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► **To cite this version:**

Minjia Shi, Wang Xuan, Patrick Solé. Two families of two-weight codes over  $\mathbb{Z}_4$ . *Designs, Codes and Cryptography*, 2020, 10.1007/s10623-020-00796-x . hal-02943701

**HAL Id: hal-02943701**

**<https://hal.science/hal-02943701>**

Submitted on 20 Sep 2020

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# Two families of two-weight codes over $\mathbb{Z}_4$

Minjia Shi<sup>1</sup> · Wang Xuan<sup>1</sup> · Patrick Solé<sup>2</sup>

Received: 21 March 2020 / Revised: 1 July 2020 / Accepted: 20 August 2020  
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## 1 Abstract

Two infinite families of  $\mathbb{Z}_4$ -codes with two nonzero Lee weights are constructed by their generator matrices. Their Gray images are nonlinear with the same weight distribution as that of the two-weight binary codes of type SU1 in the sense of (Calderbank, Kantor, 1986).

**Keywords** Two-weight codes · Projective codes · Strongly regular graphs

**Mathematics Subject Classification** 94B 05 · 05E 30

## 1 Introduction

An organic connection between two-weight codes over fields and strongly regular graphs (SRGs) was discovered in the 1970s [3], and is well-documented in [1]. A partial census of known constructions based on the arithmetic of finite fields appeared in the classic paper [2]. More recently, several new techniques of construction involve trace codes over different rings, and Gray maps [9–13,15]. Especially, in [14], the authors focused on the construction of one-weight and two-weight codes over  $\mathbb{Z}_4$ . Later, in [16], the authors considered the linearity of the constructed codes. A natural question then, is whether there are other kinds of constructions of two-weight codes over  $\mathbb{Z}_4$ .

Motivated by the works listed above, in the present paper, we revisit two-weight codes over  $\mathbb{Z}_4$  and their linearity. This alphabet has been on the forefront of research in the domain of codes over rings since the prize awarded paper [4]. More background material can be found in the recent book [7]. We give an infinite family of two-weight projective codes over  $\mathbb{Z}_4$  by their explicit generator matrices. Their Gray images are proved to be nonlinear with the same weight distribution as the two-weight binary codes of type SU1 from [2]. The coset graphs of the dual codes are shown to be SRGs, and determined completely. Thus they produce the

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Communicated by P. Charpin.

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23 same SRG's as those given by the SU1 codes, but described as Cayley graphs on a different  
24 abelian group.

25 This note is organized in the following way. The next section contains notations and defini-  
26 tions. Section 3 is dedicated to preliminary material. Section 4 derives the main construction.  
27 Section 5 studies the linearity of the Gray images of our codes. Section 6 determines the  
28 SRGs attached to the  $\mathbb{Z}_4$  codes of Sect. 4. Section 7 concludes the note.

## 29 2 Notations and definitions

### 30 2.1 $\mathbb{Z}_4$ codes

31 We first recall necessary notations and definitions about codes over  $\mathbb{Z}_4$  in [17] and [7]. Let  $\mathbb{Z}_4$   
32 denote the ring of integers modulo four  $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ . A linear **code**  $C$  over  $\mathbb{Z}_4$  of length  
33  $n$  is a  $\mathbb{Z}_4$ -submodule of  $\mathbb{Z}_4^n$ . The **Lee weights** of  $0, 1, 2, 3 \in \mathbb{Z}_4$  are  $0, 1, 2, 1$  respectively. For  
34 any vector  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}_4^n$ , we define its **Lee weight** as  $W_L(\mathbf{x}) = \sum_{i=1}^n W_L(x_i)$ .

35 An  $N$ -Lee weight code is a code such that the cardinality of the set of nonzero Lee weights  
36 is  $N$ .

37 Moreover,  $A_{w_i}$  is the number of codewords of the nonzero Lee weight  $w_i$  in  $C$ , where  
38  $1 \leq i \leq N$ .

39 The order of a nonzero codeword  $\mathbf{c} \in C$ , is the smallest positive integer  $k$  such that  $k\mathbf{c} = \mathbf{0}$ .

40 Each element  $x \in \mathbb{Z}_4$  has a 2-adic expansion  $x = \alpha(x) + 2\beta(x)$ , where  $\alpha(x), \beta(x) \in \mathbb{F}_2$ .  
41 The **Gray map** from  $\mathbb{Z}_4$  to  $\mathbb{F}_2^2$  is defined by  $\Phi(x) = (\beta(x), \alpha(x) + \beta(x))$ . This map can  
42 be extended to  $\mathbb{Z}_4^n$  naturally.  $\Phi$  is a weight-preserving map from  $(\mathbb{Z}_4^n, \text{Lee weight})$  to  $(\mathbb{F}_2^{2n},$   
43 **Hamming weight**), that is to say  $W_L(\mathbf{x}) = W_H(\Phi(\mathbf{x}))$ .

44 If  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n)$  are two arbitrary elements of  $\mathbb{Z}_4^n$ , the  
45 inner product of  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{Z}_4^n$  is defined by  $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$ , and the  
46 componentwise multiplication  $*$  of  $\mathbf{x}$  and  $\mathbf{y}$  is  $\mathbf{x} * \mathbf{y} = (x_1y_1, x_2y_2, \dots, x_ny_n)$ , where the  
47 operation is performed in  $\mathbb{Z}_4$ . The **dual code** of  $C$  is defined as  $C^\perp = \{\mathbf{x} \in \mathbb{Z}_4^n | \mathbf{x} \cdot \mathbf{y} =$   
48  $0, \forall \mathbf{y} \in C\}$ . A Lee weight **projective** code  $C$  of length  $n$  over  $\mathbb{Z}_4$  is a linear code such that  
49 the minimum nonzero Lee weight of its dual code is at least three.

### 50 2.2 Graphs

51 In this note, all graphs are undirected, without loops or multiple edges. A graph is  $k$ -regular  
52 if all vertices have exactly  $k$  neighbors.

53 A  $k$ -regular graph on  $v$  vertices is a **strongly regular** graph (SRG) with parameters  
54  $(v, k, \lambda, \mu)$  if the number of common neighbors of any pair of vertices is  $\lambda$  or  $\mu$  depending  
55 on the pair being connected or not.

56 The **spectrum** of a graph is the set of eigenvalues of its adjacency matrix. A standard  
57 characterization of SRGs is that their spectrum consists of exactly three eigenvalues, namely  
58  $k$  and two eigenvalues called the **unrestricted eigenvalues** [1].

### 3 Preliminaries

If  $k = 1$ , let  $Y_k = (1)$ , when  $k \geq 2$ , define

$$Y_k = \begin{pmatrix} Y_{k-1} & Y_{k-1} & Y_{k-1} & Y_{k-1} & B_{k-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \underbrace{\mathbf{11} \cdots \mathbf{1}}_{2^{k-1}} \end{pmatrix} \tag{1}$$

where  $B_{k-1}$  is a  $(k - 1) \times 2^{k-1}$  matrix over  $2\mathbb{Z}_4$  with all different columns and  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $2^{k-2}(2^{k-1} - 1)$ ,  $i \in \mathbb{Z}_4$ .

Obviously,  $Y_k$  is a  $k \times n$  matrix over  $\mathbb{Z}_4$  where  $n = 2^{k-1}(2^k - 1)$ . Let  $\mathbf{x}_i$  and  $\mathbf{x}_j$  be two columns of  $Y_k$ , then it is easy to check that  $\mathbf{x}_i \neq m\mathbf{x}_j$ , where  $m = 1, 2, 3$  and  $1 \leq i, j \leq n$ .

Besides, if we delete  $\mathbf{0}$  (a column of  $B_k$ ) from  $B_k$ , then we get a  $k \times (2^k - 1)$  matrix  $B'_k$  over  $2\mathbb{Z}_4$ . Let  $C'$  be the code generated by  $B'_k$ . It's easy to check that  $C'$  is a special case of the code obtained in Theorem 4.5 in [14], when  $k_1 = 0, k_2 = k$ , then  $C'$  is a one-Lee weight linear code with the nonzero Lee weight  $w' = 2^k$ .

The following proposition is an alternative proof of Property 5.68 in (Handbook of Coding Theory, North Holland, 1998, WC Huffman, V. Pless eds, p. 1561).

**Proposition 3.1** *If  $C$  is the code generated by  $Y_k$ , then  $C$  is a two-Lee weight code of length  $n = 2^{k-1}(2^k - 1)$ , with the nonzero Lee weight  $w_1 = 2^{k-1}(2^k - 1)$ ,  $w_2 = 2^{2k-1}$ . Moreover,  $A_{w_1} = 4^k - 2^k$  and  $A_{w_2} = 2^k - 1$ , respectively.*

**Proof** Let

$$G_k = (Y_k \ 3Y_k \ B_k), \tag{2}$$

then  $G_k$  is a  $k \times 4^k$  matrix over  $\mathbb{Z}_4$ . Since  $B_k$  is a  $k \times 2^k$  matrix over  $2\mathbb{Z}_4$  with all different columns, then all columns of  $G_k$  are pairwise different. If we delete the  $\mathbf{0}$  from  $G_k$  (in fact we delete  $\mathbf{0}$  from  $B_k$ ), then we get a  $k \times (4^k - 1)$  matrix  $G'_k$ . Therefore,  $G'_k$  is a matrix with all nonzero different columns over  $\mathbb{Z}_4$ .

Let  $C'_k$  be the code generated by  $G'_k$ . It's easy to check that  $C'_k$  is a special case of the code obtained in Theorem 4.5 in [14], when  $k_1 = k, k_2 = 0$ , then  $C'_k$  is a one-Lee weight linear code with the nonzero Lee weight  $w' = 4^k$ .

Let  $\mathbf{c} = (\mathbf{x}, 3\mathbf{x}, \mathbf{y}) \in C'$  and the lengths of  $\mathbf{x}$  and  $\mathbf{y}$  are  $2^{k-1}(2^k - 1)$  and  $2^k - 1$ , respectively. Obviously,  $W_L(\mathbf{y}) = 2^k$  or  $0$ , and

$$W_L(\mathbf{c}) = W_L(\mathbf{x}) + W_L(3\mathbf{x}) + W_L(\mathbf{y}) = 2W_L(\mathbf{x}) + W_L(\mathbf{y}),$$

then, we have

$$W_L(\mathbf{x}) = \frac{W_L(\mathbf{c}) - W_L(\mathbf{y})}{2} = \frac{4^k - W_L(\mathbf{y})}{2}. \tag{3}$$

If  $W_L(\mathbf{y}) \neq 0$ , then  $W_L(\mathbf{x}) = w_1 = 2^{k-1}(2^k - 1)$ . Otherwise,  $W_L(\mathbf{x}) = w_2 = 2^{2k-1}$ . It is not difficult to get  $A_{w_2} = 2^k - 1$  when  $W_L(\mathbf{y}) = \mathbf{0}$ .  $\square$

**Corollary 3.2** *Let  $C$  be the code generated by  $Y_k$ ,  $\mathbf{c} \in C$  and  $\mathbf{c} \neq \mathbf{0}$ , then*

$$W_L(\mathbf{c}) = \begin{cases} 2^{2k-1} & \text{if the order of } \mathbf{c} \text{ is } 2, \\ 2^{k-1}(2^k - 1) & \text{if the order of } \mathbf{c} \text{ is } 4. \end{cases}$$

93 **Proof** Let  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_k$  be the rows of  $Y_k$ . From the proof of Proposition 3.1,  $W_L(\mathbf{r}_i) =$   
 94  $w_1 = 2^{k-1}(2^k - 1)$ , but  $W_L(2\mathbf{r}_i) = w_2 = 2^{2k-1}$ .

95 **Step 1** Let

96 
$$\mathbf{c} = p_1\mathbf{r}_1 + p_2\mathbf{r}_2 + \dots + p_k\mathbf{r}_k,$$

97 where  $p_i \in \mathbb{Z}_4, 1 \leq i \leq k$  and there exists  $p_i \neq 0$ . We can prove that  $p_i = 0$  or  $2$  for all  
 98  $1 \leq i \leq k$ , if the order of  $\mathbf{c}$  is  $2$ . By (1), let

99 
$$\mathbf{r}_k = (\mathbf{0} \ \mathbf{1} \ \mathbf{2} \ \mathbf{3} \ \overbrace{\mathbf{1} \ \mathbf{1} \ \dots \ \mathbf{1}}^{2^{k-1}}),$$

100 where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $2^{k-2}(2^{k-1} - 1), i \in \mathbb{Z}_4$ . If the order of  
 101  $\mathbf{c}$  is  $2$ , and  $p_k = 1$  or  $3$ , then  $\mathbf{c}$  contains at least a  $1$  or a  $3$  according to the definition of  
 102  $Y_k$  in (1), contradiction. Therefore,  $p_k = 0$  or  $2$  and  $p_k\mathbf{r}_k = \mathbf{0}$  or  $2\mathbf{r}_k$ . So the order of  
 103  $p_1\mathbf{r}_1 + p_2\mathbf{r}_2 + \dots + p_{k-1}\mathbf{r}_{k-1}$  is  $2$ . Similarly, by (1), we can get  $p_{k-1} = 0$  or  $2$ . Then we  
 104 can prove  $p_i = 0$  or  $2$  for all  $1 \leq i \leq k$ .

105 **Step 2** By (3), we can get if  $p_i = 2$  or  $0$ , then  $W_L(p_i\mathbf{r}_i) = 2^{2k-1}$  or  $0, 1 \leq i \leq k$ .  
 106 Similarly,

107 
$$W_L(p_1\mathbf{r}_1 + p_2\mathbf{r}_2 + \dots + p_k\mathbf{r}_k) = 2^{2k-1} = w_2,$$

108 where  $p_i = 0$  or  $2, 1 \leq i \leq k$  and there exists  $p_i \neq 0$ . By **Step 1**, these are all the codewords  
 109 of order  $2$ . Therefore, if the order of  $\mathbf{c}$  is  $2$ , then  $W_L(\mathbf{c}) = 2^{2k-1} = w_2$ . Since  $A_{w_2} = 2^k - 1$   
 110 by Proposition 3.1 and  $C$  is a two-Lee weight code, we can get if the order of  $\mathbf{c}$  is  $4$ , then  
 111  $W_L(\mathbf{c}) = 2^{k-1}(2^k - 1)$ . □

112 When  $k \geq 2$ , let

113 
$$X_k = \begin{pmatrix} Y_{k-1} & Y_{k-1} & Y_{k-1} & Y_{k-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{pmatrix}, \tag{4}$$

114 where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $2^{k-2}(2^{k-1} - 1), i \in \mathbb{Z}_4$ .

115 Similar to Proposition 3.1, we have

116 **Proposition 3.3** *If  $C$  is the code generated by  $X_k$ , then  $C$  is a two-Lee weight code of length*  
 117  *$n = 2^k(2^{k-1} - 1)$  with the nonzero Lee weight  $w_1 = 2^k(2^{k-1} - 1)$ , and  $w_2 = 2^{2k-1}$ .*  
 118 *Moreover,  $A_{w_1} = 4^k - 2^{k-1}$  and  $A_{w_2} = 2^{k-1} - 1$ , respectively.*

119 **Proof** From the structure of  $X_k$  and Proposition 3.1, we know there are  $2^{k-1} - 1$  codewords  
 120 of weight  $2^{2k-1}$  generated by the first  $k - 1$  rows. Let  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_k$  be the rows of  $X_k$ , and  
 121 denote the codeword generated by the first  $k - 1$  rows by  $\mathbf{c}$ . Without loss of generality, by  
 122 (4), let

123 
$$\mathbf{r}_k = (\mathbf{0} \ \mathbf{1} \ \mathbf{2} \ \mathbf{3}),$$

124 where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $2^{k-2}(2^{k-1} - 1), i \in \mathbb{Z}_4$ .

125 It is sufficient to prove  $W_L(\mathbf{r}_k + \mathbf{c}) = W_L(2\mathbf{r}_k + \mathbf{c}) = 2^k(2^{k-1} - 1)$ . It is easy to check  
 126 that  $W_L(\mathbf{r}_k) = W_L(2\mathbf{r}_k) = 2^k(2^{k-1} - 1)$ . Let  $\mathbf{c} = (\mathbf{c}', \mathbf{c}', \mathbf{c}', \mathbf{c}')$ ,  $n_i$  be the number of  $i$  in  $\mathbf{c}'$   
 127 and  $n'$  be the length of  $\mathbf{c}'$ . So  $n = 4n' = 2^k(2^{k-1} - 1)$ .

128 (1) If the order of  $\mathbf{c}$  is  $2$ , according to Corollary 3.2,  $W_L(\mathbf{c}) = 2^{2k-1}, n' = n_0 + n_2$ , and

129 
$$\begin{aligned} W_L(\mathbf{r}_k + \mathbf{c}) &= W_L(\mathbf{c}') + W_L(\mathbf{c}' + \mathbf{1}) + W_L(\mathbf{c}' + \mathbf{2}) + W_L(\mathbf{c}' + \mathbf{3}) \\ &= 2n_2 + n' + 2(n' - n_2) + n' = 4n' = 2^k(2^{k-1} - 1). \end{aligned}$$

130 (2) If the order of  $\mathbf{c}$  is 4,  $n' = n_0 + n_1 + n_2 + n_3$ , and

$$\begin{aligned} W_L(\mathbf{r}_k + \mathbf{c}) &= W_L(\mathbf{c}') + W_L(\mathbf{c}' + \mathbf{1}) + W_L(\mathbf{c}' + \mathbf{2}) + W_L(\mathbf{c}' + \mathbf{3}) \\ &= (n_1 + 2n_2 + n_3) + (n_0 + 2n_1 + n_2) \\ 131 \quad &+ (2n_0 + n_1 + n_3) + (n_0 + n_2 + 2n_3) \\ &= 4(n_1 + n_3 + n_2 + n_0) = 4n' = 2^k(2^{k-1} - 1). \end{aligned}$$

132 Similarly, we can also prove  $W_L(2\mathbf{r}_k + \mathbf{c}) = 2^k(2^{k-1} - 1) = w_1$ . Therefore,  $A_{w_2} = 2^{k-1} - 1$ ,  
133 and  $A_{w_1} = 4^k - 2^{k-1}$ . □

134 **Example 3.4** If  $k = 3$ , then  $n = 24$ ,  $w_1 = 24$ ,  $w_2 = 32$ , then, according to Proposition 3.3,  
135 there is a two-Lee weight code with the generator matrix:

$$\begin{pmatrix} 111102 & 111102 & 111102 & 111102 \\ 012311 & 012311 & 012311 & 012311 \\ 000000 & 111111 & 222222 & 333333 \end{pmatrix}. \tag{5}$$

### 137 4 Two-lee weight projective codes over $\mathbb{Z}_4$

138 In this section, we will give some structures of two-Lee weight projective codes over  $\mathbb{Z}_4$ .

139 **Lemma 4.1** Let  $C$  be a linear code over  $\mathbb{Z}_4$  with type  $4^{k_1}2^{k_2}$  of length  $n$ . If  $k_2 = 1$   
140 and the columns of the generator matrix  $G_{(k_1+1) \times n}$  are all distinct nonzero vectors  
141  $(c_1, c_2, \dots, c_{k_1}, c_{k_1+1})^T$ , where  $(c_1, c_2, \dots, c_{k_1})^T$  is one column of  $Y_{k_1}$ ,  $c_{k_1+1} = 0$  or  $2$ ,  
142 then  $C$  is a two-Lee weight code of length  $n = 2^{k_1}(2^{k_1} - 1)$  with the nonzero Lee weights  
143  $w_1 = 2^{k_1}(2^{k_1} - 1)$  and  $w_2 = 2^{2k_1}$ . Moreover,  $A_{w_1} = 2^{k_1}(2^{k_1+1} - 1)$  and  $A_{w_2} = 2^{k_1} - 1$ ,  
144 respectively.

145 **Proof** Without loss of generality, let

$$G_{(k_1+1) \times n} = \begin{pmatrix} Y_{k_1} & Y_{k_1} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}, \tag{6}$$

147 where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $2^{k_1-1}(2^{k_1} - 1)$ ,  $i \in 2\mathbb{Z}_4$  and  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{k_1}, \mathbf{r}_{k_1+1}$   
148 be the rows of  $G_{(k_1+1) \times n}$ . Since the columns of  $G_{(k_1+1) \times n}$  are all nonzero distinct vectors, it  
149 follows that  $n = 2^{k_1}(2^{k_1} - 1)$ . Let  $\mathbf{c} = (\mathbf{x}, \mathbf{x}) \in C$ , where  $\mathbf{x}$  is generated by  $Y_{k_1}$ . According  
150 to Proposition 3.1, then  $W_L(\mathbf{c}) = 2W_L(\mathbf{x}) = 2^{k_1}(2^{k_1} - 1)$  or  $2^{2k_1}$ . Denote the number of  $i$   
151 in  $\mathbf{x}$  by  $n_i$ , and the length of  $\mathbf{x}$  by  $n'$ . Obviously,  $n = 2n' = 2^{k_1}(2^{k_1} - 1)$ .

152 (1) If the order of  $\mathbf{x}$  is 4,  $n' = n_0 + n_1 + n_2 + n_3$ , then we have

$$\begin{aligned} W_L(\mathbf{r}_{k_1+1} + \mathbf{c}) &= W_L(\mathbf{x}) + W_L(\mathbf{x} + \mathbf{2}) = n_1 + 2n_2 + n_3 + 2n_0 + n_1 + n_3 \\ 153 \quad &= 2(n_0 + n_1 + n_2 + n_3) = 2n' = 2^{k_1}(2^{k_1} - 1). \end{aligned}$$

154 (2) If the order of  $\mathbf{x}$  is 2,  $n' = n_0 + n_2$ ,  $n_1 = n_3 = 0$ , then we have

$$\begin{aligned} W_L(\mathbf{r}_{k_1+1} + \mathbf{c}) &= W_L(\mathbf{x}) + W_L(\mathbf{x} + \mathbf{2}) = 2n_2 + 2n_0 \\ 155 \quad &= 2n' = 2^{k_1}(2^{k_1} - 1). \end{aligned}$$

156 Hence,  $C$  is a two-Lee weight code with the nonzero Lee weights  $w_1 = 2^{k_1}(2^{k_1} - 1)$ , and  
157  $w_2 = 2^{2k_1}$ . Obviously,  $A_{w_2}$  does not change comparing with  $Y_{k_1}$ , i.e.  $A_{w_2} = 2^{k_1} - 1$ , thus,  
158  $A_{w_1} = 2^{k_1}(2^{k_1+1} - 1)$ . □

**Theorem 4.2** Let  $C$  be a linear code over  $\mathbb{Z}_4$  with type  $4^{k_1}2^{k_2}$  of length  $n$ . If the columns of the generator matrix  $G_{(k_1+k_2) \times n}$  are all distinct nonzero vectors  $(c_1, c_2, \dots, c_{k_1}, c_{k_1+1}, \dots, c_{k_1+k_2})^T$ , where  $(c_1, c_2, \dots, c_{k_1})^T$  is one column of  $Y_{k_1}$ ,  $c_i = 0$  or  $2$ ,  $k_1 + 1 \leq i \leq k_1 + k_2$ , then  $C$  is a two-Lee weight code of length  $n = 2^{k_1+k_2-1}(2^{k_1} - 1)$  with the nonzero Lee weights  $w_1 = 2^{k_1+k_2-1}(2^{k_1} - 1)$  and  $w_2 = 2^{2k_1+k_2-1}$ . Moreover,  $A_{w_1} = 2^{k_1}(2^{k_1+k_2} - 1)$  and  $A_{w_2} = 2^{k_1} - 1$ , respectively.

**Proof** If  $k_2 = 1$ , the result follows from Lemma 4.1. Assume the result is valid for any  $k_2 = m$ , we denote the generator matrix and the nonzero weights by  $G_{k_1,m}$  and by  $w'_1 = 2^{k_1+m-1}(2^{k_1} - 1)$ ,  $w'_2 = 2^{2k_1+m-1}$ , respectively, then for  $k_2 = m + 1$ , without loss of generality, the generator matrix  $G_{k_1,m+1}$  of  $C$  can be expressed as

$$G_{k_1,m+1} = \begin{pmatrix} G_{k_1,m} & G_{k_1,m} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}, \tag{7}$$

where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $2^{k_1+m-1}(2^{k_1} - 1)$ ,  $i \in 2\mathbb{Z}_4$ . It's easy to verify that  $C$  is a two-Lee weight code with  $w_1 = 2w'_1 = 2^{k_1+m}(2^{k_1} - 1)$  and  $w_2 = 2w'_2 = 2^{2k_1+m}$ . Similar to the proof of Lemma 4.1, we can get  $A_{w_2} = 2^{k_1} - 1$  and  $A_{w_1} = 2^{k_1}(2^{k_1+k_2} - 1)$ . By induction hypothesis, the results are valid.  $\square$

**Example 4.3** If  $k_1 = 2, k_2 = 1$ , then  $n = 12, w_1 = 12, w_2 = 16$ , then, according to Theorem 4.2, there is a two-Lee weight code with the generator matrix:

$$G_{2,1} = \left( \begin{array}{ccc|ccc} 111102 & 111102 & & & & \\ 012311 & 012311 & & & & \\ 222222 & 000000 & & & & \end{array} \right). \tag{8}$$

**Example 4.4** If  $k_1 = 2, k_2 = 2$ , then  $n = 24, w_1 = 24, w_2 = 32$ , then, according to Theorem 4.2, there is a two-Lee weight code with the generator matrix:

$$G_{2,2} = \left( \begin{array}{cccc|cccc} 111102111102 & 111102111102 & & & & & & \\ 012311012311 & 012311012311 & & & & & & \\ 222222000000 & 222222000000 & & & & & & \\ 222222222222 & 000000000000 & & & & & & \end{array} \right). \tag{9}$$

**Corollary 4.5** Let  $C$  be a linear code over  $\mathbb{Z}_4$  with type  $4^{k_1}2^{k_2}$  of length  $n$ . When  $k_1 \geq 2$ , the columns of the generator matrix  $G'_{(k_1+k_2) \times n}$  are all distinct nonzero vectors  $(c_1, c_2, \dots, c_{k_1}, c_{k_1+1}, \dots, c_{k_1+k_2})^T$ , where  $(c_1, c_2, \dots, c_{k_1})^T$  is one column of  $X_{k_1}$  defined above,  $c_i = 0$  or  $2$ ,  $k_1 + 1 \leq i \leq k_1 + k_2$ , then  $C$  is a two-Lee weight code of length  $n = 2^{k_1+k_2}(2^{k_1-1} - 1)$  with the nonzero Lee weights  $w_1 = 2^{k_1+k_2}(2^{k_1-1} - 1)$ ,  $w_2 = 2^{2k_1+k_2-1}$ . Moreover,  $A_{w_1} = 4^{k_1}2^{k_2} - 2^{k_1-1}$  and  $A_{w_2} = 2^{k_1-1} - 1$ , respectively.

**Proof** Assume that  $k_1 \geq 2$ , then according to Theorem 4.2, we can construct a code  $C'$  with type  $4^{k_1-1}2^{k_2}$ , and denote its generator matrix by  $G'$ . By the definition of  $X_k$  in (4), without loss of generality, let

$$G'_{(k_1+k_2) \times n} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ G' & G' & G' & G' \end{pmatrix}, \tag{10}$$

where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $2^{k_1+k_2-2}(2^{k_1-1} - 1)$ ,  $i \in \mathbb{Z}_4$ . It is easily seen that  $C$  is a two-Lee weight code with the nonzero weight  $w_1 = 2^{k_1+k_2}(2^{k_1-1} - 1)$  and  $w_2 = 2^{2k_1+k_2-1}$ . By induction hypothesis,  $C$  is a two-Lee weight code over  $\mathbb{Z}_4$ .  $\square$

193 **Example 4.6** If  $k_1 = 2, k_2 = 2$ , then  $n = 16, w_1 = 32, w_2 = 16$ , according to Corollary 4.5,  
 194 there is a two-Lee weight code with the generator matrix:

$$195 \quad G'_{2,2} = \begin{pmatrix} 1111 & 1111 & 1111 & 1111 \\ 0123 & 0123 & 0123 & 0123 \\ 0000 & 0000 & 2222 & 2222 \\ 0000 & 2222 & 0000 & 2222 \end{pmatrix}. \quad (11)$$

196 **Example 4.7** If  $k_1 = 3, k_2 = 1$ , then  $n = 48, w_1 = 48, w_2 = 64$ , according to Corollary 4.5,  
 197 there is a two-Lee weight code with the generator matrix  $G'_{3,1}$ :

$$198 \quad \left( \begin{array}{cccc|cccc} 111102 & 111102 & 111102 & 111102 & 111102 & 111102 & 111102 & 111102 \\ 012311 & 012311 & 012311 & 012311 & 012311 & 012311 & 012311 & 012311 \\ 000000 & 111111 & 222222 & 333333 & 000000 & 111111 & 222222 & 333333 \\ 222222 & 222222 & 222222 & 222222 & 000000 & 000000 & 000000 & 000000 \end{array} \right). \quad (12)$$

199 Next, we are ready to give the sufficient and necessary conditions for a two-Lee weight  
 200 linear code to be projective over  $\mathbb{Z}_4$ .

201 **Lemma 4.8** Let  $C$  be a linear code over  $\mathbb{Z}_4$  with type  $4^{k_1}2^{k_2}$  of length  $n$  and with the generator  
 202 matrix  $G$ . Then  $C$  is projective if and only if the following conditions hold:

- 203 (1) every column of  $G$  contains 1 or 3;
- 204 (2) any two columns of  $G$  are not multiple of each other by  $\pm 1$ .

205 **Proof** Let  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  be the columns of  $G$ . Let us first prove the sufficiency. Let  $C^\perp$  be  
 206 the dual code of  $C$ . If  $\mathbf{c} \in C^\perp$ , then we only need to prove  $W_L(\mathbf{c}) \geq 3$ . Obviously,  $W_L(\mathbf{c})$   
 207 cannot be 1 because all columns are nonzero. Assume  $W_L(\mathbf{c}) = 2$ . Obviously,  $\mathbf{c}$  cannot be  
 208 written as  $\mathbf{c} = (0, \dots, 0, 2, 0, \dots, 0)$  due to (1). Then we have 2 cases:

209 **Case 1** Without loss of generality, let  $\mathbf{c} = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$ , where  
 210  $c_i = c_j = 1, i < j$ . Then  $\mathbf{x}_i + \mathbf{x}_j = \mathbf{0}$ . Thus,  $\mathbf{x}_j = 3\mathbf{x}_i, \mathbf{x}_i$  and  $\mathbf{x}_j$  are in proportion by the  
 211 unit 3, contradiction.

212 **Case 2** Without loss of generality, let  $\mathbf{c} = (0, \dots, 0, 1, 0, \dots, 0, 3, 0, \dots, 0)$ , where  
 213  $c_i = 1, c_j = 3, i < j$ . Then  $\mathbf{x}_i + \mathbf{x}_j = \mathbf{0}$ . Thus,  $\mathbf{x}_j = \mathbf{x}_i$ , contradiction.

214 Therefore,  $W_L(\mathbf{c}) \geq 3$ . By definition,  $C$  is projective.

215 Now we begin to prove the necessity. If (1) doesn't hold, let  $\mathbf{x}_j \in 2\mathbb{Z}_4^{k_1+k_2}$ , then the  
 216 codeword  $\mathbf{c} = (0, \dots, 0, 2, 0, \dots, 0) \in C^\perp$ , where  $c_j = 2$ . If (2) doesn't hold,  $\mathbf{x}_i = \pm\mathbf{x}_j$   
 217 for some pair of indices  $i < j$ .

218 **Case 1** If  $\mathbf{x}_i = \mathbf{x}_j$ , we can construct  $\mathbf{c}' = (0, \dots, 0, 1, 0, \dots, 0, 3, 0, \dots, 0)$ , where  
 219  $c'_i = 1, c'_j = 3, i < j$  and  $\mathbf{c}' \in C^\perp$ .

220 **Case 2** If  $\mathbf{x}_i = 3\mathbf{x}_j$ , then  $\mathbf{c}'' = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0) \in C^\perp$ , where  $c''_i =$   
 221  $c''_j = 1$ .

222 But the Lee weight of  $\mathbf{c}, \mathbf{c}'$  and  $\mathbf{c}''$  is 2. Contradict with  $w_L(C^\perp) = 3$ . □

223 **Theorem 4.9** Let  $C$  be a linear code over  $\mathbb{Z}_4$  with type  $4^{k_1}2^{k_2}$  of length  $n$  obtained in Theo-  
 224 rem 4.2, then  $C$  is projective.

225 **Proof** Denote the generator matrix of  $C$  by  $G_{k_1,k_2}$ , then it is sufficient to prove that  $G_{k_1,k_2}$   
 226 satisfies Lemma 4.8.

227 **Step 1** Prove  $Y_{k_1}$  defined in (1) can make it. The result is obvious.

228 **Step 2** Use induction for  $k_2$ . If  $k_2 = 1$ , we know the generator matrix has the form  
 229 (6) in Lemma 4.1. And by Step 1 and Lemma 4.8, we can easily prove that the code is



projective. Assume the result is valid for any  $k_2 = m$ , then for  $k_2 = m + 1$ , according to (7) in Theorem 4.2, without loss of generality, the generator matrix  $G_{k_1, m+1}$  of  $C$  is

$$G_{k_1, m+1} = \begin{pmatrix} G_{k_1, m} & G_{k_1, m} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}, \quad (13)$$

where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $2^{k_1+m-1}(2^{k_1}-1)$ ,  $i \in 2\mathbb{Z}_4$ . So, all the columns of  $G_{k_1, m+1}$  contain a 1 or a 3. If there are two columns in proportion by  $\pm 1$ , then there are two columns of  $G_{k_1, m}$  in proportion by  $\pm 1$ , a contradiction. So the result is valid for  $k_2 = m + 1$ . By induction hypothesis,  $G_{k_1, k_2}$  satisfies Lemma 4.8 for any  $k_1, k_2 \geq 1$ , then  $C$  is projective by Lemma 4.8.  $\square$

Similar to the proof of Theorem 4.9, we have the following corollary.

**Corollary 4.10** *The linear code obtained in Corollary 4.5 is projective.*

## 5 Gray images

For  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}_4^n$ , define  $\alpha(\mathbf{x}) = (\alpha(x_1), \alpha(x_2), \dots, \alpha(x_n))$ , where  $\alpha(0) = \alpha(2) = 0$  and  $\alpha(1) = \alpha(3) = 1$ .

**Lemma 5.1** (See [4]) *Let  $C$  be a linear code over  $\mathbb{Z}_4$ ,  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$  be a set of generators of  $C$ , then  $\Phi(C)$  is linear if and only if  $2\alpha(\mathbf{x}_i) * \alpha(\mathbf{x}_j) \in C$  for all  $i, j$ , satisfying  $1 \leq i \leq j \leq m$ .*

Let  $q = p^m$ , where  $p$  is prime and  $m$  is a positive integer. According to [2], an  $[n, k]$  code  $C$  over  $GF(q)$  is a  $k$ -dimensional subspace of  $GF(q)^n$ . Next proposition is from the classical paper in [2] about the survey of two weight codes over finite fields.

**Proposition 5.2** (See [2]) *Example SU 1: Let  $\Omega$  be the complement of a  $t$ -dimensional subspace of  $GF(q)^k$  where  $1 \leq t \leq k - 1$ . If  $v$  is a non-zero vector then  $|v^\perp \cap \Omega| = q^{k-1} - q^t$  or  $q^{k-1} - q^{t-1}$ . For a two-weight code with Example SU1, if the field is  $GF(q)$ , its parameters are (see FIGURE 1a and FIGURE 2a in [2])*

$$n = \frac{q^l - q^t}{q - 1}, k = l, w_1 = q^{l-1} - q^{t-1}, w_2 = q^{l-1}, A_{w_1} = q^l - q^{l-t}, A_{w_2} = q^{l-t} - 1.$$

where  $1 \leq t \leq l - 1$ .

**Proposition 5.3** *The Gray images of the codes of Theorem 4.2 and Corollary 4.5 are binary distance invariant codes with the weights given  $w_1 = q^{l-1} - q^{t-1}$  and  $w_2 = q^{l-1}$ .*

Note that we do not claim these binary codes to be linear.

Here is a table about the parameters of the Gray images  $\Phi(C_1)$ ,  $\Phi(C_2)$  and  $\Phi(C_3)$ , where  $C_1$  and  $C_2$  are the codes obtained in Theorem 4.2 and Corollary 4.5 in this paper, respectively, and  $C_3$  is the code obtained in Theorem 5.6 in [14].

**Remark 5.4** If  $k_1 = 1$ ,  $\Phi(C_1)$  and  $\Phi(C_3)$  have the same parameters. Besides, due to Corollary 6.8 in [14], when  $k_1 = 1$ ,  $\Phi(C_1)$  reaches the Plotkin bound and Griesmer bound.

**Remark 5.5** It can be seen that the values of  $l$  for  $\Phi(C_1)$ ,  $\Phi(C_2)$  and  $\Phi(C_3)$  are all  $2k_1 + k_2$ . But the values of  $t$  are  $k_1 + k_2$ ,  $k_1 + k_2 + 1$ ,  $2k_1 + k_2 - 1$  for  $\Phi(C_1)$ ,  $\Phi(C_2)$  and  $\Phi(C_3)$ , respectively. Therefore, these three families of codes lead to non-equivalent two-Lee weight linear codes.

**Table 1** The parameters of  $\Phi(C_1)$ ,  $\Phi(C_2)$  and  $\Phi(C_3)$

	$n$	$w_1$	$w_2$	$A_{w_1}$	$A_{w_2}$
SU 1	$\frac{q^l - q^t}{q-1}$	$q^{l-1} - q^{t-1}$	$q^{l-1}$	$q^l - q^{l-t}$	$q^{l-t} - 1$
$\Phi(C_1)$	$2^{k_1+k_2}(2^{k_1} - 1)$	$2^{k_1+k_2-1}(2^{k_1} - 1)$	$2^{2k_1+k_2-1}$	$4^{k_1}2^{k_2} - 2^{k_1}$	$2^{k_1} - 1$
$\Phi(C_2)$	$2^{k_1+k_2}(2^{k_1} - 2)$	$2^{k_1+k_2}(2^{k_1-1} - 1)$	$2^{2k_1+k_2-1}$	$4^{k_1}2^{k_2} - 2^{k_1-1}$	$2^{k_1-1} - 1$
$\Phi(C_3)$	$2^{2k_1+k_2-1}$	$2^{2k_1+k_2-2}$	$2^{2k_1+k_2-1}$	$4^{k_1}2^{k_2} - 2$	1

266 Finally, we will consider the linearity of the image codes in Theorem 4.2 and Corollary 4.5.

267 **Theorem 5.6** *If the columns of the generator matrix of  $C$  satisfy the conditions of Theorem 4.2,*  
 268 *then  $\Phi(C)$  is linear when  $k_1 = 1$  and nonlinear when  $k_1 \geq 2$ .*

269 **Proof** When  $k_1 = 1$ , the result can be obtained in Corollary 6.8 in [14]. When  $k_1 \geq 2$ , denote  
 270 the generator matrix of  $C$  by  $G_{k_1, k_2}$  and let  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{k_1}, \dots, \mathbf{r}_{k_1+k_2}$  be the rows of  $G_{k_1, k_2}$ ,  
 271 where  $\mathbf{r}_j$  is a vector over  $2\mathbb{Z}_4$ ,  $k_1 + 1 \leq j \leq k_1 + k_2$ . By Theorem 4.2,

$$272 \quad G_{k_1+1, k_2} = \begin{pmatrix} G_{k_1, k_2} & G_{k_1, k_2} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}, \quad (14)$$

273 where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $2^{k_1+k_2-1}(2^{k_1} - 1)$ ,  $i \in 2\mathbb{Z}_4$ . Obviously, if  
 274 either  $\mathbf{r}_1$  or  $\mathbf{r}_2$  is from the last  $k_2$  rows, it's easy to check that  $2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) = \mathbf{0} \in C$ .

275 (1) If  $k_1 = k + 1 \geq 2$ ,  $k_2 = 0$ , then  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_k$  and  $\mathbf{r}_{k+1}$  become the rows of  $Y_{k+1}$ . By  
 276 the definition of  $Y_k$ , we can get

$$277 \quad Y_{k+1} = \begin{pmatrix} Y_k & Y_k & Y_k & Y_k & B_k \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1}' \end{pmatrix}$$

278 where  $k \geq 1$ ,  $B_k$  is a  $k \times 2^k$  matrix over  $2\mathbb{Z}_4$  with all different columns,  $\mathbf{i}$  is a row vector  
 279  $(i, i, \dots, i)$  of length  $2^{k-1}(2^k - 1)$ ,  $i \in \mathbb{Z}_4$  and  $\mathbf{1}'$  is the row vector  $(1, 1, \dots, 1)$  of length  
 280  $2^k$ .

281 Without loss of generality, let

$$282 \quad \mathbf{r}_1 = \underbrace{(0 \ 0 \ \dots \ 0 \ 1 \ 1 \ \dots \ 1 \ 2 \ 2 \ \dots \ 2 \ 3 \ 3 \ \dots \ 3 \ 1 \ 1 \ \dots \ 1)}_{2^{k-1}(2^k-1) \ 2^{k-1}(2^k-1) \ 2^{k-1}(2^k-1) \ 2^{k-1}(2^k-1) \ 2^k} \quad (15)$$

283 be the last row of  $Y_{k+1}$ , and

$$284 \quad \mathbf{r}_2 = (\mathbf{x} \ \mathbf{x} \ \mathbf{x} \ \mathbf{x} \ \mathbf{b}),$$

285 where  $\mathbf{x}$  is a row of  $Y_k$ , and  $\mathbf{b}$  is a row of  $B_k$ . Then

$$286 \quad \begin{aligned} 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) &= 2(\alpha(\mathbf{0}) \ \alpha(\mathbf{1}) \ \alpha(\mathbf{2}) \ \alpha(\mathbf{3}) \ \alpha(\mathbf{1}')) * (\alpha(\mathbf{x}) \ \alpha(\mathbf{x}) \ \alpha(\mathbf{x}) \ \alpha(\mathbf{x}) \ \alpha(\mathbf{b})) \\ &= 2(\mathbf{0} \ \alpha(\mathbf{x}) \ \mathbf{0} \ \alpha(\mathbf{x}) \ \mathbf{0}), \end{aligned}$$

287 we get

$$288 \quad W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2)) = 4W_L(\alpha(\mathbf{x})).$$

289 Since the length of  $\mathbf{x}$  is  $2^{k-1}(2^k - 1)$ , then

$$290 \quad 4W_L(\alpha(\mathbf{x})) \leq 4 \times 2^{k-1}(2^k - 1) = 2^{k+1}(2^k - 1).$$

By Proposition 3.1,  $C$  is a two-Lee weight linear code with the nonzero Lee weights  $2^k(2^{k+1} - 1)$ , and  $2^{2k+1}$ . And the nonzero smallest Lee weight of  $C$  is  $2^k(2^{k+1} - 1) > 2^{k+1}(2^k - 1)$ . Thus,  $2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) \notin C$ . By Lemma 5.1,  $\Phi(C)$  is nonlinear when  $k_1 \geq 2, k_2 = 0$ .

(2) If  $k_2 \geq 1$ , then the first  $k_1$  rows of  $G_{k_1, k_2}$  can be written as:

$$\underbrace{Y_{k_1} \ Y_{k_1} \ \dots \ Y_{k_1}}_{2^{k_2}},$$

and let

$$\mathbf{r}'_i = \underbrace{(\mathbf{r}_i \ \mathbf{r}_i \ \dots \ \mathbf{r}_i)}_{2^{k_2}},$$

where  $i = 1$  or  $2$ . Since  $k_1 = k + 1$ , we have

$$\begin{aligned} W_L(2\alpha(\mathbf{r}'_1) * \alpha(\mathbf{r}'_2)) &= 2^{k_2} W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2)) = 2^{k_2+2} W_L(\alpha(\mathbf{x})) \\ &\leq 2^{k_2+2} \times 2^{k-1}(2^k - 1) = 2^{k_1+k_2}(2^{k_1-1} - 1). \end{aligned}$$

By Theorem 4.2,  $C$  is a two-Lee weight code with the nonzero Lee weights  $2^{k_1+k_2-1}(2^{k_1} - 1)$  and  $2^{2k_1+k_2-1}$ . And the nonzero smallest Lee weight is  $2^{k_1+k_2-1}(2^{k_1} - 1) > 2^{k_1+k_2}(2^{k_1-1} - 1)$ . Thus,  $2\alpha(\mathbf{r}'_1) * \alpha(\mathbf{r}'_2) \notin C$ . By Lemma 5.1,  $\Phi(C)$  is nonlinear when  $k_1 \geq 2, k_2 \geq 0$ .  $\square$

**Example 5.7** Consider the code defined in Example 4.3. Let  $\mathbf{r}_1$  and  $\mathbf{r}_2$  denote the first and second rows of  $G_{2,1}$  in (8), respectively. Then the Lee weight of

$$\begin{aligned} 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) &= 2\alpha(111102111102) * \alpha(012311012311) \\ &= 2(111100111100) * (010111010111) \\ &= (020200020200) \end{aligned}$$

is 8 which can not be obtained in  $C$  since  $C$  is a two-Lee weight code with  $w_1 = 12, w_2 = 16$ .

**Theorem 5.8** If the columns of the generator matrix of  $C$  satisfy the conditions of Corollary 4.5, then  $\Phi(C)$  is linear when  $k_1 = 2$  and nonlinear when  $k_1 \geq 3$ .

**Proof** Denote the generator matrix of  $C$  by  $G_{k_1, k_2}$ . Let  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{k_1}, \dots, \mathbf{r}_{k_1+k_2}$  be the rows of  $G_{k_1, k_2}$ , where  $\mathbf{r}_j$  is a vector over  $2\mathbb{Z}_4, k_1 + 1 \leq j \leq k_1 + k_2$ .

(1) By Corollary 4.5, if  $k_1 = 2, k_2 = 1$ , the generator matrix is:

$$G_{2,1} = \begin{pmatrix} 1111 & 1111 \\ 0123 & 0123 \\ 2222 & 0000 \end{pmatrix}.$$

So, if  $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_j$  are the rows of  $G_{2, k_2}, 3 \leq j \leq k_2 + 2$ , we can get  $\mathbf{r}_1$  is all 1 vector. Therefore,  $\alpha(\mathbf{r}_1) = \mathbf{1}$  and  $\alpha(\mathbf{r}_j) = \mathbf{0}$ . Besides,

$$\begin{aligned} 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) &= 2\mathbf{r}_2 \in C, \quad 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_j) = \mathbf{0} \in C, \\ 2\alpha(\mathbf{r}_2) * \alpha(\mathbf{r}_j) &= \mathbf{0} \in C, \quad 2\alpha(\mathbf{r}_j) * \alpha(\mathbf{r}_k) = \mathbf{0} \in C, \end{aligned}$$

where  $3 \leq j, k \leq 2 + k_2$ . By Lemma 5.1,  $\Phi(C)$  is linear when  $k_1 = 2, k_2 \geq 0$ .

(2) If  $k_1 \geq 3$ , by (10) in the proof of Corollary 4.5,

$$G_{k_1, k_2} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ G'_{k_1-1, k_2} & G'_{k_1-1, k_2} & G'_{k_1-1, k_2} & G'_{k_1-1, k_2} \end{pmatrix},$$

319 where  $G'_{k_1-1,k_2}$  is the generator matrix of the code  $C'$  obtained in Theorem 4.2, and  $\mathbf{i}$  is the  
 320 row vector  $(i, i, \dots, i)$  of length  $2^{k_1+k_2-2}(2^{k_1-1} - 1)$ ,  $i \in \mathbb{Z}_4$ . Without loss of generality, let

321 
$$\mathbf{r}_1 = (\mathbf{0} \ \mathbf{1} \ \mathbf{2} \ \mathbf{3}) \text{ and } \mathbf{r}_2 = (\mathbf{r}_0 \ \mathbf{r}_0 \ \mathbf{r}_0 \ \mathbf{r}_0)$$

322 where  $\mathbf{r}_0$  is one row of  $G'_{k_1-1,k_2}$ . Then

323 
$$2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) = 2(\mathbf{0} \ \alpha(\mathbf{r}_0) \ \mathbf{0} \ \alpha(\mathbf{r}_0)).$$

324 Since the length of  $\mathbf{r}_0$  is  $2^{k_1+k_2-2}(2^{k_1-1} - 1)$ ,

325 
$$W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2)) = 4W_L(\alpha(\mathbf{r}_0)) \leq 2^{k_1+k_2}(2^{k_1-1} - 1). \tag{16}$$

326 If  $\mathbf{0} \neq 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) \in C$ , since  $C$  is a two-Lee weight code by Corollary 4.5,  $W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2)) = 2^{k_1+k_2}(2^{k_1-1} - 1)$  or  $2^{2k_1+k_2-1}$ . Thus

328 
$$2^{k_1+k_2}(2^{k_1-1} - 1) \leq W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2)) = 4W_L(\alpha(\mathbf{r}_0)). \tag{17}$$

329 Therefore, combining (16) and (17), we have

330 
$$W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2)) = 4W_L(\alpha(\mathbf{r}_0)) = 2^{k_1+k_2}(2^{k_1-1} - 1),$$

331 which forces  $\alpha(\mathbf{r}_0)$  to be all 1 vector. Thus we have  $W_L(\mathbf{r}_0) = 2^{k_1+k_2-2}(2^{k_1-1} - 1)$ ,  
 332  $W_L(2\mathbf{r}_0) = 2^{k_1+k_2-1}(2^{k_1-1} - 1)$  and then  $W_L(2\mathbf{r}_0) = 2W_L(\mathbf{r}_0)$ . However,  $\mathbf{r}_0$  is one row of  
 333  $G'_{k_1-1,k_2}$ , the generator matrix obtained in Theorem 4.2, and the two Lee weights  $w_1$  and  
 334  $w_2$  can not satisfy  $w_2 = 2w_1$ , contradiction. Thus there are at least 2 rows  $\mathbf{r}_1, \mathbf{r}_2$  satisfying  
 335  $2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) \notin C$ . Namely,  $\Phi(C)$  is nonlinear when  $k_1 \geq 3, k_2 \geq 0$ .  $\square$

336 **Example 5.9** Consider the code defined in Example 4.7. Let  $\mathbf{r}_1$  and  $\mathbf{r}_2$  denote the first and  
 337 second rows of  $G'_{3,1}$  in (12), respectively. Then the Lee weight of

338 
$$\begin{aligned} 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) &= 2\alpha(111102111102111102111102111102111102111102111102111102111102) \\ &\quad * \alpha(012311012311012311012311012311012311012311012311012311012311) \\ &= (020200020200020200020200020200020200020200020200020200020200) \end{aligned}$$

339 is 32 which can not be obtained in  $C$  since  $C$  is a two-Lee weight code with  $w_1 = 48$ ,  
 340  $w_2 = 64$ .

341 **6 Graphs**

342 We recall the classic lemma from [12].

343 **Lemma 6.1** *If  $C$  is a  $\mathbb{Z}_4$ -code of length  $n$  and weights  $w_1, w_2$  the coset graph of  $C^\perp$  is an*  
 344 *SRG on  $|C|$  vertices, degree  $2n$ , unrestricted eigenvalues  $2n - 2w_1$  and  $2n - 2w_2$ .*

345 From this classic lemma, and the characterization of SRGs given in §2.2, the following  
 346 two results follow immediately. Their proofs are omitted.

347 **Theorem 6.2** (1) *If  $C$  is the code of Theorem 4.2 then the coset graph of  $C^\perp$  is an SRG on*  
 348  *$|C|$  vertices, degree  $2n$ , and unrestricted eigenvalues 0 and  $-2^{k_1+k_2}$ .* (2) *If  $C$  is the code*  
 349 *of Corollary 4.5 then the coset graph of  $C^\perp$  is an SRG on  $|C|$  vertices, degree  $2n$ , and*  
 350 *unrestricted eigenvalues 0 and  $-2^{k_1+k_2+1}$ .*

## Remarks:

- Following [1, §1.1.3] we see that these graphs are complete multipartite of type  $K_{a \times m}$  with  $m = 2^{k_1+k_2}$  for Theorem 4.2 or  $m = 2^{k_1+k_2-1}$  for Corollary 4.5, and, in both cases,  $a = \frac{|C|}{m}$ .
- The graph corresponding to the code of Example 4.6 is bipartite complete, since the code is an Hadamard code of a kind studied in [5,6].

## 7 Conclusion

In this paper, we have constructed two infinite families of projective two-weight  $\mathbb{Z}_4$ -codes. Further, their Gray images are nonlinear, and have the same weight distributions as a family of two-weight codes of [2], the SU1 family. In small values of parameters these codes can be implemented in the computer package Magma. It is a worthwhile project to check which families of binary two-weight codes from [2] admit  $\mathbb{Z}_4$ -linear analogues.

**Acknowledgements** The authors thank Denis Krotov for helpful discussions. This research is supported by the National Natural Science Foundation of China (61672036), the Excellent Youth Foundation of Natural Science Foundation of Anhui Province (1808085J20), the Academic Fund for Outstanding Talents in Universities (gxbjZD03).

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