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Exact detectability and exact observability of discrete-time linear stochastic systems with periodic coefficients

Vasile Dragan^{a,b} Samir Aberkane^{c,d}

^a*Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Research Unit 2, POBox 1-764, RO-014700, Bucharest, Romania*

^b*Academy of the Romanian Scientists*

^c*Université de Lorraine, CRAN, UMR 7039, Campus Sciences, BP 70239 Vandoeuvre-les-Nancy Cedex, 54506, France*

^d*CNRS, CRAN, UMR 7039, France*

Abstract

This paper is devoted to the problem of detectability and observability of a class of time-varying stochastic systems. The main contribution of this note is to propose a PBH-type test which is equivalent to the detectability of the considered stochastic systems in the sense that all unstable modes produce some non-zero output. Then, we apply the obtained results in order to derive a criterion of exponential stability in mean square sense as well as existence conditions for the stabilizing solution of a class of discrete-time time-varying Riccati equations.

Key words: Discrete-time stochastic systems; periodic coefficients; exact detectability; exact observability; PBH type criterion; characteristic multipliers..

1 Introduction

It is a well established fact that the detectability and the observability notions play a key role in several domains from modern control theory both in the deterministic and stochastic cases. It is also well known that the characterization of the detectability and the observability in the stochastic framework offers some very specific challenges when compared to their deterministic case [5,7,9,10,14]. For example, in the deterministic case, one defines a system to be detectable if all its unstable modes produce a non-zero output and shows that the system detectability is equivalent to the dual system being stabilizable which is also equivalent to the existence of an asymptotically stable linear dynamic state observer. Moreover, detectability could be equivalently characterized via an algebraic criterion: the so called Popov-Belevich-Hautus test (PBH-test). Such equivalent

characterizations don't hold in the stochastic framework. More specifically, we know that (see [5]):

- in the stochastic case the property that all unstable modes produce some non-zero output is only a necessary but not sufficient condition for stabilizability of the dual system,
- the stabilizability of the dual system does not give rise to a realizable observer equation,
- a PBH-type test for stochastic systems is only a necessary condition for stabilizability of the dual system.

Several authors devoted their efforts to address some of the above raised points. We cite for example here [5,9,10,14] where the authors succeeded to show the equivalence between the stochastic version of the PBH-test and the *natural* concept of detectability of the system in the sense that all unstable modes produce some non-zero outputs. Exact observability together with the corresponding PBH criterion was first introduced in [13], and then developed in [5] and [14] for exact detectability of linear time-invariant Itô systems. Unlike the aforementioned papers, where only the time invariant case was considered, recently, in [15] a class of discrete-time time-varying stochastic linear systems

* This paper was not presented at any IFAC meeting. Corresponding author S. Aberkane. Tel. +33 (0)3 72 74 53 46. Fax +XXXIX-VI-mmmxxv.

Email addresses: Vasile.Dragan@imar.ro (Vasile Dragan), samir.aberkane@univ-lorraine.fr (Samir Aberkane).

is studied. Two kinds of detectability for this class of stochastic linear systems are defined and characterized, namely uniform detectability and exact detectability. In the present paper, we consider also the case of linear discrete-time time-varying stochastic systems. The time-varying nature of the system coefficients is assumed to be periodic. Different from [15] where the coefficients matrices of the output are deterministic, in the present paper, the coefficients matrices of the output are allowed to be affected by perturbations modeled by sequences of independent random variables. We first propose a definition for detectability of this class of systems, namely *exact detectability*. The concept introduced here is in adequacy with its deterministic counterpart. The central contribution of this note is then to propose a PBH-type test for this class of time-varying systems which is equivalent to the exact detectability of the considered stochastic systems. The proof of such result relies on the use of spectral theory of positive operators. We then apply the obtained results in order to obtain a criterion of exponential stability in mean square sense as well as existence conditions for the stabilizing solution of a class of discrete-time Riccati equations with periodic coefficients. It is worth pointing out that the PBH criterion for exact detectability allows us to prove the criterion for exponential stability in mean square without any additional assumptions as in [15]. The results reported in this paper can be viewed as a discrete-time time-varying counterpart of [5]. Even though we follow a similar philosophy as in [5], it is important to mention here that the time-varying nature of the considered problem imposes some hard specific challenges. Obtaining a PBH-type test for detectability in this context is far from being trivial.

The rest of the paper is organized as follows: In Section 2 we give the problem setting. In Section 3, we introduce some preliminary results. In Section 4 and Section 5 we give the main results of the paper. Section 6 is devoted to show several consequences and applications of the proposed main results.

2 Problem Description

Consider the discrete-time linear stochastic system:

$$\begin{cases} x(t+1) = (A_0(t) + \sum_{k=1}^r A_k(t)w_k(t))x(t) \\ y(t) = (C_0(t) + \sum_{k=1}^r C_k(t)w_k(t))x(t) \end{cases} \quad (1)$$

where $A_k(t+\theta) = A_k(t)$, $C_k(t+\theta) = C_k(t)$, for all $0 \leq k \leq r$, $t \in \mathbb{Z}_+ = \{0, 1, \dots\}$ where $\theta \geq 1$ is a fixed integer. In (1) $w(t) = \left(w_1(t) \dots w_r(t) \right)^T$ are independent random vectors with zero mean and $\mathbb{E} [w(t)w^T(t)] = I_r$. In this paper, \mathbb{E} stands for the mathematical expectation.

If $(t_0, x_0) \in \mathbb{Z}_+ \times \mathbb{R}^n$ we denote $x(t; t_0, x_0)$ the solution of (1) with the initial condition $x(t_0; t_0, x_0) = x_0$. The

corresponding output is

$$y(t; t_0, x_0) = (C_0(t) + \sum_{k=1}^r C_k(t)w_k(t))x(t; t_0, x_0), t \geq t_0.$$

Definition 2.1 a) We say that the system (1) is exact detectable at time $t_0 \in \mathbb{Z}_+$ if for any initial state $x_0 \in \mathbb{R}^n$ for which $y(t; t_0, x_0) = 0$, a.s. for any $t \geq t_0$, we have $\lim_{t \rightarrow \infty} \mathbb{E} [|x(t; t_0, x_0)|^2] = 0$.

b) We say that the system (1) is exact detectable if it is exact detectable at any time $t_0 \in \mathbb{Z}_+$.

Remark 1 In the stochastic framework there are many ways to define the concept of stabilizability and detectability (see for example [2,3,7,9,10,12] to cite only a few). The most known concept of detectability is the so called stochastic detectability. In this case, the system (1) is named stochastic detectable if there exist matrices $H(t) \in \mathbb{R}^{n \times n}$ such that the closed-loop systems:

$$x(t+1) = \left[A_0(t) + H(t)C_0(t) + \sum_{k=1}^r w_k(t)(A_k(t) + H(t)C_k(t)) \right] x(t) \quad (2)$$

is exponentially stable in mean square sense.

Noting that the closed-loop system (2) may be rewritten in the form:

$$x(t+1) = \left(A_0(t) + \sum_{k=1}^r w_k(t)A_k(t) \right) x(t) + H(t)y(t)$$

we may deduce easily that if system (1) is stochastic detectable then it is exact detectable. We will show in Section 5 (via an example) that the converse implication is not always true.

Definition 2.2 a) We say that the system (1) is exact observable at time $t_0 \in \mathbb{Z}_+$ if $y(t; t_0, x_0) = 0$, a.s. for all $t \geq t_0$ if and only if $x_0 = 0$.

b) The system (1) is exact observable if it is exact observable at any time $t_0 \in \mathbb{Z}_+$.

Remark 2 From Definition 2.1 and Definition 2.2 it follows that if the system (1) is exact observable then it is exact detectable.

Our goal is to derive a PBH-type criterion to test if the system (1) is or not exact detectable and is or not exact observable, respectively. In the developments from this note, we adapt to the discrete-time periodic case the approach from [5] for the continuous time-invariant case.

3 Few preliminary issues

Let $\mathcal{S}_n \subset \mathbb{R}^{n \times n}$ be the linear subspace of real symmetric matrices. Equipped with the inner product:

$$\langle X, Y \rangle = \text{Tr}[XY] \quad (3)$$

\mathcal{S}_n becomes a finite dimensional Hilbert space. Additionally, \mathcal{S}_n is an ordered linear space, ordered by the ordering relation induced by the convex cone \mathcal{S}_n^+ of positive semidefinite matrices.

If $x(t)$ is a solution of the systems (1), then we set $X(t) \triangleq \mathbb{E}[x(t)x^T(t)]$. By direct calculation one obtains that $X(\cdot)$ is a solution of the following initial value problem (IVP):

$$\begin{cases} X(t+1) = \mathcal{L}(t)[X(t)] \\ X(t_0) = x_0 x_0^T \end{cases} \quad (4)$$

where we denoted

$$\mathcal{L}(t)[X] = \sum_{k=0}^r A_k(t) X A_k^T(t) \quad (5)$$

for all $X \in \mathcal{S}_n$. The operator $\mathcal{L}(t)$ is the extended Lyapunov-type operator (or the extended Stein operator) associated to the discrete-time linear stochastic system (1). One can see from (5) that $\mathcal{L}(t)$ is a positive operator *i.e.* $\mathcal{L}(t)\mathcal{S}_n^+ \subset \mathcal{S}_n^+$. The periodicity property of the coefficients $A_k(\cdot)$, $0 \leq k \leq r$, leads to the periodicity property of the sequence of operators $\{\mathcal{L}(t)\}_{t \geq 0}$ *i.e.* $\mathcal{L}(t+\theta) = \mathcal{L}(t)$, for all $t \in \mathbb{Z}_+$. Even if the system (1) works only for $t \in \mathbb{Z}_+$, the sequence of operators $\{\mathcal{L}(t)\}_{t \geq 0}$ can be extended by periodicity to the whole set \mathbb{Z} .

Let $T(t, s) \triangleq \mathcal{L}(t-1)\mathcal{L}(t-2)\cdots\mathcal{L}(s)$ if $t > s$ and $T(t, s) = \mathcal{I}_{\mathcal{S}_n}$ if $t = s$. $T(t, s)$ is a linear evolution operator on the space \mathcal{S}_n defined by the discrete-time linear equation (4). The solutions of the equation (4) satisfy $X(t) = T(t, s)[X(s)]$, $\forall t \geq s$. We set:

$$\mathbb{T}_\theta(t) \triangleq T(t+\theta, t) \quad (6)$$

for all $t \in \mathbb{Z}$. $\mathbb{T}_\theta(t)$ will be named the *monodromy operator* associated to the linear equation (4), or equivalently to $\mathcal{L}(\cdot)$.

According with the terminology used in periodic linear systems, the eigenvalues of the monodromy operator $\mathbb{T}_\theta(t)$ will be named *characteristic multipliers*. One may show that the spectrum $\sigma(\mathbb{T}_\theta(t))$ of the monodromy operator does not depend upon t . Since $\mathbb{T}_\theta(t)$ is a positive operator, then based on [8] one may deduce that the spectral radius ρ_θ of the monodromy operator is an eigenvalue of $\mathbb{T}_\theta(t)$. Moreover, there exists an eigenvector $Y \in \mathcal{S}_n^+$ corresponding to this eigenvalue. In the developments from this paper, an important role will be played by the eigenvalues of the monodromy operator which have corresponding eigenvectors in \mathcal{S}_n^+ . We denote $\sigma^+(\mathbb{T}_\theta(t)) = \left\{ \lambda \in \sigma(\mathbb{T}_\theta(t)) \mid \exists Y \in \mathcal{S}_n^+ : \mathbb{T}_\theta(t)[Y] = \lambda Y, Y \neq 0 \right\}$. Hence, $\rho_\theta \in \sigma^+(\mathbb{T}_\theta(t))$ and $\sigma^+(\mathbb{T}_\theta(t)) \subset \mathbb{R}$, but, in general, it does not contain all real eigenvalues of the monodromy operator.

Now we consider the deterministic counterpart of system (1) given by:

$$\begin{cases} x(t+1) = A_0(t)x(t) \\ y(t) = C_0(t)x(t). \end{cases} \quad (7)$$

In this case, the linear operator (5) becomes:

$$\mathcal{L}_0(t)[X] = A_0(t)X A_0^T(t) \quad (8)$$

and the discrete-time linear equation (4) takes the form:

$$X(t+1) = A_0(t)X(t)A_0^T(t). \quad (9)$$

The linear evolution operator $T_0(t, s)$ defined by the linear equation (9) is given by:

$$T_0(t, s)[X] = \Phi_{A_0}(t, s)X\Phi_{A_0}^T(t, s) \quad (10)$$

where $\Phi_{A_0}(t, s) = A_0(t-1)A_0(t-2)\cdots A_0(s)$ if $t > s$ and $\Phi_{A_0}(t, s) = I$ if $t = s$.

The monodromy operator associated to the linear equation (9) is given by:

$$\mathbb{T}_{\theta 0}(t)[Y] = \Psi_{A_0}(t)Y\Psi_{A_0}^T(t) \quad (11)$$

for all $Y \in \mathcal{S}_n$, where $\Psi_{A_0}(t) \triangleq \Phi_{A_0}(t+\theta, t)$ is the monodromy matrix associated to the linear equation $x(t+1) = A_0(t)x(t)$. The reason behind considering the above deterministic counterpart will be clarified in the next section.

4 A PBH-type criterion for exact detectability

Before addressing the problem of exact detectability/observability in the stochastic framework and formulating our main results, we would like first to introduce a result related to the deterministic framework. Such a result gives a positive operator characterization of exact detectability of deterministic θ -periodic systems. We believe that such result has not been reported yet in the literature.

4.1 The deterministic case

In the case of system (7), Definition 2.1 becomes:

Definition 4.1 *a) We say that the system (7) is exact detectable if for some $(t_0, x_0) \in \mathbb{Z}_+ \times \mathbb{R}^n$ we have $y(t; t_0, x_0) = C_0(t)\Phi_{A_0}(t, t_0)x_0 = 0$ for all $t \geq t_0$, then $\lim_{t \rightarrow \infty} \Phi_{A_0}(t, t_0)x_0 = 0$.*

b) We say that the system (7) is detectable if there exists a periodic sequence of period θ , $H(\cdot)$ such that the system:

$$x(t+1) = (A_0(t) + H(t)C_0(t))x(t)$$

is exponentially stable.

Remark 3 Specializing Remark 1 to the deterministic case one obtains that if the system (7) is detectable in the sense of Definition 4.1 b than it is exact detectable (in the sense of definition 4.1 a).

The next result is an extended version of Proposition 4.13 from [1].

Proposition 4 The following are equivalent:

- i) system (7) is detectable;
- ii) for any characteristic multiplier λ of $A_0(\cdot)$ with $|\lambda| \geq 1$ the equalities:

$$\Psi_{A_0}(t)\eta = \lambda\eta \quad (12a)$$

$$C_0(j)\Phi_{A_0}(j, t)\eta = 0 \quad (12b)$$

for all $t \leq j \leq t + \theta - 1$ implies that $\eta = 0$;

- iii) for any characteristic multiplier $\mu \in \sigma^+(\mathbb{T}_{\theta_0}(t))$ with $\mu \geq 1$, there does not exist $Y \in \mathcal{S}_n^+$, with $Y \neq 0$, which satisfies:

$$\mathbb{T}_{\theta_0}(t)[Y] = \mu Y \quad (13a)$$

$$C_0(j)\Phi_{A_0}(j, t)Y\Phi_{A_0}^T(j, t) = 0, \quad t \leq j \leq t + \theta - 1 \quad (13b)$$

Proof 4.1 The equivalence $i) \Leftrightarrow ii)$ is proved in Proposition 4.13 from [1]. Du to page limitation, we will not develop the proof of the equivalence $ii) \Leftrightarrow iii)$. This proof can be done adapting to the discrete-time periodic case the proof of Lemma 4 from [5].

4.2 The stochastic case

The main result of this subsection is given by the following Theorem.

Theorem 5 Under the considered assumptions the following are equivalent:

- i) the system (1) is exact detectable at time t_0 ;
- ii) there does not exist characteristic multipliers $\mu \geq 1$ associated to the discrete-time linear equation (4) and non zero symmetric matrices $Y \in \mathcal{S}_n^+$ which satisfy the equations:

$$\mathbb{T}_{\theta}(t_0)[Y] = \mu Y \quad (14a)$$

$$C_k(j)T(j, t_0)[Y] = 0 \quad (14b)$$

where $t_0 \leq j \leq t_0 + \theta - 1, 0 \leq k \leq r$.

Proof 4.2 See Appendix A.

Combining the results proved in Proposition 4 and Theorem 5 we obtain:

Corollary 6 In the case of deterministic discrete-time linear systems (7) with periodic coefficients, the following are equivalent:

- (i) The system (7) is exact detectable.
- (ii) For any $t_0 \in \mathbb{Z}_+$ and any characteristic multiplier $1 \leq \mu \in \sigma^+(\mathbb{T}_{\theta_0}(t_0))$, there does not exist $Y \in \mathcal{S}_n^+$, with $Y \neq 0$, which satisfies:

$$\mathbb{T}_{\theta_0}(t_0)[Y] = \mu Y$$

$$C_0(j)\Phi_{A_0}(j, t_0)Y\Phi_{A_0}^T(j, t_0) = 0, \quad t_0 \leq j \leq t_0 + \theta - 1.$$

- (iii) The system (7) is detectable.

Proof 4.3 $(i) \rightarrow (ii)$ follows from the implication $(i) \rightarrow (ii)$ from Theorem 5 specialized to the case of system (7). The implication $(ii) \rightarrow (iii)$ follows from $(iii) \rightarrow (i)$ of Proposition 4. The implication $(iii) \rightarrow (i)$ follows from Remark 3.

Remark 7 According to the result proved in Theorem 5, in order to conclude if the system (1) is or not exact detectable at time t_0 , we need to check for each characteristic multiplier $\mu \geq 1$ if the equations (14) have or not nonzero solutions $Y \in \mathcal{S}_n^+$. It follows from the periodicity property of the coefficients of the system (1) that equations (14) need to be solved only for $t_0 \in \{\hat{t}, \hat{t} + 1, \dots, \hat{t} + \theta - 1\}$ for some $\hat{t} \in \mathbb{Z}_+$.

We describe in what follows a numerical procedure that allows us to compute the characteristic multipliers of the monodromy operator $\mathbb{T}_{\theta}(t_0)$ and the corresponding eigenvectors. Using similar arguments as in [13] and by generalizing them to our setting, let us define $\mathcal{M}_{\mathbb{T}_{\theta}(t_0)}$ as the matrix induced by $\mathbb{T}_{\theta}(t_0)$. Let $Y = (y_{ij})_{n \times n} \in \mathcal{S}_n$. We denote by:

$$\tilde{Y} \triangleq \begin{pmatrix} y_{11} & \cdots & y_{1n} & y_{22} & \cdots & y_{2n} & \cdots & y_{n-1, n-1} & \cdots & y_{n-1, n} & y_{n, n} \end{pmatrix}^T$$

the column vector formed by different elements of Y . Hence one can show that (14a) is equivalent to the following conventional matrix characteristic equation:

$$\mathcal{M}_{\mathbb{T}_{\theta}(t_0)}\tilde{Y} = \mu\tilde{Y}$$

For example, let us consider the following randomly generated 2-periodic matrices:

$$A_0(0) = \begin{pmatrix} -0.6482 & 1.0123 \\ 0.4999 & 0.4174 \end{pmatrix}, \quad A_0(1) = \begin{pmatrix} -0.0657 & 1.0103 \\ 0.6707 & 0.2737 \end{pmatrix}$$

$$A_1(0) = \begin{pmatrix} -1.1283 & 0.2255 \\ 0.3748 & -0.2133 \end{pmatrix}, \quad A_1(1) = \begin{pmatrix} -0.1186 & 0.7718 \\ -1.1584 & -1.0241 \end{pmatrix}$$

Hence by using the above described procedure one obtains: $\sigma(\mathbb{T}_{\theta}(t_0)) = \{3.3486 \ 0.4180 \ 0.6842\}$. The corresponding eigenvectors are given by:

$$Y_1 = \begin{pmatrix} 0.5617 & 0.3962 \\ 0.3962 & 0.7263 \end{pmatrix}, Y_2 = \begin{pmatrix} 0.2146 & -0.0410 \\ -0.0410 & -0.9758 \end{pmatrix}$$

$$Y_3 = \begin{pmatrix} -0.1040 & -0.2083 \\ -0.2083 & 0.9725 \end{pmatrix}$$

5 A PBH-type criterion for exact observability

In the following theorem, we give a PBH-type criterion that allows us to characterize the exact observability at time instant t_0 of system (1).

Theorem 8 *Under the considered assumptions the following are equivalent:*

- i) the system (1) is exact observable at time t_0 ;
- ii) there does not exist characteristic multipliers $\mu \geq 0$ associated to the discrete-time linear equation (4) and non zero symmetric matrices $Y \in \mathcal{S}_n^+$ which satisfy the equations:

$$\mathbb{T}_\theta(t_0)[Y] = \mu Y, \quad (15a)$$

$$C_k(t_0)T(t, t_0)[Y] = 0 \quad (15b)$$

for all $t_0 \leq t \leq t_0 + \theta - 1$, $0 \leq k \leq r$.

Proof 5.1 *i) \Rightarrow ii). Let us assume by contrary that there exist a characteristic multiplier $\mu \geq 0$ associated to the discrete-time linear equation (4) and a nonzero symmetric matrix $Y \in \mathcal{S}_n^+$ that satisfy (15). Let $X(t; t_0, Y)$ be the solution of the IVP:*

$$\begin{cases} X(t+1) = \mathcal{L}(t)[X(t)] \\ X(t_0) = Y \end{cases} \quad t \geq t_0$$

$\mathcal{L}(\cdot)$ being the linear operator (5) associated to the discrete-time linear system (1). As in the proof of the implication i) \Rightarrow ii) from Theorem 5 one shows that:

$$X(t; t_0, Y) = \mu^{\lfloor \frac{t-t_0}{\theta} \rfloor} T\left(t; t_0 + \left[\frac{t-t_0}{\theta}\right]\theta\right)[Y] \quad (16)$$

for all $t \geq t_0$. Particularly, for $t = t_0 + k\theta$, we have: $X(t_0 + k\theta; t_0, Y) = \mu^k Y$, $\forall k = 0, 1, \dots$. Also, the analogous of (51) is:

$$C_k(t)X(t; t_0, Y)C_k^T(t) = 0 \quad (17)$$

$\forall t \geq t_0$, $0 \leq k \leq r$. Let:

$$Y = \sum_{l=1}^{\nu} x_l x_l^T \quad (18)$$

be the decomposition of Y where $x_l \in \mathbb{R}^n$ and $\nu = \text{rank}(Y)$. We have $X(t; t_0, Y) = \sum_{l=1}^{\nu} X(t; t_0, x_l x_l^T)$. As in the proof of Theorem 5 one shows that:

$$0 \leq X(t; t_0, x_l x_l^T) \leq X(t; t_0, Y), \quad \forall 1 \leq l \leq \nu, t \geq t_0.$$

This allows us to deduce via (17) that:

$$C_k(t)X(t; t_0, x_l x_l^T)C_k^T(t) = 0, \quad 0 \leq k \leq r, 1 \leq l \leq \nu, t \geq t_0.$$

This is equivalent to: $\mathbb{E}[y(t; t_0, x_l)y^T(t; t_0, x_l)] = 0$, $1 \leq l \leq \nu$, $t \geq t_0$, which leads to $\mathbb{E}[|y(t; t_0, x_l)|^2] = 0$, or equivalently:

$$y(t; t_0, x_l) = 0 \text{ a.s. } \forall t \geq t_0, 1 \leq l \leq \nu. \quad (19)$$

If i) is true, then (19) yields $x_l = 0$, $1 \leq l \leq \nu$. In this case (18) becomes $Y = 0$, which contradicts our supposition that $Y \neq 0$. Thus we have shown that the implication i) \Rightarrow ii) holds.

To prove the implication ii) \Rightarrow i), let us assume that for some $t_0 \in \mathbb{Z}_+$ we have:

$$y(t; t_0, x_0) = 0 \text{ a.s. } \forall t \geq t_0. \quad (20)$$

We have to show that $x_0 = 0$. Let $X(t; t_0, x_0 x_0^T) := \mathbb{E}[x(t; t_0, x_0)x^T(t; t_0, x_0)]$. One sees that $t \rightarrow X(t; t_0, x_0 x_0^T)$ is the solution of the IVP (4). Hence it has the representation:

$$X(t; t_0, x_0 x_0^T) = T(t, t_0)[x_0 x_0^T], \quad \forall t \geq t_0 \geq 0.$$

From (20) together with $X(t; t_0, x_0 x_0^T) \geq 0$, $t \geq t_0$, we deduce that $C_k(t)X(t; t_0, x_0 x_0^T) = 0$ or equivalently:

$$C_k(t)T(t, t_0)[x_0 x_0^T] = 0, \quad 0 \leq k \leq r, t \geq t_0. \quad (21)$$

As in the proof of Theorem 5 we consider the closed linear subspace $\mathcal{H} \subset \mathcal{S}_n$ defined by $\mathcal{H} = \mathcal{H}_+ - \mathcal{H}_+$ where \mathcal{H}_+ is the closed, solid, convex cone containing the sequence $\{X(t_0 + l\theta; t_0, x_0 x_0^T)\}_{l \geq 0}$. Moreover both \mathcal{H} and \mathcal{H}_+ are invariant with respect to the monodromy operator $\mathbb{T}_\theta(t_0)$. Following similar lines as in the proof of the implication ii) \Rightarrow i) from Theorem 5, one shows that the spectral radius $\tilde{\rho}_\theta$ of the restriction $\tilde{\mathbb{T}}_\theta(t_0) \triangleq \mathbb{T}_\theta(t_0)|_{\mathcal{H}}$ is $\tilde{\rho}_\theta = 0$ if ii) from the statement is true. Hence, $\tilde{\mathbb{T}}_\theta(t_0)$ is the null operator. This allows us to deduce that:

$$\begin{aligned} \mathbb{T}_\theta(t_0)[X(t_0 + l\theta; t_0, x_0 x_0^T)] &= \tilde{\mathbb{T}}_\theta(t_0)[X(t_0 + l\theta; t_0, x_0 x_0^T)] \\ &= 0 \quad \forall l \in \mathbb{Z}_+. \end{aligned}$$

Particularly, for $l = 0$, we have:

$$\mathbb{T}_\theta(t_0)[x_0 x_0^T] = 0. \quad (22)$$

From (22) and (21) written for $t_0 \leq t \leq t_0 + \theta - 1$ we deduce that $Y = x_0 x_0^T$ and $\mu = 0$ satisfy (15).

If ii) is true, we conclude that $x_0 x_0^T$ must be zero which is equivalent to $x_0 = 0$. Thus the proof is complete

Another concept of observability for discrete-time stochastic systems of type (1) is the so called *uniform stochastic observability*. For rigorous definition and characterization of uniform stochastic observability we refer for example to Chapter 4 from [7]. According to the Definition 4.4 and assertion iv) from Theorem 4.5 in [7], the system (1) is uniform stochastic observable if there exist $\delta > 0$ and $\tau_0 \in \mathbb{Z}_+$, $\tau_0 \geq 1$, such that:

$$\sum_{t=t_0}^{t_0+\tau_0} \mathbb{E}[\Phi^T(t, t_0) \tilde{C}(t) \Phi(t, t_0)] \geq \delta I_n \quad (23)$$

for all $t_0 \in \mathbb{Z}_+$, where $\Phi(t, t_0)$ is the fundamental matrix solution of (1), i.e., $\Phi(t, t_0) = \mathcal{A}(t-1)\mathcal{A}(t-2)\cdots\mathcal{A}(t_0)$ if $t > t_0$ and $\Phi(t, t_0) = I_n$ if $t = t_0$ with $\mathcal{A}(t) \triangleq A_0(t) + \sum_{k=1}^r A_k(t)w_k(t)$, $t \in \mathbb{Z}_+$, $\tilde{C}(t) = \sum_{k=0}^r C_k^T(t)C_k(t)$.

The next result gives the relation between the uniform stochastic observability and exact observability notions. It shows that the concept of exact observability is wider than the concept of uniform stochastic observability.

Corollary 9 *If the system (1) is uniform stochastic observable then it is exact observable*

Proof 5.2 *If for some $t_0 \geq 0$ we have $y(t; t_0, x_0) = 0$, a.s., $t \geq t_0$ then (23) yields $0 = \sum_{t=t_0}^{t_0+\tau_0} \mathbb{E}[|y(t; t_0, x_0)|^2] \geq \delta|x_0|^2$ which leads to $x_0 = 0$. Hence the proof is complete.*

By using the above result, let us show via a numerical example that exact detectability is wider than the concept of stochastic detectability.

Example 5.1 *Consider the following particular case of system (1) with $n = 1$, $r = 1$:*

$$\begin{cases} x(t+1) = (a_0 + a_1 w_1(t))x(t) \\ y(t) = cx(t) \end{cases} \quad (24)$$

where $a_0, a_1, c \in \mathbb{R}$ with $c \neq 0$ and $|a_1| \geq 1$. The system (24) is uniform stochastic observable. It follows from Corollary 9 that it is exact observable. One can also show that system (24) is not stochastic detectable (see Example 4.5 from [7]). According with Remark 2, system (24) is exact detectable.

6 Several consequences of exact detectability

6.1 A criterion for exponential stability in mean square

We introduce the following discrete-time linear backward equation:

$$Z(t) = \mathcal{L}^*(t)[Z(t+1)] + \sum_{k=0}^r C_k^T(t)C_k(t) \quad (25)$$

where $\mathcal{L}^*(t)$ is the adjoint of the operator $\mathcal{L}(t)$ defined in (5) with respect to the inner product (3).

Proposition 10 *Assume that the system (1) is exact detectable. Then the following are equivalent:*

- i) *the discrete-time backward linear equation (25) has a solution $Z(\cdot)$ which is θ -periodic and positive semidefinite at any $t \in \mathbb{Z}_+$;*
- ii) *there exist $\beta \geq 1$, $\delta \in (0, 1)$ with the property that the solutions of the system (1) satisfy:*

$$\mathbb{E}[|x(t; t_0, x_0)|^2] \leq \beta \delta^{t-t_0} |x_0|^2, \quad \forall t \geq t_0 \geq 0, \quad x_0 \in \mathbb{R}^n.$$

Proof 6.1 *To prove the implication i) \Rightarrow ii) it is sufficient to show that the spectral radius ρ_θ of the monodromy operator $\mathbb{T}_\theta(t)$ satisfies: $\rho_\theta < 1$.*

Since $\rho_\theta \in \sigma^+(\mathbb{T}_\theta(t))$ it follows that there exists $0 \neq Y \in \mathcal{S}_n^+$ such that:

$$\mathbb{T}_\theta(t)[Y] = \rho_\theta Y. \quad (26)$$

On the other hand, if i) holds, then (25) has a solution $Z(\cdot)$ which is θ -periodic and $Z(t) \geq 0$, $\forall t \in \mathbb{Z}_+$. This solution has the representation:

$$\begin{aligned} Z(t) &= \mathbb{T}_\theta^*(t)[Z(t+\theta)] + \\ &+ \sum_{j=t}^{t+\theta-1} \sum_{k=0}^r T^*(j, t)[C_k^T(j)C_k(j)], \quad \forall t \in \mathbb{Z}_+ \end{aligned} \quad (27)$$

According to (3) and (26)-(27), we have:

$$\begin{aligned} 0 &\leq \text{Tr} \left[\sum_{j=t}^{t+\theta-1} \sum_{k=0}^r T^*(j, t)[C_k^T(j)C_k(j)]Y \right] = \\ &= \left\langle \sum_{j=t}^{t+\theta-1} \sum_{k=0}^r T^*(j, t)[C_k^T(j)C_k(j)], Y \right\rangle = \\ &= \langle Z(t), Y \rangle - \langle \mathbb{T}_\theta^*(t)[Z(t+\theta)], Y \rangle = \\ &= \langle Z(t), Y \rangle - \langle Z(t+\theta), \mathbb{T}_\theta(t)[Y] \rangle = \\ &= \langle Z(t), Y \rangle - \rho_\theta \langle Z(t+\theta), Y \rangle \end{aligned} \quad (28)$$

If we assume by contrary that $\rho_\theta \geq 1$, then (28) together with the fact that $Z(t) = Z(t+\theta)$, $\forall t \in \mathbb{Z}_+$ yield:

$$\sum_{j=t}^{t+\theta-1} \sum_{k=0}^r \text{Tr}[T^*(j, t)[C_k^T(j)C_k(j)]Y] = 0 \quad (29)$$

Since $T^(j, t)[C_k^T(j)C_k(j)] \geq 0$ and $Y \geq 0$ we obtain that (29) is equivalent to:*

$$\begin{aligned} \text{Tr}[T^*(j, t)[C_k^T(j)C_k(j)]Y] &= \langle T^*(j, t)[C_k^T(j)C_k(j)], Y \rangle \\ &= 0 \end{aligned}$$

$\forall 0 \leq k \leq r, t \leq j \leq t + \theta - 1$. Further we may write:

$$\begin{aligned} 0 &= \left\langle C_k^T(j)C_k(j), T(j, t)[Y] \right\rangle = \\ &= \text{Tr}[C_k^T(j)C_k(j)T(j, t)[Y]] = \\ &= \text{Tr}[C_k(j)T(j, t)[Y]C_k^T(j)] \end{aligned} \quad (30)$$

$\forall 0 \leq k \leq r, t \leq j \leq t + \theta - 1$. Since $T(j, t)[Y] \geq 0$, we deduce from (30) that

$$C_k(j)T(j, t)[Y] = 0, \quad \forall 0 \leq k \leq r, \quad t \leq j \leq t + \theta - 1 \quad (31)$$

Employing Theorem 5 we deduce that under the assumption of exact detectability of the system (1), the equations (26) and (31) cannot be satisfied simultaneously by a matrix $Y \geq 0$ and $Y \neq 0$. It follows that the supposition $\rho_\theta \geq 1$ is false and hence $\rho_\theta < 1$. This shows that *i*) \Rightarrow *ii*) is true.

The implication *ii*) \Rightarrow *i*) follows immediately by applying Theorem 2.5 from [7] in the case of equation (25). Thus the proof is complete.

Remark 11 The PBH criterion for exact detectability allowed us to state and to prove the criterion for exponential stability in mean square from Proposition 10 under less conservative assumptions than those from Theorem 3.2 and Theorem 3.3 from [15].

6.2 The existence of the stabilizing solution of a class of discrete-time Riccati equations with periodic coefficients

Let us consider the backward nonlinear equations:

$$\begin{aligned} X(t) &= \sum_{k=0}^r A_k^T(t)X(t+1)A_k(t) - \\ &- \left(L(t) + \sum_{k=0}^r A_k^T(t)X(t+1)B_k(t) \right) \times \\ &\times \left(R(t) + \sum_{k=0}^r B_k^T(t)X(t+1)B_k(t) \right)^{-1} \times \\ &\times \left(L^T(t) + \sum_{k=0}^r B_k^T(t)X(t+1)A_k(t) \right) + M(t) \end{aligned} \quad (32)$$

where $A_k(\cdot), B_k(\cdot), 0 \leq k \leq r, M(\cdot), L(\cdot), R(\cdot)$ are matrix valued periodic sequences of period $\theta \geq 1$. Additionally we assume that:

$$\begin{cases} R(t) = R^T(t) > 0 \\ M(t) = M^T(t) \\ M(t) - L(t)R^{-1}(t)L^T(t) \geq 0 \end{cases} \quad (33)$$

$\forall t \in \mathbb{Z}_+$. A solution $\tilde{X}(\cdot)$ of (32) is named stabilizing solution if the closed-loop system:

$$\begin{aligned} x(t+1) &= \left(A_0(t) + B_0(t)\tilde{F}(t) + \right. \\ &\left. + \sum_{k=1}^r w_k(t) \left(A_k(t) + B_k(t)\tilde{F}(t) \right) \right) x(t) \end{aligned} \quad (34)$$

is ESMS, where:

$$\begin{aligned} \tilde{F}(t) &= - \left(R(t) + \sum_{k=0}^r B_k^T(t)X(t+1)B_k(t) \right)^{-1} \times \\ &\times \left(L^T(t) + \sum_{k=0}^r B_k^T(t)X(t+1)A_k(t) \right), \quad t \in \mathbb{Z}_+ \end{aligned} \quad (35)$$

Let us consider the auxiliary system:

$$\begin{cases} x(t+1) = \left[A_0(t) - B_0(t)R^{-1}(t)L^T(t) + \right. \\ \left. + \sum_{k=1}^r w_k(t) \left(A_k(t) - B_k(t)R^{-1}(t)L^T(t) \right) \right] x(t) \\ y(t) = C(t)x(t) \end{cases} \quad (36)$$

with $C(t) = (M(t) - L(t)R^{-1}(t)L^T(t))^{\frac{1}{2}}$.

Proposition 12 Under the considered assumptions, if the system (36) is exact detectable, then any θ -periodic and positive semi-definite solution of (32) is a stabilizing solution.

Proof 6.2 Let $\tilde{X}(t), t \in \mathbb{Z}_+$ be a θ -periodic solution of (32) and $\tilde{X}(t) \geq 0$. By direct calculation one obtains that (32) satisfied by $\tilde{X}(\cdot)$ may be rewritten as:

$$\begin{aligned} \tilde{X}(t) &= \mathcal{L}_{\tilde{F}}^*(t)[\tilde{X}(t+1)] + M(t) - L(t)R^{-1}(t)L^T(t) + \\ &+ \left(\tilde{F}(t) + R^{-1}(t)L^T(t) \right)^T R(t) \left(\tilde{F}(t) + R^{-1}(t)L^T(t) \right) \end{aligned} \quad (37)$$

where $\mathcal{L}_{\tilde{F}}^*(t)$ is the adjoint operator of the linear operator $\mathcal{L}_{\tilde{F}}(t)$ defined by:

$$\begin{aligned} \mathcal{L}_{\tilde{F}}(t)[Y] &= \sum_{k=0}^r \left[A_k(t) + B_k(t)\tilde{F}(t) \right] Y \times \\ &\times \left[A_k(t) + B_k(t)\tilde{F}(t) \right]^T \end{aligned} \quad (38)$$

$\forall Y \in \mathcal{S}_n$ and $\tilde{F}(t)$ are the gain matrices associated to $\tilde{X}(\cdot)$ by (35). One sees that (37) is the backward linear

equation of type (25) associated to the system:

$$\begin{cases} x(t+1) = \left[A_0(t) + B_0(t)\tilde{F}(t) + \right. \\ \left. + \sum_{k=1}^r w_k(t) \left(A_k(t) + B_k(t)\tilde{F}(t) \right) \right] x(t) \\ y(t) = \left(\begin{array}{c} (M(t) - L(t)R^{-1}(t)L^T(t))^{\frac{1}{2}} \\ R^{\frac{1}{2}}(t) \left(\tilde{F}(t) + R^{-1}(t)L^T(t) \right) \end{array} \right) x(t) \end{cases} \quad (39)$$

The conclusion may be obtained applying Proposition 10 in the case of system (39). To this end we have to show that the system (39) is exact detectable.

Let $x(t; t_0, x_0)$ be a solution of the system (39) with the property that the corresponding output $y(t; t_0, x_0)$ satisfies:

$$y(t; t_0, x_0) = 0 \quad a.s. \quad \forall t \geq t_0 \geq 0 \quad (40)$$

This means that

$$\begin{aligned} C(t)x(t; t_0, x_0) &= (M(t) - L(t)R^{-1}(t)L^T(t))^{\frac{1}{2}} \times \\ &\times x(t; t_0, x_0) = 0 \end{aligned} \quad (41)$$

and

$$R^{\frac{1}{2}}(t) \left(\tilde{F}(t) + R^{-1}(t)L^T(t) \right) x(t; t_0, x_0) = 0, \quad a.s. \quad \forall t \geq t_0 \quad (42)$$

Since $R(t) > 0$, (42) yields:

$$\tilde{F}(t)x(t; t_0, x_0) = -R^{-1}(t)L^T(t)x(t; t_0, x_0) \quad a.s. \quad \forall t \geq t_0 \quad (43)$$

Substituting (43) in the first equation from (39) written for $x(t)$ replaced by $x(t; t_0, x_0)$ we obtain that $x(\cdot; t_0, x_0)$ is a solution of (36). From (41) together with exact detectability of the system (36) we deduce that:

$$\lim_{t \rightarrow \infty} \mathbb{E} [|x(t; t_0, x_0)|^2] = 0. \quad (44)$$

Finally, (40) and (44) allow us to conclude that (39) is exact detectable. Thus the proof is complete.

It is well known that an equation of type (32) is closely related with a linear quadratic optimal control problem described by the controlled system:

$$\begin{cases} x(t+1) = A_0(t)x(t) + B_0(t)u(t) + \\ + \sum_{k=1}^r w_k(t) (A_k(t)x(t) + B_k(t)u(t)) \\ x(t_0) = x_0 \end{cases} \quad (45)$$

and the performance criterion:

$$\begin{aligned} \mathcal{J}(t_0, x_0; u) &= \mathbb{E} \left[\sum_{t=t_0}^{\infty} (x^T(t)M(t)x(t) + 2x^T(t)L(t)u(t) \right. \\ &\left. + u^T(t)R(t)u(t)) \right]. \end{aligned} \quad (46)$$

Now we are in position to prove:

Theorem 13 Assume:

- a) the system (45) is stochastically stabilizable;
- b) the weights matrices in (46) satisfy the conditions (33);
- c) the system (36) is exact detectable.

Under these conditions the backward nonlinear equation (32) has a unique stabilizing solution $\tilde{X}(\cdot)$ which is periodic of period θ and positive semi-definite. Additionally, $\mathcal{J}(t_0, x_0; u) \geq \mathcal{J}(t_0, x_0; \tilde{u}) = x_0^T \tilde{X}(t_0)x_0$, for all admissible control $u(\cdot)$, where:

$$\tilde{u}(t) = \tilde{F}(t)\tilde{x}(t) \quad (47)$$

$\tilde{F}(t)$ being computed as in (35) and $\tilde{x}(t)$ is the solution of the closed-loop system (34).

Proof 6.3 Applying Theorem 5.9 from [7] we deduce that under the assumptions a) and b) the equation (32) has at least two periodic solutions of period θ . Further, from Proposition 12 from above, we deduce that if the assumption c) is fulfilled, then the periodic solution is a stabilizing solution. From Corollary 5.2. from [7] we obtain that the stabilizing solution is unique. The proof of the optimality of the control (47) is standard and is omitted.

7 Conclusion

In this paper, we have proposed a PBH-type conditions which are equivalent to the exact detectability/observability of stochastic systems. We showed also that the derived PBH-test is an appropriate algebraic criterion to deal with generalized Lyapunov and Riccati equations. It remains as a challenge for future research to derive similar algebraic conditions in order to test the exact detectability/observability of stochastic discrete-time time-varying systems with arbitrary time variation (not necessarily periodic).

8 Appendix

A. Proof of the Theorem 5

i)⇒ii). Let us assume by contrary that there exist a characteristic multiplier $\mu \geq 1$ associated to the discrete-time linear equation (4) and a non zero symmetric matrix $Y \in \mathcal{S}_n^+$ satisfying the equations (14). Let $X(s; t_0, Y)$, $s \geq t_0$ be the solution of the IVP:

$$X(s+1) = \mathcal{L}(s)[X(s)], \quad s \geq t_0, \quad X(t_0) = Y \quad (48)$$

$\mathcal{L}(s)$ being the linear operator introduced by (5) written for s in place of t . We have $X(s; t_0, Y) = T(s, t_0)[Y]$, $\forall s \geq t_0$. From the definition of the evolution operator $T(s, t_0)$ one sees immediately that: $T(s, t_0) = T(s, \tau)T(\tau, t_0) \quad \forall s \geq \tau \geq t_0$ and

$T(\tau + j\theta, t_0 + j\theta) = T(\tau, t_0) \quad \forall \tau \geq t_0, j \in \mathbb{Z}$. These allow us to obtain:

$$\begin{aligned} X(s; t_0, Y) &= \left(T \left(s, t_0 + \left[\frac{s-t_0}{\theta} \right] \theta \right) \times \right. \\ &\quad \times T \left(t_0 + \left[\frac{s-t_0}{\theta} \right] \theta, t_0 + \left[\frac{s-t_0}{\theta} \right] \theta - \theta \right) \cdots \\ &\quad \times T(t_0 + \theta, t_0) \left. \right) [Y] \\ &= T \left(s, t_0 + \left[\frac{s-t_0}{\theta} \right] \theta \right) (T_\theta(t_0))^{\left[\frac{s-t_0}{\theta} \right]} [Y] \end{aligned}$$

where $\left[\frac{s-t_0}{\theta} \right]$ is the integer part of $\frac{s-t_0}{\theta}$, i.e. the integer number which satisfies:

$$\left[\frac{s-t_0}{\theta} \right] \leq \frac{s-t_0}{\theta} < \left[\frac{s-t_0}{\theta} \right] + 1 \quad (49)$$

Further (14a) leads to:

$$X(s; t_0, Y) = \mu^{\left[\frac{s-t_0}{\theta} \right]} T \left(s, t_0 + \left[\frac{s-t_0}{\theta} \right] \theta \right) [Y] \quad (50)$$

for all $s \geq t_0$ such that $t_0 + \left[\frac{s-t_0}{\theta} \right] \theta \leq s < \left[\frac{s-t_0}{\theta} \right] \theta + \theta$. We set $j \triangleq s - \left[\frac{s-t_0}{\theta} \right] \theta$. From the previous inequality we deduce that $t_0 \leq j \leq t_0 + \theta - 1$. Employing (50) together with the periodicity of $C_k(\cdot)$ we get:

$$\begin{aligned} C_k(s)X(s; t_0, Y) &= \mu^{\left[\frac{s-t_0}{\theta} \right]} C_k(s) T \left(s, t_0 + \left[\frac{s-t_0}{\theta} \right] \theta \right) [Y] \\ &= \mu^{\left[\frac{s-t_0}{\theta} \right]} C_k \left(s - \left[\frac{s-t_0}{\theta} \right] \theta \right) \times \\ &\quad \times T \left(s - \left[\frac{s-t_0}{\theta} \right] \theta, t_0 \right) [Y] \\ &= \mu^{\left[\frac{s-t_0}{\theta} \right]} C_k(j) T(j, t_0) [Y] \end{aligned}$$

where $j \in \{t_0, t_0 + 1, \dots, t_0 + \theta - 1\}$. From (14b) we may infer that $C_k(s)X(s; t_0, Y) = \mu^{\left[\frac{s-t_0}{\theta} \right]} C_k(j) T(j, t_0) [Y] = 0, \forall s \geq t_0, 0 \leq k \leq r$. Thus we have shown that:

$$C_k(s)X(s; t_0, Y)C_k^T(s) = 0, \quad \forall s \geq t_0, 0 \leq k \leq r. \quad (51)$$

Since $Y \in \mathcal{S}_n^+$ it follows that it can be written as $Y = \sum_{l=1}^\nu x_l x_l^T$ where $x_l \in \mathbb{R}^n$ and $\nu = \text{rank } Y$. Having in mind that (48) is a linear equation, we deduce that:

$$X(s; t_0, Y) = \sum_{l=1}^\nu X(s; t_0, x_l x_l^T) \quad (52)$$

From $x_l x_l^T \leq Y$ we obtain:

$$\begin{aligned} 0 \leq X(s; t_0, x_l x_l^T) &= T(s, t_0) [x_l x_l^T] \leq T(s, t_0) [Y] = \\ &= X(s; t_0, Y) \end{aligned} \quad (53)$$

for all $s \geq t_0$, because $T(s, t_0)$ is a positive operator. Let $x(s; t_0, x_l)$ be the solution of the IVP:

$$\begin{cases} x(s+1) = (A_0(s) + \sum_{k=1}^r w_k(s)A_k(s))x(s) \\ x(t_0) = x_l. \end{cases}$$

By direct calculation one obtains that:

$$\mathbb{E} [x(s; t_0, x_l) x^T(s; t_0, x_l)] = X(s; t_0, x_l x_l^T) \quad (54)$$

$s \geq t_0, 1 \leq l \leq \nu$. If

$y(s; t_0, x_l) \triangleq (C_0(s) + \sum_{k=1}^r w_k(s)C_k(s))x(s; t_0, x_l)$, then from (51)-(54) we deduce that

$$\begin{aligned} \mathbb{E} [y(s; t_0, x_l) y^T(s; t_0, x_l)] &= \\ &= \sum_{k=0}^r C_k(s) X(s; t_0, x_l x_l^T) C_k^T(s) = 0 \end{aligned}$$

This allows us to deduce that:

$$y(s; t_0, x_l) = 0, \text{ a.s. for all } s \geq t_0, 1 \leq l \leq \nu \quad (55)$$

Since in (50) $\mu \geq 1$ we have that at least one of the terms from the right hand side of (52) does not tend to zero when s tends to ∞ . Let $l_0 \in \{1, 2, \dots, \nu\}$ be such that $X(s; t_0, x_{l_0} x_{l_0}^T) \not\rightarrow 0$ when $s \rightarrow \infty$. Thus from (54) and (55) written for $l = l_0$, it follows that $\mathbb{E} [|x(s; t_0, x_{l_0})|^2] \not\rightarrow 0$ when $s \rightarrow \infty$, but $y(s; t_0, x_{l_0}) = 0$ a.s. $s \geq t_0$. This fact is in contradiction with the exact detectability at time t_0 of system (1). Thus the proof of the implication i) \Rightarrow ii) is completed.

Now we prove the implication ii) \Rightarrow i). Let us assume that for a certain initial data $(t_0, x_0) \in \mathbb{Z}_+ \times \mathbb{R}^n$ we have:

$$y(t; t_0, x_0) = 0, \text{ a.s., } \forall t \geq t_0 \quad (56)$$

where $y(t; t_0, x_0) = (C_0(t) + \sum_{k=1}^r w_k(t)C_k(t))x(t; t_0, x_0)$. We have to prove that $\lim_{t \rightarrow \infty} \mathbb{E} [|x(t; t_0, x_0)|^2] = 0$ if ii) from the statement is true. We set:

$$X(t) \triangleq \mathbb{E} [x(t; t_0, x_0) x^T(t; t_0, x_0)] \quad (57)$$

and we remark that $X(\cdot)$ is the solution of the IVP (4). Furthermore (56) yields:

$$C_k(t)X(t) = 0, \quad \forall t \geq t_0 \quad (58)$$

because from (57) we have that $X(t) \geq 0, \forall t \geq t_0$. Let \mathcal{X} be the set of all finite sums of the terms of the sequence $\{X(t_0 + \ell\theta)\}_{\ell \geq 0}$. So, $Y \in \mathcal{X}$ if and only if:

$$Y = \sum_{j=1}^N \alpha_j X(t_0 + \ell_j \theta) \quad (59)$$

where $\alpha_j > 0$ are arbitrary. In (59), N is not prefixed. It depends upon Y . One sees that $\mathcal{X} \subset \mathcal{S}_n^+$ is a pointed convex cone. We show now that:

$$\mathbb{T}_\theta(t_0)\mathcal{X} \subset \mathcal{X}, \quad (60a)$$

$$\mathcal{X} \subset \ker C_k(t_0), \quad 0 \leq k \leq r. \quad (60b)$$

Indeed, if $Y \in \mathcal{X}$ then we obtain via (59) that:

$$\begin{aligned} \mathbb{T}_\theta(t_0)[Y] &= \sum_{j=1}^N \alpha_j \mathbb{T}_\theta(t_0)[X(t_0 + \ell_j \theta)] \\ &= \sum_{j=1}^N \alpha_j T(t_0 + (\ell_j + 1)\theta, t_0 + \ell_j \theta)[X(t_0 + \ell_j \theta)] \\ &= \sum_{j=1}^N \alpha_j X(t_0 + (\ell_j + 1)\theta) \in \mathcal{X}. \end{aligned}$$

Thus we have shown that (60a) is true. Further (58) allows us to abstain:

$$\begin{aligned} C_k(t_0)Y &= \sum_{j=1}^N \alpha_j C_k(t_0)X(t_0 + \ell_j \theta) \\ &= \sum_{j=1}^N \alpha_j C_k(t_0 + \ell_j \theta)X(t_0 + \ell_j \theta) = 0 \end{aligned}$$

which shows that (60b) holds. Let $\mathcal{H}_+ \triangleq \text{Cl}(\mathcal{X})$ be the closer of the set \mathcal{X} with respect to the topology induced on \mathcal{S}_n by the norm defined by the inner product (3). Hence, \mathcal{H}_+ is a closed pointed convex cone and $\mathcal{H}_+ \subset \mathcal{S}_n^+$. Additionally (60a) and (60b) can be extended in a natural way to \mathcal{H}_+ , that is:

$$\mathbb{T}_\theta(t_0)\mathcal{H}_+ \subset \mathcal{H}_+, \quad (61a)$$

$$\mathcal{H}_+ \subset \ker C_k(t_0), \quad 0 \leq k \leq r. \quad (61b)$$

We set:

$$\mathcal{H} = \mathcal{H}_+ - \mathcal{H}_+. \quad (62)$$

This means that \mathcal{H} consists of all matrices of the form $Y = Y_1 - Y_2$ where Y_1, Y_2 are arbitrary in \mathcal{H}_+ . One sees that $\mathcal{H} \subset \mathcal{S}_n$ is a closed linear subspace. Hence, the restriction of the inner product (3) induces a Hilbert space structure on the vector space \mathcal{H} . From (61a) we deduce that $\mathbb{T}_\theta(t_0)\mathcal{H} \subset \mathcal{H}$. Let $\tilde{\mathbb{T}}_\theta(t_0) \triangleq \mathbb{T}_\theta(t_0)|_{\mathcal{H}}$ be the restriction of $\mathbb{T}_\theta(t_0)$ to the subspace \mathcal{H} . Additionally from (61a) we deduce that $\tilde{\mathbb{T}}_\theta(t_0)$ is a positive linear operator on the finite dimensional ordered Hilbert space $(\mathcal{H}, \mathcal{H}_+)$. From (62) it follows that \mathcal{H}_+ is a reproducing convex cone in \mathcal{H} . Employing Lemma 3.1.3 from [4] we may conclude that \mathcal{H}_+ is a closed, solid, convex cone. Let $\tilde{\rho}_\theta$ be the spectral radius of the linear operator $\tilde{\mathbb{T}}_\theta(t_0)$. Applying Theorem 2.6 from [6] to the adjoint operator $\tilde{\mathbb{T}}_\theta^*(t_0)$ defined on the ordered Hilbert space $(\mathcal{H}, \mathcal{H}_+^*)$ we

deduce that there exists $0 \neq \tilde{Y} \in (\mathcal{H}_+^*)^* \subset \mathcal{S}_n^+$ such that

$$\tilde{\mathbb{T}}_\theta(t_0)[\tilde{Y}] = \tilde{\rho}_\theta \tilde{Y}. \quad (63)$$

We show that:

$$C_k(j)T(j, t_0)[\tilde{Y}] = 0, \quad t_0 \leq j \leq t_0 + \theta - 1. \quad (64)$$

For $j = t_0$, (64) is true because in this case it becomes (61b). If $\tilde{Y} \in \mathcal{X}$ then based on (59) we may write:

$$\begin{aligned} C_k(j)T(j, t_0)[\tilde{Y}] &= \sum_{i=1}^N \alpha_i C_k(j)T(j, t_0)[X(t_0 + \ell_i \theta)] \\ &= \sum_{i=1}^N \alpha_i C_k(j + \ell_i \theta) \times \\ &\quad \times T(j + \ell_i \theta, t_0 + \ell_i \theta)[X(t_0 + \ell_i \theta)] \\ &= \sum_{i=1}^N \alpha_i C_k(j + \ell_i \theta)X(j + \ell_i \theta) = 0 \end{aligned}$$

where for the last equality we have used (58). Thus we have shown that (64) is true when $\tilde{Y} \in \mathcal{X}$. Since \mathcal{X} is densely inclosed in \mathcal{H}_+ it follows that (64) remains true in the case when $\tilde{Y} \in \mathcal{H}_+$. Taking into account that $0 \neq \tilde{Y} \in \mathcal{H}_+ \subset \mathcal{S}_n^+$ satisfies (63) and (64) we may conclude via ii) from the statement that $\tilde{\rho}_\theta < 1$. Hence, $\|X(t_0 + k\theta)\| = \|\mathbb{T}_\theta^k(t_0)[X(t_0)]\| \leq \beta \tilde{\rho}_\theta^k \|X(t_0)\|, \forall k \geq 0$. If $t \geq t_0$ we may infer that $\|X(t)\| \leq \tilde{\beta} \tilde{\rho}_\theta^{\lceil \frac{t-t_0}{\theta} \rceil} \|X(t_0)\|$, where $\tilde{\beta} \geq \beta e^{\|\mathcal{L}(\cdot)\|_\infty \theta}$. Thus we have obtained from (57) that $\lim_{t \rightarrow \infty} \mathbb{E}[|X(t; t_0, X_0)|^2] = 0$ because $\tilde{\rho}_\theta < 1$. Thus the proof is complete.

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