Commuting Double Sugeno Integrals

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Double Sugeno Integrals and the Commutation Problem

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1 Introduction and Motivation: collective decision making under possibilistic uncertainty

In various applications where information fusion or multifactorial evaluation is needed, an aggregation process is carried out as a two-stepped procedure whereby several local fusion operations are performed in parallel and then the results are merged into a global result. It may sometimes be natural to demand that the result does not depend on the order with which we perform the aggregation steps because there is no reason to perform either of the steps first.

For instance, in a multi-person multi-aspect decision problem, each alternative is evaluated by a matrix of ratings where the rows represent evaluations by persons and the columns represent evaluations by criteria. One may, for each row, merge the ratings according to each column with some aggregation operation and form as such the global rating of each person, and then merge the persons opinions using another aggregation operation. On the other hand, one may decide first to merge the ratings in each column using the aggregation operation, thus forming the global ratings according to each criterion, and then merge these social evaluations across the criteria with aggregation operation. The same considerations apply when we consider several agents under uncertainty and sharing the same knowledge. Should we average out the uncertainty for each agent prior to merging the personal evaluations (i.e., follows the so-called *ex-ante* approach), or should we average out the common uncertainty after merging the personal evaluations for each possible state of affairs (i.e., adopt an *ex-post* approach)?

Even if we find it natural that the two procedures should deliver the same results in any sensible approach, the problem is that this natural outcome is not mathematically obvious at all. When the two procedures yield the same results, the aggregation operations are said to commute.

In decision under risk for instance, the *ex-ante* and *ex-post* approaches are equivalent (the aggregations commute) iff the preferences are considered with an utilitarian view [?][?]:

the expected utility of the sum is equal to the sum of the sum of the expected utilities. With an egalitarian collective utility function this is no longer the case, which leads to a timing effect: the *ex-ante* approach (minimum of the expected utilities) is not equivalent to the *ex-post* one (the expected utility of the minimum of the utilities). [**?**][**?**] prove representation theorems stating that in a classical decision-theoretic setting commutation occurs if and only if the two aggregation are weighted averages, that is the weighted average of expected utilities is the same as the expected collective utility.

More recently, Ben Amor et al. [**?**] [**?**] have reconsider the problem in the setting of qualitative decision theory under uncertainty. They have proved that commuting alternatives to weighted average operations exist, namely qualitative possibilistic integrals [**?**], that is Sugeno integrals with respect to possibility or necessity measures, respectively corresponding to optimistic and pessimistic possibilistic integrals. Namely pessimistic possibilistic integrals commute, as well as optimistic ones, but a pessimistic possibilistic integral generally does not commute with an optimistic one.

The question considered in this paper is whether there exist other uncertainty measures, in the qualitative setting of Sugeno integrals, for which this commutation result holds, replacing pessimistic and optimistic utility functionals by Sugeno integrals with respect to general capacities. In other terms, we care considering the problem of commutation in *double* Sugeno integrals.

The paper is organized as follows. plan re ecrire, Section **??** provides necessary and sufficient condition for the commutation Sugeno integrals and finally lays bare the form of capacities that ensure commutation.

2 Background

Consider a set $\mathcal{X} = \{x_1, \cdots, x_n\}$ and L a totally ordered scale with top 1, bottom 0, and the order-reversing operation denoted by $1 - (\cdot)$ (it is involutive and such that $1 - 1 = 0$ and $1 - 0 = 1$). A decision $\mathfrak{u}$ to be evaluated is represented by a function $u : \mathcal{X} \to L$ where $u(x_i)$ is, for instance, the degree of utility of the decision in state x_i.

2.1 A refresher on 1D Sugeno integral

In the definition of Sugeno integral the relative likelihood or importance of subsets of states is represented by a capacity (or fuzzy measure), which is a set function $\mu : 2^{\mathcal{X}} \to L$ that satisfies $\mu(\emptyset) = 0$, $\mu(\mathcal{X}) = 1$ and $A \subseteq B$ implies $\mu(A) \leq \mu(B)$. The conjugate capacity of μ is defined by $\mu^c(A) = 1 - \mu(A^c)$ where A^c is the complement of A. The Sugeno integral is originally defined in [**?**, **?**].

Definition 1 *The Sugeno integral (S-integral for short) of function u with respect to a capacity μ is defined by:*

$S_\mu(u) = \max_{\alpha \in L} \min(\alpha, \mu(u \geq \alpha))$, *where* $\mu(u \geq \alpha) = \mu(\{x_i \in \mathcal{X} | u(x_i) \geq \alpha\})$.

For instance, suppose that μ is a necessity measure N, i.e., a capacity such that $N(A \cap B) = \min(N(A), N(B))$. N is entirely defined by a function $\pi : \mathcal{X} \to L$, called the possibility distribution associated to N, namely by: $N(A) = \min_{i \notin A} 1 - \pi_i$. The conjugate of a necessity measure is a possibility measure Π: $\Pi(A) = \max_{i \in A} \pi_i$. We have $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$ and $\Pi(A) = 1 - N(\overline{A})$ where $\overline{A}$ is the complementary of A. We thus get the following special cases of the Sugeno integral:

$$S_\Pi(u) = MAX_\Pi(u) = \max_{x_i \in \mathcal{X}} \min(\pi_i, u(x_i)) \tag{1}$$

$$S_N(u) = MIN_\Pi(u) = \min_{x_i \in \mathcal{X}} \max(1 - \pi_i, u(x_i)). \tag{2}$$

These are the weighted maximum and minimum operations that are used in qualitative decision making under uncertainty (they are called optimistic and pessimistic qualitative utility respectively). In this interpretation, $\pi(x_i)$ measures to what extent x_i is a possible state, $S_N(u)$ (resp. $S_\Pi(u)$) evaluates to what extent it is certain (resp. possible) that u is a good decision.

A Sugeno integral can be equivalently written under various forms [**?**, **?**], especially:

$$S_\mu(u) = \max_{A \subseteq \mathcal{X}} \min(\mu(A), \min_{x_i \in A} u(x_i)) = \min_{A \subseteq \mathcal{X}} \max(\mu(A^c), \max_{x_i \in A} u(x_i)) \tag{3}$$

The S-integral can be expressed in a non-redundant format by means of the qualitative Möbius transform of μ:

$$\mu_\#(T) = \begin{cases} \mu(T) \text{ if } \mu(T) > \max_{x \in T} \ \mu(T \setminus \{x\}) \text{ (focal sets)} \\ 0 \text{ otherwise} \end{cases}$$

as

$$S_\mu(\vec{u}) = \max_{T \subseteq \mathcal{X} : \mu_\#(T) > 0} \min(\mu_\#(T), \min_{(x_i \in T} u(x_i))$$

Indeed, $\mu_\#$ contains the minimal information to reconstruct the capacity μ as $\mu(A) = \max_{T \subseteq A} \mu_\#(T)$. Subsets T of $\mathcal{X}$ for which $\mu_\#(T) > 0$ are called focal sets of μ (the set of focal sets of μ is denoted $\mathcal{F}(\mu)$). As a matter of fact, it is clear that the qualitative Moebius transform of a possibility measure coincides with its possibility distribution: $\Pi_\#(A) = \pi(s)$ if $A = \{s\}$ and 0 otherwise.

Lastly, the S-integral can be expressed with respect by Boolean capacities (i.e., of capacities that take their values in $\{0, 1\}$), namely to the λ-*cut* of μ. Given a capacity μ on $\mathcal{X}$, for all $\lambda > 0, \lambda \in L$, let $\mu_\lambda : 2^{\mathcal{X}} \to \{0, 1\}$ (the λ-cut of μ) be a Boolean capacity defined by

$$\mu_\lambda(A) = \begin{cases} 1 & \text{if } \mu(A) \geq \lambda \\ 0 & \text{otherwise.} \end{cases},$$

for all $A \subseteq \mathcal{X}$. It is clear that the capacity μ can be reconstructed from the μ_λ's as follows:

$$\mu(A) = \max_{\lambda > 0} \min(\lambda, \mu_\lambda(A)).$$

Observe that the focal sets of a Boolean capacity μ_λ form an antichain of subsets (there is no inclusion between them).

We can express S-integrals with respect to μ by means of the cuts of μ:

Proposition 1 $S_\mu(u) = \max_{\lambda>0} \min(\lambda, S_{\mu_\lambda}(u))$

Proof:

$$\begin{aligned}
S_\mu(u) &= \max_{A \subseteq \mathcal{X}} \min(\max_{\lambda>0} \min(\lambda, \mu_\lambda(A)), \min_{i \in A} u(x_i)) \\
&= \max_{A \subseteq \mathcal{X}} \max_{\lambda>0} \min(\lambda, \mu_\lambda(A), \min_{i \in A} u(x_i)) \\
&= \max_{\lambda>0} \max_{A \subseteq \mathcal{X}} \min(\lambda, \mu_\lambda(A), \min_{i \in A} u(x_i)) \\
&= \max_{\lambda>0} \min(\lambda, \max_{A \subseteq \mathcal{X}} \min(\mu_\lambda(A), \min_{i \in A} u(x_i))
\end{aligned}$$

Note that the expression $S_\mu(u) = \max_{\alpha \in L} \min(\alpha, \mu(u \geq \alpha))$ uses cuts of the utility function. It can be combined with Proposition **??** to yield:

$$S_\mu(u) = \max_{\alpha, \lambda \in L} \min(\alpha, \lambda, \mu_\lambda(u \geq \alpha)). \tag{4}$$

This expression can be simplified as follows

Proposition 2 $S_\mu(u) = \max_{\lambda \in L} \min(\lambda, \mu_\lambda(u \geq \lambda))$.

Proof: Note that $\mu_\lambda(u \geq \alpha)$ does not increase with α nor λ. Suppose then that $S_\mu(u) = \min(\alpha^*, \lambda^*, \mu_{\lambda^*}(u \geq \alpha^*))$. If $\mu_{\lambda^*}(u \geq \alpha^*) = 1$, and $\alpha^* > \lambda^*$, then notice that $\mu_{\alpha^*}(u \geq \alpha^*) = 1$ as well. Likewise, if $\alpha^* < \lambda^*, \mu_{\lambda^*}(u \geq \lambda^*)) = 1$. If $\mu_{\lambda^*}(u \geq \alpha^*) = 0$, this is also true for $\mu_\lambda(u \geq \alpha)$ with $\alpha > \alpha^*$ and $\lambda > \lambda^*$. So we can assume $\alpha = \lambda$ in equation (**??**). $\square$

2.2 Sugeno integrals as (lattice) polynomial functions

Recall that a (*lattice*) *polynomial function* on L is any map $p\colon L^n \to L$ which can be obtained as a composition of the lattice operations $\wedge$ and $\vee$, the projections $u(x_i)$ and the constant functions $c \in L$. As observed in [**?**], Sugeno integrals exactly coincide with those polynomial functions $q : L^n \to L$ which are *idempotent* polynomial functions, that is, which satisfy $q(c, \ldots, c) = c$, for every $c \in Y$. In fact, it suffices to verify this identity for $c \in \{0, 1\}$, that is, $q(\mathbf{1}_X) = 1$ and $q(\mathbf{1}_\emptyset) = 0$ where $\mathbf{1}_A$ denotes characteristic functions. As shown by Goodstein [**?**], polynomial functions over bounded distributive lattices (in particular, over bounded chains) have very neat normal form representations, namely disjunctive and conjunctive normal forms given, for idempotent ones, by (**??**).

In particular every idempotent polynomial function $p\colon L^n \to L$ is uniquely determined by its restriction to $\{0,1\}^n$. Also, since every lattice polynomial function is order-preserving, the coefficients in the disjunctive normal form are monotone increasing as well, i.e., $p(\mathbf{1}_I) \leq p(\mathbf{1}_J)$ whenever $I \subseteq J$, i.e., they yield the capacity at work in the S-integral. Moreover, a function $f\colon \{0,1\}^n \to L$ can be extended to a polynomial function over Y if and only if it is order-preserving. These results only point out that

$$S_\mu(u) = \mu(A) \text{ if } u = \mathbf{1}_A.$$

3 2D capacities

A 2D fuzzy measure is simply a fuzzy measure μ on $\mathcal{X} \times \mathcal{Y}$. The definition of a (simple) S integral on such a capacity follows directly

$$S_\mu(u) = \max_{T \subseteq \mathcal{X} \times \mathcal{Y}} \min(\mu(T), \min_{(i,j) \in T} u(x_i, y_j))$$

A Sugeno integral based on a 2D capacity will be called a 2D integral. Before exploring the properties of this type of integral we need to explore 2D capacities, an int particular the conditions under which the 2D-measure can be reconstructed via its projections.

3.1 Projection of a 2D capacities

We can define the projections of a 2D fuzzy measure μ on $\mathcal{Y}$ and on $\mathcal{X}$as follows:

Definition 2 $(\mu)^{\downarrow}_{\mathcal{X}}(A) = \mu(A \times \mathcal{Y})$. $(\mu)^{\downarrow}_{\mathcal{Y}}(B) = \mu(\mathcal{X} \times B)$.

Let $R^{\downarrow}_{\mathcal{Y}} = \{y \in \mathcal{Y}, \exists x \in \mathcal{X},\ \text{s.t.}(x,y) \in R$ be the projection of $R \subseteq \mathcal{X} \times \mathcal{Y}$ on $\mathcal{Y}$.

One may try to compute the focal sets of the projections.

Proposition 3 *The focal sets of $(\mu)^{\downarrow}_{\mathcal{Y}}$ are among the projections on $\mathcal{Y}$ of the focal sets of μ.*

Proof: Consider a set $R \subseteq \mathcal{X} \times \mathcal{Y}$ which is not a focal set of μ. Then there is a focal $T \subset R$ such that $\mu(T) = \mu(R)$, and then $T^{\downarrow}_{\mathcal{Y}} \subseteq R^{\downarrow}_{\mathcal{Y}}$. So $(\mu)^{\downarrow}_{\mathcal{Y}}(T^{\downarrow}_{\mathcal{Y}}) \leq (\mu)^{\downarrow}_{\mathcal{Y}}(R^{\downarrow}_{\mathcal{Y}})$. So $R^{\downarrow}_{\mathcal{Y}}$ is not focal for $(\mu)^{\downarrow}_{\mathcal{Y}}$. By contraposition, the focal sets of $(\mu)^{\downarrow}_{\mathcal{Y}}$ are projections of the focal sets of μ. $\Box$

Note that the projection of a focal set of μ on $\mathcal{Y}$ is not always focal for $(\mu)^{\downarrow}_{\mathcal{Y}}$.

Example 1 *For instance, suppose μ has only two focal sets, say $R = \{(x_2, y_1), (x_2, y_2)\}$ and $T = \{(x_1, y_1), (x_2, y_1)\}$, with $\mu(T) > \mu(R)$. R and T are not nested but $T^{\downarrow}_{\mathcal{Y}} = \{y_1\} \subset R^{\downarrow}_{\mathcal{Y}} == \{y_1, y_2\}$. $\mu^{\downarrow}_{\mathcal{Y}}(T^{\downarrow}_{\mathcal{Y}}) = \mu(\mathcal{X} times\{y_1\}) = \mu(\{(x_1, y_1), (x_2, y_1)\}) = \mu(T)$ and $\mu^{\downarrow}_{\mathcal{Y}}(R^{\downarrow}_{\mathcal{Y}}) = \mu(\mathcal{X} times\{y_1, y_2\}) = \mu(\mathcal{X} \times \mathcal{Y}) = \max(\mu(R), \mu(T)) = \mu(T)$. So, there is a strict subset of $T^{\downarrow}_{\mathcal{Y}}$ which receives the same μ value: $T^{\downarrow}_{\mathcal{Y}}$ is the projection of a focal set of μ, but is not a focal set of the projection of μ.*

In fact the projection of μ can be defined from the focal sets of μ as

$$(\mu)^{\downarrow}_{\mathcal{Y}}(B) = \mu(\mathcal{X} \times B) = \max_{R \in \mathcal{F}(\mu): R^{\downarrow}_{\mathcal{Y}} \subseteq A} \mu_{\#}(R)$$

Let us call *a sufficient fragment* of μ a weighted set $\{(R_i, \mu^*(R_i)), i = 1, \dots, \}$ such that $\mu(R) = \max_{R_i \subseteq R} \mu^*(R_i)$. Clearly $\mathcal{F}(\mu)$ is (the smallest) sufficient fragment of μ and any family $\mathcal{S} \supset \mathcal{F}$, with $\mu^*(R_i) = \mu(R_i), R_i \in \mathcal{S}$ is sufficient. The projections $R^{\downarrow}_{\mathcal{Y}}$ of focal sets of μ are a sufficient fragment of $(\mu)^{\downarrow}_{\mathcal{Y}}$.b One gets the focal sets of $(\mu)^{\downarrow}_{\mathcal{Y}}$ by deleting from it the sets B' for which there is a subset $B \subsetneq B'$ such that $\mu(\mathcal{X} \times B) = \mu(\mathcal{X} \times B')$.

3.2 Joint capacities

Let us now the reverse way, and build a joint capacity of $\mathcal{X} \times \mathcal{Y}$ from the knowledge of two local capacities, γ and δ on $\mathcal{X}$ and $\mathcal{Y}$ respectively, it is always possible to build a 2D capacity μ on $\mathcal{X} \times \mathcal{Y}$ from $\mu_{\mathcal{X}}$ and μ_Y as follows: for any $A \in \mathcal{X}, B \times \mathcal{Y}$, $\mu(A \times B) = \mu_{\mathcal{X}}(A) \otimes \mu_Y(B)$, where it is expected to have (in view of the definition of projection), $0 \otimes 0 = 1 \otimes 0 = 0 \otimes 1 = 0, 1 \otimes 1$ and $\otimes$ is increasing for each argument so $\otimes$ is a conjunction. Because we work in a qualitative context (which motivates the use of Sugeno integrals), $\otimes$ is the minimum. So, we define the *joint capacity* obtained from γ on $\mathcal{X}$ and δ on $\mathcal{Y}$ by

$$\mu_{\gamma \times \delta}(R) = \max_{A,B: A \times B \subseteq R} \min(\gamma(A), \delta(B)).$$

definition alternative :

$$\mu_{\gamma \times \delta}(R) = \min(\gamma(R^{\downarrow \mathcal{X}}), \delta(R^{\downarrow \mathcal{Y}})).$$

ca ne changerat pas grand chose ?

Note that $\mu_{\gamma \times \delta}(A \times B) = \min(\gamma(A), \delta(B))$. It is clear that the focal sets of $\mu_{\gamma \times \delta}$ are Cartesian products, and they are among the Cartesian products $A \times B$ where A is a focal of γ and B is a focal set of δ. But the Cathesian product of a focal set A of γ and a focal set B of δ is not necessarilty a focal set of $\mu_{\gamma \times \delta}$. It can indeed contain the product of two other focals sets A' and B' of γ and δ, respectively.

Proposition 4 *Any focal set of $\gamma \times \delta$ is the Cartesian product of a focal set of γ and a focal set of δ*

Proof:

missing $\square$

Example 2 *Suppose γ and δ are necessity measures with focal sets $\{A, \mathcal{X}\}$ and $\{B, \mathcal{Y}\}$ respectively, with $\gamma(A) = a$ and $\delta(B) = b$ where $a > b$ (i.e. $N(A) = a$ and $N(B) = b$). $\gamma \times \delta$ is the necessity measure s.t. $N(A \times B) = min(a, b) = b$, $N(A \times \mathcal{Y}) = min(a, 1) = a$, $N(\mathcal{X} \times B) = min(1, b) = b$ and of course $N(\mathcal{X} \times \mathcal{Y}) = min(1, b) = 1$. Then $\mu_{\gamma\times\delta}$ has focal sets $A \times B$ with weight b, $A \times \mathcal{Y}$ with weight a and $\mathcal{X} \times \mathcal{Y}$ with weight 1, while $\mathcal{X} \times B$ is not focal (it contains the focal set $A \times B$) which as the same $\mu_{\gamma\times\delta}$.*

It can be checked $(\mu_{\gamma\times\delta})^{\downarrow}_{\mathcal{X}} = \gamma$, $(\mu_{\gamma\times\delta})^{\downarrow}_{\mathcal{Y}} = \delta$.

The carthesian products of the focal elements of for a family $\{(A \times B, \min(\gamma(A), \delta(B)) : A \in \mathcal{F}(\gamma), B \in \mathcal{F}(\delta)\}$ which contains the focal sets of $\gamma \times \delta$, and maybe more sets - i.e. it is sufficient for $\mu_{\gamma\times\delta}$.

It is easy to check that the product is performed without any modification of the information contained in γ and δ

Proposition 5 *The projections of $\mu_{\gamma\times\delta}$ are γ and δ*

Proof: First let $\gamma(2^{\mathcal{X}}) = \{1 > a_1 \cdots > a_p\}$ and $\gamma(2^{\mathcal{Y}}) = \{1 > b_1 \cdots > b_q\}$. The weights of the joint capacity are $\{\min(a_i, b_j) : i = 1, \ldots p, j = 1, \ldots q\} = \gamma(2^{\mathcal{X}}) \cup \gamma(2^{\mathcal{Y}})$ since both sets contain 1. No information about weights is lost. Now it is easy to see that $(\mu_{\gamma\times\delta})^{\downarrow}_{\mathcal{Y}}(B_j) = \mu_{\gamma\times\delta}(\mathcal{X} \times B_j) = \min(1, b_j) = b_j$, and so on. So we recover $\delta_{\#}$ on $\mathcal{Y}$ and it is a sufficient fragment of the projection (due to Proposition **??**). $\square$

3.3 Separable 2D capacities

Now we consider the opposite problem: When does a capacity contain the same amount of information as its projections?

Definition 3 *μ is a separable capacity if it is the joint of its projections, namely, $\mu = \mu_{(\mu)^{\downarrow}_{\mathcal{X}} \times (\mu)^{\downarrow}_{\mathcal{Y}}}$*

Conjecture 1 *if μ is a separable capacity then for any R, $\mu(R) = \min((\mu)^{\downarrow}_{\mathcal{X}}(R^{\downarrow}_{\mathcal{X}}), (\mu)^{\downarrow}_{\mathcal{Y}}(R^{\downarrow}_{\mathcal{Y}})$*

Let us try to find separability conditions.

Proposition 6 *If μ is separable then all its focal sets are Cartesian products.*

This is a direct consequence of Proposition **??** that shows that the focal elements of a joint are Cartesian products

The converse of Proposition **??** is false. The fact that the EF of μ are Cartesian products (and even disjoint products) does not guarantee the separability of μ

Example 3 *Suppose $\mathcal{F}(\mu)$ has just three focal sets $A \times B$, $A' \times B$, $A \times B'$, $A' \times B'$ with $A \neq A'$ and $B \neq B'$ with no inclusion. All are Cartesian products.*

Let us denote $\mu_{\#}(A \times B) = \alpha_1$ $\mu_{\#}(A' \times B) = \beta_1$ and $\mu_{\#}(A \times B')) = \beta_2$, $\mu_{\#}(A' \times B') = \alpha_2$ and suppose that both alpha$_1$ and alpha$_2$ are than β_1 and β_2 $(\max(\alpha_1, \alpha_2) \leq \min(\beta_1, \beta_2)$. For instance, with $A = \{x_1\}$, $A' = \{x_2\}$, $B = \{y_1\}$, $B' = \{y_2\}$ and μ, is the possibility measure defined by $0 < \pi((x_1, y_1)) = \pi((x_2, y_2)) < \pi((x_2, y_1)) = \pi((x_1, y_2)) = 1$.

Because there is no inclusion, the focal sets of $(\mu)^{\downarrow}_{\mathcal{X}}$ are A ($\mu_{\#} = \beta_1$) and A' ($\mu_{\#} = \beta_2$) and the focal sets of $(\mu)^{\downarrow}_{\mathcal{Y}}$ are B ($\mu_{\#} = \beta_1$) and B' ($\mu_{\#} = \beta_2$). For the same reason, the focal sets of $\mu_{(\mu)^{\downarrow}_{\mathcal{X}} \times (\mu)^{\downarrow}_{\mathcal{Y}}}$ are $A \times B$ (β_1) $A' \times B$' (β_2), $A \times B'$ $(\min(\beta_1, \beta_2))$ and $A' \times B$ $(\min(\beta_1, \beta_2))$. We recover the same focal elements, and they are products, but

$$\mu_{(\mu)^{\downarrow}_{\mathcal{X}} \times (\mu)^{\downarrow}_{\mathcal{Y}}}(A \times B) > \mu(A \times B).$$

So, $\mu_{(\mu)^{\downarrow}_{\mathcal{X}} \times (\mu)^{\downarrow}_{\mathcal{Y}}} \neq \mu$.

We can also show that

Proposition 7 *If μ is separable then $\mu(A \times B) = \min((\mu)^{\downarrow}_{\mathcal{X}}(A), (\mu)^{\downarrow}_{\mathcal{Y}}(B))$ for all focal sets $A \times B \in \mathcal{X} \times \mathcal{Y}$ of μ.*

But the converse is also false

Example 4 *Suppose that μ has two focals R and $\mathcal{X} \times \mathcal{Y}$, with $\mu(R) = a < 1$ and ${}^{\downarrow}_{\mathcal{X}} = \mathcal{X}$ and $R^{\downarrow}_{\mathcal{Y}} = \mathcal{Y}$. For instance the necessity measure $\mu(\{(x_1, y_1), (x_1, y_2), (x_2, y_2)\} = a < 1$ (and of course $\mu(\mathcal{X} \times \mathcal{Y}) = 1$).*

It is clear that the only focal set of $(\mu)^{\downarrow}_{\mathcal{X}}$ is $\mathcal{X}$ (with weight 1) and the only focal set of $(\mu)^{\downarrow}_{\mathcal{Y}}$ is $\mathcal{Y}$ (with weight 1) . Then of course $\mu(A \times B) = \min((\mu)^{\downarrow}_{\mathcal{X}}(A), (\mu)^{\downarrow}_{\mathcal{Y}}(B))$ for all focal sets $A \times B \in \mathcal{X} \times \mathcal{Y}$ of μ.

But μ is not separable. Indeed, since one of its focal sets, R is not a Cartesian product: the joint capacity obtained from its projections ($\mu_{(\mu)^{\downarrow}_{\mathcal{X}} \times (\mu)^{\downarrow}_{\mathcal{Y}}}$) is not equal to μ. In particular, $\mu_{(\mu)^{\downarrow}_{\mathcal{X}} \times (\mu)^{\downarrow}_{\mathcal{Y}}}(R) = 0$.

However, if none of the (necessary) conditions above is sufficent for defining separable capacities, their conjunction is equivalent to define capacities which are the joint of their projection, i.e. separable capacities:

Proposition 8 *A 2D capacity μ is separable if and only if all its focal sets are Cartesian products and $\mu(A \times B) = \min((\mu)^{\downarrow}_{\mathcal{X}}(A), (\mu)^{\downarrow}_{\mathcal{Y}}(B))$ for all focal sets $A \times B \in \mathcal{X} \times \mathcal{Y}$ of μ.*

Proof: We know that the projections of the focal sets are sufficient fragments of the projections of μ. So we can write $\mu_{(\mu)^{\downarrow}_{\mathcal{X}} \times (\mu)^{\downarrow}_{\mathcal{Y}}}(R) = \max_{A \times B \in \mathcal{F}(\mu): A \times B \subseteq R} \min((\mu)^{\downarrow}_{\mathcal{X}}(A), (\mu)^{\downarrow}_{\mathcal{Y}}(B)) = \max_{A \times B \in \mathcal{F}(\mu): A \times B \subseteq R} = \mu(R)$ since all focal sets of μ are Cartesian products. □

It is an open problem to describe the constraints on weights of focal sets of μ that are characteristic of its separability. We can do it in the special case of Boolean capacities.

Proposition 9 *A Boolean 2D capacity μ is separable if and only if its focal sets are pq Cartesian products of the form $\{A_i \times B_j : i = 1 \dots p, j = 1 \dots p\}$ such that both families $\{A_i : i = 1 \dots p\}$ and $\{B_j : j = 1 \dots q\}$ are antichains.*

Proof: If the requested conditions are required it is clear that $\{A_i \times B_j : i = 1 \dots p, j = 1 \dots p\}$ form an anti-chain as well, so they are indeed focal sets for μ. The focals of $(\mu)^{\downarrow}_{\mathcal{X}}$ are clearly all of $\{A_i : i = 1 \dots p\}$, and those of $(\mu)^{\downarrow}_{\mathcal{Y}}$ are clearly all of $\{B_j : j = 1 \dots q\}$. Now, $\mu_{(\mu)^{\downarrow}_{\mathcal{X}} \times (\mu)^{\downarrow}_{\mathcal{Y}}}(A_i \times B_j) = \min((\mu)^{\downarrow}_{\mathcal{X}}(A_i), (\mu)^{\downarrow}_{\mathcal{Y}}(B_j)) = 1$. The joint capacity has clearly no other focal sets than $\{A_i \times B_j : i = 1 \dots p, j = 1 \dots p\}$. So it is μ.
Conversely suppose $\{A_i : i = 1 \dots p\}$ is not an antichain. Then there are i, k with $A_i \subset A_k$. Then A_i is not a focal for $(\mu)^{\downarrow}_{\mathcal{X}}$ and so there is no Cartesian product of the form $A_i \times B_j$ that is focal for $\mu_{(\mu)^{\downarrow}_{\mathcal{X}} \times (\mu)^{\downarrow}_{\mathcal{Y}}}$. So the joint of the projections of μ is not μ. □

TODO Donner la preuve de ce qui suit Finally, we can notice that a 2D capacity is separable if and only if its cuts are separable Boolean 2D capacities.

4 Double Sugeno integrals

In the previous section, S-integrals are assumed to be defined on one-dimensional sets. In the present section we consider the case of an S-integral on a Cartesian product of two sets $\mathcal{X}$ and $\mathcal{Y}$. There are two ways of computing them: either we start from two capacities $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ compute a *double* S-integral, or we define a unique integral based on a two-dimensional capacity, i.e., a capacity on the Cartesian product $\mathcal{X} \times \mathcal{Y}$. In the first case, there is a commutation problem, according to whether we start integrating over $\mathcal{X}$ or over $\mathcal{Y}$ - this is the topic of the second part of this report. In the second case it is interesting to compare the 2D S-integral based on a 2D capacity with the double S-integrals obtained using the projections of the 2D capacity. This topic is developed in the following sections.

4.1 Definition

We consider two sets: $\mathcal{X} = \{x_1, \cdots, x_n\}$, $\mathcal{Y} = \{y_1, \cdots, y_p\}$ We have two fuzzy measures: $\mu_{\mathcal{X}}$ on $\mathcal{X}$ and $\mu_{\mathcal{Y}}$ on $\mathcal{Y}$. Consider a function $u : \mathcal{X} \times \mathcal{Y} \to L$.

In the context of decision under uncertainty, the set $\mathcal{X}$ is a set of possible states of nature, equipped with a capacity $\mu_{\mathcal{X}}$ describing knowledge about the actual state of nature. The other set $\mathcal{Y}$ is a set of agents sharing this knowledge. Then the value $u(x, y)$ is the worth of decision $\mathfrak{u}$ according to agent y when the state of nature is x.

In the context of multiple criteria decision making, the set of possible states of nature is replaced by a finite set of criteria $\mathcal{C} = \{1, \cdots, n\}$ and alternatives are evaluated according to each criterion: by vectors $(u_1, \cdots, u_n) \in L^n$ of local evaluations, where L is the evaluation scale which is a bounded totally ordered set. Then in the multiagent setting, the same question arises, whether we aggregate across criteria first or across agents first. Then the value $u(x, y)$ is the worth of $\mathfrak{u}$ according to agent y and criterion x. The capacity $\mu_{\mathcal{X}}$ describes the importance of each group of criteria.

In both cases the capacity $\mu_{\mathcal{Y}}$ on $\mathcal{Y}$ evaluates the importance of groups of agents. We use the terminology of decision under uncertainty in the following - as a running example, suppose that we evaluate the collective worth of a decision characterized by function u according to all agents and given some common knowledge $\mu_{\mathcal{X}}$ about the states of the world.

For each agent y_j, we denote by $u(\cdot, y_j) = (u(x_1, y_j), \cdots, u(x_n, y_j))$ the vector of utilities of y_j in the different states and for each state x_i, $u(x_i, \cdot) = (u(x_i, y_1), \cdots, u(x_i, y_p))$ the vector assigning to each agent its utility value if the state of nature is x_i.

There are two types of combination, according to whether we apply S-integral on $\mathcal{X}$ first, or on $\mathcal{Y}$ first (as presented in Figure **??**):

- $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u))$ is the Sugeno integral, according to μ_X of the vector $S_{\mu_{\mathcal{Y}}} = (S_{\mu_{\mathcal{Y}}}(u(x_1, \cdot)), \ldots, S_{\mu_{\mathcal{Y}}}(u(x_n, \cdot)))$
- $S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(u))$ is the Sugeno integral, according to μ_Y of the vector $\vec{S}_{\mu_{\mathcal{X}}} = (S_{\mu_{\mathcal{X}}}(u(\cdot, y_1)), \ldots, S_{\mu_{\mathcal{X}}}(u(\cdot, y_p)))$

That is to say:

$$\begin{aligned} S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) &= \max_{A \subseteq \mathcal{X}} \min(\mu_{\mathcal{X}}(A), \min_{x \in A} S_{\mu_{\mathcal{Y}}}(u(x, \cdot))) \\ &= \max_{A \subseteq \mathcal{X}} \min(\mu_{\mathcal{X}}(A), \min_{x \in A} \max_{B \subseteq \mathcal{Y}} \min(\mu_{\mathcal{Y}}(B), \min_{y \in B}(u(x, y)))) \end{aligned}$$

$$\begin{aligned} S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(u)) &= \max_{B \subseteq \mathcal{Y}} \min(\mu_{\mathcal{Y}}(B), \min_{y \in B} S_{\mu_{\mathcal{X}}}(u(\cdot, y))) \\ &= \max_{B \subseteq \mathcal{Y}} \min(\mu_{\mathcal{Y}}(B), \min_{y \in B} \max_{A \subseteq \mathcal{X}} \min(\mu_{\mathcal{X}}(A), \min_{x \in A}(u(x, y)))) \end{aligned}$$

We call integrals of the form $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u))$ and $S_{\mu_{\mathcal{Y}}}(S\mu_{\mathcal{X}}(u))$ *double S-integrals.*

A result worth mentioning is a Fubini theorem for S-integrals, which is a special case of a result proved in [**?**] :

Proposition 10 *If* $R = A \times B$ *then* $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(\mathbf{1}_R)) = \min(\mu_{\mathcal{X}}(A), \mu_{\mathcal{Y}}(B))$

Proof: $\mathbf{1}_R$ is the characteristic function of $A \times B$. The double integral then reads $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(\mathbf{1}_R)) = \max_{C \subseteq \mathcal{X}} \min(\mu_{\mathcal{X}}(C), \min_{x \in C} \max_{C' \subseteq \mathcal{Y}} \min(\mu_{\mathcal{Y}}(C'), \min_{y \in C'}(\mathbf{1}_R(x, y))))$

because $\mathbf{1}_R(x, y) = 1$ if $(x, y) \in A \times B$, $u(x, y) = 0$ we get.:
$S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) == \max_{C \subseteq A} \min(\mu_{\mathcal{X}}(C), \min_{x \in C} \max_{C' \subseteq B} \min(\mu_{\mathcal{Y}}(C'), \min_{y \in C'}(1)))$

Then
$S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) = \max_{C \subseteq A} \min(\mu_{\mathcal{X}}(C), \min_{x \in C} \max_{C' \subseteq B} \min(\mu_{\mathcal{Y}}(C'), 1))$
$= \max_{C \subseteq A} \min(\mu_{\mathcal{X}}(C), \min_{x \in C} \mu_{\mathcal{Y}}(B)))$
$= \max_{C \subseteq A} \min(\mu_{\mathcal{X}}(C), \mu_{\mathcal{Y}}(B))) = \min(\mu_{\mathcal{X}}(A), \mu_{\mathcal{Y}}(B)))$ $\square$

$$
\begin{array}{ccc}
\begin{array}{cc}
 & \begin{array}{ccc} y_1 & \cdots & y_p \end{array} \\
\begin{array}{c} x_1 \\ \\ x_n \end{array} & \begin{pmatrix} u(x_1, y_1) & \dots & u(x_1, y_p) \\ \vdots & \ddots & \vdots \\ u(x_n, y_1) & \dots & u(x_n, y_p) \end{pmatrix}
\end{array}
& \begin{array}{c} \to \\ \vdots \\ \to \end{array} &
\begin{pmatrix} S_{\mu_{\mathcal{Y}}}(u(x_1, \cdot)) \\ \vdots \\ S_{\mu_{\mathcal{Y}}}(u(x_n, \cdot)) \end{pmatrix} \\
 & & \downarrow \\
\downarrow \ \dots \ \downarrow & & S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u(x_1, \cdot)), \cdots, S_{\mu_{\mathcal{Y}}}(u(x_n, \cdot))) \\
 & & \\
(S_{\mu_{\mathcal{X}}}(u(\cdot, y_1)), \cdots, S_{\mu_{\mathcal{X}}}(u(\cdot, y_p))) & \to & S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(u(\cdot, y_1)), \cdots, S_{\mu_{\mathcal{X}}}(u(\cdot, y_p))
\end{array}
$$

Figure 1: Two double utility functionals with Sugeno integral

The natural question is to find the conditions under which the equality $S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(u)) = S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u))$ holds. As a matter of fact, let us first study it in the context of possibilistic measures.

4.2 Possibilistic double S-integrals

Consider the cases when $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are possibility or necessity measures. In the framework of possibilistic decision under uncertainty, $\mathcal{X} = \{x_1, \cdots, x_n\}$ is a set of states, and a possibility distribution π captures the common knowledge of the agents: π_i is the possibility degree to be in state x_i. $\mathcal{Y} = \{y_1, \cdots, y_p\}$ is the set of agents. The weight vector $w = (w_1, \cdots, w_p) \in [0, 1]^p$ is modeled as a possibility distribution on $\mathcal{Y}$ where w_j is the importance of agent y_j. The attractiveness of decision $\mathfrak{u}$ for agent y_j in the different states is captured by utility function $u(\cdot, y_j) : \mathcal{X} \to [0, 1]$.

There are two possible approaches for egalitarist aggregations of pessimistic decision-makers, and two possible approaches for egalitarist aggregations of pessimistic decision-makers [**?**].

ex-post pessimistic approach

$$U^{-min}_{post}(\pi, w, u) = \min_{x_i \in \mathcal{X}} \max(1 - \pi_i, \min_{y_j \in \mathcal{Y}} \max(u(x_i, y_j), 1 - w_j)).$$

ex-ante pessimistic approach

$$U^{-min}_{ante}(\pi, w, u) = \min_{j \in \mathcal{Y}} \max(1 - w_j, \min_{x_i \in \mathcal{X}} \max(u(x_i, y_j), 1 - \pi_i)).$$

ex-post optimistic approach

$$U^{+min}_{post}(\pi, w, u) = \max_{x_i \in \mathcal{X}} \min(\pi_i, \min_{y_j \in \mathcal{Y}} \max(u(x_i, y_j), 1 - w_j)).$$

ex-ante optimistic approach

$$U^{+min}_{ante}(\pi, w, u) = \min_{j \in \mathcal{Y}} \max(1 - w_j, \max_{x_i \in \mathcal{X}} \min(u(x_i, y_j), \pi_i)).$$

It can be checked that the first two quantities are of the form

$$U^{-min}_{post}(\pi, w, u) = S_{N_{\mathcal{X}}}(S_{N_{\mathcal{Y}}}(u))$$

$$U^{-min}_{ante}(\pi, w, u) = S_{N_{\mathcal{Y}}}(S_{N_{\mathcal{X}}}(u))$$

Essghaier et al. [**?**] show that the two expressions are equal to $\min_{x_i \in \mathcal{X}, y_j \in \mathcal{Y}} \max(1 - \pi_i, u(x_i, y_j), 1 - w_j))$, i.e., $S_{N_{\mathcal{X}}}(S_{N_{\mathcal{Y}}}(u)) = S_{N_{\mathcal{Y}}}(S_{N_{\mathcal{X}}}(u))$.

In the optimistic case, qualitative decision theory prescribes the use of a Sugeno integral based on a possibility measure on $\mathcal{X}$

$$U^{+min}_{post}(\pi, w, u) = S_{\Pi_{\mathcal{X}}}(S_{N_{\mathcal{Y}}}(u))$$

$$U^{+min}_{ante}(\pi, w, u) = S_{N_{\mathcal{Y}}}(S_{\Pi_{\mathcal{X}}}(u)).$$

The two integrals do no coincide: [**?**] have shown that we only have the inequality $U^{+min}_{ante}(\pi, w, u) \geq U^{+min}_{post}(\pi, w, u)$ with no equality in general The following counter example shows that the latter inequality can be strict - no commutation result can be obtained when one of the capacity is a necessity measure and the other one a possibility measure:

Example 5 Counterexample

Let $\mathcal{X} = \{x_1, x_2\}$*,* $\pi_i = 1$*, and* $w_i = 1, \forall i = 1,2$*,* $\mathcal{Y} = \{y_1, y_2\}$*,* $u(x_1, y_1) = u(x_2, y_2) = 1$ *and* $u(x_2, y_1) = u(x_1, y_2) = 0$*.*

We have U^{+min}_{post} *as*

$\max(\min(1, \min(\max(1-1,1), \max(1-1,0)), \min(1, \min(\max(1-1,0), \max(1-1,1))) = 0.$

But $U^{+min}_{ante}(\pi, w, u)$ *is computed as*
$\min(\max(1-1, \max(\max(1,1), \max(0,1)), \max(1-1, \max(\max(0,1), \max(1,1)))) = 1.$

In summary, double S-integrals may commute, e.g. when the two capacities are necessity measures (and the same commutation result is obtained if we consider possibility measures, namely, $S_{\Pi_\mathcal{X}}(S_{\Pi_\mathcal{Y}}(u)) = S_{\Pi_\mathcal{Y}}(S_{\Pi_\mathcal{X}}(u)))$, but not always - the previous example shows that the difference between the two double S-integrals can be maximal.

4.3 From double Sugeno integrals to 2D Sugeno integrals

Let us now consider the problem of relating double Sugeno integrals and *2D S-integral:*, i.e. Sugeno integrals based on a two dimensional capacity:

$$S_\mu(u) = \max_{T \subseteq \mathcal{X} \times \mathcal{Y}} \min(\mu(T), \min_{(i,j) \in T} u(x_i, y_j))$$

In the current section, we investigate the relationships between $S_\mu(u)$ and 2D S-integrals $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u))$ and $S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u))$. In particular, we study to what extent a double S-integral based on capacities $\mu_\mathcal{X}$ and μ_Y can be understood as a 2D S-integral build on a particular two dimensional capacity the projections of which are $\mu_\mathcal{X}$ and μ_Y. We also show that even if $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u))$ and $S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u))$ are equal for some function u these double S-integrals may differ from the S-integral based on the joint 2D capacity $\mu_\mathcal{X} \times \mu_\mathcal{Y}$.

Double integrals are 2D S-integrals

It is easy to see that a double S-integral is actually an idempotent lattice polynomial $p : L^{|\mathcal{X}|+|\mathcal{Y}|} \to L$, since it is made using $\min, \max$, constants and variable terms of the form $u(x_i, x_j)$, and $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_{\mathcal{X}\times\mathcal{Y}})) = 1, S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_\emptyset)) = 0$. Using results on lattice polynomials, we conclude that *a double S-integral is always a 2D S-integral*, based on a capacity that maps each $R \subseteq \mathcal{X} \times \mathcal{Y}$ to L defined by $\mu(R) = p(\mathbf{1}_R)$. Namely noticing that

$$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R(x_1, \cdot)), \cdots, S_{\mu_\mathcal{Y}}(\mathbf{1}_R(x_n, \cdot))) = S_{\mu_\mathcal{X}}(f_R)$$

where $f_R(x_i) = \mu_{\mathcal{Y}}(\{y \in \mathcal{Y} : x_i R y\})$, is a monotonic function of R only, i.e. a 2D capacity κ, we can write any double S-integral as

$$S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) = \max_{R \subseteq \mathcal{X}\times\mathcal{Y}} \min(\kappa(R), \min_{(x_i,y_j)\in R} u(x_i, y_j)) \tag{5}$$

with $\kappa(R) = S_{\mu_{\mathcal{X}}}(f_R)$ for each $R \subseteq \mathcal{X} \times \mathcal{Y}$. The double S-integral is thus the 2D S-integral based on the so-defined 2D capacity κ. This result was independently proved very recently by Halas et al. [**?**], but it also straightforwardly follows from considerations in [**?**, **?**].

la μ 2D qui capture la double est elle unique ? yes ?

The following example shows that de 2D capacity κ is not necessarily separable. c 'est bien a ??

Example 6 *(continued): We can lay bare the 2D capacity involved in the definition of $U^{+min}_{ante}(\pi, w, u) = S_N(S_\Pi(u))$. It is easy to see that $S_N(S_\Pi(\mathbf{1}_R)) = 0$ whenever $R = \{(x_i, y_j)\}, i, j \in \{1, 2\}$ (singletons), but $S_N(S_\Pi(\mathbf{1}_R)) = 1$ for $R = \{x_i\} \times \mathcal{Y}, i = 1, 2$ and for $R = \{(x_1, y_1), (x_2, y_2)\}, \{(x_1, y_2), (x_2, y_1)\}$ but is 0 for the two other two-elements subsets of $\mathcal{X} \times \mathcal{Y}$. So, the 2D capacity underlying $S_N(S_\Pi(\mathbf{1}_R))$ has four focal sets.*

On the other hand, capacity $\Pi_{\mathcal{X}} \times N_{\mathcal{Y}}$ has only two focal sets $R = \{x_i\} \times \mathcal{Y}, i = 1, 2$.

Hence, the 2D capacity underlying $S_N(S_\Pi(\mathbf{1}_R))$ is not equal to $\Pi_{\mathcal{X}} \times N_{\mathcal{Y}}$.

This shows that the 2D capacity underlying the 2D reprentation of a double integral is not necessary separable. ????

Are 2D S-integrals Double Sugeno integrals ?

We have seen that Double Sugeno integrals can be captured by 2D integrals. Can 2D S-integrals also be captured by double Sugeno integrals ? The answer is no in the general case. It is indeed easy to find that some 2D capacity the integrals on which cannot be represented by a Double integral

Proposition 11 *There exists some 2D capacities μ and some functions u such that $\forall \mu_{\mathcal{X}}, \mu_{\mathcal{Y}}, S_{\mu_X} S_{\mu_Y}(u) \neq S_\mu(u)$*

Proof: Consider the 2D capacity with only three focal elements, $\{x_1, y_1\}$, $\{x_1, y_2\}$ and $\{x_2, y_1\}$ which all receive the degree 1, and suppose that there exist $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ such as $\forall u, S_{\mu_X} S_{\mu_Y}(u) = S_\mu(u)$.

Consider first the utility function $u(x, y) = 1$ if $x = x_2$ and $y = y_2$ and $u(x, y) = 0$ otherwise (the caracteristic function of $\{(x_2, y_2)\}$. $S_\mu(u) = \mu(\{(x_2, y_2)\}) = 0$ because no focal element is contained in $\{(x_2, y_2)\}$. On the other hand, $S_{\mu_X} S_{\mu_Y}(u) = \max(\min(\mu_{\mathcal{X}}(x_1), 0), \min(\mu_{\mathcal{X}}(\{x_2\}), 1), \min(\mu_{\mathcal{X}}(\{x_1, x_2\}, 0))) = mu_{\mathcal{X}}(x_2)$. So, $\mu_{\mathcal{X}}(\{x_2\}) = 0$.

Consider now function $u'(x,y) = 1$ if $x = x_2$ and $y = y_1$ and $u(x',y) = 0$ otherwise. $S_\mu(u') = \mu(\{(x_2,y_1)\}) = 1$ because $\{(x_2,y_1)\}$ is a focal elements of degree 1. On the other hand, $S_{\mu_X}S_{\mu_Y}(u') = \max(\min(\mu_\mathcal{X}(x_1),0), \min(\mu_\mathcal{X}(\{x_2\}),1), \min(\mu_\mathcal{X}(\{x_1,x_2\},0))) = mu_\mathcal{X}(x_2)$. So, $\mu_\mathcal{X}(\{x_2\}) = 1$, which contradicts $\mu_\mathcal{X}(\{x_2\}) = 0$.

By symmery, there exist no $\mu_\mathcal{X}$ $\mu_\mathcal{Y}$ such as $\forall u, S_{\mu_Y}S_{\mu_X}(u) = S_\mu(u)$.

□

Of course, for some μ, the representation by a double integral is possible, e.g. when μ is the product of two possibility measures. In this example, μ is separable. But it is not always the case - example **??** presents a 2D integral that can be represented by a 2D integral but is not separable. We can nevertheless show that:

Proposition 12 *If there exists μ_X on $\mathcal{X}$ and $\mu_\mathcal{Y}$ on $\mathcal{Y}$ such that $\forall u, S_{\mu_X}(S_{\mu_Y}(u)) = S_\mu(u)$, then μ_X and $\mu_\mathcal{Y}$ are the projections of μ on $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\forall A, B, \mu(A \times B) = \min(\mu_\mathcal{X}(A), \mu_\mathcal{Y}(B))$*

Proof: Suppose that there exist $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ such as $\forall u, S_{\mu_X}S_{\mu_Y}(u) = S_\mu(u)$. Consider any pair of subsets $A \in \mathcal{X}$ and B of $\mathcal{Y}$ and let u' be its caracteristic the function of $A \times B$, i.e. $u(x,y) = 1$ if $(x,y) \in A \times B$, $u(x,y) = 0$ otherwise

First of all, $S_\mu(u) = \mu(A \times B)$. From proposition **??** $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = \min(\mu_\mathcal{X}(A), \mu_\mathcal{Y}(B)))$

So, $S_{\mu_\mathcal{X}}S_{\mu_\mathcal{Y}}(u) = S_\mu(u)$ implies $\mu(A \times B) = \min(\mu_\mathcal{X}(A), \mu_\mathcal{Y}(B)$.

Setting $A = \mathcal{X}$ we get $\mu(A \times \mathcal{Y}) = \min(\mu_\mathcal{X}(A), \mu_Y(\mathcal{X})) = \mu_\mathcal{X}(A)$ $\mu_\mathcal{X})$ is the projection of μ on $\mathcal{X}$. Setting $B = \mathcal{Y}$ we get $\mu(\mathcal{X} \times B) = \min(\mu_\mathcal{X}(\mathcal{X}), \mu_Y(B)) = \mu_\mathcal{Y}(B)$ $\mu_\mathcal{Y}$ is the projection of μ on $\mathcal{Y}$.

□

Conjecture 2 *If there exists μ_X and $\mu_\mathcal{Y}$ such that $\forall u, S_\mu(u) = S_{\mu_X}(S_{\mu_Y}(u)) = S_{\mu_Y}(S_{\mu_X}(u))$, then μ is separable*

Conjecture 3 *If there exists μ_X and $\mu_\mathcal{Y}$ such that $\forall u, S_{\mu_X}(S_{\mu_Y}(u)) = S_{\mu_Y}(S_{\mu_X}(u))$, then $S_{\mu_X}(S_{\mu_Y}(u)) = S_{\mu_Y}(S_{\mu_X}(u)) = S_\mu$*

Of course, separability is not a sufficient condition : we have shall see in example **??** a separable capacity the 2D integral on which cannot be represented by a double integral.

2D S-integrals for separable capacities

If μ is a separable capacity we have that

$$S_\mu(u) = \max_{(A \times B) \in \mathcal{F}(\mu)} \min(\min(\mu_\mathcal{X}(A), \mu_\mathcal{Y}(B), \min_{(i,j) \in A \times B} u(x_i, y_j))$$

The question is whether any 2D S-integral with respect to a separable capacity can be expressed in the form of a double S-integral as proposed above on figure 1 ? The following results answer by the negative.

Proposition 13 *We consider μ a separable fuzzy measure on $\mathcal{X} \times \mathcal{Y}$. We denote $(\mu)^{\downarrow}_{\mathcal{X}}$ and $(\mu)^{\downarrow}_{\mathcal{Y}}$ the projections on $\mathcal{Y}$ and $\mathcal{X}$ respectively.*

We have $S_\mu(u) \leq S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}((u(x_1, \cdot)), \ldots, S_{\mu_{\mathcal{X}}}(u(x_n, \cdot))))$.

Proof: Note that G is a focal set of $(\mu)^{\downarrow}_{\mathcal{Y}}$ and F a focal set of $(\mu)^{\downarrow}_{\mathcal{X}}$, if and only if $G \times F$ is a focal set of μ.

$$\begin{aligned}
&S_\mu(u) = \max_{C \subseteq \mathcal{X} \times \mathcal{Y}} \min(\mu(C), \min_{(i,j) \in C} u(x_i, y_j)) \\
&= \max_{A,B} \min(\mu_{\mathcal{Y}}(A), \mu_{\mathcal{Y}}(B), \min_{i \in A, j \in B} u(x_i, y_j)) \\
&= \max_A \min(\mu_{\mathcal{X}}(A), \max_B \min(\mu_{\mathcal{Y}}(B), \min_{i \in A, j \in B} u(x_i, y_j))) \\
&= \max_A \min(\mu_{\mathcal{X}}(A), \max_B \min_{i \in A} \min(\mu_{\mathcal{Y}}(B), \min_{j \in B} u(x_i, y_j))) \\
&\leq \max_A \min(\mu_{\mathcal{X}}(A), \min_{i \in A} \max_B \min(\mu_{\mathcal{Y}}(B), \min_{j \in B} u(x_i, y_j))) \\
&= S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}((u(x_1, \cdot)), \ldots, S_{\mu_{\mathcal{X}}}(u(x_n, \cdot)))).
\end{aligned}$$

Note that we have turned $\max_B \min_{i \in A}$ into $\min_{i \in A} \max_B$, which induces the inequality. □

Symmetrically, we get $S_\mu(u) \leq S_{\mu_{\mathcal{Y}}}(S_{\mathcal{X}}(u(\cdot, y_1)), \cdots, S_{\mathcal{Y}}(u(\cdot, y_p)))$.

The inequalities in Proposition **??** may be strict, and this even if μ is separable (see examples **??** and **??** in the following).

Let us exemplify the above situation in the setting of possibility and necessity measures. Consider the case of a 2D capacity on $\mathcal{Y} \times \mathcal{X}$ whose projection on $\mathcal{Y}$ is a necessity measure $N_{\mathcal{Y}}$ with possibility distribution w, and whose projection on $\mathcal{X}$ is a possibility measure $\Pi_{\mathcal{X}}$ with possibility distribution π.

This captures for instance the problem of multigent decision making under uncertainty, with n agents and n states, with two possibility distributions $w_1 = 1 \geq w_2 \cdots \geq w_n$ (importance weights) and $\pi_1 = 1 \geq \pi_2 \cdots \geq \pi_n$ (plausibilities). Let $B_j = \{y_1, \ldots, y_j\} \subseteq \mathcal{Y}$ and $\alpha_j = N_{\mathcal{Y}}(B_j)$, i.e., $\alpha_j = 1 - w_{j+1}$. Note that $N_{\mathcal{Y}}(B) = \max_{B_j \subseteq B} \alpha_j = \min_{y_j \notin B} 1 - w_j$ and the marginal S-integral on $\mathcal{Y}$ takes the two forms:

$$S_N(u(x_i, \cdot)) = \max_{B_j \subseteq B} \min(\alpha_j, \min_{y_k \in B_j} u(x_i, y_k)) = \min_{i=1,\ldots,n} \max(1 - w_j, u(x_i, y_j)).$$

The joint necessity-possibility function $\mu = \Pi_{\mathcal{X}} \times N_{\mathcal{Y}}$ is defined by the focal sets $\{x_i\} \times B_j$ and Moebius masses $\mu_\#(\{x_i\} \times B_j) = \min(\pi_i, \alpha_j), i, j = 1, \ldots, n$.

It can be checked that $(\mu)^{\downarrow}_{\mathcal{Y}} = N_{\mathcal{Y}}$ and $(\mu)^{\downarrow}_{\mathcal{X}} = \Pi_{\mathcal{X}}$, as prescribed by Proposition **??** . Indeed, $(\mu)^{\downarrow}_{\mathcal{Y}}(B) = \mu(\mathcal{X} \times B) = \max_{B_i \subseteq B} \max_{x_i \in \mathcal{X}} \min(\pi_i, \alpha_j) = \max_{B_j \subseteq B} \alpha_j$. Idem for the other.

The 2D S-integral reads:

$$S_\mu(u) = \max_{i,j=1,\ldots,n} \min(\min(\pi_i, \alpha_j), \min_{y_k \in B_j} u(x_i, y_k))$$

and also be expressed as

$$S_\mu(u) = \max_{j=1,\ldots,n} \min(\alpha_j, \max_{i=1,\ldots,n} \min(\pi_i, \min_{y_k \in B_j} u(x_i, y_k))) \tag{6}$$

or as

$$S_\mu(u) = \max_{i=1,\ldots,n} \min(\pi_i, \max_{j=1,\ldots,n} \min(\alpha_j, \min_{y_k \in A_j} u(x_i, y_k))) = S_{(\mu)^\downarrow_\mathcal{X}}(\vec{S}_{(\mu)^\downarrow_\mathcal{Y}}(u))$$

Thus, if a 2D capacity μ is the joint of a possibility measure on $\mathcal{X}$ and a necessity measure on $\mathcal{Y}$, then $S_\mu(u) = S_{(\mu)^\downarrow_\mathcal{X}}(S_{(\mu)^\downarrow_\mathcal{Y}}(u))$, which is the value U^{+min}_{post} for u.

But what about $S_{(\mu)^\downarrow_\mathcal{Y}}(S_{(\mu)^\downarrow_\mathcal{X}}(u))$, i.e., of the U^{+min}_{ante} utility of u ? Essghaier et al. [**?**] have shown that $U^{+min}_{post}(\pi, w, u) \leq U^{+min}_{ante}(\pi, w, u)$, which can be retrieved as particular case of Proposition **??**. These authors have shown that the inequality can be strict and even extreme. Let us reconsider it in the light of 2D S-integrals.

Example 7 *:* $\mathcal{X} = \{x_1, x_2\}$, $\pi_i = 1$, *and* $w_i = 1, \forall i = 1, 2$, $\mathcal{Y} = \{y_1, y_2\}$, $u(x_1, y_1) = u(x_2, y_2) = 1$ *and* $u(x_2, y_1) = u(x_1, y_2) = 0$. *So* $u = \mathbf{1}_{\{(x_1,y_1),(x_2,y_2)\}}$, *which is the characteristic function of the set* $\{(x_1, y_1), (x_2, y_2)\}$

The joint capacity, μ, *is a capacity whose projections are the vacuous necessity measure on* $\mathcal{Y}$ *and the vacuous possibility measure on* $\mathcal{X}$. *Note that on* $\mathcal{Y}$ *there is a single focal set* $\mathcal{Y}$ *while focal sets on* $\mathcal{X}$ *are all singletons. Hence focal sets of* μ *are of the form* $\{x_i\} \times \mathcal{Y}, x_i \in \mathcal{X}$. *Since* u *is the characteristic function of the set* $\{(1, 1), (2, 2)\}$ *that contains none of them* $S_\mu(u) = \mu(\{(x_1, y_1), (x_2, y_2)\}) = 0$.

We have seen that U^{+min}_{post} *is equal to*

$$S_\Pi(S_N(u)) = S_\mu(u) = \max(\min(\min(1, 1), \min(1, 0)), \min(\min(1, 1), \min(0, 1))) = 0,$$

which coincides with value of $S_\mu(u)$.

On the other hand, $U^{+min}_{ante}(\pi, w, u) = 1$ *as already seen.*

Since $U^{+min}_{ante}(\pi, w, u) = S_{N_\mathcal{Y}}(S_{\Pi_\mathcal{X}}(u))$, *and* $U^{+min}_{post} = S_\Pi(S_N(u)) = S_\mu(u))$, *it indicates that it may happen that* $S_\mu(u)) < S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u))$

We can even give an example where two double Sugeno integrals of a Boolean function on $\mathcal{X} \times \mathcal{Y}$ are equal but strictly greater than the corresponding 2D integral, using a separable 2D capacity. but the two double Sugeno integrals of a Boolean function on $\mathcal{X} \times \mathcal{Y}$ are equal and strictly greater than the 2D integral.

Example 8 *Consider* $|\mathcal{X}| = |\mathcal{Y}| = 3$. *Let* $R = \{(x_1, y_3), (x_1, y_2), (x_2, y_2), (x_2, y_1), (x_3, y_1)\}$. *Note that* R *is symmetric. Now consider the 2D Boolean capacity* μ *with the 4 focal sets* $\{x_1, x_2\} \times \{y_1, y_2\}$, $\{x_1, x_2\} \times \{y_2, y_3\}$, $\{x_2, x_3\} \times \{y_2, y_3\}$, $\{x_2, x_3\} \times \{y_1, y_2\}$. *Note that it is separable, and* $(\mu)^\downarrow_\mathcal{X}$ *has focals* $\{x_1, x_2\}$ *and* $\{x_2, x_3\}$, $(\mu)^\downarrow_\mathcal{Y}$ *has focals* $\{y_1, y_2\}$ *and* $\{y_2, y_3\}$. *They are the same capcity on the two sets* $\mathcal{X}$ *and* $\mathcal{Y}$

It is clear that $\mu(R) = 0$ (R contains no focal sets). However, consider the double integrals $S_{(\mu)^{\downarrow}_{\mathcal{X}}}(S_{(\mu)^{\downarrow}_{\mathcal{Y}}}(1_R))$ and $S_{(\mu)^{\downarrow}_{\mathcal{Y}}}(S_{(\mu)^{\downarrow}_{\mathcal{X}}}(1_R))$. They are equal because they are the same capacity and R is symmetric.

Note that $Ry_1 = \{x_2, x_3\}$, $Ry_2 = \{x_1, x_2\}$, $Ry_3 = \{x_1\}$. We have

$$\begin{aligned} S_{(\mu)^{\downarrow}_{\mathcal{Y}}}(S_{(\mu)^{\downarrow}_{\mathcal{X}}}(1_R)) &= \max_{B \in \mathcal{F}(\mathcal{Y})} \min((\mu)^{\downarrow}_{\mathcal{Y}}(B), \min_{y_i \in B} (\mu)^{\downarrow}_{\mathcal{X}}(Ry_i)) \\ &= \max(\min((\mu)^{\downarrow}_{\mathcal{Y}}(\{y_1, y_2\}), \min((\mu)^{\downarrow}_{\mathcal{X}}(\{x_2, x_3\}), (\mu)^{\downarrow}_{\mathcal{X}}(\{x_1, x_2\})), \\ &\qquad \min((\mu)^{\downarrow}_{\mathcal{Y}}(\{y_2, y_3\}), \min((\mu)^{\downarrow}_{\mathcal{X}}(\{x_1, x_2\}), (\mu)^{\downarrow}_{\mathcal{X}}(\{x_1\}))) \\ &= \max(\min(1, \min(1, 1)), \min(1, \min(1, 0))) = 1. \end{aligned}$$

So we have a case where

$$S_{(\mu)^{\downarrow}_{\mathcal{X}}}(S_{(\mu)^{\downarrow}_{\mathcal{Y}}}(1_R)) = S_{(\mu)^{\downarrow}_{\mathcal{Y}}}(S_{(\mu)^{\downarrow}_{\mathcal{X}}}(1_R)) = 1 > \mu(R) = 0.$$

dans l'exemple precedent (ex **??**) , on n'a pas $S_{\mathcal{X}}(S_{\mathcal{Y}})(u) = S_{\mathcal{Y}}(S_{\mathcal{X}})(u)$ pour tout u ; il ne s'oppose pas aux conjectures

5 Commutation of Sugeno integrals

In this section, given two capacities on finite sets $\mu_{\mathcal{X}}$ on $\mathcal{X}$ and $\mu_{\mathcal{Y}}$ on $\mathcal{Y}$, we check for necessary and sufficient conditions under the following identity holds, namely:

$$S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u(x_1, \cdot)), \cdots, S_{\mu_{\mathcal{Y}}}(u(x_n, \cdot))) = S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}((u(\cdot, y_1)), \ldots, S_{\mu_{\mathcal{X}}}(u(\cdot, y_p)))$$

Or for short $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(u)) = S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(u))$.

We then say that the S-integrals *commute* and write $S_{\mu_{\mathcal{Y}}} \perp S_{\mu_{\mathcal{X}}}$. This question can be considered from two points of view: for which functions u do S-integrals commute for all capacities on $\mathcal{X}$ and $\mathcal{Y}$? For which capacities do the S-integrals commute for all functions u ? The first question is considered by Narukawa and Torra [**?**] for more general fuzzy integrals, and the second one by Behrisch et al. [**?**], albeit in the larger setting of distributive lattices, for general lattice polynomials. However it is of interest to prove these results for S-integrals valued on chains, as they become more palatable and an explicit description of capacities ensuring commutation is obtained. In particular the question is whether commutation holds for other pairs of capacities than possibility measures and necessity measures, a case handled in [**?**].

5.1 Necessary and sufficient conditions for commutation

We first show that commutation holds for all functions $u : \mathcal{X} \times \mathcal{Y} \to L$ whatever the capacities if and only if commutation holds for all Boolean-valued functions $u : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$, that is relations $R \subseteq \mathcal{X} \times \mathcal{Y}$.

Proposition 14 *$S_{\mu_\mathcal{Y}} \perp S_{\mu_\mathcal{X}}$ if and only if $\forall R \subseteq \mathcal{X} \times \mathcal{Y}$,*

$$S_{\mu_\mathcal{X}}(\mu_\mathcal{Y}(x_1R), \cdots, \mu_\mathcal{Y}(x_nR)) = S_{\mu_\mathcal{Y}}(\mu_\mathcal{X}(Ry_1), \ldots, \mu_\mathcal{X}(Ry_p))$$

where $x_iR = \{y \in \mathcal{Y} : x_iRy\}$, $Ry_j = \{x \in \mathcal{Y} : xRy_j\}$.

Proof: We have shown above (equation (**??**)) that any double S-integral $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u))$ is actually a 2D S-integral associated to the capacity $\mu_{\mathcal{X}\mathcal{Y}}$ defined by

$$\mu_{\mathcal{X}\mathcal{Y}}(R) = S_{\mu_\mathcal{X}}(\mu_\mathcal{Y}(x_1R), \ldots, \mu_\mathcal{Y}(x_nR))$$

Likewise $S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u))$ is actually a 2D S-integral associated to the capacity $\mu_{\mathcal{Y}\mathcal{X}}$ defined by

$$\mu_{\mathcal{Y}\mathcal{X}}(R) = S_{\mu_\mathcal{Y}}(\mu_\mathcal{X}(Ry_1), \ldots, \mu_\mathcal{X}(Ry_p))$$

So it becomes clear that the equality $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u))$ holds if and only if $S_{\mu_\mathcal{X}}(\mu_\mathcal{Y}(x_1R), \ldots, \mu_\mathcal{Y}(x_nR)) = S_{\mu_\mathcal{Y}}(\mu_\mathcal{X}(Ry_1), \ldots, \mu_\mathcal{X}(Ry_p))$ for all $R \subseteq \mathcal{X} \times \mathcal{Y}$.

□

So if commutation holds for all relations, it holds for all functions and conversely.

Another result worth mentioning is a Fubini theorem for S-integrals, which s a special case of a result proved in :

Proposition 15 *If $R = A \times B$, commutation always holds, i.e.,*

$$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(\mathbf{1}_R)) = \min(\mu_\mathcal{X}(A), \mu_\mathcal{Y}(B))$$

Proof: Indeed it then reads $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = \vee_{S \in \mathcal{X}} \mu_\mathcal{X}(S) \wedge \wedge_{x \in S} \mu_\mathcal{Y}(xR) = \vee_{S \subseteq A} \mu_\mathcal{X}(S) \wedge \wedge_{x \in S} \mu_\mathcal{Y}(B) = \min(\mu_\mathcal{X}(A), \mu_\mathcal{Y}(B))$. □

In that case there is coincidence between $\mu_{\mathcal{X}\mathcal{Y}}$ and $\mu_{\mathcal{Y}\mathcal{X}}$ with the 2D capacity $\mu_\mathcal{X} \wedge \mu_\mathcal{Y}(R)$ on such Cartesian products. As a consequence we have the following decomposability result, where double S-integrals based on marginal capacities are equal to the 2D S-integral:

Corollary 1 *If $u(x, y) = \min(u_\mathcal{X}(x), u_\mathcal{Y}(y))$, commutation holds, i.e.,*

$$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u)) = \min(S_{\mu_\mathcal{X}}(u_\mathcal{X}), S_{\mu_\mathcal{Y}}(u_\mathcal{Y})) = S_{\min(\mu_\mathcal{X}, \mu_\mathcal{Y})}(u).$$

Proof: Note that by assumption $R = \{(x, y) : u(x, y) \geq \lambda\}$ = is of the form, $S_\lambda \times T_\lambda$, where $S_\lambda = \{x : u_\mathcal{X} \geq \lambda\}$ and $T_\lambda = \{y : u_\mathcal{Y} \geq \lambda\}$. Hence

$$\begin{aligned} S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(u)) &= \vee_{\lambda \in L} \min(\mu_\mathcal{X}(S_\lambda), \mu_\mathcal{Y}(T_\lambda), \lambda) \\ &= \min(\vee_{\lambda \in L} \min(\mu_\mathcal{X}(S_\lambda), \lambda), \vee_{\lambda \in L} \min(\mu_\mathcal{Y}(T_\lambda), \lambda)) \\ &= \min(S_{\mu_\mathcal{X}}(u_\mathcal{X}), S_{\mu_\mathcal{Y}}(u_\mathcal{Y}) \end{aligned}$$

□

Finally we shall prove the main theorem of this section, that is

Theorem 1 *$S_{\mu_\mathcal{Y}} \perp S_{\mu_\mathcal{X}}$ if and only if $\forall A_1, A_2 \subseteq \mathcal{X}, \forall B_1, B_2 \subseteq \mathcal{Y}$,:*

$$\mu_\mathcal{X}(A_1 \cap A_2) \vee \mu_\mathcal{Y}(B_1) \vee \mu_\mathcal{Y}(B_2) \geq \mu_\mathcal{X}(A_1)\mu_\mathcal{X}(A_2)\mu_\mathcal{Y}(B_1 \cup B_2)$$
$$\mu_\mathcal{Y}(B_1 \cap B_2) \vee \mu_\mathcal{X}(A_1) \vee \mu_\mathcal{X}(A_2) \geq \mu_\mathcal{Y}(B_1)\mu_\mathcal{Y}(B_2)\mu_\mathcal{X}(A_1 \cup A_2).$$

Proof:[Outline] The proof is inspired by a paper on the commutation of polynomials on distributive lattices [**?**], and requires several lemmas. Our proof is easier to read and simpler, though. First we use Proposition **??** to restrict to Boolean functions (relations R) on $\mathcal{X} \times \mathcal{Y}$ without loss of generality. Then we show that commutation is equivalent to a certain identity for relations R of the form $(A_1 \times B_1) \cup (A_2 \times B_2)$ (Lemma **??**). We show this identity implies the two inequalities of the theorem (Lemmas **??** then **??**), then that these inequalities can be extended to more than just pairs of sets (Lemma **??**). Finally we show that these extended inequalities imply the commutation condition (Lemma **??**). $\square$

First, we prove a counterpart of Lemma 3.4 in [**?**]:

Lemma 1 *$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(\mathbf{1}_R))$ for $R = (A_1 \times B_1) \cup (A_2 \times B_2)$ if and only if the* 2-rectangle condition *holds, i.e.*

$$\mu_\mathcal{X}(A_1 \cap A_2)\mu_\mathcal{Y}(B_1 \cup B_2) \vee \mu_\mathcal{X}(A_1)\mu_\mathcal{Y}(B_1) \vee \mu_\mathcal{X}(A_2)\mu_\mathcal{Y}(B_2) \vee \mu_\mathcal{X}(A_1 \cup A_2)\mu_\mathcal{Y}(B_1)\mu_\mathcal{Y}(B_2)$$
$$=$$
$$\mu_\mathcal{Y}(B_1 \cap B_2)\mu_\mathcal{X}(A_1 \cup A_2) \vee \mu_\mathcal{X}(A_1)\mu_\mathcal{Y}(B_1) \vee \mu_\mathcal{X}(A_2)\mu_\mathcal{Y}(B_2) \vee \mu_\mathcal{Y}(B_1 \cup B_2)\mu_\mathcal{X}(A_1)\mu_\mathcal{X}(A_2)$$

(were we omit the sign $\wedge$, for convenience)

Proof: First note that if $R = (A_1 \times B_1) \cup (A_2 \times B_2)$, then xR is of the form

- $xR = B_1$ if $x \in A_1 \setminus A_2$ and $xR = B_2$ if $x \in A_2 \setminus A_1$
- $xR = B_1 \cup B_2$ if $x \in A_1 \cap A_2$
- $xR = \emptyset$ otherwise.

Now $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = \vee_{S \subseteq \mathcal{X}} \mu_\mathcal{X}(S) \wedge \wedge_{x \in S} \mu_\mathcal{Y}(xR) = \vee_{S \subseteq \mathcal{X}} \phi(S)$ for short. We compute the term $\wedge_{x \in S} \mu_\mathcal{Y}(xR)$ according to the position of S with respect to A_1 and A_2:

- If $S \subseteq A_1 \cap A_2$ then $\wedge_{x \in S} \mu_\mathcal{Y}(xR) = \mu_\mathcal{Y}(B_1 \cup B_2)$ so,
  $\vee_{S \subseteq A_1 \cap A_2} \phi(S) = \mu_\mathcal{X}(A_1 \cap A_2) \wedge \mu_\mathcal{Y}(B_1 \cup B_2)$.
- If $S \subseteq A_1$ and $S \not\subseteq A_2$, then $\wedge_{x \in S} \mu_\mathcal{Y}(xR) = \mu_\mathcal{Y}(B_1)$, so,
  $\vee_{S \subseteq A_1, S \not\subseteq A_2} \phi(S) = \mu_\mathcal{X}(A_1) \wedge \mu_\mathcal{Y}(B_1)$.
- If $S \subseteq A_2$ and $S \not\subseteq A_1$, then $\wedge_{x \in S} \mu_\mathcal{Y}(xR) = \mu_\mathcal{Y}(B_2)$, so,
  $\vee_{S \subseteq A_2, S \not\subseteq A_1} \phi(S) = \mu_\mathcal{X}(A_2) \wedge \mu_\mathcal{Y}(B_2)$.

- If $S \subseteq A_1 \cup A_2$ and $S \nsubseteq A_1, S \nsubseteq A_2$, then $\wedge_{x \in S} \mu_{\mathcal{Y}}(xR) = \mu_{\mathcal{Y}}(B_1) \wedge \mu_{\mathcal{Y}}(B_2)$, so, $\vee_{S \subseteq A_1 \cup A_2, S \nsubseteq A_1, S \nsubseteq A_2} \phi(S) = \mu_{\mathcal{X}}(A_1 \cup A_2) \wedge \mu_{\mathcal{Y}}(B_1) \wedge \mu_{\mathcal{Y}}(B_2)$.

- If $S \nsubseteq A_1 \cup A_2$, then $\wedge_{x \in S} \mu_{\mathcal{Y}}(xR) = 0$ and $\vee_{S \nsubseteq A_1 \cup A_2} \phi(S) = 0$.

This is the left-hand side of the 2-rectangle condition. The other side is obtained in the same way when computing $S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(\mathbf{1}_R))$. □

The following is a counterpart of Lemma 3.6 in [**?**]

Lemma 2 *The 2-rectangle condition of Lemma* **??** *implies the two following properties*

$$\mu_{\mathcal{X}}(A_1 \cap A_2)\mu_{\mathcal{Y}}(B_1 \cup B_2) \vee [\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)(\mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{Y}}(B_2))] = \mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)\mu_{\mathcal{Y}}(B_1 \cup B_2)$$
$$\mu_{\mathcal{Y}}(B_1 \cap B_2)\mu_{\mathcal{X}}(A_1 \cup A_2) \vee [\mu_{\mathcal{Y}}(B_1)\mu_{\mathcal{Y}}(B_2)(\mu_{\mathcal{X}}(A_1) \vee \mu_{\mathcal{X}}(A_2))] = \mu_{\mathcal{Y}}(B_1)\mu_{\mathcal{Y}}(B_2)\mu_{\mathcal{X}}(A_1 \cup A_2).$$

Proof: To get the first equality the idea (from [**?**]) is to compute the conjunction of each side of the 2-rectangle condition with $\mu_{\mathcal{X}}(A_1) \wedge \mu_{\mathcal{X}}(A_2)$ (applying distributivity). Indeed consider each factor of the 2-rectangle condition conjuncted with $\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)$ (omitting $\wedge$):

- On the left-hand side, we get $\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)\mu_{\mathcal{X}}(A_1 \cap A_2)\mu_{\mathcal{Y}}(B_1 \cup B_2)$ which is $\mu_{\mathcal{X}}(A_1 \cap A_2)\mu_{\mathcal{Y}}(B_1 \cup B_2)$ since $\mu_{\mathcal{X}}(A_1 \cap A_2) \le \mu_{\mathcal{X}}(A_1) \wedge \mu_{\mathcal{X}}(A_2)$ by monotonicity of $\mu_{\mathcal{X}}$.

- The second term becomes $\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)\mu_{\mathcal{Y}}(B_1)$.

- The third term becomes $\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)\mu_{\mathcal{Y}}(B_2)$.

- The fourth term becomes $\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)\mu_{\mathcal{X}}(A_1 \cup A_2)\mu_{\mathcal{Y}}(B_1)\mu_{\mathcal{Y}}(B_2)$ equal to $\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)\mu_{\mathcal{Y}}(B_1)\mu_{\mathcal{Y}}(B_2)$ from monotonicity again, but this term is subsumed via disjunction with the above second and third terms.

- The right hand side of the 2-rectangle condition is handled similarly. The first term becomes $\mu_{\mathcal{Y}}(B_1 \cap B_2)\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)$ due to monotonicity again.

- The second and third terms on the right-hand side are the same as in the left-hand side, but here they subsumed by the first term.

- The last term remains the same, i.e., $\mu_{\mathcal{Y}}(B_1 \cup B_2)\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)$ but it subsumes the first term.

We thus get the first equality. The second equality is obtained likewise, by conjunction of each side of the equality with the term $\mu_{\mathcal{Y}}(B_1) \wedge \mu_{\mathcal{Y}}(B_2)$. □

The following lemma simplifies the two obtained equalities into simpler inequalities.

Lemma 3 *The two equalities in Lemma **??** are equivalent to the two inequalities*

$$\mu_{\mathcal{X}}(A_1 \cap A_2) \vee \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{Y}}(B_2) \geq \mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)\mu_{\mathcal{Y}}(B_1 \cup B_2) \tag{7}$$

$$\mu_{\mathcal{Y}}(B_1 \cap B_2) \vee \mu_{\mathcal{X}}(A_1) \vee \mu_{\mathcal{X}}(A_2) \geq \mu_{\mathcal{Y}}(B_1)\mu_{\mathcal{Y}}(B_2)\mu_{\mathcal{X}}(A_1 \cup A_2). \tag{8}$$

Proof: Let us apply distributivity to the right-hand side of the first equality in Lemma **??**:
$[\mu_{\mathcal{X}}(A_1 \cap A_2) \wedge \mu_{\mathcal{Y}}(B_1 \cup B_2)] \vee [(\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)) \wedge (\mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{Y}}(B_2))]$. We get a conjunction of disjunctive terms of the form

- $\mu_{\mathcal{X}}(A_1 \cap A_2) \vee (\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)) = \mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)$ (monotonicity).
- $\mu_{\mathcal{X}}(A_1 \cap A_2) \vee \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{Y}}(B_2)$
- $\mu_{\mathcal{Y}}(B_1 \cup B_2) \vee \mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)$
- $\mu_{\mathcal{Y}}(B_1 \cup B_2) \vee \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{Y}}(B_2) = \mu_{\mathcal{Y}}(B_1 \cup B_2)$ (monotonicity).

It is clear that the conjunction of these terms absorbs the third term, and the first equality in Lemma **??** reduces to the equality
$\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)\mu_{\mathcal{Y}}(B_1 \cup B_2)(\mu_{\mathcal{X}}(A_1 \cap A_2) \vee \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{Y}}(B_2)) = \mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)\mu_{\mathcal{Y}}(B_1 \cup B_2)$, which is equivalent to the inequality

$$\mu_{\mathcal{X}}(A_1 \cap A_2) \vee \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{Y}}(B_2) \geq \mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)\mu_{\mathcal{Y}}(B_1 \cup B_2).$$

The second inequality is proved likewise, exchanging A and B, $\mathcal{X}$ and $\mathcal{Y}$. $\square$

The two inequalities (**??**) and (**??**) extend to more than two pairs of sets, namely:

Lemma 4 *(**??**) and (**??**) imply:*

$$\mu_{\mathcal{X}}(\cap_{i=1}^{k} A_i) \vee \vee_{j=1}^{\ell} \mu_{\mathcal{Y}}(B_j) \geq \wedge_{i=1}^{k} \mu_{\mathcal{X}}(A_i) \wedge \mu_{\mathcal{Y}}(\cup_{j=1}^{\ell} B_j) \tag{9}$$

$$\mu_{\mathcal{Y}}(\cap_{j=1}^{\ell} B_j) \vee \vee_{i=1}^{k} \mu_{\mathcal{X}}(A_i) \geq \wedge_{j=1}^{\ell} \mu_{\mathcal{Y}}(B_j) \wedge \mu_{\mathcal{X}}(\cup_{i=1}^{k} A_i). \tag{10}$$

Proof: Inequality (**??**) holds for $k = \ell = 2$ (this is (**??**)). Suppose that inequality (**??**) holds for $i = 1, \ldots k-1$ and $\ell = 2$. We can write, by assumption:

$$\mu_{\mathcal{X}}(\cap_{i=1}^{k-1} A_i) \vee \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{Y}}(B_2) \geq \wedge_{i=1}^{k-1} \mu_{\mathcal{X}}(A_i) \wedge \mu_{\mathcal{Y}}(B_1 \cup B_2)$$

Moreover we can write (**??**) for $A = \cap_{i=1}^{k-1} A_i, A_k, B_1, B_2$. So we can write the inequality

$$\mu_{\mathcal{X}}(\cap_{i=1}^{k} A_i) \vee \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{Y}}(B_2) \geq \mu_{\mathcal{X}}(\cap_{i=1}^{k-1} A_i) \wedge \mu_{\mathcal{X}}(A_k) \wedge \mu_{\mathcal{Y}}(B_1 \cup B_2)$$

Suppose $\mu_{\mathcal{X}}(\cap_{i=1}^{k-1} A_i) \geq \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{Y}}(B_2)$. So the first inequality reduces to

$$\mu_{\mathcal{X}}(\cap_{i=1}^{k-1} A_i) \ \geq \wedge_{i=1}^{k-1} \mu_{\mathcal{X}}(A_i) \wedge \mu_{\mathcal{Y}}(B_1 \cup B_2).$$

Then we can replace $\mu_\mathcal{X}(\cap_{i=1}^{k-1} A_i)$ by $\wedge_{i=1}^{k-1}\mu_\mathcal{X}(A_i) \wedge \mu_\mathcal{Y}(B_1 \cup B_2)$ in the second inequality, and get (**??**). Otherwise, $\mu_\mathcal{X}(\cap_{i=1}^{k-1} A_i) \leq \mu_\mathcal{Y}(B_1) \vee \mu_\mathcal{Y}(B_2)$, and the first inequality reads

$$\mu_\mathcal{Y}(B_1) \vee \mu_\mathcal{Y}(B_2) \geq \wedge_{i=1}^{k-1}\mu_\mathcal{X}(A_i) \wedge \mu_\mathcal{Y}(B_1 \cup B_2)$$

so we have

$$\mu_\mathcal{X}(\cap_{i=1}^{k} A_i) \vee \mu_\mathcal{Y}(B_1) \vee \mu_\mathcal{Y}(B_2) = \mu_\mathcal{Y}(B_1) \vee \mu_\mathcal{Y}(B_2) \geq (\wedge_{i=1}^{k-1}\mu_\mathcal{X}(A_i)) \wedge \mu_\mathcal{X}(A_k) \wedge \mu_\mathcal{Y}(B_1 \cup B_2),$$

which is (**??**) again. Proving that the inequality (**??**) holds for $k = 2, \ell > 2$ is similar. We can write (**??**) for $B = \cap_{i=1}^{\ell-1} B_j, B_\ell, A_1, A_2$. So we can write the two inequalities

$$\mu_\mathcal{X}(A_1 \cap A_2) \vee \vee_{j=1}^{\ell-1}\mu_\mathcal{Y}(B_j) \geq \mu_\mathcal{X}(A_1)\mu_\mathcal{X}(A_2)\mu_\mathcal{Y}(\cup_{j=1}^{\ell-1} B_j)$$
$$\mu_\mathcal{X}(A_1 \cap A_2) \vee \mu_\mathcal{Y}(\cup_{j=1}^{\ell-1} B_j) \vee \mu_\mathcal{Y}(B_\ell) \geq \mu_\mathcal{X}(A_1)\mu_\mathcal{X}(A_2)\mu_\mathcal{Y}(\cup_{j=1}^{\ell} B_j)$$

If $\mu_\mathcal{Y}(\cup_{j=1}^{\ell-1} B_j) \leq \mu_\mathcal{X}(A_1)\mu_\mathcal{X}(A_2)$ then the first inequality reduces to $\mu_\mathcal{X}(A_1 \cap A_2) \vee \vee_{j=1}^{\ell-1}\mu_\mathcal{Y}(B_j) \geq \mu_\mathcal{Y}(\cup_{j=1}^{\ell-1} B_j)$, and replacing the latter term by its upper bound in the second inequality gives the expected result. If $\mu_\mathcal{Y}(\cup_{j=1}^{\ell-1} B_j) > \mu_\mathcal{X}(A_1)\mu_\mathcal{X}(A_2)$ then the first inequality reduces to

$$\mu_\mathcal{X}(A_1 \cap A_2) \vee \vee_{j=1}^{\ell-1}\mu_\mathcal{Y}(B_j) \geq \mu_\mathcal{X}(A_1)\mu_\mathcal{X}(A_2),$$

which implies (**??**), adding the disjunction with term $\mu_\mathcal{Y}(B_\ell)$ on the left-hand side, and conjuncting with term $\mu_\mathcal{Y}(\cup_{j=1}^{\ell} B_j)$ on the right-hand side. So, the inequality (**??**) holds for any $k > 2, \ell > 2$. The inequality (**??**) is proved in a similar way, exchanging A and B, X and Y $\square$

Lemma 5 *If $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ satisfy the two inequalities (**??**) and (**??**), then $S_{\mu_\mathcal{X}} \perp S_{\mu_\mathcal{Y}}$*

Proof: First notice that the inequalities (**??**) and (**??**) can be written in the style of Lemma **??**. The first one reads:

$$[\mu_\mathcal{X}(\cap_{i=1}^{k} A_i)\mu_\mathcal{Y}(\cup_{j=1}^{\ell} B_j)] \vee [\wedge_{i=1}^{k}\mu_\mathcal{X}(A_i) \vee_{j=1}^{\ell} \mu_\mathcal{Y}(B_j)] = \wedge_{i=1}^{k}\mu_\mathcal{X}(A_i)\mu_\mathcal{Y}(\cup_{j=1}^{\ell} B_j).$$

We have to prove that

$$\vee_{S \subseteq \mathcal{X}}\mu_\mathcal{X}(S) \wedge \wedge_{x \in S}\mu_\mathcal{Y}(xR)) \geq \vee_{T \subseteq \mathcal{Y}}\mu_\mathcal{Y}(T) \wedge \wedge_{y \in T}\mu_\mathcal{X}(Ry)).$$

Consider the term $\mu_\mathcal{Y}(T) \wedge \wedge_{y \in T}\mu_\mathcal{X}(Ry))$ that we identify with the right-hand side of the above equality. This equality then reads:

$$\begin{aligned}\mu_\mathcal{Y}(T) \wedge \wedge_{y \in T}\mu_\mathcal{X}(Ry) &= [\mu_\mathcal{X}(S_T)\mu_\mathcal{Y}(T)] \vee [\wedge_{y \in T}\mu_\mathcal{X}(Ry) \wedge \vee_{t \in T}\mu_\mathcal{Y}(\{t\})] \\ &= [\mu_\mathcal{X}(S_T)\mu_\mathcal{Y}(T)] \vee \vee_{t \in T}[\wedge_{y \in T}\mu_\mathcal{X}(Ry) \wedge \mu_\mathcal{Y}(\{t\})].\end{aligned}$$

where $S_T = \cap_{y \in T} Ry$. Now we can prove that

- $\mu_{\mathcal{Y}}(T) \leq \wedge_{x\in S_T}\mu_{\mathcal{Y}}(xR)$. Indeed $S_T = \cap_{y\in T}Ry$ if and only if $S_T \times T \subseteq R$ if and only if $T = \cap_{x\in S_T}xR$. So, the term $\mu_{\mathcal{X}}(S_T)\mu_{\mathcal{Y}}(T)$ is upper bounded by $\vee_{S\subseteq\mathcal{X}}\mu_{\mathcal{X}}(S) \wedge \wedge_{x\in S}\mu_{\mathcal{Y}}(xR))$.

- The same holds for the term $\wedge_{y\in T}\mu_{\mathcal{X}}(Ry) \wedge \mu_{\mathcal{Y}}(\{t\})$. Indeed

  - as $t \in T$, $\wedge_{y\in T}\mu_{\mathcal{X}}(Ry) \leq \mu_{\mathcal{X}}(Rt)$, choosing $y = t$.
  - Let $x \in Rt$. Then $\mu_{\mathcal{Y}}(\{t\}) \leq \mu_{\mathcal{Y}}(xR)$ since $t \in xR$ as well.

  So, $\wedge_{y\in T}\mu_{\mathcal{X}}(Ry)\wedge\mu_{\mathcal{Y}}(\{t\}) \leq \mu_{\mathcal{X}}(Rt)\wedge\mu_{\mathcal{Y}}(xR), \forall x \in Rt$. So, $\wedge_{y\in T}\mu_{\mathcal{X}}(Ry)\wedge\mu_{\mathcal{Y}}(\{t\}) \leq \mu_{\mathcal{X}}(Rt) \wedge \wedge_{x\in Rt}\mu_{\mathcal{Y}}(xR)$ that is also upper bounded by $\vee_{S\subseteq\mathcal{X}}\mu_{\mathcal{X}}(S) \wedge \wedge_{x\in S}\mu_{\mathcal{Y}}(xR))$. We thus get $S_{\mu_{\mathcal{X}}}(S_{\mu_{\mathcal{Y}}}(\mathbf{1}_R)) \geq S_{\mu_{\mathcal{Y}}}(S_{\mu_{\mathcal{X}}}(\mathbf{1}_R))$

the converse inequality $\vee_{S\subseteq\mathcal{X}}\mu_{\mathcal{X}}(S) \wedge \wedge_{x\in S}\mu_{\mathcal{Y}}(xR)) \leq \vee_{T\subseteq\mathcal{Y}}\mu_{\mathcal{Y}}(T) \wedge \wedge_{y\in T}\mu_{\mathcal{X}}(Ry))$ can be proved likewise, by symmetry, using (**??**). □

The above Lemmas yield a necessary and sufficient condition expressed in Theorem 1 for the commutation of two S-integrals applied to any function $u : \mathcal{X} \times \mathcal{Y} \to L$, and based on capacities $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$. As these S-integrals are entirely characterized by these capacities, we will say that the two capacities commute.

5.2 Commuting capacities

In this subsection, we try to characterize all pairs of commuting capacities. We already know that two possibility measures, as well as two necessity measures commute, while a possibility measure does not commute with a necessity measure.

The case of Boolean capacities is of interest as it will be instrumental to address the general case:

Lemma 6 *If $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are Boolean capacities and the two inequalities (**??**) and (**??**) hold for all $A_1, A_2 \subseteq \mathcal{X}$ and for all $B_1, B_2 \subseteq \mathcal{Y}$, then $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are are both necessity measures or possibility measures or one of them is a Dirac measure.*

Proof: Suppose $\mu_{\mathcal{X}}$ is not a necessity measure and $\mu_{\mathcal{Y}}$ is not a possibility measure. Then $\exists A_1, A_2 \subseteq \mathcal{X}, \mu_{\mathcal{X}}(A_1 \cap A_2) < \mu_{\mathcal{X}}(A_1) \wedge \mu_{\mathcal{X}}(A_2)$, and $\exists B_1, B_2 \subseteq \mathcal{Y}, \mu_{\mathcal{Y}}(B_1 \cup B_2) > \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{X}}(B_2)$. In the Boolean case it reads $\mu_{\mathcal{X}}(A_1 \cap A_2) = 0, \mu_{\mathcal{X}}(A_1) = \mu_{\mathcal{X}}(A_2) = 1, \mu_{\mathcal{Y}}(B_1 \cup B_2) = 1, \mu_{\mathcal{Y}}(B_1) = \mu_{\mathcal{Y}}(B_2) = 0$. Then inequality (**??**) is violated as $\mu_{\mathcal{X}}(A_1 \cap A_2) \vee \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{Y}}(B_2) = 0$ and $\mu_{\mathcal{X}}(A_1)\mu_{\mathcal{X}}(A_2)\mu_{\mathcal{Y}}(B_1 \cup B_2) = 1$. The second inequality (**??**) is violated by choosing $A_1, A_2 \subseteq \mathcal{X}, B_1, B_2 \subseteq \mathcal{Y}$, such that $\mu_{\mathcal{Y}}(B_1 \cap B_2) = 0, \mu_{\mathcal{Y}}(B_1) = \mu_{\mathcal{Y}}(B_2) = 1, \mu_{\mathcal{X}}(A_1 \cup A_2) = 1\mu_{\mathcal{X}}(A_1) = \mu_{\mathcal{X}}(A_2) = 0$, assuming $\mu_{\mathcal{Y}}$ is not a necessity measure and $\mu_{\mathcal{X}}$ is not a possibility measure. Obeying the two inequalities (**??**) and (**??**) enforces the following constraints in the Boolean case

$$\mu_{\mathcal{Y}} \text{ possibility measure or } \mu_{\mathcal{X}} \text{ necessity measure}$$
$$\text{and}$$
$$\mu_{\mathcal{Y}} \text{ necessity measure or } \mu_{\mathcal{X}} \text{ possibility measure}$$

It leads to possibility measures on both sets $\mathcal{X}$ and $\mathcal{Y}$, or necessity measures (known cases where commuting occurs). Alternatively, if we enforce $\mu_{\mathcal{Y}}$ to be a possibility measure and a necessity measure, it is a Dirac function on $\mathcal{Y}$, and any capacity on the other space. □

Corollary 2 *S-integrals wrt Boolean capacities $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ commute if and only if they are both necessity measures or possibility measures or one of them is a Dirac measure.*

We can strengthen the result to the case when only one of the capacities is Boolean:

Proposition 16 *If one of $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ is boolean, S-integrals commute if and only if they are both necessity measures or possibility measures or one of them is a Dirac measure.*

Proof: Suppose $\mu_{\mathcal{X}}$ is Boolean and is not a necessity measure and $\mu_{\mathcal{Y}}$ is not a possibility measure. Then $\exists A_1, A_2 \subseteq \mathcal{X}, \mu_{\mathcal{X}}(A_1 \cap A_2) < \mu_{\mathcal{X}}(A_1) \wedge \mu_{\mathcal{X}}(A_2)$, and $\exists B_1, B_2 \subseteq \mathcal{Y}, \mu_{\mathcal{Y}}(B_1 \cup B_2) > \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{X}}(B_2)$. For $\mu_{\mathcal{X}}$, it reads $\mu_{\mathcal{X}}(A_1 \cap A_2) = 0, \mu_{\mathcal{X}}(A_1) = \mu_{\mathcal{X}}(A_2) = 1$. Then the 2-rectangle condition fails since it reads $0 \vee \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{X}}(B_2) \vee \mu_{\mathcal{Y}}(B_1) \wedge \mu_{\mathcal{X}}(B_2) = \mu_{\mathcal{X}}(B_2) \vee \mu_{\mathcal{Y}}(B_1) < \mu_{\mathcal{Y}}(B_1 \cap B_2) \vee \mu_{\mathcal{Y}}(B_1) \vee \mu_{\mathcal{X}}(B_2) \vee \mu_{\mathcal{Y}}(B_1 \cup B_2) = \mu_{\mathcal{Y}}(B_1 \cup B_2)$. The rest of the reasoning is as above. □

Note that to violate (**??**) it is enough that neither $\mu_{\mathcal{X}}$ nor $\mu_{\mathcal{Y}}$ are possibility and necessity measures, and moreover for A_1, A_2, B_1, B_2 where, say $\mu_{\mathcal{X}}$ violates the axiom of necessities and $\mu_{\mathcal{Y}}$ violates the axiom of possibilities, we have that $\mu_{\mathcal{X}}(A_1)$ and $\mu_{\mathcal{X}}(A_2)$ are both greater than each of $\mu_{\mathcal{Y}}(B_1), \mu_{\mathcal{Y}}(B_2)$ and moreover $\mu_{\mathcal{Y}}(B_1 \cup B_2) > \mu_{\mathcal{Y}}(A_1 \cap A_2)$. Then the integrals will not commute.

In the non-Boolean case, we can give examples of commuting capacities that are neither only possibility measures, nor only necessity measures nor a Dirac function contrary to the Boolean case of Corollary **??**. Suppose $\mu_{\mathcal{Y}}$ is a possibility measure. Then inequality (**??**) trivially holds. As $\mu_{\mathcal{Y}}$ is a possibility measure, we have $\mu_{\mathcal{Y}}(B_1) = \mu_{\mathcal{Y}}(y_1)$ for $y_1 \in B_1$ and $y_2 \in B_2$, and the other inequality reads

$$\forall y_1 \neq y_2 \in \mathcal{Y}, \mu_{\mathcal{X}}(A_1) \vee \mu_{\mathcal{X}}(A_2) \geq \mu_{\mathcal{Y}}(y_1)\mu_{\mathcal{Y}}(y_2)\mu_{\mathcal{X}}(A_1 \cup A_2).$$

as $\mu_{\mathcal{Y}}(B_1 \cap B_2) = 0$ in this case (when $y_1 = y_2$ then the inequality $\mu_{\mathcal{Y}}(y_1) \vee \mu_{\mathcal{X}}(A_1) \vee \mu_{\mathcal{X}}(A_2) \geq \mu_{\mathcal{Y}}(y_1) \wedge \mu_{\mathcal{X}}(A_1 \cup A_2)$. is trivial). The most demanding case is when $\mu_{\mathcal{Y}}(y_1) = 1$ and $\mu_{\mathcal{Y}}(y_2)$ is the possibility degree π_2 of the second most plausible element in $\mu_{\mathcal{Y}}$. It is then equivalent to $\mu_{\mathcal{X}}(A_1) \vee \mu_{\mathcal{X}}(A_2) \geq \pi_2 \wedge \mu_{\mathcal{X}}(A_1 \cup A_2)$, which enforces a possibility measure for $\mu_{\mathcal{X}}$ only if $\pi_2 \geq \mu_{\mathcal{X}}(A)$ for all $A \subseteq \mathcal{X}$, namely $\pi_2 = 1$.

Example 9 *Let $\mathcal{X} = \{x_1, x_2\}; \mathcal{Y} = \{y_1, y_2\}$. Then let $\mu_\mathcal{X}(x_1) = \alpha, \mu_\mathcal{X}(x_2) = \alpha$, $\mu_\mathcal{Y}(y_1) = 1, \mu_\mathcal{Y}(y_2) = \alpha$, so a constant capacity and a possibility measure.*

We can check that

- $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = \mu_\mathcal{X}(x_1)\mu_\mathcal{Y}(x_1R) \vee \mu_\mathcal{X}(2)\mu_\mathcal{Y}(x_2R) \vee \mu_\mathcal{Y}(x_1R)\mu_\mathcal{Y}(x_2R)$
- $S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}\mathbf{1}_R)) = \mu_\mathcal{Y}(y_1)\mu_\mathcal{X}(Ry_1)) \vee \mu_\mathcal{Y}(y_2)\mu_\mathcal{Y}(Ry_2) \vee \mu_\mathcal{Y}(Ry_1)\mu_\mathcal{Y}(Ry_2)$

For instance: $R = \{(x_1, y_1), (x_2, y_2)\}$ then
$\mu_\mathcal{X}(Ry_1) = \alpha, \mu_\mathcal{X}(Ry_2) = \alpha$ so you get $S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(\mathbf{1}_R)) = \alpha$.
$\mu_\mathcal{Y}(x_1R) = 1, \mu_\mathcal{Y}(x_2R) = \alpha$ so you get $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = \alpha$. Likewise if $R = \{(x_1, y_1), (x_1, y_2), (x_2, y_1)\}$, then

- $\mu_\mathcal{X}(Ry_1) = \mu_\mathcal{X}(\mathcal{X}) = 1; \mu_\mathcal{X}(Ry_2)) = \mu_\mathcal{X}(\{x_1\}) = \alpha$;
- $\mu_\mathcal{Y}(x_1R)\mu_\mathcal{Y}(\mathcal{Y}) = 1 = \mu_\mathcal{Y}(x_2R) = \mu_\mathcal{Y}(\{y_1\})$

Then
$S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = (\alpha \wedge 1) \vee (\alpha \wedge 1) \vee (1 \vee 1) = 1$
$S_{\mu_\mathcal{Y}}(S_{\mu_\mathcal{X}}(\mathbf{1}_R)) = (1 \wedge 1) \vee (\alpha) \wedge \alpha \vee (1 \wedge \alpha) = 1$.
Note that we do have that $\mu_\mathcal{X}(A_1) \vee \mu_\mathcal{X}(A_2) \geq \alpha \wedge \mu_\mathcal{X}(A_1 \cup A_2)$, as $\mu_\mathcal{X}(A_1) \wedge \mu_\mathcal{X}(A_2) \geq \alpha$.
So the commutation is expected in this case. □

In the following we lay bare the pairs of capacities that commute by applying the result of Corollary **??** to cuts of the capacities. We first prove that for Boolean functions on $\mathcal{X} \times \mathcal{Y}$, the double S-integrals are completely defined by the cuts of the involved capacities, thus generalizing Proposition **??** to double S-integrals.

Proposition 17 $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(u)) = \max_{\lambda>0} \min(\lambda, S_{\mu_{\mathcal{X}\lambda}}(S_{\mu_{\mathcal{Y}\lambda}}(u)))$ *when $u = \mathbf{1}_R$.*

Proof:: For simplicity we denote $\mu_\mathcal{X}$ by μ and $\mu_\mathcal{Y}$ by ν

$$\begin{aligned} S_\mu(S_\nu(u)) &= \max_{A \subseteq \mathcal{X}} \min(\mu(A), \min_{x \in A} S_\nu(u(x, \cdot))) \\ &= \max_{A \subseteq \mathcal{X}} \min(\max_{\lambda>0} \min(\lambda, \mu_\lambda(A)), \min_{x \in A} \max_{\alpha>0} \min(\alpha, S_{\nu_\alpha}(u(x, \cdot)))) \end{aligned}$$

Note that

$$\min_{x \in A} \max_{\alpha>0} \min(\alpha, S_{\nu_\alpha}(u(x, \cdot))) \geq \max_{\alpha>0} \min_{x \in A} \min(\alpha, S_{\nu_\alpha}(u(x, \cdot))).$$

Then we have

$$
\begin{aligned}
S_\mu(S_\nu(u)) &\geq \max_{A\subseteq\mathcal{X}} \min(\max_{\lambda>0}\min(\lambda,\mu_\lambda(A)), \max_{\alpha>0}\min_{x\in A}\min(\alpha, S_{\nu_\alpha}(u(x,\cdot))))\\
&= \max_{\lambda>0}\max_{A\subseteq\mathcal{X}} \min(\min(\lambda,\mu_\lambda(A)), \max_{\alpha>0}\min_{x\in A}\min(\alpha, S_{\nu_\alpha}(u(x,\cdot))))\\
&= \max_{\lambda>0}\min(\lambda, \max_{A\subseteq\mathcal{X}}\min(\mu_\lambda(A)), \max_{\alpha>0}\min(\alpha, \min_{x\in A} S_{\nu_\alpha}(u(x,\cdot))))\\
&= \max_{\lambda>0,\alpha>0}\min(\lambda,\alpha, \max_{A\subseteq\mathcal{X}}\min(\mu_\lambda(A), \min_{x\in A} S_{\nu_\alpha}(u(x,\cdot))))\\
&= \max_{\lambda>0,\alpha>0}\min(\lambda,\alpha, S_{\mu_\lambda}(S_{\nu_\alpha}(u))
\end{aligned}
$$

Suppose the maximum is attained for $\alpha^* \neq \lambda^*$ then since $\alpha \geq \beta \Rightarrow \nu_\alpha(A) \leq \nu_\beta(A)$, decreasing α to $\min(\alpha^*,\lambda^*)$ will increase $S_{\nu_\alpha}(u(x,\cdot))$, and decreasing λ to $\min(\alpha^*,\lambda^*)$ will increase $S_{\mu_\lambda}(S_{\nu_\alpha}(u))$. So we can assume $\alpha^* = \lambda^*$. Let us restrict to the case when $u(x,y)$ is Boolean and is thus a relation $R \subseteq \mathcal{X}\times\mathcal{Y}$. In this case $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = \max_{A\subseteq\mathcal{X}}\min(\mu_\mathcal{X}(A), \min_{x\in A}\mu_\mathcal{Y}(xR))$ and we are led to study conditions for the equality $\min_{x\in A}\max_{\alpha>0}\min(\alpha,\nu_\alpha(xR)) = \max_{\alpha>0}\min_{x\in A}\min(\alpha,\nu_\alpha(xR))$.

Let $\alpha^*, \hat{x}$ be optima for $\min(\alpha,\nu_\alpha(xR))$ on the right hand side, that is,

$$\max_{\alpha>0}\min_{x\in A}\min(\alpha,\nu_\alpha(xR)) = \min(\alpha^*,\nu_\alpha(\hat{x}R)).$$

It means that $\forall x\in A, \nu_{\alpha^*}(xR) = 1$, and due to monotonicity, $\forall x\in A, \forall\alpha\leq\alpha^*, \nu_\alpha(xR) = 1$. However, $\forall\alpha>\alpha^*, \exists x\in A, \nu_\alpha(xR) = 0$.

Hence $\min_{x\in A}\max_{\alpha>0}\min(\alpha,\nu_\alpha(xR)) = \min_{x\in A}\max_{\alpha^*\geq\alpha>0}\min(\alpha,\nu_\alpha(xR)) \leq \alpha^*$.

So $\min_{x\in A}\max_{\alpha>0}\min(\alpha,\nu_\alpha(xR)) \leq \max_{\alpha>0}\min_{x\in A}\min(\alpha,\nu_\alpha(xR))$, and we get the equality since we already have the converse inequality. □

We know that commutation between integrals holds for functions $u(x,y)$ if it holds for relations. The above result shows that commutation between capacities will hold if and only if it will hold for their cuts, to which we can apply Corollary **??**.

Corollary 3 *Capacities $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ commute if and only if their cuts $\mu_{\mathcal{X}\lambda}$ and $\mu_{\mathcal{Y}\lambda}$ commute for all $\lambda\in L$.*

This is a clear consequence of Proposition **??** since $S_{\mu_{\mathcal{X}\lambda}}(S_{\mu_{\mathcal{Y}\lambda}}(\mathbf{1}_R)) = S_{\mu_{\mathcal{Y}\lambda}}(S_{\mu_{\mathcal{X}\lambda}}(\mathbf{1}_R))$ for all $\lambda\in L$ and $R\subseteq\mathcal{X}\times\mathcal{Y}$ is equivalent to $S_{\mu_\mathcal{X}}(S_{\mu_\mathcal{Y}}(\mathbf{1}_R)) = S_{\mu_\mathcal{Y}}(S_{\mu_{\mathcal{X}\lambda}}(\mathbf{1}_R))$ for all $R\subseteq\mathcal{X}\times\mathcal{Y}$, which by Proposition **??**, is equivalent to commutation of S-integrals w.r.t. $\mu_\mathcal{X}$ and $\mu_\mathcal{Y}$ for all 2-place functions u.

For instance, in the above example the commutation is obvious because the 1-cut of $\mu_\mathcal{X}$ is a necessity measure (with focal set $\mathcal{X}$) and $\mu_\mathcal{Y}$ is a Dirac function on y_1. And the α-cut of $\mu_\mathcal{X}$ is the vacuous possibility measure, as well as the α-cut of $\mu_\mathcal{Y}$. More generally we can claim that

Corollary 4 *Capacities $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ commute if and only if each their cuts $\mu_{\mathcal{X}\lambda}$ and $\mu_{\mathcal{Y}\lambda}$ are two possibility measures, two necessity measures, or one of them is a Dirac measure, for each $\lambda \in L$.*

It is interesting to define the focal sets of the cuts of a capacity μ on $\mathcal{X}$. Namely:

Proposition 18 *If $\mathcal{F}(\mu)$ denote the focal sets of μ, then the focal sets of μ_λ from the family*

$$\mathcal{F}(\mu_\lambda) = \min_{\subseteq}\{E \subseteq \mathcal{X} : \mu_{\#}(E) \geq \lambda\}$$

that is the smallest sets for inclusion in the family of focal sets of μ with weight at least λ.

Indeed the focal sets of a Boolean capacity form an antichain, that is, they are not nested, and if $\mu_{\#}(E) > \mu_{\#}(F) \geq \lambda$, while $F \subset E$, then E is not focal for μ_λ.

Proposition 19 *For any capacity μ on $\mathcal{X}$,*

1. *μ_λ is a necessity measure if and only if there is a single focal set E with $\mu_{\#}(E) \geq \lambda$ such that for all focal sets F in $\mathcal{F}(\mu)$ with weights $\mu_{\#}(F) \geq \lambda$, we have $E \subset F$.*

2. *μ_λ is a possibility measure if and only if there is a set S of singletons with $\mu_{\#}(x_i) \geq \lambda$ such that for all focal sets F in $\mathcal{F}(\mu)$ with weights $\mu_{\#}(F) \geq \lambda$, we have $S \cap F \neq \emptyset$.*

3. *μ_λ is a Dirac measure if and only if there is a focal singleton $\{x\}$ with $\mu_{\#}(\{x\}) \geq \lambda$ such that for all focal sets F in $\mathcal{F}(\mu)$ with weights $\mu_{\#}(F) \geq \lambda$, we have $x \in F$.*

Proof:

1. The condition does ensure that E is the only focal set of μ_λ hence it is a necessity measure. If the condition does not hold it is clear that μ_λ has more than one focal set, hence is a not a necessity measure.

2. The condition does ensure that the focal sets of μ_λ are the singletons in S, hence it is a possibility measure. If the condition does not hold it is clear that μ_λ has a focal set that is not a singleton, hence is not a possibility measure.

3. The condition implies that μ_λ is both a possibility and a necessity measure, hence a Dirac measure. If it is not satisfied, either μ_λ has more than one focal set or its focal set is not a singleton.

$\square$

Note that if μ_λ is a possibility measure with focal sets that are the singletons of S and $\alpha < \lambda$ then μ_α cannot be a necessity measure, since if a set E is focal for μ_α, it must be disjoint from S so that $\mathcal{F}(\mu_\lambda)$ contains all singletons of S and E at least. So we have the following claim: if $\forall \lambda \in L$ μ_λ is either a possibility measure or a necessity measure, there is a threshold value θ such that $\forall \lambda \leq \theta$ μ_λ is a possibility measure (possibly a Dirac measure), and $\forall \lambda > \theta$ μ_λ is a necessity measure. We are then in a position to state the main result of this section

Theorem 2 *Two capacities $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ commute if and only if there exist at most two thresholds $\theta_N \leq \theta_\Pi \in L$ such that*

- *For $1 \geq \lambda > \theta_N$, the λ-cuts of $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are necessity measures*
- *For $\theta_N \geq \lambda > \theta_\Pi$, the λ-cut of one of $\mu_{\mathcal{X}}, \mu_{\mathcal{Y}}$ is a Dirac measure, the other one being any Boolean capacity.*
- *For $\theta_\Pi \geq \lambda$, the λ-cuts of $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are possibility measures.*

Proof: We apply Corollary **??**, noticing that if the λ-cut of $\mu_{\mathcal{X}}$ is a possibility measure, its λ'-cuts for $\lambda' < \lambda$ cannot be necessity measures. □

So, without loss of generality, if $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ commute, the set of focal sets $\mathcal{F}(\mu_{\mathcal{X}})$ is partitioned in $\mathcal{F}_N(\mu_{\mathcal{X}}) \cup \mathcal{F}_\Pi(\mu_{\mathcal{X}})$, where

- $\mathcal{F}_N(\mu_{\mathcal{X}}) = \{E \in \mathcal{F}(\mu_{\mathcal{X}}) : \mu_{\mathcal{X}\#}(E) > \theta_N\}$ is nested, say $E_p \subset \ldots E_1$
- $\mathcal{F}_\Pi(\mu_{\mathcal{X}}) = \{E \in \mathcal{F}(\mu_{\mathcal{X}}) : \mu_{\mathcal{X}\#}(E) \leq \theta_N\}$ contains only singletons,
- $\exists x \in E_p, \mu_{\mathcal{X}\#}(\{x\}) = \theta_N$ (to ensure that no set in $\mathcal{F}_N(\mu_{\mathcal{X}})$ is focal for the λ-cut of $\mu_{\mathcal{X}}$ when $\lambda \leq \theta_N$).

while the set of focal sets $\mathcal{F}(\mu_{\mathcal{Y}})$ is partitioned in $\mathcal{F}_N(\mu_{\mathcal{Y}}) \cup \mathcal{F}_D(\mu_{\mathcal{Y}}) \cup \mathcal{F}_\Pi(\mu_{\mathcal{Y}})$, where

- $\mathcal{F}_N(\mu_{\mathcal{Y}}) = \{E \in \mathcal{F}(\mu_{\mathcal{Y}}) : \mu_{\mathcal{Y}\#}(E) > \theta_N\}$ is nested
- $\mathcal{F}_\Pi(\mu_{\mathcal{Y}}) = \{E \in \mathcal{F}(\mu_{\mathcal{Y}}) : \mu_{\mathcal{Y}\#}(E) \leq \theta_\Pi\}$ contains only singletons
- $\forall E \notin \mathcal{F}_\Pi(\mu_{\mathcal{Y}}), \exists y \in \mathcal{Y}, \mu_{\mathcal{Y}\#}(\{y\}) = \theta_\Pi$ such that $y \in E$ (to ensure that no set outside of $\mathcal{F}_\Pi(\mu_{\mathcal{Y}})$ is focal for the λ-cut of $\mu_{\mathcal{Y}}$ when $\lambda \leq \theta_\Pi$)
- the focal sets in $\mathcal{F}_D(\mu_{\mathcal{Y}}) =$ are not constrained otherwise.

We can exchange $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ above. Moreover, $\mathcal{F}_D(\mu_{\mathcal{Y}}) = \emptyset$ if $\theta_\Pi = \theta_N$.

We can try to express commuting capacities in closed form. Let $N^\mu_{\mathcal{X}}$ be the necessity measure such that $N^\mu_{\mathcal{X}\#}(E) = \mu_{\mathcal{X}\#}(E), E \in \mathcal{F}_N(\mu_{\mathcal{X}})$ (likewise for $N^\mu_{\mathcal{Y}}$), $\Pi_{\mathcal{X}}$ be the possibility measure such that

$$\pi^\mu_{\mathcal{X}}(x) = \begin{cases} 1 & \text{if } \mu_{\mathcal{X}\#}(\{x\}) = \theta_N, \\ \mu_{\mathcal{X}\#}(\{x\}) & \text{if } \mu_{\mathcal{X}\#}(\{x\}) < \theta_N. \end{cases}$$

Let $\theta_D = \max\{\mu_{\mathcal{Y}\#}(E) : E \in \mathcal{F}_D(\mu_{\mathcal{X}})\} \leq \theta_N$. Let $\delta^\mu_{\mathcal{Y}}$ be the capacity with qualitative Moebius transform defined by

$$\delta^\mu_{\#}(E) = \begin{cases} 1 & \text{if } \mu_{\mathcal{Y}\#}(E) = \theta_D, E \in \mathcal{F}_D(\mu_{\mathcal{X}}) \\ \mu_{\mathcal{Y}\#}(E) & \text{if } \mu_{\mathcal{Y}\#}(E) < \theta_D E \in \mathcal{F}_D(\mu_{\mathcal{X}}), \\ 0 & \text{otherwise.} \end{cases}$$

Finally let $\Pi_{\mathcal{Y}}$ be the possibility measure such that

$$\pi^\mu_{\mathcal{Y}}(y) = \begin{cases} 1 & \text{if } \mu_{\mathcal{X}\#}(\{y\}) = \theta_\Pi, \\ \mu_{\mathcal{Y}\#}(\{y\}) & \text{if } \mu_{\mathcal{Y}\#}(\{y\}) < \theta_\pi. \end{cases}$$

Then (up to an exchange between $\mathcal{X}$ and $\mathcal{Y}$) $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ commute if they are of the form

$$\mu_{\mathcal{X}}(A) = \max(N^\mu_{\mathcal{X}}(A), \min(\theta_N, \Pi^\mu_{\mathcal{X}}(A))); \quad \mu_{\mathcal{Y}}(B) = \max(N^\mu_{\mathcal{Y}}(B), \min(\theta_D, \delta^\mu_{\mathcal{Y}}(A)), \min(\theta_\Pi, \Pi^\mu_{\mathcal{Y}}(B)))$$

Example 10 *: we can find the condition for commutation on $\{x_1, x_2\} \times \{y_1, y_2\}$ where in general $\mu_{\mathcal{X}}(x_1) = \alpha_1, \mu_{\mathcal{X}}(x_2) = \alpha_2$, $\mu_{\mathcal{Y}}(y_1) = \beta_1, \mu_{\mathcal{Y}}(y_2) = \beta_2$. Note that cuts of capacity on two-element sets can only be Boolean possibility or necessity measures. So the capacities will commute except if there is $\lambda \in L$ such that the cut of $\mu_{\mathcal{X}}$ is a possibility measure and the cut of $\mu_{\mathcal{Y}}$ is a necessity measure. So commutation will hold only if*

- *$\mu_{\mathcal{X}}$ is a possibility measure with $\alpha_1 > \alpha_2$ and $\mu_{\mathcal{Y}}$ is a necessity measure with mass $\beta_1 > \beta_2 = 0$ with $\beta_1 > \alpha_2$*
- *$\mu_{\mathcal{X}}$ is a capacity ($1 > \alpha_1 \geq \alpha_2$) and $\mu_{\mathcal{Y}}$ a possibility measure with $\beta_1 = 1 > \beta_2$, where $\alpha_1 > \beta_2$.*
- *$\mu_{\mathcal{X}}$ is a capacity ($1 > \alpha_1 \geq \alpha_2$) then $\mu_{\mathcal{Y}}$ is a necessity measure with mass $\beta_1 > \beta_2 = 0$ with $\beta_1 \geq \alpha_2$.*
- *$\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ are genuine capacities ($1 > \alpha_1 \geq \alpha_2; 1 > \beta_1 \geq \beta_2$), then $\max(\alpha_1, \alpha_2) \geq \min(\beta_1, \beta_2)$ and $\max(\beta_1, \beta_2) \geq \min(\alpha_1, \alpha_2)$.*

*It can be checked that the latter condition $\max(\alpha_1, \alpha_2) \geq \min(\beta_1, \beta_2)$ and $\max(\beta_1, \beta_2) \geq \min(\alpha_1, \alpha_2)$ covers all 4 cases. To check that this is correct, note that the only cases when the cuts are a possibility vs. a necessity measure are when $\max(\alpha_1, \alpha_2) < \min(\beta_1, \beta_2)$ or $\max(\beta_1, \beta_2) < \min(\alpha_1, \alpha_2)$ (take λ in the interval). Note that this is the case in the counterexample from [**?**] since then $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 0$. However the commutation condition is clearly satisfied in Example **??**.*

6 Conclusion

In this paper, we have found necessary and sufficient conditions for two Sugeno integrals on distinct universes to commute. We have seen that even if we do not have to restrict to pairs of possibility or of necessity measures, the commuting capacities are essentially built from gluing possibility and necessity measures, except when the cut of one of them trivializes into a Dirac measure. An essential lesson is that, like with fuzzy sets under the maxmin operations, we can reason about qualitative capacities using cuts, and that the double fuzzy integrals are cutworthy in the sense of De Baets and Kerre [**?**]. These results open the way to a generalization of the qualitative counterpart of Harsanyi [**?**] theorem for expected utility studied in [**?**], namely provide axioms over acts or possibilistic lotteries that justify a commuting double Sugeno integral to evaluate social utility. It is also interesting to apply this result to qualitative game theory since the commutation can be used to define a kind of qualitative counterpart to Nash equilibrium.

It remains to be studied conditions under which where the two double S-integrals are equal the 2D S-integral based on the joint capacity, and study whether there are examples where the two double S-integrals are equal but differ from the 2D one based on the joint capacity. We have seen that when $\mu_\mathcal{X}$ is a possibility measure $\Pi_\mathcal{X}$ and $\mu_\mathcal{Y}$ is a necessity measure $N_\mathcal{X}$, the double S-integral $S_{\Pi_\mathcal{X}}(S_{N_\mathcal{Y}}(u))$ is equal to the 2D S-integral $S_{\min(\Pi_\mathcal{X},N_\mathcal{Y}}$ while it differs from $S_{N_\mathcal{Y}}(S_{\Pi_\mathcal{X}}(u))$. The latter is a 2D S-integral with respect to a 2D capacity that is not a joint one. This state of facts suggests that to evaluate a collective utility function, $S_{\Pi_\mathcal{X}}(S_{N_\mathcal{Y}}(u))$ is better behaved since the 2D capacity associated to this double S-integral is built from $\Pi_\mathcal{X}$ and $N_\mathcal{Y}$ only. This line of thought needs further investigation.

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