum_{k=1}^{\eta r^2} \mathbf{P}(\bar{S}_k \leq 0, -S_k \in [br, br + 1]), \end{aligned}$$

which by (A.7) and by (A.19) for $(-S_n)_{n \geq 0}$ is bounded by

$$\sum_{j=1}^{\eta r^2} \frac{C_{51}}{\sqrt{j}ar} \frac{\eta}{b^2} \leq C_{30}(a, b) \eta^{3/2}. \quad \square$$

Proof of (A.27). Let

$$\mathbf{E}_{(A.27)} := \mathbf{E} \left[g \left(\sum_{i=1}^n e^{-S_i} \right) \mathbf{1}_{\{\underline{S}_n \geq -\alpha, \bar{S}_n \leq x\sqrt{n}, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h]\}} \right].$$

Let $\delta \in (0, 1/2)$.

$$\mathbf{E}_{(A.27)} = \mathbf{E} \left[g \left(\sum_{i=1}^n e^{-S_i} \right) \mathbf{1}_{\{\underline{S}_n \geq -\alpha, \underline{S}_{[n^\delta, n]} \geq n^{\delta/6}, \bar{S}_{n^\delta} \leq n^\delta, \bar{S}_n \leq x\sqrt{n}, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h]\}} \right] + \text{Error}_{(A.27)}$$

where

$$\begin{aligned} \text{Error}_{(A.27)} &\leq \|g\|_\infty \mathbf{E} \left[\mathbf{1}_{\{\underline{S}_n \geq -\alpha, \underline{S}_{[n^\delta, n]} \leq n^{\delta/6}, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h]\}} \right] \\ &\quad + \|g\|_\infty \mathbf{E} \left[\mathbf{1}_{\{\underline{S}_n \geq -\alpha, \bar{S}_{n^\delta} \geq n^\delta, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h]\}} \right] \end{aligned}$$

First, let us check that $\text{Error}_{(A.27)} = o_n(\frac{1}{n})$. On the one hand, by (A.17),

$$\begin{aligned} \mathbf{E} \left[\mathbf{1}_{\{\underline{S}_n \geq -\alpha, \bar{S}_{n^\delta} \geq n^\delta, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h]\}} \right] &\leq \mathbf{P}(\bar{S}_{n^\delta} \geq n^\delta) \\ &\leq \sum_{k=1}^{n^\delta} \mathbf{P}(S_k \geq n^\delta) \leq e^{-C_{21}n^\delta} = o_n\left(\frac{1}{n}\right). \end{aligned}$$

On the other hand, by considering the first time hitting $\underline{S}_{[n^\delta, n]}$,

$$\begin{aligned} &\mathbf{E} \left[\mathbf{1}_{\{\underline{S}_n \geq -\alpha, \underline{S}_{[n^\delta, n]} \leq n^{\delta/6}, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h]\}} \right] \\ &\leq \sum_{j=n^\delta}^{n-1} \mathbf{E} \left[\mathbf{1}_{\{\underline{S}_j \geq -\alpha, S_j \leq n^{\delta/6}\}} \mathbf{P}(\underline{S}_{n-j} \geq 0, S_{n-j} \in [y\sqrt{n}+z-t, y\sqrt{n}+z-t+h]) \Big|_{t=S_j} \right] \\ &\leq \sum_{j=n^\delta}^{n-\sqrt{n}} \frac{C_5(1+\alpha)(1+\alpha+n^{\delta/6})^2}{j^{3/2}} \frac{C_5(1+h)(1+y\sqrt{n}+z+\alpha+h)}{(n-j)^{3/2}} \\ &\quad + \sum_{j=n-\sqrt{n}}^{n-1} \frac{C_5(1+\alpha)(1+\alpha+n^{\delta/6})^2}{j^{3/2}} \mathbf{P}(S_{n-j} \geq y\sqrt{n}+z-n^{\delta/6}) \end{aligned}$$

where the last inequality follows from (A.2). By (A.17),

$$\mathbf{E} \left[\mathbf{1}_{\{\underline{S}_n \geq -\alpha, \underline{S}_{[n^\delta, n]} \leq n^{\delta/6}, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h]\}} \right] \leq C_{52}(1+\alpha)^4 n^{-\delta/6-1} + C_{52}(1+\alpha)^3 n^{\delta/3-3/2} e^{-c_{53}\sqrt{n}}$$

which is $o_n(\frac{1}{n})$. It remains to prove the convergence of

$$\mathbf{E}_{(A.27)}^+ := \mathbf{E} \left[g\left(\sum_{i=1}^n e^{-S_i}\right) \mathbf{1}_{\{\underline{S}_n \geq -\alpha, \underline{S}_{[n^\delta, n]} \geq n^{\delta/6}, \bar{S}_{n^\delta} \leq n^\delta, \bar{S}_n \leq x\sqrt{n}, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h]\}} \right]. \quad (A.33)$$

As $\underline{S}_{[n^\delta, n]} \geq n^{\delta/6}$ and g is uniformly continuous on any compact set of $[0, \infty)$,

$$g\left(\sum_{i=1}^n e^{-S_i}\right) = g\left(\sum_{i=1}^{n^\delta} e^{-S_i}\right) + o_n(1). \quad (A.34)$$

In fact, we need to work on $\{\sum_{i=1}^n e^{-S_i} \leq K\}$ with $K > 0$ fixed. It is easy to check that

$$\begin{aligned} &\mathbf{E} \left[\left(\sum_{i=1}^n e^{-S_i}\right) \mathbf{1}_{\{\underline{S}_n \geq -\alpha, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h]\}} \right] \\ &\leq \sum_{i=1}^{n-1} \mathbf{E}_\alpha [e^{\alpha-S_i} \mathbf{1}_{\{\underline{S}_i \geq 0\}} \mathbf{P}_{S_i}(\underline{S}_{n-i} \geq 0, S_{n-i} \in [y\sqrt{n}+z, y\sqrt{n}+z+h])] + e^{-y\sqrt{n}-z} \\ &\leq \sum_{i=1}^{n-1} \frac{C_{54}(1+\alpha)(1+y\sqrt{n}+z+h)e^\alpha}{i^{3/2}(n-i)^{3/2}} \leq \frac{C_{55}}{n}. \end{aligned}$$

So,

$$\mathbf{E}_{(A.27)} = \mathbf{E} \left[g \left(\sum_{i=1}^n e^{-S_i} \right) \mathbf{1}_{\{\underline{S}_n \geq -\alpha, \bar{S}_n \leq x\sqrt{n}, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h], \sum_{i=1}^n e^{-S_i} \leq K\}} \right] + \frac{o_K(1)}{n}.$$

Let us work directly with (A.34). By (A.2), it is clear that

$$\begin{aligned} & o_n(1) \mathbf{P} \left(\underline{S}_n \geq -\alpha, \underline{S}_{[n^\delta, n]} \geq n^{\delta/6}, \bar{S}_{n^\delta} \leq n^\delta, \bar{S}_n \leq x\sqrt{n}, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h] \right) \\ & \leq o_n(1) \mathbf{P} \left(\underline{S}_n \geq -\alpha, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h] \right) \\ & \leq o_n(1) \frac{C_5(1+\alpha)(1+h)(1+y\sqrt{n}+z+\alpha+h)}{n^{3/2}} = o_n\left(\frac{1}{n}\right). \end{aligned}$$

It then follows that

$$\begin{aligned} \mathbf{E}_{(A.27)}^+ &= \mathbf{E} \left[g \left(\sum_{i=1}^{n^\delta} e^{-S_i} \right) \mathbf{1}_{\{\underline{S}_n \geq -\alpha, \underline{S}_{[n^\delta, n]} \geq n^{\delta/6}, \bar{S}_{n^\delta} \leq n^\delta, \bar{S}_n \leq x\sqrt{n}, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h]\}} \right] + o_n\left(\frac{1}{n}\right) \\ &= \mathbf{E} \left[g \left(\sum_{i=1}^{n^\delta} e^{-S_i} \right) \mathbf{1}_{\{\underline{S}_n \geq -\alpha, \bar{S}_{n^\delta} \leq n^\delta, \bar{S}_n \leq x\sqrt{n}, S_n \in [y\sqrt{n}+z, y\sqrt{n}+z+h]\}} \right] + o_n\left(\frac{1}{n}\right) \\ &= \mathbf{E} \left[g \left(\sum_{i=1}^{n^\delta} e^{-S_i} \right) \mathbf{1}_{\{\underline{S}_{n^\delta} \geq -\alpha, \bar{S}_{n^\delta} \leq n^\delta\}} \mathbf{P}_{S_{n^\delta}} \left(\underline{S}_{n-n^\delta} \geq -\alpha, \bar{S}_{n-n^\delta} \leq x\sqrt{n}, S_{n-n^\delta} - y\sqrt{n} - z \in [0, h] \right) \right] + o_n\left(\frac{1}{n}\right). \end{aligned} \tag{A.35}$$

where the last equality is obtained by Markov property at time n^δ . For $t = S_{n^\delta} \in [-\alpha, n^\delta]$, one sees that

$$\begin{aligned} & \mathbf{P}_t \left(\underline{S}_{n-n^\delta} \geq -\alpha, \bar{S}_{n-n^\delta} \leq x\sqrt{n}, S_{n-n^\delta} - y\sqrt{n} - z \in [0, h] \right) \\ &= \mathbf{P}_{t+\alpha} \left(\bar{S}_{n-n^\delta} \leq x\sqrt{n} + \alpha | \underline{S}_{n-n^\delta} \geq 0, S_{n-n^\delta} - y\sqrt{n} - z + \alpha \in [0, h] \right) \\ & \quad \times \mathbf{P}_{t+\alpha} \left(\underline{S}_{n-n^\delta} \geq 0, S_{n-n^\delta} - y\sqrt{n} - z \in [\alpha, \alpha + h] \right). \end{aligned}$$

By (5.3) of [CC13],

$$\mathbf{P}_{t+\alpha} \left(\underline{S}_{n-n^\delta} \geq 0, S_{n-n^\delta} - y\sqrt{n} - z \in [\alpha, \alpha + h] \right) = \frac{\mathbf{c}_+}{\sigma n} \mathcal{R}(t + \alpha) \left(\psi\left(\frac{y}{\sigma}\right) h + o_n(1) \right),$$

where the constant $\mathbf{c}_+ = \lim_{n \rightarrow \infty} \sqrt{n} \mathbf{P}(\underline{S}_n \geq 0) \in (0, \infty)$. Moreover, in the spirit of Theorem 2.4 of [CC13], we can say that

$$\mathbf{P}_{t+\alpha} \left(\bar{S}_{n-n^\delta} \leq x\sqrt{n} + \alpha | \underline{S}_{n-n^\delta} \geq 0, S_{n-n^\delta} - y\sqrt{n} - z + \alpha \in [0, h] \right) \rightarrow \mathbf{P}(\bar{R}_1 \leq \frac{x}{\sigma} | R_1 = \frac{y}{\sigma})$$

uniformly for (x, y) in a compact set of $(0, \infty)^2$. In fact, in Theorem 2.4 of [CC13], the *Hypothesis 2.2* is needed for the density of increments. However, in this work, as we consider $\{S_n \in [y, y + h]\}$ instead of $\{S_n = y\}$, the *Hypothesis 2.2* is not necessary. As a result,

$$\begin{aligned} & \mathbf{P}_t \left(\underline{S}_{n-n^\delta} \geq -\alpha, \bar{S}_{n-n^\delta} \leq x\sqrt{n}, S_{n-n^\delta} - y\sqrt{n} - z \in [0, h] \right) |_{t=S_{n^\delta}} \\ &= \frac{\mathbf{c}_+}{n\sigma} \mathcal{R}(S_{n^\delta} + \alpha) \psi\left(\frac{y}{\sigma}\right) h \mathbf{P}(\bar{R}_1 \leq \frac{x}{\sigma} | R_1 = \frac{y}{\sigma}) (1 + o_n(1)). \end{aligned}$$

Plugging it into (A.35) yields that

$$\begin{aligned}
\mathbf{E}_{(A.27)}^+ &= \mathbf{E} \left[g \left(\sum_{i=1}^{n^\delta} e^{-S_i} \mathbf{1}_{\{\underline{S}_{n^\delta} \geq -\alpha, \bar{S}_{n^\delta} \leq n^\delta\}} \frac{\mathbf{c}_+}{n\sigma} \mathcal{R}(S_{n^\delta} + \alpha) \psi\left(\frac{y}{\sigma}\right) h \mathbf{P}(\bar{R}_1 \leq \frac{x}{\sigma} | R_1 = \frac{y}{\sigma}) (1 + o_n(1)) \right) \right] + o_n\left(\frac{1}{n}\right) \\
&= \mathbf{E}_\alpha \left[g \left(\sum_{i=1}^{n^\delta} e^{\alpha - S_i} \mathbf{1}_{\{\underline{S}_{n^\delta} \geq 0\}} \mathcal{R}(S_{n^\delta}) \right) \frac{\mathbf{c}_+}{n\sigma} \psi\left(\frac{y}{\sigma}\right) h \mathbf{P}(\bar{R}_1 \leq \frac{x}{\sigma} | R_1 = \frac{y}{\sigma}) (1 + o_n(1)) \right] + o_n\left(\frac{1}{n}\right) \\
&= \mathcal{R}(\alpha) \mathbf{E}_\alpha \left[g \left(\sum_{i=1}^{n^\delta} e^{\alpha - \zeta_i} \right) \frac{\mathbf{c}_+}{n\sigma} \psi\left(\frac{y}{\sigma}\right) h \mathbf{P}(\bar{R}_1 \leq \frac{x}{\sigma} | R_1 = \frac{y}{\sigma}) (1 + o_n(1)) \right] + o_n\left(\frac{1}{n}\right)
\end{aligned}$$

where $(\zeta_i)_{i \geq 0}$ is a Markov chain taking values in \mathbb{R}_+ , satisfying $\mathbf{P}_\alpha(\zeta_0 = \alpha) = 1$, with transition probability $p(x, dy) = \mathbf{1}_{\{y > 0\}} \frac{\mathcal{R}(y)}{\mathcal{R}(x)} \mathbf{P}_x(S_1 \in dy)$. It is known that for any $\delta \in (0, 1/2)$ small, \mathbf{P}_α -a.s., $\zeta_n \geq n^{1/2-\delta}$ for $n \gg 1$. So $\mathcal{H}_\infty = \sum_{i=0}^\infty e^{-\zeta_i}$ is a positive random variable taking values in \mathbb{R}_+ . It is obvious that

$$\sum_{i=1}^{n^\delta} e^{\alpha - \zeta_i} \rightarrow \sum_{i=1}^\infty e^{\alpha - \zeta_i} = e^\alpha \mathcal{H}_\infty - 1.$$

As g is bounded, one obtains (A.27) by dominated convergence. \square

Proof of (A.29). Let

$$\mathbf{E}_{(A.29)} := \mathbf{E} \left[F(e^{b\sqrt{n} - (\bar{S}_n - S_n)}, \sum_{i=0}^n e^{S_i - \bar{S}_n}) \mathbf{1}_{\{\underline{S}_n \geq -\alpha, \bar{S}_n \geq a\sqrt{n} + a_n, \max_{0 \leq k \leq n} (\bar{S}_k - S_k) \leq a\sqrt{n} + a'_n, \bar{S}_n - S_n \in [b\sqrt{n} - K, b\sqrt{n} + K]\}} \right].$$

By considering the first time hitting \bar{S}_n , we have

$$\begin{aligned}
\mathbf{E}_{(A.29)} &= \sum_{j=1}^{n-1} \mathbf{E} \left[F(e^{b\sqrt{n} - (\bar{S}_n - S_n)}, \sum_{i=0}^n e^{S_i - \bar{S}_n}) \mathbf{1}_{\{\underline{S}_n \geq -\alpha, \bar{S}_{j-1} < S_j, S_j = \bar{S}_n \geq a\sqrt{n} + a_n\}} \right. \\
&\quad \left. \times \mathbf{1}_{\{\max_{0 \leq k \leq n} (\bar{S}_k - S_k) \leq a\sqrt{n} + a'_n, \bar{S}_n - S_n \in [b\sqrt{n} - K, b\sqrt{n} + K]\}} \right]. \quad (A.36)
\end{aligned}$$

By Markov property at time j , this is equal to $\sum_{j=1}^{n-1} \sum_{\ell=-K/h}^{K/h-1} \mathbf{E}_{(A.29)}(j, n, \ell)$ where

$$\begin{aligned}
\mathbf{E}_{(A.29)}(j, n, \ell) &:= \mathbf{E} \left[F(e^{b\sqrt{n} + R_{n-j}}, \sum_{i=0}^j e^{S_i - S_j} + \sum_{k=1}^{n-j} e^{R_k}) \mathbf{1}_{\{\underline{S}_j \geq -\alpha, \bar{S}_{j-1} < S_j, S_j \geq a\sqrt{n} + a_n, \max_{0 \leq k \leq j} (\bar{S}_k - S_k) \leq a\sqrt{n} + a'_n\}} \right. \\
&\quad \left. \times \mathbf{1}_{\{\max_{k \leq n-j} R_k \leq 0, \min_{0 \leq k \leq n-j} (-R_k) \leq (a\sqrt{n} + a'_n) \wedge (\alpha + S_j), -R_{n-j} \in [b\sqrt{n} + \ell h, b\sqrt{n} + \ell h + h]\}} \right] \quad (A.37)
\end{aligned}$$

with $(R_k)_{k \geq 0}$ is an independent copy of the random walk $(S_k)_{k \geq 0}$. First, let us prove that for $n \gg 1$,

$$\sum_{j \leq \varepsilon n \text{ or } j \geq n - \varepsilon n} \sum_{\ell = -K/h}^{K/h-1} n \mathbf{E}_{(A.29)}(j, n, \ell) = o_\varepsilon(1)$$

For $j \leq \varepsilon n$, similarly to (6.28), by (A.17), (A.7) and (A.2) one has

$$\begin{aligned} & \sum_{j \leq \varepsilon n} \sum_{\ell = -K/h}^{K/h-1} \mathbf{E}_{(A.29)}(j, n, \ell) \leq \sum_{j \leq n^{3/4}} \mathbf{P}(S_j \geq a\sqrt{n} + a_n) \\ & \quad + \sum_{j = n^{3/4}}^{\delta n} \mathbf{P}(\underline{S}_j \geq -\alpha, \bar{S}_j = S_j \geq a\sqrt{n} + a_n) \mathbf{P}(\bar{S}_{n-j} \leq 0, -S_{n-j} \in [b\sqrt{n} - K, b\sqrt{n} + K]) \\ & \leq e^{-C_{56}n^{1/4}} + \sum_{j = n^{3/4}}^{\varepsilon n} \frac{C_{57}(1 + \alpha)}{\sqrt{j}(a\sqrt{n} + a_n)} \frac{(1 + b\sqrt{n} + K)}{(n - j)^{3/2}} = \frac{o_n(1) + o_\varepsilon(1)}{n}. \end{aligned}$$

For $j \geq n - \varepsilon n$, by (A.6) and (A.19), one has

$$\begin{aligned} & \sum_{j = n - \varepsilon n}^n \sum_{\ell = -K/h}^{K/h-1} \mathbf{E}_{(A.29)}(j, n, \ell) \\ & \leq \sum_{j=1}^{\varepsilon n} \mathbf{P}(\underline{S}_{n-j} \geq -\alpha, \bar{S}_{n-j} = S_{n-j} \geq a\sqrt{n} + a_n) \mathbf{P}(\bar{S}_j \leq 0, -S_j \in [b\sqrt{n} - K, b\sqrt{n} + K]) \\ & \leq \frac{C_{58}(1 + \alpha)(1 + 2K)}{n} \varepsilon = \frac{o_\varepsilon(1)}{n}. \end{aligned}$$

Thus, it remains to study $\sum_{j = \varepsilon n}^{n - \varepsilon n} \sum_{\ell = -K/h}^{K/h-1} n \mathbf{E}_{(A.29)}(j, n, \ell)$. Recall that $F(x, y) = \frac{x}{y} e^{-x/y}$ with $x > 0$ and $y \geq 1$. So, for any fixed $h > 0$

$$\sup_{x > 0, y \geq 1} |F(xe^h, y) - F(x, y)| \leq 2(e^h - 1) \text{ and } \sup_{x > 0, y \geq 1} |F(x, y + h) - F(x, y)| \leq 2h.$$

Therefore, on $-R_{n-j} \in [b\sqrt{n} + \ell h, b\sqrt{n} + \ell h + h)$,

$$F(e^{b\sqrt{n} + R_{n-j}}, \sum_{i=0}^j e^{S_i - S_j} + \sum_{k=1}^{n-j} e^{R_k}) = F(e^{-\ell h}, \sum_{i=0}^j e^{S_i - S_j} + \sum_{k=1}^{n-j} e^{R_k}) + o_h(1).$$

Moreover, let $(S_k^{(-)})_{k \geq 0}$ be the random walk distributed as the reflected walk $-S$, and independent of S . Observe that for $\ell \in [-K, K]$ with $K \geq 1/h$ fixed integer

$$\begin{aligned} \mathbf{E}_{(A.29)}(j, n, \ell) & = \mathbf{E} \left[\mathbf{1}_{\{S_j \geq -\alpha, \bar{S}_{j-1} < S_j, S_j \geq a\sqrt{n} + a_n, \max_{0 \leq k \leq j} (\bar{S}_k - S_k) \leq a\sqrt{n} + a_n\}} \right. \\ & \quad \left. \times \mathbf{E} \left[(F(e^{-\ell h}, t + \sum_{k=1}^{n-j} e^{-S_k^{(-)}}) + o_h(1)) \mathbf{1}_{\{S_{n-j}^{(-)} \geq 0, \bar{S}_{n-j}^{(-)} \leq (a\sqrt{n} + a_n) \wedge (\alpha + s), S_{n-j}^{(-)} - c\sqrt{n} \in [\ell h, \ell h + h]\}} \right] \Big|_{t = \sum_{i=0}^j e^{S_i - S_j}, s = S_j} \right]. \end{aligned}$$

By use of (A.27) for $S^{(-)}$, one sees that $\varepsilon n \leq j \leq n - \varepsilon n$ with $\varepsilon \in (0, 1/2)$, for $n \gg 1$,

$$\begin{aligned} & (n - j) \mathbf{E} \left[(F(e^{-\ell h}, t + \sum_{k=1}^{n-j} e^{-S_k^{(-)}}) + o_h(1)) \mathbf{1}_{\{S_{n-j}^{(-)} \geq 0, \bar{S}_{n-j}^{(-)} \leq (a\sqrt{n} + a_n) \wedge (\alpha + s), S_{n-j}^{(-)} - b\sqrt{n} \in [\ell h, \ell h + h]\}} \right] \\ & = \frac{c-h}{\sigma} \mathbf{E} \left[(F(e^{-\ell h}, t + \mathcal{H}_\infty^{(-)} - 1) + o_n(1)) \right] \mathcal{C}_0(a \wedge \frac{s}{\sqrt{n}} - b, b) + o_n(1), \end{aligned}$$

where $\mathbf{c}_- := \lim_{n \rightarrow \infty} \sqrt{n} \mathbf{P}(\bar{S}_n \leq 0)$ and $\mathcal{H}_\infty^{(-)} := \sum_{k=0}^{\infty} e^{-\zeta_k^{(-)}}$ with $(\zeta_k^{(-)})_{k \geq 0}$ the Markov chain obtained from the reflected walk. It follows that

$$n\mathbf{E}_{(\text{A.29})}(j, n, \ell) = \frac{n}{n-j} \mathbf{E} \left[\mathbf{1}_{\{S_j \geq -\alpha, \bar{S}_{j-1} < S_j, S_j \geq a\sqrt{n} + a_n, \max_{0 \leq k \leq j} (\bar{S}_k - S_k) \leq a\sqrt{n} + a'_n\}} \right] \\ \times \left(\frac{\mathbf{c}_- h}{\sigma} \mathbf{E}[F(e^{-\ell h}, \sum_{i=0}^j e^{S_i - S_j} + \mathcal{H}_\infty^{(-)} - 1) + o_n(1)] \mathcal{C}_0(a-b, b) + o_n(1) \right)$$

which by Fact A.2 is equal to

$$\frac{n}{(n-j)j} \mathcal{C}_{\frac{a\sqrt{n}}{\sqrt{j}}, \frac{a\sqrt{n}}{\sqrt{j}}} \mathcal{R}(\alpha) \frac{\mathbf{c}_- h}{\sigma} \mathbf{E}[(F(e^{-\ell h}, \mathcal{H}_\infty + \mathcal{H}_\infty^{(-)} - 1)] \mathcal{C}_0(a-b, b) + o_n(\frac{1}{n}).$$

This leads to

$$\sum_{j=\varepsilon n}^{n-\varepsilon n} \sum_{\ell=-K/h}^{K/h-1} n\mathbf{E}_{(\text{A.29})}(j, n, \ell) \\ = \sum_{j=\varepsilon n}^{n-\varepsilon n} \sum_{\ell=-K}^K \frac{n}{j(n-j)} \mathcal{C}_{\frac{a\sqrt{n}}{\sqrt{j}}, \frac{a\sqrt{n}}{\sqrt{j}}} \mathcal{R}(\alpha) \frac{\mathbf{c}_- h}{\sigma} \mathbf{E}[(F(e^{-\ell h}, \mathcal{H}_\infty + \mathcal{H}_\infty^{(-)} - 1)] \mathcal{C}_0(a-b, b) + o_n(1) \\ = \int_\varepsilon^{1-\varepsilon} \mathcal{C}_{\frac{a}{\sqrt{t}}, \frac{a}{\sqrt{t}}} \frac{dt}{t(1-t)} \frac{\mathbf{c}_- \mathcal{R}(\alpha)}{\sigma} \mathcal{C}_0(a-b, b) \int_{-K}^K \mathbf{E}[(F(e^{-s}, \mathcal{H}_\infty + \mathcal{H}_\infty^{(-)} - 1)] ds + o_n(1) + o_n(1)Kh$$

Letting $n \rightarrow \infty$ then letting $h \rightarrow 0$ and $\varepsilon \rightarrow 0$, we conclude (A.29). \square

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