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Two families of two-weight codes over $\mathbb{Z}_4$

Minjia Shi[1] · Wang Xuan[1] · Patrick Solé[2]

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Abstract
Two infinite families of $\mathbb{Z}_4$-codes with two nonzero Lee weights are constructed by their generator matrices. Their Gray images are nonlinear with the same weight distribution as that of the two-weight binary codes of type SU1 in the sense of (Calderbank, Kantor, 1986).

Keywords Two-weight codes · Projective codes · Strongly regular graphs

Mathematics Subject Classification 94B 05 · 05E 30

1 Introduction

An organic connection between two-weight codes over fields and strongly regular graphs (SRGs) was discovered in the 1970s [3], and is well-documented in [1]. A partial census of known constructions based on the arithmetic of finite fields appeared in the classic paper [2]. More recently, several new techniques of construction involve trace codes over different rings, and Gray maps [9–13,15]. Especially, in [14], the authors focused on the construction of one-weight and two-weight codes over $\mathbb{Z}_4$. Later, in [16], the authors considered the linearity of the constructed codes. A natural question then, is whether there are other kinds of constructions of two-weight codes over $\mathbb{Z}_4$.

Motivated by the works listed above, in the present paper, we revisit two-weight codes over $\mathbb{Z}_4$ and their linearity. This alphabet has been on the forefront of research in the domain of codes over rings since the prize awarded paper [4]. More background material can be found in the recent book [7]. We give an infinite family of two-weight projective codes over $\mathbb{Z}_4$ by their explicit generator matrices. Their Gray images are proved to be nonlinear with the same weight distribution as the two-weight binary codes of type SU1 from [2]. The coset graphs of the dual codes are shown to be SRGs, and determined completely. Thus they produce the

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Revised Proof

same SRG's as those given by the SU1 codes, but described as Cayley graphs on a different abelian group.

This note is organized in the following way. The next section contains notations and definitions. Section 3 is dedicated to preliminary material. Section 4 derives the main construction. Section 5 studies the linearity of the Gray images of our codes. Section 6 determines the SRGs attached to the $\mathbb{Z}_4$ codes of Sect. 4. Section 7 concludes the note.

2 Notations and definitions

2.1 $\mathbb{Z}_4$ codes

We first recall necessary notations and definitions about codes over $\mathbb{Z}_4$ in [17] and [7]. Let $\mathbb{Z}_4$ denote the ring of integers modulo four $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. A linear **code** C over $\mathbb{Z}_4$ of length n is a $\mathbb{Z}_4$-submodule of $\mathbb{Z}_4^n$. The **Lee weights** of $0, 1, 2, 3 \in \mathbb{Z}_4$ are $0, 1, 2, 1$ respectively. For any vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_4^n$, we define its **Lee weight** as $W_L(\mathbf{x}) = \sum_{i=1}^{n} W_L(x_i)$. An N-Lee weight code is a code such that the cardinality of the set of nonzero Lee weights is N.

Moreover, A_{w_i} is the number of codewords of the nonzero Lee weight w_i in C, where $1 \leq i \leq N$.

The order of a nonzero codeword $\mathbf{c} \in \mathbf{C}$, is the smallest positive integer k such that $k\mathbf{c} = \mathbf{0}$.

Each element $x \in \mathbb{Z}_4$ has a 2-adic expansion $x = \alpha(x) + 2\beta(x)$, where $\alpha(x), \beta(x) \in \mathbb{F}_2$. The **Gray map** from $\mathbb{Z}_4$ to $\mathbb{F}_2^2$ is defined by $\Phi(x) = (\beta(x), \alpha(x) + \beta(x))$. This map can be extended to $\mathbb{Z}_4^n$ naturally. Φ is a weight-preserving map from ($\mathbb{Z}_4^n$, Lee weight) to ($\mathbb{F}_2^{2n}$, Hamming weight), that is to say $W_L(\mathbf{x}) = W_H(\Phi(\mathbf{x}))$.

If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are two arbitrary elements of $\mathbb{Z}_4^n$, the inner product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{Z}_4^n$ is defined by $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, and the componentwise multiplication $*$ of $\mathbf{x}$ and $\mathbf{y}$ is $\mathbf{x} * \mathbf{y} = (x_1y_1, x_2y_2, \ldots, x_ny_n)$, where the operation is performed in $\mathbb{Z}_4$. The **dual code** of C is defined as $C^{\perp} = \{\mathbf{x} \in \mathbb{Z}_4^n | \mathbf{x} \cdot \mathbf{y} = 0, \forall\, \mathbf{y} \in C\}$. A Lee weight **projective** code C of length n over $\mathbb{Z}_4$ is a linear code such that the minimum nonzero Lee weight of its dual code is at least three.

2.2 Graphs

In this note, all graphs are undirected, without loops or multiple edges. A graph is k-regular if all vertices have exactly k neighbors.

A k-regular graph on v vertices is a **strongly regular** graph (SRG) with parameters (v, k, λ, μ) if the number of common neighbors of any pair of vertices is λ or μ depending on the pair being connected or not.

The **spectrum** of a graph is the set of eigenvalues of its adjacency matrix. A standard characterization of SRGs is that their spectrum consists of exactly three eigenvalues, namely k and two eigenvalues called the **unrestricted eigenvalues** [1].

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3 Preliminaries

If $k = 1$, let $Y_k = (1)$, when $k \geq 2$, define

$$Y_k = \begin{pmatrix} Y_{k-1} & Y_{k-1} & Y_{k-1} & Y_{k-1} & B_{k-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \underbrace{1\,1\cdots 1}_{2^{k-1}} \end{pmatrix} \tag{1}$$

where B_{k-1} is a $(k-1) \times 2^{k-1}$ matrix over $2\mathbb{Z}_4$ with all different columns and $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length $2^{k-2}(2^{k-1} - 1)$, $i \in \mathbb{Z}_4$.

Obviously, Y_k is a $k \times n$ matrix over $\mathbb{Z}_4$ where $n = 2^{k-1}(2^k - 1)$. Let $\mathbf{x}_i$ and $\mathbf{x}_j$ be two columns of Y_k, then it is easy to check that $\mathbf{x}_i \neq m\mathbf{x}_j$, where $m = 1, 2, 3$ and $1 \leq i, j \leq n$.

Besides, if we delete $\mathbf{0}$ (a column of B_k) from B_k, then we get a $k \times (2^k - 1)$ matrix B'_k over $2\mathbb{Z}_4$. Let C' be the code generated by B'_k. It's easy to check that C' is a special case of the code obtained in Theorem 4.5 in [14], when $k_1 = 0$, $k_2 = k$, then C' is a one-Lee weight linear code with the nonzero Lee weight $w' = 2^k$.

The following proposition is an alternative proof of Property 5.68 in (Handbook of Coding Theory, North Holland, 1998, WC Huffman, V. Pless eds, p. 1561).

Proposition 3.1 *If C is the code generated by Y_k, then C is a two-Lee weight code of length $n = 2^{k-1}(2^k - 1)$, with the nonzero Lee weight $w_1 = 2^{k-1}(2^k - 1)$, $w_2 = 2^{2k-1}$. Moreover, $A_{w_1} = 4^k - 2^k$ and $A_{w_2} = 2^k - 1$, respectively.*

Proof Let

$$G_k = \begin{pmatrix} Y_k & 3Y_k & B_k \end{pmatrix}, \tag{2}$$

then G_k is a $k \times 4^k$ matrix over $\mathbb{Z}_4$. Since B_k is a $k \times 2^k$ matrix over $2\mathbb{Z}_4$ with all different columns, then all columns of G_k are pairwise different. If we delete the $\mathbf{0}$ from G_k (in fact we delete $\mathbf{0}$ from B_k), then we get a $k \times (4^k - 1)$ matrix G'_k. Therefore, G'_k is a matrix with all nonzero different columns over $\mathbb{Z}_4$.

Let C'_k be the code generated by G'_k. It's easy to check that C'_k is a special case of the code obtained in Theorem 4.5 in [14], when $k_1 = k$, $k_2 = 0$, then C'_k is a one-Lee weight linear code with the nonzero Lee weight $w' = 4^k$.

Let $\mathbf{c} = (\mathbf{x}, 3\mathbf{x}, \mathbf{y}) \in C'$ and the lengths of $\mathbf{x}$ and $\mathbf{y}$ are $2^{k-1}(2^k - 1)$ and $2^k - 1$, respectively. Obviously, $W_L(\mathbf{y}) = 2^k$ or 0, and

$$W_L(\mathbf{c}) = W_L(\mathbf{x}) + W_L(3\mathbf{x}) + W_L(\mathbf{y}) = 2W_L(\mathbf{x}) + W_L(\mathbf{y}),$$

then, we have

$$W_L(\mathbf{x}) = \frac{W_L(\mathbf{c}) - W_L(\mathbf{y})}{2} = \frac{4^k - W_L(\mathbf{y})}{2}. \tag{3}$$

If $W_L(\mathbf{y}) \neq 0$, then $W_L(\mathbf{x}) = w_1 = 2^{k-1}(2^k - 1)$. Otherwise, $W_L(\mathbf{x}) = w_2 = 2^{2k-1}$. It is not difficult to get $A_{w_2} = 2^k - 1$ when $W_L(\mathbf{y}) = \mathbf{0}$. □

Corollary 3.2 *Let C be the code generated by Y_k, $\mathbf{c} \in C$ and $\mathbf{c} \neq \mathbf{0}$, then*

$$W_L(\mathbf{c}) = \begin{cases} 2^{2k-1} & \textit{if the order of } \mathbf{c} \textit{ is } 2, \\ 2^{k-1}(2^k - 1) & \textit{if the order of } \mathbf{c} \textit{ is } 4. \end{cases}$$

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Proof Let $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k$ be the rows of Y_k. From the proof of Proposition 3.1, $W_L(\mathbf{r}_i) = w_1 = 2^{k-1}(2^k - 1)$, but $W_L(2\mathbf{r}_i) = w_2 = 2^{2k-1}$.

Step 1 Let

$$\mathbf{c} = p_1\mathbf{r}_1 + p_2\mathbf{r}_2 + \cdots + p_k\mathbf{r}_k,$$

where $p_i \in \mathbb{Z}_4$, $1 \le i \le k$ and there exists $p_i \neq 0$. We can prove that $p_i = 0$ or 2 for all $1 \le i \le k$, if the order of $\mathbf{c}$ is 2. By (1), let

$$\mathbf{r}_k = (\mathbf{0}\ \mathbf{1}\ \mathbf{2}\ \mathbf{3}\ \overbrace{1\ 1\ \ldots\ 1}^{2^{k-1}}),$$

where $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length $2^{k-2}(2^{k-1} - 1)$, $i \in \mathbb{Z}_4$. If the order of $\mathbf{c}$ is 2, and $p_k = 1$ or 3, then $\mathbf{c}$ contains at least a 1 or a 3 according to the definition of Y_k in (1), contradiction. Therefore, $p_k = 0$ or 2 and $p_k\mathbf{r}_k = \mathbf{0}$ or $2\mathbf{r}_k$. So the order of $p_1\mathbf{r}_1 + p_2\mathbf{r}_2 + \cdots + p_{k-1}\mathbf{r}_{k-1}$ is 2. Similarly, by (1), we can get $p_{k-1} = 0$ or 2. Then we can prove $p_i = 0$ or 2 for all $1 \le i \le k$.

Step 2 By (3), we can get if $p_i = 2$ or 0, then $W_L(p_i\mathbf{r}_i) = 2^{2k-1}$ or 0, $1 \le i \le k$. Similarly,

$$W_L(p_1\mathbf{r}_1 + p_2\mathbf{r}_2 + \cdots + p_k\mathbf{r}_k) = 2^{2k-1} = w_2,$$

where $p_i = 0$ or 2, $1 \le i \le k$ and there exists $p_i \neq 0$. By **Step 1**, these are all the codewords of order 2. Therefore, if the order of $\mathbf{c}$ is 2, then $W_L(\mathbf{c}) = 2^{2k-1} = w_2$. Since $A_{w_2} = 2^k - 1$ by Proposition 3.1 and C is a two-Lee weight code, we can get if the order of $\mathbf{c}$ is 4, then $W_L(\mathbf{c}) = 2^{k-1}(2^k - 1)$. □

When $k \ge 2$, let

$$X_k = \begin{pmatrix} Y_{k-1} & Y_{k-1} & Y_{k-1} & Y_{k-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{pmatrix}, \tag{4}$$

where $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length $2^{k-2}(2^{k-1} - 1)$, $i \in \mathbb{Z}_4$.

Similar to Proposition 3.1, we have

Proposition 3.3 *If C is the code generated by X_k, then C is a two-Lee weight code of length $n = 2^k(2^{k-1} - 1)$ with the nonzero Lee weight $w_1 = 2^k(2^{k-1} - 1)$, and $w_2 = 2^{2k-1}$. Moreover, $A_{w_1} = 4^k - 2^{k-1}$ and $A_{w_2} = 2^{k-1} - 1$, respectively.*

Proof From the structure of X_k and Proposition 3.1, we know there are $2^{k-1} - 1$ codewords of weight 2^{2k-1} generated by the first $k - 1$ rows. Let $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k$ be the rows of X_k, and denote the codeword generated by the first $k - 1$ rows by $\mathbf{c}$. Without loss of generality, by (4), let

$$\mathbf{r}_k = (\mathbf{0}\ \mathbf{1}\ \mathbf{2}\ \mathbf{3}),$$

where $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length $2^{k-2}(2^{k-1} - 1)$, $i \in \mathbb{Z}_4$.

It is sufficient to prove $W_L(\mathbf{r}_k + \mathbf{c}) = W_L(2\mathbf{r}_k + \mathbf{c}) = 2^k(2^{k-1} - 1)$. It is easy to check that $W_L(\mathbf{r}_k) = W_L(2\mathbf{r}_k) = 2^k(2^{k-1} - 1)$. Let $\mathbf{c} = (\mathbf{c}', \mathbf{c}', \mathbf{c}', \mathbf{c}')$, n_i be the number of i in $\mathbf{c}'$ and n' be the length of $\mathbf{c}'$. So $n = 4n' = 2^k(2^{k-1} - 1)$.

(1) If the order of $\mathbf{c}$ is 2, according to Corollary 3.2, $W_L(\mathbf{c}) = 2^{2k-1}$, $n' = n_0 + n_2$, and

$$\begin{aligned} W_L(\mathbf{r}_k + \mathbf{c}) &= W_L(\mathbf{c}') + W_L(\mathbf{c}' + \mathbf{1}) + W_L(\mathbf{c}' + \mathbf{2}) + W_L(\mathbf{c}' + \mathbf{3}) \\ &= 2n_2 + n' + 2(n' - n_2) + n' = 4n' = 2^k(2^{k-1} - 1). \end{aligned}$$

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(2) If the order of $\mathbf{c}$ is 4, $n' = n_0 + n_1 + n_2 + n_3$, and

$$\begin{aligned} W_L(\mathbf{r}_k + \mathbf{c}) &= W_L(\mathbf{c}') + W_L(\mathbf{c}' + \mathbf{1}) + W_L(\mathbf{c}' + \mathbf{2}) + W_L(\mathbf{c}' + \mathbf{3}) \\ &= (n_1 + 2n_2 + n_3) + (n_0 + 2n_1 + n_2) \\ &\quad + (2n_0 + n_1 + n_3) + (n_0 + n_2 + 2n_3) \\ &= 4(n_1 + n_3 + n_2 + n_0) = 4n' = 2^k(2^{k-1} - 1). \end{aligned}$$

Similarly, we can also prove $W_L(2\mathbf{r}_k + \mathbf{c}) = 2^k(2^{k-1} - 1) = w_1$. Therefore, $A_{w_2} = 2^{k-1} - 1$, and $A_{w_1} = 4^k - 2^{k-1}$. □

Example 3.4 If $k = 3$, then $n = 24$, $w_1 = 24$, $w_2 = 32$, then, according to Proposition 3.3, there is a two-Lee weight code with the generator matrix:

$$\begin{pmatrix} 111102 & 111102 & 111102 & 111102 \\ 012311 & 012311 & 012311 & 012311 \\ 000000 & 111111 & 222222 & 333333 \end{pmatrix}. \tag{5}$$

4 Two-lee weight projective codes over $\mathbb{Z}_4$

In this section, we will give some structures of two-Lee weight projective codes over $\mathbb{Z}_4$.

Lemma 4.1 *Let C be a linear code over $\mathbb{Z}_4$ with type $4^{k_1}2^{k_2}$ of length n. If $k_2 = 1$ and the columns of the generator matrix $G_{(k_1+1)\times n}$ are all distinct nonzero vectors $(c_1, c_2, \ldots, c_{k_1}, c_{k_1+1})^T$, where $(c_1, c_2, \ldots, c_{k_1})^T$ is one column of Y_{k_1}, $c_{k_1+1} = 0$ or 2, then C is a two-Lee weight code of length $n = 2^{k_1}(2^{k_1} - 1)$ with the nonzero Lee weights $w_1 = 2^{k_1}(2^{k_1} - 1)$ and $w_2 = 2^{2k_1}$. Moreover, $A_{w_1} = 2^{k_1}(2^{k_1+1} - 1)$ and $A_{w_2} = 2^{k_1} - 1$, respectively.*

Proof Without loss of generality, let

$$G_{(k_1+1)\times n} = \begin{pmatrix} Y_{k_1} & Y_{k_1} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}, \tag{6}$$

where $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length $2^{k_1-1}(2^{k_1} - 1)$, $i \in 2\mathbb{Z}_4$ and $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{k_1}, \mathbf{r}_{k_1+1}$ be the rows of $G_{(k_1+1)\times n}$. Since the columns of $G_{(k_1+1)\times n}$ are all nonzero distinct vectors, it follows that $n = 2^{k_1}(2^{k_1} - 1)$. Let $\mathbf{c} = (\mathbf{x}, \mathbf{x}) \in C$, where $\mathbf{x}$ is generated by Y_{k_1}. According to Proposition 3.1, then $W_L(\mathbf{c}) = 2W_L(\mathbf{x}) = 2^{k_1}(2^{k_1} - 1)$ or 2^{2k_1}. Denote the number of i in $\mathbf{x}$ by n_i, and the length of $\mathbf{x}$ by n'. Obviously, $n = 2n' = 2^{k_1}(2^{k_1} - 1)$.

(1) If the order of $\mathbf{x}$ is 4, $n' = n_0 + n_1 + n_2 + n_3$, then we have

$$\begin{aligned} W_L(\mathbf{r}_{k_1+1} + \mathbf{c}) &= W_L(\mathbf{x}) + W_L(\mathbf{x} + \mathbf{2}) = n_1 + 2n_2 + n_3 + 2n_0 + n_1 + n_3 \\ &= 2(n_0 + n_1 + n_2 + n_3) = 2n' = 2^{k_1}(2^{k_1} - 1). \end{aligned}$$

(2) If the order of $\mathbf{x}$ is 2, $n' = n_0 + n_2$, $n_1 = n_3 = 0$, then we have

$$\begin{aligned} W_L(\mathbf{r}_{k_1+1} + \mathbf{c}) &= W_L(\mathbf{x}) + W_L(\mathbf{x} + \mathbf{2}) = 2n_2 + 2n_0 \\ &= 2n' = 2^{k_1}(2^{k_1} - 1). \end{aligned}$$

Hence, C is a two-Lee weight code with the nonzero Lee weights $w_1 = 2^{k_1}(2^{k_1} - 1)$, and $w_2 = 2^{2k_1}$. Obviously, A_{w_2} does not change comparing with Y_{k_1}, *i.e.* $A_{w_2} = 2^{k_1} - 1$, thus, $A_{w_1} = 2^{k_1}(2^{k_1+1} - 1)$. □

Revised Proof

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Theorem 4.2 *Let C be a linear code over $\mathbb{Z}_4$ with type $4^{k_1}2^{k_2}$ of length n. If the columns of the generator matrix $G_{(k_1+k_2)\times n}$ are all distinct nonzero vectors $(c_1, c_2, \ldots, c_{k_1}, c_{k_1+1}, \ldots, c_{k_1+k_2})^T$, where $(c_1, c_2, \ldots, c_{k_1})^T$ is one column of Y_{k_1}, $c_i = 0$ or 2, $k_1 + 1 \le i \le k_1 + k_2$, then C is a two-Lee weight code of length $n = 2^{k_1+k_2-1}(2^{k_1} - 1)$ with the nonzero Lee weights $w_1 = 2^{k_1+k_2-1}(2^{k_1} - 1)$ and $w_2 = 2^{2k_1+k_2-1}$. Moreover, $A_{w_1} = 2^{k_1}(2^{k_1+k_2} - 1)$ and $A_{w_2} = 2^{k_1} - 1$, respectively.*

Proof If $k_2 = 1$, the result follows from Lemma 4.1. Assume the result is valid for any $k_2 = m$, we denote the generator matrix and the nonzero weights by $G_{k_1,m}$ and by $w_1' = 2^{k_1+m-1}(2^{k_1} - 1)$, $w_2' = 2^{2k_1+m-1}$, respectively, then for $k_2 = m + 1$, without loss of generality, the generator matrix $G_{k_1,m+1}$ of C can be expressed as

$$G_{k_1,m+1} = \begin{pmatrix} G_{k_1,m} & G_{k_1,m} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}, \tag{7}$$

where $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length $2^{k_1+m-1}(2^{k_1} - 1)$, $i \in 2\mathbb{Z}_4$. It's easy to verify that C is a two-Lee weight code with $w_1 = 2w_1' = 2^{k_1+m}(2^{k_1} - 1)$ and $w_2 = 2w_2' = 2^{2k_1+m}$. Similar to the proof of Lemma 4.1, we can get $A_{w_2} = 2^{k_1} - 1$ and $A_{w_1} = 2^{k_1}(2^{k_1+k_2} - 1)$. By induction hypothesis, the results are valid. □

Example 4.3 If $k_1 = 2, k_2 = 1$, then $n = 12, w_1 = 12, w_2 = 16$, then, according to Theorem 4.2, there is a two-Lee weight code with the generator matrix:

$$G_{2,1} = \left(\begin{array}{c|c} 111102 & 111102 \\ 012311 & 012311 \\ 222222 & 000000 \end{array}\right). \tag{8}$$

Example 4.4 If $k_1 = 2, k_2 = 2$, then $n = 24, w_1 = 24, w_2 = 32$, then, according to Theorem 4.2, there is a two-Lee weight code with the generator matrix:

$$G_{2,2} = \left(\begin{array}{c|c} 111102111102 & 111102111102 \\ 012311012311 & 012311012311 \\ 222222000000 & 222222000000 \\ 222222222222 & 000000000000 \end{array}\right). \tag{9}$$

Corollary 4.5 *Let C be a linear code over $\mathbb{Z}_4$ with type $4^{k_1}2^{k_2}$ of length n. When $k_1 \ge 2$, the columns of the generator matrix $G'_{(k_1+k_2)\times n}$ are all distinct nonzero vectors $(c_1, c_2, \ldots, c_{k_1}, c_{k_1+1}, \ldots, c_{k_1+k_2})^T$, where $(c_1, c_2, \ldots, c_{k_1})^T$ is one column of X_{k_1} defined above, $c_i = 0$ or 2, $k_1 + 1 \le i \le k_1 + k_2$, then C is a two-Lee weight code of length $n = 2^{k_1+k_2}(2^{k_1-1} - 1)$ with the nonzero Lee weights $w_1 = 2^{k_1+k_2}(2^{k_1-1} - 1)$, $w_2 = 2^{2k_1+k_2-1}$. Moreover, $A_{w_1} = 4^{k_1}2^{k_2} - 2^{k_1-1}$ and $A_{w_2} = 2^{k_1-1} - 1$, respectively.*

Proof Assume that $k_1 \ge 2$, then according to Theorem 4.2, we can construct a code C' with type $4^{k_1-1}2^{k_2}$, and denote its generator matrix by G'. By the definition of X_k in (4), without loss of generality, let

$$G'_{(k_1+k_2)\times n} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ G' & G' & G' & G' \end{pmatrix}, \tag{10}$$

where $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length $2^{k_1+k_2-2}(2^{k_1-1} - 1)$, $i \in \mathbb{Z}_4$. It is easily seen that C is a two-Lee weight code with the nonzero weight $w_1 = 2^{k_1+k_2}(2^{k_1-1} - 1)$ and $w_2 = 2^{2k_1+k_2-1}$. By induction hypothesis, C is a two-Lee weight code over $\mathbb{Z}_4$. □

Revised Proof

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Example 4.6 If $k_1 = 2, k_2 = 2$, then $n = 16, w_1 = 32, w_2 = 16$, according to Corollary 4.5, there is a two-Lee weight code with the generator matrix:

$$G'_{2,2} = \begin{pmatrix} 1111\ 1111\ 1111\ 1111 \\ 0123\ 0123\ 0123\ 0123 \\ 0000\ 0000\ 2222\ 2222 \\ 0000\ 2222\ 0000\ 2222 \end{pmatrix}. \tag{11}$$

Example 4.7 If $k_1 = 3, k_2 = 1$, then $n = 48, w_1 = 48, w_2 = 64$, according to Corollary 4.5, there is a two-Lee weight code with the generator matrix $G'_{3,1}$:

$$\left(\begin{array}{l|l} 111102\ 111102\ 111102\ 111102 & 111102\ 111102\ 111102\ 111102 \\ 012311\ 012311\ 012311\ 012311 & 012311\ 012311\ 012311\ 012311 \\ 000000\ 111111\ 222222\ 333333 & 000000\ 111111\ 222222\ 333333 \\ 222222\ 222222\ 222222\ 222222 & 000000\ 000000\ 000000\ 000000 \end{array}\right). \tag{12}$$

Next, we are ready to give the sufficient and necessary conditions for a two-Lee weight linear code to be projective over $\mathbb{Z}_4$.

Lemma 4.8 *Let C be a linear code over $\mathbb{Z}_4$ with type $4^{k_1}2^{k_2}$ of length n and with the generator matrix G. Then C is projective if and only if the following conditions hold:*

(1) *every column of G contains 1 or 3;*
(2) *any two columns of G are not multiple of each other by ± 1.*

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be the columns of G. Let us first prove the sufficiency. Let $C^\perp$ be the dual code of C. If $\mathbf{c} \in C^\perp$, then we only need to prove $W_L(\mathbf{c}) \geq 3$. Obviously, $W_L(\mathbf{c})$ cannot be 1 because all columns are nonzero. Assume $W_L(\mathbf{c}) = 2$. Obviously, $\mathbf{c}$ cannot be written as $\mathbf{c} = (0, \ldots, 0, 2, 0, \ldots, 0)$ due to (1). Then we have 2 cases:

Case 1 Without loss of generality, let $\mathbf{c} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$, where $c_i = c_j = 1, i < j$. Then $\mathbf{x}_i + \mathbf{x}_j = \mathbf{0}$. Thus, $\mathbf{x}_j = 3\mathbf{x}_i$, $\mathbf{x}_i$ and $\mathbf{x}_j$ are in proportion by the unit 3, contradiction.

Case 2 Without loss of generality, let $\mathbf{c} = (0, \ldots, 0, 1, 0, \ldots, 0, 3, 0, \ldots, 0)$, where $c_i = 1, c_j = 3, i < j$. Then $\mathbf{x}_i + \mathbf{x}_j = \mathbf{0}$. Thus, $\mathbf{x}_j = \mathbf{x}_i$, contradiction.

Therefore, $W_L(\mathbf{c}) \geq 3$. By definition, C is projective.

Now we begin to prove the necessity. If (1) doesn't hold, let $\mathbf{x}_j \in 2\mathbb{Z}_4^{k_1+k_2}$, then the codeword $\mathbf{c} = (0, \ldots, 0, 2, 0, \ldots, 0) \in C^\perp$, where $c_j = 2$. If (2) doesn't hold, $\mathbf{x}_i = \pm\mathbf{x}_j$ for some pair of indices $i < j$.

Case 1 If $\mathbf{x}_i = \mathbf{x}_j$, we can construct $\mathbf{c}' = (0, \ldots, 0, 1, 0, \ldots, 0, 3, 0, \ldots, 0)$, where $c'_i = 1, c'_j = 3, i < j$ and $\mathbf{c}' \in C^\perp$.

Case 2 If $\mathbf{x}_i = 3\mathbf{x}_j$, then $\mathbf{c}'' = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \in C^\perp$, where $c''_i = c''_j = 1$.

But the Lee weight of $\mathbf{c}$, $\mathbf{c}'$ and $\mathbf{c}''$ is 2. Contradict with $w_L(C^\perp) = 3$. □

Theorem 4.9 *Let C be a linear code over $\mathbb{Z}_4$ with type $4^{k_1}2^{k_2}$ of length n obtained in Theorem 4.2, then C is projective.*

Proof Denote the generator matrix of C by G_{k_1,k_2}, then it is sufficient to prove that G_{k_1,k_2} satisfies Lemma 4.8.

Step 1 Prove Y_{k_1} defined in (1) can make it. The result is obvious.

Step 2 Use induction for k_2. If $k_2 = 1$, we know the generator matrix has the form (6) in Lemma 4.1. And by Step 1 and Lemma 4.8, we can easily prove that the code is

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projective. Assume the result is valid for any $k_2 = m$, then for $k_2 = m + 1$, according to (7) in Theorem 4.2, without loss of generality, the generator matrix $G_{k_1,m+1}$ of C is

$$G_{k_1,m+1} = \begin{pmatrix} G_{k_1,m} & G_{k_1,m} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}, \tag{13}$$

where $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length $2^{k_1+m-1}(2^{k_1} - 1)$, $i \in 2\mathbb{Z}_4$. So, all the columns of $G_{k_1,m+1}$ contain a 1 or a 3. If there are two columns in proportion by ± 1, then there are two columns of $G_{k_1,m}$ in proportion by ± 1, a contradiction. So the result is valid for $k_2 = m + 1$. By induction hypothesis, G_{k_1,k_2} satisfies Lemma 4.8 for any $k_1, k_2 \geq 1$, then C is projective by Lemma 4.8. □

Similar to the proof of Theorem 4.9, we have the following corollary.

Corollary 4.10 *The linear code obtained in Corollary 4.5 is projective.*

5 Gray images

For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_4^n$, define $\alpha(\mathbf{x}) = (\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n))$, where $\alpha(0) = \alpha(2) = 0$ and $\alpha(1) = \alpha(3) = 1$.

Lemma 5.1 (*See* [4]) *Let C be a linear code over $\mathbb{Z}_4$, $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}$ be a set of generators of C, then $\Phi(C)$ is linear if and only if $2\alpha(\mathbf{x_i}) * \alpha(\mathbf{x_j}) \in C$ for all i, j, satisfying $1 \leq i \leq j \leq m$.*

Let $q = p^m$, where p is prime and m is a positive integer. According to [2], an $[n, k]$ code C over $GF(q)$ is a k-dimensional subspace of $GF(q)^n$. Next proposition is from the classical paper in [2] about the survey of two weight codes over finite fields.

Proposition 5.2 (*See* [2]) *Example SU 1: Let Ω be the complement of a t-dimensional subspace of $GF(q)^k$ where $1 \leq t \leq k-1$. If v is a non-zero vector then $|v^{\perp} \cap \Omega| = q^{k-1} - q^t$ or $q^{k-1} - q^{t-1}$. For a two-weight code with Example SU1, if the field is $GF(q)$, its parameters are (see FIGURE 1a and FIGURE 2a in [2])*

$$n = \frac{q^l - q^t}{q - 1}, k = l, w_1 = q^{l-1} - q^{t-1}, w_2 = q^{l-1}, A_{w_1} = q^l - q^{l-t}, A_{w_2} = q^{l-t} - 1.$$

where $1 \leq t \leq l - 1$.

Proposition 5.3 *The Gray images of the codes of Theorem 4.2 and Corollary 4.5 are binary distance invariant codes with the weights given $w_1 = q^{l-1} - q^{t-1}$ and $w_2 = q^{l-1}$.*

Note that we do not claim these binary codes to be linear.

Here is a table about the parameters of the Gray images $\Phi(C_1)$, $\Phi(C_2)$ and $\Phi(C_3)$, where C_1 and C_2 are the codes obtained in Theorem 4.2 and Corollary 4.5 in this paper, respectively, and C_3 is the code obtained in Theorem 5.6 in [14].

Remark 5.4 If $k_1 = 1$, $\Phi(C_1)$ and $\Phi(C_3)$ have the same parameters. Besides, due to Corollary 6.8 in [14], when $k_1 = 1$, $\Phi(C_1)$ reaches the Plotkin bound and Griesmer bound.

Remark 5.5 It can be seen that the values of l for $\Phi(C_1)$, $\Phi(C_2)$ and $\Phi(C_3)$ are all $2k_1 + k_2$. But the values of t are $k_1 + k_2$, $k_1 + k_2 + 1$, $2k_1 + k_2 - 1$ for $\Phi(C_1)$, $\Phi(C_2)$ and $\Phi(C_3)$, respectively. Therefore, these three families of codes lead to non-equivalent two-Lee weight linear codes.

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Table 1 The parameters of $\Phi(C_1)$, $\Phi(C_2)$ and $\Phi(C_3)$

	n	w_1	w_2	A_{w_1}	A_{w_2}
SU 1	$\frac{q^l-q^t}{q-1}$	$q^{l-1}-q^{t-1}$	q^{l-1}	q^l-q^{l-t}	$q^{l-t}-1$
$\Phi(C_1)$	$2^{k_1+k_2}(2^{k_1}-1)$	$2^{k_1+k_2-1}(2^{k_1}-1)$	$2^{2k_1+k_2-1}$	$4^{k_1}2^{k_2}-2^{k_1}$	$2^{k_1}-1$
$\Phi(C_2)$	$2^{k_1+k_2}(2^{k_1}-2)$	$2^{k_1+k_2}(2^{k_1-1}-1)$	$2^{2k_1+k_2-1}$	$4^{k_1}2^{k_2}-2^{k_1-1}$	$2^{k_1-1}-1$
$\Phi(C_3)$	$2^{2k_1+k_2-1}$	$2^{2k_1+k_2-2}$	$2^{2k_1+k_2-1}$	$4^{k_1}2^{k_2}-2$	1

Finally, we will consider the linearity of the image codes in Theorem 4.2 and Corollary 4.5.

Theorem 5.6 *If the columns of the generator matrix of C satisfy the conditions of Theorem 4.2, then $\Phi(C)$ is linear when $k_1 = 1$ and nonlinear when $k_1 \geq 2$.*

Proof When $k_1 = 1$, the result can be obtained in Corollary 6.8 in [14]. When $k_1 \geq 2$, denote the generator matrix of C by G_{k_1,k_2} and let $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{k_1}, \ldots, \mathbf{r}_{k_1+k_2}$ be the rows of G_{k_1,k_2}, where $\mathbf{r}_j$ is a vector over $2\mathbb{Z}_4$, $k_1+1 \leq j \leq k_1+k_2$. By Theorem 4.2,

$$G_{k_1+1,k_2} = \begin{pmatrix} G_{k_1,k_2} & G_{k_1,k_2} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}, \tag{14}$$

where $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length $2^{k_1+k_2-1}(2^{k_1}-1)$, $i \in 2\mathbb{Z}_4$. Obviously, if either $\mathbf{r}_1$ or $\mathbf{r}_2$ is from the last k_2 rows, it's easy to check that $2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) = \mathbf{0} \in C$.

(1) If $k_1 = k+1 \geq 2$, $k_2 = 0$, then $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k$ and $\mathbf{r}_{k+1}$ become the rows of Y_{k+1}. By the definition of Y_k, we can get

$$Y_{k+1} = \begin{pmatrix} Y_k & Y_k & Y_k & Y_k & B_k \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1}' \end{pmatrix}$$

where $k \geq 1$, B_k is a $k \times 2^k$ matrix over $2\mathbb{Z}_4$ with all different columns, $\mathbf{i}$ is a row vector $(i, i, \ldots, i)$ of length $2^{k-1}(2^k-1)$, $i \in \mathbb{Z}_4$ and $\mathbf{1}'$ is the row vector $(1, 1, \ldots, 1)$ of length 2^k.

Without loss of generality, let

$$\mathbf{r}_1 = (\overbrace{0\,0\,\ldots\,0}^{2^{k-1}(2^k-1)}\overbrace{1\,1\,\ldots\,1}^{2^{k-1}(2^k-1)}\overbrace{2\,2\,\ldots\,2}^{2^{k-1}(2^k-1)}\overbrace{3\,3\,\ldots\,3}^{2^{k-1}(2^k-1)}\overbrace{1\,1\,\ldots\,1}^{2^k}) \tag{15}$$

be the last row of Y_{k+1}, and

$$\mathbf{r}_2 = (\mathbf{x}\,\mathbf{x}\,\mathbf{x}\,\mathbf{x}\,\mathbf{b}),$$

where $\mathbf{x}$ is a row of Y_k, and $\mathbf{b}$ is a row of B_k. Then

$$\begin{aligned} 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) &= 2(\alpha(\mathbf{0})\,\alpha(\mathbf{1})\,\alpha(\mathbf{2})\,\alpha(\mathbf{3})\,\alpha(\mathbf{1}') * (\alpha(\mathbf{x})\,\alpha(\mathbf{x})\,\alpha(\mathbf{x})\,\alpha(\mathbf{x})\,\alpha(\mathbf{b})) \\ &= 2(\mathbf{0}\,\alpha(\mathbf{x})\,\mathbf{0}\,\alpha(\mathbf{x})\,\mathbf{0}), \end{aligned}$$

we get

$$W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2)) = 4W_L(\alpha(\mathbf{x})).$$

Since the length of $\mathbf{x}$ is $2^{k-1}(2^k-1)$, then

$$4W_L(\alpha(\mathbf{x})) \leq 4 \times 2^{k-1}(2^k-1) = 2^{k+1}(2^k-1).$$

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By Proposition 3.1, C is a two-Lee weight linear code with the nonzero Lee weights $2^k(2^{k+1}-1)$, and 2^{2k+1}. And the nonzero smallest Lee weight of C is $2^k(2^{k+1}-1) > 2^{k+1}(2^k-1)$. Thus, $2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) \notin C$. By Lemma 5.1, $\Phi(C)$ is nonlinear when $k_1 \geq 2$, $k_2 = 0$.

(2) If $k_2 \geq 1$, then the first k_1 rows of G_{k_1,k_2} can be written as:

$$\underbrace{Y_{k_1}\ Y_{k_1}\ \ldots Y_{k_1}}_{2^{k_2}},$$

and let

$$\mathbf{r}'_i = \underbrace{(\mathbf{r}_i\ \mathbf{r}_i\ \ldots\ \mathbf{r}_i)}_{2^{k_2}},$$

where $i = 1$ or 2. Since $k_1 = k+1$, we have

$$\begin{aligned} W_L(2\alpha(\mathbf{r}'_1) * \alpha(\mathbf{r}'_2)) &= 2^{k_2} W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2)) = 2^{k_2+2} W_L(\alpha(\mathbf{x})) \\ &\leq 2^{k_2+2} \times 2^{k-1}(2^k-1) = 2^{k_1+k_2}(2^{k_1-1}-1). \end{aligned}$$

By Theorem 4.2, C is a two-Lee weight code with the nonzero Lee weights $2^{k_1+k_2-1}(2^{k_1}-1)$ and $2^{2k_1+k_2-1}$. And the nonzero smallest Lee weight is $2^{k_1+k_2-1}(2^{k_1}-1) > 2^{k_1+k_2}(2^{k_1-1}-1)$. Thus, $2\alpha(\mathbf{r}'_1) * \alpha(\mathbf{r}'_2) \notin C$. By Lemma 5.1, $\Phi(C)$ is nonlinear when $k_1 \geq 2$, $k_2 \geq 0$. □

Example 5.7 Consider the code defined in Example 4.3. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ denote the first and second rows of $G_{2,1}$ in (8), respectively. Then the Lee weight of

$$\begin{aligned} 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) &= 2\alpha(111102111102) * \alpha(012311012311) \\ &= 2(111100111100) * (010111010111) \\ &= (020200020200) \end{aligned}$$

is 8 which can not be obtained in C since C is a two-Lee weight code with $w_1 = 12$, $w_2 = 16$.

Theorem 5.8 *If the columns of the generator matrix of C satisfy the conditions of Corollary 4.5, then $\Phi(C)$ is linear when $k_1 = 2$ and nonlinear when $k_1 \geq 3$.*

Proof Denote the generator matrix of C by G_{k_1,k_2}. Let $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{k_1}, \ldots, \mathbf{r}_{k_1+k_2}$ be the rows of G_{k_1,k_2}, where $\mathbf{r}_j$ is a vector over $2\mathbb{Z}_4$, $k_1+1 \leq j \leq k_1+k_2$.

(1) By Corollary 4.5, if $k_1 = 2$, $k_2 = 1$, the generator matrix is:

$$G_{2,1} = \begin{pmatrix} 1111\ 1111 \\ 0123\ 0123 \\ 2222\ 0000 \end{pmatrix}.$$

So, if $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_j$ are the rows of G_{2,k_2}, $3 \leq j \leq k_2+2$, we can get $\mathbf{r}_1$ is all 1 vector. Therefore, $\alpha(\mathbf{r}_1) = \mathbf{1}$ and $\alpha(\mathbf{r}_j) = \mathbf{0}$. Besides,

$$\begin{aligned} 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) = 2\mathbf{r}_2 \in C,\ 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_j) &= \mathbf{0} \in C, \\ 2\alpha(\mathbf{r}_2) * \alpha(\mathbf{r}_j) = \mathbf{0} \in C,\quad 2\alpha(\mathbf{r}_j) * \alpha(\mathbf{r}_k) &= \mathbf{0} \in C, \end{aligned}$$

where $3 \leq j, k \leq 2+k_2$. By Lemma 5.1, $\Phi(C)$ is linear when $k_1 = 2$, $k_2 \geq 0$.

(2) If $k_1 \geq 3$, by (10) in the proof of Corollary 4.5,

$$G_{k_1,k_2} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ G'_{k_1-1,k_2} & G'_{k_1-1,k_2} & G'_{k_1-1,k_2} & G'_{k_1-1,k_2} \end{pmatrix},$$

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where G'_{k_1-1,k_2} is the generator matrix of the code C' obtained in Theorem 4.2, and $\mathbf{i}$ is the row vector $(i, i, \ldots, i)$ of length $2^{k_1+k_2-2}(2^{k_1-1}-1)$, $i \in \mathbb{Z}_4$. Without loss of generality, let

$$\mathbf{r}_1 = (\mathbf{0\ 1\ 2\ 3}) \text{ and } \mathbf{r}_2 = (\mathbf{r}_0\ \mathbf{r}_0\ \mathbf{r}_0\ \mathbf{r}_0)$$

where $\mathbf{r}_0$ is one row of G'_{k_1-1,k_2}. Then

$$2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) = 2(\mathbf{0}\ \alpha(\mathbf{r}_0)\ \mathbf{0}\ \alpha(\mathbf{r}_0)).$$

Since the length of $\mathbf{r}_0$ is $2^{k_1+k_2-2}(2^{k_1-1}-1)$,

$$W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2)) = 4W_L(\alpha(\mathbf{r}_0)) \leq 2^{k_1+k_2}(2^{k_1-1}-1). \tag{16}$$

If $\mathbf{0} \neq 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) \in C$, since C is a two-Lee weight code by Corollary 4.5, $W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2)) = 2^{k_1+k_2}(2^{k_1-1}-1)$ or $2^{2k_1+k_2-1}$. Thus

$$2^{k_1+k_2}(2^{k_1-1}-1) \leq W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2)) = 4W_L(\alpha(\mathbf{r}_0)). \tag{17}$$

Therefore, combining (16) and (17), we have

$$W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2)) = 4W_L(\alpha(\mathbf{r}_0)) = 2^{k_1+k_2}(2^{k_1-1}-1),$$

which forces $\alpha(\mathbf{r}_0)$ to be all 1 vector. Thus we have $W_L(\mathbf{r}_0) = 2^{k_1+k_2-2}(2^{k_1-1}-1)$, $W_L(2\mathbf{r}_0) = 2^{k_1+k_2-1}(2^{k_1-1}-1)$ and then $W_L(2\mathbf{r}_0) = 2W_L(\mathbf{r}_0)$. However, $\mathbf{r}_0$ is one row of G'_{k_1-1,k_2}, the generator matrix obtained in Theorem 4.2, and the two Lee weights w_1 and w_2 can not satisfy $w_2 = 2w_1$, contradiction. Thus there are at least 2 rows $\mathbf{r}_1, \mathbf{r}_2$ satisfying $2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) \notin C$. Namely, $\Phi(C)$ is nonlinear when $k_1 \geq 3, k_2 \geq 0$. □

Example 5.9 Consider the code defined in Example 4.7. Let $\mathbf{r}_1$ annd $\mathbf{r}_2$ denote the first and second rows of $G'_{3,1}$ in (12), respectively. Then the Lee weight of

$$\begin{aligned} 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_2) &= 2\alpha(111102111102111102111102111102111102111102111102) \\ &\quad * \alpha(012311012311012311012311012311012311012311012311) \\ &= (020200020200020200020200020200020200020200020200) \end{aligned}$$

is 32 which can not be obtained in C since C is a two-Lee weight code with $w_1 = 48$, $w_2 = 64$.

6 Graphs

We recall the classic lemma from [12].

Lemma 6.1 *If C is a $\mathbb{Z}_4$-code of length n and weights w_1, w_2 the coset graph of $C^\perp$ is an SRG on $|C|$ vertices, degree $2n$, unrestricted eigenvalues $2n - 2w_1$ and $2n - 2w_2$.*

From this classic lemma, and the characterization of SRGs given in §2.2, the following two results follow immediately. Their proofs are omitted.

Theorem 6.2 (1) *If C is the code of Theorem 4.2 then the coset graph of $C^\perp$ is an SRG on $|C|$ vertices, degree $2n$, and unrestricted eigenvalues 0 and $-2^{k_1+k_2}$.* (2) *If C is the code of Corollary 4.5 then the coset graph of $C^\perp$ is an SRG on $|C|$ vertices, degree $2n$, and unrestricted eigenvalues 0 and $-2^{k_1+k_2+1}$.*

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Remarks:

- Following [1, §1.1.3] we see that these graphs are complete multipartite of type $K_{a\times m}$ with $m = 2^{k_1+k_2}$ for Theorem 4.2 or $m = 2^{k_1+k_2-1}$ for Corollary 4.5, and, in both cases, $a = \frac{|C|}{m}$.
- The graph corresponding to the code of Example 4.6 is bipartite complete, since the code is an Hadamard code of a kind studied in [5,6].

7 Conclusion

In this paper, we have constructed two infinite families of projective two-weight $\mathbb{Z}_4$-codes. Further, their Gray images are nonlinear, and have the same weight distributions as a family of two-weight codes of [2], the SU1 family. In small values of parameters these codes can be implemented in the computer package Magma. It is a worthwhile project to check which families of binary two-weight codes from [2] admit $\mathbb{Z}_4$-linear analogues.

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References

1. Brouwer A.E., van Maldeghem H.: Strongly regular graphs. https://homepages.cwi.nl/~aeb/math/srg/rk3/srgw.pdf. Accessed 20 Feb 2020.
2. Calderbank A.R., Kantor W.M.: The geometry of two-weight codes. Bull. Lond. Math. Soc. **18**, 97–122 (1986).
3. Delsarte P.: Weights of linear codes and strongly regular normed spaces. Discret. Math. **3**, 47–64 (1972).
4. Hammons A.R., Kumar P.V., Calderbank A.R., Sloane N.J.A., Solé P.: The $\mathbb{Z}_4$-linearity of kerdock, preparata, goethals and related codes. IEEE Trans. Inf. Theory **40**, 301–319 (1994).
5. Krotov, D. S.: $\mathbb{Z}_4$-linear perfect codes, Diskretn. Anal. Issled. Oper., Ser. 1, **7**(4), 78–90 (2000). arXiv:0710.0198 (English translation at arXiv:0710.0198).
6. Krotov D.S.: $\mathbb{Z}_4$-Linear Hadamard and Extended Perfect Codes, WCC2001. In: D. Augot, C. Carlet (eds.) International Workshop on Coding and Cryptography (Paris, France, 8–12 January 2001), Electronic Notes in Discrete Mathematics, vol. 6, 107–112. Elsevier B. V., Amsterdam (2001). arXiv:0710.0199.
7. Shi M.J., Alahmadi A., Solé P.: Codes and Rings: Theory and Practice. Academic Press, New York (2017).
8. Shi M.J., Chen L.: Construction of two-Lee weight codes over $\mathbb{F}_p + v\mathbb{F}_p + v^2\mathbb{F}_p$. Int. J. Comput. Math. **93**(3), 415–424 (2016).
9. Shi M.J., Guan Y., Solé P.: Two new families of two-weight codes. IEEE Trans. Inf. Theory **63**(10), 6240–6246 (2017).
10. Shi M.J., Liu Y., Solé P.: Optimal two-weight codes from trace codes over $\mathbb{F}_2 + u\mathbb{F}_2$. IEEE Commun. Lett. **20**(12), 2346–2349 (2016).
11. Shi M.J., Wu R.S., Liu Y., Solé P.: Two and three weight codes over $\mathbb{F}_p + u\mathbb{F}_p$. Cryptogr. Commun. **9**(5), 637–646 (2017).
12. Shi M.J., Sepasdar Z., Alahmadi A., Solé P.: On two-weight Z_{2^k} -codes. Des. Codes Cryptogr. **86**(6), 1201–1209 (2018).
13. Shi M.J., Solé P.: Optimal p-ary codes from one-weight codes and two-weight codes over $\mathbb{F}_p + v\mathbb{F}_p$. J. Syst. Sci. Complex. **28**(3), 679–690 (2015).
14. Shi M.J., Wang Y.: Optimal binary codes from one-Lee weight codes and two-Lee weight projective codes over $\mathbb{Z}_4$. J. Syst. Sci. Complex. **27**(4), 795–810 (2014).
15. Shi M.J., Wang C.C., Wu R.S., Hu Y., Chang Y.Q.: One-weight and two-weight $\mathbb{Z}_2Z_2[u, v]$-additive codes. Cryptogr. Commun. **12**(3), 443–454 (2020).
16. Shi M.J., Xu L.L., Yang G.: A note on one weight and two weight projective $\mathbb{Z}_4$-codes. IEEE Trans. Inf. Theory **63**(1), 177–182 (2017).

Revised Proof

Journal: 10623 Article No.: 0796 TYPESET DISK LE CP Disp.:2020/9/3 Pages: 13 Layout: Small

17. Wan Z.X.: Quaternary Codes. World Scientific, Singapore (1997).

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Revised Proof

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