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# On maximal regularity for the second order Cauchy problems in Hilbert space

Mahdi Achache \*

## Abstract

We consider the problem of maximal regularity for semilinear non-autonomous second order Cauchy problems

$$\begin{cases} u''(t) + \mathcal{B}(t)u'(t) + \mathcal{A}(t)u(t) = F(t, u, u') & t\text{-a.e.} \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \quad (0.1)$$

Here, the time dependent operator  $\mathcal{A}(t)$  is bounded from the Hilbert space  $\mathcal{V}$  to its dual space  $\mathcal{V}'$  and  $\mathcal{B}(t)$  is associated with a sesquilinear form  $\mathfrak{b}(t, \cdot, \cdot)$  with domain  $\mathcal{V}$ . We prove maximal  $L^2$ -regularity results and other regularity properties for the solutions of the above equation under minimal regularity assumptions on the operators and the inhomogeneous term  $F$ . One of our main results shows that maximal  $L^2$ -regularity holds if the operators are piecewise  $H^{\frac{1}{2}}$  with respect to  $t$ . This regularity assumption is optimal and provides the best positive result on this problem.

**keywords:** Damped wave equation, maximal regularity, non-autonomous evolution equations.

**Mathematics Subject Classification (2010):** 35K90, 35K45, 47D06.

## 1 Introduction

The present paper is a continuation of [1],[3] which are devoted to maximal regularity for non-autonomous evolution equations governed by forms. The aim of this article is to study non-autonomous second order evolution equations governed by forms.

Let  $(\mathcal{H}, (\cdot, \cdot), \|\cdot\|)$  be a separable Hilbert space over  $\mathbb{R}$  or  $\mathbb{C}$ . We consider another separable Hilbert space  $\mathcal{V}$  which is densely and continuously embedded into  $\mathcal{H}$ . We denote by  $\mathcal{V}'$  the (anti-) dual space of  $\mathcal{V}$  so that

$$\mathcal{V} \hookrightarrow_d \mathcal{H} \hookrightarrow_d \mathcal{V}'.$$

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Hence there exists a constant  $C > 0$  such that

$$\|u\| \leq C\|u\|_{\mathcal{V}} \quad (u \in \mathcal{V}),$$

where  $\|\cdot\|_{\mathcal{V}}$  denotes the norm of  $\mathcal{V}$ . Similarly,

$$\|\psi\|_{\mathcal{V}'} \leq C\|\psi\| \quad (\psi \in \mathcal{H}).$$

We denote by  $\langle, \rangle$  the duality  $\mathcal{V}'$ - $\mathcal{V}$  and note that  $\langle \psi, v \rangle = (\psi, v)$  if  $\psi, v \in \mathcal{H}$ . In this paper we consider maximal regularity for second order Cauchy problems. We focus on the damped wave equation.

We consider a family of sesquilinear forms

$$\mathfrak{b} : [0, \tau] \times \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C},$$

such that

- [H1]:  $D(\mathfrak{b}(t)) = \mathcal{V}$  (constant form domain),
- [H2]:  $|\mathfrak{b}(t, u, v)| \leq M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$  ( $\mathcal{V}$ -uniform boundedness),
- [H3]:  $\operatorname{Re} \mathfrak{b}(t, u, u) + \nu\|u\|^2 \geq \delta\|u\|_{\mathcal{V}}^2$  ( $\forall u \in \mathcal{V}$ ) for some  $\delta > 0$  and some  $\nu \in \mathbb{R}$  (uniform quasi-coercivity).

We denote by  $B(t), \mathcal{B}(t)$  the usual operators associated with  $\mathfrak{b}(t)$  (as operators on  $\mathcal{H}$  and  $\mathcal{V}'$ ). Recall that  $u \in \mathcal{H}$  is in the domain  $D(B(t))$  if there exists  $h \in \mathcal{H}$  such that for all  $v \in \mathcal{V}$ :  $\mathfrak{b}(t, u, v) = (h, v)$ . We then set  $B(t)u := h$ . The operator  $\mathcal{B}(t)$  is a bounded operator from  $\mathcal{V}$  into  $\mathcal{V}'$  such that  $\mathcal{B}(t)u = \mathfrak{b}(t, u, \cdot)$ . The operator  $B(t)$  is the part of  $\mathcal{B}(t)$  on  $\mathcal{H}$ .

It is a classical fact that  $-B(t)$  and  $-\mathcal{B}(t)$  are both generators of holomorphic semigroups  $(e^{-rB(t)})_{r \geq 0}$  and  $(e^{-r\mathcal{B}(t)})_{r \geq 0}$  on  $\mathcal{H}$  and  $\mathcal{V}'$ , respectively. The semigroup  $e^{-rB(t)}$  is the restriction of  $e^{-r\mathcal{B}(t)}$  to  $\mathcal{H}$ . In addition,  $e^{-rB(t)}$  induces a holomorphic semigroup on  $\mathcal{V}$  (see, e.g., Ouhabaz [36, Chapter 1]). Let  $\mathcal{A}(t) \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$  for all  $t \in [0, \tau]$  such that

$$\|\mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \leq M, \quad M > 0.$$

Given a function  $f$  defined on  $[0, \tau]$  with values either in  $\mathcal{H}$  or in  $\mathcal{V}'$  we consider the second order evolution equation

$$\begin{cases} u''(t) + \mathcal{B}(t)u'(t) + \mathcal{A}(t)u(t) = f(t) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \quad (1.1)$$

This is an abstract damped non-autonomous wave equation and our aim is to prove well-posedness and maximal  $L^2$ -regularity in  $\mathcal{H}$ .

**Definition 1.1.** Let  $X = \mathcal{H}$  or  $\mathcal{V}'$ . We say that Problem (1.1) has maximal  $L^2$ -regularity in  $X$ , if for all  $f \in L^2(0, \tau; X)$  and all  $(u_0, u_1)$  in the trace space (see Sections ?? and ?? for more details) there exists a unique  $u \in H^2(0, \tau; X) \cap H^1(0, \tau; \mathcal{V})$  which satisfies (1.1) in the  $L^2$ -sense.

Thus we are able to define the non-autonomous maximal  $L^2$ -regularity of a second order problem in a manner similar to maximal  $L^2$ -regularity of the first order non-autonomous problem  $u'(t) + \mathcal{A}(t)u(t) = f(t)$ ,  $u(0) = u_0$ . Due to applications to many parabolic partial differential equations, the maximal  $L^2$ -regularity ( $L^p$ -regularity,  $p \in (1, \infty)$ ) of first order problems has been studied intensively in recent years, both from the abstract and the applied point of view. We mention only two accounts of this theory ([3], [8]) and refer to the references therein. The maximal  $L^2$ -regularity in  $\mathcal{V}'$  was first considered by Lions [32] (p. 151). He assumes that  $\mathcal{A}(t)$  is associated with a sesquilinear form  $\mathfrak{a}(t)$  which satisfies the same properties as  $\mathfrak{b}(t)$  together with an additional regularity assumption on the forms  $t \rightarrow \mathfrak{a}(t, u, v)$  and  $t \rightarrow \mathfrak{b}(t, u, v)$  for every fixed  $u, v \in \mathcal{V}$ . Dautray-Lions [19] (p.667) proved maximal  $L^2$ -regularity in  $\mathcal{V}'$  without the regularity assumption by taking  $f \in L^2(0, \tau; \mathcal{H})$  and considering mainly symmetric forms. Recently, Batty, Chill, Srivastava [15] proved maximal  $L^p$ -regularity for general forms  $\mathcal{B}(\cdot)$  and  $\mathcal{A}(\cdot)$  for the case  $u_0 = u_1 = 0$  and  $t \mapsto \|\mathcal{A}(\cdot)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \in L^p(0, \tau)$  by reducing the problem to a first order non-autonomous Cauchy problem. Dier-Ouhabaz [22] proved maximal  $L^2$ -regularity in  $\mathcal{V}'$  for  $u_0 \in \mathcal{V}, u_1 \in \mathcal{H}$  and  $\mathcal{A}(t)$  is also associated with a  $\mathcal{V}$ -bounded quasi-coercive non-autonomous form  $\mathfrak{a}(t)$ . By using the result of the first order problem as in [15] and the fixed point argument Achache [1] improve the result in [15] by proving maximal  $L^p$ -regularity in  $\mathcal{V}'$  for  $u_0$  and  $u_1$  not necessarily 0 and  $t \mapsto t^{1-\frac{1}{p}} \|\mathcal{A}(\cdot)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \in L^p(0, \tau)$ .

More interesting is the question of second order maximal regularity in  $\mathcal{H}$ , i.e. whether the solution  $u$  of (1.1) is in  $H^2(0, \tau; \mathcal{H})$  provided that  $f \in L^2(0, \tau; \mathcal{H})$ . A first answer to this question was giving by Batty, Chill, Srivastava [15] in the particular case  $\mathcal{B}(\cdot) = k\mathcal{A}(\cdot)$  for some constant  $k$  and that  $\mathcal{A}(\cdot)$  has the maximal regularity in  $\mathcal{H}$ . By using the form method, Dier and Ouhabaz [22], proved maximal  $L^2$ -regularity in  $\mathcal{H}$  without the rather strong assumption  $\mathcal{B}(\cdot) = k\mathcal{A}(\cdot)$ , but  $\mathcal{A}(t)$  is also associated with  $\mathcal{V}$ -bounded quasi-coercive form  $\mathfrak{a}(t)$  and  $t \rightarrow \mathfrak{a}(t, u, v), \mathfrak{b}(t, u, v)$  are symmetric and Lipschitz continuous for all  $u, v \in \mathcal{V}$ . Achache [1] extend the results in [22] in three directions. The first one is to consider general forms which may not be symmetric. The second direction is to deal with maximal  $L^p$ -regularity, for all  $p \in (1, \infty)$ . The third, is to assume less regularity on the operators  $\mathcal{A}(t), \mathcal{B}(t)$  with respect to  $t$  ( $t \mapsto \mathcal{A}(t), \mathcal{B}(t) \in C^{\frac{1}{2}+\varepsilon}, \varepsilon > 0$ ).

In the present paper we solve the problem of maximal  $L^2$ -regularity in  $\mathcal{H}$  under minimal assumptions on the operators. One of our main result shows that for forms satisfying the uniform Kato square root property and an in-

tegrability condition (see Theorem 3.10 for more general and precise statements), if  $t \mapsto \mathcal{A}(t), \mathcal{B}(t)$  is piecewise in the Sobolev space  $\dot{H}^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$  maximal  $L^2$ -regularity in  $\mathcal{H}$  is satisfied. The initial data  $u_0, u_1$  are arbitrary in  $\mathcal{V}$ . This result is optimal and provides the best positive result on this problem. This result allows us also to study the semilinear linear equation

$$\begin{cases} u''(t) + \mathcal{B}(t)u'(t) + \mathcal{A}(t)u(t) = F(t, u, u') & t\text{-a.e.} \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases}$$

Here  $u_0, u_1$  are arbitrary in  $\mathcal{V}$ ,  $F$  satisfies some continuity property (see Theorem 5.3 for more details) and  $F(\cdot, 0, 0) \in L^2(0, \tau; \mathcal{H})$ . We prove that there exists a unique solution to this equation, moreover  $u \in H^2(0, \tau; \mathcal{H}) \cap H^1(0, \tau; \mathcal{V})$  and we have the apriori estimate

$$\|u\|_{H^2(0, \tau; \mathcal{H}) \cap H^1(0, \tau; \mathcal{V})} \leq C \left[ \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} + \|F(\cdot, 0, 0)\|_{L^2(0, \tau; \mathcal{H})} \right].$$

This work is structured as follows. In the second section we prove some results and preparatory lemma, while in the section 3 we prove our results for maximal  $L^2$ -regularity to the considered linear non-autonomous second order Cauchy problem (1.1) and by induction, our approach allows to consider Cauchy problems of order  $N$  for any  $N \geq 3$ . In this section we prove also a decay estimate in time for the solution of (1.1). We discuss the optimality of our results in Section 4. We prove our results for the semilinear equation in section 5 and several examples are given in Section 6.

**Notation.** We denote by  $\mathcal{L}(E, F)$  (or  $\mathcal{L}(E)$ ) the space of bounded linear operators from  $E$  to  $F$  (from  $E$  to  $E$ ). The spaces  $L^p(a, b; E)$  and  $W^{k,p}(a, b; E)$  or  $H^k(a, b; E)$  if  $p = 2$  denote respectively the Lebesgue and usual Sobolev spaces of order  $k$  of function on  $(a, b)$  with values in  $E$ . For  $u \in W^{1,p}(a, b; E)$  we denote the first weak derivative by  $u'$  and for  $u \in W^{2,p}(a, b; E)$  the second derivative by  $u''$ . Recall that the norms of  $\mathcal{H}$  and  $\mathcal{V}$  are denoted by  $\|\cdot\|$  and  $\|\cdot\|_{\mathcal{V}}$ . The scalar product of  $\mathcal{H}$  is  $(\cdot, \cdot)$  and the duality  $\mathcal{V}'$ - $\mathcal{V}$  is  $\langle \cdot, \cdot \rangle$ . We denote by  $m!$  the factorial of  $m$ .

Finally, we denote by  $C, C'$  or  $C_0, C_1, c, \dots$  all inessential constants. Their values may change from line to line.

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## 2 Preparatory lemmas

In this section we prove several estimates which will play an important role in the proofs of the main results. Before we start let us point out that we

may assume without loss of generality that assumption [H3] is satisfied with  $\nu = 0$ , that is the forms are coercive with constant  $\delta > 0$  independent of  $t$ . The reason is that the maximal regularity of

$$\begin{cases} v''(t) + \mathcal{B}(t)v'(t) + \mathcal{A}(t)v(t) = f(t) & t\text{-a.e.} \\ v(0) = u_0, \quad v'(0) = u_1 + \gamma u_0 \end{cases} \quad (2.1)$$

is equivalent to the same property for

$$\begin{cases} u''(t) + (\mathcal{B}(t) + 2\gamma I)u'(t) + (\mathcal{A}(t) + \gamma\mathcal{B}(t) + \gamma^2 I)u(t) = e^{-\gamma t}f(t) & t\text{-a.e.} \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \quad (2.2)$$

for all  $\gamma \in \mathbb{C}$ . This can be seen by observing that for  $g(t) := e^{-\gamma t}f(t)$ , then  $u(t) = e^{-\gamma t}v(t)$  and clearly  $v \in H^2(0, \tau; \mathcal{H}) \cap H^1(0, \tau; \mathcal{V})$  if and only if  $u \in H^2(0, \tau; \mathcal{H}) \cap H^1(0, \tau; \mathcal{V})$  (and obviously  $f \in L^2(0, \tau; \mathcal{H})$  if and only if  $g \in L^2(0, \tau; \mathcal{H})$ ).

We deduce that we may replace  $\mathcal{B}(t)$  by  $\mathcal{B}(t) + \gamma$  and  $\mathcal{A}(t)$  by  $\mathcal{A}(t) + \gamma\mathcal{B}(t) + \gamma^2 I$ . Noting that for  $\gamma > 0$  big enough ( $\gamma > \max\{\frac{M}{\delta}, \nu\}$ ) and  $t \in [0, \tau]$ , we have that  $\mathcal{C}(t) = \mathcal{A}(t) + \gamma\mathcal{B}(t) + \gamma^2 I$  is associated with a  $\mathcal{V}$ -bounded coercive form  $\mathfrak{c}(t)$  (i.e., it satisfies [H3] with  $\nu = 0$ ). In fact, let  $u \in \mathcal{V}$ . We get

$$\begin{aligned} \operatorname{Re} \mathfrak{c}(t, u, u) &= \operatorname{Re} \langle \mathcal{A}(t)u, u \rangle + \gamma \operatorname{Re} \mathfrak{b}(t, u, u) + \gamma^2 \|u\|^2 \\ &\geq -\|\mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \|u\|_{\mathcal{V}}^2 + \gamma \delta \|u\|_{\mathcal{V}}^2 + (\gamma^2 - \gamma \nu) \|u\|^2 \\ &= -M \|u\|_{\mathcal{V}}^2 + \gamma \delta \|u\|_{\mathcal{V}}^2 + (\gamma^2 - \gamma \nu) \|u\|^2 \\ &\geq (\gamma \delta - M) \|u\|_{\mathcal{V}}^2. \end{aligned}$$

In particular, we may suppose that  $B(t)$  and  $\mathcal{B}(t)$  (resp.  $\mathcal{A}(t)$ ) are invertibles. We will do so in the sequel without mentioning it.

From [4] [Theorem III 4.10.2] we have the following lemma

**Lemma 2.1.** *Let  $E_1, E_2$  be two Banach spaces such that  $E_2 \subseteq E_1$ . Then*

$$W^{1,p}(0, \tau; E_1) \cap L^p(0, \tau; E_2) \hookrightarrow C([0, \tau]; (E_1, E_2)_{1-\frac{1}{p}, p}).$$

From [36] (Theorem 1.52 and Theorem 1.55), we have the following lemma which point out that the constants involved in the estimates are uniform with respect to  $t$

**Lemma 2.2.** *For any  $t \in [0, \tau]$ , the operators  $-B(t)$  and  $-\mathcal{B}(t)$  generate strongly continuous analytic semigroups of angle  $\gamma = \frac{\pi}{2} - \arctan(\frac{M}{\delta})$  on  $\mathcal{H}$  and  $\mathcal{V}'$ , respectively. In addition, there exist constant  $C_\theta$ , independent of  $t$ , such that*

$$\|(z + \mathcal{B}(t))^{-1}\|_{\mathcal{L}(Y)} \leq \frac{C_\theta}{1 + |z|} \text{ for all } z \in \Sigma_{\pi-\theta} \text{ with fixed } \theta < \gamma.$$

Here,  $Y = \mathcal{H}, \mathcal{V}$  or  $\mathcal{V}'$ .

All of the previous estimates holds for the adjoint operator  $\mathcal{B}(t)^*$ .

For  $f \in L^2(0, \tau; \mathcal{H})$  and for almost every  $t \in [0, \tau]$  we define the operator  $L$  by

$$L(f)(t) := B(t) \int_0^t e^{-(t-s)B(t)} f(s) ds.$$

Our aim is to prove  $L \in \mathcal{L}(L^2(0, \tau; \mathcal{H}))$ . It is proved in [27] that  $L$  is bounded on  $L^p(0, \tau; \mathcal{H})$  for all  $p \in (1, \infty)$  provided  $t \mapsto \mathfrak{a}(t, \cdot, \cdot)$  is  $C^\epsilon$  for some  $\epsilon > 0$  (or similarly,  $t \mapsto \mathcal{A}(t)$  is  $C^\epsilon$  on  $[0, \tau]$  with values in  $\mathcal{L}(\mathcal{V}, \mathcal{V}')$ ). The proof for the case  $p = 2$  is based on vector-valued pseudo-differential operators. The extension from  $p = 2$  to  $p \in (1, \infty)$  uses Hörmander's almost  $L^1$ -condition for singular integral operators. Here we give a direct proof for the case  $p = 2$  which does not appeal to pseudo-differential operators. It is essentially based on Lions' theorem and the holomorphic functional calculus for the sectorial operator.

**Lemma 2.3.** *Assume that  $\mathcal{B}(\cdot) \in \dot{H}^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ . Then  $L \in \mathcal{L}(L^2(0, \tau; \mathcal{V}'))$ .*

*Proof.* By Lions' theorem we obtain that for all  $f \in L^2(0, \tau; \mathcal{V}')$  there exists a unique  $u \in MR(\mathcal{V}', \mathcal{V}) := H^1(0, \tau; \mathcal{V}') \cap L^2(0, \tau; \mathcal{V})$  be the solution to the Cauchy problem

$$\begin{cases} u'(t) + \mathcal{B}(t)u(t) = f(t) \\ u(0) = 0. \end{cases} \quad (2.3)$$

In addition,

$$\|u\|_{MR(\mathcal{V}', \mathcal{V})} \leq C(\delta, M) \|f\|_{L^2(0, \tau; \mathcal{V}')} \quad (2.4)$$

Set  $v(s) = e^{-(t-s)B(t)}u(s)$  where  $0 \leq s \leq t \leq \tau$ . We remark that  $v(t) = u(t)$ ,  $v(0) = 0$  and

$$\begin{aligned} v'(s) &= B(t)e^{-(t-s)B(t)}u(s) + e^{-(t-s)B(t)}u'(s) \\ &= e^{-(t-s)B(t)}(\mathcal{B}(t) - \mathcal{B}(s))u(s) + e^{-(t-s)B(t)}f(s). \end{aligned}$$

Since  $u(t) = \int_0^t v'(s) ds$ , we have

$$\begin{aligned} \mathcal{B}(t)u(t) &= \mathcal{B}(t) \int_0^t e^{-(t-s)B(t)}(\mathcal{B}(t) - \mathcal{B}(s))u(s) ds + \mathcal{B}(t) \int_0^t e^{-(t-s)B(t)}f(s) ds \\ &:= (S(\mathcal{B}(\cdot)u))(t) + L(f)(t). \end{aligned}$$

Therefore,  $L(f)(t) = (I - S)(\mathcal{B}(\cdot)u)$ .

Noting that

$$(S(g))(t) := \mathcal{B}(t) \int_0^t e^{-(t-s)B(t)} (\mathcal{B}(t) - \mathcal{B}(s)) \mathcal{B}(s)^{-1} g(s) ds,$$

where  $g \in L^2(0, \tau; \mathcal{V}')$ .

Using the analyticity of the semigroup and Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \|(S(g))(t)\|_{\mathcal{V}'} &\leq \int_0^t \|\mathcal{B}(t) e^{-(t-s)B(t)}\|_{\mathcal{L}(\mathcal{V}')} \|\mathcal{B}(t) - \mathcal{B}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \|\mathcal{B}(s)^{-1}\|_{\mathcal{L}(\mathcal{V}', \mathcal{V})} \|g(s)\|_{\mathcal{V}'} ds \\ &\lesssim \int_0^t \frac{\|\mathcal{B}(t) - \mathcal{B}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}}{t-s} \|g(s)\|_{\mathcal{V}'} ds. \\ &\lesssim \left( \int_0^t \frac{\|\mathcal{B}(t) - \mathcal{B}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}^2}{(t-s)^2} ds \right)^{\frac{1}{2}} \|g\|_{L^2(0, \tau; \mathcal{V}')} . \end{aligned}$$

So

$$\|(S(g))\|_{L^2(0, \tau; \mathcal{V}')} \lesssim \|\mathcal{B}\|_{\dot{H}^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))} \|g\|_{L^2(0, \tau; \mathcal{V}')}.$$

Therefore

$$\begin{aligned} \|L(f)\|_{L^2(0, \tau; \mathcal{V}')} &\lesssim \left[ \|(S(\mathcal{B}(\cdot)u))\|_{L^2(0, \tau; \mathcal{V}')} + \|\mathcal{B}(\cdot)u\|_{L^2(0, \tau; \mathcal{V}')} \right] \\ &\lesssim \left( \|\mathcal{B}\|_{\dot{H}^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))} + 1 \right) \|u\|_{L^2(0, \tau; \mathcal{V})}. \end{aligned}$$

Now, we use the estimate (2.4) to get  $L \in \mathcal{L}(L^2(0, \tau; \mathcal{V}'))$  and

$$\|L(f)\|_{L^2(0, \tau; \mathcal{V}')} \lesssim \|f\|_{L^2(0, \tau; \mathcal{V}')}.$$

□

For almost every  $s \in [0, \tau]$  we define the operator

$$L_1(f)(s) := \mathcal{B}(s)^* \int_s^\tau e^{-(t-s)\mathcal{B}^*(s)} f(t) dt$$

where  $f \in L^2(0, \tau; \mathcal{V}')$ .

**Lemma 2.4.** *If  $t \rightarrow \mathcal{B}(t)$  is measurable, then for all  $f \in L^2(0, \tau; \mathcal{V}')$  there exists a unique  $u \in MR(\mathcal{V}', \mathcal{V})$  be the solution to the retrograde problem*

$$\begin{cases} u'(s) - \mathcal{B}(s)^* u(s) = f(s) \\ u(\tau) = 0. \end{cases} \quad (2.5)$$

*Proof.* We define the multiplication operator  $\mathbb{B}$  on  $L^2(0, \tau; \mathcal{V}')$  by  $D(\mathbb{B}) = L^2(0, \tau; \mathcal{V})$  and  $(\mathbb{B}u)(t) = \mathcal{B}(t)u(t)$ , for  $t \in [0, \tau]$  and  $u \in L^2(0, \tau; \mathcal{V})$ . We define also the differentiation operator  $\partial$  on  $L^2(0, \tau; \mathcal{V}')$  with domain

$$D(\partial) := \{v \in H^1(0, \tau; \mathcal{V}') \text{ s.t } v(0) = 0\},$$

by  $(\partial u)(t) = u'(t)$ .

To prove our result it is enough to prove

$$\partial(\partial - \mathbb{B}^*)^{-1} \in \mathcal{L}(L^2(0, \tau; \mathcal{V}')).$$

By Lions' theorem we have  $\partial + \mathbb{B}$  is invertible and

$$(\partial + \mathbb{B})^{-1} \in \mathcal{L}(L^2(0, \tau; \mathcal{V}'), L^2(0, \tau; \mathcal{V})).$$

So  $(\partial - \mathbb{B}^*)^{-1} \in \mathcal{L}(L^2(0, \tau; \mathcal{V}'), L^2(0, \tau; \mathcal{V}))$  and

$$\partial(\partial - \mathbb{B}^*)^{-1} = I + \mathbb{B}^*(\partial - \mathbb{B}^*)^{-1} \in \mathcal{L}(L^2(0, \tau; \mathcal{V}')).$$

Then we get the desired result. We can also use the Lax-Milgram lemma to prove this result.  $\square$

**Lemma 2.5.** *If  $\mathcal{B}(\cdot) \in \dot{H}^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ , we have  $L_1 \in \mathcal{L}(L^2(0, \tau; \mathcal{V}'))$ .*

*Proof.* For the proof we use the maximal regularity for the retrograde adjoint problem in  $\mathcal{V}'$

$$\begin{cases} u'(s) - \mathcal{B}(s)^* u(s) = f(s) \\ u(\tau) = 0 \end{cases} \quad (2.6)$$

We set  $v(t) = e^{-(t-s)\mathcal{B}(s)^*} u(t)$ , with  $0 \leq s \leq t \leq \tau$ . Noting that  $u(s) = v(s)$ ,  $v(\tau) = 0$  and

$$v'(t) = e^{-(t-s)\mathcal{B}(s)^*} (\mathcal{B}(t)^* - \mathcal{B}(s)^*) u(t) + e^{-(t-s)\mathcal{B}(s)^*} f(t).$$

Thus,

$$\mathcal{B}(s)^* u(s) = \mathcal{B}(s)^* \int_s^\tau e^{-(t-s)\mathcal{B}(s)^*} (\mathcal{B}(t)^* - \mathcal{B}(s)^*) u(t) dt + L_1(f)(s).$$

Now we follow the same strategy of proof of Lemma 2.3 to get the desired result.  $\square$

**Lemma 2.6.** *For all  $f \in L^2(0, \tau; \mathcal{H})$ ,  $s \in (0, \tau)$  one has*

$$L^*(f)(s) = \int_s^\tau \mathcal{B}^*(t) e^{-(t-s)\mathcal{B}^*(t)} f(t) dt.$$

*Proof.* Let  $f, g \in L^2(0, \tau; \mathcal{H})$ . We obtain by Fubini's theorem

$$\begin{aligned} \int_0^\tau (L(g)(t), f(t)) dt &= \int_0^\tau (B(t) \int_0^t e^{-(t-s)B(t)} g(s) ds, f(t)) dt \\ &= \int_0^\tau \int_0^t (B(t) e^{-(t-s)B(t)} g(s), f(t)) ds dt \\ &= \int_0^\tau \int_s^\tau (g(s), B(t)^* e^{-(t-s)B(t)^*} f(t)) dt ds \\ &= \int_0^\tau (g(s), \int_s^\tau B(t)^* e^{-(t-s)B(t)^*} f(t) dt) ds. \end{aligned}$$

Therefore

$$L^*(f)(s) = \int_s^\tau \mathcal{B}^*(t) e^{-(t-s)\mathcal{B}^*(t)} f(t) dt, s \in (0, \tau).$$

□

**Lemma 2.7.** *If  $\mathcal{B}(\cdot) \in \dot{H}^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ , we have  $L^* \in \mathcal{L}(L^2(0, \tau; \mathcal{V}'))$ .*

*Proof.* Let  $s \in [0, \tau]$  and  $t \in [s, \tau]$ . Choose a contour  $\Gamma$  in the positive half-plane and write by the holomorphic functional calculus for the sectorial operators  $\mathcal{B}^*(t), \mathcal{B}^*(s)$

$$\mathcal{B}^*(t) e^{-t\mathcal{B}^*(t)} - \mathcal{B}^*(s) e^{-t\mathcal{B}^*(s)} = \frac{1}{2\pi i} \int_\Gamma \lambda e^{-t\lambda} (\lambda I - \mathcal{B}^*(t))^{-1} (\mathcal{B}^*(t) - \mathcal{B}^*(s)) (\lambda I - \mathcal{B}^*(s))^{-1} d\lambda.$$

By taking the norm in  $\mathcal{L}(\mathcal{V}')$  we have by Lemma 2.1

$$\begin{aligned} &\|\mathcal{B}^*(t) e^{-t\mathcal{B}^*(t)} - \mathcal{B}^*(s) e^{-t\mathcal{B}^*(s)}\|_{\mathcal{L}(\mathcal{V}')} \\ &\leq C \int_0^\infty |\lambda| e^{-t|\cos \gamma||\lambda|} \|(\lambda I - \mathcal{B}^*(t))^{-1}\|_{\mathcal{L}(\mathcal{V}')} \\ &\times \|(\lambda I - \mathcal{B}^*(s))^{-1}\|_{\mathcal{L}(\mathcal{V}; \mathcal{V}')} d|\lambda| \|\mathcal{B}^*(t) - \mathcal{B}^*(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \\ &\leq C_1 \int_0^\infty e^{-t|\cos \gamma||\lambda|} d|\lambda| \|\mathcal{B}^*(t) - \mathcal{B}^*(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} . \end{aligned}$$

Since,

$$\int_0^\infty e^{-t|\cos \gamma||\lambda|} d|\lambda| = \frac{1}{|\cos \gamma|t},$$

we obtain

$$\|\mathcal{B}^*(t) e^{-(t-s)\mathcal{B}^*(t)} - \mathcal{B}^*(s) e^{-(t-s)\mathcal{B}^*(s)}\|_{\mathcal{L}(\mathcal{V}')} \lesssim \frac{\|\mathcal{B}^*(s) - \mathcal{B}^*(t)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}}{t-s}.$$

Hence,

$$\|L^*(f)(s) - L_1(f)(s)\|_{\mathcal{V}'} \lesssim \int_s^\tau \frac{\|\mathcal{B}^*(s) - \mathcal{B}^*(t)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}}{t-s} \|f(t)\|_{\mathcal{V}'} dt.$$

The Cauchy-Schwarz inequality gives

$$\|(L^* - L_1)(f)\|_{L^2(0,\tau;\mathcal{V}')} \lesssim \|\mathcal{B}^*\|_{\dot{H}^{\frac{1}{2}}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|f\|_{L^2(0,\tau;\mathcal{V}')}.$$

Then

$$\begin{aligned} \|L^*(f)\|_{L^2(0,\tau;\mathcal{V}')} &\leq 2\|[L^* - L_1](f)\|_{L^2(0,\tau;\mathcal{V}')} + 2\|L_1(f)\|_{L^2(0,\tau;\mathcal{V}')} \\ &\lesssim \left(\|\mathcal{B}\|_{\dot{H}^{\frac{1}{2}}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))} + 1\right) \|f\|_{L^2(0,\tau;\mathcal{V}')} \end{aligned}$$

□

**Proposition 2.8.** *If  $\mathcal{B}(\cdot) \in \dot{H}^{\frac{1}{2}}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))$ , we obtain  $L \in \mathcal{L}(L^2(0,\tau;\mathcal{H}))$ .*

*Proof.* Since by Lemma 2.3,  $L \in \mathcal{L}(L^2(0,\tau;\mathcal{V}'))$  and by Lemma 2.7,  $L^* \in \mathcal{L}(L^2(0,\tau;\mathcal{V}'))$ , one has  $L \in \mathcal{L}(L^2(0,\tau;\mathcal{V}))$ . Then by interpolation we get  $L \in \mathcal{L}(L^2(0,\tau;\mathcal{H}))$ . □

**Lemma 2.9.** *Assume that,*

$$\begin{aligned} \int_0^\tau \frac{\|\mathcal{B}(t) - \mathcal{B}(0)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}^2}{t} dt &< \infty, \\ \int_0^\tau \frac{\|\mathcal{A}(t) - \mathcal{A}(0)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}^2}{t} dt &< \infty. \end{aligned}$$

Let  $u_1, u_0 \in \mathcal{V}$ , then the operators

$$R_1 u_1(t) = B(t) e^{-tB(t)} u_1$$

$$R_2 u_0(t) = e^{-tB(t)} \mathcal{A}(t) u_0$$

are bounded in  $L^2(0,\tau;\mathcal{H})$ .

*Proof.* For the operator  $R_1$  we refer to [3][Lemma 4.7]. It remains to prove that  $R_2 \in \mathcal{L}(\mathcal{V}, L^2(0,\tau;\mathcal{H}))$ . Indeed, observing that

$$\begin{aligned} R_2 u_0(t) &= e^{-tB(t)} \mathcal{A}(t) u_0 \\ &= e^{-tB(t)} (\mathcal{A}(t) - \mathcal{A}(0)) u_0 + e^{-tB(t)} \mathcal{A}(0) u_0. \end{aligned} \tag{2.7}$$

For the first term in the RHS of (2.7), we have

$$\begin{aligned} &\int_0^\tau \|e^{-tB(t)} (\mathcal{A}(t) - \mathcal{A}(0)) u_0\|^2 dt \\ &\leq \int_0^\tau \|e^{-tB(t)}\|_{\mathcal{L}(\mathcal{V},\mathcal{H})}^2 \|\mathcal{A}(t) - \mathcal{A}(0)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}^2 \|u_0\|_{\mathcal{V}}^2 dt \\ &\leq C \int_0^\tau \frac{\|\mathcal{A}(t) - \mathcal{A}(0)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}^2}{t} dt \|u_0\|_{\mathcal{V}}^2. \end{aligned}$$

We write

$$e^{-t\mathcal{B}(t)}\mathcal{A}(0)u_0 = e^{-t\mathcal{B}(t)}\mathcal{A}(0)u_0 - e^{-tB(0)}\mathcal{A}(0)u_0 + e^{-tB(0)}\mathcal{A}(0)u_0.$$

The functional calculus for the sectorial operators  $B(t), B(0)$  gives

$$\|e^{-t\mathcal{B}(t)}\mathcal{A}(0)u_0 - e^{-tB(0)}\mathcal{A}(0)u_0\| \leq c \frac{\|\mathcal{B}(t) - \mathcal{B}(0)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}^2}{t^{\frac{1}{2}}} \|u_0\|_{\mathcal{V}}.$$

Clearly,

$$e^{-t\mathcal{B}(0)}\mathcal{A}(0)u_0 = B(0)^{\frac{1}{2}}e^{-tB(0)}\mathcal{B}(0)^{-\frac{1}{2}}\mathcal{A}(0)u_0.$$

Hence, by Lemma 3.4 we have

$$\begin{aligned} & \int_0^\tau \|e^{-t\mathcal{B}(0)}\mathcal{A}(0)u_0\|^2 dt \\ &= \int_0^\tau \|B(0)^{\frac{1}{2}}e^{-tB(0)}\mathcal{B}(0)^{-\frac{1}{2}}\mathcal{A}(0)u_0\|^2 dt \\ &\leq c\|\mathcal{B}(0)^{-\frac{1}{2}}\mathcal{A}(0)u_0\|^2. \end{aligned}$$

Since  $D(B(0)^{\frac{1}{2}}) = \mathcal{V}$ , we have  $D(\mathcal{B}(0)^{\frac{1}{2}}) = \mathcal{H}$  and  $\mathcal{B}(0)^{-\frac{1}{2}} \in \mathcal{L}(\mathcal{V}', \mathcal{H})$  (see Lemma ??).

Therefore

$$\int_0^\tau \|e^{-t\mathcal{B}(0)}\mathcal{A}(0)u_0\|^2 dt \leq C\|u_0\|_{\mathcal{V}}^2.$$

Thus,  $\|R_2 u_0\|_{L^2(0, \tau; \mathcal{H})} \leq C\|u_0\|_{\mathcal{V}}$  □

For  $0 \leq t \leq \tau$  define the operator

$$L_t f = \int_0^t e^{-(t-r)B(t)} f(r) dr, f \in L_2(0, \tau; \mathcal{H}).$$

In the following lemma we prove that the assumption (3.3) is necessary to get  $L_t \in \mathcal{L}(L^2(0, t; \mathcal{H}); \mathcal{V})$  for all  $t \in [0, \tau]$ .

**Lemma 2.10.** *1- Suppose (3.3). Then for all  $f \in L_2(0, \tau; \mathcal{H})$ ,  $0 \leq t \leq \tau$ ,*

$$\|L_t f\|_{\mathcal{V}} \leq C\|f\|_{L_2(0, t; \mathcal{H})}.$$

*2- If  $\mathcal{V} \subsetneq D(B(t)^{\frac{1}{2}})$  for some  $t \in (0, \tau]$ , it follows that  $L_t \notin \mathcal{L}(L^2(0, t; \mathcal{H}); \mathcal{V})$ .*

*Proof.* For the first assertion we refer to [3][Lemma 4.1]. Take  $v \in \mathcal{H}$ . Then

$$\begin{aligned} (L_t f; v) &= \int_0^t (f(s); e^{-(t-s)B(t)*} v) ds \\ &= \int_0^t (f(s); L_1^*(t, s)v) ds. \end{aligned}$$

So the adjoint operator of  $L_t$  is defined in  $L^2(0, t; \mathcal{H})$  by  $L_1^*(t, s)v = e^{-(t-s)A(t)^*}v$ . If  $\mathcal{V} \subsetneq D(B(t)^{\frac{1}{2}})$ , we want to prove that  $L_t \notin \mathcal{L}(L^2(0, t; \mathcal{H}); \mathcal{V})$ . Hence, it is equivalent to prove  $L_1^*(t, \cdot) \notin \mathcal{L}(\mathcal{V}'; L^2(0, t; \mathcal{H}))$ . Let  $v \in \mathcal{V}'$  and set  $u = \mathcal{B}(t)^{*-1}v \in \mathcal{V}$ . Therefore, by the definition of the real interpolation space (see e.g. [33], Proposition 5.1.1) we get for  $t \in (0, \tau]$

$$\begin{aligned} \|u\|^2 + \|L_1^*(t, \cdot)v\|_{L^2(0, t; \mathcal{H})}^2 &= \|u\|^2 + \int_0^t \|B(s)^*e^{-sB(s)^*}u\|^2 ds \\ &\simeq \|u\|_{D(B(t)^{\frac{1}{2}})}^2. \end{aligned}$$

Since  $\mathcal{V} \subsetneq D(B(t)^{\frac{1}{2}})$  we have by [1][Lemma 6.4] and [35][Theorem 1] that  $D(B(t)^{\frac{1}{2}}) \subsetneq \mathcal{V}$  and  $\mathcal{V} \setminus D(B(t)^{\frac{1}{2}})$  is non empty. Hence, there exists a  $u \in \mathcal{V} \setminus D(B(t)^{\frac{1}{2}})$  and for  $v = \mathcal{B}(t)^*u \in \mathcal{V}'$  we get  $L_1^*(t, \cdot)v$  is not bounded in  $L^2(0, t; \mathcal{H})$ . Therefore  $L_t \notin \mathcal{L}(L^2(0, t; \mathcal{H}); \mathcal{V})$ .  $\square$

### 3 Maximal Regularity under Fractional Sobolev Regularity in time

In this section we prove our main result on maximal regularity for the linear equation.

Let us define the spaces

$$\begin{aligned} MR(\mathcal{V}, \mathcal{H}) &:= \{u \in H^2(0, \tau; \mathcal{H}) \cap W^{1, \infty}(0, \tau; \mathcal{V}) : \mathcal{B}(\cdot)u' + \mathcal{A}(\cdot)u \in L^2(0, \tau; \mathcal{H})\}. \\ Tr(\mathcal{V}, \mathcal{H}) &:= \{(u(0), u'(0)) : u \in MR(\mathcal{V}, \mathcal{H})\}, \end{aligned}$$

endowed with norms respectively

$$\begin{aligned} \|u\|_{MR(\mathcal{V}, \mathcal{H})} &:= \|u''\|_{L^2(0, \tau; \mathcal{H})} + \|u\|_{W^{1, \infty}(0, \tau; \mathcal{V})} \\ &\quad + \|\mathcal{B}(\cdot)u'(\cdot) + \mathcal{A}(\cdot)u(\cdot)\|_{L^2(0, \tau; \mathcal{H})}. \end{aligned}$$

$$\begin{aligned} \|(u(0), u'(0))\|_{Tr(\mathcal{V}, \mathcal{H})} &:= \inf\{\|v\|_{MR(\mathcal{V}, \mathcal{H})} : \\ &\quad v \in MR(\mathcal{V}, \mathcal{H}), v(0) = u(0), v'(0) = u'(0)\}. \end{aligned}$$

From [1][Theorem 2.6] we have the following theorem

**Theorem 3.1.** *Assume that  $t \rightarrow \mathbf{b}(t)$  is measurable. Then for all  $f \in L^2(0, \tau; \mathcal{V}')$  and  $(u_0, u_1) \in (\mathcal{V}, \mathcal{H})$ , there exists a unique solution  $u \in H^2(0, \tau; \mathcal{V}') \cap H^1(0, \tau; \mathcal{V})$  to the problem*

$$\begin{cases} u''(t) + \mathcal{B}(t)u'(t) + \mathcal{A}(t)u(t) = f(t) & t\text{-a.e.} \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \quad (3.1)$$

Moreover, there exists a positive constant  $C$  independent of  $u_0, u_1$  and  $f$  such that the following estimate holds

$$\|u\|_{H^2(0,\tau;\mathcal{V}') \cap H^1(0,\tau;\mathcal{V})} \leq C \left[ \|(u_0, u_1)\|_{(\mathcal{V}, \mathcal{H})} + \|f\|_{L^2(0,\tau;\mathcal{V})} \right]. \quad (3.2)$$

**Remark 3.2.** Noting that

$$H^2(0, \tau; \mathcal{V}') \cap H^1(0, \tau; \mathcal{V}) \hookrightarrow C^1([0, \tau]; \mathcal{H}) \cap C([0, \tau]; \mathcal{V}).$$

Indeed, let  $v \in H^2(0, \tau; \mathcal{V}') \cap H^1(0, \tau; \mathcal{V})$ , it follows by the Sobolev embedding that  $v \in C([0, \tau]; \mathcal{V})$  and since  $v' \in H^1(0, \tau; \mathcal{V}') \cap L^2(0, \tau; \mathcal{V}')$  we obtain by Lemma 2.1 that  $v \in C^1([0, \tau]; \mathcal{H})$ . Thus, since  $u \in H^2(0, \tau; \mathcal{V}') \cap H^1(0, \tau; \mathcal{V})$ , both  $u(0)$  and  $u'(0)$  make sense.

Following [3](Definition 3.4), we introduce the following definition

**Definition 3.3.** We say that  $(B(t))_{t \in [0, \tau]}$  satisfies the uniform Kato square root property if  $D(B(t)^{\frac{1}{2}}) = \mathcal{V}$  for all  $t \in [0, \tau]$  and there are  $c_1, c^1 > 0$  such that for all  $v \in \mathcal{V}$

$$c_1 \|v\|_{\mathcal{V}} \leq \|B(t)^{\frac{1}{2}} v\| \leq c^1 \|v\|_{\mathcal{V}}. \quad (3.3)$$

The uniform Kato square root property is obviously satisfied for symmetric forms. It is also satisfied for uniformly elliptic operators (not necessarily symmetric)

$$B(t) = - \sum_{k,l=1}^d \partial_k (a_{kl}(t, x) \partial_l)$$

on  $L^2(\mathbb{R}^d)$  since  $\|\nabla u\|_2$  is equivalent to  $\|B(t)^{\frac{1}{2}} u\|_2$  with constants depending only on the dimension and the ellipticity constants, see [11].

The next lemma shows the quadratic estimate for  $B(t)$  with constant independent of  $t$ . Here we assume the uniform Kato square root property. Quadratic estimates are an important tool in harmonic analysis and we will use them at several places in the proofs of maximal regularity.

**Lemma 3.4.** Suppose in addition to [H1]-[H3] (with  $\nu = 0$ ) that the uniform Kato square root property is satisfied. Then there exists a constant  $C$  such that for every  $t \in [0, \tau]$

$$\int_0^\tau \|B(t)^{\frac{1}{2}} e^{-sB(t)} x\|^2 ds \leq C \|x\|^2 \quad (3.4)$$

for all  $x \in \mathcal{H}$ .

**Definition 3.5.** Let  $X$  be a Banach space and  $\alpha \in (0, 1)$ ,

- A Bochner measurable function  $f : [0, \tau] \rightarrow X$ , lies in the homogeneous fractional Sobolev space  $\dot{H}^\alpha(0, \tau; X)$  provided

$$\|f\|_{\dot{H}^\alpha(0,\tau;X)}^2 := \int_0^\tau \int_0^\tau \frac{\|f(t) - f(s)\|_X^2}{|t - s|^{2\alpha+1}} ds dt < \infty.$$

- A Bochner measurable function  $f : [0, \tau] \rightarrow X$ , lies in the space  $\dot{H}_0^{\frac{1}{2}}(0, \tau; X)$  provided

$$\|f\|_{\dot{H}_0^{\frac{1}{2}}(0, \tau; X)}^2 := \sup_{t \in [0, \tau]} \int_0^\tau \frac{\|f(t) - f(s)\|_X^2}{|t - s|} ds + \int_0^\tau \int_0^\tau \frac{\|f(t) - f(s)\|_X^2}{|t - s|^2} ds dt < \infty.$$

We shall say that  $f$  is piecewise in  $\dot{H}^\alpha(I; X)$  (resp.  $\dot{H}_0^{\frac{1}{2}}(I; X)$ ) if there exists  $t_0 < t_1 < \dots < t_n$  such that  $I = \cup_i [t_i, t_{i+1}]$  and the restriction of  $f$  to each sub-interval  $(t_i, t_{i+1})$  is in  $\dot{H}^\alpha(t_i, t_{i+1}; X)$  (resp.  $\dot{H}_0^{\frac{1}{2}}(t_i, t_{i+1}; X)$ ).

Let  $\mathbf{b}(t) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$  for  $0 \leq t \leq \tau$  be a family of forms satisfying [H1]-[H3] and let  $B(t)$  and  $\mathcal{B}(t)$  be the associated operators on  $\mathcal{H}$  and  $\mathcal{V}'$ , respectively. We shall need the following property.

Given  $\varepsilon > 0$ , there exists  $\tau_0 = 0 < \tau_1 < \dots < \tau_n = \tau$  such that

$$\sup_{t \in (\tau_{i-1}, \tau_i)} \int_{\tau_{i-1}}^{\tau_i} \frac{\|\mathcal{B}(t) - \mathcal{B}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}^2}{|t - s|} ds < \varepsilon. \quad (3.5)$$

Note that this assumption is satisfied in many cases. Suppose for example that  $t \mapsto \mathbf{b}(t, u, v)$  is  $C^\alpha$  for some  $\alpha > 0$  in the sense that

$$|\mathbf{b}(t, u, v) - \mathbf{b}(s, u, v)| \leq M|t - s|^\alpha \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}} \quad (3.6)$$

for some positive constant  $M$  and all  $u, v \in \mathcal{V}$ . Then clearly

$$\|\mathcal{B}(t) - \mathcal{B}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \leq M|t - s|^\alpha$$

and this implies (3.5). More generally, if  $\omega_i$  denotes the modulus of continuity of  $\mathcal{B}(\cdot)$  on the interval  $(\tau_{i-1}, \tau_i)$  then (3.5) is satisfied if

$$\int_{\tau_{i-1}}^{\tau_i} \frac{\omega_i(r)^2}{r} dr < \varepsilon. \quad (3.7)$$

**Proposition 3.6.** *Let  $v \in L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})$  and  $\mathcal{C}(\cdot) \in L^\infty(\tau_{i-1}, \tau_i; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ . We define the operator  $Q_{\mathcal{C}(\tau_{i-1}, \tau_i)}$  by*

$$(Q_{\mathcal{C}(\tau_{i-1}, \tau_i)} v)(t) = \int_{\tau_{i-1}}^t e^{-(t-s)B(t)} (\mathcal{C}(t) - \mathcal{C}(s)) v(s) ds,$$

where  $t \in [\tau_{i-1}, \tau_i]$ . Assume that  $\mathcal{C}(\cdot)$  satisfies the condition (3.5) for  $\varepsilon$  small enough. Therefore,  $Q_{\mathcal{C}(\tau_{i-1}, \tau_i)} \in \mathcal{L}(L^\infty(\tau_{i-1}, \tau_i; \mathcal{V}))$  and

$$\|Q_{\mathcal{C}(\tau_{i-1}, \tau_i)}\|_{\mathcal{L}(L^\infty(\tau_{i-1}, \tau_i; \mathcal{V}))} < 1.$$

*Proof.* For  $v \in L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})$ , we have for  $t \in (\tau_{i-1}, \tau_i)$

$$\begin{aligned}
\|(Q_{\mathcal{C}(\tau_{i-1}, \tau_i)} v)(t)\|_{\mathcal{V}} &= \sup_{\|w\|_{\mathcal{V}'}=1} \left\langle \int_{\tau_{i-1}}^t e^{-(t-s)\mathcal{B}(t)} (\mathcal{C}(t) - \mathcal{C}(s)) v(s) ds, w \right\rangle \\
&= \sup_{\|w\|_{\mathcal{V}'}=1} \int_{\tau_{i-1}}^t \left\langle e^{-\frac{(t-s)}{2}\mathcal{B}(t)} (\mathcal{C}(t) - \mathcal{C}(s)) v(s), \mathcal{B}(t)^{* \frac{1}{2}} e^{-\frac{(t-s)}{2}\mathcal{B}(t)^*} \mathcal{B}(t)^{* - \frac{1}{2}} w \right\rangle ds \\
&\leq \sup_{\|w\|_{\mathcal{V}'}=1} \int_{\tau_{i-1}}^t \|\mathcal{C}(t) - \mathcal{C}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \|e^{-\frac{(t-s)}{2}\mathcal{B}(t)}\|_{\mathcal{L}(\mathcal{V}', \mathcal{H})} \\
&\quad \times \|\mathcal{B}(t)^{* \frac{1}{2}} e^{-\frac{(t-s)}{2}\mathcal{B}(t)^*} \mathcal{B}(t)^{* - \frac{1}{2}} w\| ds \|v\|_{L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})} \\
&\leq \sup_{t \in [\tau_{i-1}, \tau_i]} \left( \int_{\tau_{i-1}}^t \|\mathcal{C}(t) - \mathcal{C}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}^2 \|e^{-\frac{(t-s)}{2}\mathcal{B}(t)}\|_{\mathcal{L}(\mathcal{V}', \mathcal{H})}^{\frac{1}{2}} ds \right)^{\frac{1}{2}} \\
&\quad \times \sup_{\|w\|_{\mathcal{V}'}=1} \left( \int_{\tau_{i-1}}^t \|\mathcal{B}(t)^{* \frac{1}{2}} e^{-\frac{(t-s)}{2}\mathcal{B}(t)^*} \mathcal{B}(t)^{* - \frac{1}{2}} w\|^2 ds \right)^{\frac{1}{2}} \|v\|_{L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})} \\
&\leq C \sup_{t \in [\tau_{i-1}, \tau_i]} \left( \int_{\tau_{i-1}}^t \frac{\|\mathcal{C}(t) - \mathcal{C}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}^2}{|t-s|} ds \right)^{\frac{1}{2}} \\
&\quad \times \sup_{\|w\|_{\mathcal{V}'}=1} \left( \int_{\tau_{i-1}}^t \|\mathcal{B}(t)^{* \frac{1}{2}} e^{-\frac{(t-s)}{2}\mathcal{B}(t)^*} \mathcal{B}(t)^{* - \frac{1}{2}} w\|^2 ds \right)^{\frac{1}{2}} \|v\|_{L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})} \\
&\leq C_2 \sup_{t \in [\tau_{i-1}, \tau_i]} \left( \int_{\tau_{i-1}}^t \frac{\|\mathcal{C}(t) - \mathcal{C}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}^2}{|t-s|} ds \right)^{\frac{1}{2}} \|v\|_{L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})}.
\end{aligned}$$

Then

$$\|(Q_{\mathcal{C}(\tau_{i-1}, \tau_i)} v)\|_{L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})} \leq C_2 \sup_{t \in [\tau_{i-1}, \tau_i]} \left( \int_{\tau_{i-1}}^t \frac{\|\mathcal{C}(t) - \mathcal{C}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}^2}{|t-s|} ds \right)^{\frac{1}{2}} \|v\|_{L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})}.$$

Using (3.5), we get for  $\varepsilon < \frac{1}{C_2}$

$$\begin{aligned}
\|Q_{\mathcal{C}(\tau_{i-1}, \tau_i)}\|_{\mathcal{L}(L^\infty(\tau_{i-1}, \tau_i; \mathcal{V}))} &\leq C_2 \varepsilon \|v\|_{L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})} \\
&< \|v\|_{L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})}.
\end{aligned}$$

This finishes the proof. □

In the following proposition we set  $Q_{\mathcal{C}} = Q_{\mathcal{C}(0, \tau)}$

**Proposition 3.7.** *Let  $v \in L^\infty(0, \tau; \mathcal{V})$  and assume that  $\mathcal{C} \in \dot{H}^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ . Then  $B(\cdot)Q_{\mathcal{C}}v \in L^2(0, \tau; \mathcal{H})$ .*

*Proof.* For  $v \in L^\infty(0, \tau; \mathcal{V})$ , we obtain

$$\begin{aligned}
& \| (B(\cdot)Q_C v) \|_{L^2(0, \tau; \mathcal{H})} \\
&= \sup_{\|g\|_{L^2(0, \tau; \mathcal{H})}=1} \left| \int_0^\tau \int_0^t (B(t)e^{-(t-s)B(t)}(\mathcal{C}(t) - \mathcal{C}(s))v(s); g(t)) ds dt \right| \\
&= \sup_{\|g\|_{L^2(0, \tau; \mathcal{H})}=1} \left| \int_0^\tau \int_0^t (B(t)^{\frac{1}{2}}e^{-\frac{(t-s)}{2}B(t)}(\mathcal{C}(t) - \mathcal{C}(s))v(s); B(t)^{* \frac{1}{2}}e^{-\frac{(t-s)}{2}B(t)^*}g(t)) ds dt \right| \\
&\lesssim \sup_{\|g\|_{L^2(0, \tau; \mathcal{H})}=1} \int_0^\tau \int_0^t \frac{\|(\mathcal{C}(t) - \mathcal{C}(s))\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}}{|t-s|} \\
&\times \|B(t)^{* \frac{1}{2}}e^{-\frac{(t-s)}{2}B(t)^*}g(t)\| ds dt \|v\|_{L^\infty(0, \tau; \mathcal{V})} \\
&\lesssim \left( \int_0^\tau \int_0^t \frac{\|(\mathcal{C}(t) - \mathcal{C}(s))\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}^2}{|t-s|^2} ds dt \right)^{\frac{1}{2}} \\
&\times \sup_{\|g\|_{L^2(0, \tau; \mathcal{H})}=1} \left( \int_0^\tau \int_0^t \|B(t)^{* \frac{1}{2}}e^{-(t-s)B(t)^*}g(t)\|^2 ds dt \right)^{\frac{1}{2}} \|v\|_{L^\infty(0, \tau; \mathcal{V})} \\
&\lesssim \|\mathcal{C}\|_{\dot{H}^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))} \|v\|_{L^\infty(0, \tau; \mathcal{V})}.
\end{aligned}$$

□

**Lemma 3.8.** Assume that  $\mathcal{C} \in \dot{H}^{\frac{1}{2}+\epsilon}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ , with  $\epsilon > 0$ . Then the condition (3.5) is satisfied.

*Proof.* Let  $\tau_0 = 0 < \tau_1 < \dots < \tau_n = \tau$  be a subdivision of the interval  $[0, \tau]$ . Using [[?], p. 745, (6.8)], we get for  $t \in [\tau_{i-1}, \tau_i]$

$$\int_{\tau_{i-1}}^t \frac{\|\mathcal{C}(t) - \mathcal{C}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}^2}{|t-s|^{1+\epsilon}} ds \leq C \|\mathcal{C}\|_{\dot{H}^{\frac{1}{2}+\epsilon}(\tau_{i-1}, \tau_i; \mathcal{L}(\mathcal{V}, \mathcal{V}'))}^2.$$

Therefore,

$$\sup_{t \in [\tau_{i-1}, \tau_i]} \int_{\tau_{i-1}}^t \frac{\|\mathcal{C}(t) - \mathcal{C}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}^2}{|t-s|} ds \leq C |\tau_{i-1} - \tau_i|^\epsilon \|\mathcal{C}\|_{\dot{H}^{\frac{1}{2}+\epsilon}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))}^2.$$

Then if we choose  $|\tau_{i-1} - \tau_i|$  small enough we get the desired result. □

It is interesting to know whether  $u'(t) \in \mathcal{V}$  for all  $t \in [0, \tau]$ . This is indeed the case if  $t \mapsto \mathbf{a}(t) \in C^{\frac{1}{2}+\epsilon}$ ,  $\epsilon > 0$ . This is proved in [1]. In the following proposition we prove that the solution of the equation (3.1) lies in the space  $W^{1, \infty}(0, \tau; \mathcal{V})$  and  $u'(t) \in \mathcal{V}$  for all  $t \in [0, \tau]$ .

**Proposition 3.9.** Let  $f \in L^2(0, \tau; \mathcal{H})$  and  $(u_0, u_1) \in \mathcal{V} \times \mathcal{V}$ . Assume that (3.5) and  $\mathcal{A}(\cdot) \in \dot{H}_0^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ . Then  $u \in W^{1, \infty}(0, \tau; \mathcal{V})$  and  $u'(t) \in \mathcal{V}$  for all  $t \in [0, \tau]$ . Moreover,

$$\|u'\|_{L^\infty(0, \tau; \mathcal{V})} \leq C [\|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} + \|f\|_{L^2(0, \tau; \mathcal{H})}]. \quad (3.8)$$

*Proof.* Let  $f \in L^2(0, \tau; \mathcal{H})$  and  $(u_0, u_1) \in \mathcal{V} \times \mathcal{V}$ . A direct application of Theorem 3.10 gives  $u \in H^2(0, \tau; \mathcal{V}') \cap H^1(0, \tau; \mathcal{V})$  and

$$\|u\|_{H^2(0, \tau; \mathcal{V}') \cap H^1(0, \tau; \mathcal{V})} \leq C \left[ \|u_0\|_{\mathcal{V}} + \|u_1\| + \|f\|_{L^2(0, \tau; \mathcal{H})} \right].$$

Let  $\tau_0 = 0 < \tau_1 < \dots < \tau_n = \tau$  be a subdivision of the interval  $[0, \tau]$ . From [1](26) we have for  $t \in [0, \tau_1]$

$$\begin{aligned} u'(t) &= -\mathcal{B}(t)^{-1} \mathcal{A}(t) u(t) + e^{-tB(t)} \mathcal{B}(t)^{-1} \mathcal{A}(t) u(0) \\ &\quad + e^{-tB(t)} u'(0) + \int_0^t e^{-(t-s)B(t)} (\mathcal{B}(t) - \mathcal{B}(s)) u'(s) ds \\ &\quad - \int_0^t e^{-(t-s)B(t)} [\mathcal{A}(s) - \mathcal{A}(t)] u(s) ds \\ &\quad - \int_0^t e^{-(t-s)B(t)} \mathcal{B}(t)^{-1} \mathcal{A}(t) u'(s) ds + \int_0^t e^{-(t-s)B(t)} f(s) ds. \end{aligned}$$

Then

$$\begin{aligned} (I - Q_{\mathcal{B}(0, \tau_1)})(u')(t) &= -\mathcal{B}(t)^{-1} \mathcal{A}(t) u(t) + e^{-tB(t)} \mathcal{B}(t)^{-1} \mathcal{A}(t) u_0 \\ &\quad + (Q_{\mathcal{A}(0, \tau_1)} u)(t) + e^{-tB(t)} u_1 \\ &\quad - \int_0^t e^{-(t-s)B(t)} \mathcal{B}(t)^{-1} \mathcal{A}(t) u'(s) ds + L_t f. \end{aligned}$$

We remark that

$$\begin{aligned} \|\mathcal{B}(t)^{-1} \mathcal{A}(t) u(t)\|_{\mathcal{V}} &\leq C \|u(t)\|_{\mathcal{V}}. \\ \|e^{-(t)B(t)} \mathcal{B}(t)^{-1} \mathcal{A}(t) u(0)\|_{\mathcal{V}} &\leq C_1 \|u_0\|_{\mathcal{V}}. \\ \|e^{-(t)B(t)} u_1\|_{\mathcal{V}} &\leq C_2 \|u_1\|_{\mathcal{V}} \\ \left\| \int_0^t e^{-(t-s)B(t)} \mathcal{B}(t)^{-1} \mathcal{A}(t) u'(s) ds \right\|_{\mathcal{V}} &\leq C_3 \|u'\|_{L^2(0, t; \mathcal{V})}. \end{aligned}$$

Now, we choose  $\tau_1$  small enough such that  $\|Q_{\mathcal{B}(0, \tau_1)}\|_{\mathcal{L}(L^\infty(0, \tau_1; \mathcal{V}))} < 1$ . Therefore,  $I - Q_{\mathcal{B}(0, \tau_1)}$  is invertible on  $L^\infty(0, \tau_1; \mathcal{V})$  and

$$\begin{aligned} \|u'\|_{L^\infty(0, \tau_1; \mathcal{V})} &\lesssim \|(1 - Q_{\mathcal{B}(0, \tau_1)})^{-1}\|_{\mathcal{L}(L^\infty(0, \tau_1; \mathcal{V}))} \left[ \|u\|_{L^\infty(0, \tau_1; \mathcal{V})} + \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} \right. \\ &\quad \left. + \|Q_{\mathcal{A}(0, \tau_1)}\|_{\mathcal{L}(L^\infty(0, \tau_1; \mathcal{V}))} \|u\|_{L^\infty(0, \tau_1; \mathcal{V})} + \|u'\|_{L^2(0, \tau_1; \mathcal{V})} + \|f\|_{L^2(0, \tau_1; \mathcal{H})} \right]. \end{aligned}$$

Therefore

$$\begin{aligned} &\|u'\|_{L^\infty(0, \tau_1; \mathcal{V})} \\ &\leq C \left[ \|u\|_{C([0, \tau_1]; \mathcal{V})} + \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} + \|u'\|_{L^2(0, \tau_1; \mathcal{V})} + \|f\|_{L^2(0, \tau_1; \mathcal{H})} \right] \\ &\leq C_1 \left[ \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} + \|f\|_{L^2(0, \tau_1; \mathcal{H})} \right]. \end{aligned}$$

Hence,  $u' \in L^\infty(0, \tau_1; \mathcal{V}) \cap C([0, \tau_1]; \mathcal{H})$ . By Lemma [3](Lemma 3.7) one has  $u'(t) \in \mathcal{V}$  for all  $t \in [0, \tau_1]$ . In particular,  $u'(\tau_1) \in \mathcal{V}$ .

Fix  $\tau_1 \leq t \leq \tau_2$ . We get from the equation in (2.2)

$$\begin{aligned} & \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} u''(s) ds + \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} \mathcal{B}(s) u'(s) ds \\ & + \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} \mathcal{A}(s) u(s) ds = \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} g(s) ds. \end{aligned} \quad (3.9)$$

Noting that (3.9) is equivalent to

$$\begin{aligned} & \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} u''(s) ds + \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} \mathcal{B}(s) u'(s) ds \\ & + \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} (\mathcal{A}(s) - \mathcal{A}(t)) u(s) ds \\ & + \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} \mathcal{A}(t) u(s) ds \\ & = \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} f(s) ds. \end{aligned} \quad (3.10)$$

Integrating by parts, we obtain

$$\begin{aligned} \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} u''(s) ds &= u'(t) - e^{-(t-\tau_1)\mathcal{B}(t)} u'(\tau_1) \\ &\quad - \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} \mathcal{B}(t) u'(s) ds \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} \mathcal{A}(t) u(s) ds &= \mathcal{A}(t) u(t) - e^{-(t-\tau_1)\mathcal{B}(t)} \mathcal{A}(t) u(\tau_1) \\ &\quad - \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} \mathcal{B}(t)^{-1} \mathcal{A}(t) u'(s) ds. \end{aligned} \quad (3.12)$$

Combining (3.11) with (3.12) and (3.10), we have

$$\begin{aligned} u'(t) &= e^{-(t-\tau_1)\mathcal{B}(t)} u'(\tau_1) + \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} (\mathcal{B}(t) - \mathcal{B}(s)) u'(s) ds \\ &+ \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} (\mathcal{A}(t) - \mathcal{A}(s)) u(s) ds - \mathcal{B}(t)^{-1} \mathcal{A}(t) u(t) \\ &+ e^{-(t-\tau_1)\mathcal{B}(t)} \mathcal{B}(t)^{-1} \mathcal{A}(t) u(\tau_1) + \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} \mathcal{B}(t)^{-1} \mathcal{A}(t) u'(s) ds \\ &+ \int_{\tau_1}^t e^{-(t-s)\mathcal{B}(t)} f(s) ds. \end{aligned} \quad (3.13)$$

Therefore

$$\begin{aligned}
(I - Q_{\mathcal{B}(\tau_1, \tau_2)})u'(t) &= e^{-(t-\tau_1)B(t)}u'(\tau_1) + (Q_{\mathcal{A}(\tau_1, \tau_2)}u)(t) - \mathcal{B}(t)^{-1}\mathcal{A}(t)u(t) \\
&+ e^{-(t-\tau_1)B(t)}\mathcal{B}(t)^{-1}\mathcal{A}(t)u(\tau_1) + \int_{\tau_1}^t e^{-(t-s)B(t)}\mathcal{B}(t)^{-1}\mathcal{A}(t)u'(s) ds \\
&+ L_t(I_{(\tau_1, \tau_2)}f).
\end{aligned} \tag{3.14}$$

We remark that

$$\begin{aligned}
\|e^{-(t-\tau_1)B(t)}u'(\tau_1)\|_{\mathcal{V}} &\leq C\|u'(\tau_1)\|_{\mathcal{V}}. \\
\|\mathcal{B}(t)^{-1}\mathcal{A}(t)u(t)\|_{\mathcal{V}} &\leq C_1\|u\|_{C([\tau_1, \tau_2], \mathcal{V})}. \\
\|e^{-(t-\tau_1)B(t)}\mathcal{B}(t)^{-1}\mathcal{A}(t)u(\tau_1)\|_{\mathcal{V}} &\leq C_2\|u\|_{C([\tau_1, \tau_2], \mathcal{V})} \\
\left\| \int_{\tau_1}^t e^{-(t-s)B(t)}\mathcal{B}(t)^{-1}\mathcal{A}(t)u'(s) ds \right\|_{\mathcal{V}} &\leq C_3\|u'\|_{L^2(\tau_1, \tau_2; \mathcal{V})}.
\end{aligned}$$

Now, we choose  $\tau_2$  such that  $\tau_1 - \tau_2$  small enough to get  $\|Q_{\mathcal{B}(\tau_1, \tau_2)}\|_{\mathcal{L}(L^\infty(\tau_1, \tau_2; \mathcal{V}))} < 1$ . Then,  $I - Q_{\mathcal{B}(\tau_1, \tau_2)}$  is invertible on  $L^\infty(\tau_1, \tau_2; \mathcal{V})$ .

Therefore

$$\begin{aligned}
&\|u'\|_{L^\infty(\tau_1, \tau_2; \mathcal{V})} \\
&\leq C \left[ \|u\|_{C([\tau_1, \tau_2]; \mathcal{V})} + \|u'(\tau_1)\|_{\mathcal{V}} + \|u'\|_{L^2(\tau_1, \tau_2; \mathcal{V})} + \|f\|_{L^2(\tau_1, \tau_2; \mathcal{H})} \right] \\
&\leq C_1 \left[ \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} + \|f\|_{L^2(0, \tau_2; \mathcal{H})} \right].
\end{aligned}$$

Hence,  $u' \in L^\infty(\tau_1, \tau_2; \mathcal{V}) \cap C([\tau_1, \tau_2]; \mathcal{V})$ . One has by [3] (Lemma 3.7)  $u'(t) \in \mathcal{V}$  for all  $t \in [\tau_1, \tau_2]$ .

Now repeat the same strategy. We work on  $[\tau_{i-1}, \tau_i]$  and argue exactly as before. We obtain (3.8) on each sub-intervals  $[\tau_{i-1}, \tau_i]$ . This implies (3.8) on  $[0, \tau]$  for arbitrary  $\tau > 0$  and finishes the proof.  $\square$

The following theorem is the main result of this section

**Theorem 3.10.** *Let  $f \in L^2(0, \tau; \mathcal{H})$ ,  $u_0, u_1 \in \mathcal{V}$ . Assume that  $\mathcal{B}(\cdot) \in \dot{H}^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ , (3.5) and  $\mathcal{A}(\cdot) \in \dot{H}_0^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ . Then (3.1) has maximal  $L^2$ -regularity in  $\mathcal{H}$ .*

*Moreover, there exists  $N > 0$  such that*

$$\|u\|_{MR(\mathcal{V}, \mathcal{H})} \leq N \left[ \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} + \|f\|_{L^2(0, \tau; \mathcal{H})} \right]. \tag{3.15}$$

**Remark 3.11.** *The regularity assumption on  $\mathcal{B}(t)$  (resp.  $\mathcal{A}(t)$ ) can be weakened considerably. Indeed, piecewise in  $\dot{H}^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$  (resp.  $\dot{H}_0^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ ) is sufficient.*

*Proof.* Let  $f \in L^2(0, \tau; \mathcal{H})$  and  $(u_0, u_1) \in (\mathcal{V}, \mathcal{V})$ . Then by Theorem 3.1 there exists a unique  $u \in H^2(0, \tau; \mathcal{V}') \cap H^1(0, \tau; \mathcal{V})$  solution to Problem (3.1). In addition

$$\|u\|_{H^2(0, \tau; \mathcal{V}') \cap H^1(0, \tau; \mathcal{V})} \leq C \left[ \|(u_0, u_1)\|_{(\mathcal{V}, \mathcal{H})} + \|f\|_{L^2(0, \tau; \mathcal{V}')} \right].$$

By integration by parts (see [1](26)) we have

$$\begin{aligned} \mathcal{B}(t)u'(t) + \mathcal{A}(t)u(t) &= e^{-tB(t)}\mathcal{A}(t)u(0) \\ &+ B(t)e^{-tB(t)}u'(0) + B(t) \int_0^t e^{-(t-s)B(t)}(\mathcal{B}(t) - \mathcal{B}(s))u'(s) ds \\ &+ B(t) \int_0^t e^{-(t-s)B(t)}[\mathcal{A}(t) - \mathcal{A}(s)]u(s) ds \\ &+ \int_0^t e^{-(t-s)B(t)}\mathcal{A}(t)u'(s)ds + B(t) \int_0^t e^{-(t-s)B(t)}f(s) ds. \end{aligned} \quad (3.16)$$

Hence,

$$\begin{aligned} \mathcal{B}(t)u'(t) + \mathcal{A}(t)u(t) &= R_2u_0(t) + R_1u_1(t) + (\mathcal{B}(\cdot)Q_{\mathcal{B}(0, \tau)}u')(t) + (\mathcal{B}(\cdot)Q_{\mathcal{A}(0, \tau)}u)(t) \\ &+ \int_0^t e^{-(t-s)B(t)}\mathcal{A}(t)u'(s) ds + (Lf)(t). \end{aligned}$$

We remark that

$$\begin{aligned} \|t \mapsto \int_0^t e^{-(t-s)B(t)}\mathcal{A}(t)u'(s) ds\|_{L^2(0, \tau; \mathcal{H})} &\leq M \|t \mapsto e^{-tB(t)}\|_{L^1(0, \tau; \mathcal{L}(\mathcal{V}', \mathcal{H}))} \|u'\|_{L^2(0, \tau; \mathcal{V})} \\ &\leq 2M\sqrt{\tau} \|u'\|_{L^2(0, \tau; \mathcal{V})}. \end{aligned}$$

Then, in light of Propositions 2.8, 3.7, 3.9 and Lemma 2.9 we obtain

$$\begin{aligned} \|\mathcal{B}(\cdot)u' + \mathcal{A}(\cdot)u\|_{L^2(0, \tau; \mathcal{H})} &\leq \left[ \|R_2u_0\|_{L^2(0, \tau; \mathcal{H})} + \|R_1u_1\|_{L^2(0, \tau; \mathcal{H})} + \|Lf\|_{L^2(0, \tau; \mathcal{H})} \right. \\ &+ \|t \mapsto \int_0^t e^{-(t-s)B(t)}\mathcal{A}(t)u'(s) ds\|_{L^2(0, \tau; \mathcal{H})} + \|\mathcal{B}(\cdot)Q_{\mathcal{A}(0, \tau)}u\|_{L^2(0, \tau; \mathcal{H})} \\ &+ \left. \|\mathcal{B}(\cdot)Q_{\mathcal{B}(0, \tau)}u'\|_{L^2(0, \tau; \mathcal{H})} \right] \\ &\leq C \left[ \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} + \|u'\|_{W^{1, \infty}(0, \tau; \mathcal{V})} + \|f\|_{L^2(0, \tau; \mathcal{H})} \right] \\ &\leq C_1 \left[ \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} + \|f\|_{L^2(0, \tau; \mathcal{H})} \right]. \end{aligned}$$

Therefore  $\mathcal{B}(\cdot)u' + \mathcal{A}(\cdot)u \in L^2(0, \tau; \mathcal{H})$  and hence  $u'' = f - (\mathcal{B}(\cdot)u' + \mathcal{A}(\cdot)u) \in L^2(0, \tau; \mathcal{H})$ . Then  $u \in MR(\mathcal{V}, \mathcal{H})$ .  $\square$

The following proposition gives a characterization of the trace space  $Tr(\mathcal{V}, \mathcal{H})$ .

**Proposition 3.12.** *Assume the same hypothesis of Theorem 3.10 we get*

$$Tr(\mathcal{V}, \mathcal{H}) = \mathcal{V} \times \mathcal{V} \text{ with equivalent norms.}$$

*Proof.* First we prove the injection  $\mathcal{V} \times \mathcal{V} \hookrightarrow Tr(\mathcal{V}, \mathcal{H})$ . Indeed, let  $(u_0, u_1) \in \mathcal{V} \times \mathcal{V}$  then by Theorem 3.10 there exists  $u \in MR(\mathcal{V}, \mathcal{H})$  be the unique solution to (3.1) such that  $f = 0$  and  $u(0) = u_0, u'(0) = u_1$ . Hence,  $(u_0, u_1) \in Tr(\mathcal{V}, \mathcal{H})$  and by (5.3) we get

$$\|(u_0, u_1)\|_{Tr(\mathcal{V}, \mathcal{H})} \leq \|u\|_{MR(\mathcal{V}, \mathcal{H})} \leq C[\|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}}]$$

and so the first injection holds.

For the second injection " $\hookleftarrow$ " let us take  $(u_0, u_1) \in Tr(\mathcal{V}, \mathcal{H})$ . Then there exists  $u \in MR(\mathcal{V}, \mathcal{H})$  such that  $u(0) = u_0, u'(0) = u_1$ . Since

$$u \in W^{1,\infty}(0, \tau; \mathcal{V}) \cap H^2([0, \tau]; \mathcal{H}) \hookrightarrow C([0, \tau]; \mathcal{V}) \cap C^1([0, \tau]; \mathcal{H}),$$

we have  $u(0) = u_0 \in \mathcal{V}$  and by [3][Lemma 3.7] we get  $u'(t) \in \mathcal{V}$  for all  $t \in [0, \tau]$ . One has  $u'(0) = u_1 \in \mathcal{V}$  and

$$\begin{aligned} [\|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}}] &\leq \inf\{\|u\|_{W^{1,\infty}(0,\tau;\mathcal{V})} : u \in MR(\mathcal{V}, \mathcal{H}), u(0) = u_0, u'(0) = u_1\} \\ &\leq \inf\{\|u\|_{MR(\mathcal{V}, \mathcal{H})} : u \in MR(\mathcal{V}, \mathcal{H}), u(0) = u_0, u'(0) = u_1\} \\ &= \|(u_0, u_1)\|_{Tr(\mathcal{V}, \mathcal{H})}. \end{aligned}$$

Therefore  $Tr(\mathcal{V}, \mathcal{H}) \hookrightarrow \mathcal{V} \times \mathcal{V}$ . □

Let  $u_0 \in \mathcal{V}, u_1 \in [\mathcal{H}, \mathcal{V}]_{\beta}, \beta \in [0, 1]$ . In the following proposition we prove optimal estimates for the decay in time of solutions to (3.1)

**Proposition 3.13.** *Assume that  $\mathfrak{b}(t) = \mathfrak{b}(0), \mathcal{A}(t) = \mathcal{A}(0)$  for all  $t \in [0, \tau]$ . Let  $u_0 \in \mathcal{V}, u_1 \in [\mathcal{H}, \mathcal{V}]_{\beta}, \beta \in [0, 1]$  and  $u$  be the solution to (3.1) with  $f = 0$ . We have for all  $t > 0$*

$$\|u''(t)\| = \|\mathcal{B}(0)u'(t) + \mathcal{A}(0)u(t)\| \leq C_1 \left( \frac{1}{t^{1-\frac{\beta}{2}}} + 1 \right) \|u_1\|_{[\mathcal{H}, \mathcal{V}]_{\beta}} + C_2 \left( \frac{1}{t^{\frac{1}{2}}} + 1 \right) \|u_0\|_{\mathcal{V}}.$$

*Proof.* Let  $u_0 \in \mathcal{V}, u_1 \in [\mathcal{H}, \mathcal{V}]_{\beta}, \beta \in [0, 1]$  and  $u$  be the solution to (1.1) with  $f = 0$ . A direct application of Theorem 3.10 gives  $u \in H^2(0, \tau; \mathcal{V}') \cap H^1(0, \tau; \mathcal{V})$  and

$$\|u\|_{H^2(0,\tau;\mathcal{V}') \cap H^1(0,\tau;\mathcal{V})} \leq C[\|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}}].$$

From (3.16) we obtain

$$\begin{aligned} \mathcal{B}(0)u'(t) + \mathcal{A}(0)u(t) &= e^{-tB(0)}\mathcal{A}(0)u_0 + B(0)e^{-tB(0)}u_1 \\ &\quad - \int_0^t e^{-(t-s)B(0)}\mathcal{A}(0)u'(s) ds. \end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathcal{B}(0)u'(t) + \mathcal{A}(0)u(t)\| &\leq \|e^{-tB(0)}\mathcal{A}(0)u_0\| + \|B(0)e^{-tB(0)}u_1\| \\
&\quad + \left\| \int_0^t e^{-(t-s)B(0)}\mathcal{A}(0)u'(s) ds \right\| \\
&\leq \|e^{-tB(0)}\|_{\mathcal{L}(\mathcal{V};\mathcal{H})} \|u_0\|_{\mathcal{V}} + \|B(0)e^{-tB(0)}u_1\| \\
&\quad + \left\| \int_0^t e^{-(t-s)B(0)}\mathcal{A}(0)u'(s) ds \right\|.
\end{aligned}$$

We remark that

$$\begin{aligned}
\|B(0)e^{-tB(0)}\|_{\mathcal{L}(\mathcal{V};\mathcal{H})} &\leq \frac{C}{t^{\frac{1}{2}}}, \\
\|B(0)e^{-tB(0)}\|_{\mathcal{L}(\mathcal{H})} &\leq \frac{C_1}{t}.
\end{aligned}$$

We have by interpolation

$$\|B(0)e^{-tB(0)}u_1\| \leq \frac{C_\beta}{t^{1-\frac{\beta}{2}}} \|u_1\|_{[\mathcal{H},\mathcal{V}]_\beta}.$$

We write

$$\int_0^t e^{-(t-s)B(0)}\mathcal{A}(0)u'(s) ds = B(0)^{\frac{1}{2}} \int_0^t e^{-(t-s)B(0)} B(0)^{-\frac{1}{2}} \mathcal{A}(0)u'(s) ds.$$

Hence,

$$\begin{aligned}
\left\| \int_0^t e^{-(t-s)B(0)}\mathcal{A}(0)u'(s) ds \right\| &= \|B(0)^{\frac{1}{2}} \int_0^t e^{-(t-s)B(0)} B(0)^{-\frac{1}{2}} \mathcal{A}(0)u'(s) ds\| \\
&\leq \|B(0)^{\frac{1}{2}}\|_{\mathcal{L}(\mathcal{V},\mathcal{H})} \left\| \int_0^t e^{-(t-s)B(0)} B(0)^{-\frac{1}{2}} \mathcal{A}(0)u'(s) ds \right\|_{\mathcal{V}} \\
&= c^1 \|L_t(B(0)^{-\frac{1}{2}} \mathcal{A}(0)u')\|_{\mathcal{V}} \\
&\leq C \|B(0)^{-\frac{1}{2}} \mathcal{A}(0)u'\|_{L^2(0,\tau;\mathcal{H})} \leq C' \|u'\|_{L^2(0,\tau;\mathcal{V})}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|u''(t)\| = \|\mathcal{B}(0)u'(t) + \mathcal{A}(0)u(t)\| &\leq \frac{C}{t^{\frac{1}{2}}} \|u_0\|_{\mathcal{V}} + \frac{C_\beta}{t^{1-\frac{\beta}{2}}} \|u_1\|_{[\mathcal{H},\mathcal{V}]_\beta} \\
&\quad + C' \|u'\|_{L^2(0,\tau;\mathcal{V})} \\
&\leq C_1 \left( \frac{1}{t^{1-\frac{\beta}{2}}} + 1 \right) \|u_1\|_{[\mathcal{H},\mathcal{V}]_\beta} + C_2 \left( \frac{1}{t^{\frac{1}{2}}} + 1 \right) \|u_0\|_{\mathcal{V}}.
\end{aligned}$$

□

For higher order equations we have

**Theorem 3.14.** *Let  $(\mathcal{A}_i(t))_{t \in [0, \tau], i \in [1, N-1]}$ ,  $N \in \mathbb{N}^*$  such that  $\mathcal{A}_i(t) \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$  for all  $i \in [1, N-1]$  and  $\|\mathcal{A}_i(t)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \leq M$ . We suppose that  $(\mathcal{B}(t))_{t \in [0, \tau]} \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$  is associated with  $\mathcal{V}$ -bounded quasi-coercive forms. We assume in addition  $\mathcal{B}(\cdot) \in \dot{H}^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ , (3.5), (3.3) and  $\mathcal{A}_i(\cdot) \in \dot{H}_0^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ . Then for all  $f \in L^2(0, \tau; \mathcal{H})$  and  $u_0, \dots, u_{N-1} \in \mathcal{V}$  there exists a unique  $u \in H^N(0, \tau; \mathcal{H}) \cap W^{N-1, \infty}(0, \tau; \mathcal{V})$  such that  $u^{(N-1)}(t) \in \mathcal{V}$  for all  $t \in [0, \tau]$ , be the solution to the problem*

$$\begin{cases} u^{(N)}(t) + \mathcal{B}(t)u^{(N-1)}(t) + \mathcal{A}_{N-1}(t)u^{(N-2)}(t) + \dots + \mathcal{A}_1(t)u(t) = f(t) \quad t\text{-a.e.} \\ u^{(N-1)}(0) = u_{N-1}, \dots, u(0) = u_0. \end{cases} \quad (3.17)$$

In addition, there exists a positive constant  $C$  independent of  $u_0, \dots, u_{N-1}$  and  $f$  such that

$$\|u\|_{H^N(0, \tau; \mathcal{H}) \cap W^{N-1, \infty}(0, \tau; \mathcal{V})} \leq C \left( \sum_{i=0}^{N-1} \|u_i\|_{\mathcal{V}} + \|f\|_{L^2(0, \tau; \mathcal{H})} \right).$$

*Proof.* We give only the main ideas of the proof, the details are left to the reader. We shall prove the theorem by induction. Indeed, in case  $N = 1$  the result follows from [3] (Theorem 2.2). The theorem holds for  $N = 2$  by Theorem 3.10. Now, we assume that the theorem is true at order  $N-1$  where  $N$  is an arbitrary positive integer. By integration and following the same strategy of proof as in [1][Theorem 2.6] we prove maximal  $L^2$ -regularity in  $\mathcal{V}'$  for the Cauchy problem (3.17) and we have  $u \in H^N(0, \tau; \mathcal{V}') \cap H^{N-1}(0, \tau; \mathcal{V})$ . Let  $\gamma > 0$  and set  $v(t) = e^{-\gamma t}u(t)$ ,  $\mathcal{A}_N(t) = \mathcal{B}(t)$ . By Leibniz's rule and using the equation (3.17) we get that  $v$  is the solution to the problem

$$\begin{cases} v^{(N)}(t) + (\mathcal{A}_N(t) + N\gamma I)v^{(N-1)}(t) + \sum_{j=0}^{N-2} \mathcal{C}_j(t)v^{(j)}(t) = e^{-\gamma t}f(t) \quad t\text{-a.e.} \\ v^{(k)}(0) = v_k = \sum_{j=0}^k C_j^k(-\gamma)^{k-j}u_j, \quad k \in [0, N-1], \end{cases} \quad (3.18)$$

where  $\mathcal{C}_j(t) = \left( \sum_{m=j}^{N-1} (-1)^{N+1-m} C_m^N C_j^m \right) \gamma^{N-j} I + \sum_{m=j}^{N-1} C_j^m \gamma^{m-j} \mathcal{A}_{m+1}(t)$ , for all  $j \in [0, N-1]$  and  $C_j^m = \frac{m!}{j!(m-j)!}$ . Here  $v^{(j)}$  is the derivative of order  $j$ .

We assume now that  $\gamma > \frac{|\nu|}{N}$ , then  $\mathcal{C}_{N-1}(t) = \mathcal{A}_N(t) + N\gamma I$  is associated with  $\mathcal{V}$ -bounded coercive form for all  $t \in [0, \tau]$ .

By performing an integration by parts (see [1] [p.29] ) we obtain

$$\begin{aligned}
v^{(N-1)}(t) = & -\mathcal{C}_{N-1}(t)^{-1} \left[ \mathcal{C}_{N-2}(t)v^{(N-2)}(t) + \dots + \mathcal{C}_0(t)v(t) \right] \\
& + e^{-t\mathcal{C}_{N-1}(t)}v_{N-1} + e^{-t\mathcal{C}_{N-1}(t)}\mathcal{C}_{N-1}(t)^{-1} \left( \mathcal{C}_1(t)v_0 + \dots + \mathcal{C}_{N-2}(t)v_{N-2} \right) \\
& + \int_0^t e^{-(t-s)\mathcal{C}_{N-1}(t)} \left( \mathcal{C}_{N-1}(t) - \mathcal{C}_{N-1}(s) \right) v^{(N-1)}(s) ds \\
& + \int_0^t e^{-(t-s)\mathcal{C}_{N-1}(t)} \left( \mathcal{C}_{N-2}(t) - \mathcal{C}_{N-2}(s) \right) v^{(N-2)}(s) ds \\
& + \dots + \int_0^t e^{-(t-s)\mathcal{C}_{N-1}(t)} \left( \mathcal{C}_0(t) - \mathcal{C}_0(s) \right) v(s) ds \\
& + \int_0^t e^{-(t-s)\mathcal{C}_{N-1}(t)} \mathcal{C}_{N-1}(t)^{-1} \left( \mathcal{C}_0(t)v'(s) + \dots + \mathcal{C}_{N-2}(t)v^{(N-1)}(s) \right) ds \\
& + \int_0^t e^{-(t-s)\mathcal{C}_{N-1}(t)} e^{-\gamma s} f(s) ds. \tag{3.19}
\end{aligned}$$

We now proceed analogously to the proof of Proposition 3.9 to get  $v^{(N-1)} \in L^\infty(0, \tau; \mathcal{V})$  and  $v^{(N-1)}(t) \in \mathcal{V}$  for all  $t \in [0, \tau]$ .

From (3.19) we can see that

$$\begin{aligned}
& \mathcal{C}_{N-1}(t)v^{(N-1)}(t) + \mathcal{C}_{N-2}(t)v^{(N-2)}(t) + \dots + \mathcal{C}_0(t)v(t) \\
& = R_1 v_{N-1}(t) + e^{-t\mathcal{C}_{N-1}(t)} \left( \mathcal{C}_1(t)v_0 + \dots + \mathcal{C}_{N-2}(t)v_{N-2} \right) \\
& + \mathcal{C}_{N-1}(t)(Q_{\mathcal{C}_{N-1}}v^{(N-1)})(t) + \dots + \mathcal{C}_{N-1}(t)(Q_{\mathcal{C}_0}v)(t) \\
& + \int_0^t e^{-(t-s)\mathcal{C}_{N-1}(t)} \left( \mathcal{C}_0(t)v'(s) + \dots + \mathcal{C}_{N-2}(t)v^{(N-1)}(s) \right) ds + (L(e^{-\gamma \cdot} f))(t).
\end{aligned}$$

We now proceed analogously to the proof of Theorem 3.10 to get  $v \in H^N(0, \tau; \mathcal{H}) \cap W^{N-1, \infty}(0, \tau; \mathcal{V})$ . Hence,  $u \in H^N(0, \tau; \mathcal{H}) \cap W^{N-1, \infty}(0, \tau; \mathcal{V})$  and this finishes the proof.  $\square$

## 4 Optimality of the results

**Definition 4.1.** Let  $X$  be a Banach space, let  $p, q \in [1, \infty]$ , and  $\alpha \in (0, 1)$ . A Bochner measurable function  $f : [0, \tau] \rightarrow X$  is in the homogeneous Besov space  $\dot{B}_q^{\alpha, p}(0, \tau; X)$  if

$$\|f\|_{\dot{B}_q^{\alpha, p}(0, \tau; X)}^q = \int_0^\tau \frac{1}{l^{\alpha q}} \left( \int_l^\tau \|f(t) - f(t-l)\|_X^p dt \right)^{\frac{q}{p}} \frac{dl}{l} < \infty.$$

We note that  $\dot{B}_2^{\alpha, 2}(0, \tau; X) = \dot{H}^\alpha(0, \tau; X)$  and  $\dot{H}^\alpha(0, \tau; X) \subset \dot{B}_q^{\alpha, 2}(0, \tau; X)$  for all  $q > 2$ .

**Proposition 4.2.** *Let  $p \in [1, \infty)$  and  $q \in (2, \infty)$ . There exists  $\mathbf{a}, \mathbf{b}$  coercive, symmetric, non-autonomous forms with  $\mathcal{A}(\cdot), \mathcal{B}(\cdot) \in \dot{B}_q^{\frac{1}{2}, p}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$  such that (3.1) does not have maximal  $L^2$ -regularity in  $\mathcal{H}$ .*

**Remark 4.3.** *Since  $\dot{H}^{\frac{1}{2}}(0, \tau; X) \subset \dot{B}_q^{\frac{1}{2}, 2}(0, \tau; X) \subset \dot{H}^{\frac{1}{2}-\epsilon}(0, \tau; X)$ ,  $\epsilon > 0$  for all  $q > 2$ . Then the regularity assumption in Theorem 3.10 is optimal and provides the best positive result on this problem.*

*Proof.* Let  $p \in [1, \infty), q \in (2, \infty)$ . According to [8], there exist a Gelfand triple  $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$  and a coercive, symmetric, non-autonomous forms  $\mathbf{a}(t)$  with  $\mathcal{A}(\cdot) \in \dot{B}_q^{\frac{1}{2}, p}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$  that does not satisfy the maximal  $L^2$ -regularity in  $\mathcal{H}$ . The example in [8] proves that there exists  $f \in L^\infty(0, \tau; \mathcal{H})$  such that the solution  $v$  to the first order Cauchy problem

$$\begin{cases} v'(t) + \mathcal{A}(t)v(t) = f(t), & t\text{-a.e.} \\ v(0) = 0. \end{cases}$$

is in  $H^1(0, \tau; \mathcal{V}') \cap L^\infty(0, \tau; \mathcal{V})$ , but  $v'(t) \notin \mathcal{H}$  for all  $t \in [0, \tau]$ . Now, we put  $\mathbf{b}(t) = \mathbf{a}(t) + I$  and  $u(t) = \int_0^t e^{-(t-s)}v(s)ds$ . Consequently,  $u(t) + u'(t) = v(t)$  and so  $u'(t) + u''(t) = v'(t)$ .

We get by Theorem 3.1 that  $u \in H^2(0, \tau; \mathcal{V}') \cap H^1(0, \tau; \mathcal{V})$  is the unique solution to the problem

$$\begin{cases} u''(t) + (\mathcal{A}(t) + I)u'(t) + \mathcal{A}(t)u(t) = f(t) & t\text{-a.e.} \\ v(0) = 0, \quad v'(0) = 0. \end{cases} \quad (4.1)$$

Note that  $u \in H^2(0, \tau; \mathcal{H})$  if and only if  $v \in H^1(0, \tau; \mathcal{H})$ . But  $v \notin H^1(0, \tau; \mathcal{H})$ , hence,  $u \notin H^2(0, \tau; \mathcal{H})$  and so Problem (4.1) does not have maximal  $L^2$ -regularity in  $\mathcal{H}$ .  $\square$

## 5 Semilinear equation

Let  $F(t, x, y) : (0, \tau) \times \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$  and  $F_0(t) = F(t, 0, 0)$ . Assume that  $F_0 \in L^2(0, \tau; \mathcal{H})$  and  $F(\cdot, x, y)$  satisfies the following continuity property: for any  $\epsilon > 0$  there exists a constant  $N_\epsilon > 0$  such that

$$\begin{aligned} & \|F(\cdot, u, u') - F(\cdot, v, v')\|_{L^2(0, \tau; \mathcal{H})}^2 \\ & \leq \epsilon \|u - v\|_{MR(\mathcal{V}, \mathcal{H})}^p + N_\epsilon \left[ \|u' - v'\|_{L^2(0, \tau; \mathcal{H})}^2 + \|u - v\|_{L^2(0, \tau; \mathcal{V})}^2 \right], \end{aligned} \quad (5.1)$$

for any  $u, v \in MR(\mathcal{V}, \mathcal{H})$ .

**Example 5.1.** *If we assume that  $\|F(t, x_1, x_2) - F(t, y_1, y_2)\| \leq K[\|x_1 - y_1\|_{\mathcal{V}} + \|x_2 - y_2\|_{\mathcal{V}}]$ ,  $K > 0, x_1, x_2, y_1, y_2 \in \mathcal{V}, t \in (0, \tau)$  then the conditions*

(5.1) is satisfied. Indeed, let  $u, v \in MR(\mathcal{V}, \mathcal{H})$ . One has by Lemma 5.2

$$\begin{aligned}
\|F(., u, u') - F(., v, v')\|_{L^2(0, \tau; \mathcal{H})}^2 &\leq K^2 \|u - v\|_{H^1(0, \tau; \mathcal{V})}^2 \\
&= \frac{K^2}{\delta} \left[ \delta \|u - v\|_{H^1(0, \tau; \mathcal{V})}^2 \right] \\
&\leq 2 \frac{K^2}{\delta} \int_0^\tau \operatorname{Re} (\mathcal{A}(t)(u - v)(t) + \mathcal{B}(t)(u - v)'(t), (u - v)'(t)) dt \\
&\quad + L \frac{K^2}{\delta} \|u - v\|_{L^2(0, \tau; \mathcal{V})}^2 \\
&\leq \epsilon \|\mathcal{A}(\cdot)(u - v) + \mathcal{B}(\cdot)(u - v)'\|_{L^2(0, \tau; \mathcal{H})}^2 \\
&\quad + N_\epsilon \left[ \|u' - v'\|_{L^2(0, \tau; \mathcal{H})}^2 + \|u - v\|_{L^2(0, \tau; \mathcal{V})}^2 \right] \\
&\leq \epsilon \|u - v\|_{MR(\mathcal{V}, \mathcal{H})}^2 + N_\epsilon \left[ \|u' - v'\|_{L^2(0, \tau; \mathcal{H})}^2 + \|u - v\|_{L^2(0, \tau; \mathcal{V})}^2 \right],
\end{aligned}$$

where  $N_\epsilon = \sup\{\frac{4K^4}{\delta^2\epsilon}, L\frac{K^2}{\delta}\}$ , and in the second inequality we have used the basic inequality  $ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$ ,  $a, b, \epsilon > 0$ .

**Lemma 5.2.** Let  $u \in H^1(0, \tau; \mathcal{V})$ . We have

$$\begin{aligned}
\delta \left[ \|u\|_{L^2(0, \tau; \mathcal{V})}^2 + \|u'\|_{L^2(0, \tau; \mathcal{V})}^2 \right] &\leq 2 \int_0^\tau \operatorname{Re} \langle \mathcal{A}(t)u(t) + \mathcal{B}(t)u'(t), u'(t) \rangle dt \\
&\quad + L \|u\|_{L^2(0, \tau; \mathcal{V})}^2,
\end{aligned}$$

with  $L = 2(\frac{M^2}{\delta} + \delta)$ .

*Proof.* Let  $u \in H^1(0, \tau; \mathcal{V})$ , we get

$$\begin{aligned}
&\int_0^\tau \operatorname{Re} \langle \mathcal{A}(t)u(t) + \mathcal{B}(t)u'(t), u'(t) \rangle dt \\
&= \int_0^\tau \operatorname{Re} \langle \mathcal{B}(t)u'(t); u'(t) \rangle dt \\
&\quad + \int_0^\tau \operatorname{Re} \langle \mathcal{A}(t)u(t); u'(t) \rangle dt \\
&\geq \delta \int_0^\tau \|u'(t)\|_{\mathcal{V}}^2 dt - M \int_0^\tau \|u(t)\|_{\mathcal{V}} \|u'(t)\|_{\mathcal{V}} dt \\
&\geq \delta \int_0^\tau \|u'(t)\|_{\mathcal{V}}^2 dt - \frac{M^2}{\delta} \int_0^\tau \|u(t)\|_{\mathcal{V}}^2 dt - \frac{\delta}{2} \int_0^\tau \|u'(t)\|_{\mathcal{V}}^2 dt,
\end{aligned}$$

where in the last inequality we have used the basic inequality  $2ab \leq \frac{a^2}{\delta} + \delta b^2$ ,  $a, b > 0$ . Therefore

$$\begin{aligned}
\delta \left[ \|u\|_{L^2(0, \tau; \mathcal{V})}^2 + \|u'\|_{L^2(0, \tau; \mathcal{V})}^2 \right] &\leq 2 \int_0^\tau \operatorname{Re} \langle \mathcal{A}(t)u(t) + \mathcal{B}(t)u'(t), u'(t) \rangle dt \\
&\quad + 2\left(\frac{M^2}{\delta} + \delta\right) \|u\|_{L^2(0, \tau; \mathcal{V})}^2,
\end{aligned}$$

□

In the following theorem we assume the same assumptions with Theorem 3.10 for  $\mathcal{B}(\cdot)$  and  $\mathcal{A}(\cdot)$ .

**Theorem 5.3.** *Let  $f \in L^2(0, \tau; \mathcal{H})$ ,  $u_0, u_1 \in \mathcal{V}$ . Assume that (5.1). Then the semilinear equation*

$$\begin{cases} u''(t) + \mathcal{B}(t)u'(t) + \mathcal{A}(t)u(t) = F(t, u, u') & t\text{-a.e.} \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \quad (5.2)$$

*admits a unique solution  $u \in MR(\mathcal{V}, \mathcal{H})$ .*

*Moreover, there is  $C > 0$  independent of  $u_0, u_1, F_0$  such that*

$$\|u\|_{MR(\mathcal{V}, \mathcal{H})} \leq C \left[ \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} + \|F_0\|_{L^2(0, \tau; \mathcal{H})} \right]. \quad (5.3)$$

*Proof.* Let  $u_0, u_1 \in \mathcal{V}$ . Since by Proposition 3.12,  $Tr(\mathcal{V}, \mathcal{H}) = \mathcal{V} \times \mathcal{V}$ , then there exists  $v \in MR(\mathcal{V}, \mathcal{H})$  (with minimal norm) such that  $v(0) = u_0, v'(0) = u_1$  and

$$\|v\|_{MR(\mathcal{V}, \mathcal{H})} = \left[ \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} \right]. \quad (5.4)$$

We introduce the subspace

$$MR_0(\mathcal{V}, \mathcal{H}) := \{w \in MR(\mathcal{V}, \mathcal{H}) : w(0) = 0, w'(0) = 0\}.$$

We equip this subspace with the norm  $w \rightarrow \|w''\|_{L^2(0, \tau; \mathcal{H})} + \|w'\|_{L^\infty(0, \tau; \mathcal{V})} + \|\mathcal{B}(\cdot)w' + \mathcal{A}(\cdot)w\|_{L^2(0, \tau; \mathcal{H})}$ . For  $w \in MR_0(\mathcal{V}, \mathcal{H})$  we define the function

$$G(t, w, w') = F(t, w + v, w' + v') - \left( v''(t) + \mathcal{B}(t)v'(t) + \mathcal{A}(t)v(t) \right), \quad t \in (0, \tau).$$

It easy to see that  $G$  satisfies the condition (5.1),  $t \mapsto G(t, w, w') \in L^2(0, \tau; \mathcal{H})$ ,  $G(t, 0, 0) = F(t, v, v') - \left( v''(t) + \mathcal{B}(t)v'(t) + \mathcal{A}(t)v(t) \right)$ . Moreover,

$$\begin{aligned} \|G(\cdot, 0, 0)\|_{L^2(0, \tau; \mathcal{H})} &\leq \|F(\cdot, v, v') - F(\cdot, 0, 0)\|_{L^2(0, \tau; \mathcal{H})} + \|F(\cdot, 0, 0)\|_{L^2(0, \tau; \mathcal{H})} \\ &\quad + \|v'' + \mathcal{B}(\cdot)v' + \mathcal{A}(\cdot)v\|_{L^2(0, \tau; \mathcal{H})} \\ &\leq C_1 \|v\|_{MR(\mathcal{V}, \mathcal{H})} + \|F_0\|_{L^2(0, \tau; \mathcal{H})} \\ &\leq C \left[ \|F_0\|_{L^2(0, \tau; \mathcal{H})} + \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} \right]. \end{aligned} \quad (5.5)$$

For  $w \in MR_0(\mathcal{V}, \mathcal{H})$  consider the linear equation

$$\begin{cases} z''(t) + \mathcal{B}(t)z'(t) + \mathcal{A}(t)z(t) = G(t, w, w') & t\text{-a.e.} \\ z(0) = 0, \quad z'(0) = 0. \end{cases} \quad (5.6)$$

By Theorem 3.10, this equation admits a unique solution  $z \in MR_0(\mathcal{V}, \mathcal{H})$ . We define

$$\begin{aligned} S : MR_0(\mathcal{V}, \mathcal{H}) &\rightarrow MR_0(\mathcal{V}, \mathcal{H}) \\ w &\mapsto z. \end{aligned}$$

For  $w_1, w_2 \in MR_0(\mathcal{V}, \mathcal{H})$  we have by Theorem 3.10 and (5.1)

$$\begin{aligned} &\|Sw_1 - Sw_2\|_{MR_0(\mathcal{V}, \mathcal{H})}^2 \\ &\leq N\|G(\cdot, w_1, w'_1) - G(\cdot, w_2, w'_2)\|_{L^2(0, \tau; \mathcal{H})}^2 \\ &\leq N\epsilon\|w_1 - w_2\|_{MR_0(\mathcal{V}, \mathcal{H})}^2 + NN_\epsilon\left[\|w'_1 - w'_2\|_{L^2(0, \tau; \mathcal{H})}^2 + \|w_1 - w_2\|_{L^2(0, \tau; \mathcal{V})}^2\right] \\ &\leq N\epsilon\|w_1 - w_2\|_{MR_0(\mathcal{V}, \mathcal{H})}^p \\ &+ NN_\epsilon\left[\int_0^\tau s \int_0^s \|w''_1(r) - w''_2(r)\|^2 dr ds + \int_0^\tau s \int_0^s \|w'_1(r) - w'_2(r)\|_{\mathcal{V}}^2 dr ds\right] \\ &\leq N\epsilon\|w_1 - w_2\|_{MR_0(\mathcal{V}, \mathcal{H})}^p \\ &+ NN_\epsilon\tau\left[\int_0^\tau \int_0^s \|w''_1(r) - w''_2(r)\|^2 + \|w'_1(r) - w'_2(r)\|_{\mathcal{V}}^2 dr ds\right] \\ &\leq N\epsilon\|w_1 - w_2\|_{MR_0(\mathcal{V}, \mathcal{H})}^p \\ &+ NN_\epsilon\tau\left[\int_0^\tau \|(w_1 - w_2)''\|_{L^2(0, s; \mathcal{H})}^2 + \|(w_1 - w_2)'\|_{L^2(0, s; \mathcal{V})}^2\right. \\ &\quad \left.+ \|\mathcal{B}(\cdot)(w_1 - w_2)' + \mathcal{A}(\cdot)(w_1 - w_2)\|_{L^2(0, s; \mathcal{H})}^2 ds\right]. \end{aligned}$$

Set  $K_0 := N\epsilon$  and  $K_1 := 2NN_\epsilon\tau$ . Then repeating the above inequality and using the identity

$$\int_0^t \int_0^{s_1} \dots \int_0^{s_{n-1}} ds_n \dots ds_1 = \frac{1}{n!} t^n,$$

we obtain

$$\begin{aligned} \|S^n w_1 - S^n w_2\|_{MR_0(\mathcal{V}, \mathcal{H})}^2 &\leq \sum_0^n \binom{n}{k} K_0^{n-k} (K_1 \tau)^k \frac{1}{k!} \|w_1 - w_2\|_{MR_0(\mathcal{V}, \mathcal{H})}^2 \\ &\leq (2K_0)^n \left[ \max_{k=0, \dots, n} \left( \frac{(K_0^{-1} \tau K_1)^k}{k!} \right) \right] \|w_1 - w_2\|_{MR_0(\mathcal{V}, \mathcal{H})}^2. \end{aligned}$$

For the second inequality we use  $\sum_0^n \binom{n}{k} = 2^n$ . Note that  $\left[ \max_{k=0, \dots, n} \left( \frac{(K_0^{-1} \tau K_1)^k}{k!} \right) \right]$

is bounded for all  $n \in \mathbb{N}^*$ .

Now, we choose  $\epsilon < \frac{1}{4N}$ , which gives  $K_0 < \frac{1}{4}$  and  $n$  sufficiently large to get

$$\begin{aligned} \|S^n w_1 - S^n w_2\|_{MR_0(\mathcal{V}, \mathcal{H})}^p &< \frac{1}{2^n} \left[ \max_{k=1, \dots, n} \left( \frac{(K_0^{-1} \tau K_1)^k}{k!} \right) \right] \|w_1 - w_2\|_{MR_0(\mathcal{V}, \mathcal{H})}^p \\ &< \|w_1 - w_2\|_{MR_0(\mathcal{V}, \mathcal{H})}^p. \end{aligned}$$

Then  $S^n$  is a contraction on  $MR_0(\mathcal{V}, \mathcal{H})$  and this yields the existence and uniqueness of a solution  $z \in MR_0(\mathcal{V}, \mathcal{H})$  to the equation

$$\begin{cases} z''(t) + \mathcal{B}(t)z'(t) + \mathcal{A}(t)z(t) = G(t, z, z) & t\text{-a.e.} \\ z(0) = 0, \quad z'(0) = 0. \end{cases}$$

It remains to prove the apriori estimate. Indeed, from the linear equation and (5.1) we have for all  $\epsilon > 0$

$$\begin{aligned} \|z\|_{MR_0(\mathcal{V}, \mathcal{H})}^2 &\leq N\|G(\cdot, z, z')\|_{L^2(0, \tau; \mathcal{H})}^2 \\ &\leq 2N\|G(\cdot, z, z') - G(\cdot, 0, 0)\|_{L^2(0, \tau; \mathcal{H})}^2 + 2N\|G(\cdot, 0, 0)\|_{L^2(0, \tau; \mathcal{H})}^2 \\ &\leq 2N\epsilon\|z\|_{MR_0(\mathcal{V}, \mathcal{H})}^2 + 2NN_\epsilon\left[\|z\|_{L^2(0, \tau; \mathcal{V})}^2 + \|z'\|_{L^2(0, \tau; \mathcal{H})}^2\right] \\ &\quad + 2N\|G(\cdot, 0, 0)\|_{L^2(0, \tau; \mathcal{H})}^2 \\ &\leq 2N\epsilon\|z\|_{MR_0(\mathcal{V}, \mathcal{H})}^2 + NN_\epsilon\tau\int_0^\tau\left[\|z'\|_{L^2(0, s; \mathcal{V})}^2 + \|z''\|_{L^2(0, s; \mathcal{H})}^2\right]ds \\ &\quad + 2N\|G(\cdot, 0, 0)\|_{L^2(0, \tau; \mathcal{H})}^2. \end{aligned}$$

Taking  $\epsilon = \frac{1}{4N}$  and applying Gronwall's lemma gives that there exists  $C' > 0$  such that

$$\|z\|_{MR_0(\mathcal{V}, \mathcal{H})} \leq C'\|G(\cdot, 0, 0)\|_{L^2(0, \tau; \mathcal{H})}. \quad (5.7)$$

Now, we set  $u = z + v$ . Then  $u$  is the unique solution to (5.2) and from (5.4), (5.5) and (5.7) we obtain

$$\begin{aligned} \|u\|_{MR(\mathcal{V}, \mathcal{H})} &\leq \|z\|_{MR(\mathcal{V}, \mathcal{H})} + \|v\|_{MR(\mathcal{V}, \mathcal{H})} \\ &\leq C'\|G(\cdot, 0, 0)\|_{L^2(0, \tau; \mathcal{H})} + \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}} \\ &\leq C\left[\|F_0\|_{L^2(0, \tau; \mathcal{H})} + \|u_0\|_{\mathcal{V}} + \|u_1\|_{\mathcal{V}}\right]. \end{aligned}$$

□

## 6 Applications

This section is devoted to some applications of the results given in the previous sections. We give examples illustrating the theory without seeking for generality.

### 6.1 Laplacian with time dependent Robin boundary conditions.

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^d$ , with Lipschitz boundary  $\Gamma$ . Denote by  $\sigma$  the  $(d-1)$ -dimensional Hausdorff measure on  $\Gamma$ . Let

$$\beta_1, \beta_2 : [0, \tau] \times \Gamma \rightarrow \mathbb{R}$$

be bounded measurable functions such that

$$\int_0^\tau \int_0^\tau \frac{\|\beta_i(t, \cdot) - \beta_i(s, \cdot)\|_{L^\infty(\partial\Omega)}^2}{|t-s|^{1+2\alpha}} ds dt < \infty \quad (6.1)$$

for some  $\alpha > \frac{1}{2}$ . In particular,  $\beta_i(\cdot, x) \in \dot{H}^\alpha(0, \tau)$ ,  $i = 1, 2$ . We consider the forms  $\mathfrak{a}, \mathfrak{b}$

$$\mathfrak{a}, \mathfrak{b} : [0, \tau] \times H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$$

defined by

$$\mathfrak{a}(t, u, v) = \int_\Omega \nabla u \nabla v dx + \int_\Gamma \beta_1(t, \cdot) uv d\sigma$$

and

$$\mathfrak{b}(t, u, v) = \int_\Omega \nabla u \nabla v dx + \int_\Gamma \beta_2(t, \cdot) uv d\sigma.$$

The forms  $\mathfrak{a}, \mathfrak{b}$  are  $H^1(\Omega)$ -bounded, quasi-coercive and symmetric. The first statement follows readily from the continuity of the trace operator and the boundedness of  $\beta_i$ ,  $i = 1, 2$ . The second one is a consequence of the inequality

$$\int_\Gamma |u|^2 d\sigma \leq \varepsilon \|u\|_{H^1(\Omega)}^2 + C_\varepsilon \|u\|_{L^2(\Omega)}^2$$

which is valid for all  $\varepsilon > 0$  ( $C_\varepsilon$  is a constant depending on  $\varepsilon$ ). Note that this is a consequence of compactness of the trace as an operator from  $H^1(\Omega)$  into  $L^2(\Gamma, d\sigma)$ .

Let  $\mathcal{A}(t)$  be the operator associated with  $\mathfrak{a}(t, \cdot, \cdot)$  and  $\mathcal{B}(t)$  the operator associated with  $\mathfrak{b}(t, \cdot, \cdot)$ . Note that the part  $A(t)$  in  $H := L^2(\Omega)$  of  $\mathcal{A}(t)$  is interpreted as (minus) the Laplacian with time dependent Robin boundary conditions:

$$\partial_\nu v + \beta_1(t, \cdot)v = 0 \text{ on } \Gamma.$$

Here we use the following weak definition of the normal derivative. Let  $v \in H^1(\Omega)$  such that  $\Delta v \in L^2(\Omega)$ . Let  $h \in L^2(\Gamma, d\sigma)$ . Then  $\partial_\nu v = h$  by definition if  $\int_\Omega \nabla v \nabla w dx + \int_\Omega \Delta v \nabla w dx = \int_\Gamma h w d\sigma$  for all  $w \in H^1(\Omega)$ . Based on this definition, the domain of  $A(t)$  is the set

$$D(A(t)) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \partial_\nu u + \beta_1(t, \cdot)u = 0\}$$

and for  $u \in D(A(t))$  the operator is given by  $A(t)u := -\Delta u$ . The same definition for the operator  $B(t)$ .

Note that for any  $\varepsilon > 0$

$$\begin{aligned} & |\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| \\ &= \left| \int_{\partial\Omega} [\beta_1(t, \cdot) - \beta_1(s, \cdot)] uv d\sigma \right| \\ &\leq \|\beta_1(t, \cdot) - \beta_1(s, \cdot)\|_{L^\infty(\partial\Omega)} \|u\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)} \|v\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)}, \end{aligned}$$

where we used the fact that the trace operator is bounded from  $H^{\frac{1}{2}+\varepsilon}(\Omega)$  into  $L_2(\partial\Omega)$ .

Let  $F(t, x, y) : (0, \tau) \times \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$  and  $F_0(t) = F(t, 0, 0)$ . Assume that  $F_0 \in L^2(0, \tau; \mathcal{H})$  and  $F$  satisfies the following continuity property:

$$\|F(t, x_1, y_1) - F(t, x_2, y_2)\|_{L^2(\Omega)} \leq K \left[ \|x_1 - x_2\|_{H^1(\Omega)} + \|y_1 - y_2\|_{H^1(\Omega)} \right],$$

where  $K > 0, x_1, y_1, x_2, y_2 \in H^1(\Omega), t \in (0, \tau)$ . In the next proposition we suppose that  $w_1 \in H^1(\Omega), w_0 \in H^1(\Omega)$ .

**Proposition 6.1.** *There exists a unique solution to the problem*

$$\begin{cases} w''(t) - \Delta w'(t) - \Delta w(t) = F(t, w, w') & t\text{-a.e.} \\ w(0) = w_0, \quad w'(0) = w_1, \\ \partial_\nu(w'(t) + w(t)) + \beta_2(t, \cdot)w'(t) + \beta_1(t, \cdot)w(t) = 0 & \text{on } \Gamma, \end{cases} \quad (6.2)$$

where  $w \in H^2(0, \tau; L^2(\Omega)) \cap W^{1,\infty}(0, \tau; H^1(\Omega))$  and  $w'(t) \in \mathcal{V}$  for all  $t \in [0, \tau]$ .

The proposition follows from Theorem 5.3.

Maximal  $L^2$ -regularity for the Laplacian with time dependent Robin boundary condition with  $\beta_1 = \beta_2$  and  $w_0 = w_1 = 0$  was previously proved in [15] and maximal  $L^2$ -regularity with  $t \rightarrow \beta_1(t, \cdot), \beta_2(t, \cdot) \in C^1$  was proved in [22], for  $t \rightarrow \beta_1(t, \cdot), \beta_2(t, \cdot) \in C^{\frac{1}{2}+\varepsilon}$  the result was proved in [1].

## 6.2 Elliptic operators on $\mathbb{R}^d$

Let  $\mathcal{H} = L^2(\mathbb{R}^d)$  and  $\mathcal{V} = H^1(\mathbb{R}^d)$ . Suppose that  $a_{jk}^l \in L^\infty(I \times \mathbb{R}^d)$ , where  $I = [0, \tau]$  and  $j, k \in (1, \dots, d), l \in (1, 2)$  and there exists a constant  $\alpha > 0$ , such that

$$\sum_{j,k=1}^d a_{jk}^l(t, x) \xi_j \overline{\xi_k} \geq \delta |\xi|^2 \quad (t \in I, x \in \mathbb{R}^d, \xi \in \mathbb{C}^d).$$

We put  $A^l(t, x) = (a_{jk}^l(t, x))_{1 \leq j, k \leq d}$ . We define the forms

$$\mathfrak{a}^l(t, u, v) = \sum_{j,k=1}^d \int_{\mathbb{R}^d} a_{jk}^l(t, x) \partial_j u \overline{\partial_k v} dx$$

with domain  $\mathcal{V} = H^1(\mathbb{R}^d)$ . For each  $t$ , the corresponding operator is formally given by

$$A(t)^l = - \sum_{j,k=1}^d \partial_j (a_{jk}^l(t, x) \partial_k) = -\operatorname{div} \left( (a_{jk}^l(t, \cdot))_{jk} \nabla \right).$$

Next we assume that  $A^l(t, x) \in H^{\frac{1}{2}}(0, \tau; L^\infty(\mathbb{C}^{d^2}))$ . We note that

$$\|\mathcal{A}(t)^l - \mathcal{A}(s)^l\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \leq M' \|A^l(t, \cdot) - A^l(s, \cdot)\|_{L^\infty(\mathbb{C}^{d^2})}$$

for some constant  $M'$ . This implies that  $\mathcal{A}(\cdot)^l \in H^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ . We assume in addition that each  $a_{jk}^l$  is Hölder continuous of order  $\alpha$  for some  $\alpha > 0$  with

$$|a_{jk}^l(t, x) - a_{jk}^l(s, x)| \leq c|t - s|^\alpha$$

for a.e.  $x \in \mathbb{R}^d$ . This assumption implies in particular (3.5). We could also weaken this assumption by formulating it in terms of the modulus of continuity, see (3.7).

We are now allowed to apply Theorem 3.10. We obtain the following proposition

**Proposition 6.2.** *Let  $u_0, u_1 \in H^1(\mathbb{R}^d)$  and  $f \in L^2(0, \tau; L^2(\mathbb{R}^d))$ . There exists a unique  $u \in H^2(I; L^2(\mathbb{R}^d)) \cap W^{1,\infty}(I; H^1(\mathbb{R}^d))$  such that*

$$\begin{cases} u''(t) - \operatorname{div}\left((a_{jk}^1(t, \cdot))_{jk} \nabla u'(t)\right) - \operatorname{div}\left((a_{jk}^2(t, \cdot))_{jk} \nabla u(t)\right) = f & t\text{-a.e.} \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \quad (6.3)$$

As we already mentioned before, the uniform Kato square root property required in Theorem 3.10 is satisfied in this setting, see [11](Theorem 6.1). Then Proposition 6.2 follows from Theorem 3.10.

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