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# The Flow of Power-Law Fluids in Concentric Annuli: a full analytical approximate solution

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## Abstract

10 This work deals with the zero shear rate (maximum velocity) position parameter  $\lambda$  of a steady laminar axial annular flow of power-law fluids especially polymeric ones.  $\lambda$  is involved in the shear rate, velocity profile and the flow rate calculations, which are essential for studies such as viscous dissipation, convective heat transfer and pressure drop prediction in annuli. However, the analytical explicit expression of  $\lambda$  remains unsolved despite  
15  $\lambda$  so that it can be used for parametric investigations, identifications, sensitivity analysis etc. in terms of industrial or laboratory applications.

*Keywords: pseudo-plastic fluid; analytical expression; explicit expression; annular flow; power law*

20

## 1 Introduction

The concentric or eccentric annular geometry with moving or static walls is used in numerous industrial fields such as drilling of oil wells, circulating muds [1], food processing [2], plastics processing [3] and heat transfer equipment [4]. The channel can be helical or curved in some  
25 applications for example bio-fluid mechanics or chemical reactors. Combined with different flow behaviors such as Bingham, power-law or viscoelastic models, the problem is sometimes difficult to solve analytically. This paper deals with the use of pseudo-plastic flow behavior in a concentric annular geometry. The mastery of the annulus flow in the industry and laboratories requires the simplest expression of the flow equation in order to help designing  
30 geometries and allow parametric investigations between independent parameters [5].

The most simple and representative behavior law used in polymer processing models is the so-called power law [6–8], which has the advantage to require only 2 parameters as shown in the equation (1),

$$\eta = K\dot{\gamma}^{n-1} \quad (1)$$

with  $\eta$  being the viscosity,  $K$  the consistency coefficient,  $\dot{\gamma}$  the absolute value of shear rate and  $n$  the power-law index ( $0 < n < 1$  for pseudo-plastic materials such as polymer [9], blood[10] and whipped cream[11]).

The shortcoming of the power-law model to describe the viscosity at the zero shear rate was pointed out by Frederickson [12]. Nevertheless Bird [13] and McEachern [14] showed that this model describes well the rheological behavior of a laminar axial flow in annuli by comparing it to some experimental data, especially in the shear rate ranges used in polymer processing. Escudier & al. [1,15] further confirmed the use of the power-law model with some experimental data and sensitivity analysis.

The first analysis of flow through annulus was done by Volarovich & Gutkin [16], Laird [17] and in the case of a power-law fluid by Frederickson & Bird [18]. The first analytical relation between the flow rate and the pressure drop was achieved by Frederickson & Bird [18] in an integral form. The results of Frederickson & Bird were substantiated by the measurement of Tiu & Bhattacharyya [19]. However, Frederickson & Bird's solution gave an analytical explicit expression either for integer values of  $1/n$  ratio with a cumbersome power series or for a thin annular slit case. The expression of the thin annular slit case has proved to be surprisingly accurate [20–22] for a large range of  $n$  and  $\sigma$ : the ratio between the inner and the outer radius  $R$  of an annular flow (figure 1). Later, Bird & al. [9] improved this expression by refining the approximation.

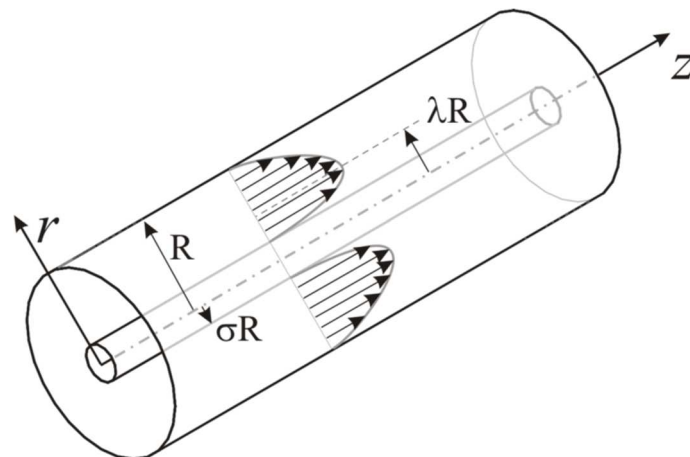


Figure 1 : Description of variables and parameters of the annuli flow

- 55 In 1979, Hanks & Larsen [23] solved analytically Frederickson & Bird's integral. They obtained a general analytical explicit flow rate expression depending on the lambda value: the ratio between the zero shear rate (maximum velocity) position  $\lambda R$  and the outer radius  $R$  (figure 1). Nevertheless, Hanks & Larsen's solution requires the use of a numerical procedure in order to calculate the value of lambda.
- 60 Several analytical expressions (2), (3) and (4) of lambda exist for respectively  $n = 1$ ,  $n = 0$  and  $n$  tending to infinity [9,18].

$$\lambda(n = 1, \sigma) = \sqrt{\frac{1}{2} \frac{\sigma^2 - 1}{\ln(\sigma)}} \quad (2)$$

$$\lambda(n = 0, \sigma) = \sqrt{\sigma} \quad (3)$$

$$\lim_{n \rightarrow +\infty} \lambda(n, \sigma) = \frac{1}{2}(1 + \sigma) \quad (4)$$

For a more general solution, Frederickson & Bird [18] proposed tabulated values of lambda which were obtained from the numerical integration and from the interpolation between the non-integer values of the  $1/n$  ratio. Many authors like McEachern, Pinho and Daprà [14,24,25] used numerical calculations to determine lambda. Hanks & Larsen [23] also used a table of computed values for lambda depending on the  $n$  and  $\sigma$  parameters. Wein & al. [26] found an analytic differential equation for lambda, but the result is a recursive formula based on approximate values determined by the tangent method. Ilicali & Engez [2] proposed to use the Newtonian case ( $n = 1$ ) lambda value in the Hanks & Larsen's flow rate expression, when the radius ratio  $\sigma$  is greater than 0.3 (figure 1). However, their experiments were performed on materials with a power-law index  $n$  ranging from 0.62 to 0.97 only. Based on the equations (2) and (3), David & al. [27] found a pseudo-plastic lambda's approximate expression (5) as one of the most recent and accurate lambda models.

$$\lambda_{David}(n, \sigma) = \sqrt{\sigma} + n^{0.37+0.28(1-n)^2+\frac{1}{100\sigma}} \left( \sqrt{\frac{1}{2} \frac{\sigma^2 - 1}{\ln(\sigma)}} - \sqrt{\sigma} \right) \quad (5)$$

75 With the same numerical approach, David & Filip [27–29] also obtained other approximate expressions for a dilatant fluid's lambda and for the flow rate. All expressions of lambda are either limited to a small range of  $(n, \sigma)$  values or cannot be fully demonstrated with analytical methods. Besides, semi-analytical solutions constructed with numerical values are cumbersome for analytical use.

In this paper we present a mathematical procedure to obtain a new analytical lambda for pseudo-plastic fluids. The proposed expression of lambda is simple enough in order to allow the parametric analysis and the identification process. A comparison is carried out between the new lambda's expression and numerical values. The precision of the flow rate calculated with our lambda expression is also evaluated against some other solutions for the validation of our model in a large range of power-law index  $n$  and radius ratio  $\sigma$ .

## 2 Pseudo-plastic Fluid Flow in an annuli Channel

The case studied is an incompressible steady laminar axial viscous flow in an annular duct without taking inertia terms into account. The momentum conservation equation in the flow can be written as equation (6),

$$\frac{dp}{dz} = \frac{\tau}{r} + \frac{d\tau}{dr} \quad (6)$$

where  $p$  is the pressure and  $\tau$  is the shear stress. At the zero shear rate (maximum velocity) position  $r = \lambda R$ , the shear stress equals to zero (figure 1). With this boundary condition, equation (6) can be integrated and becomes equation (7).

$$\tau = -\frac{1}{2} \frac{dp}{dz} \left( \frac{\lambda^2 R^2}{r} - r \right) \quad (7)$$

Since we assume that the flow follows the pseudo-plastic power law, the shear stress is defined by (8),

$$\tau = \eta \frac{dw}{dr} = K \left| \frac{dw}{dr} \right|^{n-1} \frac{dw}{dr} \quad (8)$$

where  $w$  is the axial velocity with  $\dot{\gamma} = |dw/dr|$ .

### 2.1 Kinetic equations of the annular flow

By combining equations (7) and (8), we obtain the shear rate profile as equations (9) and (10) which should be written into 2 parts to avoid negative sign problems with  $0 < n \leq 1$ .

$$\frac{dw}{dr} = \left[ -\frac{1}{2K} \frac{dp}{dz} \left( \frac{\lambda^2 R^2}{r} - r \right) \right]^{\frac{1}{n}} ; \sigma R \leq r \leq \lambda R \quad (9)$$

$$\frac{dw}{dr} = - \left[ -\frac{1}{2K} \frac{dp}{dz} \left( r - \frac{\lambda^2 R^2}{r} \right) \right]^{\frac{1}{n}} ; \lambda R \leq r \leq R \quad (10)$$

In order to obtain the velocity profile as equations (11) and (12), we integrate the shear rate equations (9) and (10).

$$w_i(r) = -\frac{dp}{dz} \left| \frac{1}{2K} \right|^{\frac{1}{n}} \left| \frac{dp}{dz} \right|^{\frac{1}{n}-1} \int_{\sigma R}^r \left( \frac{\lambda^2 R^2}{\zeta} - \zeta \right)^{\frac{1}{n}} d\zeta + w_i(r = \sigma R); \sigma R \leq r \leq \lambda R \quad (11)$$

$$w_o(r) = -\frac{dp}{dz} \left| \frac{1}{2K} \right|^{\frac{1}{n}} \left| \frac{dp}{dz} \right|^{\frac{1}{n}-1} \int_r^R \left( \zeta - \frac{\lambda^2 R^2}{\zeta} \right)^{\frac{1}{n}} d\zeta + w_o(r = R); \lambda R \leq r \leq R \quad (12)$$

100 Since no-slip wall conditions are considered at the positions  $r = R$  and  $r = \sigma R$  (figure 1), we have  $w_i(r = \sigma R) = w_o(r = R) = 0$ . Thanks to the continuity of the velocity profile, the condition  $w_i(r = \lambda R) = w_o(r = \lambda R)$  i.e. relation (13) should be self-satisfied.

$$\int_{\sigma R}^{r=\lambda R} \left( \frac{\lambda^2 R^2}{\zeta} - \zeta \right)^{\frac{1}{n}} d\zeta = \int_{r=\lambda R}^R \left( \zeta - \frac{\lambda^2 R^2}{\zeta} \right)^{\frac{1}{n}} d\zeta \quad (13)$$

According to the scientific literature, relation (13) is by far the simplest expression to compute lambda numerically.

105 By integrating the velocity profile, we obtained the exact analytical expression (14) of the flow rate as described by Hanks & Larsen [23] in 1979.

$$Q_{annular} = \frac{\pi n}{3n+1} \left( -\frac{1}{2K} \frac{dp}{dz} \right)^{\frac{1}{n}} R^{\frac{1+3n}{n}} \left[ (1 - \lambda^2)^{\frac{1}{n}+1} - \sigma^{1-\frac{1}{n}} (\lambda^2 - \sigma^2)^{\frac{1}{n}+1} \right] \quad (14)$$

We recall that the exact solution of the flow rate for a Newtonian fluid is written as (15).

$$Q_{annular,Newt} = \frac{\pi}{8\eta} \left( -\frac{dp}{dz} \right) R^4 \left[ \left( 1 - \sigma^4 + \frac{(\sigma^2 - 1)^2}{\ln(\sigma)} \right) \right] \quad (15)$$

It should be noted that when  $\sigma$  and  $\lambda$  tend to 0, the expression (14) becomes the flow rate expression (16) for a circular tube.

$$Q_{cylinder} = \frac{\pi n}{3n+1} \left( -\frac{1}{2K} \frac{dp}{dz} \right)^{\frac{1}{n}} R^{\frac{1+3n}{n}} \quad (16)$$

110 With all these equations from shear rate to flow rate, we can see that  $\lambda(n, \sigma)$  is involved in every analytical calculation of an annular flow.

### 3 Analytical approximate solution for the lambda expression

The mathematical procedure presented in this section consists in obtaining a new analytical explicit lambda expression. We start with equation (13) as we change the variable  $s = r/\lambda R$ ,

115 we obtain the expression (17).

$$\int_{\frac{\sigma}{\lambda}}^1 \left(\frac{1}{s} - s\right)^{\frac{1}{n}} ds = \int_1^{\frac{1}{\lambda}} \left(s - \frac{1}{s}\right)^{\frac{1}{n}} ds \quad (17)$$

Considering that  $U = U(n, \sigma) = 1/\lambda$ , we obtain equation (18).

$$\int_{\sigma U}^1 \left(\frac{1}{s} - s\right)^{\frac{1}{n}} ds = \int_1^U \left(s - \frac{1}{s}\right)^{\frac{1}{n}} ds \quad (18)$$

Differentiating (18) with respect to  $\sigma$ , we see that  $U$  satisfies the ordinary differential equation (19), with the initial data  $\lim_{\sigma \rightarrow 1} U(n, \sigma) = 1$ .

$$-\frac{\partial(\sigma U)}{\partial \sigma} (1 - \sigma^2 U^2)^{\frac{1}{n}} \sigma^{-\frac{1}{n}} = \frac{\partial U}{\partial \sigma} (U^2 - 1)^{\frac{1}{n}} \quad (19)$$

Multiplying (19) by  $2U$  and setting  $V = U^2 = 1/\lambda^2$ , the calculation leads to the expression (20),

$$\frac{\partial V}{\partial \sigma} = \frac{-2V}{\sigma^{\frac{1}{n}} \left(\frac{V-1}{1-\sigma^2 V}\right)^{\frac{1}{n}} + \sigma} \quad (20)$$

with the initial data  $\lim_{\sigma \rightarrow 1} V(n, \sigma) = 1$  and therefore  $\lim_{\sigma \rightarrow 1} \frac{\partial V}{\partial \sigma}(n, \sigma) = -1$  according to L'Hospital's rule [30] as shown in equation (21).

$$\lim_{\sigma \rightarrow 1} \frac{V-1}{1-\sigma^2 V} = -\frac{\lim_{\sigma \rightarrow 1} \frac{\partial V}{\partial \sigma}(n, \sigma)}{2 + \lim_{\sigma \rightarrow 1} \frac{\partial V}{\partial \sigma}(n, \sigma)} \quad (21)$$

For the sake of simplicity, we set  $W = \frac{V-1}{1-\sigma^2 V}$  (or equivalently  $V = \frac{1+W}{1+\sigma^2 W}$ ). Since we know that  $V = 1/\lambda^2$  and  $W = \frac{1-\lambda^2}{\lambda^2 - \sigma^2} = \frac{-\tau(r=R)}{\sigma\tau(r=\sigma R)}$ , the latter can also be considered as a parameter proportional to the ratio between the outer and the inner walls' shear stress. As a consequence of introducing  $W$ , our explicit expression (22) takes a form which is as simple as the Newtonian lambda's equation (2) without any approximation.

$$\lambda(n, \sigma) = \sqrt{\frac{1 + \sigma^2 W(n, \sigma)}{1 + W(n, \sigma)}} \quad (22)$$

In order to look into the limiting values of  $W = \frac{V-1}{1-\sigma^2 V}$ , we use the relation (21) and obtain  $\lim_{\sigma \rightarrow 1} W(n, \sigma) = 1$ . By injecting the relation  $W = \frac{V-1}{1-\sigma^2 V}$  into (20), we see that the function  $W$  satisfies the differential equation (23)

$$\frac{\partial W}{\partial \sigma} = \frac{2}{1-\sigma^2} \frac{(W+1)[(\sigma W)^{1+\frac{1}{n}} - 1]}{(\sigma W)^{\frac{1}{n}} + \sigma} \quad (23)$$

and thus  $\lim_{\sigma \rightarrow 1} \frac{\partial W}{\partial \sigma}(n, \sigma) = -\frac{1+n}{1+2n}$  thanks to L'Hospital's rule again. In addition, we should have  $W(n=0, \sigma) = \sigma^{-1}$  according to the equation (3), knowing that the expression (2) of the Newtonian lambda is a special case where the differential equation of lambda gives a natural logarithmic function because of  $n = 1$ .

135 Then, we can calculate the first terms of the Taylor's expansion of  $W(n, \sigma)$  with  $\sigma$  in a neighborhood of 1, by approximating  $W$  with  $W_0$  following the relation (24),

$$W_0 = \sigma^{-b_1} = 1 + b_1(1 - \sigma) + O(1 - \sigma)^2 \quad (24)$$

where  $b_1 = -\lim_{\sigma \rightarrow 1} \frac{\partial W}{\partial \sigma}(n, \sigma) = \frac{1+n}{1+2n}$ . A new analytical explicit expression (25) of  $\lambda$  is achieved.

$$\lambda_{D\&N} = \sqrt{\frac{1 + \sigma^2 \sigma^{-\frac{1+n}{1+2n}}}{1 + \sigma^{-\frac{1+n}{1+2n}}}} \quad (25)$$

140 In terms of errors, we obtain the following relation (26) for  $\sigma$  close to 1 as a by-product of relation (24).

$$|\lambda - \lambda_{D\&N}| \leq C(1 - \sigma)^2 \quad (26)$$

Mathematically, we can expect our Deterre & Nicoleau's model  $\lambda_{D\&N}$  to be precise for  $\sigma$  close to 1 or even for a larger range of  $\sigma$  like the Bird & al.'s pressure drop-flow rate expression [9,20–22] and for materials with a small power-law index  $n$  within the whole range of  $\sigma$ . Some comparisons will be carried out in the next section to verify its precision. It should also be noted that  $W_0(n, \sigma)$  can take forms with more terms other than  $\sigma^{f(n)}$  to improve the accuracy in a neighborhood of any values of  $\sigma$ .

145

#### 4 Validation procedure for the analytical lambda expression

For the validation of our model, we are going to compare it with numerical lambda values in a large range of  $n$  and  $\sigma$ . Another criterion is the precision of the flow rate calculation with our lambda compared to other solutions.

150

##### 4.1 Comparison criteria with the numerical lambda values

The numerical lambda  $[\lambda_{num}]$  values are computed from the equation (13) by minimizing  $|w_i(\lambda) - w_o(\lambda)|$  with the trust region reflective algorithm in Matlab software. For comparing the David & al.'s lambda values  $[\lambda_{David}]$  and ours  $[\lambda_{D\&N}]$  to the numerical ones, the error  $\varepsilon_\lambda$  is calculated according the following definition (27),

155



$$\varepsilon_\lambda = \frac{\lambda_i - \lambda_{num}}{1 - \sigma} \quad (27)$$

which represents the zero shear rate position's deviation  $\Delta\lambda R$  over the width of the slit  $(1 - \sigma)R$  while  $\lambda_i = \lambda_{D\&N}$  for our case and  $\lambda_i = \lambda_{David}$  for the David & al.'s case.

#### 4.2 Comparison criteria with analytical or semi-analytical historical flow rate models

The exact flow rate expression has been shown in equation (14). Ilicali & Engez [2] proposed to use it with the Newtonian lambda's expression (2) even for calculations of pseudo-plastic materials. Bird & al. [9] obtained the expression (28), with an approximation for  $\sigma$  in a neighborhood of 1.

$$Q_{Bird} = \pi \left( \frac{n}{2n+1} \right) \left( -\frac{1}{2K} \frac{dp}{dz} \right)^{\frac{1}{n}} R^{\frac{1+3n}{n}} (1-\sigma)^{2+\frac{1}{n}} \frac{(1+\sigma)}{2} \quad (28)$$

Here for the comparison, we take the exact expression (14) with the numerical lambda  $[\lambda_{num}]$ , the Newtonian one  $[\lambda_{Newt}]$ , our expression  $[\lambda_{D\&N}]$  and the David & al.'s one  $[\lambda_{David}]$ . The approximate expression (28), which has the same approach (for  $\sigma$  close to 1) as our expression of lambda, is also taken into account for comparison. Since the annular flow rate expression (14) and the circular tube's flow rate expression (16) share the same multiplicative factor, we introduce a dimensionless flow rate  $\tilde{Q}$  as described in the relation (29) to avoid using arbitrary geometry and pressure drop values in the discussion of results.

$$\tilde{Q} = \frac{Q_{annular}}{Q_{cylinder}} \quad (29)$$

According to its definition,  $\tilde{Q}$  varies from 1 to 0 when the inner radius  $\sigma R$  changes. By calculating  $\frac{Q_{annular}}{Q_{cylinder}}$  and  $\frac{Q_{Bird}}{Q_{cylinder}}$  we obtain equations (30) and (31) which we will compare in the next section,

$$\tilde{Q} = (1 - \lambda^2)^{\frac{1}{n}+1} - \sigma^{1-\frac{1}{n}} (\lambda^2 - \sigma^2)^{\frac{1}{n}+1} \quad (30)$$

$$\tilde{Q}_{Bird} = \left( \frac{3n+1}{2n+1} \right) (1-\sigma)^{2+\frac{1}{n}} \frac{(1+\sigma)}{2} \quad (31)$$

where we will use  $\tilde{Q}$  with numerical lambda values as the reference flow rate. And the flow rate errors are calculated as described in equations (32) and (33).

$$\varepsilon_{Q,j} = \frac{\tilde{Q}(\lambda_i, n, -\sigma) - \tilde{Q}(\lambda_{num}, n, -\sigma)}{\tilde{Q}(\lambda_{num}, n, -\sigma)} \quad (32)$$

$$\varepsilon_{Q,Bird} = \frac{\tilde{Q}_{Bird}(n, -\sigma) - \tilde{Q}(\lambda_{num}, n, -\sigma)}{\tilde{Q}(\lambda_{num}, n, -\sigma)} \quad (33)$$

175  $[\lambda_i = \lambda_{Newt}; \varepsilon_{Q,j} = \varepsilon_{Q,I\&E}]$ ,  $[\lambda_i = \lambda_{D\&N}; \varepsilon_{Q,j} = \varepsilon_{Q,D\&N}]$ ,  $[\lambda_i = \lambda_{David}; \varepsilon_{Q,j} = \varepsilon_{Q,David}]$  and  $\varepsilon_{Q,Bird}$  stand for the flow rate errors calculated with Ilicali & Engez's flow rate approach [2], with our Deterre & Nicoleau's lambda model, with David & al.'s lambda model [27] and with Bird & al.'s flow rate model [9]. We mention that the exact solution for the Newtonian case  $\tilde{Q}(\lambda_{Newt}, n = 1, \sigma)$  is included in the Ilicali & Engez's flow rate approach.

### 180 4.3 Validation of the Deterre & Nicoleau's analytical approximate model

This section compares our lambda values to the numerical ones. Also, comparisons are carried out between the flow rate calculated with our lambda values and some other solutions. The results are presented for a large set of  $\sigma$  values ranging from 0.05 to 0.95 with a step of 0.05 and with an additional  $\sigma = 0.01$  and 0.99 for extremities.  $n = 0.1, 0.3, 0.5$  and 1 in order  
185 to cover the full range of rheological behaviors of polymer melts (0.3 to 1 for most thermoplastics and more rarely with  $n$  close to 0.1 for rubber compounds)[6].

#### 4.3.1 Comparison of the zero shear rate position

The comparison between the lambda values is presented in figure 2. The analytical Newtonian lambda  $[\lambda_{Newt}]$  calculated with equation (2) is also plotted and it superimposes with good  
190 accuracy on the numerical Newtonian lambda  $[\lambda_{num}$  with  $n=1$ ]. Furthermore, the numerical lambda values  $[\lambda_{num}]$  have been verified via a comparison with the Hanks & Larsen's table [23].

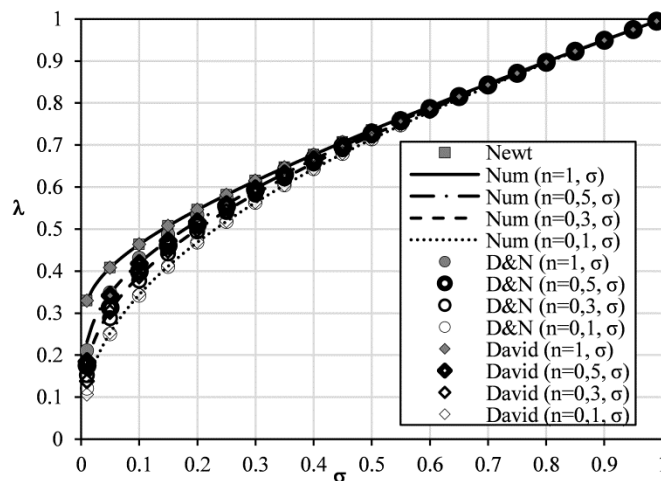


Figure 2 : Precision and validity range for the zero shear rate position  $\lambda_{D\&N}$  and  $\lambda_{David}$

The advantage of using our model or the David & al.'s one has already been shown when it is  
195 compared to solutions using the Newtonian lambda for all values of  $n$ . To quantify the

deviation, errors between our model and the numerical lambda values are calculated according to the equation (27) and presented in figure 3.

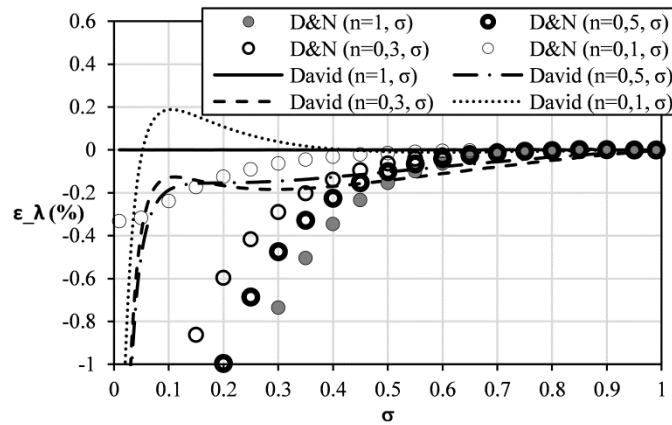


Figure 3 : Variation of errors  $\varepsilon_\lambda$  for our model and the David & al.'s model in the range of  $-1\% < \varepsilon_\lambda < 0.6\%$

200 We can see in figure 3 that our model has better or at least the same degree of accuracy as the David & al.'s model when  $\sigma$  is bigger than 0.5. For  $\sigma$  varying from 0.3 to 0.99, the precision of the David & al.'s model is better (especially for  $n = 1$ , where it turns into the exact Newtonian lambda expression), but the absolute values of our lambda's errors remain less than 0.8%. It should also be noted that the smaller  $n$  is, the smaller our model's  $|\varepsilon_\lambda|$  are. For  
 205 small values of  $\sigma$  (less than 0.3), our model's  $|\varepsilon_\lambda|$  begins to increase rapidly whereas it is not the case for the David & al.'s model until  $\sigma < 0.1$ .

The zero shear rate position parameter  $\lambda$ , is an internal parameter for the calculation of the flow's physical quantities such as shear rate, velocity and flow rate. In the following section, we will quantify the improvement in the precision of the flow rate thanks to a more precise  
 210 description of lambda via a comparison of solutions found in literature.

#### 4.3.2 Comparison of the prediction ability of the flow rate

To verify the use of the Newtonian lambda proposed by Ilicali & Engez [2] in the exact flow rate expression, the result of  $\tilde{Q}(\lambda_{Newt}, n, \sigma)$  has been plotted in figure 4 and compared to the reference data  $\tilde{Q}(\lambda_{Num}, n, \sigma)$ . The comparisons of  $\tilde{Q}(\lambda_{D\&N}, n, \sigma)$ ,  $\tilde{Q}(\lambda_{David}, n, \sigma)$  and  
 215  $\tilde{Q}_{Bird}(n, \sigma)$  to  $\tilde{Q}(\lambda_{Num}, n, \sigma)$  are also plotted in figure 4.

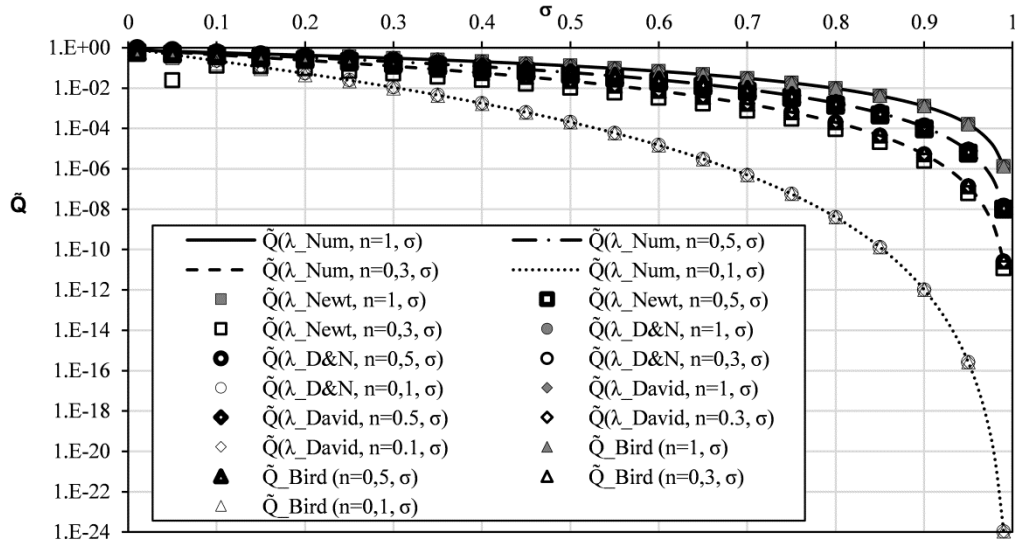


Figure 4 : Comparison of the dimensionless flow rate to the reference  $\tilde{Q}(\lambda_{Num}, n, \sigma)$  calculated with the numerical lambda values

Once again, the curves have a good agreement for  $n = 1$  thanks to the precision of the numerical lambda values. However, Ilicali & Engez's flow rate values  $\tilde{Q}(\lambda_{Newt}, n, \sigma)$  for  $n =$  220  $0.1$  are all negative in our range of investigation and thus not shown in figure 4. The value of  $\tilde{Q}(\lambda_{Newt}, n = 0.3, \sigma = 0.1)$  is also negative and doesn't make sense as a flow rate value in such a configuration. We mention that when  $\sigma$  tends to 0,  $\tilde{Q}$  tends to 1 (cylinder flow) according its definition. These results show that the use of the Newtonian lambda in the exact flow rate expression are safely precise for materials with  $n$  close to 1, but it can also be less 225 satisfactory for those with small values of  $n$ .

The overall precisions of the other models are greatly improved compared to  $\tilde{Q}(\lambda_{Newt}, n, \sigma)$  in figure 4. Both  $\tilde{Q}(\lambda_{D\&N}, n, \sigma)$  and  $\tilde{Q}_{Bird}(n, \sigma)$  have a good ability to describe the flow rate in a large range of  $n$  and  $\sigma$ , and not only when  $\sigma$  is close to 1.

$\varepsilon_{Q,I\&E}$ ,  $\varepsilon_{Q,D\&N}$ ,  $\varepsilon_{Q,David}$  and  $\varepsilon_{Q,Bird}$  are plotted in figure 5 for further investigations.  $\varepsilon_{Q,I\&E}$  for 230  $n \leq 0.3$  are under -50% and not shown in figure 5.

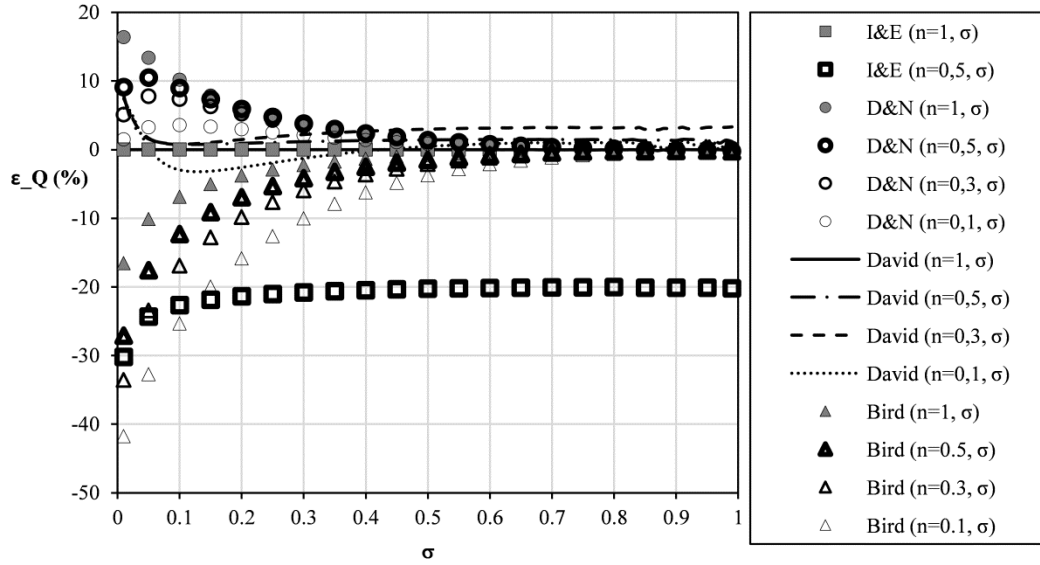


Figure 5 : Variation of errors  $\varepsilon_Q$  calculated with the Ilicali & Engez's approach, our lambda model, the David & al.'s lambda model and the Bird & al.'s flow rate model

We can see in figure 5 that when  $n$  decreases, the absolute values of errors  $\varepsilon_{Q,I\&E}$  Ilicali & Engez's flow rate approach [2] works only for materials with big power-law index values.

235 David & al.'s model gives the best overall precision for the whole range of our investigation, then comes our model in which the least precise case of  $|\varepsilon_{Q,D\&N}|$  with  $n = 1$  has the same degree of accuracy as the most precise case of  $|\varepsilon_{Q,Bird}|$  with  $n = 1$ . As in the section 4.3.1, here we can draw the same conclusion that for  $\sigma > 0.5$ , our model can provide the best overall precision compared to the other solutions.

## 240 5 Conclusion

We have presented in this paper an analytical explicit expression of lambda with a mathematically argued procedure. The accuracy of the values and the derivatives from  $\sigma$  are also analytically ensured for small  $n$  values or for  $\sigma$  close to 1. Through the comparison of some numerical values of lambda, we proved that our expression can perform a good  
 245 precision in a large range of  $\sigma$  for all pseudo-plastic materials. The errors are less than 0.8% for  $0.3 \leq \sigma < 1$ . And we have confirmed that the smaller  $n$  is, the smaller the errors are for all  $\sigma$  values. Compared to the David & al.'s lambda model which still has the best overall precision so far among those expressions of lambda, our model has at least the same degree of accuracy or a better one for  $\sigma > 0.5$ .

250 By comparing the flow rate calculation with our lambda model to some other solutions, several advantages of using our model have also been observed. We can calculate the flow

rate of all pseudo-plastic materials with our lambda model, and that is not the case of the Ilicali & Engez's approach [2]. Compared to the Bird & al.'s flow rate calculation, the use of our lambda model in the flow rate expression has a better precision in a larger range of  $\sigma$ . The calculation with the David & al.'s lambda model provides the best overall precision, but the solution obtained by using our model has better accuracy for bigger  $\sigma$  values.

In addition to the flow rate, having an expression of lambda allows us to calculate the shear rate and the velocity profile more easily for viscous dissipation and heat convection studies. It should be noted that the error from the lambda estimation could be amplified by other calculations due to their sensitivity to the lambda value; such is the case of the flow rate especially for small  $n$  values close to 0 or for  $\sigma$  values close to 1 (figure 3 compared to figure 5). Not only the error of our expression is mathematically predictable, but also its form is simple enough for analytical use such as parametric investigations, sensitivity analysis or development of a reduced order model.

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