CELLULARIZATION FOR EXCEPTIONAL SPHERICAL SPACE FORMS AND APPLICATION TO FLAG MANIFOLDS

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Abstract. Given a finite group acting freely and isometrically on a Euclidean sphere, it is natural to look for an equivariant cellular decomposition of the sphere. In this way, one obtains a cellular decomposition of the quotient manifold, called a *spherical space form*.

In this paper, we apply the method developed by the first and thirs authors, using the so-called orbit polytope, to construct an explicit equivariant cellular decomposition of the $(4n-1)$-sphere with respect to the binary octahedral and the binary icosahedral groups, and describe the associated cellular homology chain complex. As an application, we find free resolutions of the constant module over these groups and deduce their integral cohomology.

Moreover, we use the octahedral case to describe a cellular structure on the flag manifold of $\mathbb{R}^3$ that is equivariant with respect to the Weyl group action.

Contents

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0. Introduction

If we have a finite group acting on a topological manifold, it is natural to look for an equivariant cellular decomposition. Finding such a cellular structure, one obtains an equivariant chain complex. This can be expressed as a chain complex of free modules over the group algebra of the considered group. Hence, this provides not only information about the space, but also on the group itself. For instance, such a complex allows us to fully derive which type of representation is carried by the homology of the space. Such an equivariant chain complex allows to lift the action at the level of the derived category of integral modules over the group. For instance, we may calculate derived functors.

However, not so many cases have been explicitly worked out. Finite groups acting freely and isometrically on the 3-sphere provide an interesting class of examples. It turns out that all such groups were entirely classified by Milnor in [Mil57] and there is five families of them: the quaternionic groups, the metacyclic groups, the generalized tetrahedral groups, the binary octahedral and icosahedral groups. So far, explicit equivariant cellular structures were found for the quaternionic groups ([MNdMS13]), the metacyclic groups ([FGMNS13] and [CS17]) and the generalized tetrahedral groups ([FGMNS16]). According to Milnor's classification, the only remaining cases are the binary octahedral group P_{48} and the binary icosahedral group P_{120}. These two cases are solved in the present paper.

Each one of these cases has its own interest. Recall that a *flag* in $V := \mathbb{C}^3$ is a pair (V_1, V_2) of subspaces of $\mathbb{C}^3$, where V_1 is a line and V_2 is a plane containing V_1. The set of such flags is the *flag manifold* $\mathcal{F}$ of type A_2. It is a closed algebraic subvariety of $\mathbb{P}(V) \times \mathbb{P}(V^*)$. Its real points $\mathcal{F}(\mathbb{R})$ form the set of real flags in $\mathbb{R}^3$, and is a compact smooth manifold, called the *real flag manifold of type* A_2. It turns out that, if we denote by W the Weyl group of $\mathcal{F}$ (see Remark 3.4.1), then $W \simeq \mathfrak{S}_3$ acts freely on $\mathcal{F}(\mathbb{R})$ and we shall see (in the Proposition 3.4.4) that we have a diffeomorphism of quotient spaces

$$\mathcal{F}(\mathbb{R})/W \simeq \mathbb{S}^3/P_{48}.$$

This will provide an $\mathfrak{S}_3$-equivariant cellular decomposition of $\mathcal{F}(\mathbb{R})$, once a P_{48}-equivariant cellular structure on $\mathbb{S}^3$ is known. This is actually the motivation of this paper. On the other hand, the icosahedral case provides a P_{120}-equivariant cellular decomposition of the universal cover of the *Poincaré homology sphere*, see Theorem 4.3.2 and Remark 4.3.5.

Our strategy is based on the ideas of [CS17]. Given a finite group G acting freely on $\mathbb{S}^n$, one looks at the *orbit polytope* $\mathscr{P}_G$ of the considered group, that is, the convex hull of an orbit of points in $\mathbb{S}^n$. Then, the group acts freely on the *boundary* $\partial\mathscr{P}_G$ of $\mathscr{P}_G$ (see Lemma 1.2.1) and an equivariant cellular decomposition is found by determining a *fundamental domain* $\mathscr{D}_G$ for the action on $\partial\mathscr{P}_G$ and by finding a cellular decomposition of $\mathscr{D}_G$ (with smaller open faces as cells). Then, the cellular decomposition of $\partial\mathscr{P}_G$ is obtained by translating the one of $\mathscr{D}_G$. Next, one has to notice that the natural projection $\partial\mathscr{P}_G \to \mathbb{S}^n$ defined by $x \mapsto x/|x|$ is a G-equivariant homeomorphism and then, one obtains a fundamental domain for the action of G on $\mathbb{S}^n$ and an equivariant cellular decomposition. Furthermore, the fact that the decomposition comes from a decomposition of some polytopal complex with open faces as cells, implies that the resulting decomposition of $\mathbb{S}^n$ is regular (that is the closure of a cell is homeomorphic to a closed ball) and the boundaries are then easily computed. The main tool is the theorem 1.2.2, which essentially says that we can find representatives for the action of G on the *facets* of $\mathscr{P}_G$ such that their union is a fundamental domain. We proceed by determining such representatives for both P_{48} and P_{120} (with the precious help of GAP ([Gro19]) and the Maple package Convex ([Fra]) for computations).

The main results of this paper may be summed up in the following statements, which combines Theorems 3.3.4 and 4.3.4.

Theorem. *Every sphere $\mathbb{S}^{4n-1}$, endowed with the natural free and isometric action of P_{48} (resp. of P_{120}), admits an explicit equivariant cell decomposition. As a consequence, the* *associated cellular homology chain complex is explicitly given in terms of matrices with entries in the group algebras* $\mathbb{Z}[P_{48}]$ *and* $\mathbb{Z}[P_{120}]$, *respectively.*

The crucial case is $\mathbb{S}^3$. Then one may prove the result inductively, using curved joins. As a consequence, one obtains the following result, which combines Corollaries 3.3.5, 4.3.6, 3.3.7 and 4.3.7.

Corollary. *One may give an explicit free* 4*-periodic resolution of the trivial module* $\mathbb{Z}$ *over* $\mathbb{Z}[P_{48}]$ *and over* $\mathbb{Z}[P_{120}]$. *In particular, one can compute the cohomology modules* $H^*(P_{48}, \mathbb{Z})$ *and* $H^*(P_{120}, \mathbb{Z})$.

It should be noted that such resolutions were already given in [TZ08]. Our approach however has the advantage of being more conceptual and geometric. Moreover, using the first result above we can derive the following consequence (see Theorems 3.4.6, 3.4.8 and Corollary 3.4.7):

Theorem. *The real flag manifold of type* A_2 *admits an explicit equivariant cell decomposition, with respect to its Weyl group* $\mathfrak{S}_3$. *In particular, its cellular homology chain complex is explicitly given in terms of matrices with entries in* $\mathbb{Z}[\mathfrak{S}_3]$ *and the isomorphism type of the* $\mathbb{Z}[\mathfrak{S}_3]$*-module* $H^*(\mathcal{F}(\mathbb{R}), \mathbb{Z})$ *is determined.*

Let us outline the content of the article. In the Section 1, after a quick reminder on polytopes, we introduce orbit polytopes and study some of their properties. Most importantly, we explain how to obtain a polytopal fundamental domain for the boundary of an orbit polytope, and hence for the sphere, using the radial projection. Most of those results appeared in [CS17], we recall them for the convenience of the reader.

In the Section 2, we review some basic facts about quaternions and we identify P_{48} and P_{120} (which we prefer to denote by $\mathcal{O}$ and $\mathcal{I}$, respectively) with explicit finite subgroups of the sphere of unit quaternions. This has the advantage of describing simultaneously their action on the sphere, since they act by left multiplication. These facts are contained in [LT09]. Coxeter and Moser also described those groups by generators and relations. We give an explicit isomorphism between the two descriptions. Finally, after some remarks on their fixed-point free representations, we introduce the *spherical space forms.*

In Sections 3, 4 and in the Appendix A, we apply the orbit polytope techniques to the cases where G is $\mathcal{O}$, $\mathcal{I}$ or $\mathcal{T}$ acting on $\mathbb{S}^3$; the later having been treated previously in [FGMNS16]. In particular, we explicitly describe a fundamental domain for the boundary of the polytope, and we use it to determine a G-equivariant cellular decomposition of $\mathbb{S}^3$. Moreover, we compute the resulting cellular homology chain complexes (which are bounded complexes of free $\mathbb{Z}[G]$-modules). Finally, we generalize this to $\mathbb{S}^{4n-1}$ and use the resulting equivariant cellular decomposition to obtain an explicit 4-periodic free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$ and recover the integral cohomology of G. Moreover, in Section 3, the application to the real flag manifold of type A_2 is derived.

1. Orbit polytopes

The following section gives the main tools for determining fundamental domains for finite groups acting isometrically on the sphere $\mathbb{S}^3$, by using their orbit polytopes. The presentation, the results and the proofs of this section are taken from [CS17], we reproduce it here to set the notations and for the sake of self-containement. For general properties of polytopes, the reader is refered to [Zie95].

1.1. **Some general facts on real polytopes.**

First of all, we shall give the very basic vocabulary about polytopes. Recall that, for $x, y \in \mathbb{R}^n$, we denote by $\langle x, y\rangle$ their usual Euclidean scalar product, and by $|x| := \sqrt{\langle x, x\rangle}$ the norm of x. We also denote by $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \ ; \ |x| = 1\}$ the $(n-1)$-dimensional sphere and by $\mathbb{D}^n := \{x \in \mathbb{R}^n \ ; \ |x| \leq 1\}$ the n-dimensional disc.

To a set of points X in $\mathbb{R}^n$, one can associate its *convex hull* defined by

$$\mathrm{conv}(X) := \bigcap_{\substack{X \subset C \subset \mathbb{R}^n \\ C \text{ convex}}} C.$$

It is the smallest convex set containing X. It can also be seen as the set of convex combinations of points of X, i.e.

$$\mathrm{conv}(X) = \left\{ \sum_{x \in X} \lambda_x x \ ; \ \sum_{x \in X} \lambda_x = 1, \ \lambda_x \geq 0, \ \forall x \text{ and } \lambda_x = 0 \text{ for all but a finite number of } x \in X \right\}.$$

A *polytope* $\mathscr{P}$ in $\mathbb{R}^n$ is the convex hull of a finite set of points that is, there exist $x_1, \ldots, x_n \in \mathbb{R}^n$ such that $\mathscr{P} = \mathrm{conv}(\{x_1, \ldots, x_n\})$. The *dimension* $\dim(\mathscr{P})$ of $\mathscr{P}$ is the dimension of the affine subspace generated by the x_i's. Notice that a polytope can also be defined as a bounded set given by intersection of a finite number of closed half-spaces (see [Zie95]).

A face of $\mathscr{P}$ is the intersection of $\mathscr{P}$ with an affine hyperplane $\mathcal{H}$ such that $\mathscr{P}$ is entirely contained in one of the closed half-spaces defined by $\mathcal{H}$. More precisely, given a linear form f on $\mathbb{R}^n$, we say that the inequality $f(x) \leq c$ ($c \in \mathbb{R}$) is *valid* in $\mathscr{P}$ if it is satisfied for every x in $\mathscr{P}$. Then, a *face* of $\mathscr{P}$ is any set of the form

$$F := \mathscr{P} \cap f^{-1}(c),$$

with $f(x) \leq c$ a valid inequality in $\mathscr{P}$. In this case, we call f a *defining functional* for F, we say that $f(x) \leq c$ is a *defining inequality* for F and we call the hyperplane $\mathcal{H} := f^{-1}(c)$ a *defining hyperplane* for F. Note that, in general, a face can have infinitely many defining inequalities and hyperplanes and that there is no natural choice among them.

A *proper face* of $\mathscr{P}$ is a face F such that $F \neq \mathscr{P}$. The *dimension* of a face F is the dimension of the affine space it generates. The faces of $\mathscr{P}$ of dimension 0, 1 or $\dim \mathscr{P} - 1$ are called *vertices*, *edges* and *facets*, respectively. The *boundary* $\partial\mathscr{P}$ of $\mathscr{P}$ is the union of all the faces of $\mathscr{P}$ of dimension smaller than $\dim \mathscr{P}$. A point of $\mathscr{P}$ is said to be an *interior point* if it doesn't belong to $\partial\mathscr{P}$. The set of d-faces of $\mathscr{P}$ (i.e. of d-dimensional faces of $\mathscr{P}$) is denoted by $\mathscr{P}_d$. Usually, we denote also $\mathrm{vert}(\mathscr{P}) := \mathscr{P}_0$.

Any polytope $\mathscr{P}$ is the convex hull of its vertices. Moreover, if F is a face of $\mathscr{P}$ and if $\mathcal{H}$ is a defining hyperplane for F, then one has $F = \mathcal{H} \cap \mathscr{P} = \mathrm{conv}(\mathcal{H} \cap \mathrm{vert}(\mathscr{P}))$, so a face is the convex hull of its vertices and is itself a polytope. When we want to stress the vertices of F, we write $F = [v_1, \ldots, v_n]$ if $\{v_1, \ldots, v_n\} = \mathrm{vert}(F) = F \cap \mathrm{vert}(\mathscr{P})$. Note also that if 0 is an interior point of a polytope $\mathscr{P}$, then any proper face of $\mathscr{P}$ has a defining inequality of the form $f(x) \leq 1$.

Despite the non-uniqueness of the defining inequalities and hyperplanes, it should be noted that any facet of $\mathscr{P}$ has a unique defining hyperplane if $\mathscr{P} \subset \mathbb{R}^n$ and $\dim \mathscr{P} = n$. Furthermore, in this case and if 0 is an interior point of $\mathscr{P}$, then any facet has a unique defining inequality of the form $f(x) \leq 1$.

We shall now state two easy lemmas that we will use in the following sections.

Lemma 1.1.1. ([CS17], Lemma 2.1)
Let $\mathcal{V}$ be a finite subset of the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Then

$$\mathrm{vert}(\mathrm{conv}(\mathcal{V})) = \mathcal{V}.$$

Lemma 1.1.2. ([CS17, Lemma 2.2])
Let $\mathcal{V}$ be a finite set spanning $\mathbb{R}^n$. Then, a convex combination $\sum_{v\in\mathcal{V}} \lambda_v v$ is an interior point of $\operatorname{conv}(\mathcal{V})$ *if and only if $\lambda_v > 0$ for all $v \in \mathcal{V}$.*

1.2. **Finite group acting freely on $\mathbb{S}^n$, orbit polytope and fundamental domains.**

Let G be a finite group acting freely and isometrically on a sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and let $\mathscr{P} := \operatorname{conv}(G \cdot v_0)$ be the *orbit polytope* associated to some point $v_0 \in \mathbb{S}^{n-1}$. In order to construct an equivariant cellular decomposition of the sphere with respect to the group action, it will prove to be useful to first determine a fundamental domain for the action of G on $\partial\mathscr{P}$.

Recall that, if a group G acts on a topological space X, then a *fundamental domain* for the action of G on X is a subset $\mathcal{D}$ of X satisfying the following two properties:

(1) for $g \neq h \in G$, the set $g\mathcal{D} \cap h\mathcal{D}$ has empty interior,
(2) the translates of $\mathcal{D}$ cover X, i.e. $X = \bigcup_{g\in G} g\mathcal{D}$.

By Lemma 1.1.1, we have $\operatorname{vert}(\mathscr{P}) = G \cdot v_0$ and it is clear that $\mathscr{P}$ is G-invariant, the G-action on $\mathscr{P}$ being given by linear extension to convex combinations of points in $G \cdot v_0$.

We now state a preliminary result to the main theorem 1.2.2:

Lemma 1.2.1. ([CS17, Lemma 6.1, Proposition 6.2 and Corollary 6.3])
If F and F' are distinct proper faces of $\mathscr{P}$ of the same dimension, then $F \cap gF'$ has empty interior for every $1 \neq g \in G$.
Assume that the orbit $G \cdot v_0$ spans $\mathbb{R}^n$, then the group G acts freely on the set $\mathscr{P}_d$ of d-dimensional faces of $\mathscr{P}$, for every $0 \leq d < \dim(\mathscr{P})$. Moreover, the origin 0 is an interior point of $\mathscr{P}$ and we have a G-equivariant homeomorphism

$$\begin{array}{ccc} \partial\mathscr{P} & \xrightarrow{\sim} & \mathbb{S}^{n-1} \\ x & \mapsto & x/|x| \end{array}$$

We can now state the main result of this section, which we shall intensively use in the sequel.

Theorem 1.2.2. ([CS17, Theorem 6.4])
Let G be a finite group acting freely by isometries on a sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, let $v_0 \in \mathbb{S}^{n-1}$ be a point and $\mathscr{P}_G := \operatorname{conv}(G \cdot v_0)$ the associated orbit polytope. Assume that the orbit $G \cdot v_0$ spans $\mathbb{R}^n$.

For any system of representatives $F_1, \ldots, F_r$ for the (free) action of G on the set of facets of $\mathscr{P}_G$, if the union $\bigcup_i F_i$ is connected, then it is a fundamental domain for the action of G on $\partial\mathscr{P}_G$.

Furthermore, there exists such a system.

We finish this section by giving a simple but useful fact.

Proposition 1.2.3. *Let G be a finite group acting freely by isometries on $\mathbb{S}^{n-1}$, let $v_0 \in \mathbb{S}^{n-1}$ be a point and $\mathscr{P} := \operatorname{conv}(G \cdot v_0)$ the associated orbit polytope.*

Given distinct facets $F_1, \ldots, F_r$ of $\mathscr{P}$, form their union $\mathcal{D} := \bigcup_{i=1}^r F_i$, consider the subset V of G defined by $\operatorname{vert}(\mathcal{D}) = V \cdot v_0$ and assume that $v_0 \in \bigcap_{i=1}^r \operatorname{vert}(F_i)$. If $V \cap V^{-1} = \{1\}$, then the F_i's belong to distinct G-orbits.

If moreover $\mathbb{R}^n = \operatorname{span}(G \cdot v_0)$ and $r|G| = |\mathscr{P}_{n-1}|$, then $\mathcal{D}$ is a fundamental domain for the action of G on $\partial\mathscr{P}$.

Proof. Suppose that there are $1 \leq i \neq j \leq r$ and $g \in G$ such that $F_j = gF_i$. Since $v_0 \in \operatorname{vert}(F_i)$, we get $gv_0 \in g\operatorname{vert}(F_i) = \operatorname{vert}(gF_i) = \operatorname{vert}(F_j)$, so $g \in V$. On the other hand, $v_0 \in \operatorname{vert}(F_j) = g\operatorname{vert}(F_i)$, hence $g^{-1}v_0 \in \operatorname{vert}(F_i)$, that is $g^{-1} \in V$. Therefore $g \in V \cap V^{-1}$, so $g = 1$ and thus $F_i = F_j$, a contradiction.

Now, the equation $r|G| = |\mathscr{P}_{n-1}|$ ensures that $F_1, \ldots, F_r$ is a system of representatives of facets and the condition $v_0 \in \bigcap_i \operatorname{vert}(F_i)$ shows that $\mathcal{D}$ is connected, hence the second statement follows from the theorem 1.2.2. $\square$

1.3. **The curved join.**

Here, we shall define the notion of *curved join*, which allows one to describe the fundamental domain for $\partial\mathscr{P}_G$ as a subset of the sphere. It will also be used to treat the higher dimensional cases of $\mathbb{S}^{4n-1}$, once we know the case of $\mathbb{S}^3$. The treatment here is taken from [FGMNS13, §2.4].

Let $w \in \mathbb{S}^1$. We can write $w = (\cos\theta, \sin\theta)$ for some $\theta \in [0, 2\pi[$. Given two points $w_1 = (\cos\theta_1, \sin\theta_1)$ and $w_2 = (\cos\theta_2, \sin\theta_2)$, with $(w_1, w_2) \in \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$, the vectors $\widetilde{w_1} = (\cos\theta_1, \sin\theta_1, 0, 0)$ and $\widetilde{w_2} = (0, 0, \cos\theta_2, \sin\theta_2)$ are orthogonal. We denote the shortest (unitary) geodesic arc from w_1 to w_2 in $\mathbb{S}^3$ by

$$w_1 * w_2 = [w_1, w_2].$$

This arc can be explicitly parametrized by

$$w_1 * w_2 = \left\{(\cos t\cos\theta_1, \cos t\sin\theta_1, \sin t\cos\theta_2, \sin t\sin\theta_2),\ 0 \leq t \leq \frac{\pi}{2}\right\}.$$

Now, for any two subsets W_1 and W_2 with $W_1 \times W_2 \subset \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C} \times \mathbb{C}$, we can form their *curved join* by

$$W_1 * W_2 := \bigcup_{w_i \in W_i} w_1 * w_2.$$

It is the projection, under the map $x \mapsto \frac{x}{|x|}$, of $\operatorname{conv}(W_1 \cup W_2)$. In particular, this shows that the curved join is associative. Furthermore, one can also define $W_1 * W_2$ for all subsets W_i of $\mathbb{S}^3$, as soon as there is no $x \in W_1$ such that $-x \in W_2$.

For example, one has

$$\mathbb{S}^1 * \mathbb{S}^1 = \mathbb{S}^3.$$

This generalizes as follows: identifying $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ and given the standard orthonormal basis $\{e_1, \ldots, e_{2m}\}$ of $\mathbb{R}^{2m}$, for each $2 \leq r \leq 2m$, denote by Π_r the plane generated by $\{e_{r-1}, e_r\}$. Suppose $\Pi_{r_1} \cap \Pi_{r_2} = 0$ and let W_1 and W_2 be subsets of the unit circles of Π_{r_1} and Π_{r_2}, respectively. Then, one can define the curved join $W_1 * W_2$ as above. In particular, we denote by Σ_k the unit circle lying in the k^{th} copy of $\mathbb{C}$ in $\mathbb{C}^m$ and we have the following equality

$$\mathbb{S}^{2m-1} = \Sigma_1 * \Sigma_2 * \cdots * \Sigma_m.$$

We can represent the two half spheres $\mathbb{S}^1 * \mathbb{S}^1_\pm$ (where $\mathbb{S}^1_\pm$ denotes the north/south hemisphere of $\mathbb{S}^1$) as in Figure 1, where the framing is given by the geodesic lines joining the endpoints of the basis vectors e_j.

One can also consider the *open* curved join, simply defined as

$$w_1 \mathbin{\mathring{*}} w_2 =]w_1, w_2[:= [w_1, w_2] \setminus \{w_1, w_2\}$$

and

$$W_1 \mathbin{\mathring{*}} W_2 := \bigcup_{w_i \in W_i} w_1 \mathbin{\mathring{*}} w_2.$$

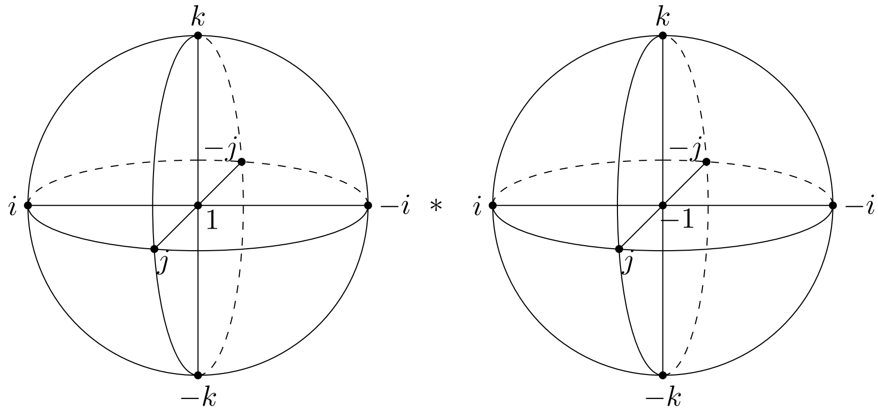

FIGURE 1. Representing $\mathbb{S}^3$ as the curved join $\mathbb{S}^1 * \mathbb{S}^1$

It is clear that the curved join is homeomorphic to the usual join

$$J(X,Y) = \frac{X \times [0,1] \times Y}{(\{x\} \times \{0\} \times Y) \cup (X \times \{1\} \times \{y\})}.$$

However, the curved join $X * Y$ is not isometric to the usual join $J(X,Y)$. The metric on the curved join is the one of $\mathbb{S}^{n-1}$ and the segments are segments of geodesics. This is a crucial fact when we describe the natural action appearing in the definition of the spherical space forms. More precisely, let G be a finite group acting freely and isometrically on $\mathbb{S}^{n-1}$ and let $h \in \mathbb{N}^*$. Then, we can make G act diagonally on $\mathbb{S}^{hn-1}$. Under the identification $\mathbb{S}^{hn-1} = \mathbb{S}^{(h-1)n-1} * \mathbb{S}^{n-1}$, this coincides with the action

$$g \cdot (x *_t y) = gx *_t gy$$

where $t \in [0,1]$ parametrizes the shortest unitary geodesic from x et y.

To compute the boundaries, we shall need the following technical result:

Lemma 1.3.1. ([FGMNS13, Lemma 2.5]) *We have the following Leibniz formula for the oriented boundary of a curved join*

$$\partial(X * Y) = \partial X * Y - (-1)^{\dim X} X * \partial Y.$$

In fact, we will use the following general lemma, allowing to recursively determine a fundamental domain and an equivariant cellular decomposition on $\mathbb{S}^{hn-1}$, once we know one on $\mathbb{S}^{n-1}$.

More precisely, let G be a finite group acting freely and isometrically on $\mathbb{S}^{n-1}$. Assume that $\mathcal{D}$ is a fundamental domain for the action on $\mathbb{S}^{n-1}$ and that $\widetilde{L}$ is a cellular decomposition of $\mathcal{D}$. We obtain an equivariant cell decomposition $\widetilde{K} = G \cdot \widetilde{L}$ of $\mathbb{S}^{n-1}$ and $L = \widetilde{K}/G$ is a cellular decomposition of $\mathbb{S}^{n-1}/G$. Assume further that $\widetilde{Z}$ is a subcomplex of $\widetilde{L}$ that is a minimal decomposition of $\mathcal{D}$ by lifts of the cells of L.

Let $h \in \mathbb{N}^*$ and consider the diagonal action of G on $\mathbb{S}^{hn-1}$. Then, a fundamental domain for this action on $\mathbb{S}^{hn-1}$ is given by

$$\mathcal{D}' := \mathbb{S}^{(h-1)n-1} * \mathcal{D}.$$

Furthermore, one can construct an equivariant cellular decomposition $\widetilde{K'}$ of $\mathbb{S}^{hn-1}$ and a minimal cellular decomposition $\widetilde{L}'$ of $\mathcal{D}'$ as follows:

- The $(h-1)n-1$-skeleton of $\widetilde{L}'$ is $\widetilde{L}'_{(h-1)n-1} = \widetilde{K}$,

- The $(h-1)n$-skeleton of $\widetilde{L}'$ is obtained by attaching $k_0(h-1)n$-cells to $\widetilde{K}$, where k_0 is the number of 0-cells $\widetilde{e}_l^0$ of $\widetilde{Z}$ and the corresponding attaching map is given by the parametrization of the curved join $\widetilde{K} * \widetilde{e}_l^0$,
- The $(h-1)n+1$-skeleton of $\widetilde{L}'$ is obtained by attaching $k_1(h-1)n+1$-cells to the $(h-1)n$-skeleton of $\widetilde{L}'$, where k_1 is the number of 1-cells $\widetilde{e}_l^1$ of $\widetilde{Z}$ and the attaching map is given by the parametrization of $\widetilde{L}'_{(h-1)n} * \widetilde{e}_l^1$,
- We carry on this procedure up to dimension $hn-1$.

We can summarize this in the following result.

Lemma 1.3.2. ([FGMNS13, Lemma 4.1])
If G is a finite group acting freely and isometrically on $\mathbb{S}^{n-1}$, if $\mathcal{D}$ is a fundamental domain for this action and if $\widetilde{\mathrm{L}}$ is a cellular decomposition of $\mathcal{D}$, with associated equivariant cellular decomposition $\widetilde{K} = G \cdot \widetilde{\mathrm{L}}$ of $\mathbb{S}^{n-1}$, then for every $h \in \mathbb{N}^*$, the subset

$$\mathcal{D}' := \mathbb{S}^{(h-1)n-1} * \mathcal{D}$$

is a fundamental domain for the diagonal action of G on $\mathbb{S}^{hn-1}$ and the above construction gives a cell decomposition $\widetilde{L}'$ of $\mathcal{D}'$, with associated equivariant cell decomposition $\widetilde{K'} := G\cdot\widetilde{L}'$ of $\mathbb{S}^{hn-1}$.

2. Binary polyhedral groups

2.1. **Real quaternions.**

Before introducing the binary polyhedral groups, let us recall some basic facts on the quaternion algebra. The material presented here is very standard, see for example [LT09].

The algebra of quaternions was discovered by William Hamilton in 1843. It can be defined as a two-dimensional module $\mathbb{H}$ over $\mathbb{C}$ with basis $\{1, j\}$ and multiplication given by the rules $j^2 = -1$ and $ji = -ij$ (1 is neutral of course). If we restrict the scalars to $\mathbb{R}$, then we obtain a four-dimensional algebra with basis elements $1, i, j, k = ij$ satisfying the symmetrical relations

$$i^2 = j^2 = k^2 = ijk = -1.$$

Given $q = a + bi + cj + dk \in \mathbb{H}$, we can define its *conjugate* $\overline{q} := a - bi - cj - dk$, its *norm* $\mathrm{N}(q) := q\overline{q} = \overline{q}q = a^2 + b^2 + c^2 + d^2$ and its *trace* $\mathrm{Tr}(q) := q + \overline{q} = 2a$. Note that, if $q \neq 0$, then $q\,\dot{}\,\overline{q}/\mathrm{N}(q) = \overline{q}/\mathrm{N}(q)\,\dot{}\,q = 1$, so $\mathbb{H}$ is a division algebra and we immediately check that $Z(\mathbb{H}) = \mathbb{R}$. Moreover, conjugation is an anti-involution of algebras and the trace is a symmetrizing linear form on $\mathbb{H}$.

We have the following result.

Proposition 2.1.1. ([LT09, Proposition 5.1])

1. *The algebra $\mathbb{H}$ is a $\mathbb{C}$-vector space with basis $\{1, j\}$. Furthermore, if $q = \alpha + \beta j \in \mathbb{H}$, with $\alpha, \beta \in \mathbb{C}$, then $\overline{q} = \overline{\alpha} - \beta j$, $\mathrm{N}(q) = |\alpha|^2 + |\beta|^2$ and $\mathrm{Tr}(q) = \alpha + \overline{\alpha}$.*
2. *$\mathrm{N}(qr) = \mathrm{N}(q)\mathrm{N}(r)$ for any $q, r \in \mathbb{H}$,*
3. *$q^2 - \mathrm{Tr}(q)q + \mathrm{N}(q) = 0$ for any $q \in \mathbb{H}$.*

For $q \in \mathbb{H}$, denote by $\mathrm{L}(q)$ the $\mathbb{R}$-linear endomorphism of $\mathbb{H}$ given by left multiplication by q and by $\mathrm{R}(q)$ the $\mathbb{R}$-linear endomorphism of $\mathbb{H}$ given by right multiplication by $\overline{q}$.

Lemma 2.1.2. ([LT09, Proposition 5.2])
For $q \in \mathbb{H}$, we have $\det \mathrm{L}(q) = \det \mathrm{R}(q) = \mathrm{N}(q)^2$.

The subgroup $\{q \in \mathbb{H} \ ; \ \mathrm{N}(q) = 1\} \leq \mathbb{H}^*$ of *unit quaternions* is in fact the 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$. In particular, $\mathbb{S}^3$ is endowed with the structure of a compact Lie group, and the map $q \mapsto \mathrm{R}(q)$ gives an isomorphism $\mathbb{S}^3 \to SU_2(\mathbb{C})$. Explicitly, if $q = \alpha + \beta j \in \mathbb{H}$, with $\alpha, \beta \in \mathbb{C}$, then the matrix of $\mathrm{R}(q)$ in the basis $\{1, j\}$ is given by $\begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ -\beta & \alpha \end{pmatrix} \in SU_2(\mathbb{C})$.

Lemma 2.1.3. ([LT09, Proposition 5.3])
For non-zero elements of $\mathbb{H}$, the following conditions are equivalent

(i) q and r have the same norm and trace,
(ii) there is some $h \in \mathbb{S}^3$ such that $q = hrh^{-1}$.

The norm function is the square of the Euclidean norm on $\mathbb{H}$, regarded as $\mathbb{R}^4$, and the associated Euclidean inner product is given by

$$\langle q, r \rangle := \frac{1}{2}\mathrm{Tr}(q\overline{r}) = \frac{1}{2}\left(\mathrm{N}(q+r) - \mathrm{N}(q) - \mathrm{N}(r)\right).$$

Denote by V the space of *pure quaternions*, spanned by i, j and k. By the Proposition 2.1.1, (2) and lemma 2.1.2, we have that, for $q \in \mathbb{S}^3$, the linear transformations $\mathrm{R}(q)$ and $\mathrm{L}(q)$ belong to $SO_4(\mathbb{R})$. For $q \in \mathbb{S}^3$, define the transformation $\mathrm{B}(q) := (\mathrm{L}(q)\mathrm{R}(q))_{|V}$, i.e.

$$\begin{array}{rcccc} \mathrm{B}(q) & : & V & \to & V \\ & & v & \mapsto & qvq^{-1} \end{array}$$

This is well-defined since, using Proposition 2.1.1 (3), a quaternion q is in V if and only if $q^2 \in \mathbb{R}_-$, so that $\mathrm{B}(q)(V) \subset V$. Then, $\mathrm{B}(q) \in SO_3(\mathbb{R})$ and, more precisely, one has the following short exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathbb{S}^3 \xrightarrow{\mathrm{B}} SO_3(\mathbb{R}) \longrightarrow 1 \ .$$

This will be used to determine the finite subgroups of $\mathbb{S}^3$.

2.2. Classification of the finite subgroups of $\mathbb{H}^*$.

In this section we construct all the possible finite subgroups of $\mathbb{H}^*$, up to conjugation. Here again, the main reference is [LT09]. The first trivial thing to note is that, if $q \in \mathbb{H}^*$ has finite order (in particular, if it belongs to a finite subgroup), then $\mathrm{N}(q) \in \mathbb{R}_+^*$ has finite order, so it must be equal to 1 and so $q \in \mathbb{S}^3$. Furthermore, the images of finite subgroups of $\mathbb{S}^3$ under B are finite subgroups of $SO_3(\mathbb{R})$. Those are the cyclic and dihedral groups, as well as the rotation groups of the Platonic solids, namely $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$.

The binary dihedral groups.

For $m \in \mathbb{N}^*$, define $\zeta_m := e^{\frac{2i\pi}{m}}$ and let $\mathcal{C}_m := \langle \zeta_m \rangle$ be the cyclic group generated by ζ_m and let $\mathcal{BD}_m := \langle \zeta_m, j \rangle$ be the subgroup of $\mathbb{H}^*$ generated by ζ_m and j. For $\alpha \in \mathcal{C}_m$, one has $j\alpha j^{-1} = \overline{\alpha} = \alpha^{-1}$ and so we have $\mathcal{C}_m \trianglelefteq \mathcal{BD}_m$. Furthermore, if m is odd then $\mathcal{BD}_m = \mathcal{BD}_{2m}$ and if m is even, the subgroup $\mathcal{C}_m$ of $\mathcal{BD}_m$ has index 2. Therefore, $\mathcal{BD}_{2m}$ has order $4m$ and $x^2 = -1$ for all $x \in \mathcal{BD}_{2m} \setminus \mathcal{C}_{2m}$. The quotient $\mathcal{BD}_{2m}/\{\pm 1\} \simeq \mathcal{D}_m$ is the dihedral group of order $2m$ and any group isomorphic to $\mathcal{BD}_{2m}$ is called a *binary dihedral group*. Sometimes, thay are also called *dicyclic* or *metacyclic* groups. When m is a power of 2, they can be called *generalized quaternion groups* and $\mathcal{Q}_8 := \mathcal{BD}_4 = \{\pm 1, \pm i, \pm j, \pm k\}$ is usually known as the *quaternion group*. Note that we have the following (non-split) short exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathcal{BD}_{2m} \xrightarrow{\mathrm{B}} \mathcal{D}_m \longrightarrow 1\ .$$

The binary polyhedral groups.

Since the groups $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$ have even orders, there are subgroups of $\mathbb{S}^3$ denoted by $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{I}$ fitting in the following short exact sequences

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathcal{T} \xrightarrow{\mathrm{B}} \mathfrak{A}_4 \longrightarrow 1\ ,$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathcal{O} \xrightarrow{\mathrm{B}} \mathfrak{S}_4 \longrightarrow 1\ ,$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathcal{I} \xrightarrow{\mathrm{B}} \mathfrak{A}_5 \longrightarrow 1\ .$$

These groups are named after the fact that $\mathfrak{A}_4$ is the group of positive isometries of a regular tetrahedron, $\mathfrak{S}_4$ is the one of a regular cube (and its dual: the regular octahedron) and $\mathfrak{A}_5$ is the one of a regular icosahedron (and its dual: the regular dodecahedron). We now want to construct these groups.

From Lemma 2.1.3, we know that two elements of $\mathbb{S}^3$ are conjugate if and only if they have the same trace. The knowledge of the trace of elements of small order can be useful. We record this in the following table:

order	3	4	5	6	8	10
trace	-1	0	$-\varphi,\ \varphi^{-1}$	1	$\pm\sqrt{2}$	$\varphi,\ -\varphi^{-1}$

where $\varphi := \frac{1+\sqrt{5}}{2}$.

Then, we see that the element

$$\varpi := \frac{-1+i+j+k}{2}$$

has order 3 and direct calculations show that ϖ normalizes $\mathcal{Q}_8$. Hence, the group

$$\mathcal{T} := \langle \mathcal{Q}_8, \varpi \rangle$$

has order 24 and the 16 elements of $\mathcal{T} \setminus \mathcal{Q}_8$ have the form $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$. The group $\mathcal{T}$ is the *binary tetrahedral group* and we have $\mathcal{T} = \langle i, \varpi \rangle$.

Next, the element

$$\gamma := \frac{1+i}{\sqrt{2}}$$

has order 8 and normalizes both $\mathcal{Q}_8$ and $\mathcal{T}$. Hence the group

$$\mathcal{O} := \langle \mathcal{T}, \gamma \rangle$$

is of order 48 (since $\gamma^2 = i$) and is called the *binary octahedral group* and we have $\mathcal{O} = \langle \varpi, \gamma \rangle$. The set $\mathcal{O} \setminus \mathcal{T}$ consists of the 24 elements $\frac{1}{\sqrt{2}}(\pm u \pm v)$ where $u \neq v \in \{1, i, j, k\}$.

Finally, the element

$$\sigma := \frac{\varphi^{-1} + i + \varphi j}{2}$$

is of order 5. The group

$$\mathcal{I} := \langle \mathcal{T}, \sigma \rangle$$

is the *binary icosahedral group* and we have $\mathcal{I} = \langle i, \sigma \rangle$. Here, some work has to be done if one wants to see that $\mathcal{I}$ is finite and to compute its order. In fact, we have

Proposition 2.2.1. ([LT09, §4.2])
The binary polyhedral groups $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{I}$ have order

$$|\mathcal{T}| = 24, \ |\mathcal{O}| = 48, \ |\mathcal{I}| = 120.$$

We can now state the following result:

Theorem 2.2.2. (Stringham, 1881, [LT09, Theorem 5.12])
Every finite subgroup of $\mathbb{H}^*$ is conjugate in $\mathbb{S}^3$ to one of the following groups

- *a cyclic group $\mathcal{C}_m$ of order m,*
- *a binary dihedral group $\mathcal{BD}_{2m}$ of order $4m$,*
- *the binary tetrahedral group $\mathcal{T}$ of order 24,*
- *the binary octahedral group $\mathcal{O}$ of order 48,*
- *the binary icosahedral group $\mathcal{I}$ of order 120.*

Remark 2.2.3. (1) *Using the homomorphisms* B *and* R *defined above and the previous result, one can obtain (up to conjugation) the finite subgroups of $SO_3(\mathbb{R})$ and $SU_2(\mathbb{C})$ and describe their generators, see* [LT09, Theorem 5.13 and Theorem 5.14].

(2) *One can prove that $\mathcal{T} \simeq SL_2(\mathbb{F}_3)$, $\mathcal{I} \simeq SL_2(\mathbb{F}_5)$ and that $\mathcal{O} \leq SL_2(\mathbb{F}_7)$. Moreover, $\mathcal{T}$ is an index 2 subgroup of both $GL_2(\mathbb{F}_3)$ and $\mathcal{O}$, but these two are non-isomorphic. See* [LT09, §5.1] *for more details.*

2.3. **Presentations of the binary polyhedral groups and spherical space forms.**

In [CM72, §6.4], Coxeter and Moser introduce the following presentations

$$\langle \ell, m, n\rangle := \left\langle r, s, t \mid r^\ell = s^m = t^n = rst \right\rangle,$$

and they prove that this group is finite if and only if $\ell mn \left(\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} - 1\right) > 0$, that is, if and only if

$$(\ell, m, n) \in \{(1, k, k), \ (2, 2, k), \ (2, 3, 3), \ (2, 3, 4), \ (2, 3, 5)\}.$$

These correspond to finite subgroups of $\mathbb{H}^*$. Note that, in these presentations, one has of course $r = t^{n-1}s^{-1}$. Let us give some explicit isomorphisms:

Lemma 2.3.1. *For $m \in \mathbb{N}^*$, one has the following isomorphisms*

$$\begin{array}{ccc} \langle 1, m, m\rangle & \to & \mathcal{C}_{2m} \\ s & \mapsto & \zeta_{2m} \end{array} \qquad \begin{array}{ccc} \langle 2, 2, m\rangle & \to & \mathcal{BD}_{2m} \\ s & \mapsto & j \\ t & \mapsto & \zeta_{2m} \end{array}$$

$$\begin{array}{ccc} \langle 2, 3, 3\rangle & \to & \mathcal{T} \\ s & \mapsto & \frac{1}{2}(1+i+j+k) \\ t & \mapsto & \frac{1}{2}(1+i+j-k) \end{array}$$

$$\begin{array}{ccc} \langle 2, 3, 4\rangle & \to & \mathcal{O} \\ s & \mapsto & \frac{1}{2}(1+i+j+k) \\ t & \mapsto & \frac{1}{\sqrt{2}}(1+i) \end{array} \qquad \begin{array}{ccc} \langle 2, 3, 5\rangle & \to & \mathcal{I} \\ s & \mapsto & \frac{1}{2}(1+i+j+k) \\ t & \mapsto & \frac{1}{2}(\varphi+\varphi^{-1}i+j) \end{array}$$

Proof. In every case, the images given for s and t clearly satisfy the required relations, giving a well defined morphism from $\langle \ell, m, n\rangle$, which is surjective using the descriptions of the groups given above. So it suffices to determine the order of $\langle \ell, m, n\rangle$ and this can be done by coset enumeration (see [Rot95, Chapter 11,§3]), applying the Todd-Coxeter algorithm to the cyclic subgroup $\langle t\rangle$. We omit the details. □

Using the $SU_2(\mathbb{C})$-matrix form of a quaternion, we introduce the following faithfull representations (defined by the images of the generators s and t)

$$\rho_{\mathcal{G}} : \mathcal{G} \to SU_2(\mathbb{C}),\ \forall \mathcal{G} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$$

defined by

$$\begin{cases} \rho_{\mathcal{T}}(s) := \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix}, \\ \\ \rho_{\mathcal{T}}(t) := \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ -1+i & 1+i \end{pmatrix} \end{cases} \qquad \begin{cases} \rho_{\mathcal{O}}(s) := \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix}, \\ \\ \rho_{\mathcal{O}}(t) := \frac{1}{\sqrt{2}}\begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix} \end{cases}$$

and

$$\begin{cases} \rho_{\mathcal{I}}(s) := \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix}, \\ \\ \rho_{\mathcal{I}}(t) := \frac{1}{2}\begin{pmatrix} \varphi - \varphi^{-1}i & 1 \\ -1 & \varphi + \varphi^{-1}i \end{pmatrix} \end{cases}$$

These correspond to the natural representations of $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{I}$ on $\mathbb{C}^2$ and are irreducible fixed point-free representations of $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{I}$, respectively. In fact, following [Ste14, Lemmae 2.15, 2.16 and 2.17] (or [Wol67, Lemmae 7.1.3, 7.1.5 and 7.1.7]), $\rho_{\mathcal{T}}$ (resp. $\rho_{\mathcal{O}}$, $\rho_{\mathcal{I}}$) is the only irreducible complex representation of $\mathcal{T}$ without fixed points (resp. one of the only two irreducible complex representations of $\mathcal{O}$, $\mathcal{I}$ without fixed points, the other one beeing also of degree 2, with a conjugate of $\mathrm{im}\,(\rho_{\mathcal{O}})$, $\mathrm{im}\,(\rho_{\mathcal{I}})$ as its image). This gives an explicit way to make these group act freely on $\mathbb{S}^3 \subset \mathbb{R}^4$. Consider more generally, for $\mathcal{G} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$, the $2n$-dimensional representation

$$\rho_{\mathcal{G}}^n := \bigoplus_{i=1}^{n} \rho_{\mathcal{G}} : \mathcal{G} \to SU_{2n}(\mathbb{C}).$$

This gives an action of $\mathcal{G}$ on $\mathbb{S}^{4n-1} \subset \mathbb{R}^{4n} = \mathbb{C}^{2n}$. According to [Wol67, §7.4], this is (up to equivariant isometry) the only way $\mathcal{G}$ may act isometrically and freely on $\mathbb{S}^{4n-1}$.

Definition 2.3.2. *Let $n \in \mathbb{N}^*$ and $\mathcal{G} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$ be a binary polyhedral group. The quotient space of the free action given by $\rho_{\mathcal{G}}^n$ on $\mathbb{S}^{4n-1}$, defined by*

$$\mathsf{P}_{\mathcal{G}}^{4n-1} := \mathbb{S}^{4n-1}/\mathrm{im}\,(\rho_{\mathcal{G}}^n)$$

is called a polyhedral spherical space form*.*

3. The octahedral case

In the following two sections, we let both $\mathcal{O}$ and $\mathcal{I}$ act (freely) by (quaternion) multiplication on the left on $\mathbb{S}^3$.

3.1. **Fundamental domain.**

We use Theorem 1.2.2 to find a fundamental domain for $\mathcal{O}$ on $\mathbb{S}^3$. To this end, we first introduce the *orbit polytope* in $\mathbb{R}^4$

$$\mathscr{P} := \mathscr{P}_{\mathcal{O}} := \operatorname{conv}(\mathcal{O}).$$

Then, we know that $\mathcal{O}$ acts freely on the set $\mathscr{P}_3$ of facets of $\mathscr{P}$ and by Theorem 1.2.2, it suffices to find a set of representatives in $\mathscr{P}_3$ such that their union is connected; this will be a fundamental domain for the action on $\partial\mathscr{P}$, which we can transport to the sphere $\mathbb{S}^3$ using the equivariant homeomorphism $\partial\mathscr{P} \to \mathbb{S}^3$, $x \mapsto x/|x|$.

The 4-polytope $\mathscr{P}$ has 48 vertices, 336 edges, 576 faces and 288 facets and is known as the *disphenoidal 288-cell*; it is dual to the bitruncated cube. Since $\mathcal{O}$ acts freely on $\mathscr{P}_3$, there must be exactly six orbits in $\mathscr{P}_3$. We introduce the following elements of $\mathcal{O}$, also expressed in terms of the generators s and t:

$$\begin{cases} \omega_0 := \frac{1+i+j+k}{2} = s, \\ \omega_i := \frac{1-i+j+k}{2} = t^{-1}st^{-1}, \\ \omega_j := \frac{1+i-j+k}{2} = s^{-1}t^2, \\ \omega_k := \frac{1+i+j-k}{2} = t^{-1}st. \end{cases} \quad \text{and} \quad \begin{cases} \tau_i := \frac{1+i}{\sqrt{2}} = t, \\ \tau_j := \frac{1+j}{\sqrt{2}} = t^{-1}s, \\ \tau_k := \frac{1+k}{\sqrt{2}} = st^{-1}. \end{cases}$$

Next, we may find explicit representatives for the $\mathcal{O}$-orbits of $\mathscr{P}_3$.

Proposition 3.1.1. *The following tetrahedra (in $\mathbb{R}^4$)*

$$\begin{cases} \Delta_1 := [1, \tau_i, \tau_j, \omega_0], \\ \Delta_2 := [1, \tau_j, \tau_k, \omega_0], \\ \Delta_3 := [1, \tau_k, \tau_i, \omega_0], \end{cases} \quad \textit{and} \quad \begin{cases} \Delta_4 := [1, \tau_i, \omega_k, \tau_j], \\ \Delta_5 := [1, \tau_j, \omega_i, \tau_k], \\ \Delta_6 := [1, \tau_i, \omega_j, \tau_k] \end{cases}$$

form a system of representatives of $\mathcal{O}$-orbits of facets of $\mathscr{P}_{\mathcal{O}}$. Furthermore, the subset of $\mathscr{P}_{\mathcal{O}}$ defined by

$$\mathscr{D}_{\mathcal{O}} := \bigcup_{i=1}^{6} \Delta_i$$

is a (connected) polytopal complex and is a fundamental domain for the action of $\mathcal{O}$ on $\partial\mathscr{P}_{\mathcal{O}}$.

Proof. First, we have to find the facets of $\mathscr{P}_{\mathcal{O}}$ by giving the defining inequalities. To do this, let the symmetric group $\mathfrak{S}_4$ act naturally on $\mathbb{R}^4$, by permuting the coordinates and let $\mathbb{Z}_2^4 := (\mathbb{Z}/2\mathbb{Z})^4$ act by changing signs. The group $\mathfrak{S}_4$ acts by permutation on $\mathbb{Z}_2^4$ and the resulting morphism $\mathfrak{S}_4 \to \operatorname{Aut}(\mathbb{Z}_2^4)$ defines a semidirect product $\mathbb{Z}_2^4 \rtimes \mathfrak{S}_4$, which acts on $\mathbb{R}^4$. We check that the following 288 inequalities

$$\langle v, x\rangle \leq 1,$$

with

$$v \in (\mathbb{Z}_2^4 \rtimes \mathfrak{A}_4) \cdot \left\{ \begin{pmatrix} 3-2\sqrt{2} \\ \sqrt{2}-1 \\ \sqrt{2}-1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2-\sqrt{2} \\ 2-\sqrt{2} \\ 2\sqrt{2}-2 \\ 0 \end{pmatrix} \right\}$$

are valid for $\mathscr{P}_{\mathcal{O}}$. Then, the facets are given by the equalities $\langle v, x\rangle = 1$ and we find their vertices by looking at vertices of $\mathscr{P}_{\mathcal{O}}$ that satisfy these equalities. We find

$$\mathrm{vert}(\mathscr{D}_{\mathcal{O}}) = \{1, \tau_i, \tau_j, \tau_k, \omega_i, \omega_j, \omega_k, \omega_0\}.$$

Now, since $\mathbb{R}^4 = \mathrm{span}(\mathcal{O})$ and $\mathrm{vert}(\mathscr{D}_{\mathcal{O}}) \cap \mathrm{vert}(\mathscr{D}_{\mathcal{O}})^{-1} = \{1\}$, Proposition 1.2.3 ensures that $\mathscr{D}_{\mathcal{O}}$ is indeed a fundamental domain for $\partial\mathscr{P}_{\mathcal{O}}$. □

Remark 3.1.2. *The recipe used to find these tetrahedra is quite simple. First, choose Δ_1 in some $\mathcal{O}$-orbit of $\partial\mathscr{P}_3$ and containing 1 as a vertex. Then, we arbitrarily choose another orbit and look at the dimensions of the intersections of Δ_1 with the facets of this second orbit. It turns out that there is exactly one facet (namely Δ_2) for which the intersection has dimension 2, so we take as second facet Δ_2 and continue further until we obtain representatives for the six orbits. Hence, we notice that a lot of different fundamental domains can be produced in this way. The calculations were tremendeously simplified by the use of the Maple package "Convex" (see* [Fra]*) and quaternionic multiplication, as implemented in GAP (see* [Gro19]*).*

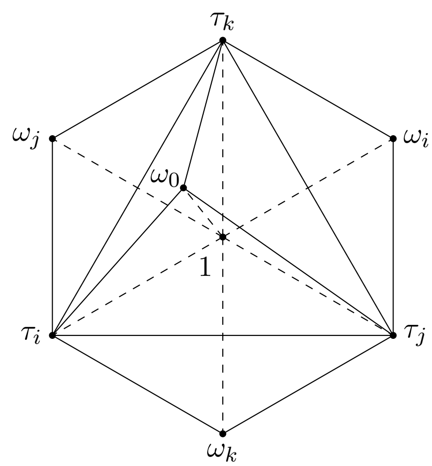

FIGURE 2. The six tetrahedra inside $\mathscr{D}_{\mathcal{O}}$

3.2. Associated $\mathcal{O}$-cellular decomposition of $\partial\mathscr{P}_{\mathcal{O}}$.

We shall now examine the combinatorics of the polytopal complex $\mathscr{D} := \mathscr{D}_{\mathcal{O}}$ constructed in the previous subsection to obtain a cellular decomposition of it. Since $\mathscr{D}$ is a fundamental domain for $\mathcal{O}$ on $\partial\mathscr{P}_{\mathcal{O}}$, translating the cells will give an equivariant decomposition of $\partial\mathscr{P}_{\mathcal{O}}$ and projecting to $\mathbb{S}^3$ will give the desired equivariant cellular structure on the sphere.

The facets of $\mathscr{D}$ are the ones of the six tetrahedra Δ_i, except the ones that are contained in some intersection $\Delta_i \cap \Delta_j$. We obtain the following facets

$$\begin{aligned}\mathscr{D}_2 = \{&[1,\tau_j,\omega_i],[1,\omega_i,\tau_k],[1,\tau_k,\omega_j],[1,\omega_j,\tau_i],[1,\tau_i,\omega_k],[1,\omega_k,\tau_j],[\tau_j,\omega_i,\tau_k],\\ &[\tau_k,\omega_j,\tau_i],[\tau_i,\omega_k,\tau_j],[\tau_i,\tau_j,\omega_0],[\tau_j,\tau_k,\omega_0],[\tau_k,\tau_i,\omega_0]\}.\end{aligned}$$

We notice the following relations among facets of $\mathscr{D}$

$$\begin{cases} \tau_i\cdot[1,\tau_j,\omega_i] = [\tau_i,\omega_0,\tau_k], \\ \tau_i\cdot[1,\omega_i,\tau_k] = [\tau_i,\tau_k,\omega_j], \end{cases} \quad \begin{cases} \tau_j\cdot[1,\tau_i,\omega_j] = [\tau_j,\omega_k,\tau_i], \\ \tau_j\cdot[1,\omega_j,\tau_k] = [\tau_j,\tau_i,\omega_0], \end{cases} \quad \begin{cases} \tau_k\cdot[1,\tau_j,\omega_k] = [\tau_k,\omega_i,\tau_j], \\ \tau_k\cdot[1,\omega_k,\tau_i] = [\tau_k,\tau_j,\omega_0]. \end{cases}$$

These are the only relations linking facets, hence, we may gather facets two by two and define the following 2-cells

$$\begin{cases} e_1^2 :=]\tau_j, 1, \omega_i[\cup]1, \omega_i[\cup]1, \omega_i, \tau_k[, \\ e_2^2 :=]\tau_i, 1, \omega_j[\cup]1, \omega_j[\cup]1, \omega_j, \tau_k[, \\ e_3^2 :=]\tau_i, 1, \omega_k[\cup]1, \omega_k[\cup]1, \omega_k, \tau_j[, \end{cases}$$

recalling that, for a polytope $[v_1, \ldots, v_n] := \mathrm{conv}(v_1, \ldots, v_n)$, we denote by $]v_1, \ldots, v_n[$ its *interior*, namely its maximal face.

Now, define the following 1-cells

$$\begin{cases} e_1^1 :=]1, \tau_i[, \\ e_2^1 :=]1, \tau_j[, \\ e_3^1 :=]1, \tau_k[. \end{cases}$$

If we add vertices of $\mathscr{D}$ and its interior, which is formed by only one cell e^3 by construction, then we may cover all of $\mathscr{D}$ with these cells and some of their translates. Thus, we have obtained the

Lemma 3.2.1. *Consider the following sets of cells in* $\mathscr{D}_{\mathcal{O}}$

$$\begin{cases} E^0_{\mathscr{D}} := \{1,\ \tau_i,\ \tau_j,\ \tau_k,\ \omega_i,\ \omega_j,\ \omega_k\}, \\ E^1_{\mathscr{D}} := \{e_1^1,\ \tau_j e_1^1,\ \tau_k e_1^1,\ \omega_i e_1^1,\ e_2^1,\ \tau_i e_2^1,\ \tau_k e_2^1,\ \omega_j e_2^1,\ e_3^1,\ \tau_i e_3^1,\ \tau_j e_3^1,\ \omega_k e_3^1\}, \\ E^2_{\mathscr{D}} := \{e_1^2,\ \tau_i e_1^2,\ e_2^2,\ \tau_j e_2^2,\ e_3^2,\ \tau_k e_3^2\}, \\ E^3_{\mathscr{D}} := \{e^3\} \end{cases}$$

Then, one has the following cellular decomposition of the fundamental domain

$$\mathscr{D}_{\mathcal{O}} = \coprod_{\substack{0 \leq j \leq 3 \\ e \in E^j_{\mathscr{D}}}} e.$$

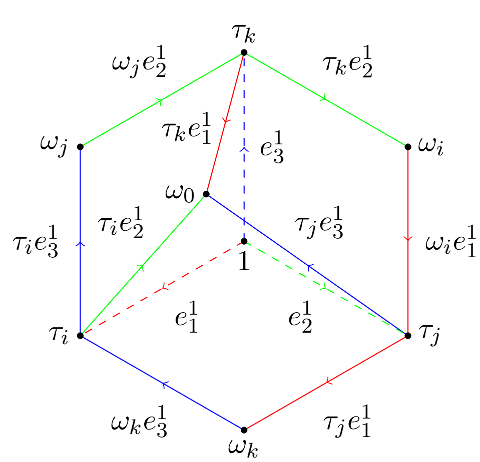

FIGURE 3. The 1-skeleton of $\mathscr{D}_{\mathcal{O}}$

Then, combining Proposition 3.1.1 and Lemma 3.2.1, yields the

Proposition 3.2.2. *Letting $E^0 := \{1\}$, $E^1 := \{e_i^1,\ i = 1,2,3\}$, $E^2 := \{e_i^2,\ i = 1,2,3\}$ and $E^3 := \{e^3\}$ with the above notations, we have the following $\mathcal{O}$-equivariant cellular decomposition of $\partial\mathscr{P}_\mathcal{O}$*

$$\partial\mathscr{P}_\mathcal{O} = \coprod_{\substack{0\leq j\leq 3\\ e\in E^j, g\in\mathcal{O}}} ge.$$

As a consequence, using the homeomorphism $\phi : \partial\mathscr{P}_\mathcal{O} \xrightarrow{\sim} \mathbb{S}^3$ given by $x \mapsto x/|x|$, we obtain the following $\mathcal{O}$-equivariant cellular decomposition of the sphere

$$\mathbb{S}^3 = \coprod_{\substack{0\leq j\leq 3\\ e\in E^j, g\in\mathcal{O}}} g\phi(e).$$

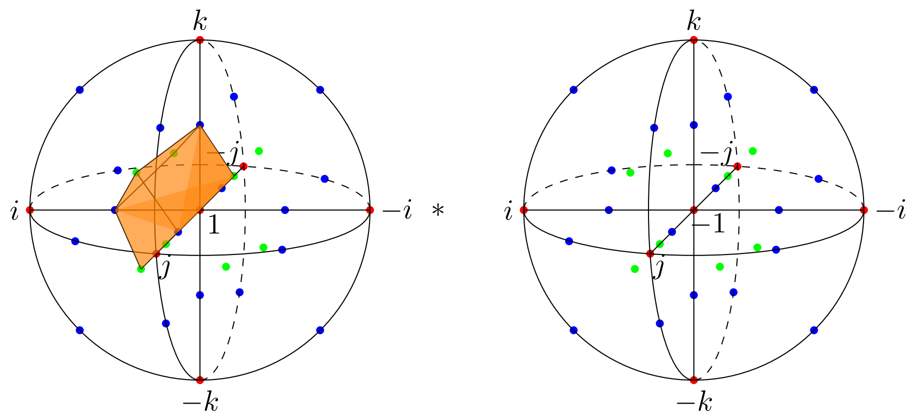

Figure 4. The fundamental domain $\mathscr{D}_\mathcal{O}$
Legend : The red points belong to $\mathcal{Q}_8$, the green ones belong to $\mathcal{T}\setminus\mathcal{Q}_8$ and the blue ones to $\mathcal{O}\setminus\mathcal{T}$.

We now have to compute the boundaries of the cells and the resulting cellular homology chain complex. We choose to orient the 3-cell e^3 directly, and the 2-cells undirectly:

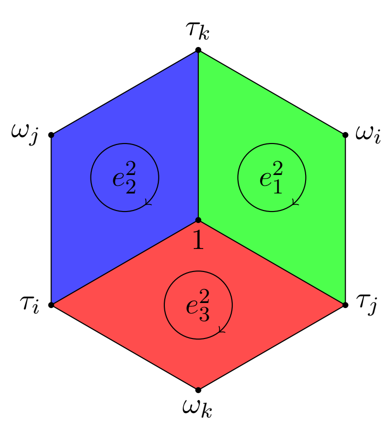

Figure 5. Orientation of the 2-cells

The induced orientations seen in $\mathscr{D}_{\mathcal{O}}$ can be visualized as follows

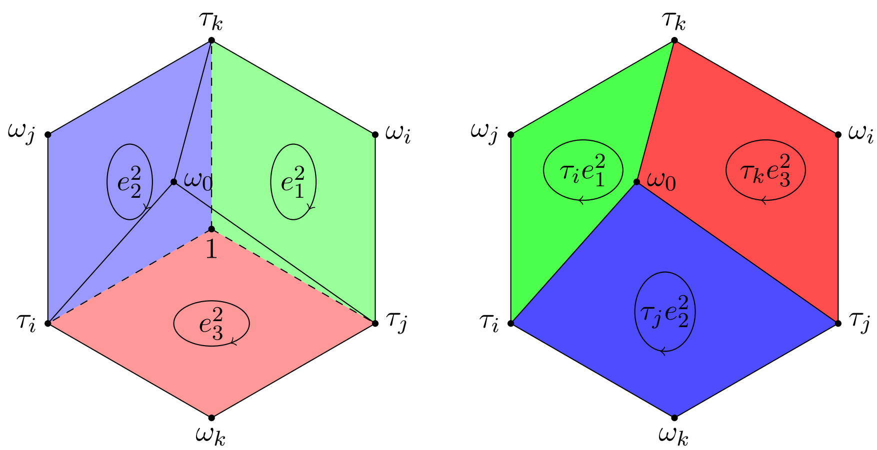

FIGURE 6. The fundamental domain with its 2-cells

These orientations allow us to easily compute the boundaries of the representing cells e_v^u and give the resulting chain complex of free left $\mathbb{Z}[\mathcal{O}]$-modules.

Proposition 3.2.3. *The cellular homology complex of $\partial\mathscr{P}_{\mathcal{O}}$ associated to the cellular structure given in the Proposition 3.2.2 is a chain complex of free left $\mathbb{Z}[\mathcal{O}]$-modules isomorphic to*

$$\mathcal{K}_{\mathcal{O}} := \Big(\mathbb{Z}[\mathcal{O}] \xrightarrow{d_3} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{d_2} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{d_1} \mathbb{Z}[\mathcal{O}] \Big),$$

where the d_i's are given, in the canonical bases, by right multiplication by the following matrices

$$d_1 = \begin{pmatrix} \tau_i - 1 \\ \tau_j - 1 \\ \tau_k - 1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} \omega_i & \tau_k - 1 & 1 \\ 1 & \omega_j & \tau_i - 1 \\ \tau_j - 1 & 1 & \omega_k \end{pmatrix}, \quad d_3 = \begin{pmatrix} 1 - \tau_i & 1 - \tau_j & 1 - \tau_k \end{pmatrix}.$$

To conclude this section, we can give an instructive picture (see Figure 7), in which all the tetrahedra in $(\mathscr{P}_{\mathcal{O}})_3$ containing 1 as a vertex are displayed. In this picture, we put the points $\omega_h^{\pm}$ (with $h = 0, i, j, k$) at the centers of the facets of the octahedron[1]. The tetrahedra in question are constructed in the following way: one chooses an edge of the octahedron and the center of a face which is adjacent to this edge. The resulting four vertices (including 1) are vertices of the corresponding tetrahedron.

This representation will be useful when we study the application to the flag manifold $A_2(\mathbb{R})$. We may also display the domain $\mathscr{D}_{\mathcal{O}}$ inside an octahedron as in Figure 8.

[1]The points in gray are on the background of the figure

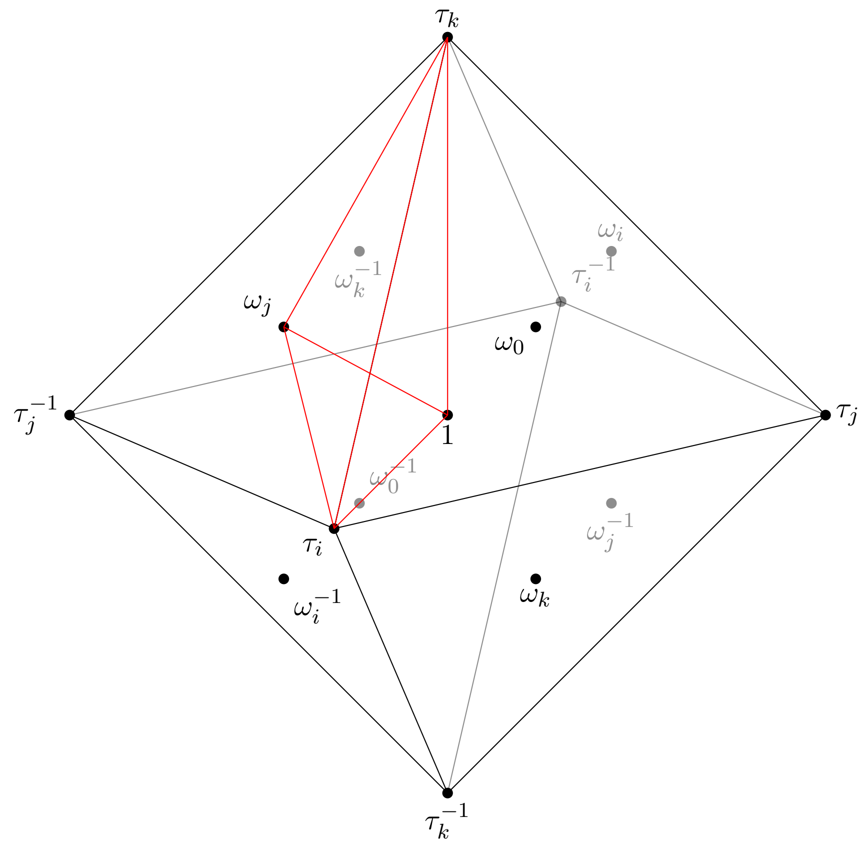

FIGURE 7. One of the twenty-four facets of $\mathscr{P}_{\mathcal{O}}$ containing 1

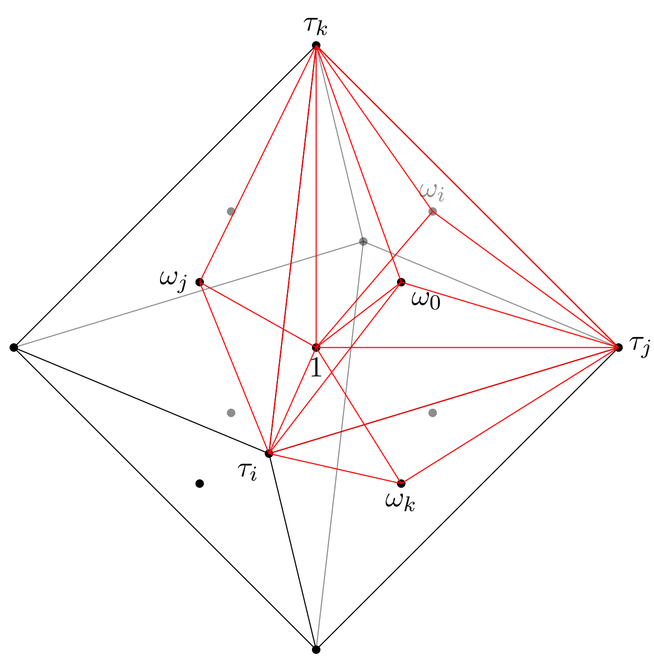

FIGURE 8. The fundamental domain for $\partial\mathscr{P}_{\mathcal{O}}$ inside an octahedron

3.3. **The case of spheres and free resolution of the trivial $\mathcal{O}$-module.**

As said in the introduction of this paper, once we know a fundamental domain for the group on $\partial\mathscr{P}_G$ (hence for $\mathbb{S}^3$), we can obtain one on higher dimensional spheres using the *curved join*. In this section, we shall describe the fundamental domain obtained above in $\mathbb{S}^3$ in terms of curved join and give a fundamental domain on $\mathbb{S}^{4n-1}$. This will come with an equivariant cellular structure on $\mathbb{S}^{4n-1}$ for which we can give the cellular homology complex. This has a particular consequence : letting $n \to \infty$, one can obtain in this way a 4-periodic free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$.

We first have to describe the fundamental domain in $\mathbb{S}^3$, using the curved join. This is done in the following result

Proposition 3.3.1. *The following subset of $\mathbb{S}^3$ is a fundamental domain for the action of $\mathcal{O}$*

$$\begin{aligned}\mathscr{F}_{\mathcal{O},3} := (\omega_i * 1 * \tau_j * \tau_k) \cup (1 * \tau_j * \tau_k * \omega_0) \cup (\omega_j * 1 * \tau_k * \tau_i)\\ \cup (1 * \tau_k * \tau_i * \omega_0) \cup (\omega_k * 1 * \tau_i * \tau_j) \cup (1 * \tau_i * \tau_j * \omega_0).\end{aligned}$$

Proof. According to Lemma 1.2.1, it suffices to show that $\mathscr{F}_{\mathcal{O},3} = p(\mathscr{D}_{\mathcal{O}})$, where $p : \partial\mathscr{P}_{\mathcal{O}} \xrightarrow{\sim} \mathbb{S}^3$ is the equivariant projection $x \mapsto x/|x|$. But this is a straightforward calculation which we omit. □

This way of writing the fundamental domains allows one to describe the cells of the resulting $\mathcal{O}$-cellular decomposition of $\mathbb{S}^3$. This gives the following result, which is just a reformulation of Proposition 3.2.2, in terms of curved join and cells in the sphere $\mathbb{S}^3$:

Theorem 3.3.2. *The sphere $\mathbb{S}^3$ admits a $\mathcal{O}$-equivariant cellular decomposition with the following cells as orbit representatives*

$$\widetilde{e}^0 := 1 * \emptyset = \{1\},$$

$$\begin{cases} \widetilde{e}^1_1 := 1 \mathbin{\mathring{*}} \tau_i =]1, \tau_i[, \\ \widetilde{e}^1_2 := 1 \mathbin{\mathring{*}} \tau_j =]1, \tau_j[, \\ \widetilde{e}^1_3 := 1 \mathbin{\mathring{*}} \tau_k =]1, \tau_k[\end{cases} \qquad \begin{cases} \widetilde{e}^2_1 := (1 \mathbin{\mathring{*}} \omega_i \mathbin{\mathring{*}} \tau_j) \cup (1 \mathbin{\mathring{*}} \omega_i) \cup (1 \mathbin{\mathring{*}} \omega_i \mathbin{\mathring{*}} \tau_k), \\ \widetilde{e}^2_2 := (1 \mathbin{\mathring{*}} \omega_j \mathbin{\mathring{*}} \tau_k) \cup (1 \mathbin{\mathring{*}} \omega_j) \cup (1 \mathbin{\mathring{*}} \omega_j \mathbin{\mathring{*}} \tau_i), \\ \widetilde{e}^2_3 := (1 \mathbin{\mathring{*}} \omega_k \mathbin{\mathring{*}} \tau_i) \cup (1 \mathbin{\mathring{*}} \omega_k) \cup (1 \mathbin{\mathring{*}} \omega_k \mathbin{\mathring{*}} \tau_j) \end{cases}$$

$$\widetilde{e}^3 := \mathring{\mathscr{F}}_{\mathcal{O},3}$$

Furthermore, the associated cellular homology complex is a chain complex of free left $\mathbb{Z}[\mathcal{O}]$-modules isomorphic to

$$\mathcal{K}_{\mathcal{O}} := \left(\mathbb{Z}[\mathcal{O}] \xrightarrow{d_3} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{d_2} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{d_1} \mathbb{Z}[\mathcal{O}] \right),$$

where the d_i's are given, in the canonical bases, by right multiplication by the following matrices

$$d_1 = \begin{pmatrix} \tau_i - 1 \\ \tau_j - 1 \\ \tau_k - 1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} \omega_i & \tau_k - 1 & 1 \\ 1 & \omega_j & \tau_i - 1 \\ \tau_j - 1 & 1 & \omega_k \end{pmatrix}, \quad d_3 = \begin{pmatrix} 1 - \tau_i & 1 - \tau_j & 1 - \tau_k \end{pmatrix}.$$

For the higher dimensional case, combining lemma 1.3.2 and Proposition 3.3.1 yields

Proposition 3.3.3. *Letting $\mathcal{O}$ act on $\mathbb{S}^{4n-1}$ ($n \geq 1$) via the representation $\rho^n_{\mathcal{O}}$, the following subset of $\mathbb{S}^{4n-1}$ is a fundamental domain for the action*

$$\mathscr{F}_{\mathcal{O},4n-1} := \Sigma_1 * \Sigma_2 * \cdots * \Sigma_{2(n-1)} * \mathscr{F}_{\mathcal{O},3},$$

with $\mathscr{F}_{\mathcal{O},3}$ inside $\Sigma_{2n-1} * \Sigma_{2n}$.

We can now describe the resulting equivariant cellular decomposition on $\mathbb{S}^{4n-1}$ using lemma 1.3.2 and Theorem 3.3.2. It only remains to consider the boundary of the cells $\widetilde{e}^{4q}$ for $q > 0$. But it follows from the fact that $\widetilde{e}^{4q} = \mathbb{S}^{4q-1} * \widetilde{e}^{4q-1}$, hence its boundary is given by all the cells in $\mathbb{S}^{4q-1}$, that is, all the orbits under $\mathcal{O}$. This gives the following result, which we prefer to state using the vocabulary of universal covering spaces. We denote by $C(\widetilde{K}, \mathbb{Z}[G])$ the chain complex of finitely generated free (left) $\mathbb{Z}[G]$-modules given by the cellular homology complex of the universal covering space $\widetilde{K}$ of a finite CW-complex K with the fundamental group G acting by covering transformations.

Theorem 3.3.4. *The chain complex $\mathcal{C}(\mathsf{P}_{\mathcal{O}}^{4n-1}, \mathbb{Z}[\mathcal{O}])$ of the universal covering space of the octahedral space forms $\mathsf{P}_{\mathcal{O}}^{4n-1}$ with the fundamental group acting by covering transformations is isomorphic to the following complex of left $\mathbb{Z}[\mathcal{O}]$-modules:*

$$0 \longrightarrow \mathbb{Z}[\mathcal{O}] \xrightarrow{d_{4n-1}} \mathbb{Z}[\mathcal{O}]^3 \longrightarrow \ldots \longrightarrow \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{d_2} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{d_1} \mathbb{Z}[\mathcal{O}] \longrightarrow 0\ ,$$

where the boundaries are given, in the canonical bases, by the following matrices ($q \geq 1$)

$$d_{4q-3} = \begin{pmatrix} \tau_i - 1 \\ \tau_j - 1 \\ \tau_k - 1 \end{pmatrix}, \qquad d_{4q-2} = \begin{pmatrix} \omega_i & \tau_k - 1 & 1 \\ 1 & \omega_j & \tau_i - 1 \\ \tau_j - 1 & 1 & \omega_k \end{pmatrix},$$

$$d_{4q-1} = \begin{pmatrix} 1 - \tau_i & 1 - \tau_j & 1 - \tau_k \end{pmatrix}, \qquad d_{4q} = \left(\textstyle\sum_{g \in \mathcal{O}} g\right).$$

In particular, the complex is exact in middle terms, i.e.

$$\forall 0 < i < 4n-1,\ H_i(\mathcal{C}(\mathsf{P}_{\mathcal{O}}^{4n-1}, \mathbb{Z}[\mathcal{O}])) = 0$$

and we have

$$H_0(\mathcal{C}(\mathsf{P}_{\mathcal{O}}^{4n-1}, \mathbb{Z}[\mathcal{O}])) = H_{4n-1}(\mathcal{C}(\mathsf{P}_{\mathcal{O}}^{4n-1}, \mathbb{Z}[\mathcal{O}])) = \mathbb{Z}.$$

Proof. The computation of the complex follows from lemma 1.3.2 and the previous discussion. The claims on its homology follow, $\mathbb{S}^{4n-1}$ being the universal covering space of $\mathsf{P}_{\mathcal{O}}^{4n-1}$. ☐

Next, adding the augmentation morphism $\varepsilon : \mathbb{Z}[\mathcal{O}] \to \mathbb{Z}$ defined by $\varepsilon\left(\sum_{g \in \mathcal{O}} a_g g\right) := \sum_{g \in \mathcal{O}} a_g$ and letting $n \to \infty$ yields the

Corollary 3.3.5. *The following chain complex*

$$\ldots \longrightarrow \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{d_{4q-3}} \mathbb{Z}[\mathcal{O}] \xrightarrow{d_{4q-4}} \ldots \longrightarrow \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{d_2} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{d_1} \mathbb{Z}[\mathcal{O}] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0\ ,$$

with boundaries d_i as in the Theorem 3.3.4, is a 4-periodic resolution of the constant module $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{O}]$.

We are now able to compute the group cohomology of $\mathcal{O}$ using this result. But first, let us recall a basic fact

Lemma 3.3.6. *(1) If G is a finite group acting freely and cellularily on a CW-complex X and $\mathcal{K}_\bullet$ is the cellular homology chain complex of X, as a complex of free (left) $\mathbb{Z}[G]$-modules, then the induced cellular homology complex of X/G is given by $\mathcal{K} \otimes_{\mathbb{Z}[G]} \mathbb{Z}$.*

(2) If $f : \mathbb{Z}[G]^m \to \mathbb{Z}[G]^n$ is a homomorphism of left $\mathbb{Z}[G]$-modules, identified with its matrix in the canonical bases, then the matrix of the induced homomorphism $f \otimes_{\mathbb{Z}[G]} id_{\mathbb{Z}} : \mathbb{Z}^m \to \mathbb{Z}^n$ is given by the matrix $\varepsilon(f)$, computed term by term.

Proof. The first statement is obvious, by definition of the cellular structure on X/G and the second one is a direct calculation. □

Corollary 3.3.7. *The group cohomology of $\mathcal{O}$ with integer coefficients is given as follows:*

$$\forall q \geq 1, \quad \begin{cases} H^q(\mathcal{O},\mathbb{Z}) = \mathbb{Z} & \text{if } \; q = 0, \\ H^q(\mathcal{O},\mathbb{Z}) = \mathbb{Z}/48\mathbb{Z} & \text{if } \; q \equiv 0 \pmod 4, \\ H^q(\mathcal{O},\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} & \text{if } \; q \equiv 2 \pmod 4, \\ H^q(\mathcal{O},\mathbb{Z}) = 0 & \text{otherwise} \end{cases}$$

Proof. In view of Lemma 3.3.6, is suffices to compute $\mathcal{C}(\mathsf{P}^{\infty}_{\mathcal{O}},\mathbb{Z}[\mathcal{O}])\otimes_{\mathbb{Z}[\mathcal{O}]}\mathbb{Z}$, with $\mathcal{C}(\mathsf{P}^{\infty}_{\mathcal{O}},\mathbb{Z}[\mathcal{O}])$ the complex given in Corollary 3.3.5. Computing the matrices $\varepsilon(d_i)$ leads to the following complex

$$\ldots \longrightarrow \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 48} \mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z} \xrightarrow{\times 48} \mathbb{Z} \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0\,.$$

Hence, determining the elementary divisors of the only non-trivial matrix occuring in it gives the following homology

$$\forall q \in \mathbb{N}, \quad \begin{cases} H_0(\mathcal{O},\mathbb{Z}) = \mathbb{Z}, \\ H_{4q-4}(\mathcal{O},\mathbb{Z}) = 0 = H_{4q-2}(\mathcal{O},\mathbb{Z}), \text{ with } q > 1 \text{ in the first equality}, \\ H_{4q-3}(\mathcal{O},\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}, \\ H_{4q-1}(\mathcal{O},\mathbb{Z}) = \mathbb{Z}/48\mathbb{Z} \end{cases}$$

and the result then follows from the universal coefficients theorem. □

Remark 3.3.8. *In* [TZ08, Proposition 4.7]*, Tomoda and Zvengrowski give an explicit resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{O}]$. They use the following presentation*

$$\mathcal{O} = \langle T, U \mid TU^2T = U^2,\; TUT = UTU \rangle$$

from [CM72]*. As we would like to work with presentations, we use the following isomorphism*

$$\begin{array}{ccc} \left\langle T, U \;\middle|\; \begin{array}{c} TU^2T = U^2 \\ TUT = UTU \end{array} \right\rangle & \xrightarrow{\sim} & \mathcal{O} \\ T & \longmapsto & (1+i)/\sqrt{2} \\ U & \longmapsto & (1+j)/\sqrt{2} \end{array}$$

Then, the Tomoda-Zvengrowski complex reads

$$\mathcal{K}^{TZ}_{\mathcal{O}} = \left(\mathbb{Z}[\mathcal{O}] \xrightarrow{\delta_3} \mathbb{Z}[\mathcal{O}]^2 \xrightarrow{\delta_2} \mathbb{Z}[\mathcal{O}]^2 \xrightarrow{\delta_1} \mathbb{Z}[\mathcal{O}] \right),$$

with

$$\delta_1 = \begin{pmatrix} T-1 \\ U-1 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 1+TU-U & T-1-UT \\ 1+TU^2 & T-U-1+TU \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} 1-TU & U-1 \end{pmatrix}.$$

On the other hand, the differentials d_i of the complex $\mathcal{K}_{\mathcal{O}}$ from Proposition 3.2.3 are given, through the above presentation, by

$$d_1 = \begin{pmatrix} T-1 \\ U-1 \\ TUT^{-1}-1 \end{pmatrix}, \ d_2 = \begin{pmatrix} UT^{-1} & TUT^{-1}-1 & 1 \\ 1 & U^{-1}T & T-1 \\ U-1 & 1 & UT \end{pmatrix}, \ d_3 = \begin{pmatrix} 1-T & 1-U & 1-TUT^{-1} \end{pmatrix}.$$

We claim that the complexes $\mathcal{K}_{\mathcal{O}}$ and $\mathcal{K}_{\mathcal{O}}^{TZ}$ are homotopy equivalent. This observation relies on elementary operations on matrix rows and columns. Write $Z := U^4 = T^4$ for the only non trivial element of $Z(\mathcal{O})$. For short, define

$$P := \begin{pmatrix} -Z & 0 & 0 \\ Z(1-T) & TUT & -U^2 \\ -U^{-3}T & -TUT & 0 \end{pmatrix}, \quad Q := \begin{pmatrix} 0 & -TUT & 0 \\ -TUT & 0 & 0 \\ U^2-TUT & U^2T & 1 \end{pmatrix},$$

then $P, Q \in GL_3(\mathbb{Z}[\mathcal{O}])$ and

$$P^{-1} = \begin{pmatrix} -Z & 0 & 0 \\ U^{-1} & 0 & -(TUT)^{-1} \\ U^{-2}(T-1)+U^{-1}T & -U^{-2} & -U^{-2} \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 0 & -(TUT)^{-1} & 0 \\ -(TUT)^{-1} & 0 & 0 \\ UT^{-1} & TUT^{-1}-1 & 1 \end{pmatrix}.$$

Now, we have the following relations

$$-Q^{-1}d_1TUT = \begin{pmatrix} T-1 \\ U-1 \\ 0 \end{pmatrix}, \quad P^{-1}d_2Q = \begin{pmatrix} 0 & 0 & -Z \\ 1+TU-U & T-1-UT & 0 \\ 1+TU^2 & T-U-1+TU & 0 \end{pmatrix},$$

$$U^{-2}d_3P = \begin{pmatrix} 0 & 1-TU & U-1 \end{pmatrix}.$$

Hence, we have an isomorphism

$$\mathcal{K}_{\mathcal{O}} \simeq \mathcal{K}_{\mathcal{O}}^{TZ} \oplus \left(0 \longrightarrow \mathbb{Z}[\mathcal{O}] \xrightarrow{1} \mathbb{Z}[\mathcal{O}] \longrightarrow 0 \right)$$

and then, $\mathcal{K}_{\mathcal{O}}$ is indeed homotopy equivalent to $\mathcal{K}_{\mathcal{O}}^{TZ}$.

3.4. **Application to the real flag manifold of type A_2.**

The $\mathcal{O}$-equivariant cellular structure of $\mathbb{S}^3$ may be used to obtain a cellular decomposition of the real points of the flag manifold $SU_3(\mathbb{C})/T$ of type A_2. The elementary facts concerning Lie groups we use here can be found in [Bum13] or [FH91].

Given a *maximal torus* T in a simply connected compact semisimple Lie group G, one can consider the *Weyl group* $W := N_G(T)/T$. It is a finite Coxeter group ([Bum13, Proposition 15.8 and Theorem 25.1]), which acts by right multiplication on the *flag manifold* G/T. For instance, in type A_{n-1}, we have $G = SU_n(\mathbb{C})$ and we can take T to be the group of diagonal matrices in $SU_n(\mathbb{C})$ (in fact, any other maximal torus is conjugate to this one, see [Bum13, Theorem 16.5]). In this case, one has $W \simeq \mathfrak{S}_n$. This group has Coxeter presentation

$$\begin{aligned} W = \mathfrak{S}_n &= \left\langle s_1, \ldots, s_{n-1} \mid s_i^2 = 1,\ (s_is_{i+1})^3 = 1,\ (s_is_j)^2 = 1,\ \forall |i-j| > 1 \right\rangle \\ &= \left\langle s_1, \ldots, s_{n-1} \mid s_i^2 = 1,\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1},\ s_is_j = s_js_i,\ \forall |i-j| > 1 \right\rangle \end{aligned}$$

and a representative $\dot{s_i}$ for the *reflection* s_i in $N_{SU_n(\mathbb{C})}(T)$ can be taken as a block matrix (with $(i-1)$ ones before the matrix s):

$$\dot{s_i} := \mathrm{diag}(1, \ldots, 1, s, 1 \ldots, 1), \quad \text{with} \quad s := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now, one can observe that, if $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ is a *reduced word* in W, then the element $\dot{w} := \dot{s_{i_1}}\dot{s_{i_2}}\cdots\dot{s_{i_k}} \in N_G(T)$ does not depend on the chosen word for w and for $g \in G$, the action of w on g is given by multiplication $g \cdot w := g\dot{w}$.

An another hand, the *Iwasawa decomposition* (see [Bum13, Theorem 26.4]) gives a diffeomorphism $G/T \simeq G^{\mathbb{C}}/B$, with $G^{\mathbb{C}}$ a *complexification* of G (that is, a complex algebraic group with Lie algebra $\mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$) and B a *Borel subgroup* of $G^{\mathbb{C}}$ containing T. This provides G/T with a structure of complex algebraic variety. Hence, one may talk about *real points* of G/T. We use the standard notation $X(\mathbb{R})$ to denote the set of real points of an algebraic variety X. In this context, one has the *Bruhat decomposition* of $G^{\mathbb{C}}$ (see [Bum13, Chapter 27]), which provides a cellular structure on G/T, with cells indexed by W. However, this cellular structure is not W-equivariant. So, a natural question is to find a W-equivariant cellular decomposition of G/T.

Remark 3.4.1. *In type A_{n-1}, that is if $G = SU_n(\mathbb{C})$ and if T is the group diagonal matrices in $SU_n(\mathbb{C})$, then one may take $G^{\mathbb{C}} = SL_n(\mathbb{C})$ and B the Borel subgroup of upper-triangular matrices in $SL_n(\mathbb{C})$. We denote by $\mathcal{F}_n$ the set of* flags *in $\mathbb{C}^n$, that is*

$$\mathcal{F}_n := \{V_\bullet := (V_1, \dots, V_{n-1})\ ;\ V_i \leq \mathbb{C}^n,\ V_i \subset V_{i+1},\ \dim V_i = i\}.$$

Moreover, the group $SL_n(\mathbb{C})$ acts on $\mathcal{F}_n$ by simply letting $g \cdot V_\bullet := (g(V_1), \dots, g(V_{n-1}))$ for $g \in SL_n(\mathbb{C})$ and $V_\bullet \in \mathcal{F}_n$. Denote by $V_0 := (\mathrm{span}(e_1), \mathrm{span}(e_1, e_2), \dots, \mathrm{span}(e_1, \dots, e_{n-1})) \in \mathcal{F}_n$ the canonical flag, where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{C}^n$, then one has a bijection

$$\begin{array}{ccc} G^{\mathbb{C}}/B & \to & \mathcal{F}_n \\ gB & \mapsto & g \cdot V_0 \end{array}$$

and this endows $\mathcal{F}_n$ with the structure of a complex algebraic variety.

Furthermore, it is easy to see that the real points $\mathcal{F}_n(\mathbb{R})$ of $\mathcal{F}_n$ is the set of real flags *in $\mathbb{R}^n$ and we have*

$$\mathcal{F}_n(\mathbb{R}) \simeq SO_n(\mathbb{R})/T(\mathbb{R})$$

and $T(\mathbb{R})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1}$.

The case $G = SU_2(\mathbb{C})$ (i.e. in type A_1) is fairly trivial, since $SU_2(\mathbb{C})/T \simeq \mathbb{S}^2$ and $W = \mathfrak{S}_2 = \{1, s\}$ acts as the antipode on $\mathbb{S}^2$, so the quotient $(SU_2(\mathbb{C})/T)/\mathfrak{S}_2$ is the projective plane $\mathbb{P}^2(\mathbb{R})$ and its simplest cellular structure lifts to a W-equivariant one on $\mathbb{S}^2$, see Figure 9.

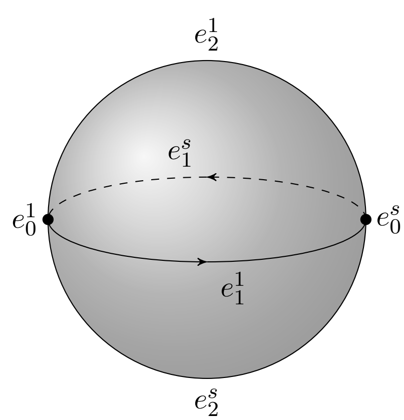

FIGURE 9. Equivariant cellular decomposition of $SU_2(\mathbb{C})/T = \mathbb{S}^2$

In this section, we treat the case of the real points of $SU_3(\mathbb{C})/T$, using the octahedral spherical space form.

First of all, we have to identify spaces and actions. We begin with a trivial lemma.

Lemma 3.4.2. *Let G be a finite group acting freely by diffeomorphisms on a manifold X and $N \trianglelefteq G$ be a normal subgroup of G. Then, G/N acts freely on the quotient manifold X/N and the projection $X \twoheadrightarrow X/G$ induces a natural diffeomorphism*

$$(X/N)\Big/(G/N) \xrightarrow{\sim} X/G.$$

We will apply this lemma to $G = \mathcal{O}$, $N = \mathcal{Q}_8$ and $X = \mathbb{S}^3$. One has to be careful at this point: we let $\mathcal{O}$ act on $\mathbb{S}^3$ on the *left*, whereas $W = \mathfrak{S}_3$ acts on $\mathcal{F}(\mathbb{R})$ on the *right*. Since there is no natural way to make W act on the left on $\mathcal{F}(\mathbb{R})$, we shall let $\mathcal{O}$ act on the right on $\mathbb{S}^3$ by multiplication. It is clearly straightforward to adapt our results to this case. For instance, we replace $\Delta_i =: \mathrm{conv}(q_1, q_2, q_3, q_4)$ by $\widehat{\Delta_i} := \mathrm{conv}(q_1^{-1}, q_2^{-1}, q_3^{-1}, q_4^{-1})$ and $\mathscr{F}_{\mathcal{O},3}$ by $\widehat{\mathscr{F}_{\mathcal{O},3}} := \mathrm{pr}(\widehat{\mathscr{D}_{\mathcal{O}}})$ where $\mathrm{pr}(x) = \frac{x}{|x|}$ is the usual projection and $\widehat{\mathscr{D}_{\mathcal{O}}} := \bigcup_i \widehat{\Delta_i}$ and we can do the same for the cells in $\mathbb{S}^3$. Briefly, we just have to replace every quaternion appearing in sections 3.1, 3.2 and 3.3 by its inverse and left multiplications by right multiplications.

Now, denote by $\mathcal{F} := SU_3(\mathbb{C})/T$ the flag manifold. Remark that the set of real points of $SU_3(\mathbb{C})$ is the manifold $SO_3(\mathbb{R})$ and that the real points of T are diagonal matrices of SU_3 with real coefficients, hence

$$T(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\},$$

and $T(\mathbb{R})$ is thus isomorphic to the Klein four-group. Therefore, one has a diffeomorphism

$$\mathcal{F}(\mathbb{R}) \simeq SO_3(\mathbb{R})/T(\mathbb{R}).$$

Recall the surjective homomorphism $\mathrm{B} : \mathbb{S}^3 \twoheadrightarrow SO_3(\mathbb{R})$ from section 2.1, with kernel $\{\pm 1\}$. We have a surjective homomorphism

$$\widetilde{\phi} : \mathbb{S}^3 \overset{\mathrm{B}}{\twoheadrightarrow} SO_3(\mathbb{R}) \twoheadrightarrow SO_3(\mathbb{R})/T(\mathbb{R}) \simeq \mathcal{F}(\mathbb{R}).$$

Now, it is clear that $\mathrm{B}^{-1}(T(\mathbb{R})) = \{\pm 1, \pm i, \pm j, \pm k\} = \mathcal{Q}_8$. The lemma 3.4.2 applied to $G = \mathcal{Q}_8$, $N := \{\pm 1\} = Z(\mathcal{Q}_8)$ and $X = \mathbb{S}^3$ yields the following Lemma.

Lemma 3.4.3. *Denoting by $\mathcal{F} := SU_3(\mathbb{C})/T$ the flag manifold of type A_2, the above defined map $\widetilde{\phi}$ induces a diffeomorphism*

$$\phi : \mathbb{S}^3/\mathcal{Q}_8 \xrightarrow{\sim} \mathcal{F}(\mathbb{R}).$$

Now, one has $W = \mathfrak{S}_3 = \left\langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = 1,\ s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta \right\rangle$ (the notation s_α, s_β makes reference to the simple roots α and β of the root system of type A_2). The reflections s_α ans s_β can be represented in $SO_3(\mathbb{R})$ by the following matrices

$$\dot{s_\alpha} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \dot{s_\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

These matrices may be obtained from $\mathbb{S}^3$ using B:

$$\dot{s_\alpha} = \mathrm{B}\left(\frac{1+k}{\sqrt{2}}\right),\ \dot{s_\beta} = \mathrm{B}\left(\frac{1+i}{\sqrt{2}}\right),$$

and this induces a well-defined isomorphism

$$\begin{array}{cccc} \sigma & : & \mathcal{O}/\mathcal{Q}_8 & \xrightarrow{\sim} & \mathfrak{S}_3 \\ & & (1+i)/\sqrt{2} & \longmapsto & s_\beta \\ & & (1+k)/\sqrt{2} & \longmapsto & s_\alpha \end{array}$$

Therefore, recalling that $\mathfrak{S}_3 = N_{SU_3(\mathbb{C})}(T)/T = (N_{SO_3(\mathbb{R})}(SO_3(\mathbb{R})\cap T))/(SO_3(\mathbb{R})\cap T)$ acts on $\mathcal{F}(\mathbb{R})$ by multiplication on the right by a representative matrix, one obtains the following relation

$$\forall (x,g) \in \mathbb{S}^3 \times \mathcal{O},\ \phi(\overline{x}) \cdot \sigma(\overline{g}) = \widetilde{\phi}(xg).$$

Henceforth, using the lemma 3.4.2, one obtains the following result:

Proposition 3.4.4. *The diffeomorphism ψ from the Lemma 3.4.3 induces a diffeomorphism*

$$\overline{\phi} : \mathbb{S}^3/\mathcal{O} \xrightarrow{\sim} \mathcal{F}(\mathbb{R})/\mathfrak{S}_3.$$

In particular, $\mathcal{O}$-equivariant cellular structure on $\mathbb{S}^3$ defined in the Theorem 3.3.2 induces an $\mathfrak{S}_3$-equivariant cellular structure on the real flag manifold $\mathcal{F}(\mathbb{R})$.

Corollary 3.4.5. *The fundamental groups of the real flag manifold $\mathcal{F}(\mathbb{R})$ and of its quotient space by $\mathfrak{S}_3$ are given by*

$$\pi_1(\mathcal{F}(\mathbb{R}),*) = \mathcal{Q}_8 \quad \textit{and} \quad \pi_1(\mathcal{F}(\mathbb{R})/\mathfrak{S}_3,*) = \mathcal{O}.$$

We are now in a position to state and prove the principal result of this section:

Theorem 3.4.6. *The real flag manifold $\mathcal{F}(\mathbb{R}) = SO_3(\mathbb{R})/T(\mathbb{R})$ admits an $\mathfrak{S}_3$-equivariant cellular decomposition with orbit representatives cells given by*

$$\mathfrak{e}^i_j := \phi\left(\pi_{\mathcal{Q}_8}\left((e^i_j)^{-1}\right)\right),$$

where $\pi_{\mathcal{Q}_8} : \mathbb{S}^3 \to \mathbb{S}^3/\mathcal{Q}_8$ is the natural projection, $\phi : \mathbb{S}^3/\mathcal{Q}_8 \to \mathcal{F}(\mathbb{R})$ is the $\mathfrak{S}_3$-equivariant diffeomorphism from the Proposition 3.4.3 and e^i_j are the cells of the $\mathcal{O}$-equivariant cellular decomposition from the Theorem 3.3.2.

Furthermore, the associated cellular homology complex is a chain complex of free right $\mathbb{Z}[\mathfrak{S}_3]$-modules isomorphic to

$$\mathcal{K}_{\mathfrak{S}_3} := \left(\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{d_3} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{d_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{d_1} \mathbb{Z}[\mathfrak{S}_3] \right),$$

where the d_i's are given, in the canonical bases, by left multiplication by the following matrices

$$d_1 = \begin{pmatrix} 1-s_\beta & 1-w_0 & 1-s_\alpha \end{pmatrix}, \quad d_2 = \begin{pmatrix} s_\alpha s_\beta & 1 & w_0 - 1 \\ s_\alpha - 1 & s_\alpha s_\beta & 1 \\ 1 & s_\beta - 1 & s_\alpha s_\beta \end{pmatrix}, \quad d_3 = \begin{pmatrix} 1-s_\beta \\ 1-w_0 \\ 1-s_\alpha \end{pmatrix}.$$

Proof. This only relies on Proposition 3.4.4 and the fact that $((e^i_j)^{-1})_{i,j}$ is an $\mathcal{O}$-equivariant cell decomposition of $\mathbb{S}^3$, the group $\mathcal{O}$ acting by right multiplication on the sphere. Next, we have to determine the images of the points of $\mathcal{O}$ we used to construct $\widehat{\mathscr{F}_{\mathcal{O},3}}$ under the projection

$$\pi^{\mathcal{O}} : \mathcal{O} \twoheadrightarrow \mathcal{O}/\mathcal{Q}_8 = \mathfrak{S}_3.$$

Recall that, denoting by s_α and s_β the simple reflections in the Weyl group $W = \mathfrak{S}_3$, we have

$$\mathfrak{S}_3 = \left\langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = 1,\ s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta \right\rangle = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha\}$$

and we denote by $w_0 := s_\alpha s_\beta s_\alpha$ the *longest element* of $\mathfrak{S}_3$. We compute

$$\begin{cases} \pi^{\mathcal{O}}(\tau_i) = s_\beta,\ \pi^{\mathcal{O}}(\tau_j) = w_0,\ \pi^{\mathcal{O}}(\tau_k) = s_\alpha, \\ \\ \pi^{\mathcal{O}}(\omega_i) = \pi^{\mathcal{O}}(\omega_j) = \pi^{\mathcal{O}}(\omega_k) = s_\beta s_\alpha,\ \pi^{\mathcal{O}}(\omega_0) = s_\alpha s_\beta. \end{cases}$$

Thus, the resulting cellular homology chain complex can be computed from the one in the Theorem 3.3.2, replacing each coefficient $q \in \mathcal{O}$ in d_i by $\pi^{\mathcal{O}}(q^{-1})$ and transposing the matrices. $\square$

Corollary 3.4.7. *The integral homologies of $\mathcal{F}(\mathbb{R})$ and of the quotient $\mathcal{F}(\mathbb{R})/\mathfrak{S}_3$ are respectively given by*

$$H_n(\mathcal{F}(\mathbb{R}),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n=0 \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } n=1 \\ 0 & \text{if } n=2 \\ \mathbb{Z} & \text{if } n=3 \\ 0 & \text{if } n\geq 4 \end{cases} \quad \text{and} \quad H_n(\mathcal{F}(\mathbb{R})/\mathfrak{S}_3,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n=0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n=1 \\ 0 & \text{if } n=2 \\ \mathbb{Z} & \text{if } n=3 \\ 0 & \text{if } n\geq 4 \end{cases}$$

We can now deduce the action of $\mathfrak{S}_3$ on the cohomology of $\mathcal{F}(\mathbb{R})$. Since $\mathfrak{S}_3$ acts on the right of $\mathcal{F}(\mathbb{R})$ and since cohomology is a contravariant functor, $\mathfrak{S}_3$ acts on the left on $H^*(\mathcal{F}(\mathbb{R}),\mathbb{Z})$.

First of all, define the integral representation

$$\mathbf{2} : \mathfrak{S}_3 \to GL_2(\mathbb{Z})$$

by

$$\mathbf{2}(s_\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{2}(s_\beta) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Then, $\mathbf{2}$ is an integral form of the 2-dimensional irreducible complex representation of $\mathfrak{S}_3$.

Writing $V := \mathbb{Z}^2$, the representation $\mathbf{2}$ may be identified with a morphism of algebras $\mathbb{Z}[\mathfrak{S}_3] \to \mathrm{End}_{\mathbb{Z}}(V)$ and it is straightforward to check that the reduction mod 2 homomorphism $\pi : V \to (\mathbb{Z}/2\mathbb{Z})^2 = V \otimes \mathbb{F}_2$ induces a well-defined homomorphism of algebras $\pi_* : \mathrm{End}_{\mathbb{Z}}(V) \to \mathrm{End}_{\mathbb{Z}}(V \otimes \mathbb{F}_2)$. Thus, we have an induced $\mathbb{Z}$-representation $\overline{\mathbf{2}} := \pi_*\mathbf{2}$ of $(\mathbb{Z}/2\mathbb{Z})^2$ that fits into a commutative diagram

$$\begin{tikzcd} \mathbb{Z}[\mathfrak{S}_3] \arrow[rr, "\overline{\mathbf{2}}"] \arrow[dr, "\mathbf{2}"'] & & \mathrm{End}_{\mathbb{Z}}(V \otimes \mathbb{F}_2) \\ & \mathrm{End}_{\mathbb{Z}}(V) \arrow[ur, "\pi_*"'] & \end{tikzcd}$$

The same statement holds for the dual representation $\mathbf{2}^*$ and we have $\overline{\mathbf{2}^*} \simeq \overline{\mathbf{2}}$. This is just the irreducible $\mathbb{F}_2$-representation $\mathbf{2} \otimes \mathbb{F}_2$, but with coefficients lifted to $\mathbb{Z}$ via $\mathbb{Z} \twoheadrightarrow \mathbb{F}_2$.

For convenience, we consider $\mathbb{Z}[\mathfrak{S}_3]$ as a graded algebra concentrated in degree zero.

Theorem 3.4.8. *The cohomology ring $H^*(\mathcal{F}(\mathbb{R}),\mathbb{Z})$ of $\mathcal{F}(\mathbb{R})$ is a graded commutative left $\mathbb{Z}[\mathfrak{S}_3]$-algebra such that*

* *For $i \neq 0,2,3$, one has $H^i(\mathcal{F}(\mathbb{R}),\mathbb{Z}) = 0$,*
* *The modules $H^0(\mathcal{F}(\mathbb{R}),\mathbb{Z})$ and $H^3(\mathcal{F}(\mathbb{R}),\mathbb{Z})$ are trivial,*
* *The module $H^2(\mathcal{F}(\mathbb{R}),\mathbb{Z})$ is the representation $\overline{\mathbf{2}}$.*

Moreover, the action of $\mathfrak{S}_3$ on $\mathcal{F}(\mathbb{R})$ preserves the orientation.

Proof. Let

$$\sigma := \sum_{w \in \mathfrak{S}_3} w$$

and recall the cellular homology complex

$$\mathcal{K}_{\mathfrak{S}_3} = \left(\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{d_3} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{d_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{d_1} \mathbb{Z}[\mathfrak{S}_3] \right),$$

with

$$d_1 = \begin{pmatrix} 1-s_\beta & 1-w_0 & 1-s_\alpha \end{pmatrix}, \quad d_2 = \begin{pmatrix} s_\alpha s_\beta & 1 & w_0 - 1 \\ s_\alpha - 1 & s_\alpha s_\beta & 1 \\ 1 & s_\beta - 1 & s_\alpha s_\beta \end{pmatrix}, \quad d_3 = \begin{pmatrix} 1-s_\beta \\ 1-w_0 \\ 1-s_\alpha \end{pmatrix}.$$

We can directly compute

$$H_0(\mathcal{F}(\mathbb{R}), \mathbb{Z}) = \operatorname{coker} d_1 = \mathbb{Z}\langle 1 \rangle \simeq \mathbb{Z}$$

and

$$H_3(\mathcal{F}(\mathbb{R}), \mathbb{Z}) = \ker d_3 = \mathbb{Z}\langle \sigma \rangle \simeq \mathbb{Z}.$$

We determine an orientation of $\mathcal{F}(\mathbb{R})$ by choosing as fundamental class

$$[\mathcal{F}(\mathbb{R})] := \sigma.$$

Thus, for $w \in \mathfrak{S}_3$ one has $[\mathcal{F}(\mathbb{R})] \cdot w = [\mathcal{F}(\mathbb{R})]$ and so, the right action of $\mathfrak{S}_3$ on $\mathcal{F}(\mathbb{R})$ preserves the orientation. Denoting by

$$\mathcal{D}^i := ([\mathcal{F}(\mathbb{R})] \cap -) : H^i(\mathcal{F}(\mathbb{R}), \mathbb{Z}) \xrightarrow{\sim} H_{3-i}(\mathcal{F}(\mathbb{R}), \mathbb{Z})$$

the associated Poincaré duality, the naturality theorem (see [Mun84, Theorem 67.2]) yields

$$w_* \mathcal{D}^i w^* = \mathcal{D}^i.$$

For a right $\mathfrak{S}_3$-set X, we naturally write X^{op} for the left $\mathfrak{S}_3$-set X endowed with the action $w \cdot x := xw^{-1}$. Then, the last equation becomes a reformulation of the property

$$\mathcal{D}^i \in \operatorname{Hom}_{\mathbb{Z}[\mathfrak{S}_3]} \left(H^i(\mathcal{F}(\mathbb{R}), \mathbb{Z}), H_{3-i}(\mathcal{F}(\mathbb{R}), \mathbb{Z})^{op} \right)$$

and the left modules $H^i(\mathcal{F}(\mathbb{R}), \mathbb{Z})$ and $H_{3-i}(\mathcal{F}(\mathbb{R}), \mathbb{Z})^{op}$ are thus isomorphic.

Now, we have to see why the right modules $H_0(\mathcal{F}(\mathbb{R}), \mathbb{Z})$ and $H_3(\mathcal{F}(\mathbb{R}), \mathbb{Z})$ are trivial. For the first case, notice that

$$d_1 \begin{pmatrix} -s_\alpha s_\beta \\ 1 \\ 0 \end{pmatrix} = 1 - s_\alpha s_\beta \quad \text{and} \quad d_1 \begin{pmatrix} s_\alpha - s_\alpha s_\beta \\ 1 - s_\alpha \\ 0 \end{pmatrix} = 1 - s_\beta s_\alpha,$$

hence, we have that $1 - w \in \operatorname{im}(d_1)$ for every $w \in \mathfrak{S}_3$ and so, $\mathfrak{S}_3$ acts trivially on $H_0(\mathcal{F}(\mathbb{R}), \mathbb{Z})$. The same holds for $H_3(\mathcal{F}(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}\langle \sigma \rangle$.

It remains to show that $H_1(\mathcal{F}(\mathbb{R}), \mathbb{Z})^{op} \simeq \overline{\mathbf{2}}$. Denote respectively by x and y the classes of $\begin{pmatrix} 1+s_\beta \\ 0 \\ 0 \end{pmatrix} \in \ker d_1$ and $\begin{pmatrix} s_\alpha + s_\beta s_\alpha \\ 0 \\ 0 \end{pmatrix} \in \ker d_1$ in $H_1(\mathcal{F}(\mathbb{R}), \mathbb{Z})$. Then we have $H_1(\mathcal{F}(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}\langle x, y \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and since

$$x + y + \begin{pmatrix} s_\alpha s_\beta + w_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix} = d_2 \begin{pmatrix} 1 + 2s_\alpha - s_\beta s_\alpha + s_\alpha s_\beta \\ 1 + s_\alpha + s_\beta \\ -1 - s_\beta - s_\beta s_\alpha \end{pmatrix}$$

we get

$$y \cdot s_\beta = \begin{pmatrix} s_\alpha s_\beta + w_0 \\ 0 \\ 0 \end{pmatrix} = -x - y.$$

Next, it is easy to compute that $x \cdot s_\alpha = y$, $x \cdot s_\beta = x$ and $y \cdot s_\alpha = x$. These equations mean that, with respect to the generating set $\{x, y\}$ of the torsion $\mathbb{Z}$-module $H_1(\mathcal{F}(\mathbb{R}), \mathbb{Z})^{op}$, the matrices of the action of s_α and s_β are given by

$$\operatorname{Mat}_{\{x,y\}}(s_\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \operatorname{Mat}_{\{x,y\}}(s_\beta) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

and these are indeed the matrices defining $\overline{\mathbf{2}}$. □

From this and the universal coefficients theorem, the Brauer character table of $\mathfrak{S}_3$ over $\mathbb{F}_2$ being well-known (see [Ser98, Part III]), we deduce the following corollary:

Corollary 3.4.9. *The mod 2 cohomology ring $H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$ is a graded commutative left $\mathbb{F}_2[\mathfrak{S}_3]$-algebra, semisimple as a module, satisfying*

- *For $i \geq 4$, one has $H^i(\mathcal{F}(\mathbb{R}), \mathbb{F}_2) = 0$,*
- *The modules $H^0(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$ and $H^3(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$ are trivial,*
- *The modules $H^1(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$ and $H^2(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$ are isomorphic to the unique two-dimensional irreducible representation $\mathbf{2} \otimes \mathbb{F}_2$.*

Finally, using Figure 7, we can describe the 3-cells in a more combinatorial way. More precisely, one can describe all the curved tetrahedra having a given element $w \in \mathfrak{S}_3$ in its boundary. By right multiplication by w^{-1}, we may assume that $w = 1$. First consider the octahedron as in Figure 7, with vertices (and centers of faces) given by the images of the ones of 7 under the projection $\pi^{\mathcal{O}} : \mathcal{O} \twoheadrightarrow \mathfrak{S}_3$ as in the figure 10. A curved tetrahedron containing 1 can be described in the following way:

(1) Choose a face F of the octahedron,
(2) Choose an edge of F,
(3) The curved tetrahedron has its vertices given by the center of F, the two vertices of the chosen edge of F and 1.

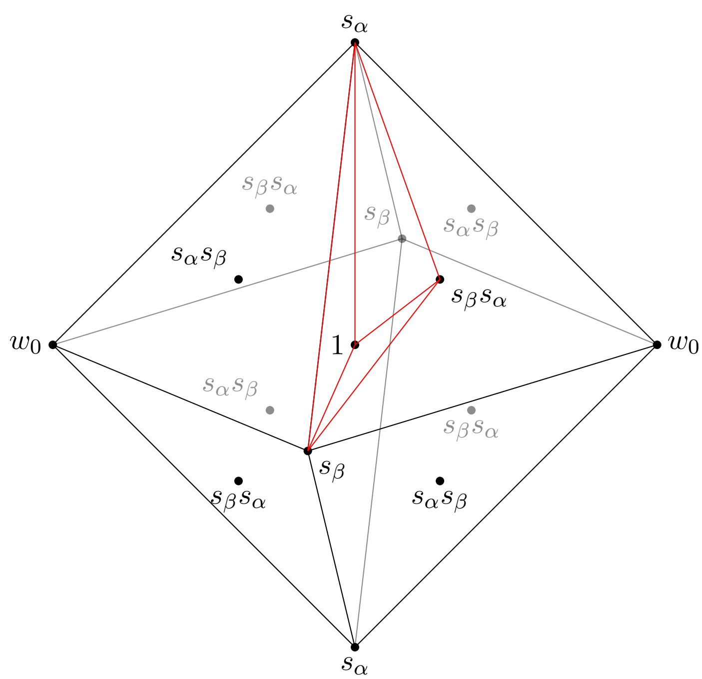

FIGURE 10. A curved tetrahedron in $\mathcal{F}(\mathbb{R})$ containing 1 in its boundary

Remark 3.4.10. *Note that in this representation, many different cells can have the same vertices. For instance, the 1-cell formed by the edge linking 1 to the w_0 on the right, and then from the other copy of w_0 on the left, back to one is not a trivial path in $\mathcal{F}(\mathbb{R})$. In fact, it corresponds to the element j of the group $Q_8 \simeq \pi_1(\mathcal{F}(\mathbb{R}), 1)$.*

4. The icosahedral case

4.1. **Fundamental domain.**

We shall use for the binary icosahedral group $\mathcal{I}$ of order 120 exactly the same method as for $\mathcal{O}$. First, we are looking for a fundamental domain for $\mathcal{I}$ in $\mathbb{S}^3$. To do this, we consider the orbit polytope in $\mathbb{R}^4$

$$\mathscr{P} := \mathscr{P}_{\mathcal{I}} := \mathrm{conv}(\mathcal{I}).$$

This polytope has 120 vertices, 720 edges, 1200 faces and 600 facets and is known as the *600-cell* (or the *hexacosichoron*, or even the *tetraplex*). Since $\mathcal{I}$ acts freely on $\mathscr{P}_3$, there must be exactly five orbits in $\mathscr{P}_3$. Here again, we consider some elements of $\mathcal{I}$, also expressed in terms of the Coxeter generators s and t and with $\varphi := (1+\sqrt{5})/2$:

$$\begin{cases} \sigma_i^+ := \frac{\varphi+\varphi^{-1}i+j}{2} = t, \\ \sigma_i^- := \frac{\varphi+\varphi^{-1}i-j}{2} = st^{-2}, \end{cases} \quad \begin{cases} \sigma_j^+ := \frac{\varphi+\varphi^{-1}j-k}{2} = ts^{-1}t, \\ \sigma_j^- := \frac{\varphi-\varphi^{-1}j-k}{2} = s^{-1}t, \end{cases} \quad \begin{cases} \sigma_k^+ := \frac{\varphi+i+\varphi^{-1}k}{2} = st^{-1}, \\ \sigma_k^- := \frac{\varphi+i-\varphi^{-1}k}{2} = s^{-1}t^2. \end{cases}$$

As for $\mathcal{O}$, we may find explicit representatives for the $\mathcal{I}$-orbits of $\mathscr{P}_3$:

Proposition 4.1.1. *The following tetrahedra (in $\mathbb{R}^4$)*

$$\begin{cases} \Delta_1 := [1, \sigma_k^-, \sigma_k^+, \sigma_i^+], \\ \Delta_2 := [1, \sigma_k^-, \sigma_i^+, \sigma_j^+], \\ \Delta_3 := [1, \sigma_k^-, \sigma_j^+, \sigma_j^-], \\ \Delta_4 := [1, \sigma_k^-, \sigma_j^-, \sigma_i^-], \\ \Delta_5 := [1, \sigma_k^-, \sigma_i^-, \sigma_k^+] \end{cases}$$

form a system of representatives of $\mathcal{I}$-orbits of facets of $\mathscr{P}_{\mathcal{I}}$. Furthermore, the subset of $\mathscr{P}_{\mathcal{I}}$ defined by

$$\mathscr{D}_{\mathcal{I}} := \bigcup_{i=1}^{5} \Delta_i$$

is a (connected) polytopal complex and is a fundamental domain for the action of $\mathcal{I}$ on $\partial\mathscr{P}_{\mathcal{I}}$.

Proof. First, we have to find the facets of $\mathscr{P}_{\mathcal{I}}$ by giving the defining inequalities. Here again, the semidirect product $\mathbb{Z}_2^4 \rtimes \mathfrak{S}_4$ acts on $\mathbb{R}^4$ in a natural way and we check that the following 600 inequalities

$$\langle v, x \rangle \leq 1,$$

with

$$v \in (\mathbb{Z}_2^4 \rtimes \mathfrak{A}_4) \cdot \left\{ \begin{pmatrix} 4-2\varphi \\ 4-2\varphi \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2-\varphi \\ 2-\frac{3}{\varphi} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\varphi-3 \\ \frac{3}{\varphi}-1 \\ \varphi-1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\varphi-3 \\ 2\varphi-3 \\ 2\varphi-3 \\ 1 \end{pmatrix}, \begin{pmatrix} \varphi-1 \\ \varphi-1 \\ \varphi-1 \\ 2-\frac{3}{\varphi} \end{pmatrix}, \begin{pmatrix} 2-\varphi \\ 2-\varphi \\ 2-\varphi \\ \frac{3}{\varphi}-1 \end{pmatrix}, \begin{pmatrix} 2\varphi-3 \\ 2-\varphi \\ \varphi-1 \\ 4-2\varphi \end{pmatrix} \right\},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$, are valid for $\mathscr{P}_{\mathcal{I}}$. Then, the facets are given by the equalities $\langle v, x \rangle = 1$ and we find their vertices by looking at vertices of $\mathscr{P}_{\mathcal{I}}$ that satisfy these equalities. We find

$$\mathrm{vert}(\mathscr{D}_{\mathcal{I}}) = \{1, \sigma_i^{\pm}, \sigma_j^{\pm}, \sigma_k^{\pm}\}$$

and since $\mathrm{vert}(\mathscr{D}_{\mathcal{I}}) \cap \mathrm{vert}(\mathscr{D}_{\mathcal{I}})^{-1} = \{1\}$, the Proposition 1.2.3 shows that $\mathscr{D}_{\mathcal{I}}$ is a fundamental domain. ☐

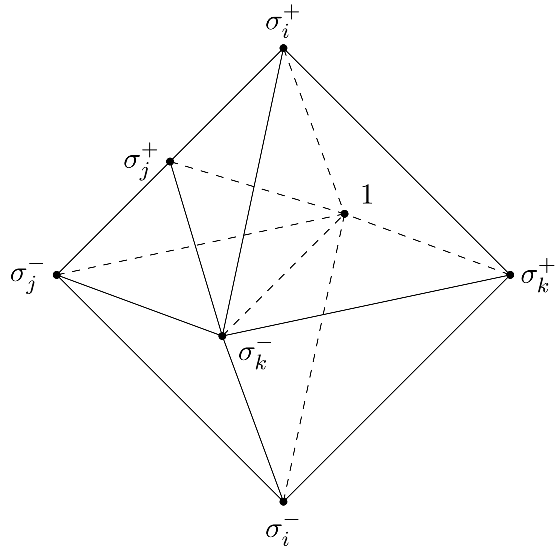

FIGURE 11. The five tetrahedra inside $\mathscr{D}_{\mathcal{I}}$

4.2. **Associated $\mathcal{I}$-cellular decomposition of $\partial\mathscr{P}_{\mathcal{I}}$.**

Here also, we investigate the combinatorics of the polytopal fundamental domain $\mathscr{D} := \mathscr{D}_{\mathcal{I}}$ constructed above to obtain a cellular decomposition of it. This will give a cellular structure on $\partial\mathscr{P}_{\mathcal{I}}$ and projecting to $\mathbb{S}^3$ gives the desired cellular structure.

The facets of $\mathscr{D}$ are the ones of the five tetrahedra Δ_i, except the ones that are contained in some intersection $\Delta_i \cap \Delta_j$. We obtain the following facets

$$\mathscr{D}_2 = \{[1,\sigma_i^-,\sigma_k^+],[1,\sigma_k^+,\sigma_i^+],[1,\sigma_i^+,\sigma_j^+],[1,\sigma_j^+,\sigma_j^-],[1,\sigma_j^-,\sigma_i^-],$$
$$[\sigma_k^-,\sigma_i^-,\sigma_k^+],[\sigma_k^-,\sigma_k^+,\sigma_i^+],[\sigma_k^-,\sigma_i^+,\sigma_j^+],[\sigma_k^-,\sigma_j^+,\sigma_j^-],[\sigma_k^-,\sigma_j^-,\sigma_i^-]\}.$$

We remark the following relations among them

$$\begin{cases} \sigma_j^+ \cdot [1,\sigma_i^-,\sigma_k^+] = [\sigma_j^+,\sigma_j^-,\sigma_k^-], \\ \sigma_j^- \cdot [1,\sigma_k^+,\sigma_i^+] = [\sigma_j^-,\sigma_i^-,\sigma_k^-], \\ \sigma_i^- \cdot [1,\sigma_i^+,\sigma_j^+] = [\sigma_i^-,\sigma_k^+,\sigma_k^-], \end{cases} \quad \text{and} \quad \begin{cases} \sigma_k^+ \cdot [1,\sigma_j^+,\sigma_j^-] = [\sigma_k^+,\sigma_i^+,\sigma_k^-], \\ \sigma_i^+ \cdot [1,\sigma_j^-,\sigma_i^-] = [\sigma_i^+,\sigma_j^+,\sigma_k^-] \end{cases}$$

These are the only relations linking facets, hence we may define the following 2-cells

$$\begin{cases} e_1^2 :=]1, \sigma_j^-, \sigma_i^-[, \\ \\ e_2^2 :=]1, \sigma_i^-, \sigma_k^+[, \end{cases} \quad \text{and} \quad \begin{cases} e_3^2 :=]1, \sigma_k^+, \sigma_i^+[, \\ \\ e_4^2 :=]1, \sigma_i^+, \sigma_j^+[, \\ \\ e_5^2 :=]1, \sigma_j^+, \sigma_j^-[\end{cases}$$

Now, define the following 1-cells

$$e_1^1 :=]1, \sigma_k^+[, \quad e_2^1 :=]1, \sigma_i^+[, \quad e_3^1 :=]1, \sigma_j^+[, \quad e_4^1 :=]1, \sigma_j^-[, \quad e_5^1 :=]1, \sigma_i^-[.$$

If we add to this the vertices of $\mathscr{D}$ and its interior, which is formed by only one cell e^3 by construction, then we may cover all of $\mathscr{D}$ with these cells and some of their translates. Thus, we have obtained the

Lemma 4.2.1. *Consider the following sets of cells in $\mathscr{D}_{\mathcal{I}}$*

$$\begin{cases} E^0_{\mathscr{D}} := \{1,\ \sigma_i^{\pm},\ \sigma_j^{\pm},\ \sigma_k^{\pm}\}, \\ \\ E^1_{\mathscr{D}} := \{e_1^1,\ \sigma_j^- e_1^1,\ \sigma_j^+ e_1^1,\ e_2^1,\ \sigma_i^- e_2^1,\ \sigma_j^- e_2^1,\ e_3^1,\ \sigma_i^- e_3^1,\ \sigma_k^+ e_3^1,\ e_4^1,\ \sigma_i^+ e_4^1,\ \sigma_k^+ e_4^1, e_5^1,\ \sigma_i^+ e_5^1,\ \sigma_j^+ e_5^1\}, \\ \\ E^2_{\mathscr{D}} := \{e_1^2,\ \sigma_i^+ e_1^2,\ e_2^2,\ \sigma_j^+ e_2^2,\ e_3^2,\ \sigma_j^- e_3^2,\ e_4^2,\ \sigma_i^- e_4^2,\ e_5^2,\ \sigma_k^+ e_5^2\}, \\ \\ E^3_{\mathscr{D}} := \{e^3\} \end{cases}$$

Then, one has the following cellular decomposition of the fundamental domain

$$\mathscr{D}_{\mathcal{I}} = \coprod_{\substack{0 \le j \le 3 \\ e \in E^j_{\mathscr{D}}}} e.$$

The 1-skeleton of $\mathscr{D}_{\mathcal{I}}$ is displayed in figure 12.

Then, combining Proposition 4.1.1 and Lemma 4.2.1, yields the

Proposition 4.2.2. *Letting $E^0 := \{1\}$, $E^1 := \{e_i^1,\ 1 \le i \le 5\}$, $E^2 := \{e_i^2,\ 1 \le i \le 5\}$ and $E^3 := \{e^3\}$ with the above notations, we have the following $\mathcal{I}$-equivariant cellular decomposition of $\partial\mathscr{P}_{\mathcal{I}}$*

$$\partial\mathscr{P}_{\mathcal{I}} = \coprod_{\substack{0 \le j \le 3 \\ e \in E^i, g \in \mathcal{I}}} ge.$$

As a consequence, using the homeomorphism $\phi : \partial\mathscr{P}_{\mathcal{I}} \xrightarrow{\sim} \mathbb{S}^3$ given by $x \mapsto x/|x|$, we obtain the following $\mathcal{I}$-equivariant cellular decomposition of the sphere

$$\mathbb{S}^3 = \coprod_{\substack{0 \le j \le 3 \\ e \in E^j, g \in \mathcal{I}}} g\phi(e).$$

We now have to compute the boundaries of the cells and the resulting cellular homology chain complex. We choose to orient the 3-cell e^3 undirectly, and the 2-cells directly:

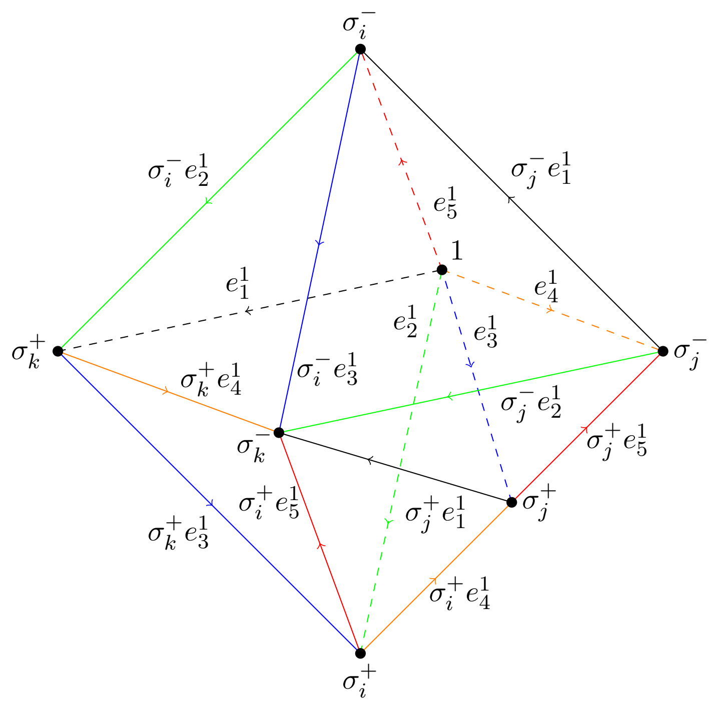

FIGURE 12. The oriented 1-skeleton of $\mathscr{D}_{\mathcal{I}}$

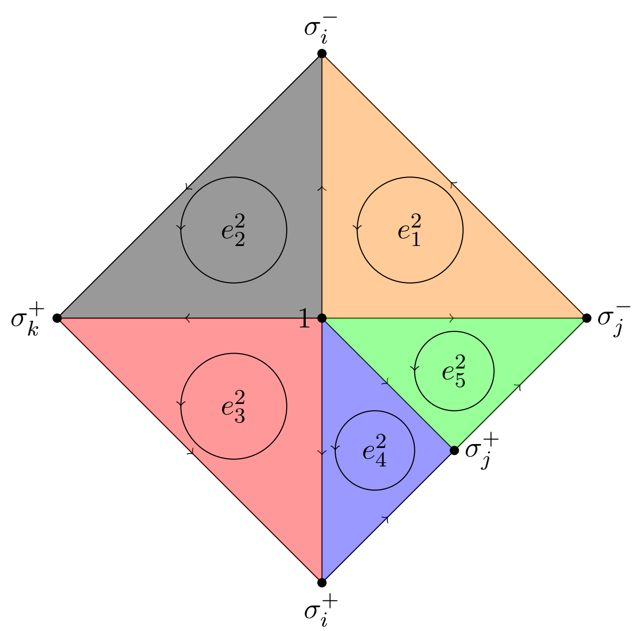

FIGURE 13. Orientation of the 2-cells

These orientations allow us to easily compute the boundaries of the representing cells e_v^u and give the resulting chain complex of free left $\mathbb{Z}[\mathcal{I}]$-modules

Proposition 4.2.3. *The cellular homology complex of $\partial\mathscr{P}_{\mathcal{I}}$ associated to the cellular structure given in the Proposition 4.2.2 is a chain complex of free left $\mathbb{Z}[\mathcal{I}]$-modules isomorphic*

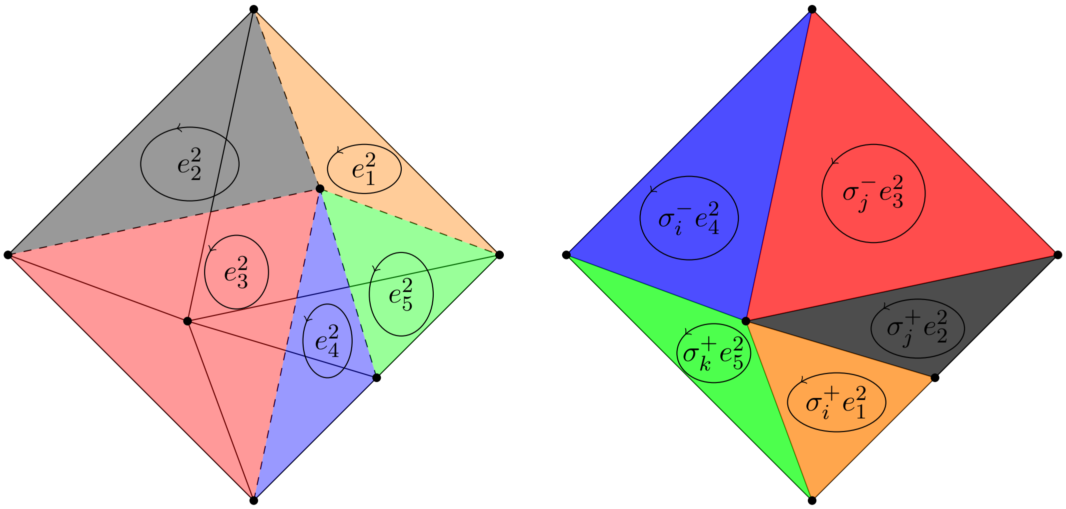

FIGURE 14. The oriented 2-skeleton of $\mathscr{D}_{\mathcal{I}}$

to

$$\mathcal{K}_{\mathcal{I}} := \left(\mathbb{Z}[\mathcal{I}] \xrightarrow{d_3} \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{d_2} \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{d_1} \mathbb{Z}[\mathcal{I}] \right),$$

where the d_i's are given, in the canonical bases, by right multiplication by the following matrices

$$d_1 = \begin{pmatrix} \sigma_k^+ - 1 \\ \sigma_i^+ - 1 \\ \sigma_j^+ - 1 \\ \sigma_j^- - 1 \\ \sigma_i^- - 1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} \sigma_j^- & 0 & 0 & 1 & -1 \\ -1 & \sigma_i^- & 0 & 0 & 1 \\ 1 & -1 & \sigma_k^+ & 0 & 0 \\ 0 & 1 & -1 & \sigma_i^+ & 0 \\ 0 & 0 & 1 & -1 & \sigma_j^+ \end{pmatrix},$$

$$d_3 = \begin{pmatrix} \sigma_i^+ - 1 & \sigma_j^+ - 1 & \sigma_j^- - 1 & \sigma_i^- - 1 & \sigma_k^+ - 1 \end{pmatrix}.$$

4.3. **The case of spheres and free resolution of the trivial $\mathcal{I}$-module.**

Here again, we shall describe the fundamental domain obtained above in $\mathbb{S}^3$ in terms of curved join and give a fundamental domain on $\mathbb{S}^{4n-1}$ and the equivariant cellular structure on that goes with it. We finish by giving a 4-periodic free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{I}]$.

We first have to describe the fundamental domain in $\mathbb{S}^3$, using the curved join.

Proposition 4.3.1. *The following subset of $\mathbb{S}^3$ is a fundamental domain for the action of $\mathcal{I}$*

$$\begin{aligned} \mathscr{F}_{\mathcal{I},3} := (1 * \sigma_k^- * \sigma_i^+ * \sigma_j^+) \cup (1 * \sigma_k^- * \sigma_j^+ * \sigma_j^-) \cup (1 * \sigma_k^- * \sigma_j^- * \sigma_i^-) \\ \cup (1 * \sigma_k^- * \sigma_i^- * \sigma_k^+) \cup (1 * \sigma_k^- * \sigma_k^+ * \sigma_i^+). \end{aligned}$$

Proof. According to Lemme 1.2.1, it suffices to show that $\mathscr{F}_{\mathcal{I},3} = p(\mathscr{D}_{\mathcal{I}})$, where $p : \partial \mathscr{P}_{\mathcal{I}} \xrightarrow{\sim} \mathbb{S}^3$ and this is a straightforward calculation which we omit again. $\square$

Then, we give the reformulation of Proposition 4.2.2, in terms of curved join and cells in the sphere $\mathbb{S}^3$.

Theorem 4.3.2. *The sphere $\mathbb{S}^3$ admits a $\mathcal{I}$-equivariant cellular decomposition with the following cells as orbit representatives*

$$\widetilde{e}^0 := 1 * \emptyset = \{1\},$$

$$\begin{cases} \widetilde{e}^1_1 := 1 \mathbin{\mathring{*}} \sigma_k^+ =]1, \sigma_k^+[, \\ \widetilde{e}^1_2 := 1 \mathbin{\mathring{*}} \sigma_i^+ =]1, \sigma_i^+[, \\ \widetilde{e}^1_3 := 1 \mathbin{\mathring{*}} \sigma_j^+ =]1, \sigma_j^+[, \\ \widetilde{e}^1_4 := 1 \mathbin{\mathring{*}} \sigma_j^- =]1, \sigma_j^-[, \\ \widetilde{e}^1_5 := 1 \mathbin{\mathring{*}} \sigma_i^- =]1, \sigma_i^-[\end{cases} \qquad \begin{cases} \widetilde{e}^2_1 := 1 \mathbin{\mathring{*}} \sigma_j^- \mathbin{\mathring{*}} \sigma_i^-, \\ \widetilde{e}^2_2 := 1 \mathbin{\mathring{*}} \sigma_i^- \mathbin{\mathring{*}} \sigma_k^+, \\ \widetilde{e}^2_3 := 1 \mathbin{\mathring{*}} \sigma_k^+ \mathbin{\mathring{*}} \sigma_i^+, \\ \widetilde{e}^2_4 := 1 \mathbin{\mathring{*}} \sigma_i^+ \mathbin{\mathring{*}} \sigma_j^+, \\ \widetilde{e}^2_5 := 1 \mathbin{\mathring{*}} \sigma_j^+ \mathbin{\mathring{*}} \sigma_j^- \end{cases}$$

$$\widetilde{e}^3 := \mathring{\mathscr{F}}_{\mathcal{I},3}.$$

Furthermore, the associated cellular homology complex is a chain complex of free left $\mathbb{Z}[\mathcal{I}]$-modules isomorphic to

$$\mathcal{K}_{\mathcal{I}} := \left(\mathbb{Z}[\mathcal{I}] \xrightarrow{d_3} \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{d_2} \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{d_1} \mathbb{Z}[\mathcal{I}] \right),$$

where the d_i's are given, in the canonical bases, by right multiplication by the following matrices

$$d_1 = \begin{pmatrix} \sigma_k^+ - 1 \\ \sigma_i^+ - 1 \\ \sigma_j^+ - 1 \\ \sigma_j^- - 1 \\ \sigma_i^- - 1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} \sigma_j^- & 0 & 0 & 1 & -1 \\ -1 & \sigma_i^- & 0 & 0 & 1 \\ 1 & -1 & \sigma_k^+ & 0 & 0 \\ 0 & 1 & -1 & \sigma_i^+ & 0 \\ 0 & 0 & 1 & -1 & \sigma_j^+ \end{pmatrix},$$

$$d_3 = \begin{pmatrix} \sigma_i^+ - 1 & \sigma_j^+ - 1 & \sigma_j^- - 1 & \sigma_i^- - 1 & \sigma_k^+ - 1 \end{pmatrix}.$$

For the higher dimensional case, combining lemma 1.3.2 and Proposition 4.3.1 yields

Proposition 4.3.3. *Letting $\mathcal{I}$ act on $\mathbb{S}^{4n-1}$ $(n \geq 1)$ via the representation $\rho_{\mathcal{I}}^n$, the following subset of $\mathbb{S}^{4n-1}$ is a fundamental domain for the action*

$$\mathscr{F}_{\mathcal{I},4n-1} := \Sigma_1 * \Sigma_2 * \cdots * \Sigma_{2(n-1)} * \mathscr{F}_{\mathcal{I},3},$$

with $\mathscr{F}_{\mathcal{I},3}$ inside $\Sigma_{2n-1} * \Sigma_{2n}$.

We can now describe the resulting equivariant cellular decomposition on $\mathbb{S}^{4n-1}$ using lemmae 1.2.1, 1.3.2 and Theorem 4.3.2. Here also, the boundary of the cells $\widetilde{e}^{4q}$ for $q > 0$ is given by all the cells in $\mathbb{S}^{4q-1}$, that is, all the orbits under $\mathcal{I}$. The following result can be proved as the Theorem 3.3.4

Theorem 4.3.4. *The chain complex $\mathcal{C}(\mathsf{P}_{\mathcal{I}}^{4n-1}, \mathbb{Z}[\mathcal{I}])$ of the universal covering space of the icosahedral space forms $\mathsf{P}_{\mathcal{I}}^{4n-1}$ with the fundamental group acting by covering transformations is isomorphic to the following complex of left $\mathbb{Z}[\mathcal{I}]$-modules:*

$$0 \longrightarrow \mathbb{Z}[\mathcal{I}] \xrightarrow{d_{4n-1}} \mathbb{Z}[\mathcal{I}]^5 \longrightarrow \ldots \longrightarrow \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{d_2} \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{d_1} \mathbb{Z}[\mathcal{I}] \longrightarrow 0,$$

where the boundaries are given, in the canonical bases, by the following matrices $(q \geq 1)$

$$d_{4q-3} = \begin{pmatrix} \sigma_k^+ - 1 \\ \sigma_i^+ - 1 \\ \sigma_j^+ - 1 \\ \sigma_j^- - 1 \\ \sigma_i^- - 1 \end{pmatrix}, \quad d_{4q-2} = \begin{pmatrix} \sigma_j^- & 0 & 0 & 1 & -1 \\ -1 & \sigma_i^- & 0 & 0 & 1 \\ 1 & -1 & \sigma_k^+ & 0 & 0 \\ 0 & 1 & -1 & \sigma_i^+ & 0 \\ 0 & 0 & 1 & -1 & \sigma_j^+ \end{pmatrix},$$

$$d_{4q-1} = \begin{pmatrix} \sigma_i^+ - 1 & \sigma_j^+ - 1 & \sigma_j^- - 1 & \sigma_i^- - 1 & \sigma_k^+ - 1 \end{pmatrix}, \quad d_{4q} = \left(\sum_{g \in \mathcal{I}} g \right).$$

In particular, the complex is exact in middle terms, i.e.

$$\forall 0 < i < 4n-1, \ H_i(\mathcal{C}(\mathsf{P}_{\mathcal{I}}^{4n-1}, \mathbb{Z}[\mathcal{I}])) = 0$$

and we have

$$H_0(\mathcal{C}(\mathsf{P}_{\mathcal{I}}^{4n-1}, \mathbb{Z}[\mathcal{I}])) = H_{4n-1}(\mathcal{C}(\mathsf{P}_{\mathcal{I}}^{4n-1}, \mathbb{Z}[\mathcal{I}])) = \mathbb{Z}.$$

Remark 4.3.5. *Using the augmentation map $\varepsilon : \mathbb{Z}[\mathcal{I}] \twoheadrightarrow \mathbb{Z}$, one can compute the complex $\mathcal{K}_{\mathcal{I}} \otimes_{\mathbb{Z}[\mathcal{I}]} \mathbb{Z}$ and since we have*

$$\det(d_2 \otimes \mathbb{Z}) = \det \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} = 1,$$

we find that $\mathbb{S}^3/\mathcal{I}$ is a homology sphere, but it is not a sphere. That is, one has $H_*(\mathbb{S}^3/\mathcal{I}, \mathbb{Z}) = H_*(\mathbb{S}^3, \mathbb{Z})$, and however $\mathbb{S}^3/\mathcal{I}$ is not homeomorphic to $\mathbb{S}^3$, since $\pi_1(\mathbb{S}^3/\mathcal{I}) = \mathcal{I} \neq 1 = \pi_1(\mathbb{S}^3)$.

This space has a long story, it is called the Poincaré homology sphere*. It can also be constructed as the link of the singularity of the complex affine variety $\{(x, y, z) \in \mathbb{C}^3 \ ; \ x^2 + y^3 + z^5 = 0\}$ near the origin, as the* Seifert bundle *or as the* dodecahedral space*. This last one corresponds to the original construction of Poincaré. For a detailed expository paper on the Poincaré homology sphere, we refer the reader to* [KS79].

Corollary 4.3.6. *The following chain complex*

$$\ldots \longrightarrow \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{d_{4q-3}} \mathbb{Z}[\mathcal{I}] \xrightarrow{d_{4q-4}} \ldots \longrightarrow \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{d_2} \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{d_1} \mathbb{Z}[\mathcal{I}] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

with boundaries d_i as in the Theorem 4.3.4, is a 4-periodic resolution of the constant module $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{I}]$.

We are now able to compute the group cohomology of $\mathcal{I}$ using this result.

Corollary 4.3.7. *The group cohomology of $\mathcal{I}$ with integer coefficients is given as follows:*

$$\forall q \in \mathbb{N}, \begin{cases} H^0(\mathcal{I}, \mathbb{Z}) = \mathbb{Z} & \text{if } q = 0, \\ H^q(\mathcal{I}, \mathbb{Z}) = \mathbb{Z}/120\mathbb{Z} & \text{if } q \equiv 0 \pmod 4 \\ H^q(\mathcal{I}, \mathbb{Z}) = 0 & \text{else} \end{cases}$$

Proof. In view of Lemma 3.3.6, it is suffices to compute $\mathcal{C}(\mathsf{P}_{\mathcal{I}}^{\infty}, \mathbb{Z}[\mathcal{I}]) \otimes_{\mathbb{Z}[\mathcal{I}]} \mathbb{Z}$, with $\mathcal{C}(\mathsf{P}_{\mathcal{I}}^{\infty}, \mathbb{Z}[\mathcal{I}])$ the complex given in Corollary 4.3.6. Computing the matrices $\varepsilon(d_i)$ leads to the following complex

$$\ldots \longrightarrow \mathbb{Z}^5 \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 120} \mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z} \xrightarrow{\times 120} \mathbb{Z} \xrightarrow{0} \mathbb{Z}^5 \xrightarrow{d} \mathbb{Z}^5 \xrightarrow{0} \mathbb{Z} \longrightarrow 0,$$

where

$$d := \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}.$$

Hence, determining the elementary divisors of d gives the following homology

$$\forall q \geq 1, \begin{cases} H_0(\mathcal{I}, \mathbb{Z}) = \mathbb{Z}, \\ \\ H_{4q-4,q>1}(\mathcal{I}, \mathbb{Z}) = H_{4q-2}(\mathcal{I}, \mathbb{Z}) = H_{4q-3}(\mathcal{I}, \mathbb{Z}) = 0, \\ \\ H_{4q-1}(\mathcal{I}, \mathbb{Z}) = \mathbb{Z}/120\mathbb{Z} \end{cases}$$

and the result then follows from the universal coefficients theorem. □

Remark 4.3.8. *The Corollary 4.3.7 agrees with the previously known result on the cohomology of $\mathcal{I}$, see* [TZ08, Theorem 4.16].

Appendix A. The tetrahedral case

Even if the case of $\mathcal{T}$ has already been treated in [FGMNS16], we can recover it by applying the above methods to this case. Note that all the groups in the tetrahedral family are studied in [CS17], but there $\mathcal{T}$ is excluded since, while it is the simplest one of the family, it is somehow different from all the other ones. Since it's always the same arguments and the case is solved, we omit the proofs.

A.1. **Fundamental domain.**

We consider the orbit polytope in $\mathbb{R}^4$

$$\mathscr{P} := \mathscr{P}_{\mathcal{T}} := \mathrm{conv}(\mathcal{T}).$$

This polytope has 24 vertices, 96 edges, 96 faces and 24 facets and is known as the *24-cells* (or the *icositetrachoron*, or even the *octaplex*). Since $\mathcal{T}$ acts freely on $\mathscr{P}_3$, there must be exactly one orbit in $\mathscr{P}_3$. We keep the notations of the Section 3 and define

$$\begin{cases} \omega_0 = \frac{1+i+j+k}{2} = s, \\ \\ \omega_i = \frac{1-i+j+k}{2} = t^{-1}s, \\ \\ \omega_j = \frac{1+i-j+k}{2} = st^{-1}, \\ \\ \omega_k = \frac{1+i+j-k}{2} = t, \\ \\ \omega_{ij} := \frac{1-i-j+k}{2} = t^{-1}. \end{cases}$$

Proposition A.1.1. *The subset of $\mathscr{P}_{\mathcal{T}}$ defined by*

$$\mathscr{D}_{\mathcal{T}} := [1, \omega_0, \omega_j, \omega_i, \omega_{ij}, k]$$

is a (connected) polytopal complex and is a fundamental domain for the action of $\mathcal{T}$ on $\partial\mathscr{P}_{\mathcal{T}}$.

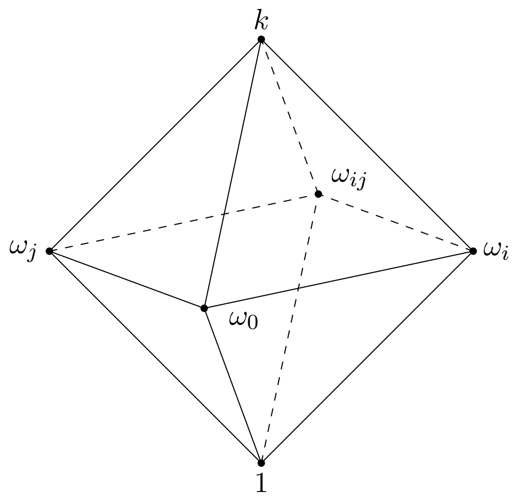

FIGURE 15. The tetrahedron $\mathscr{D}_{\mathcal{T}}$

A.2. **Associated $\mathcal{T}$-cellular decomposition of $\partial\mathscr{P}_{\mathcal{T}}$.**

Let $\mathscr{D} := \mathscr{D}_{\mathcal{T}}$. The facets of $\mathscr{D}$ are the following

$$\mathscr{D}_2 = \{[1, \omega_j, \omega_0], [1, \omega_0, \omega_i], [1, \omega_i, \omega_{ij}], [1, \omega_{ij}, \omega_j],\\ [k, \omega_j, \omega_0], [k, \omega_0, \omega_i], [k, \omega_i, \omega_{ij}], [k, \omega_{ij}, \omega_j]\}.$$

We remark the following relations among them

$$\begin{cases} \omega_{ij} \cdot [1, \omega_j, \omega_0] = [\omega_{ij}, k, \omega_i], \\ \omega_j \cdot [1, \omega_0, \omega_i] = [\omega_j, k, \omega_{ij}], \\ \omega_0 \cdot [1, \omega_i, \omega_{ij}] = [\omega_0, k, \omega_j], \\ \omega_i \cdot [1, \omega_{ij}, \omega_j] = [\omega_i, k, \omega_0]. \end{cases}$$

These are the only relations linking facets, hence we may define the following 2-cells

$$\begin{cases} e_1^2 :=]1, \omega_j, \omega_0[, \\ e_2^2 :=]1, \omega_0, \omega_i[, \\ e_3^2 :=]1, \omega_i, \omega_{ij}[, \\ e_4^2 :=]1, \omega_{ij}, \omega_j[. \end{cases}$$

Now, define the following 1-cells

$$e_1^1 :=]1, \omega_{ij}[, \quad e_2^1 :=]1, \omega_j[, \quad e_3^1 :=]1, \omega_0[, \quad e_4^1 :=]1, \omega_i[.$$

If we add to this the vertices of $\mathscr{D}$ and its interior, which is formed by only one cell e^3 by construction, then we may cover all of $\mathscr{D}$ with these cells and some of their translates. Thus, we have obtained the

Lemma A.2.1. *Consider the following sets of cells in $\mathscr{D}_{\mathcal{T}}$*

$$\begin{cases} E^0_{\mathscr{D}} := \{1,\ \omega_i, \omega_j, \omega_{ij}, \omega_0, k\}, \\ E^1_{\mathscr{D}} := \{e_1^1,\ \omega_0 e_1^1,\ \omega_i e_1^1,\ e_2^1,\ \omega_i e_2^1,\ \omega_{ij} e_2^1,\ e_3^1,\ \omega_{ij} e_3^1,\ \omega_j e_3^1,\ e_4^1,\ \omega_j e_4^1,\ \omega_0 e_4^1,\ \}, \\ E^2_{\mathscr{D}} := \{e_1^2,\ \omega_{ij} e_1^2,\ e_2^2,\ \omega_j e_2^2,\ e_3^2,\ \omega_0 e_3^2,\ e_4^2,\ \omega_i e_4^2\}, \\ E^3_{\mathscr{D}} := \{e^3\} \end{cases}$$

Then, one has the following cellular decomposition of the fundamental domain

$$\mathscr{D}_{\mathcal{T}} = \coprod_{\substack{0 \le j \le 3 \\ e \in E^j_{\mathscr{D}}}} e.$$

The 1-skeleton of $\mathscr{D}_{\mathcal{T}}$ is displayed in figure 16.

Then, combining Proposition A.1.1 and Lemma A.2.1, yields the following proposition.

Proposition A.2.2. *Letting $E^0 := \{1\}$, $E^1 := \{e_i^1,\ 1 \le i \le 4\}$, $E^2 := \{e_i^2,\ 1 \le i \le 4\}$ and $E^3 := \{e^3\}$ with the above notations, we have the following $\mathcal{T}$-equivariant cellular decomposition of $\partial\mathscr{P}_{\mathcal{T}}$*

$$\partial\mathscr{P}_{\mathcal{I}} = \coprod_{\substack{0 \le j \le 3 \\ e \in E^i, g \in \mathcal{T}}} ge.$$

As a consequence, using the homeomorphism $\phi : \partial\mathscr{P}_{\mathcal{T}} \xrightarrow{\sim} \mathbb{S}^3$ given by $x \mapsto x/|x|$, we obtain the following $\mathcal{T}$-equivariant cellular decomposition of the sphere

$$\mathbb{S}^3 = \coprod_{\substack{0 \le j \le 3 \\ e \in E^j, g \in \mathcal{T}}} g\phi(e).$$

We now have to compute the boundaries of the cells and the resulting cellular homology chain complex. We choose to orient the 3-cell e^3 directly, and the 2-cells undirectly:

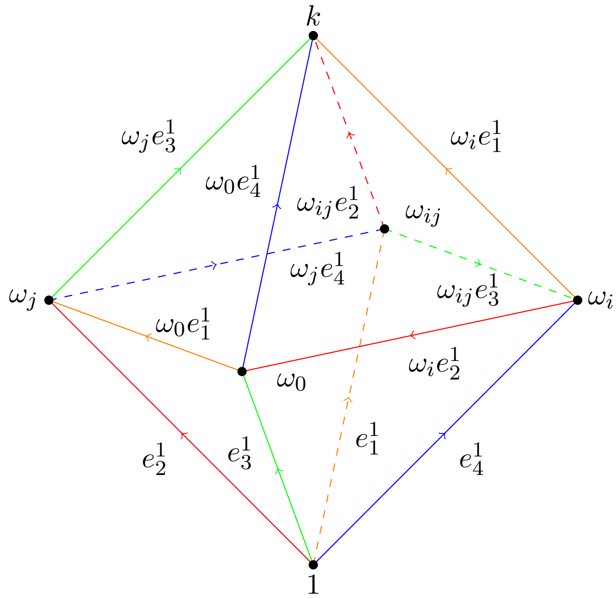

FIGURE 16. The oriented 1-skeleton of $\mathscr{D}_{\mathcal{T}}$

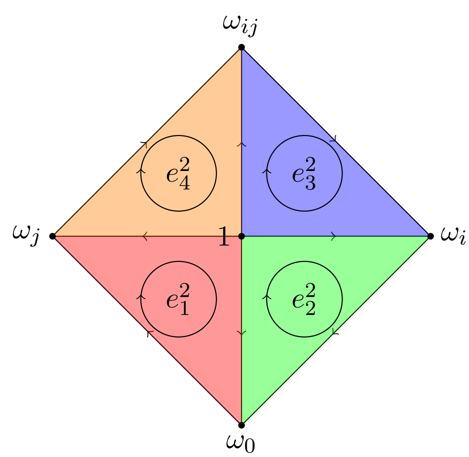

FIGURE 17. Orientation of the 2-cells

These orientations allow us to easily compute the boundaries of the representing cells e_v^u and give the resulting chain complex of free left $\mathbb{Z}[\mathcal{T}]$-modules

Proposition A.2.3. *The cellular homology complex of $\partial\mathscr{P}_{\mathcal{T}}$ associated to the cellular structure given in the Proposition A.2.2 is a chain complex of free left $\mathbb{Z}[\mathcal{T}]$-modules isomorphic to*

$$\mathcal{K}_{\mathcal{T}} := \left(\mathbb{Z}[\mathcal{T}] \xrightarrow{d_3} \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{d_2} \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{d_1} \mathbb{Z}[\mathcal{T}] \right),$$

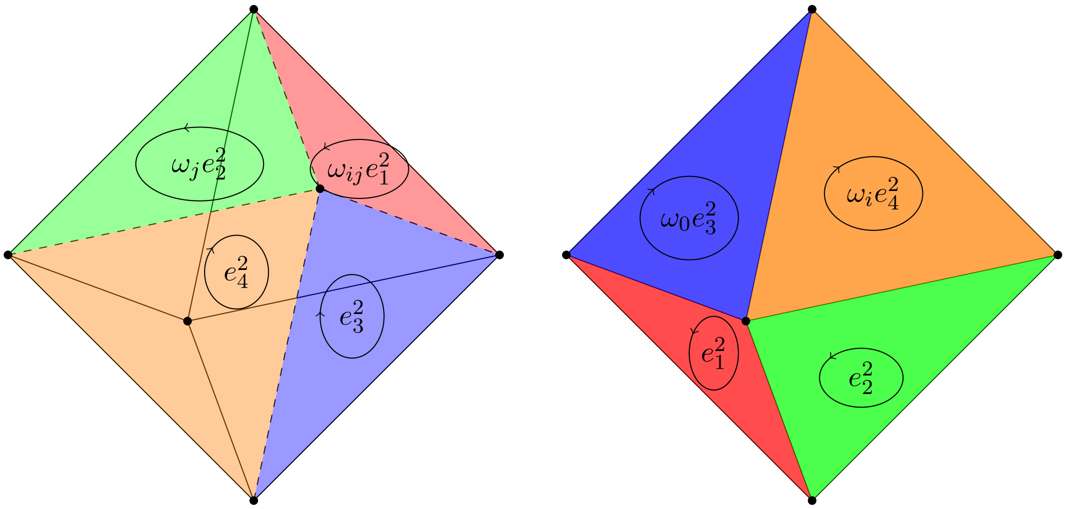

FIGURE 18. The oriented 2-skeleton of $\mathscr{D}_{\mathcal{T}}$

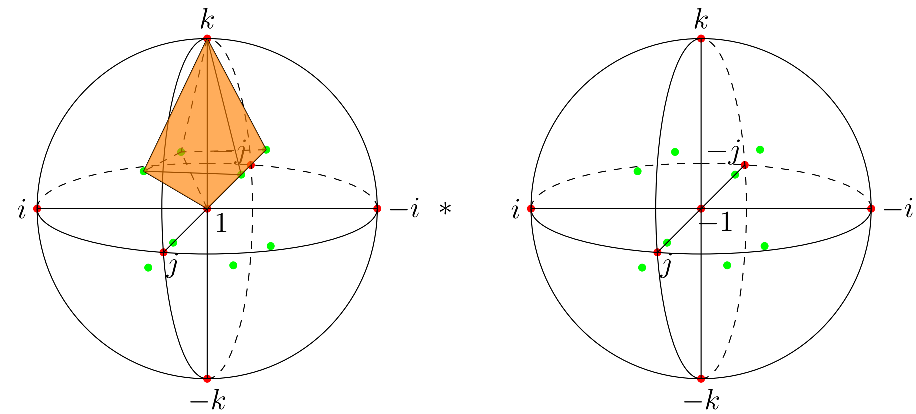

FIGURE 19. The fundamental domain $\mathscr{D}_{\mathcal{T}}$
LEGEND : The red points belong to $\mathcal{Q}_8$, while the green ones belong to $\mathcal{T} \setminus \mathcal{Q}_8$.

where the d_i's are given, in the canonical bases, by right multiplication by the following matrices

$$d_1 = \begin{pmatrix} \omega_{ij} - 1 \\ \omega_j - 1 \\ \omega_0 - 1 \\ \omega_i - 1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} \omega_0 & -1 & 1 & 0 \\ 0 & \omega_i & -1 & 1 \\ 1 & 0 & \omega_{ij} & -1 \\ -1 & 1 & 0 & \omega_j \end{pmatrix},$$
$$d_3 = \begin{pmatrix} 1 - \omega_{ij} & 1 - \omega_j & 1 - \omega_0 & 1 - \omega_i \end{pmatrix}.$$

A.3. The case of spheres and free resolution of the trivial $\mathcal{T}$-module.

Here again, we shall describe the fundamental domain obtained above in $\mathbb{S}^3$ in terms of curved join and give a fundamental domain on $\mathbb{S}^{4n-1}$ and the equivariant cellular structure on that goes with it. We finish by giving a 4-periodic free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{T}]$.

We first have to describe the fundamental domain in $\mathbb{S}^3$, using the curved join. This is done in the following result

Proposition A.3.1. *The following subset of $\mathbb{S}^3$ is a fundamental domain for the action of $\mathcal{T}$*

$$\mathscr{F}_{\mathcal{T},3} := (1 * \omega_{ij} * \omega_i * \omega_0 * \omega_j) \cup (\omega_{ij} * \omega_i * \omega_0 * \omega_j * k).$$

Then, we give the reformulation of Proposition A.2.2, in terms of curved join and cells in the sphere $\mathbb{S}^3$:

Theorem A.3.2. *The sphere $\mathbb{S}^3$ admits a $\mathcal{T}$-equivariant cellular decomposition with the following cells as orbit representatives*

$$\widetilde{e}^0 := 1 * \emptyset = \{1\},$$

$$\begin{cases} \widetilde{e}^1_1 := 1 \mathbin{\mathring{*}} \omega_{ij}, \\ \widetilde{e}^1_2 := 1 \mathbin{\mathring{*}} \omega_j, \\ \widetilde{e}^1_3 := 1 \mathbin{\mathring{*}} \omega_0, \\ \widetilde{e}^1_4 := 1 \mathbin{\mathring{*}} \omega_i \end{cases} \qquad \begin{cases} \widetilde{e}^2_1 := 1 \mathbin{\mathring{*}} \omega_j \mathbin{\mathring{*}} \omega_0, \\ \widetilde{e}^2_2 := 1 \mathbin{\mathring{*}} \omega_0 \mathbin{\mathring{*}} \omega_i, \\ \widetilde{e}^2_3 := 1 \mathbin{\mathring{*}} \omega_i \mathbin{\mathring{*}} \omega_{ij}, \\ \widetilde{e}^2_4 := 1 \mathbin{\mathring{*}} \omega_{ij} \mathbin{\mathring{*}} \omega_j \end{cases}$$

$$\widetilde{e}^3 := \mathring{\mathscr{F}}_{\mathcal{T},3}.$$

Furthermore, the associated cellular homology complex is a chain complex of free left $\mathbb{Z}[\mathcal{T}]$-modules isomorphic to

$$\mathcal{K}_{\mathcal{T}} := \left(\mathbb{Z}[\mathcal{T}] \xrightarrow{d_3} \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{d_2} \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{d_1} \mathbb{Z}[\mathcal{T}] \right),$$

where the d_i's are given, in the canonical bases, by right multiplication by the following matrices

$$d_1 = \begin{pmatrix} \omega_{ij} - 1 \\ \omega_j - 1 \\ \omega_0 - 1 \\ \omega_i - 1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} \omega_0 & -1 & 1 & 0 \\ 0 & \omega_i & -1 & 1 \\ 1 & 0 & \omega_{ij} & -1 \\ -1 & 1 & 0 & \omega_j \end{pmatrix},$$
$$d_3 = \begin{pmatrix} 1 - \omega_{ij} & 1 - \omega_j & 1 - \omega_0 & 1 - \omega_i \end{pmatrix}.$$

For the higher dimensional case, combining lemma 1.3.2 and Proposition A.3.1 yields

Proposition A.3.3. *Letting $\mathcal{T}$ act on $\mathbb{S}^{4n-1}$ ($n \geq 1$) via the representation $\rho^n_{\mathcal{T}}$, the following subset of $\mathbb{S}^{4n-1}$ is a fundamental domain for the action*

$$\mathscr{F}_{\mathcal{T},4n-1} := \Sigma_1 * \Sigma_2 * \cdots * \Sigma_{2(n-1)} * \mathscr{F}_{\mathcal{T},3},$$

with $\mathscr{F}_{\mathcal{T},3}$ inside $\Sigma_{2n-1} * \Sigma_{2n}$.

We can now describe the resulting equivariant cellular decomposition on $\mathbb{S}^{4n-1}$ using lemmae 1.2.1, 1.3.2 and Theorem A.3.2.

Theorem A.3.4. *The chain complex $\mathcal{C}(\mathsf{P}^{4n-1}_{\mathcal{T}}, \mathbb{Z}[\mathcal{T}])$ of the universal covering space of the tetrahedral space forms $\mathsf{P}^{4n-1}_{\mathcal{T}}$ with the fundamental group acting by covering transformations is isomorphic to the following complex of left $\mathbb{Z}[\mathcal{T}]$-modules:*

$$0 \longrightarrow \mathbb{Z}[\mathcal{T}] \xrightarrow{d_{4n-1}} \mathbb{Z}[\mathcal{T}]^4 \longrightarrow \dots \longrightarrow \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{d_2} \mathbb{Z}[\mathcal{T}]^3 \xrightarrow{d_1} \mathbb{Z}[\mathcal{T}] \longrightarrow 0,$$

where the boundaries are given, in the canonical bases, by the following matrices ($q \geq 1$)

$$d_{4q-3} = \begin{pmatrix} \omega_{ij}-1 \\ \omega_j - 1 \\ \omega_0 - 1 \\ \omega_i - 1 \end{pmatrix}, \quad d_{4q-2} = \begin{pmatrix} \omega_0 & -1 & 1 & 0 \\ 0 & \omega_i & -1 & 1 \\ 1 & 0 & \omega_{ij} & -1 \\ -1 & 1 & 0 & \omega_j \end{pmatrix},$$
$$d_{4q-1} = \begin{pmatrix} 1-\omega_{ij} & 1-\omega_j & 1-\omega_0 & 1-\omega_i \end{pmatrix}, \quad d_{4q} = \left(\sum_{g\in\mathcal{T}} g\right).$$

In particular, the complex is exact in middle terms, i.e.

$$\forall 0 < i < 4n-1,\ H_i(\mathcal{C}(\mathsf{P}_{\mathcal{T}}^{4n-1}, \mathbb{Z}[\mathcal{T}])) = 0$$

and we have

$$H_0(\mathcal{C}(\mathsf{P}_{\mathcal{T}}^{4n-1}, \mathbb{Z}[\mathcal{T}])) = H_{4n-1}(\mathcal{C}(\mathsf{P}_{\mathcal{T}}^{4n-1}, \mathbb{Z}[\mathcal{T}])) = \mathbb{Z}.$$

Next, letting $n \to \infty$ yields the

Corollary A.3.5. *The following chain complex*

$$\dots \longrightarrow \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{d_{4q-3}} \mathbb{Z}[\mathcal{T}] \xrightarrow{d_{4q-4}} \dots \longrightarrow \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{d_2} \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{d_1} \mathbb{Z}[\mathcal{T}] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

with boundaries d_i as in the Theorem A.3.4, is a 4-periodic resolution of the constant module $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{T}]$.

We are now able to compute the group cohomology of $\mathcal{T}$ using this result.

Corollary A.3.6. *The group cohomology of $\mathcal{T}$ with integer coefficients is given as follows:*

$$\begin{cases} H^q(\mathcal{T}, \mathbb{Z}) = \mathbb{Z} & \text{if } q = 0, \\ \forall q \geq 1, \begin{cases} H^q(\mathcal{T}, \mathbb{Z}) = \mathbb{Z}/24\mathbb{Z} & \text{if } q \equiv 0 \pmod 4, \\ H^q(\mathcal{T}, \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z} & \text{if } q \equiv 2 \pmod 4, \\ H^q(\mathcal{T}, \mathbb{Z}) = 0 & \text{else} \end{cases} \end{cases}$$

Proof. In view of Lemma 3.3.6, is suffices to compute $\mathcal{C}(\mathsf{P}_{\mathcal{T}}^{\infty}, \mathbb{Z}[\mathcal{T}]) \otimes_{\mathbb{Z}[\mathcal{T}]} \mathbb{Z}$, with $\mathcal{C}(\mathsf{P}_{\mathcal{T}}^{\infty}, \mathbb{Z}[\mathcal{T}])$ the complex given in Corollary A.3.5. Computing the matrices $\varepsilon(d_i)$ leads to the following complex

$$\dots \longrightarrow \mathbb{Z}^4 \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 24} \mathbb{Z} \longrightarrow \dots \longrightarrow \mathbb{Z} \xrightarrow{\times 24} \mathbb{Z} \xrightarrow{0} \mathbb{Z}^4 \xrightarrow{d} \mathbb{Z}^4 \xrightarrow{0} \mathbb{Z} \longrightarrow 0,$$

where

$$d = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 \end{pmatrix}.$$

Hence, determining the elementary divisors of the only non-trivial matrix occuring in it gives the following homology

$$\forall q \in \mathbb{N}, \begin{cases} H_0(\mathcal{T}, \mathbb{Z}) = \mathbb{Z}, \\ H_{4q-4, q>1}(\mathcal{T}, \mathbb{Z}) = H_{4q-2}(\mathcal{T}, \mathbb{Z}) = 0, \\ H_{4q-3}(\mathcal{T}, \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}, \\ H_{4q-1}(\mathcal{T}, \mathbb{Z}) = \mathbb{Z}/24\mathbb{Z} \end{cases}$$

and the result then follows from the universal coefficients theorem. $\square$

A.4. **Simplicial structure and minimal resolution.**

Since we have chosen polytopal fundamental domains for $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{I}$, it is clear that we can refine our cellular decompositions to equivariant simplicial decompositions of $\mathbb{S}^3$. We will just investigate the case of $\mathcal{T}$, since the other ones can be treated in a similar way. The method is trivial: just take each one of the facets Δ_i of $\mathscr{P}$ as the 3-cells and their boundary (up to multiplication) as 2-cells.

For instance, here, take as 3-cells the following open curved joins:

$$\begin{cases} c_1^3 :=]\omega_0, 1, \omega_{ij}, \omega_j[, \\ c_2^3 :=]\omega_0, 1, \omega_i, \omega_{ij}[, \\ c_3^3 :=]\omega_0, \omega_{ij}, k, \omega_j[, \\ c_4^3 :=]k, \omega_0, \omega_i, \omega_{ij}[. \end{cases}$$

and as 2-cells the following open triangles:

$$\forall 1 \leq i \leq 4,\ c_i^2 := e_i^2 \quad \text{and} \quad \begin{cases} c_5^2 :=]\omega_0, 1, \omega_{ij}[, \\ c_6^2 :=]\omega_{ij}, \omega_j, \omega_0[, \\ c_7^2 :=]\omega_0, \omega_i, \omega_{ij}[, \\ c_8^2 :=]\omega_0, k, \omega_{ij}[. \end{cases}$$

and we may keep the 1-cells as they are, i.e. $c_i^1 := e_i^1$ for $1 \leq i \leq 4$. Then, the resulting simplicial homology complex is easily computed (for example, by orienting the 3-cells directly), just as we did above. One shall find of course a complex that is homotopy equivalent to the complex $\mathcal{K}_{\mathcal{T}}$ defined in the Theorem A.3.2. We omit the details.

We conclude by discussing the minimal resolution. Group resolution and group cohomology are purely algebraic invariants of the given group G. Under this point of view, Swan [Swan] proved the existence of a minimal periodic free resolution of $\mathbb{Z}$ over G, for a family of finite groups containing the spherical space forms group. This means a resolution with minimal $\mathbb{Z}[G]$ module's ranks. He also gave a bound for these ranks. This point has been discussed in [CS17] for the resolution over the groups $P'_{8\cdot 3^s}$ of the tetrahedral family. Here, we show how to "reduce" our resolution for $\mathcal{T}$ to the minimal one, that has ranks 1-2-2-1, compare [CS17, 10.6]. (We note that in [CS17, 10.5] there is a missprint: one should read $f_h(F^\bullet)$ instead of $\mu_h(G)$ in the statement of the proposition.) We first describe the underlying geometric idea, and next we give an explicit chain homotopy.

Geometrically, the construction is as follows: start with the cellular decomposition from Theorem A.3.2. As seen in Figure 18, the four upper triangles are sent by different group elements to the four lower triangles. It is clear that there is no way of collecting two triangles in one single 2-cell but we may proceed as follows. Pick up one triangle, say e_1^2, and one of its neighbours, say $\omega_0 e_3^2$ and set a_1 to be the union of these two triangles, namely

$$a_1 := e_1^2 + e_2^2.$$

Then, we have that $\omega_{ij} a_1 = \omega_{ij} e_1^2 + \omega_{ij} e_2^2$ and $y := \omega_{ij} e_2^2$ does not belong to the boundary of the fundamental domain $\mathscr{F}_{\mathcal{T},3}$. However, we may find an other pair of coherent triangles such that one of them is mapped to y by some group element, while the other one is mapped to some triangle in the boundary of $\mathscr{F}_{\mathcal{T},3}$. For example, take

$$a_2 := \omega_0 e_3^2 + \omega_j e_2^2.$$

Then, we have $\omega_0^{-1}a_2 = e_3^2 + y$. As a consequence,

$$\omega_0^{-1}a_2 - \omega_{ij}a_1 = e_3^2 - \omega_{ij}e_1^2$$

and this means that we can use the three 2-cells a_1, a_2 and e_4^2 to cover all the boundary of $\mathscr{F}_{\mathcal{T},3}$. We would like to add one more triangle to the first two 2-cells in order to reduce the total number to two, but we easily see that the same procedure fails. However, we may proceed in the following "dual" way. Let x be a triangle such that $\omega_0^{-1}x = e_4^2$ and $\omega_{ij}x = \omega_i e_4^2$. We can take $x :=]i, \omega_j, \omega_0[$ and then we define

$$b_1 := a_1 + x = e_1^2 + e_2^2 + x$$

and

$$b_2 := a_2 + x = \omega_0 e_3^2 + \omega_j e_2^2 + x.$$

Then, after a simple calculation, we find that

$$\begin{aligned} b_1 - b_2 + \omega_0^{-1}b_2 - \omega_{ij}b_1 &= a_1 - a_2 + \omega_0^{-1}a_2 - \omega_{ij}a_1 + \omega_0^{-1}x - \omega_{ij}x \\ &= (1-\omega_{ij})e_1^2 + (1-\omega_j)e_2^2 + (1-\omega_0)e_3^2 + (1-\omega_i)e_4^2 = d_3(e^3), \end{aligned}$$

that is, the whole boundary of $\mathscr{F}_{\mathcal{T},3}$ is obtained using only the two 2-chains b_1 and b_2.

We can then give the reduced complex. It is given by the following

$$\mathcal{K}'_{\mathcal{T}} := \left(0 \longrightarrow K'_3 \xrightarrow{d'_1} K'_2 \xrightarrow{d'_2} K'_1 \xrightarrow{d'_1} K'_0 \longrightarrow 0 \right),$$

where $K'_0 = \mathbb{Z}[\mathcal{T}]\left\langle f^0 \right\rangle$, $K'_3 = \mathbb{Z}[\mathcal{T}]\left\langle f^3 \right\rangle$, $K'_1 = \mathbb{Z}[\mathcal{T}]\left\langle f_1^1, f_2^1 \right\rangle$ and $K'_2 = \mathbb{Z}[\mathcal{T}]\left\langle f_1^2, f_2^2 \right\rangle$ and

$$\begin{cases} d'_3(f^3) = (1-\omega_{ij})f_1^2 + (1-\omega_0)f_2^2, \\ d'_2(f_1^2) = (\omega_0 + \omega_i - 1)f_1^1 + (i+1)f_2^1, \\ d'_2(f_2^2) = (1 + (-i))f_1^1 + (\omega_j - 1 + \omega_{ij})f_2^1, \\ d'_1(f_1^1) = (\omega_j - 1)f^0, \\ d'_1(f_2^1) = (\omega_i - 1)f^0, \end{cases}$$

i.e. are given in the canonical bases by right multiplication by the following matrices

$$d'_1 = \begin{pmatrix} \omega_j - 1 \\ \omega_i - 1 \end{pmatrix}, \quad d'_2 = \begin{pmatrix} \omega_0 + \omega_i - 1 & 1 + i \\ 1 + (-i) & \omega_j - 1 + \omega_{ij} \end{pmatrix}, \quad d'_3 = \begin{pmatrix} 1 - \omega_{ij} & 1 - \omega_0 \end{pmatrix}.$$

We finish by giving explicit homotopy equivalences $\varphi : \mathcal{K}_{\mathcal{T}} \to \mathcal{K}'_{\mathcal{T}}$ and $\varphi' : \mathcal{K}'_{\mathcal{T}} \to \mathcal{K}_{\mathcal{T}}$. We define $\varphi(e^i) := f^i$ and $\varphi'(f^i) := e^i$ for $i = 0, 3$ as well as

$$\begin{cases} \varphi_2(e_1^2) := f_1^2, \\ \varphi_2(e_2^2) = \varphi_2(e_4^2) := 0, \\ \varphi_2(e_3^2) := f_2^2, \end{cases} \quad \text{and} \quad \begin{cases} \varphi'_2(f_1^2) := e_1^2 + e_2^2 + \omega_0 e_4^2, \\ \varphi'_2(f_2^2) := \omega_{ij}e_2^2 + e_3^2 + e_4^2 \end{cases}$$

also

$$\begin{cases} \varphi_1(e_1^1) := f_1^1 + \omega_j f_2^1, \\ \varphi_1(e_2^1) := f_1^1, \\ \varphi_1(e_3^1) := f_2^1 + \omega_i f_1^1, \\ \varphi_1(e_4^1) := f_2^1, \end{cases} \quad \text{and} \quad \begin{cases} \varphi'_1(f_1^1) := e_2^1, \\ \varphi'_1(f_2^1) := e_4^1. \end{cases}$$

We immediately check that $\varphi \circ \varphi' = id_{\mathcal{K}'_{\mathcal{T}}}$ and we just have to show that the other composition is homotopic to $id_{\mathcal{K}_{\mathcal{T}}}$. If we define $H : \mathcal{K}_* \to \mathcal{K}_{*+1}$ by $H_0 = H_2 = 0$, $H_1(e_2^1) = H_1(e_4^1) := 0$ and $H_1(e_1^1) := e_4^2$, $H_1(e_3^1) := e_2^2$, then we have $\varphi'_1\varphi_1 = id + d_2H_1 + H_0d_1$ and $\varphi'_2\varphi_2 = id + d_3H_2 + H_1d_2$, i.e.

$$\varphi' \circ \varphi = id_{\mathcal{K}_{\mathcal{T}}} + dH + Hd$$

and φ is indeed a homotopy equivalence, with homotopy inverse φ'. Thus, we have proved that the complex $\mathcal{K}_{\mathcal{T}}$ from the Theorem A.3.2 is homotopy equivalent to the complex

$$\mathcal{K}'_{\mathcal{T}} = \left(0 \longrightarrow \mathbb{Z}[\mathcal{T}] \xrightarrow{d'_3} \mathbb{Z}[\mathcal{T}]^2 \xrightarrow{d'_2} \mathbb{Z}[\mathcal{T}]^2 \xrightarrow{d'_1} \mathbb{Z}[\mathcal{T}] \longrightarrow 0 \right)$$

defined above.

Remark 1.4.1. *Observe that this process works for the group $\mathcal{T}$ but fails for the other two groups, $\mathcal{O}$ and $\mathcal{I}$. This is not unexpected, since the resolutions determined in the present work are characterised by their geometric feature, i.e. constructed through particular orthogonal representations of the groups, and it is not likely that this characterisation would produce a minimal resolution, that in general may not be induced by a representation. Indeed, it would be interesting to investigate the possible bounds for the ranks of a free periodic resolution induced by a linear representation.*

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