



HAL
open science

On characteristic forms of positive vector bundles, mixed discriminants and pushforward identities

Siarhei Finski

► **To cite this version:**

Siarhei Finski. On characteristic forms of positive vector bundles, mixed discriminants and pushforward identities. 2020. hal-02943833v1

HAL Id: hal-02943833

<https://hal.science/hal-02943833v1>

Preprint submitted on 20 Sep 2020 (v1), last revised 12 Apr 2021 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On characteristic forms of positive vector bundles, mixed discriminants and pushforward identities.

Siarhei Finski

Abstract. We prove that Schur polynomials in Chern forms of Nakano and dual Nakano positive vector bundles are positive as differential forms. Moreover, modulo a statement about the positivity of a “double mixed discriminant” of linear operators on matrices, which preserve the cone of positive-definite matrices, we establish that Schur polynomials in Chern forms of Griffiths-positive vector bundles are weakly-positive as differential forms.

An important step in our proof is to establish a certain pushforward identity for characteristic forms, refining the determinantal formula of Kempf-Laksov for homomorphic vector bundles on the level of differential forms. In the same vein, we establish a local version of Jacobi-Trudi identity.

1 Introduction

The main goal of this paper is to study positivity of characteristic forms for positive vector bundles.

To fix the notation, let X be a smooth complex manifold of dimension n . Let (E, h^E) be a Hermitian vector bundle of rank r over X , and let $R^E := (\nabla^E)^2$ be the curvature of the Chern connection ∇^E of (E, h^E) . We consider the Chern forms $c_i(E, h^E)$, $i = 0, \dots, r$, defined by

$$\det \left(\text{Id}_E + \frac{\sqrt{-1}tR^E}{2\pi} \right) = \sum_{i=0}^r c_i(E, h^E)t^i. \quad (1.1)$$

By Chern-Weil theory, for $i = 0, \dots, r$, the Chern form $c_i(E, h^E)$ is a d -closed real (i, i) -form on X , lying in the cohomology class of the i -th Chern class of E , denoted here by $c_i(E)$.

Fix $k \in \mathbb{N}$ and denote by $\Lambda(k, r)$ the set of all partitions of k by decreasing non negative integers $\leq r$, i.e. $a \in \Lambda(k, r)$ is a sequence $r \geq a_1 \geq a_2 \geq \dots \geq a_k \geq 0$ such that

$$a_1 + \dots + a_k = k. \quad (1.2)$$

Each $a \in \Lambda(k, r)$ gives rise to a *Schur polynomial* $P_a \in \mathbb{Z}[c_1, \dots, c_r]$ of weighted degree $2k$ (with $\deg c_i = 2i$), defined through the following determinant

$$P_a(c) = \det(c_{a_i - i + j})_{i,j=1}^k, \quad (1.3)$$

where by convention $c_0 = 1$ and $c_i = 0$ if $i > r$ or $i < 0$. Schur polynomials P_a , for $a \in \Lambda(k, r)$, form a basis of the vector space of polynomials of weighted degree $2k$, and the product of two Schur polynomials is a linear combination of Schur polynomials with positive coefficients, cf. [16].

Now, for every Hermitian vector bundle (E, h^E) of rank r over X , $k \in \mathbb{N}$, $k \leq n$, and $a \in \Lambda(k, r)$, we consider a differential form on X , obtained by substitution of $c_i(E, h^E)$ for c_i in (1.3).

The resulting (k, k) -differential form, $P_a(c(E, h^E))$, which we later call a *Schur form*, is closed. We denote the associated cohomological class by $P_a(c(E))$. Interesting examples are

$$P_a(c(E)) = \begin{cases} c_k(E), & \text{for } a = k00\dots, \\ \text{Segre class } s_k(E), & \text{for } a = 11\dots10\dots0, \\ c_1(E)c_{k-1}(E) - c_k(E), & \text{for } a = k - 110\dots0. \end{cases} \quad (1.4)$$

Griffiths in [19] gave a conjectural analytic description of the cone of *numerically positive* homogeneous polynomials $P \in \mathbb{R}[c_1, \dots, c_r]$ (with $\deg c_i = 2i$), i.e. such that for any Griffiths positive vector bundle (E, h^E) over a complex manifold X (see Section 2 for a definition) and any analytical subset $V \subset X$, P satisfies

$$\int_V P(c_1(E), \dots, c_r(E)) \geq 0. \quad (1.5)$$

In [4], Bloch-Gieseker proved that Chern classes of ample vector bundles (see Section 2 for a definition) satisfy (1.5). By using this result and Kempf-Laksov [22] formula, Fulton-Lazarsfeld in [16] proved that (1.5) holds for any ample vector bundle E and any Schur polynomial P . This extends the previous works of Kleiman [23] who proved it for surfaces, Gieseker [18] who proved it for monomials of Chern classes, Usui-Tango [38], who established (1.5) for ample and globally generated E . Demailly-Peternell-Schneider in [11] extended this result to nef vector bundles on compact Kähler manifolds. Fulton-Lazarsfeld also proved in [16] that if (1.5) holds for any ample E , then P should be a linear combination with positive coefficients of Schur polynomials. This gives an answer to the question of Griffiths and provides an algebraic description (through Schur polynomials) of the cone of numerically positive polynomials. In [16, Appendix A], Fulton-Lazarsfeld proved that this description coincides with the analytic description of Griffiths.

Griffiths in [19, p. 247] proposed to refine the positivity in (1.5) in differential-geometric sense. For this, recall that a (i, i) -differential form α on X is called *weakly-positive*, see Reese-Knapp [33], if at any point $x \in X$, the restriction of α to any i -dimensional complex plane in $T_x X \otimes_{\mathbb{R}} \mathbb{C}$ gives a non negative volume form with respect to the canonical orientation class of the complex structure. Using the description of the Griffiths cone due to Fulton-Lazarsfeld [16, Appendix A], we can reformulate the question from [19, p. 247] in the following way.

Question of Griffiths. Let $P \in \mathbb{R}[c_1, \dots, c_r]$ be a non negative homogeneous linear combination of Schur polynomials. Are the forms $P(c_1(E, h), \dots, c_r(E, h))$ weakly-positive for any Griffiths positive vector bundle (E, h) over a complex manifold X ?

Studying this question is one of the main goals of this article. To formulate our first result, recall that a (i, i) -differential form α on X is called *positive*, cf. [33], if for any $(n - i, 0)$ -form β , the form $\alpha \wedge (\sqrt{-1})^{n-k} \beta \wedge \bar{\beta}$ is non negative. As name suggests, positivity is a more stronger notion than weak positivity, see [33], cf. Section 2.

Theorem 1.1. *Let (E, h^E) be a Nakano or dual Nakano positive (see Section 2 for definitions) vector bundle of rank r over a complex manifold X of dimension n . Then for any $k \in \mathbb{N}$, $k \leq n$, and $a \in \Lambda(k, r)$, the (k, k) -differential form $P_a(c(E, h^E))$ is positive.*

Remark 1.2. This theorem permits to construct positive characteristic classes for Griffiths positive vector bundles (E, h^E) , as from Demailly-Skoda [12], the induced metric on $E \otimes \det E$ is Nakano positive. See Berndtsson [3], Liu-Sun-Yang [26] for related results on ample vector bundles.

To formulate our second result, recall that for a complex vector space V , $\dim V = r$, the *mixed discriminant* $D_V : \text{End}(V)^{\otimes r} \rightarrow \mathbb{C}$ was defined by Alexandroff [1, §1] as the polarization of the determinant. In coordinates, for matrices $A^i := (a_{kl}^i)_{k,l=1}^r$, $i = 1, \dots, r$, we have

$$D_V(A^1, \dots, A^r) = \frac{1}{r!} \sum_{\sigma \in S_r} \det(a_{ik}^{\sigma(i)})_{i,k=1}^r, \quad (1.6)$$

where S_r is the permutation group on $\{1, 2, \dots, r\}$. We use the natural duality $\text{End}(V) \simeq \text{End}(V)^*$ and denote by $D_V^* : \mathbb{C} \rightarrow \text{End}(V)^{\otimes r}$ the dual of D_V .

For Hermitian vector spaces V, E , a linear operator $P : \text{End}(V) \rightarrow \text{End}(E)$ is called a *positive (semi)definite linear preserver* (the terminology is from Li-Pierce [24], see also Størmer [37]) if $P(\mathcal{C}_V) \subseteq \mathcal{C}_E$ for positive (semi)definite operators \mathcal{C}_V in $\text{End}(V)$ (resp. \mathcal{C}_E in $\text{End}(E)$).

Open problem. Let $P : \text{End}(V) \rightarrow \text{End}(E)$ be a positive semidefinite linear preserver, and $\dim V = \dim E$. Is it true that $D_E \circ P^{\otimes r} \circ D_V^* \in \mathbb{R}$ is non negative?

In the special case when P is zero on non diagonal matrices for some basis of V , Open problem holds due to the Alexandroff inequality [1, §1], stating non negativity of the mixed discriminant for positive semidefinite matrices. See also Panov [31], Bapat [2] for refinements of this inequality and Florentin-Milman-Schneider [14] for a characterization of the mixed discriminant through it.

Theorem 1.3. *The answers to Open problem and Question of Griffiths coincide.*

Now, although we couldn't find a complete proof to Open problem, we have a partial result.

Proposition 1.4. The answer to Open problem is positive under any of the additional assumptions

- a) We have $P(\mathcal{C}_V) = \mathcal{C}_E$.
- b) Among the operators $P' : \text{Hom}(V, E) \rightarrow \text{Hom}(V, E)$, $P'' : \text{Hom}(V, E^*) \rightarrow \text{Hom}(V, E^*)$, associated by the natural isomorphisms $\text{Hom}(\text{End}(V), \text{End}(E)) \simeq \text{End}(\text{Hom}(V, E))$ and $\text{Hom}(\text{End}(V), \text{End}(E)) \simeq \text{End}(\text{Hom}(V, E^*))$ to P , there is at least one positive semidefinite.
- c) Among the operators $P^{\otimes r} : \text{End}(V^{\otimes r}) \rightarrow \text{End}(E^{\otimes r})$, $(P^T)^{\otimes r} : \text{End}(V^{\otimes r}) \rightarrow \text{End}((E^*)^{\otimes r})$, for $P^T(X) = P(X)^T$, $X \in \text{End}(V)$, there is at least one positive semidefinite linear preserver.
- d) We have $\dim V = \dim E = 2$.

Let's now put Theorems 1.1, 1.3 in the context of previous results. Griffiths in [19] verified his own question for $c_2(E, h^E)$ by explicit evaluation. Bott-Chern in [6, Lemma 5.3] gave an algebraic proof of the fact the top Chern class of a Bott-Chern positive vector bundle (see Section 2 for a definition, the terminology is due to P. Li [25]) is strongly-positive (see Section 2 for a definition), and then P. Li in [25] extended the methods of Bott-Chern for all Schur forms. As we explain in Proposition 2.3, Bott-Chern positivity is equivalent to dual Nakano positivity. As strong positivity is stronger than positivity, Theorem 1.1 for dual Nakano positive metrics is then a weak version of Li's result (our methods are drastically different). Guler in [20] verified the question of Griffiths for Serge forms, see (1.4). See Diverio [13], Pingali [32], Ross-Toma [34] for related results.

Now, let's describe some applications of Theorem 1.1. Recall that very recently, Demailly in [10] proposed an elliptic system of differential equations of Hermitian-Yang-Mills type for the curvature tensor of a vector bundle with an ample determinant. This system of differential equations is designed so that the existence of a solution to it implies the existence of a dual Nakano positive Hermitian metric on the vector bundle. So if it can be proved that for any ample vector bundle a solution exists, then it would imply a strong version of Griffiths conjecture on the equivalence between ampleness and Griffiths-positivity for vector bundles. This has led Demailly to conjecture in [10, Basic question 1.7] that the ampleness for a vector bundle over a compact manifold is equivalent to the existence of a dual Nakano positive metric. Theorem 1.1 implies

Corollary 1.5. If Demailly's conjecture [10, Basic question 1.7], described above, is true, then for any ample vector bundle E of rank r over a compact manifold X , and any $k \in \mathbb{N}$, $k \leq n$, $a \in \Lambda(k, r)$, the cohomological class of $P_a(c(E))$ contains a positive form.

The last statement was proved (unconditionally on [10, Basic question 1.7]) by Xiao in [39] for $k = n - 1$ in a different way. The general case was conjectured in [39, Conjecture 1.4].

Let's now describe our strategy of the proofs of Theorems 1.1, 1.3. Similarly to [16], the proof decomposes into two separate statements. The first one is a refinement of the determinantal formula of Kempf-Laksov [22] on the level of differential forms. It expresses the Schur forms as a certain pushforward of the top Chern form of a Hermitian vector bundle obtained as a quotient of the tensor power of (E, h) . The second statement establishes the positivity of the top Chern form of a (dual) Nakano positive vector bundle and relates the positivity of the top Chern form of a Griffiths positive vector bundle to the Open problem.

This article is organized as follows. In Section 2, we recall several concepts of positivity for vector bundles and differential forms. In Section 3, we prove Theorems 1.1, 1.3 modulo some results which are established later in the article. In Section 4, we establish the refinement of Kempf-Laksov formula on the level of differential forms. In Section 5, we discuss the positivity of the top Chern form and its relation with Open problem. In Section 6, we establish Proposition 1.4. Finally, in Section 7, we use the methods developed in Section 4, to give a local version of the Jacobi-Trudi identity for holomorphic vector bundles.

Acknowledgements. Author would like to express his deepest gratitude for Jean-Pierre Demailly for numerous discussions on the subject of this article. He also thanks the colleagues and staff from Institute Fourier, University Grenoble Alps for their hospitality. This work is supported by the European Research Council grant ALKAGE number 670846 managed by J.-P. Demailly.

Notation. For a Hermitian vector bundle (F, h^F) , we denote by $R^F := (\nabla^F)^2$ the curvature of the associated Chern connection ∇^F .

2 Positivity concepts for vector bundles and differential forms

The main goal of this section is to review several notions of positivity for Hermitian vector bundles and differential forms, and to describe relations between them.

Following Hartshorne, [21, §2], we say that a vector bundle E over a complex manifold X is *ample* if for every coherent sheaf \mathcal{F} , there is an integer $n_0 > 0$, such that for every $n > n_0$, the sheaf $\mathcal{F} \otimes S^n E$ is generated as an \mathcal{O}_X -module by its global sections. According to [21, Proposition 3.2], ampleness of E is equivalent to the ampleness of the line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ over $\mathbb{P}(E^*)$.

We fix a Hermitian vector bundle (E, h^E) . We let $r := \text{rk}(E)$, $n := \dim X$. Fix some holomorphic coordinates (z_1, \dots, z_n) on X , an orthonormal frame e_1, \dots, e_r of E , and decompose the curvature as follows

$$\frac{\sqrt{-1}R^E}{2\pi} = \sum_{1 \leq j, k \leq n} \sum_{1 \leq \lambda, \mu \leq r} c_{jk\lambda\mu} \sqrt{-1} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu. \quad (2.1)$$

We say that (E, h^E) is *Griffiths positive* if the associated quadratic form

$$\tilde{\Theta}^E(v \otimes \xi) := \frac{1}{2\pi} \langle \sqrt{-1}R^E(v, \bar{v})\xi, \xi \rangle_{h^E} = \sum_{1 \leq j, k \leq n} \sum_{1 \leq \lambda, \mu \leq r} c_{jk\lambda\mu} \xi_\lambda \bar{\xi}_\mu v_j \bar{v}_k \quad (2.2)$$

takes positive values on non zero tensors $v \otimes \xi \in T_x^{1,0} X \otimes E$. Griffiths in [19] proved that Griffiths positivity implies ampleness.

Let's now construct the linear operator $P_x^E : T_x^{1,0} X \otimes E_x \rightarrow T_x^{1,0} X \otimes E_x$ by

$$P_x^E(\tau) = \sum_{1 \leq j, k \leq n} \sum_{1 \leq \lambda, \mu \leq r} c_{jk\lambda\mu} \tau_{j\lambda} \frac{\partial}{\partial z_k} \otimes e_\mu, \quad (2.3)$$

where $\tau = \sum \tau_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda$. *Nakano positivity*, see [30], demands positive definiteness of P_x^E (we endow $T_x^{1,0} X$ with the Hermitian metric making the basis $\frac{\partial}{\partial z_i}, i = 1, \dots, n$, orthonormal).

Dual Nakano positivity stipulates that the linear operator $P_x^{E*} : T_x^{1,0} X \otimes E_x^* \rightarrow T_x^{1,0} X \otimes E_x^*$, associated to P_x^E by the natural isomorphism $\text{End}(T_x^{1,0} X \otimes E_x) \simeq \text{End}(T_x^{1,0} X \otimes E_x^*)$, is positive definite. In local coordinates the operator takes form

$$P_x^{E*}(\tau^*) := \sum_{1 \leq j, k \leq n} \sum_{1 \leq \lambda, \mu \leq r} c_{jk\mu\lambda} \tau_{j\lambda}^* \frac{\partial}{\partial z_k} \otimes e_\mu^*, \quad (2.4)$$

where $\tau = \sum \tau_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda^*$ in $T_x^{1,0} X \otimes E_x^*$. Griffiths positivity is weaker than (dual) Nakano positivity, cf. Demailly [9, Proposition 6.6].

Let's reformulate the notion of Griffiths positivity in terms of *linear preservers*. We denote by $P_x^{E, \text{Hom}} : \text{End}(T_x^{1,0} X) \rightarrow \text{End}(E_x)$ the linear operator associated to P_x^E by the natural isomorphism $\text{End}(T_x^{1,0} X \otimes E_x) \simeq \text{Hom}(\text{End}(T_x^{1,0} X), \text{End}(E_x))$.

Proposition 2.1. A Hermitian vector bundle (E, h^E) is Griffiths positive if and only if at any point $x \in X$, the operator $P_x^{E, \text{Hom}}$ sends non zero positive semidefinite operators to positive definite operators. In particular, $P_x^{E, \text{Hom}}$ is a positive semidefinite linear preserver in this case.

Remark 2.2. The operator $P_x^{E, \text{Hom}}$ is a positive semidefinite linear preserver if and only if the associated operator $P_x^{E*, \text{Hom}} \in \text{Hom}(\text{End}(E_x), \text{End}(T_x^{1,0} X))$ is a positive semidefinite linear preserver, cf. [37, Proposition 1.4.3a)].

Proof. As any Hermitian positive definite matrix has an orthonormal basis of eigenvectors with positive eigenvalues, it is enough to verify that for any $v \in T_x^{1,0} X$, the positivity of the form $\tilde{\Theta}^E(\cdot \otimes v) : E_x^{\otimes 2} \rightarrow \mathbb{C}$ is equivalent to the positive definiteness of $P_x^{E, \text{Hom}}(v^* \otimes v)$. This follows directly from the definition of $P_x^{E, \text{Hom}}$. \square

Proposition 2.3. A Hermitian vector bundle (E, h^E) is dual Nakano positive if and only if at any point $x \in X$, there exists a vector space V and an element $A \in T_x^{*(1,0)} X \otimes \text{Hom}(V, E_x)$, satisfying $R_x^E = A \wedge A^*$, for the adjoint $A^* \in T_x^{*(0,1)} X \otimes \text{Hom}(E_x, V)$.

Remark 2.4. It seems that the positivity condition, requiring the existence of A as above, has first appeared in the paper of Bott-Chern [6]. P. Li [25] calls it *Bott-Chern positivity*.

Proof. In one direction, assume that there is A_x as described. Then in the notation of (2.4), $\langle P^{E^*}(\tau^*), \tau^* \rangle = \|A^* \tau^*\|^2$, so (E, h^E) is dual Nakano positive. In other direction, assume P^{E^*} is positive definite. We denote by $\tau_i \in T_x^{1,0} X \otimes E_x^*$, $i = 1, \dots, N$, the basis of eigenvectors of P^{E^*} of the norm $\sqrt{\lambda_i}$, where λ_i is the corresponding eigenvalue. Then one can take $V := \mathbb{C}^N$ and A to be the operator sending the standard basis of \mathbb{C}^N to $\tau_i^* \in T_x^{*(1,0)} X \otimes E_x$. \square

Proposition 2.5. A quotient of a dual Nakano positive vector bundle is dual Nakano positive. The analogical statement holds for Griffiths positive vector bundles.

Remark 2.6. For Nakano positive vector bundles, the quotients are not necessarily Nakano positive, cf. [9, Example 6.8, end of §VII.6].

Proof. For Griffiths positive vector bundles, it is proved in [9, Proposition VII.6.10]. For dual Nakano positive vector bundles, establishing Proposition 2.5 is equivalent to proving that a sub bundle of a Nakano negative vector bundle is a Nakano negative vector bundle. The last statement is proved in [9, Proposition VII.6.10] using the curvature formula for sub bundle. \square

Now let's study positivity for differential forms. Recall that a (i, i) -differential form α on X is called *strongly-positive*, cf. [33], if it can be represented as a linear combination with positive coefficients of the forms $\sqrt{-1}\beta_1 \wedge \bar{\beta}_1 \wedge \dots \wedge \sqrt{-1}\beta_i \wedge \bar{\beta}_i$ for some $(1, 0)$ -forms β_j , $j = 1, \dots, i$.

For $i = 0, 1, n-1, n$, all the concepts of positivity for differential forms from this article coincide. For other i , all strongly-positive (i, i) forms are positive, and all positive (i, i) forms are weakly-positive. Moreover, those are strict inclusions, see [33, Corollary 1.6].

The products of positive forms is positive and the product of strongly-positive forms is strongly-positive, [33, Corollary 1.3]. The analogical statement for weakly-positive forms is known to be false, [33, Proposition 1.5], see, however, Błocki-Pliś [5] for a related result.

Proposition 2.7 (Reese-Knapp [33, Definition 1.1, (1.4')]). A (i, i) -differential form α on X is weakly-positive if and only if for any local holomorphic chart (z_1, \dots, z_n) on X , $\alpha \wedge \sqrt{-1}dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge \sqrt{-1}dz_{n-i} \wedge d\bar{z}_{n-i}$ is a positive multiple of $\sqrt{-1}dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge \sqrt{-1}dz_n \wedge d\bar{z}_n$.

Proposition 2.8 (Reese-Knapp [33, Theorem 1.2]). A (i, i) -differential form α on X is positive if and only if for some Hermitian product on $T^{1,0}X$, the operator in $\text{End}(\Lambda^i T^{1,0}X)$, associated to α , is positive-definite.

Corollary 2.9. Let α be a (weakly-)positive differential form on a complex manifold Y , and let $\pi : Y \rightarrow B$ be a proper holomorphic map between complex manifolds. Then $\pi_*[\alpha]$ is also a (weakly-)positive differential form.

Proof. By the definition of pushforward, we have

$$\pi_*[\alpha] \wedge \bigwedge_{i=1}^{n-i} \sqrt{-1} \alpha_i \wedge \bar{\alpha}_i = \pi_* \left[\alpha \wedge \bigwedge_{i=1}^{n-i} \sqrt{-1} \pi^* \alpha_i \wedge \pi^* \bar{\alpha}_i \right]. \quad (2.5)$$

We conclude by Proposition 2.7 and (2.5). Positive forms are treated in the same way, we only need to use the definition of positive forms instead of Proposition 2.7. \square

3 Positivity of Schur forms, proofs of Theorems 1.1, 1.3

In this section we describe the proofs for Theorems 1.1, 1.3 modulo some technical statements, which will be treated in further sections. The first step of the proof is common for both theorems, and it uses a refinement of the determinantal formula of Kempf-Laksov, [22].

To start with, let's recall the original statement from [22]. We fix r, k and a partition $a \in \Lambda(k, r)$. Let E be a smooth complex vector bundle of rank r over a smooth real manifold X . Let V be a complex vector space of dimension $k + r$ and let V_X be the trivial vector bundle $X \times V$.

Fix a flag of subspaces $\{0\} \subset V_1 \subset \dots \subset V_k \subset V$ with $\dim V_i = r + i - a_i$. Consider the cone $\Omega_a(E) \subset \text{Hom}(V_X, E)$, whose fiber over $x \in X$ consists of $u \in \text{Hom}(V, E_x)$, satisfying

$$\dim(\ker u \cap V_i) \geq i. \quad (3.1)$$

We denote by $\mathbb{P}_{\text{Hom}} := \mathbb{P}(\text{Hom}(V_X, E) \oplus \mathcal{O})$ the compactification of $\text{Hom}(V_X, E)$ by the hyperplane at infinity and by $\pi : \mathbb{P}_{\text{Hom}} \rightarrow X$ the obvious projection. We denote by

$$\overline{\Omega_a(E)} = \mathbb{P}(\Omega_a(E) \oplus \mathcal{O}), \quad \pi^a : \overline{\Omega_a(E)} \rightarrow X \quad (3.2)$$

the closure of $\Omega_a(E)$ in \mathbb{P}_{Hom} and the restriction of π to $\overline{\Omega_a(E)}$. Note that $\Omega_a(E)$ is locally a trivial cone bundle with analytic fibers over X of codimension k in $\text{Hom}(V_X, E)$. We denote by $Z_{\text{Hom}(V_X, E)} \subset \mathbb{P}_{\text{Hom}}$ the image of the zero section of $\text{Hom}(V_X, E)$. There are well-defined cohomology classes

$$\{Z_{\text{Hom}(V_X, E)}\} \in H^{2(r+k)r}(\mathbb{P}_{\text{Hom}}, \mathbb{R}), \quad \{\overline{\Omega_a(E)}\} \in H^{2k}(\mathbb{P}_{\text{Hom}}, \mathbb{R}), \quad (3.3)$$

which can be seen as the cohomology classes of the induced closed currents $[Z_{\text{Hom}(V_X, E)}], [\overline{\Omega_a(E)}]$.

Theorem 3.1 (Kempf-Laksov [22]). *The following identity between cohomology classes holds*

$$P_a(c(E)) = \pi_* \left[\{Z_{\text{Hom}(V_X, E)}\} \cdot \{\overline{\Omega_a(E)}\} \right]. \quad (3.4)$$

Now, we would like to prove a refinement of Theorem 3.1 on the level of differential forms in the holomorphic setting. We assume from now on that X is a complex manifold and E is a holomorphic vector bundle. We denote by $\mathcal{O}_{\mathbb{P}_{\text{Hom}}}(-1)$ the tautological line bundle on \mathbb{P}_{Hom} , and define the hyperplane bundle $Q := (\pi^* \text{Hom}(V_X, E) \oplus \mathcal{O}) / \mathcal{O}_{\mathbb{P}_{\text{Hom}}}(-1)$ on \mathbb{P}_{Hom} .

Since $\Omega_a(E)$ is locally a trivial cone bundle with analytic fibers over X of codimension k in $\text{Hom}(V_X, E)$, for any smooth (l, l) -differential form α , $l \in \mathbb{N}^*$, on \mathbb{P}_{Hom} , one can define the pushforward $\pi_*^a[\alpha]$ as a $(\max\{l - k, 0\}, \max\{l - k, 0\})$ -differential form over X .

Theorem 3.2. *We endow E with a Hermitian metric h^E . Denote by h^Q the Hermitian metric on Q induced by the trivial metric on \mathcal{O} and h^E . The following identity of (k, k) -differential forms holds*

$$P_a(c(E, h^E)) = \pi_*^a [c_{\text{rk}(Q)}(Q, h^Q)]. \quad (3.5)$$

Remark 3.3. We say that Theorem 3.2 is a local version of Theorem 3.1, because (3.5) is a point-wise identity and it makes sense even over contractible manifolds X , whereas the original statement from [22] becomes a triviality for such X .

In Section 4, we prove Theorem 3.2 and show that it is compatible with Theorem 3.1. One ingredient of the proof is the formula for the curvature of the hyperplane bundle over the projectivization of a vector bundle due to Mourougane [29].

Now, Theorem 3.2 and Corollary 2.9 clearly suggest that it is enough to study the positivity of the top Chern form. We do so directly by giving alternative expressions for the top Chern form through tensor calculus in Section 5. Below, we give the first two main result of Section 5.

Proposition 3.4. For any Nakano or dual Nakano positive vector bundle (E, h^E) over a complex manifold X , the form $c_{\text{rk}(E)}(E, h^E)$ is positive.

Proposition 3.5. Let (E, h^E) be a Nakano positive vector bundle over a complex manifold X . Let $\Xi \subset \mathbb{P}(E)$ be a locally a trivial bundle with analytic fibers over X . Denote by π' the restriction of the projection $\pi : \mathbb{P}(E) \rightarrow X$ to Ξ . Define $Q_0 := \pi'^*E/\mathcal{O}_{\mathbb{P}(E)}(-1)$ and endow it with the induced Hermitian metric h^{Q_0} . Then the differential form $\pi'^*[c_{\text{rk}(Q_0)}(Q_0, h^{Q_0})]$ is positive.

Remark 3.6. It is known that (Q_0, h^{Q_0}) is neither Nakano, nor dual Nakano positive in general, so Proposition 3.5 doesn't follow directly from Proposition 3.4 and Corollary 2.9.

Proof of Theorem 1.1. We conserve the notation from Theorem 1.1 and Theorem 3.2. Let's treat the dual Nakano positive case first. First of all, since (E, h^E) is dual Nakano positive, the vector bundle $\pi^*(\text{Hom}(V_X, E)) \oplus \mathcal{O}$ is non strictly dual Nakano positive. From Proposition 2.5, we conclude that (Q, h^Q) is non strictly dual Nakano positive. By Theorem 3.2, Proposition 3.4 and Corollary 2.9, we prove Theorem 1.1 for dual Nakano positive vector bundles. For Nakano positive vector bundles, the result follows from Theorem 3.2 and Proposition 3.5. \square

Now, to establish Theorem 1.3, we need the following proposition. See Section 5 for a proof.

Proposition 3.7. The following statements are equivalent.

- a) For any Griffiths positive vector bundle (E, h^E) over any complex manifold X , the form $c_{\text{rk}(E)}(E, h^E)$ is weakly-positive.
- b) The answer to Open question is positive.

Proof of Theorem 1.3. By Proposition 3.7, it is enough to prove that if the answer to Open question is positive, then the answer to Griffiths question is positive as well. We use the notation from Theorem 1.3 and Theorem 3.2. As (E, h^E) is Griffiths positive, (Q, h^Q) is non strictly Griffiths positive by Proposition 2.5. By Proposition 3.7, the top Chern form of (Q, h^Q) is weakly-positive. We conclude by Corollary 2.9 and Theorem 3.2. \square

4 Refinement of the determinantal formula of Kempf-Laksov

The main goal of this section is to establish Theorem 3.2. We start by verifying that Theorem 3.2 is compatible with Theorem 3.1. We conserve the notation from Theorems 3.1, 3.2.

Since the canonical section of Q (i.e. image of $0 \oplus 1$ in Q) admits $Z_{\text{Hom}(V_X, E)}$ as its transversal zero locus, we have $c_{\text{rk}(Q)}(Q) = \{Z_{\text{Hom}(V_X, E)}\}$. But as $\Omega_a(E)$ is locally a product over X , we have

$$\pi_* \left[\{Z_{\text{Hom}(V_X, E)}\} \cdot \{\overline{\Omega_a(E)}\} \right] = \pi_*^a [c_{\text{rk}(Q)}(Q)], \quad (4.1)$$

which proves the compatibility of Theorem 3.2 with Theorem 3.1 by Chern-Weil theory.

The proof of Theorem 3.2 consists of two steps. The first step, encapsulated in Lemma 4.1, says that the right-hand side of (3.5) is a polynomial in terms of the components of the curvature of (E, h^E) with respect to some fixed basis of E . In the second step, using a topological argument, we show that this polynomial coincides with the left-hand side of (3.5).

Let's fix some further notation. We fix a point $x \in X$ and a local holomorphic frame e_1, \dots, e_r for E , orthonormal at x . For $1 \leq i, j \leq r$, we denote

$$c_{ij} := \frac{1}{2\pi} \langle (\sqrt{-1}R^E)e_i, e_j \rangle_{h^E}. \quad (4.2)$$

Lemma 4.1. For any $r \in \mathbb{N}^*$, there is a polynomial $P(d_{ij})$ in the entries of a self-adjoint matrix $D = (d_{ij})_{i,j=1}^r$, such that for any $X, (E, h^E), \pi^a, (Q, h^Q), c_{ij}$ as in Theorem 3.2 and (4.2), we have

$$\pi_*^a [c_{\text{rk}(Q)}(Q, h^Q)] = P(c_{ij}). \quad (4.3)$$

Now, for $p, N \in \mathbb{N}, p < N$, we denote by $\text{Gr}_{\mathbb{C}}(p, N)$ the complex Grassmannian, and by E' the tautological vector bundle of rank p over $\text{Gr}_{\mathbb{C}}(p, N)$. The proof of the following lemma is given in the end of this section.

Lemma 4.2. Let $k \in \mathbb{N}$ satisfy $k \leq N - p$. Then the cohomology group $H^{2k}(\text{Gr}_{\mathbb{C}}(p, N), \mathbb{C})$ is freely generated as a vector space by the monomials of $c_1(E'), \dots, c_p(E')$ of degree $2k$.

Proof of Theorem 3.2. We conserve the notation from Theorem 3.2. Let $P(d_{ij})$ be as in Lemma 4.1. Clearly, the right-hand side of (4.3) is invariant under the action of the group $U(r)$ on the matrix $(c_{ij}), 1 \leq i, j \leq r$, by conjugation, as this only amounts to choosing another frame e_1, \dots, e_r , and the left-hand side of (4.3) is independent of this choice.

This implies that the polynomial $P(d_{ij})$ is invariant under the action of the group $U(r)$ on $D := (d_{ij})$. In particular, for a diagonal matrix D , the polynomial $P(d_{ij})$ can be expressed as a linear combination of symmetric polynomials in diagonal entries of D . But any self-adjoint matrix can be diagonalized by the action of the group $U(r)$, so $P(d_{ij})$ is actually a polynomial of $\text{Tr}[D], \text{Tr}[\Lambda^2 D], \dots, \text{Tr}[\Lambda^r D]$, where $\Lambda^i D$ is the i -th wedge power of D . In other words, for any $b \in \Lambda(k, r)$, there is a coefficient $a_b \in \mathbb{R}$, which is universal in the same sense as P , so that

$$P(d_{ij}) = \sum_{b \in \Lambda(k, r)} a_b \cdot \text{Tr}[D]^{b(1)} \cdot \text{Tr}[\Lambda^2 D]^{b(2)} \cdot \dots \cdot \text{Tr}[\Lambda^r D]^{b(r)}, \quad (4.4)$$

where $b(i), i = 1, \dots, r$ is the number of times i appears in the partition b .

Recall that for any $b \in \Lambda(k, r)$, we have defined a Schur form $P_b(c(E, h^E))$ after (1.3). Clearly, proving Theorem 3.2 is now equivalent by (1.1) to proving that for any $b \in \Lambda(k, r)$, we have $a_b = c_b$, where the coefficients $c_b \in \mathbb{R}$ are defined by expanding (1.3) to

$$P_a(c(E, h^E)) = \sum_{b \in \Lambda(k, r)} c_b \cdot c_1(E, h^E)^{\wedge b(1)} \wedge \dots \wedge c_r(E, h^E)^{\wedge b(r)}. \quad (4.5)$$

Let's now establish that $a_b = c_b$. By Theorem 3.1 and the discussion after it, Lemma 4.1, Chern-Weil theory and (4.4), (4.5), on the level of cohomology, we have

$$\sum_{b \in \Lambda(k, r)} a_b \cdot c_1(E)^{b(1)} \cdot \dots \cdot c_r(E)^{b(r)} = \sum_{b \in \Lambda(k, r)} c_b \cdot c_1(E)^{b(1)} \cdot \dots \cdot c_r(E)^{b(r)}. \quad (4.6)$$

It is only left to apply (4.6) for $X := \text{Gr}_{\mathbb{C}}(r, N)$, $N > k - r$, and E the tautological r -bundle to see that Lemma 4.2 implies $a_b = c_b$. \square

Now, to establish Lemma 4.1, we need a formula for the curvature of the hyperplane bundle on the projectivization of a vector bundle due to Mourougane [29], which we recall below.

Let (F, h^F) be a Hermitian vector bundle over X of rank r' . Let $\mathcal{O}_{\mathbb{P}(F)}(-1)$ be the tautological bundle over $\mathbb{P}(F)$, $\pi_0 : \mathbb{P}(F) \rightarrow X$, and let $Q' := \pi_0^* F / \mathcal{O}_{\mathbb{P}(F)}(-1)$ be the quotient bundle. We endow Q' with the metric $h^{Q'}$ induced by h^F .

We fix a point $x \in X$, some local coordinates $\mathbf{z} := (z_1, \dots, z_n)$ on X , centered at x , and a local normal frame $f_1, \dots, f_{r'}$ of F at x , defined in a neighborhood U of x . By a *normal frame* we mean one satisfying $\langle f_i, f_j \rangle_{h^F} = \delta_{ij} - \sum_{\lambda\mu} d_{\lambda\mu ij} z_\lambda \bar{z}_\mu + O(|z|^3)$ for some constants $d_{\lambda\mu ij}$. For $1 \leq i, j \leq r'$, we denote

$$g_{ij} := \frac{1}{2\pi} \langle \sqrt{-1} R^F f_i, f_j \rangle_{h^F}. \quad (4.7)$$

The data above defines a trivialization of $U \times \mathbb{P}(\mathbb{C}^{r'}) \rightarrow \mathbb{P}(F)$ near $\pi_0^{-1}(x)$ as follows. For $\mathbf{a} := (a_1, \dots, a_{r'})$, where $a_i \in \mathbb{C}$, $1 \leq i \leq r'$, and not all a_i are equal to zero, the trivialization is given by the following map

$$(\mathbf{z}, [\mathbf{a}]) \rightarrow \left[\sum_{i=1}^{r'} a_i e_i(x) \right] \in \mathbb{P}(F). \quad (4.8)$$

Now we take $a_1 = 1$ and denote $b_i := a_i$, $2 \leq i \leq r'$, $\mathbf{b} := (b_i)$. Then (\mathbf{z}, \mathbf{b}) gives a chart for $\mathbb{P}(F)$ by (4.8). Mourougane in [29, (2.1)] proved that in this chart, at the point $(x, \mathbf{1}) := (x, [1, 0, \dots, 0]) \in \mathbb{P}(F)$, the following formula holds

$$\frac{\sqrt{-1}}{2\pi} R_{(x, \mathbf{1})}^{Q'} = \sum_{2 \leq j, k \leq r'} \left(g_{ij} + \frac{\sqrt{-1}}{2\pi} db_j \wedge d\bar{b}_k \right) \left(\frac{\partial'}{\partial b_k} \right)^* \otimes \left(\frac{\partial'}{\partial b_j} \right), \quad (4.9)$$

where $\frac{\partial'}{\partial b_j} := \frac{\partial}{\partial b_j} \otimes (f_1 + \sum b_i f_i)$ (we implicitly used an isomorphism $Q' \cong T_{\mathbb{P}(F)/X} \otimes \mathcal{O}_{\mathbb{P}(F)}(-1)$).

Proof of Lemma 4.1. First of all, we would like to extend the formula (4.9) to the whole fiber $\pi_0^{-1}(x)$. As we will not make use of an explicit formula, we will content ourselves with some

general remarks in this direction. Clearly, if the frame $f_1, \dots, f_{r'}$ is a normal basis of F at x , then by Gram-Schmidt process, there are some universal functions $p_{ij}(\mathbf{b}, \bar{\mathbf{b}})$, $2 \leq i \leq r'$, $1 \leq j \leq r'$, so that the the following local frame is also normal

$$\frac{1}{\sqrt{1 + |\mathbf{b}|^2}} \left(f_1 + \sum_{i \geq 2} b_i f_i \right), \sum_{j \geq 1} p_{2j}(\mathbf{b}, \bar{\mathbf{b}}) f_j, \dots, \sum_{j \geq 1} p_{r'j}(\mathbf{b}, \bar{\mathbf{b}}) f_j. \quad (4.10)$$

By universal we mean that those functions depend only on r' and on nothing more. Now, if we use the above fact and apply the formula (4.9), we obtain that for any $r' \in \mathbb{N}$, there are some functions $P_{ij}(\mathbf{b}, \bar{\mathbf{b}})$, $1 \leq i, j \leq r'$, and $R_{\gamma\mu}(\mathbf{b}, \bar{\mathbf{b}})$, $2 \leq \gamma, \mu \leq r'$, such that for any (F, h^F) , $(Q', h^{Q'})$ as above, in the chart (\mathbf{z}, \mathbf{b}) , the following formula holds

$$\begin{aligned} \sqrt{-1}R_{(x, \mathbf{b})}^{Q'} = & \sum_{2 \leq \alpha, \beta \leq r} \left(\sum_{1 \leq i, j \leq r} P_{ij}(\mathbf{b}, \bar{\mathbf{b}}) \cdot g_{ij} + \sum_{2 \leq \gamma, \mu \leq r} R_{\gamma\mu}(\mathbf{b}, \bar{\mathbf{b}}) \cdot \sqrt{-1}db_\gamma \wedge d\bar{b}_\mu \right) \\ & \cdot \left(\frac{\partial}{\partial b_\alpha} \right)^* \otimes \left(\frac{\partial}{\partial b_\beta} \right). \end{aligned} \quad (4.11)$$

From (4.11), applied to $F := \text{Hom}(V_X, E) \oplus \mathcal{O}$ with the metric h^F induced by h^E and the trivial metric on \mathcal{O} , we see that for any $r, k \in \mathbb{N}^*$, $a \in \Lambda(k, r)$, there are some constants a_{IJ} , $I, J \in \{1, 2, \dots, r\}^{\times k}$, such that the polynomial $P(d_{ij}) := \sum a_{IJ} \cdot d_{i_1 j_1} \cdot \dots \cdot d_{i_k j_k}$ in the entries of a self-adjoint matrix $D := (d_{ij})$ satisfies (4.3).

Moreover, a_{IJ} , $I, J \in \{1, 2, \dots, r\}^{\times k}$, can be expressed through integrals over the analytic space $\Omega_a(\mathbb{C}^r)$ (where \mathbb{C}^r is viewed as a vector bundle over a point, see (3.1) for the definition of Ω_a) of universal polynomials in functions $P_{ij}(\mathbf{b}, \bar{\mathbf{b}})$, $R_{\gamma\mu}(\mathbf{b}, \bar{\mathbf{b}})$ from (4.11). This implies that a_{IJ} are universal constants, which concludes the proof. \square

Proof of Lemma 4.2. The proof can be found in the lecture notes of Fulton [15, §2]. Here we only highlight the main steps, following closely the exposition from [15].

First of all, we recall that $\text{Gr}_{\mathbb{C}}(p, N) = M_{N \times p}^0 / GL_p(\mathbb{C})$, where $M_{N \times p}^0 \subset M_{N \times p}$ is the subset of all full rank $N \times p$ matrices and the action is given by the right multiplication.

Now, the manifold $M_{N \times p}^0$ is $2(N - p)$ -connected. Indeed, $M_{N \times p}^0$ is the complement of a closed algebraic set of codimension $N - p + 1$ in $M_{N \times p}$, and it is a general fact that $\pi_i(\mathbb{C}^N \setminus Z) = \{1\}$ for $0 < i \leq 2d - 2$ if Z is a Zariski closed subset of codimension d , cf. Fulton [15, §A.4].

Now, the cohomology ring of $\text{Gr}_{\mathbb{C}}(p, \infty)$ is freely generated by the Chern classes of the tautological vector bundle of rank p , cf. [28, Theorem 14.5]. The result now follows from the following lemma, cf. [15, §2, Proposition 2.2]: If $E \rightarrow B$ and $E' \rightarrow B'$ are two principal right G -bundles for a Lie group G , and $H^i(E) = H^i(E') = 0$ for $0 < i \leq k$, then there is a canonical isomorphism $H^i(B) \cong H^i(B')$ for $i \leq k$. \square

5 The top Chern form and the mixed discriminant

The main goal of this section is to describe the connection between the top Chern form and the mixed discriminant. In particular, we establish Propositions 3.7, 3.4, 3.5. To establish those propositions, we find alternative expressions for the top Chern form through tensor calculus.

We fix a Hermitian vector bundle (E, h^E) over a complex manifold X . We let $r := \text{rk}(E)$, $n := \dim X$, fix some holomorphic coordinates (z_1, \dots, z_n) on X , an orthonormal frame e_1, \dots, e_r of E in the neighborhood of a point $x \in X$, and decompose the curvature of (E, h^E) as in (2.1). We endow $T_x^{1,0}X$ with the Hermitian product which turns the vectors $\frac{\partial}{\partial z_i}$ into an orthonormal basis. Now, let's denote by $\tilde{c} \in \text{End}(\Lambda^r T_x^{1,0}X)$ the operator associated to $c_{\text{rk}(E)}(E, h^E)$ as in Proposition 2.8. Recall that the operators P_x^E , P_x^{E*} , $P_x^{E, \text{Hom}}$ were defined in (2.3), (2.4) and after (2.4) respectively.

Now, define the map $(T_x^{1,0}X)^{\otimes r} \rightarrow \text{Sym}^r(T_x^{1,0}X \otimes E_x)$ as follows

$$(v_1, v_2, \dots, v_r) \mapsto \sum_{\sigma \in S_r} (-1)^{[\sigma]} \cdot \bigodot_{i=1}^r (v_i \otimes e_{\sigma(i)}). \quad (5.1)$$

It is easy to see that this operator is antisymmetric with respect to the permutation of coordinates in $(T_x^{1,0}X)^{\otimes r}$. We denote by W_1 the induced operator on the antisymmetric tensors

$$W_1 : (T_x^{1,0}X)^{\wedge r} \rightarrow \text{Sym}^r(T_x^{1,0}X \otimes E_x). \quad (5.2)$$

We denote by $W_1^* : \text{Sym}^r(T_x^{1,0}X \otimes E_x) \rightarrow (T_x^{1,0}X)^{\wedge r}$ the adjoint of W_1 . Similarly, we define the operator $W_2 : (T_x^{1,0}X)^{\wedge r} \rightarrow \text{Sym}^r(T_x^{1,0}X \otimes E_x^*)$ and its adjoint W_2^* .

The main technical statements of this section are given below.

Lemma 5.1. Assume $r = n$. Then at $x \in X$, the following identity holds

$$c_r(E, h^E) = (r!)^2 \cdot \left(D_E \circ (P_x^{E, \text{Hom}})^{\otimes r} \circ D_{T_x^{1,0}X}^* \right) \cdot \sqrt{-1} dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge \sqrt{-1} dz_n \wedge d\bar{z}_n. \quad (5.3)$$

Lemma 5.2. The following identities hold

$$\tilde{c} = W_1^* \circ (P_x^E)^{\otimes r} \circ W_1 = W_2^* \circ (P_x^{E*})^{\otimes r} \circ W_2. \quad (5.4)$$

The proofs of Lemmas 5.1, 5.2 are given in the end of this section. Before this, let's see how they imply Propositions 3.4, 3.5, 3.7.

Proof of Proposition 3.4. For brevity, we restrict ourselves to Nakano positive vector bundles. By definition, it means that the operator P_x^E is positive definite. Hence $(P_x^E)^{\otimes r}$ is positive definite as well. We conclude by Lemma 5.2. \square

Proof of Proposition 3.5. Clearly, it suffices to consider smooth Ξ . Let's fix a point $(x, [e_1]) \in \Xi \subset \mathbb{P}(E)$, $h^E(e_1, e_1) = 1$. We denote by $e_2, \dots, e_k \in T_{(x, [e_1])}^{1,0}\Xi$ the orthonormal basis for the vertical vectors in $T_{(x, [e_1])}^{1,0}\Xi$. We fix a normal frame e_1, \dots, e_r , $r := \text{rk}(E)$, of E , defined in a neighborhood of x , satisfying $e_i(x) = e_i$, $i = 1, \dots, k$, and denote by b_1, \dots, b_r the associated vertical coordinates from (4.8).

We conserve the notation for the $(1, 1)$ -forms c_{ij} , $i, j = 1, \dots, r$ from (4.2). Then by (1.1) and (4.9), the restriction to the horizontal subspace (in coordinates (4.8), associated to e_1, \dots, e_r) of the contraction of $c_{\text{rk}(Q_0)}(Q_0, h^{Q_0})$ with the vertical form $dv_{\Xi} := \sqrt{-1} db_1 \wedge d\bar{b}_1 \wedge \dots \wedge \sqrt{-1} db_{k-1} \wedge d\bar{b}_{k-1}$, is given in the notations of (4.9), coincides at the point $(x, [e_1])$ with

$$\frac{1}{(2\pi)^{k-1}} \Lambda_{\Xi^V}[A] \cdot \det \begin{pmatrix} c_{k+1k+1} & \cdots & c_{k+1r} \\ \vdots & \ddots & \vdots \\ c_{rk+1} & \cdots & c_{rr} \end{pmatrix}, \quad (5.5)$$

where $\Lambda_{\Xi V}[\cdot]$ is the contraction with dv_{Ξ} , and A is the differential form, given by the determinant of the matrix $(\sqrt{-1}db_i \wedge d\bar{b}_j)_{i,j=1}^{k-1}$. An easy calculation shows that the contraction in (5.5) is equal to $(k-1)!$. Also, the matrix in (5.5) is just the restriction of the analogous matrix associated to (E, h^E) . As we assumed that (E, h^E) is Nakano positive, the matrix in (5.5) is associated to a positive definite map from (2.3). By the proof of Proposition 3.4, we conclude that the differential form (5.5) is positive. This finishes our argument by Corollary 2.9. \square

Proof of Proposition 3.7. First, let's assume that b) holds and show that a) then holds as well. By the definition of weak positivity, we may assume $\dim X = \text{rk}(E)$. Then we conclude by Proposition 2.1, Lemma 5.1 and the fact that the answer to the Open question is positive.

Now, let's assume that b) doesn't hold. We fix a semidefinite linear preserver $P : \text{End}(V) \rightarrow \text{End}(E)$, for which the quantity from Open problem is negative. Clearly, by making a small perturbation, we may assume that P sends positive semidefinite operators to positive definite operators, and the quantity from Open problem is still negative. We fix bases $e_1, \dots, e_r, v_1, \dots, v_r$ of E and V respectively. We denote

$$d_{ijkl} := \langle P(e_i^* \otimes e_j)v_k, v_l \rangle. \quad (5.6)$$

We consider a trivial Hermitian vector bundle $F = \mathbb{C}^r \times \mathbb{C}^r$ of rank r over \mathbb{C}^r , and fix the basis f_1, \dots, f_r , given by the standard basis of \mathbb{C}^r . For linear coordinates (z_1, \dots, z_r) on \mathbb{C}^r , we define

$$h^F(f_i, f_j) := \delta_{i,j} + \sum_{k,l=1}^r d_{ijkl} z_k \bar{z}_l. \quad (5.7)$$

Since P preserves positive semidefiniteness, and any Hermitian matrix can be written as a difference of two positive semidefinite matrices, P preserves the set of Hermitian matrices. From this, we see that $d_{ijkl} = \overline{d_{jilk}}$, which implies that h^F induces a Hermitian metric on F (at least in a small neighborhood of $0 \in \mathbb{C}^r$). Moreover, by using the formula for the curvature of the Chern connection, cf. [9, Theorem V.12.4], we see that the operator, constructed by the same rules as in Proposition 2.1 from the curvature of (F, h^F) at $0 \in \mathbb{C}^r$ coincides with P . From this and Proposition 2.1, we conclude that (F, h^F) is Griffiths positive in the neighborhood of 0. We conclude by the assumption on P and Lemma 5.1. \square

Proof of Lemma 5.1. Let's first establish the following identity

$$c_r(E, h^E) = \sum_{\sigma, \rho, \mu \in S_r} (-1)^{[\sigma] + [\rho] + [\mu]} \prod_{i=1}^r c_{\rho(i)\mu(i)i\sigma(i)} \cdot \bigwedge_{i=1}^r \sqrt{-1} dz_i \wedge d\bar{z}_i, \quad (5.8)$$

where by $[\sigma]$ we mean the sign of a permutation σ . We conserve the notation for the $(1, 1)$ -forms $c_{ij}, i, j = 1, \dots, r$ from (4.2). Then from (1.1), we have

$$c_r(E, h^E) = \det \begin{pmatrix} c_{11} & \cdots & c_{1r} \\ \vdots & \ddots & \vdots \\ c_{r1} & \cdots & c_{rr} \end{pmatrix}. \quad (5.9)$$

By the definition of determinant and (5.9), we infer

$$c_r(E, h^E) = \sum_{\sigma \in S_r} (-1)^{[\sigma]} \bigwedge_{i=1}^r c_{i\sigma(i)}. \quad (5.10)$$

We now expand each summand in the right-hand side of (5.10) as follows

$$\begin{aligned} \bigwedge_{i=1}^r c_{i\sigma(i)} &= \sum_{\rho, \mu \in S_r} \prod_{i=1}^r c_{\rho(i)\mu(i)i\sigma(i)} \cdot \bigwedge_{i=1}^r \sqrt{-1} dz_{\rho(i)} \wedge d\bar{z}_{\mu(i)} \\ &= \sum_{\rho, \mu \in S_r} (-1)^{[\rho]+[\mu]} \prod_{i=1}^r c_{\rho(i)\mu(i)i\sigma(i)} \cdot \bigwedge_{i=1}^r \sqrt{-1} dz_i \wedge d\bar{z}_i \end{aligned} \quad (5.11)$$

From (5.9), (5.10) and (5.11), we deduce (5.8).

Let's treat the right hand side of (5.3). From (1.6), we obtain

$$D_E \circ (P_x^{E, \text{Hom}})^{\otimes r} \circ D_{T_x^{1,0}X}^* = \frac{1}{r!} \sum_{\rho \in S_r} (-1)^{[\rho]} D_E(c^{1\rho(1)}, \dots, c^{r\rho(r)}), \quad (5.12)$$

where $c^{ij} \in \text{End}(E)$, $i, j = 1, \dots, r$ are defined by the contraction of $\frac{\sqrt{-1}}{2\pi} R^E$ with $\sqrt{-1} dz_i d\bar{z}_j$. Now, in the notations of (1.6), by renaming the permutations and using the fact that the sign of a permutation is multiplicative, we have the following identity

$$D_V(A^1, \dots, A^r) = \frac{1}{r!} \sum_{\sigma, \tau \in S_r} (-1)^{[\sigma]+[\tau]} \prod_{i=1}^r a_{\sigma(i)\tau(i)}^i. \quad (5.13)$$

Now, by (5.13), we have

$$D_E(c^{1\rho(1)}, \dots, c^{r\rho(r)}) = \frac{1}{r!} \sum_{\sigma, \tau \in S_r} (-1)^{[\sigma]+[\tau]} \prod_{i=1}^r c_{i\rho(i)\sigma(i)\tau(i)}. \quad (5.14)$$

From (5.8), (5.12) and (5.14), we conclude by renaming the permutations and using the fact that the sign of a permutation is multiplicative. \square

Proof of Lemma 5.2. Clearly, both identities are completely analogical, so we only concentrate on proving the former one. As the proof of this lemma is similar in spirit to the proof of Lemma 5.1, we will be brief, and verify that both operators agree on $\frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_r}$. Let's first give an expression for \tilde{c} . By (5.10), we deduce

$$\tilde{c} \left(\frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_r} \right) = \sum_{\sigma, \mu \in S_r} (-1)^{[\sigma]+[\mu]} \sum_{\alpha} \left(\prod_{i=1}^r c_{\mu(i)\alpha(i)i\sigma(i)} \right) \frac{\partial}{\partial z_{\alpha(1)}} \wedge \dots \wedge \frac{\partial}{\partial z_{\alpha(r)}}, \quad (5.15)$$

where the summation for α is done over the set of all maps from $\{1, \dots, r\}$ to $\{1, \dots, n\}$. Now, from (5.1), we deduce that for $v_i \in T_x^{1,0}X$, we have

$$W_1^* \left(\bigcirc_{i=1}^r (v_i \otimes e_{\sigma(i)}) \right) = \begin{cases} 0, & \text{if } \sigma \text{ is not a permutation,} \\ (-1)^{[\sigma]} v_1 \wedge \dots \wedge v_r & \text{if } \sigma \text{ is a permutation.} \end{cases} \quad (5.16)$$

Hence, by (5.1) and (5.16), we have

$$\begin{aligned} W_1^* \circ (P_x^E)^{\otimes r} \circ W_1 \left(\frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_r} \right) \\ = \sum_{\sigma, \mu \in S_r} (-1)^{[\sigma] + [\mu]} \sum_{\alpha} \left(\prod_{i=1}^r c_{i\alpha(i)\sigma(i)\mu(i)} \right) \frac{\partial}{\partial z_{\alpha(1)}} \wedge \dots \wedge \frac{\partial}{\partial z_{\alpha(r)}}, \end{aligned} \quad (5.17)$$

where the summation for α is done as in (5.15). From (5.15) and (5.17), we conclude by renaming the permutations and using multiplicativity of the sign of permutations. \square

6 Linear preservers and double mixed discriminant, a proof of Proposition 1.4

The main goal of this section is to prove Proposition 1.4. For this, we first need to recall several results from the theory of linear preservers. We fix two Hermitian vector spaces V, E of dimensions n and e respectively.

Theorem 6.1 (Choi's theorem, [7], cf. [36, Proposition 2.2]). *For a positive semidefinite linear preserver $P : \text{End}(V) \rightarrow \text{End}(E)$, the following statements are equivalent.*

a) *The induced operator $P^{\otimes k} : \text{End}(V^{\otimes k}) \rightarrow \text{End}(E^{\otimes k})$ is a positive semidefinite linear preserver for $k = \min(\dim V, \dim E)$.*

b) *For a basis v_1, \dots, v_n of V , the bloc matrix $(P(v_j^* \otimes v_i))_{i,j=1}^n$ is positive semidefinite on $E^{\otimes n}$.*

c) *There are operators $V_i : E \rightarrow V$, satisfying $P(X) = \sum V_i^* X V_i$ for any $X \in \text{End}(V)$.*

Remark 6.2. Clearly, by the definition of the operator P' from Proposition 1.4, the condition Theorem 6.1b) means that P' is positive semidefinite.

Theorem 6.3 (Schneider [35, Theorem 2]). *Assume $e = n$. For a positive semidefinite linear preserver $P : \text{End}(V) \rightarrow \text{End}(E)$, the following statements are equivalent.*

a) *The identity $P(\mathcal{C}_V) = \mathcal{C}_E$ holds.*

b) *There exists an invertible $A : E \rightarrow V$, satisfying $P(X) = A^* X A$ or $P(X) = A^* X^T A$.*

Proof of Proposition 1.4. First of all, let's treat the case d). Recall that in [19], Griffiths established that for a Griffiths positive vector bundle (E, h^E) of rank 2, the top Chern class is weakly-positive. From this and Proposition 3.7, we deduce d).

By Theorems 6.1b,c), 6.3 and Remark 6.2, we see that Proposition 1.4a) follows from Proposition 1.4b). Similarly, Theorem 6.1b,c), applied for P and P^T , reduces Proposition 1.4c) to Proposition 1.4b). Proposition 1.4b), in its turn, follows from the proofs of Lemmas 5.1, 5.2. \square

7 A local version of Jacobi-Trudi identity

The main goal of this section is to establish another pushforward identity for Schur forms, similar to Theorem 3.2, but based on Jacobi-Trudi identity.

Let's first recall the original Jacobi-Trudi identity (the terminology is due to Fulton-Pragacz [17, p. 42]). Let E be a smooth complex vector bundle of rank r over a smooth real manifold X .

Consider the flag manifold $\mathrm{Fl}_X(E)$ associated to E . A point $y \in \mathrm{Fl}_X(E)$ parametrizes a pair of $x \in X$ and a complete flag $(V_0(y), \dots, V_r(y))$ in E_x , where

$$E_x = V_0(y) \supset V_1(y) \supset \dots \supset V_r(y) = \{0\}, \quad \mathrm{codim}(V_i(y)) = i. \quad (7.1)$$

We denote by $\pi : \mathrm{Fl}_X(E) \rightarrow X$ the natural projection. On $\mathrm{Fl}_X(E)$, the flag (7.1) induces the filtration of the natural vector bundles

$$\pi^*E = V_0 \supset V_1 \supset \dots \supset V_r = \{0\}. \quad (7.2)$$

We introduce the line bundles $Q_\mu := V_{\mu-1}/V_\mu$, $1 \leq \mu \leq r$. Now for a given $k \in \mathbb{N}$ and a partition $a \in \Lambda(k, r)$, we denote by $a^T = (b_1, \dots, b_r)$, $b_i \in \mathbb{N}$, a non increasing partition of k , obtained through the transposition of the Young diagram associated to a .

Theorem 7.1 (Jacobi-Trudi identity, cf. [17, (4.1)], Manivel [27, Exercise 3.8.3]). *The following formula for Schur classes on X holds*

$$P_a(c(E)) = \pi_* \left[c_1(Q_1)^{b_1+r-1} \cdot \dots \cdot c_1(Q_r)^{b_r} \right]. \quad (7.3)$$

Now, we would like to prove a refinement of Theorem 7.1 on the level of differential forms in the holomorphic setting. We assume from now on that X is a complex manifold and E is a holomorphic vector bundle.

Theorem 7.2. *We fix a Hermitian metric h^E on E and denote by h^{Q_μ} , $\mu = 1, \dots, r$, the induced metrics on the line bundles Q_μ . The following identity between (k, k) -differential forms on X holds*

$$P_a(c(E, h^E)) = \pi_* \left[c_1(Q_1, h^{Q_1})^{\wedge(b_1+r-1)} \wedge \dots \wedge c_1(Q_r, h^{Q_r})^{\wedge b_r} \right]. \quad (7.4)$$

Let's consider one special case of Theorem 7.2, when $a = 11 \dots 10 \dots 0$, where 1 is repeated k times. By (1.4), on the left hand side of (7.3) for such a we have the Segre class, $s_k(E)$. Let's study the right-hand side. First, we have $b_1 = k$, $b_2, \dots, b_r = 0$. Now, consider the map $\pi^1 : \mathbb{P}(E^*) \rightarrow X$, and denote by H the hyperplane bundle on $\mathbb{P}(E^*)$. Recall that the flag manifold $\mathrm{Fl}_X(E)$ can be constructed inductively through the following isomorphism

$$\mathrm{Fl}_X(E) \simeq \mathrm{Fl}_{\mathbb{P}(E^*)}(H), \quad (7.5)$$

and H corresponds to V_1 in this identification. From this and the well-known isomorphism $(\pi^1)^*E/H \simeq \mathcal{O}_{\mathbb{P}(E^*)}(1)$, we see that the identity (7.3) reduces to the well-known interpretation of Segre classes $s_k(E) = \pi_*^1 [c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1))^{k+r-1}]$. Theorem 7.2 in this case gives us

$$s_k(E, h^E) = \pi_*^1 \left[c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1), h^{\mathcal{O}})^{\wedge(k+r-1)} \right], \quad (7.6)$$

where $h^{\mathcal{O}}$ is the induced metric on $\mathcal{O}_{\mathbb{P}(E^*)}(1)$.

The identity (7.6) was obtained before by Mourougane in [29, Proposition 6] by explicit calculation. It played a crucial role in the proof of positivity of the Segre forms by Guler in [20]. In our proof of Theorem 7.2, we use methods from the proof of Theorem 3.2, along with the curvature formula for the line bundles Q_μ , establishes by Demailly [8].

Proof of Theorem 7.2. Let's fix a point $y \in \text{Fl}_X(E)$ and denote by e_1^0, \dots, e_r^0 the orthonormal basis of E , which is compatible with the filtration of E associated to the flag of y in the sense that in the notations of (7.1), we have $V_i(y) = \langle e_{i+1}^0, e_i^0, \dots, e_r^0 \rangle$, $0 \leq i \leq r-1$.

Let's describe a local chart of $\text{Fl}_X(E)$ near y . For this, we fix a normal frame e_1, \dots, e_r , defined in a neighborhood of $x \in X$ and satisfying $e_i(x) = e_i^0$ (see Section 4 for a definition of the normal frame). Then for an arbitrary array of complex numbers $(z_{\lambda\mu})$, $1 \leq \lambda < \mu \leq r$, we define a local chart of $\pi^{-1}(x)$ near y as follows. Let $\xi_\mu := e_\mu + \sum_{\lambda < \mu} z_{\lambda\mu} e_\lambda$ for $1 \leq \mu \leq r$ and denote the associated flag $V_i := \langle \xi_{i+1}, \xi_i, \dots, \xi_r \rangle$, $0 \leq i \leq r-1$. Then the local chart for $\text{Fl}_X(E)$ near y is obtained by a product of this local chart on the fiber and a local chart on X near x .

We conserve the notation for the $(1, 1)$ -forms c_{ij} , $i, j = 1, \dots, r$ from (4.2). In the above chart, at point y , Demailly [8, (4.9)] established for any $\mu = 1, \dots, r$, the following curvature formula

$$c_1(Q_\mu, h^{Q_\mu}) = c_{\mu\mu} + \sum_{\mu < \lambda} \frac{\sqrt{-1}}{2\pi} dz_{\mu\lambda} \wedge d\bar{z}_{\mu\lambda} - \sum_{\lambda < \mu} \frac{\sqrt{-1}}{2\pi} dz_{\lambda\mu} \wedge d\bar{z}_{\lambda\mu}. \quad (7.7)$$

From this moment, our proof repeats the proof of Theorem 3.2, so we only highlight the main steps. In the same way as in Lemma 4.1, but based on (7.7), we establish that for any $r, k \in \mathbb{N}^*$, $a \in \Lambda(k, r)$, there is a polynomial $P'(d_{ij})$ in the entries of a self-adjoint matrix $D = (d_{ij})_{i,j=1}^r$, such that for any X , (E, h^E) as above, the right-hand side of (7.4) is equal to $P'(c_{ij})$.

In the same way as in the beginning of the proof of Theorem 3.2, we establish that there are coefficients $t_b \in \mathbb{R}$, $b \in \Lambda(k, r)$, which are universal in the same sense as P' , such that

$$P'(d_{ij}) = \sum_{b \in \Lambda(k, r)} t_b \cdot \text{Tr}[D]^{b(1)} \cdot \text{Tr}[\Lambda^2 D]^{b(2)} \cdot \dots \cdot \text{Tr}[\Lambda^r D]^{b(r)}, \quad (7.8)$$

where $b(i)$, $i = 1, \dots, r$ is the number of times i appears in the partition b . Then it is only left to prove that $t_b = c_b$ for any $b \in \Lambda(k, r)$, where c_b was defined in (4.5). This is done by Lemma 4.2 in the same way as we did in the end of Theorem 3.2. \square

To conclude, we note, however, that apart from the case considered in (7.6), the applications of Theorem 7.2 to positivity of Schur forms are not evident, as the Demailly's formula, (7.7), shows that the line bundles Q_μ , $\mu = 2, \dots, r$, are not positive or negative in general.

References

- [1] A. Alexandroff. Zur Theorie der gemischten Volumina von konvexen K"orpern. IV: Die gemischten Diskriminanten und die gemischten Volumina. *Rec. Math. Moscou, n. Ser.*, 3:227–251, 1938.
- [2] R. B. Bapat. Mixed discriminants of positive semidefinite matrices. *Linear Algebra Appl.*, 126:107–124, 1989.
- [3] B. Berndtsson. Curvature of Vector Bundles Associated to Holomorphic Fibrations. *Ann. of Math.*, 169(2):531–560, 2009.
- [4] S. Bloch and D. Gieseker. The positivity of the Chern classes of an ample vector bundle. *Invent. Math.*, 12(2):112–117, 1971.
- [5] Z. Błocki and S. Pliś. Squares of positive (p, p) -forms. *C. R. Acad. Sci. Paris*, 351(1-2):27–32, 2013.

-
- [6] R. Bott and S. S. Chern. Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections. *Acta Math.*, 114(1):71–112, 1965.
- [7] M.-D. Choi. Completely positive linear maps on complex matrices. *Linear Algebra Appl.*, 10:285–290, 1975.
- [8] J.-P. Demailly. Vanishing theorems for tensor powers of a positive vector bundle. Proc. 21st Int. Taniguchi Symp., Lect. Notes Math. 1339, 86-105, 1988.
- [9] J.-P. Demailly. *Complex Analytic and Differential Geometry*. 2012.
- [10] J.-P. Demailly. Hermitian-Yang-Mills approach to the conjecture of Griffiths on the positivity of ample vector bundles, arXiv: 2002.02677. 2020.
- [11] J.-P. Demailly, T. Peternell, and M. Schneider. Compact complex manifolds with numerically effective tangent bundles. *J. Algebr. Geom.*, 3(2):295–345, 1994.
- [12] J.-P. Demailly and H. Skoda. Relations entre les notions de positivites de P. A. Griffiths et de S. Nakano pour les fibres vectoriels. Semin. P. Lelong - H. Skoda, Analyse, Annees 1978/79, Lect. Notes Math. 822, 304-309, 1980.
- [13] S. Diverio. Segre forms and Kobayashi-Lübke inequality. *Math. Z.*, 283(3-4):1033–1047, 2016.
- [14] D. Florentin, V. D. Milman, and R. Schneider. A characterization of the mixed discriminant. *Proc. Am. Math. Soc.*, 144(5):2197–2204, 2016.
- [15] W. Fulton. Equivariant cohomology in algebraic geometry, lectures at Columbia University, notes by D. Anderson. 2007.
- [16] W. Fulton and R. Lazarsfeld. Positive Polynomials for Ample Vector Bundles. *Ann. Math.*, 118(1):35–60, 1983.
- [17] W. Fulton and P. Pragacz. *Schubert varieties and degeneracy loci*, volume 1689. Berlin: Springer, 1998.
- [18] D. Gieseker. p -ample bundles and their Chern classes. *Nagoya Math. J.*, 43:91–116, 1971.
- [19] P. A. Griffiths. *Hermitian differential geometry, Chern classes, and positive vector bundles*, pages 185 – 252. Princeton University Press, Princeton, 1969.
- [20] D. Guler. On Segre Forms of Positive Vector Bundles. *Can. Math. Bull.*, 55(1):108–113, 2012.
- [21] R. Hartshorne. Ample vector bundles. *Publ. Math. IHES*, 29:63–94, 1966.
- [22] G. Kempf and D. Laksov. The determinantal formula of Schubert calculus. *Acta Math.*, 132:153–162, 1974.
- [23] S. L. Kleiman. Ample vector bundles on algebraic surfaces. *Proc. Am. Math. Soc.*, 21:673–676, 1969.
- [24] C.-K. Li and S. Pierce. Linear preserver problems. *Am. Math. Mon.*, 108(7):591–605, 2001.
- [25] P. Li. Nonnegative Hermitian vector bundles and Chern numbers. *Math. Annalen*, 2020.
- [26] K. Liu, X. Sun, and X. Yang. Positivity and vanishing theorems for ample vector bundles. *J. Algebr. Geom.*, 22(2):303–331, 2013.

-
- [27] L. Manivel. *Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence*, volume 3. Paris: Société Mathématique de France, 1998.
- [28] J. W. Milnor and J. D. Stasheff. *Characteristic classes*, volume 76. Princeton University Press, Princeton, NJ, 1974.
- [29] C. Mourougane. Computations of Bott-Chern classes on $\mathbb{P}(E)$. *Duke Math. J.*, 124(2):389–420, 2004.
- [30] S. Nakano. On complex analytic vector bundles. *J. Math. Soc. Japan*, 7(1):1–12, 1955.
- [31] A. A. Panov. On some properties of mixed discriminants. *Math. USSR, Sb.*, 56:279–293, 1987.
- [32] V. P. Pingali. Representability of Chern–Weil forms. *Math. Z.*, 288(1):629–641, 2018.
- [33] H. Reese and A. W. Knap. Positive (p,p) forms, Wirtinger’s inequality, and currents. *Proc. Tulane Univ. Progr.*, Part A, 43-62, 1974.
- [34] J. Ross and M. Toma. Hodge-Riemann bilinear relations for Schur classes of ample vector bundles, arXiv: 1905.13636. 2019.
- [35] H. Schneider. Positive operators and an inertia theorem. *Numer. Math.*, 7:11–17, 1965.
- [36] L. Skowronek, E. Størmer, and K. Życzkowski. Cones of positive maps and their duality relations. *J. Math. Phys.*, 50(6):062106, 18, 2009.
- [37] E. Størmer. *Positive linear maps of operator algebras*. Berlin: Springer, 2013.
- [38] S. Usui and H. Tango. On numerical positivity of ample vector bundles with additional condition. *J. Math. Kyoto Univ.*, 17:151–164, 1977.
- [39] J. Xiao. On the positivity of high-degree Schur classes of an ample vector bundle. arXiv: 2007.12425. 2020.

SIARHEI FINSKI, INSTITUT FOURIER - UNIVERSITÉ GRENOBLE ALPES, FRANCE.

E-mail: finski.siarhei@gmail.com.