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Convergence of a multidimensional Glimm-like scheme for the transport of fronts.

T. Gallouët, O. Hurisse

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1 Introduction

In [7], a scheme based on the idea of Glimm [5, 1, 2, 3, 6] has been introduced in order to deal with the advection of indicator functions. The whole algorithm is very simple and it relies on a two-step approach for which a convection step is followed by a projection step. For the convection step, the approximated solution is updated using a classical scheme, as for instance the Upwind scheme which has been chosen in the present work. It should be noted that in [7] other schemes have been used. This convection step introduces some smearing of the front of the the indicator function due to the numerical diffusion of the scheme. Then, the second step sharpens the smeared profile in order to recover an approximated indicator function that for each cell of the mesh can only takes two values. This second step, or Glimm Random Update, and nick-named GRU in the present work, is performed on the basis of a random choice.

The whole scheme proposed in [7], and studied more in details here, relies on the analysis of the Upwind scheme that has been carried out in [4]. In the latter, a probabilistic analysis of the classical finite-volumes Upwind scheme is performed. The approximated solutions of the latter can be built as the expectation of a stochastic process, where the characteristics are stochastic and follow a Markov chain. It arises from this analysis that the effective order of $1/2$ of the Upwind scheme can be associated to the fluctuations of the stochastic characteristics around the average characteristic, and to the expectation operator applied to recover the Upwind scheme. The basic idea of the scheme proposed here is thus that, following Glimm's idea, selecting only one sample of this stochastic process avoid to apply an expectation operator. Doing so, we could expect to get an effective convergence rate higher than $1/2$ and we could wait a better accuracy for the approximated solutions. The GRU step can thus be seen as the random choice of one sample of an underlying stochastic process for the characteristics, whose transition probabilities are obtained by the scheme used for the first step. This scheme has been tested in [7] on several two-dimensional test cases, including structured and non-structured meshes. Actually, for all these test cases, an effective rate of convergence around 0.8 has been observed for the Upwind-GRU scheme, whereas a classical first-order scheme has an effective rate of convergence of 0.5 on such test cases. Besides this, it has also been shown that the GRU step improves the accuracy of the approximated solutions on coarse meshes. It seems that it allows to get for two-dimensional cases the same efficiency than the classical Glimm's scheme in the one-dimensional setting.

On the contrary to the Glimm's scheme, the GRU step is dedicated to a limited class of problem. But the main advantage of the GRU step is that it can be applied to multi-dimensional advection problem, as shown in [7]. We focus here on the advection of a scalar quantity Φ in a two-dimensional space whose coordinates are x and y . We assume that Φ only takes two different values at any time t and any point (x, y) , say 0 and 1 . Even if the GRU step can handle more complex advection problems, we restrict our study here to the simple case with a constant uniform advection velocity field (U_x, U_y) . Hence, the system of equations we are concerned in can be written:

$$\begin{cases} \partial_t (\Phi(t, x, y)) + U_x \partial_x (\Phi(t, x, y)) + U_y \partial_y (\Phi(t, x, y)) = 0, \\ \Phi(t = 0, x, y) = \Phi^0(x, y). \end{cases} \quad (1)$$

Moreover, the initial condition $\Phi^0(x, y)$ is chosen as an indicator function for which the domain with value 1 and the domain with the value 0 are separated by a planar front. The scheme used in this paper uses the GRU step coupled with the Upwind scheme for the convection step. It is nicknamed Upwind-GRU.

As mentioned in [7], the Upwind-GRU scheme corresponds to the Glimm's scheme for the one-dimensional counterpart of the advection problem described above. So that the proof of convergence for the Glimm's scheme is sufficient to prove the convergence of the Upwind-GRU scheme. Hence, we focus in this work on the two-dimensional case which is more specific to the GRU step. Indeed, for the particular two-dimensional problem

described above, some theoretical results can be exhibited in particular a proof that the approximated solutions provided by Upwind-GRU scheme statistically converge towards the exact solution.

The paper is organized as follows. We first describe in section 2 the whole Upwind-GRU scheme. Then, in section 3 some properties of the Upwind-GRU scheme are proposed. This first results allow to grasp the way the scheme behaves and provide some preliminary results for the next sections. Indeed, the convergence results of section 4 are based on these preliminary results. In this section, we first propose a proof of convergence for the one-dimensional case that helps understanding the proof of convergence for the two-dimensional case.

2 The Upwind-GRU scheme

We recall here the Upwind-GRU scheme as proposed initially in [7]. The latter is a two-step scheme based on the Upwind scheme and on an update that sharpens the approximated solutions. In order to describe this scheme we introduce here some (classical) notations.

For the sake of simplicity, we consider uniform and cartesian meshes composed of square-cells. Here again, we insist on the fact that the scheme can be applied to any kind of meshes, as shown in [7]. The spatial domain is then discretized using square-cells of size $h \times h$, see figure 1 for a sketch of the notations. The center of gravity of cell (i, j) is denoted by (X_i, Y_j) and we obviously have: $X_{i+1} = X_i + h$ and $Y_{i+1} = Y_i + h$. The four faces of the cell (i, j) are denoted in a classical manner by the indices: $(i - 1/2, j)$ for the left face, $(i + 1/2, j)$ for the right face, $(i, j - 1/2)$ for the lower face and $(i, j + 1/2)$ for the upper face.

Let us introduce an angle θ so that we can write the components of the velocity in the form: $U_x = U \cos(\theta)$ and $U_y = U \sin(\theta)$, where U is the norm of the velocity field. Obviously, θ and U are uniform and constant. Moreover, thanks to the symmetry of the considered meshes, we can choose $\theta \in [0, \Pi/2]$ without any loss of generality.

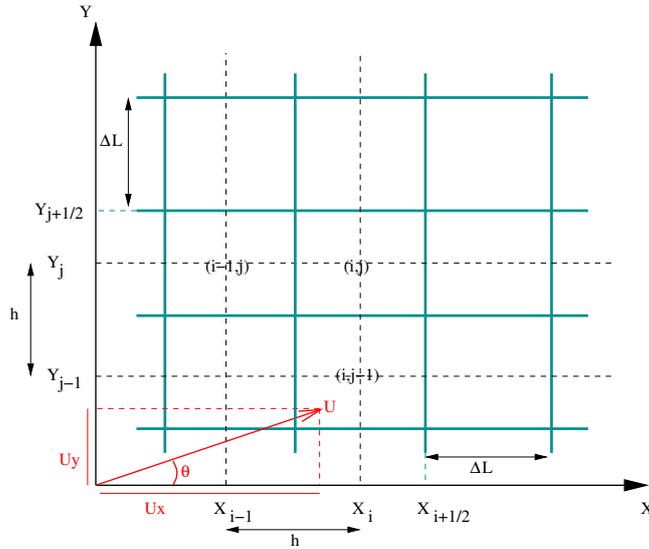


Figure 1: Some notations and definitions for the mesh and the velocity field.

2.1 First step: update according to the Upwind scheme

The Upwind scheme applied to (1) gives the update formula $\Phi_{i,j}^{n+1,*}$ at time $t^{n+1} = t^n + \Delta t$:

$$\Phi_{i,j}^{n+1,*} = \Phi_{i,j}^n - \frac{\Delta t}{h} \left(F_{i+1/2,j}^n - F_{i-1/2,j}^n \right) - \frac{\Delta t}{h} \left(F_{i,j+1/2}^n - F_{i,j-1/2}^n \right), \quad (2)$$

where the fluxes $F_{i+1/2,j}^n$, $F_{i-1/2,j}^n$, $F_{i,j+1/2}^n$ and $F_{i,j-1/2}^n$ respectively denotes the fluxes at the right, left, lower and upper faces of the cell (i, j) . Due to the specific case we are interested in, and in particular because of $\theta \in [0, \Pi/2]$, we have:

$$F_{i-1/2,j}^n = U_x \Phi_{i-1,j}^n, \quad \text{and} \quad F_{i,j-1/2}^n = U_y \Phi_{i,j-1}^n.$$

Thus, the update formula (2) gives for our specific configurations:

$$\Phi_{i,j}^{n+1,*} = \Phi_{i,j}^n - \frac{U\Delta t}{h} \left((\Phi_{i,j}^n - \Phi_{i-1,j}^n) \cos(\theta) + (\Phi_{i,j}^n - \Phi_{i,j-1}^n) \sin(\theta) \right). \quad (3)$$

It can thus be seen that the value $\Phi_{i,j}^{n+1,*}$ is a linear combination of the values $\Phi_{i,j}^n$, $\Phi_{i-1,j}^n$ and $\Phi_{i,j-1}^n$:

$$\Phi_{i,j}^{n+1,*} = \Phi_{i-1,j}^n \frac{U\Delta t}{h} \cos(\theta) + \Phi_{i,j}^n \left(1 - \frac{U\Delta t}{h} (\cos(\theta) + \sin(\theta)) \right) + \Phi_{i,j-1}^n \frac{U\Delta t}{h} \sin(\theta). \quad (4)$$

Assuming that at time t^n the values $\Phi_{i,j}^n$ lie in $[0, 1]$ for all the cells (i, j) , it can be deduced from equation (4) a sufficient CFL condition that enables that $\Phi_{i,j}^{n+1,*}$ also lie in $[0, 1]$ for all the cells (i, j) . Indeed, for a given $\theta \in [0, \Pi/2]$ and a given velocity $U \geq 0$, if Δt and h are such that:

$$\frac{U\Delta t}{h} (\cos(\theta) + \sin(\theta)) < 1,$$

then equation (4) corresponds to a barycentric combinations of values in $[0, 1]$ and thus $\Phi_{i,j}^{n+1,*}$ also lies in $[0, 1]$. In the following, we set $\beta = U\Delta t/h$ and we assume that for a given time step Δt , the mesh size h is chosen so that $\beta \in [0, (\cos(\theta) + \sin(\theta))^{-1}]$. For the sake of simplicity, we assume that Δt remains the same for all the time-iterations.

Remark. We have chosen here to deduce the mesh size h from the time step Δt according to the CFL condition, though the classical and practical way consists in adapting Δt to h (the latter arising from the mesh). However, the former choice is useful for the purpose of the proof of section 4.

2.2 Second step: the Glimm Random Update

Let us now assume that the approximated value $\Phi_{i,j}^n$ of Φ in each cell (i, j) at time t^n lies in $\{0, 1\}$. When applying the Upwind scheme (4) recalled in section 2.1, the new values $\Phi_{i,j}^{n+1,*}$ generally do not all lie in $\{0, 1\}$. The Glimm Random Update proposed in [7] simply consists in updating $\Phi_{i,j}^{n+1,*}$ according to:

$$\Phi_{i,j}^{n+1} = \begin{cases} 1, & \text{if } \omega^n \in [0, \Phi_{i,j}^{n+1,*}], \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

where ω^n is a quasi-random number that follows a uniform law in $[0, 1]$. A straightforward consequence of this update (5) is that at time t^{n+1} the approximated value $\Phi_{i,j}^{n+1}$ lies in $\{0, 1\}$ for all the cells (i, j) . **It's worth to insist on the fact that ω^n is the same for all the cells, but that ω^n is independent from all the ω^k with $k < n$.** It should be noted that this feature was introduced in [1, 2, 3] considering the Glimm scheme [5] for the one-dimensional cases. This point is essential for the scheme to behave properly, as shown in the following sections. In particular, this disables the formation of ‘‘holes’’ in the profiles, as highlighted in the beginning of section 3.2.

3 Some properties of the Upwind-GRU scheme

In this section, the scheme presented in section 2 is translated in terms of stochastic process. Indeed, the complete update of $\Phi_{i,j}^n$ into $\Phi_{i,j}^{n+1}$ involves a random choice, so that further analyses of the scheme can be done when considering the approximated values of Φ as a realization of a stochastic process. In the following, $\mathcal{P}(A)$ denotes the probability of event A according to the $[0, 1]$ uniform probability law, which is the probability law for the choice ω^n in the update (5).

3.1 Expectations and transitional probabilities

From formula (5) one get the following conditional probabilities knowing the set of values at time t^n and denoted by $\Phi^n = \{\forall(i, j), \Phi_{i,j}^n\}$:

$$\mathcal{P}(\Phi_{i,j}^{n+1} = 1 | \Phi^n) = \Phi_{i,j}^{n+1,*} \quad \text{and} \quad \mathcal{P}(\Phi_{i,j}^{n+1} = 0 | \Phi^n) = 1 - \Phi_{i,j}^{n+1,*}. \quad (6)$$

It should be noted here that it is mandatory to have $0 \leq \Phi_{i,j}^{n+1,*} \leq 1$ for the two probabilities above to be properly defined. This point highlights the importance of the CFL condition of section 2.1. Considering the probabilities (6) one can easily obtain the conditional expectation $E[\Phi_{i,j}^{n+1} | \Phi^n]$ of $\Phi_{i,j}^{n+1}$ knowing $\Phi_{i,j}^n$:

$$E[\Phi_{i,j}^{n+1} | \Phi^n] = (1 \times \mathcal{P}(\Phi_{i,j}^{n+1} = 1 | \Phi^n) + 0 \times \mathcal{P}(\Phi_{i,j}^{n+1} = 0 | \Phi^n)) = \Phi_{i,j}^{n+1,*}, \quad (7)$$

where $\Phi_{i,j}^{n+1,*}$ is given by equation (4). Then, by integrating $E[\Phi_{i,j}^{n+1}|\Phi^n]$ over the values of Φ^n , it can be straightforwardly obtained from (4) and (7) that:

$$E[\Phi_{i,j}^{n+1}] = E[\Phi_{i-1,j}^n] \frac{U\Delta t}{h} \cos(\theta) + E[\Phi_{i,j}^n] \left(1 - \frac{U\Delta t}{h} (\cos(\theta) + \sin(\theta))\right) + E[\Phi_{i,j-1}^n] \frac{U\Delta t}{h} \sin(\theta). \quad (8)$$

It can thus be observed that the expectations $E[\Phi_{i,j}^n]$ are updated according to formula (8), which corresponds to the update formula for the classical Upwind scheme without the GRU step. In that sense the Upwind-GRU scheme corresponds to one way (among other) of selecting one realization of the stochastic process underlying the classical Upwind scheme [4].

The update formula (4) implies that the value $\Phi_{i,j}^{n+1,*}$ only depends on the values $\Phi_{i,j}^n$, $\Phi_{i-1,j}^n$ and $\Phi_{i,j-1}^n$. Obviously, combining (4) and (5) we have:

$$\Phi_{i,j}^n = \Phi_{i-1,j}^n = \Phi_{i,j-1}^n \implies \Phi_{i,j}^{n+1} = \Phi_{i,j}^n.$$

Hence, the value in a cell (i, j) can change from t^n to t^{n+1} , i.e. $\Phi_{i,j}^{n+1} \neq \Phi_{i,j}^n$, only if the value of the approximated solution in at least one of its neighboring cells $(i-1, j)$ and $(i, j-1)$ differs from $\Phi_{i,j}^n$, the value in cell (i, j) . This implies that during one time-step, the approximated solution only changes in the cells that have neighboring cells which contain a different value at time t^n . An other consequence is that no “hole” is created by the scheme.

$\Phi_{i,j}^n = 0$	$\Phi_{i-1,j}^n = 0$	$\Phi_{i-1,j}^n = 1$
$\Phi_{i,j-1}^n = 0$	$\Phi_{i,j}^{n+1,*} = 0$	$\Phi_{i,j}^{n+1,*} = \beta \cos(\theta)$
$\Phi_{i,j-1}^n = 1$	$\Phi_{i,j}^{n+1,*} = \beta \sin(\theta)$	$\Phi_{i,j}^{n+1,*} = \beta(\cos(\theta) + \sin(\theta))$

Table 1: Updated value $\Phi_{i,j}^{n+1,*}$ when $\Phi_{i,j}^n = 0$ with respect to the value in the neighboring cells $(i-1, j)$ and $(i, j-1)$.

$\Phi_{i,j}^n = 1$	$\Phi_{i-1,j}^n = 0$	$\Phi_{i-1,j}^n = 1$
$\Phi_{i,j-1}^n = 0$	$\Phi_{i,j}^{n+1,*} = 1 - \beta(\cos(\theta) + \sin(\theta))$	$\Phi_{i,j}^{n+1,*} = 1 - \beta \sin(\theta)$
$\Phi_{i,j-1}^n = 1$	$\Phi_{i,j}^{n+1,*} = 1 - \beta \cos(\theta)$	$\Phi_{i,j}^{n+1,*} = 1$

Table 2: Updated value $\Phi_{i,j}^{n+1,*}$ when $\Phi_{i,j}^n = 1$ with respect to the value in the neighboring cells $(i-1, j)$ and $(i, j-1)$.

The different possible configurations due to our specific choices are gathered in tables 1 and 2. From these different cases and according to (5), we can give the following conditional probabilities that represent the transitional probabilities for the stochastic process associated with the Upwind-GRU scheme:

$$\begin{aligned} \mathcal{P}(\Phi_{i,j}^{n+1} = 1 \mid \Phi_{i,j}^n = 0, \Phi_{i-1,j}^n = 0, \Phi_{i,j-1}^n = 0) &= 0, \\ \mathcal{P}(\Phi_{i,j}^{n+1} = 1 \mid \Phi_{i,j}^n = 0, \Phi_{i-1,j}^n = 1, \Phi_{i,j-1}^n = 0) &= \beta \cos(\theta), \\ \mathcal{P}(\Phi_{i,j}^{n+1} = 1 \mid \Phi_{i,j}^n = 0, \Phi_{i-1,j}^n = 0, \Phi_{i,j-1}^n = 1) &= \beta \sin(\theta), \\ \mathcal{P}(\Phi_{i,j}^{n+1} = 1 \mid \Phi_{i,j}^n = 0, \Phi_{i-1,j}^n = 1, \Phi_{i,j-1}^n = 1) &= \beta(\cos(\theta) + \sin(\theta)), \\ \mathcal{P}(\Phi_{i,j}^{n+1} = 1 \mid \Phi_{i,j}^n = 1, \Phi_{i-1,j}^n = 0, \Phi_{i,j-1}^n = 0) &= 1 - \beta(\cos(\theta) + \sin(\theta)), \\ \mathcal{P}(\Phi_{i,j}^{n+1} = 1 \mid \Phi_{i,j}^n = 1, \Phi_{i-1,j}^n = 1, \Phi_{i,j-1}^n = 0) &= \beta \sin(\theta), \\ \mathcal{P}(\Phi_{i,j}^{n+1} = 1 \mid \Phi_{i,j}^n = 1, \Phi_{i-1,j}^n = 0, \Phi_{i,j-1}^n = 1) &= \beta \cos(\theta), \\ \mathcal{P}(\Phi_{i,j}^{n+1} = 1 \mid \Phi_{i,j}^n = 1, \Phi_{i-1,j}^n = 1, \Phi_{i,j-1}^n = 1) &= 1. \end{aligned}$$

In the relations above, $\mathcal{P}(\Phi_{i,j}^{n+1} = 1 | \Phi_{i,j}^n, \Phi_{i-1,j}^n, \Phi_{i,j-1}^n)$ represents the probability to obtain $\Phi_{i,j}^{n+1} = 1$ knowing the values $\Phi_{i,j}^n$, $\Phi_{i-1,j}^n$ and $\Phi_{i,j-1}^n$. Obviously, for any of these possible configurations we have:

$$\mathcal{P}(\Phi_{i,j}^{n+1} = 0 \mid \Phi_{i,j}^n, \Phi_{i-1,j}^n, \Phi_{i,j-1}^n) = 1 - \mathcal{P}(\Phi_{i,j}^{n+1} = 1 \mid \Phi_{i,j}^n, \Phi_{i-1,j}^n, \Phi_{i,j-1}^n).$$

3.2 Pathological cases

In this section, we aim at highlighting a pathological behavior of the Upwind-GRU scheme. For that purpose, we consider a “one-dimensional” situation. Let us choose $\theta = 0$. Let us also assume that at time t^n we have:

$$\Phi_{i,j}^n = \begin{cases} 1 & \text{if } I_a \leq i < I_b, \\ 0 & \text{otherwise,} \end{cases}$$

with $I_a < I_b$. We denote by ΔI^N the number of cells in the x -direction for which $\Phi_{i,j}^N = 1$ at iteration N , and we have $\Delta I^n = I_b - I_a$. For this very specific case, the CFL condition of section 2.1 implies that $\beta \in [0, 1]$. Therefore, we assume here that h is fixed and that Δt is chosen so that β lies in $[1/2, 1]$.

Following the results of tables 1 and 2 of section 3.1, we get that after the first step of the Upwind-GRU scheme:

$$\Phi_{i,j}^{n+1,*} = \begin{cases} \Phi_{i,j}^n & \text{if } i \notin \{I_a, I_b\}, \\ 1 - \beta & \text{if } i = I_a, \\ \beta & \text{if } i = I_b. \end{cases} \quad (9)$$

It can be noticed here that for all the cells such that $i \notin \{I_a, I_b\}$, $\Phi_{i,j}^{n+1,*} = \Phi_{i,j}^n \in \{0, 1\}$. As a consequence we also get that $\Phi_{i,j}^{n+1} = \Phi_{i,j}^n$ if $i \notin \{I_a, I_b\}$, which shows that no hole is created by the scheme in the initial profile. Moreover, because ω^n is the same for all the cells, the update (9) combined with the step (5) implies that the values $\Phi_{i,j}^{n+1}$ at time t^{n+1} do not depend on j : $\forall i, \Phi_{i,j_0}^{n+1} = \Phi_{i,j_1}^{n+1}$. These remarks imply that the quantity ΔI^N remains meaningful.

When turning to the cells $i \in \{I_a, I_b\}$, according to the second step (5), we have three cases. Since we have chosen $\beta > 1/2$, we have $0 \leq 1 - \beta < \beta \leq 1$ and these cases are:

- if $\omega^n \in [0, 1 - \beta]$, we have $\Phi_{I_a,j}^{n+1} = 1 = \Phi_{I_a,j}^n$ and $\Phi_{I_b,j}^{n+1} = 1 \neq \Phi_{I_b,j}^n$, and hence $\Delta I^{n+1} = \Delta I^n + 1$;
- if $\omega^n \in]1 - \beta, \beta]$, we have $\Phi_{I_a,j}^{n+1} = 1 = \Phi_{I_a,j}^n$ and $\Phi_{I_b,j}^{n+1} = 0 \neq \Phi_{I_b,j}^n$, and hence $\Delta I^{n+1} = \Delta I^n$;
- if $\omega^n \in]\beta, 1]$, we have $\Phi_{I_a,j}^{n+1} = 0 \neq \Phi_{I_a,j}^n$ and $\Phi_{I_b,j}^{n+1} = 0 = \Phi_{I_b,j}^n$, and hence $\Delta I^{n+1} = \Delta I^n - 1$.

When focusing on the width ΔI^{n+1} of the approximated solution, due to the uniform law for ω^n , we have the following transitional probabilities :

$$\mathcal{P}(\Delta I^{n+1} = \Delta I^n - 1) = 1 - \beta, \quad (10)$$

$$\mathcal{P}(\Delta I^{n+1} = \Delta I^n) = 2\beta - 1, \quad (11)$$

$$\mathcal{P}(\Delta I^{n+1} = \Delta I^n + 1) = 1 - \beta. \quad (12)$$

If we have $I_b = I_a + 1$, the probability to set the all cells to 0 is then equal to $(1 - \beta)$. Once all the cells are equal to zero, they will remain to zero and the initial profile is definitively lost. The probabilities (10)-(12) clearly advocate for the use of large values of β .

A more general result can be obtained. Indeed, the parameter β is constant and uniform, and the transitional probabilities (10)-(12) do thus not depend on the time-iteration. Each possible realization is a combination of the three cases mentioned above. Since the probabilities (10)-(12) do not depend on the time-iteration, the probability of one realization only depends on: the number I_- of cases (10), the number I_0 of cases (11) and the number I_+ of cases (12), whatever the order of occurrence of these cases. We thus have:

$$\mathcal{P}(I_-, I_0, I_+) = \mathcal{C}_{I_-+I_0+I_+}^{I_-, I_+} (1 - \beta)^{I_-+I_+} (2\beta - 1)^{I_0},$$

where $\mathcal{C}_n^{p,k}$ stands for the number of combinations:

$$\mathcal{C}_n^{p,k} = \frac{n!}{p!k!(n-p-k)!}.$$

From these remarks, and starting with a width $\Delta I^n \geq 1$ at time t^n , we can write the probability to set all the values to zero after p iterations. First, if $p < \Delta I^n$ it is not possible to set all the cells to zero. For the configurations with $p \geq \Delta I^n$, we have to sum the probability of all triplet $(I_-, I_0, I_+) \in \Delta_n^p$, where:

$$\Delta_n^p = \{(I_-, I_0, I_+) \in \llbracket 0, p \rrbracket^3 \text{ with } I_- + I_0 + I_+ = p \text{ and } I_- + I_+ \geq \Delta I^n\},$$

so that we get:

$$\mathcal{P}(\forall(i, j), \Phi_{i,j}^{n+p} = 0) = \begin{cases} 0 & \text{if } p < \Delta I^n, \\ \sum_{\Delta_n^p} \mathcal{P}(I_-, I_0, I_+) & \text{otherwise,} \end{cases} \quad (13)$$

The ‘‘worst case’’ in terms of simulation could be defined as the case with $p = \Delta I^n$ and $I_- = \Delta I^n$. This case corresponds to the minimal number of iterations required to set all the cells to zero. It occurs with the probability $(1 - \beta)^{\Delta I^n}$.

The same kind of results can obviously be obtained for $\beta \in [0, 1/2]$. Moreover, the cases where all the cells are set to one (instead of zero) can be treated following the same idea. The main point to notice here is that for coarse meshes, some parts of the initial solution may disappear with a non-zero probability. Nonetheless, for a given physical width of the initial solution and for a given final time, mesh refinement decreases the probability to set all the cells to zero.

4 A result of convergence for the Upwind-GRU scheme

In this section, we propose a proof of convergence of the Upwind-GRU scheme when considering the advection of a planar front through system (1). We first begin with the one-dimensional case. Due to our assumption of constant and uniform velocity, the Upwind-GRU scheme exactly coincides with the Glimm scheme in our one-dimensional case. Hence the proof of convergence of the latter is sufficient to prove the convergence of the former. Nevertheless, the proof that we propose in section 4.2 for the Upwind-GRU scheme in the two-dimensional case is an extension of the proof of section 4.1 for the one-dimensional case. Starting with the one-dimensional case may help the reader to grasp the main ingredients of the proof.

4.1 One-dimensional case

For the sake of simplicity, all the indices j that are related to the coordinate y , and the y components and coordinate are omitted. We consider here the one-dimensional counterpart of system of equations (1):

$$\begin{cases} \partial_t (\Phi(t, x)) + U \partial_x (\Phi(t, x)) = 0, \\ \Phi(t = 0, x) = \Phi^0(x), \end{cases} \quad (14)$$

where the velocity U is constant and uniform. The initial condition is:

$$\Phi^0(x) = \begin{cases} 1 & \text{if } (x - X_f^0) < 0, \\ 0 & \text{otherwise,} \end{cases} \quad (15)$$

where X_f^0 stands for a given abscissa. It represents here the position of the initial front. Thanks to our assumptions, the exact solution of system (14) with initial condition (15) can straightforwardly be written as:

$$\forall t > 0, \forall x, \Phi(t, x) = \Phi^0(x - Ut). \quad (16)$$

The first point here is to choose an approximated initial condition for the numerical scheme. Let us take the classical approximated initial solution which is equal to the value of the initial condition (15) at the center of gravity of the cell, that is:

$$\Phi_i^0 = \Phi^0(X_i). \quad (17)$$

It is an important point to be quoted that we obviously have:

$$\sum_i \int_{X_{i-1/2}}^{X_{i+1/2}} |\Phi^0(x) - \Phi_i^0| dx = o(h). \quad (18)$$

Starting with this approximated initial solution, the Upwind-GRU scheme defined in section 2 is then applied for a given time-step Δt such that the CFL condition of section 2.1 is fulfilled. The following proposition then holds.

Proposition 4.1 *With an approximated initial solution of the form (15) and (17) at time $t^0 = 0$, the transitional probabilities for the approximated solution from time t^n to time $t^{n+1} = t^n + \Delta t$, obtained with the Upwind-GRU scheme are:*

$$\begin{cases} \mathcal{P}(\forall i, \Phi_i^{n+1} = \Phi_{i-1}^n) = \beta, \\ \mathcal{P}(\forall i, \Phi_i^{n+1} = \Phi_i^n) = 1 - \beta. \end{cases} \quad (19)$$

The parameter β corresponds to the CFL number and it should belong to $]0, 1[$ for ensuring the stability of the numerical scheme.

Proof. Due to our choices for the velocity field and the initial condition there are only three possibilities among the configurations gathered in tables 1 and 2:

- case 1: $\Phi_i^0 = 0$ and $\Phi_{i-1}^0 = 1$;
- case 2: $\Phi_i^0 = 0$ and $\Phi_{i-1}^0 = 0$;
- case 3: $\Phi_i^0 = 1$ and $\Phi_{i-1}^0 = 1$.

For any cells i in the case 2 or 3, updating the approximated solution through the Upwind-GRU scheme does not change the value in the cell: $\Phi_i^1 = \Phi_i^0$. Indeed, the update through the Upwind-GRU scheme implies that a change may only occur in a cell that is in case 1. Considering the specific form for the initial condition in this section, there is a unique cell i_0 in case 1 at first iteration. We have in that cell the transitional probability:

$$\mathcal{P}(\Phi_{i_0}^1 = 1 \mid \Phi_{i_0}^0 = 0, \Phi_{i_0-1}^0 = 1) = \beta.$$

Since cells that are in case 2 or 3 remain unchanged with probability 1, the update of the approximated solution Φ^1 corresponds to a translation of the initial approximated solution Φ^0 :

- from $+h$ to the right with respect to the positive x when $\Phi_{i_0}^1 = 1$, in other words $\forall i, \Phi_i^1 = \Phi_{i-1}^0$;
- from 0 when $\Phi_{i_0}^1 = 0$, which means that $\Phi^1 = \Phi^0$.

Thus, at the end of the first iteration there is always exactly one cell which is in case 1, and all the other cells are in case 2 or 3. Moreover, we then obtain the following transitional probabilities:

$$\begin{cases} \mathcal{P}(\forall i, \Phi_i^1 = \Phi_{i-1}^0) = \beta, \\ \mathcal{P}(\forall i, \Phi_i^1 = \Phi_i^0) = 1 - \beta. \end{cases} \quad (20)$$

It should be noted that the transitional probabilities (20) do not depend on the time iteration (we recall that β is fixed in our specific case), and since the initial profile is just translated from h or 0, we deduce that the probabilities (20) remain the same for all the iterations. We thus get the transitional probabilities (19) which hold for any iteration time t^n . This ends the proof of proposition 4.1. \square

Let us consider $N > 0$ time-iterations. As mentioned above in the proof of proposition 4.1, each realization of the Upwind-GRU scheme for N iterations is thus composed of $K \in \llbracket 0, N \rrbracket$ shifts of h to the right of the approximated initial solution Φ^0 and $N - K$ standstills. The probability of each configuration is given by proposition 4.2.

Proposition 4.2 *The probability P_N^K to get exactly $K \in \llbracket 0, N \rrbracket$ shifts for N iterations with the Upwind-GRU scheme is:*

$$P_N^K = \mathcal{C}_N^K(\beta)^K (1 - \beta)^{N-K}, \quad (21)$$

where \mathcal{C}_N^K stands for the number of combinations.

Proof. As mentioned above in the proof of proposition 4.1, for each iteration we have the same transitional probabilities (19). Moreover, the random numbers ω^n involved in the second step of the Upwind-GRU scheme are independent for all the iterations. Hence each realization is independent and follows (19). Therefore, all the configurations of K shifts and $N - K$ standstills are equivalent in terms of probability. As a consequence, the configurations obtained after N iterations with the Upwind-GRU scheme follow the binomial distribution with the parameter β : $\mathcal{B}(N, \beta)$. Thus, we obtain the formula (21) for the probability P_N^K to get K shifts for N iterations. \square

Hence, the configurations obtained with the Upwind-GRU scheme follow the binomial distribution $\mathcal{B}(N, \beta)$. Since the displacement associated with each configuration is known, we can deduce E_d^N the expectation of the displacement of the approximated initial solution after N iterations.

Proposition 4.3 *The expectation of the displacement of the approximated initial solution after N iterations E_d^N is equal to the displacement of the approximated initial solution through system (14).*

Proof. For each configuration with N iterations and exactly K shifts, the displacement of the approximated initial solution is equal to Kh . We then get from proposition 4.2 that E_d^N is:

$$E_d^N = \sum_{K=0}^N (P_N^K Kh), \quad (22)$$

$$E_d^N = h \sum_{K=1}^N (K \mathcal{C}_N^K(\beta)^K (1 - \beta)^{N-K}), \quad (23)$$

$$E_d^N = h \sum_{K=1}^N (N \mathcal{C}_{N-1}^{K-1}(\beta)^K (1 - \beta)^{N-K}), \quad (24)$$

$$E_d^N = N\beta h \sum_{K=0}^{N-1} (\mathcal{C}_{N-1}^K(\beta)^K (1 - \beta)^{(N-1)-K}). \quad (25)$$

The sum on the right hand side of equation (25) is the development of the polynomial $(\beta + (1 - \beta))^{N-1}$ which is obviously equals to 1. We thus finally get:

$$E_d^N = N\beta h. \quad (26)$$

By introducing the definition of $\beta = U\Delta t/h$, we can write:

$$E_d^N = N\beta h = UN\Delta t = Ut^N. \quad (27)$$

The displacement Ut^N corresponds exactly to the displacement of the approximated initial solution between $t = 0$ and $t = t^N$ when it is considered as the initial condition for system (14). This ends the proof of proposition 4.3. \square

Thanks to proposition 4.3, we know that the Upwind-GRU scheme implies an exact average displacement of the approximated initial solution. The proof proposed in this section is based on the fact that the probability law of one realization of the displacement for a given N , based on (21), corresponds to a law that is centered on a neighborhood of the displacement E_d^N . By studying the probabilities P_N^K given by (21), we obtain proposition (4.4).

Proposition 4.4 *For any number of iterations $N > 0$, there exists an integer $0 \leq K^{N,\beta} < N$, such that $K \mapsto P_N^K$ increases for $K \leq K^{N,\beta}$ and decreases for $K > K^{N,\beta}$. Therefore, the probability P_N^K reaches a maximum for $K = K^{N,\beta}$ or $K = K^{N,\beta} + 1$. Moreover, the displacement associated with $K^{N,\beta}$ fulfills:*

$$|K^{N,\beta}h - E_d^N| < 2h. \quad (28)$$

Proof. For a given number of iterations N , simple calculus rules lead to the relation:

$$P_N^{K+1} = P_N^K \frac{N-K}{K+1} \frac{\beta}{1-\beta}.$$

Hence we get that

$$P_N^{K+1} \geq P_N^K \iff K \leq \beta N - (1 - \beta).$$

We denote $K^{N,\beta}$ the integer part of $(\beta N - (1 - \beta))$, we then have:

$$P_N^{K+1} \geq P_N^K \iff K \leq K^{N,\beta}. \quad (29)$$

From the definition of $K^{N,\beta}$, since $0 < \beta < 1$ and thanks to relation (26) from proposition 4.3, it yields that:

$$|K^{N,\beta}h - E_d^N| < 2h.$$

Hence, from (29) and (28), we deduce that $K \mapsto P_N^K$ increases for $K \leq K^{N,\beta}$ and decreases for $K > K^{N,\beta}$. Therefore, the probability P_N^K reaches a maximum for $K = K^{N,\beta}$ or $K = K^{N,\beta} + 1$. This ends the proof for proposition 4.4. \square

As an illustration of proposition 4.4, the probabilities P_N^K are plotted on figure (2) with respect to K/N for $N = 100$ and $\beta = 0.75$. In this case, we have $K^{100,0.75} = 74$ and the maximum is reached for $K = 75$. It can be observed that the probabilities P_N^K quickly tend to zero as K deviates from $K^{100,0.75}$. This feature also plays an important role in the present proof, and in particular for proposition 4.5.

Let us fix a final time T and a CFL number β . The time-step is thus deduced from T and from the number of iterations N through: $\Delta t = T/N$; and, then, the mesh size is obtained from the CFL condition: $h = TU/(N\beta)$. For a given $\eta > 0$, the probability for the displacement of the approximated initial solution to be in the interval $[E_d^N - \eta, E_d^N + \eta]$ after N time-iterations is denoted by \mathcal{P}_N^η . We can state the following proposition for \mathcal{P}_N^η .

Proposition 4.5 *Let us assume that the final time T and the CFL number $\beta \in]0, 1[$ are given, so that both h and dt are known for any number of iterations N . Then, for a given $\eta > 0$, the probability for the displacement of the approximated initial solution to be in the interval $[E_d^N - \eta, E_d^N + \eta]$ after N time-iterations, denoted by \mathcal{P}_N^η , is such that :*

$$\mathcal{P}_N^\eta \xrightarrow{N \rightarrow +\infty} 1.$$

In other words, when N tends towards $+\infty$, the displacement of the approximated initial solution lies in $[E_d^N - \eta, E_d^N + \eta]$ with a probability that tends towards 1.

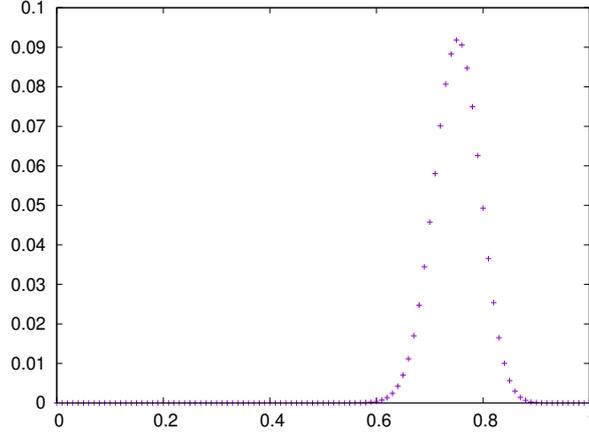


Figure 2: Probabilities P_N^K (y -axis) with respect to K/N (x -axis) for $N = 100$ and $\beta = 0.75$. The maximum is reached for $K = 75$ (with $K^{100,0.75} = 74$) and its value is $P_{100}^{75} \sim 0.0917997$.

Proof. The probability \mathcal{P}_N^η is defined as:

$$\mathcal{P}_N^\eta = \sum_{K \in \mathcal{H}_N(\eta)} P_N^K,$$

where $\mathcal{H}_N(\eta) = \{K \in \llbracket 0, N \rrbracket, |E_d^N - Kh| \leq \eta\}$ and $E_d^N = TU$. So, thanks to relation (28), there exists a number of iterations N_η for which we have:

$$\forall N \geq N_\eta, \{K^{N,\beta}, K^{N,\beta} + 1\} \subset \mathcal{H}_N(\eta).$$

From now we assume that N is greater than N_η . Let us also define K_{inf} (resp. K_{sup}) as the integer part of $(E_d^N - \eta)/h + 1$ (resp. of $(E_d^N + \eta)/h - 1$). Obviously, K_{inf} and K_{sup} belong to $\mathcal{H}_N(\eta)$. But since we have chosen $N > N_\eta$ and thanks to (29), we have the relation:

$$\forall K \notin \mathcal{H}_N(\eta), P_N^K < \max(P_N^{K_{inf}}, P_N^{K_{sup}})$$

from which we get that:

$$1 - \mathcal{P}_N^\eta = \sum_{K \notin \mathcal{H}_N(\eta)} P_N^K \leq \text{card}(\mathcal{H}'_N(\eta)) \times \max(P_N^{K_{inf}}, P_N^{K_{sup}}) \leq N \max(P_N^{K_{inf}}, P_N^{K_{sup}}), \quad (30)$$

where $\mathcal{H}'_N(\eta)$ is the complement of $\mathcal{H}_N(\eta)$: $\mathcal{H}'_N(\eta) = \{K \in \llbracket 0, N \rrbracket, |E_d^N - Kh| > \eta\}$. Thanks to our choices, we have in fact:

$$(E_d^N - \eta)/h + 1 = N \beta \left(1 - \frac{\eta}{TU}\right) + 1 \quad \text{and} \quad (E_d^N + \eta)/h - 1 = N \beta \left(1 + \frac{\eta}{TU}\right) - 1.$$

It means that for $0 < \eta < TU$, K_{inf} and K_{sup} tend to $+\infty$ as fast as N when N tends to $+\infty$. Since in the definition of P_N^K , the integer K is the exponent of $\beta \in]0, 1[$, it means that the term involving the maximum on the right hand side of (30) tends to zero faster than $1/N$ when N tends towards $+\infty$. As a consequence, we get that:

$$0 \leq 1 - \mathcal{P}_N^\eta \leq N \max(P_N^{K_{inf}}, P_N^{K_{sup}}) \xrightarrow{N \rightarrow +\infty} 0,$$

and we can conclude that the probability for the displacement of the approximated initial solution to be at a distance less than η from the exact displacement tends to 1 when the number of iterations tends towards $+\infty$. \square

As the result of proposition 4.5 remains true for any $\eta > 0$, it can be stated that the displacement due to the Upwind-GRU scheme tends towards the exact displacement with probability 1 when the number of iterations N tends towards $+\infty$. We can then conclude the whole proof with the following result.

Proposition 4.6 *Let us assume that the final time T and the CFL number $\beta \in]0, 1[$ are given, so that both h and dt are known for any number of iterations N . The approximated solutions computed with the Upwind-GRU*

scheme at time T for a CFL number β converge towards the exact solution with probability 1 when N tends towards $+\infty$ in the sense that:

$$\forall \eta > 0, P(|X_f(T) - X_f^{N,gru}| \leq \eta) \xrightarrow{N \rightarrow +\infty} 1,$$

where $X_f(T)$ is the position of the exact front at time T and $X_f^{N,gru}$ the position of the front for the approximated solution at time T for N iterations.

Proof. For a final time T and a CFL number $\beta \in]0, 1[$, both h and Δt are known for any number of iterations N through the relations: $\Delta t = T/N$ and $h = TU/(N\beta)$. The position of the approximated initial front $X_f^{0,N}$ for the approximated initial condition (17) also only depends on the number of iterations N . When considering the initial condition associated with (17):

$$\begin{cases} 1 & \text{if } (x - X_f^{0,N}) < 0, \\ 0 & \text{otherwise,} \end{cases} \quad (31)$$

the position $X_f^N(T)$ of the front at time T of the exact solution of system (1) with initial condition (31) is:

$$X_f^N(T) = X_f^{0,N} + UT.$$

Let us denote by $X_f(T)$ the position at time T of the exact front in the solution (16) of system of equations (1) for the initial condition (15):

$$X_f(T) = X_f^0 + UT.$$

We know from proposition 4.1 that for the approximated initial condition (17), the approximated solution computed by the Upwind-GRU corresponds to a translation of the approximated initial condition. We can thus define a front location $X_f^{N,gru}$ for the approximated solution computed by the Upwind-GRU scheme after N iterations.

Thanks to our choice for the approximated initial condition, we obtain that:

$$|X_f(T) - X_f^N(T)| = |X_f^{0,N} - X_f^0| \leq h.$$

We choose $\eta > 0$, then there exists N_0 such that: $N > N_0 \Rightarrow h < \eta/2$, and thus $|X_f(T) - X_f^N(T)| < \eta/2$. Hence, for $N > N_0$ we have $\{|X_f(T) - X_f^{N,gru}| \geq \eta\} \subset \{|X_f^N(T) - X_f^{N,gru}| \geq \eta/2\}$. Thanks to proposition 4.5, we know that the probability $P(|X_f^N(T) - X_f^{N,gru}| \geq \eta/2)$ tends towards zero when N tends towards $+\infty$; whereof we deduce that:

$$P(|X_f(T) - X_f^{N,gru}| \geq \eta) \xrightarrow{N \rightarrow +\infty} 0,$$

and therefore we conclude that:

$$P(|X_f(T) - X_f^{N,gru}| \leq \eta) \xrightarrow{N \rightarrow +\infty} 1.$$

Since this result holds for any $\eta > 0$, the approximated solutions computed with the Upwind-GRU scheme at time T for a CFL number β converge towards the exact solution with probability 1 when N tends towards $+\infty$. This ends the proof of proposition 4.6. \square

Finally, we have proved that the approximated solutions computed with the Upwind-GRU scheme converge towards the exact solution with probability 1 for the specific problem involving: a constant and uniform velocity and an initial condition based on the Heavyside function. It is possible to extend this result to more general initial conditions while keeping a constant and uniform velocity. Nevertheless, for these more general cases, it should be accounted for the pathological behavior described in section 3.2. It should still be noted that in the case of non constant and non-uniform velocities, the proof becomes very tricky. In particular expressing the probabilities (21) as in proposition 4.2 is no more possible since the transitional probabilities that are used in the proof of proposition 4.1 depend on the time-iteration. This is indeed the key point in the proof proposed here.

4.2 Two-dimensional case

In the following, the results of section 4.1 are extended to the two-dimensional case. Each proposition of the previous section has thus its counterpart in the present section. Let us begin by defining the initial condition:

$$\Phi^0(x, y) = \begin{cases} 1 & \text{if } (x - X_f^0) \cos(\theta) + (y - Y_f^0) \sin(\theta) < 0, \\ 0 & \text{otherwise.} \end{cases} \quad (32)$$

for system (1). We define the line $D_f^0 : (x - X_f^0) \cos(\theta) + (y - Y_f^0) \sin(\theta) = 0$ where (X_f^0, Y_f^0) corresponds to a reference point of D_f^0 , and $(\cos(\theta), \sin(\theta))$ corresponds to the unit vector which is orthogonal to D_f^0 and oriented following the velocity field (we recall that $\theta \in [0, \Pi/2]$). This initial condition corresponds to $\Phi^0(x, y) = 1$ for all the points (x, y) below D_f^0 and to $\Phi^0(x, y) = 0$ for all the points (x, y) above D_f^0 . Thanks to the assumptions detailed in section 2, the exact solution of (1) with the initial condition (32) is:

$$\Phi(t, x, y) = \begin{cases} 1 & \text{if } (x - X_f(t)) \cos(\theta) + (y - Y_f(t)) \sin(\theta) < 0, \\ 0 & \text{otherwise,} \end{cases} \quad (33)$$

where the reference point $(X_f(t), Y_f(t))$ corresponds to the translation of the reference point (X_f^0, Y_f^0) from the vector $(tU \cos(\theta), tU \sin(\theta))$, that is:

$$X_f(t) = X_f^0 + tU \cos(\theta) \quad \text{and} \quad Y_f(t) = Y_f^0 + tU \sin(\theta). \quad (34)$$

Let us now turn to the approximated solutions of (33) and (34) computed by the Upwind-GRU scheme.

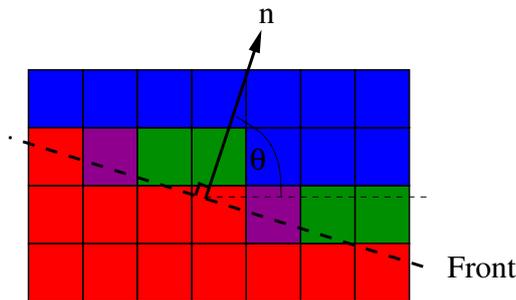


Figure 3: Example of a front on a given mesh at the beginning of a time-iteration. The front is represented by the dashed line, and n stands for its normal vector. The red cells contain the value 1, and all the other one contain the value 0. Therefore: red cells correspond to case 4, blue cells correspond to case 3, green cells correspond to case 1 and purple cells correspond to case 2

We first define the approximated initial condition $\Phi_{i,j}^0$ for the Upwind-GRU scheme. Even if several choices are possible, they have no influence on the results of this section provided that we have:

$$\sum_{(i,j)} \int_{(i,j)} |\Phi(t=0, x, y) - \Phi_{i,j}^0| \, dx dy = o(h). \quad (35)$$

In the following, we define:

$$\Phi_{i,j}^0 = \Phi^0(X_i, Y_j), \quad (36)$$

which is a classical choice fulfilling (35).

Starting from this initial condition, we then apply the Upwind-GRU scheme with a time-step Δt such that the CFL condition of section 2.1 is fulfilled. Then the following proposition holds for the transitional probabilities.

Proposition 4.7 *With an approximated initial solution of the form (32) and (36) at time $t^0 = 0$, the transitional probabilities for the approximated solution from time t^n to time $t^{n+1} = t^n + \Delta t$, obtained with the Upwind-GRU scheme are:*

$$\begin{cases} \mathcal{P}(\forall(i, j), \Phi_{i,j}^{n+1} = \Phi_{i-1,j}^n) = \beta \cos(\theta), \\ \mathcal{P}(\forall(i, j), \Phi_{i,j}^{n+1} = \Phi_{i,j-1}^n) = \beta \sin(\theta), \\ \mathcal{P}(\forall(i, j), \Phi_{i,j}^{n+1} = \Phi_{i,j}^n) = 1 - \beta(\cos(\theta) + \sin(\theta)). \end{cases} \quad (37)$$

The parameter β corresponds to the CFL number and it should belong to $]0, (\cos(\theta) + \sin(\theta))^{-1}[$ for ensuring the stability of the numerical scheme.

Proof. Due to our choices for the velocity field and the initial condition we are faced to only four of the eight configurations gathered in tables 1 and 2:

- case 1: $\Phi_{i,j}^0 = 0$, $\Phi_{i-1,j}^0 = 1$, and $\Phi_{i,j-1}^0 = 0$;

- case 2: $\Phi_{i,j}^0 = 0$, $\Phi_{i-1,j}^0 = 1$, and $\Phi_{i,j-1}^0 = 1$;
- case 3: $\Phi_{i,j}^0 = 0$, $\Phi_{i-1,j}^0 = 0$, and $\Phi_{i,j-1}^0 = 0$;
- case 4: $\Phi_{i,j}^0 = 1$, $\Phi_{i-1,j}^0 = 1$, and $\Phi_{i,j-1}^0 = 1$.

Cases 3 and 4 correspond to cells (i, j) for which the approximated value of Φ can not change at the first iteration: $\Phi_{i,j}^1 = \Phi_{i,j}^0$. We thus focus on the cells of cases 1 and 2, that are the cells just downwind the approximated front, see figure 3. For these cells, the first step Upwind-GRU scheme leads to the values:

- case 1: $\Phi_{i,j}^{1,*} = \beta \cos(\theta)$;
- case 2: $\Phi_{i,j}^{1,*} = \beta(\cos(\theta) + \sin(\theta))$.

Then, the second step of the Upwind-GRU scheme leads to:

- case 1: $\Phi_{i,j}^1 = 1$ if $\omega^0 \in [0, \beta \cos(\theta)]$, and $\Phi_{i,j}^{1,*} = 0$ otherwise;
- case 2: $\Phi_{i,j}^1 = 1$ if $\omega^0 \in [0, \beta(\cos(\theta) + \sin(\theta))]$, and $\Phi_{i,j}^{1,*} = 0$ otherwise.

Since $\theta \in [0, \Pi/2]$, we have $0 \leq \beta \cos(\theta) \leq \beta(\cos(\theta) + \sin(\theta)) \leq 1$. Moreover, ω^0 is the same for all the cells, therefore three possibilities arise:

- When $\omega^0 \in [0, \beta \cos(\theta)]$, cells in case 1 and 2 change from 0 to 1, and $\Phi_{i,j}^1$ corresponds to a translation of the initial condition of h in the direction of the positive x : $\Phi_{i,j}^1 = \Phi_{i-1,j}^0$.
- When $\omega^0 \in]\beta \cos(\theta), \beta(\cos(\theta) + \sin(\theta))]$, only the cells in case 2 change from 0 to 1, and $\Phi_{i,j}^1$ corresponds to a translation of the initial condition of h in the direction of the positive y : $\Phi_{i,j}^1 = \Phi_{i,j-1}^0$.
- When $\omega^0 \in]\beta(\cos(\theta) + \sin(\theta)), 1]$, all the cells remain unchanged, and $\Phi_{i,j}^1$ is equal to the initial condition: $\Phi_{i,j}^1 = \Phi_{i,j}^0$.

Hence we can deduce from these remarks that, at first iteration, the initial profile is simply translated with the following transitional probabilities

$$\begin{cases} \mathcal{P}(\forall(i, j), \Phi_{i,j}^1 = \Phi_{i-1,j}^0) = \beta \cos(\theta), \\ \mathcal{P}(\forall(i, j), \Phi_{i,j}^1 = \Phi_{i,j-1}^0) = \beta \sin(\theta), \\ \mathcal{P}(\forall(i, j), \Phi_{i,j}^1 = \Phi_{i,j}^0) = 1 - \beta(\cos(\theta) + \sin(\theta)). \end{cases} \quad (38)$$

Two important points have to be quoted here in order to pursue the proof. First, the update from $t = 0$ to $t = \Delta t$ preserves the initial profile in the sense that it is just translated to the left or to the top, or it remains at its initial location. The second point is that the transitional probabilities (38) only depend on the velocity field, on the time-step Δt and on the mesh-size h . Since all these parameters are uniform and constant, the probabilities (38) remain unchanged for all iterations and for all cells. Hence, the iteration from a time t^n to the time t^{n+1} exactly corresponds to the first iteration in terms of transitional probabilities, and we thus obtain the probabilities (37). \square

For the sake of readability, let us denote by $P_x = \beta \cos(\theta)$ the probability to shift the approximated solution from one cell in the direction of positive x and by $P_y = \beta \sin(\theta)$ the probability to shift the approximated solution from one cell in the direction of positive y .

Let us consider $N > 0$ time-iterations of the Upwind-GRU scheme with (32) and (36) as an initial condition. These N iterations are composed of successive x -shifts, y -shifts or standstills of the approximated initial solution. Since P_x and P_y do not depend of the iteration, all the combinations of the same number of shifts and standstills are equivalent. Therefore, the probability for each configurations is given by proposition 4.8.

Proposition 4.8 *The probability to get exactly K_x shifts along x and K_y shifts along y , with $(K_x, K_y) \in \llbracket 0, N \rrbracket^2$ and $K_x + K_y \leq N$, reads:*

$$P_N^{K_x, K_y} = \mathcal{C}_N^{K_x, K_y} (P_x)^{K_x} (P_y)^{K_y} (1 - P_x - P_y)^{N - K_x - K_y}. \quad (39)$$

where $\mathcal{C}_N^{K_x, K_y}$ stands for the number of combinations:

$$\mathcal{C}_N^{K_x, K_y} = \frac{N!}{K_x! K_y! (N - K_x - K_y)!}.$$

Proof. As mentioned above in the proof of proposition 4.7, for each iteration we have the same transitional probabilities (37). Moreover, the random numbers ω^n involved in the second step of the Upwind-GRU scheme are independent for all the iterations. Hence each realization is independent and follows (19). Therefore, all the configurations of K_x x -shifts, K_y y -shifts and $N - K$ standstills are equivalent in terms of probability. As a consequence, the configurations obtained after N iterations with the Upwind-GRU scheme follow the multinomial distribution with the parameter (P_x, P_y) : $\mathcal{B}(N, P_x, P_y)$. Thus, we obtain the formula (39) for the probability $P_N^{K_x, K_y}$ to get K_x x -shifts and K_y y -shifts for N iterations. \square

We know from the previous proposition that the different configurations that can be obtained after N iterations follow a multinomial distribution $\mathcal{B}(N, P_x, P_y)$. Since the space discretisation is known, the displacement associated with each configuration can easily be obtained. Therefore, the expectation of the displacement of the approximated initial solution after N iterations, E_d^N , can be explicitly written. The following result then holds.

Proposition 4.9 *The expectation of the displacement of the approximated initial solution after N iterations E_d^N is equal to the displacement of the approximated initial solution through system (1).*

Proof. Thanks to this probability (39), we can write explicitly the expectation E_d^N of the displacement of the initial approximated solution after N iterations by summing all the displacement weighted by their probabilities:

$$E_d^N = \sum_{0 \leq K_x + K_y \leq N} \left(P_N^{K_x, K_y} (K_x h, K_y h) \right). \quad (40)$$

The x -component $E_{d,x}^N$ of the average displacement E_d^N can be written in a more convenient form:

$$E_{d,x}^N = h \sum_{K_x=1}^N \left(\sum_{K_y=0}^{N-K_x} \left(K_x \mathcal{C}_N^{K_x, K_y} (P_x)^{K_x} (P_y)^{K_y} (1 - P_x - P_y)^{N-K_x-K_y} \right) \right).$$

Yet for $1 \leq K_x \leq N$ we have the relation:

$$K_x \mathcal{C}_N^{K_x, K_y} = N \mathcal{C}_{N-1}^{K_x-1, K_y},$$

hence the formula above for $E_{d,x}^N$ can be simplified in:

$$E_{d,x}^N = NhP_x \sum_{K_x=0}^{N-1} \left(\sum_{K_y=0}^{(N-1)-K_x} \left(K_x \mathcal{C}_{N-1}^{K_x, K_y} (P_x)^{K_x} (P_y)^{K_y} (1 - P_x - P_y)^{(N-1)-K_x-K_y} \right) \right),$$

$$E_{d,x}^N = NhP_x \sum_{0 \leq K_x + K_y \leq N-1} \left(\mathcal{C}_{N-1}^{K_x, K_y} (P_x)^{K_x} (P_y)^{K_y} (1 - P_x - P_y)^{(N-1)-K_x-K_y} \right). \quad (41)$$

The sum in the formula above corresponds to the development of the polynomial $(P_x + P_y + (1 - P_x - P_y))^{(N-1)}$ which is equal to 1. So that we finally get:

$$E_{d,x}^N = NhP_x.$$

Thanks to the definition of P_x and β , we then obtain that:

$$E_{d,x}^N = Nh\beta \cos(\theta) = N\Delta t U \cos(\theta) = t^N U_x. \quad (42)$$

Obviously, the same computations can be done for $E_{d,y}^N$, the y -component of E_d^N , and it can be found that:

$$E_{d,y}^N = t^N U_y. \quad (43)$$

As a consequence, the expectation of the displacement is equal to the exact displacement due to the velocity field (U_x, U_y) :

$$E_d^N = t^N (U_x, U_y).$$

This ends the proof of proposition (4.9). \square

At that point, we have thus proved that each approximated solution $\Phi_{i,j}^N$ computed with the Upwind-GRU scheme corresponds to the initial solution $\Phi_{i,j}^0$, translated and that the expectation of these translations is equal to the displacement of the exact solution between time $t = 0$ and time $t = t^N$. For the continuation of the proof, we need to study in details the probabilities (39). In particular, it will be proved that $(K_x, K_y) \mapsto P_N^{K_x, K_y}$ possesses a unique maximum on $\mathcal{D}_N = \{[0, N]^2, K_x + K_y \leq N\}$.

Proposition 4.10 *For any number of iterations $N > N_{\beta, \theta}$, with $N_{\beta, \theta} = \max(1, \max(1/P_x, 1/P_y) - 2)$, there exists two integers $0 \leq K_x^* < N$ and $0 \leq K_y^* < N$, such that the probability $P_N^{K_x, K_y}$ reaches a maximum for (K_x^*, K_y^*) . Moreover, the displacement associated with (K_x^*, K_y^*) fulfills:*

$$|E_{d,x}^N - hK_x^*| \leq 2 \max(P_x, (1 - P_x))h, \quad (44)$$

and:

$$|E_{d,y}^N - hK_y^*| \leq 2 \max(P_y, (1 - P_y))h. \quad (45)$$

Proof. For a given number of iterations N it can be shown by simple calculus rules the following relations between the probabilities:

$$P_N^{K_x+1, K_y} = P_N^{K_x, K_y} \frac{P_x(N - K_x - K_y)}{(1 - P_x - P_y)(K_x + 1)}; \quad (46)$$

$$P_N^{K_x, K_y+1} = P_N^{K_x, K_y} \frac{P_y(N - K_x - K_y)}{(1 - P_x - P_y)(K_y + 1)}. \quad (47)$$

From (46) and (47), one can obtain that:

$$\begin{aligned} P_N^{K_x+1, K_y} \geq P_N^{K_x, K_y} &\iff NP_x - (1 - P_x - P_y) - P_x K_y - (1 - P_y)K_x \geq 0; \\ P_N^{K_x, K_y+1} \geq P_N^{K_x, K_y} &\iff NP_y - (1 - P_x - P_y) - (1 - P_x)K_y - P_y K_x \geq 0. \end{aligned}$$

Since the probabilities P_x and P_y do not depend on the iteration, each inequality above is affine with respect to (K_x, K_y) . The associated sub-domains of \mathcal{D}_N are then separated by a line. For the first inequality, and respectively the second one, we define the lines (D_x) and respectively (D_y) :

$$\begin{aligned} (D_x) : NP_x - (1 - P_x - P_y) - P_x K_y - (1 - P_y)K_x &= 0; \\ (D_y) : NP_y - (1 - P_x - P_y) - (1 - P_x)K_y - P_y K_x &= 0. \end{aligned}$$

These two lines are different because the determinant of the matrix that contains their normal vector reads:

$$\begin{vmatrix} 1 - P_y & P_y \\ P_x & 1 - P_x \end{vmatrix} = 1 - P_x - P_y = 1 - \beta(\cos(\theta) + \sin(\theta)),$$

which is non-negative thanks to the CFL condition for β . Moreover, the two components of the normal vector of (D_x) (resp. (D_y)) have the same sign which implies that (D_x) (resp. (D_y)) has a negative slope in the (K_x, K_y) -plane. We now define the two lines that are the counterpart of (D_x) and (D_y) in the (x, y) -plane:

$$\begin{aligned} (D'_x) : NP_x - (1 - P_x - P_y) - P_x y - (1 - P_y)x &= 0; \\ (D'_y) : NP_y - (1 - P_x - P_y) - (1 - P_x)y - P_y x &= 0. \end{aligned}$$

The line (D'_x) (resp. (D'_y)) has the same normal vector and the same distance to the origin $(0, 0)$ than (D_x) (resp. (D_y)). This distance to the origin is $NP_x - (1 - P_x - P_y)$ for (D_x) and (D'_x) , and $NP_y - (1 - P_x - P_y)$ for (D_y) and (D'_y) . Let us now compute (x^*, y^*) the intersection point for (D'_x) and (D'_y) in the (x, y) -plane. It can be easily found that:

$$x^* = NP_x - (1 - 2P_x) \quad \text{and} \quad y^* = NP_y - (1 - 2P_y).$$

It is an important point to be quoted that in general x^* and y^* do not lie in $[0, N]$, this explains why (D'_x) and (D'_y) have been introduced. For small N , x^* and y^* can be non-positive. But for any given β and θ , there exist a number of iterations $N_{\beta, \theta} = \max(1, \max(1/P_x, 1/P_y) - 2)$ such that for all $N > N_{\beta, \theta}$ we have $x^* \geq 0$ and $y^* \geq 0$. Moreover, we have:

$$x^* + y^* \leq N \iff P_x + P_y \leq 1,$$

which is fulfilled thanks to the CFL condition for β . We assume from now that $N \geq N_{\beta, \theta}$. Therefore we get that $(x^*, y^*) \in \{[0, N], x^* + y^* \leq N\}$, the latter subset being the counterpart in \mathbb{R}^2 of the subset \mathcal{D}_N . By multiplying the coordinates (x^*, y^*) by h , and thanks to relations (42) and (43), we obtain the associated displacement:

$$h(x^*, y^*) = E_d^N + h(1 - 2P_x, 1 - 2P_y). \quad (48)$$

Since P_x and P_y lie in $[0, 1]$, we thus get that

$$|hx^* - E_{d,x}^N| < h; \quad (49)$$

$$|hy^* - E_{d,y}^N| < h. \quad (50)$$

Let us now turn back to the probabilities $P_N^{K_x, K_y}$. From the previous remarks, see also figure 4, we know

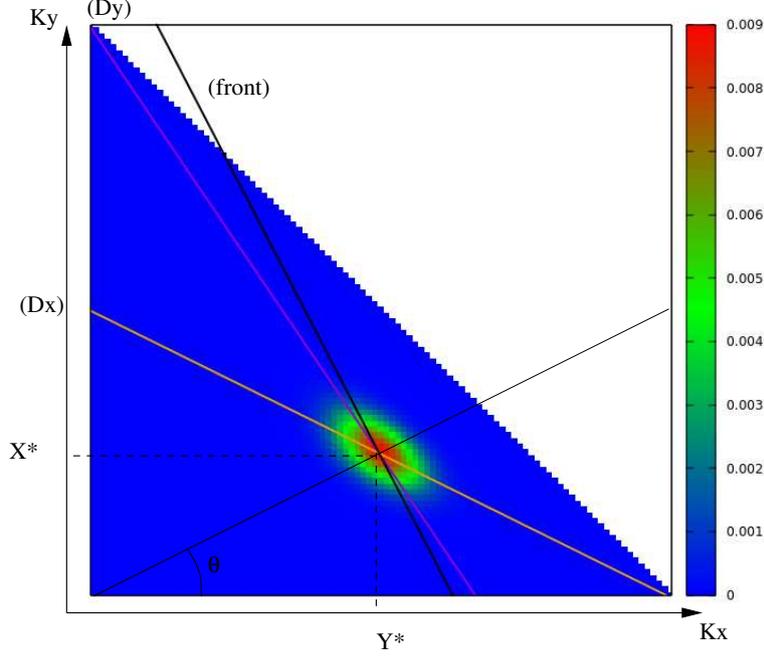


Figure 4: Probabilities $P_N^{K_x, K_y}$ with respect to K_x and K_y for $N = 100$, $\theta = 0.15 \Pi$ and $\beta = 0.75/(\cos(\theta) + \sin(\theta))$. The front (black line), the lines (D_x) (orange line) and (D_y) (purple line), and the point (x^*, y^*) are plotted.

that $K_x \mapsto P_N^{K_x, K_y}$ increases when (K_x, K_y) is below (D'_x) and decreases when (K_x, K_y) is above (D'_x) ; and that $K_y \mapsto P_N^{K_x, K_y}$ increases when (K_x, K_y) is below (D'_y) and decreases when (K_x, K_y) is above (D'_y) . The intersection point of these two lines is (x^*, y^*) , so that the maximum of $(K_x, K_y) \mapsto P_N^{K_x, K_y}$ on \mathcal{D}_N is reached at a point (K_x^*, K_y^*) such that $|x^* - K_x^*| \leq 1$ and $|y^* - K_y^*| \leq 1$. Then, using relation (48), the inequality $|x^* - K_x^*| \leq 1$ leads to:

$$\begin{aligned} -1 &\leq x^* - K_x^* \leq 1; \\ -h &\leq E_{d,x}^N + h(1 - 2P_x) - hK_x^* \leq h; \\ -2(1 - P_x)h &\leq E_{d,x}^N - hK_x^* \leq 2P_x h. \end{aligned}$$

This leads to inequality (44) and, applying the same idea for the y -component, (45) can be found. In fact, these two inequalities state that the most probable displacement (hK_x^*, hK_y^*) following $P_N^{K_x, K_y}$ tends towards the exact displacement E_d^N when the mesh size h tends towards zero (at a fixed β). This ends the proof of proposition 4.10. \square

Let us choose $\eta > 0$, and let us define the subset of \mathcal{D}_N :

$$\mathcal{H}_N(\eta) = \{(K_x, K_y) \in \mathcal{D}_N / |E_{d,x}^N - hK_x| \leq \eta \quad \text{and} \quad |E_{d,y}^N - hK_y| \leq \eta\}.$$

The subset $\mathcal{H}_N(\eta)$ contains all the possible couples (K_x, K_y) that, for N iterations, lead to a displacement which have a distance from the exact displacement less than η . Since we have a cartesian setting, the distance used here correspond to a L^1 -norm component by component. Let us fix a final time T and let us fix a CFL number β in $]0, (\cos(\theta) + \sin(\theta))^{-1}[$. For a given number of iterations N , the time-step is chosen constant: $\Delta t = T/N$. We thus get that the mesh size if fixed and reads: $h = TU/(N\beta)$.

Proposition 4.11 *Let us assume that the final time T and the CFL number $\beta \in]0, (\cos(\theta) + \sin(\theta))^{-1}[$ are given, so that both h and dt are known for any number of iterations N . Then, for a given $\eta > 0$, the probability for the*

displacement of the approximated initial solution to be in the domain $[E_{d,x}^N - \eta, E_{d,x}^N + \eta] \times [E_{d,y}^N - \eta, E_{d,y}^N + \eta]$ after N time-iterations, denoted by \mathcal{P}_N^η , is such that :

$$\mathcal{P}_N^\eta \xrightarrow{N \rightarrow +\infty} 1.$$

In other words, when N tends towards $+\infty$, the displacement of the approximated initial solution lies in $[E_{d,x}^N - \eta, E_{d,x}^N + \eta] \times [E_{d,y}^N - \eta, E_{d,y}^N + \eta]$ with a probability that tends towards 1.

Proof. Inequalities (44) and (45) imply that, for a given h , there exists $N_\eta > N_{\beta,\theta}$ such that for all $N > N_\eta$ we have $(K_x^*, K_y^*) \in \mathcal{H}_N(\eta)$. In the following, we assume that we have $N > N_\eta$.

From the previous results, see inequalities (49)-(50) and the results of monotony for $P_N^{K_x, K_y}$ on \mathcal{D}_N , we can deduce that there exists a couple $(K_{x,m}, K_{y,m})$ in $\mathcal{D}_N \setminus \mathcal{H}_N(\eta)$ such that:

$$\forall (K_x, K_y) \in \mathcal{D}_N \setminus \mathcal{H}_N(\eta), P_N^{K_x, K_y} \leq P_N^{K_{x,m}, K_{y,m}}.$$

Due to the monotony results, this maximum probability $P_N^{K_{x,m}, K_{y,m}}$ is always reached close to the frontier between $\mathcal{H}_N(\eta)$ and $\mathcal{D}_N \setminus \mathcal{H}_N(\eta)$, which means that we have:

$$\eta < |E_{d,x}^N - hK_{x,m}| \leq \eta + h, \quad \text{and} \quad \eta < |E_{d,y}^N - hK_{y,m}| \leq \eta + h. \quad (51)$$

As a consequence, by using (42) and the relation $h = TU/(N\beta)$, the first inequality of (51) gives:

$$\begin{aligned} E_{d,x}^N - \eta - h &\leq hK_{x,m} \leq E_{d,x}^N + \eta + h; \\ hNP_x - \eta - h &\leq hK_{x,m}; \\ N\beta \left(\cos(\theta) - \frac{\eta}{TU} \right) &\leq K_{x,m} + 1. \end{aligned} \quad (52)$$

In the same way, one can easily find from the second inequality of (51) that:

$$N\beta \left(\sin(\theta) - \frac{\eta}{TU} \right) \leq K_{y,m} + 1. \quad (53)$$

We can deduce from inequalities (52) and (53), and from the fact that the parameters θ , T and U are fixed, that for small enough values of η the integers $K_{x,m}$ and $K_{y,m}$ tend towards $+\infty$ linearly when N tends towards $+\infty$. According to (52) and (53), this is the case if the following sufficient condition is fulfilled: $\eta \leq \min(U_x, U_y)T$. From now we thus assume that η fulfills that condition.

The probability to have a couple (K_x, K_y) in $\mathcal{D}_N \setminus \mathcal{H}_N(\eta)$ is thus:

$$\mathcal{P}((K_x, K_y) \in \mathcal{D}_N \setminus \mathcal{H}_N(\eta)) = \sum_{(K_x, K_y) \in \mathcal{D}_N \setminus \mathcal{H}_N(\eta)} P_N^{K_x, K_y} < \text{card}(\mathcal{D}_N \setminus \mathcal{H}_N(\eta)) P_N^{K_{x,m}, K_{y,m}}.$$

Since $\text{card}(\mathcal{D}_N \setminus \mathcal{H}_N(\eta)) \leq N^2$, we get the inequality:

$$\mathcal{P}((K_x, K_y) \in \mathcal{D}_N \setminus \mathcal{H}_N(\eta)) < N^2 P_N^{K_{x,m}, K_{y,m}},$$

and by expressing explicitly $P_N^{K_{x,m}, K_{y,m}}$ we obtain:

$$\mathcal{P}((K_x, K_y) \in \mathcal{D}_N \setminus \mathcal{H}_N(\eta)) < N^2 C_N^{K_{x,m}, K_{y,m}} (P_x)^{K_{x,m}} (P_y)^{K_{y,m}} (1 - P_x - P_y)^{N - K_{x,m} - K_{y,m}}.$$

When N tends towards $+\infty$, the term $N^2 C_N^{K_{x,m}, K_{y,m}}$ on the right hand side of the inequality tends to $+\infty$ as a polynomial in N ; whereas the three other terms tend to zero as exponential terms (since P_x and P_y are in $[0, 1]$). Hence, we finally get that

$$\mathcal{P}((K_x, K_y) \in \mathcal{D}_N \setminus \mathcal{H}_N(\eta)) \xrightarrow{N \rightarrow +\infty} 0 \quad (54)$$

and obviously we have

$$\mathcal{P}((K_x, K_y) \in \mathcal{H}_N(\eta)) \xrightarrow{N \rightarrow +\infty} 1. \quad (55)$$

The latter means that for any given $\eta \leq \min(U_x, U_y)T$, the probability to have a displacement inside $\mathcal{H}_N(\eta)$ tends to 1. Due to our choice, E_d^N does not depend on N : $E_d^N = t^N (U_x, U_y) = T(U_x, U_y)$. Thus the frontier of the domain $\mathcal{H}_N(\eta)$ which is given by h and E_d^N , does not depend on N . We can then deduce from (55) that:

$$\mathcal{P}_N^\eta = \mathcal{P}(|hK_x - TU_x| < \eta \text{ and } |hK_y - TU_y| < \eta) = \mathcal{P}((K_x, K_y) \in \mathcal{H}_N(\eta)) \xrightarrow{N \rightarrow +\infty} 1. \quad (56)$$

Since the limit (56) holds for all $0 < \eta \leq \min(U_x, U_y)T$, we can conclude that the approximated displacement associated with the Upwind-GRU scheme tends towards the exact displacement with probability 1 when the number of iterations tends towards $+\infty$ (i.e. when the time-step Δt or the mesh-size h tend towards 0). This point achieved the proof of proposition 4.11. \square

In the following, we assume that the final time T and the CFL number $\beta \in]0, (\cos(\theta) + \sin(\theta))^{-1}[$ are given, so that both h and Δt are known for any number of iterations N . At the beginning of the section the line D_f^0 has been introduced. It represents the position of the initial exact front. We then define three other fronts.

- $D_f(t)$ stands for the exact front at time t for system (1) with the initial condition (32). In other words, $D_f(t)$ corresponds to the translation with the displacement $U \times t$ of the initial exact front D_f^0 . The initial exact front D_f^0 is determined by the reference point (X_f^0, Y_f^0) and the normal vector n . Thanks to system (1), the front $D_f(t)$ is determined by the same normal vector n and by the reference point $(X_f(t), Y_f(t))$ which has been defined at the beginning of this section.
- From the approximated initial condition (36), we define an approximated initial front $D_f^{0,N}$ which obviously depends on the number of iterations N (i.e. on the mesh size). By definition of the approximated initial solution, we know that the front $D_f^{0,N}$ is also determined by the normal vector n and by a reference point $(X_f^{0,N}, Y_f^{0,N})$ that also depends on the number of iterations N (i.e. on the mesh size).
- After N iterations of the Upwind-GRU scheme, the approximated initial front $D_f^{0,N}$ is displaced and denoted by $D_f^{N,gru}$.

From these definitions and from the previous results, we can then state the following proposition for the front $D_f^{N,gru}$.

Proposition 4.12 *Let us assume that the final time T and the CFL number $\beta \in]0, (\cos(\theta) + \sin(\theta))^{-1}[$ are given, so that both h and Δt are known for any number of iterations N . The approximated front $D_f^{N,gru}$ obtained after N iterations of the Upwind-GRU scheme from the approximated initial front $D_f^{0,N}$ is determined by the normal vector n and by the reference point $(X_f^{N,gru}, Y_f^{N,gru})$, which is the translation of the reference point $(X_f^{0,N}, Y_f^{0,N})$ by the Upwind-GRU scheme.*

Proof. It is an important point to be recalled here that the same random number ω^k is used for all the cells of iteration k . Therefore, one time-step of the Upwind-GRU scheme corresponds to a translation $(+h, 0)$ or $(0, +h)$ of the approximated solution, see proposition 4.7. Consequently, for each time-step the slope of the approximated front remains unchanged, and so remains unchanged the normal vector to the approximated front. We then straightforwardly get that the normal vector to $D_f^{0,N}$ is n . It is also straightforward to get that $(X_f^{N,gru}, Y_f^{N,gru})$, the translation of the reference point $(X_f^{0,N}, Y_f^{0,N})$ after N iterations of the Upwind-GRU scheme belongs to the approximated front $D_f^{0,N}$. This ends the proof of proposition (4.12) \square

Since the result of proposition 4.11 remains true for any $\eta > 0$, it can be stated that the displacement due to the Upwind-GRU scheme tends towards the exact displacement with probability 1 when the number of iterations N tends towards $+\infty$. We can then conclude the whole proof with the following theorem.

Theorem 4.1 *Let us assume that the final time T and the CFL number $\beta \in]0, (\cos(\theta) + \sin(\theta))^{-1}[$ are given, so that both h and Δt are known for any number of iterations N . The approximated solutions computed with the Upwind-GRU scheme at time T converge towards the exact solution with probability 1 when N tends towards $+\infty$ in the sense that:*

$$\forall \eta > 0, P\left(d_n\left(D_f(T), D_f^{N,gru}\right) \leq \eta\right) \xrightarrow{N \rightarrow +\infty} 1,$$

where $D_f(T)$ stands for the exact front at time T and $D_f^{N,gru}$ stands for the approximated front obtained after N iterations of the Upwind-GRU scheme. The distance $d_n(\cdot, \cdot)$ denotes the distance between two lines with the same normal vector n .

Proof. For a final time T and a CFL number $\beta \in]0, 1[$, both h and dt are known for any number of iterations N through the relations: $\Delta t = T/N$ and $h = TU/(N\beta)$. Thanks to proposition (4.12), we know that both fronts $D_f(T)$ and $D_f^{N,gru}$ have the same normal vector n , thus the distance d_n can be defined as:

$$d_n(D_a, D_b) = |((X_a, Y_a) - (X_b, Y_b)) \cdot n|,$$

where the points (X_a, Y_a) and (X_b, Y_b) respectively belong to the lines D_a and D_b . When considering the initial condition associated with (36):

$$\begin{cases} 1 & \text{if } \left((x, y) - (X_f^{0,N}, Y_f^{0,N}) \right) \cdot n < 0, \\ 0 & \text{otherwise,} \end{cases} \quad (57)$$

the position of the reference point $(X_f^N(T), Y_f^N(T))$ of the front at time T of the exact solution of system (1) with initial condition (57) is:

$$(X_f^N(T), Y_f^N(T)) = (X_f^{0,N}, Y_f^{0,N}) + UT.$$

$(X_f(T), Y_f(T))$ denotes the position at time T of the exact front in the solution (33) of system of equations (1) for the initial condition (32), we then have:

$$(X_f(T), Y_f(T)) = (X_f^0, Y_f^0) + UT.$$

We know from proposition 4.7 that for the approximated initial condition (36), the approximated solution computed by the Upwind-GRU corresponds to a translation of the approximated initial condition. As stated by proposition (4.12), the approximated front $D_f^{N,gru}$ can be determined by the normal n and by the location of the reference point $(X_f^{N,gru}, Y_f^{N,gru})$ which corresponds to the approximated displacement of $(X_f^{0,N}, Y_f^{0,N})$ after N iterations of the Upwind-GRU scheme.

Thanks to our choice for the approximated initial condition, we obtain that:

$$\left| \left((X_f(T), Y_f(T)) - (X_f^N(T), Y_f^N(T)) \right) \cdot n \right| = \left| \left((X_f^{0,N}, Y_f^{0,N}) - (X_f^0, Y_f^0) \right) \cdot n \right| \leq h.$$

We choose $\eta > 0$, then there exists N_0 such that: $N > N_0 \Rightarrow h < \eta/2$, and thus:

$$\left| \left((X_f(T), Y_f(T)) - (X_f^N(T), Y_f^N(T)) \right) \cdot n \right| < \eta/2.$$

Hence, for $N > N_0$ we have:

$$\left\{ \left| \left((X_f(T), Y_f(T)) - (X_f^{N,gru}, Y_f^{N,gru}) \right) \cdot n \right| \geq \eta \right\} \subset \left\{ \left| \left((X_f^N(T), Y_f^N(T)) - (X_f^{N,gru}, Y_f^{N,gru}) \right) \cdot n \right| \geq \eta/2 \right\}. \quad (58)$$

Thanks to proposition 4.11, we know that:

$$P \left(\left| \left((X_f^N(T), Y_f^N(T)) - (X_f^{N,gru}, Y_f^{N,gru}) \right) \cdot n \right| \geq \eta/2 \right) \xrightarrow{N \rightarrow +\infty} 0;$$

whereof from (58) we deduce that:

$$P \left(\left| \left((X_f(T), Y_f(T)) - (X_f^{N,gru}, Y_f^{N,gru}) \right) \cdot n \right| \geq \eta \right) \xrightarrow{N \rightarrow +\infty} 0.$$

Therefore we can conclude that:

$$P \left(\left| \left((X_f(T), Y_f(T)) - (X_f^{N,gru}, Y_f^{N,gru}) \right) \cdot n \right| \leq \eta \right) \xrightarrow{N \rightarrow +\infty} 1,$$

which reads using the distance d_n :

$$P \left(d_n \left(D_f(T), D_f^{N,gru} \right) \leq \eta \right) \xrightarrow{N \rightarrow +\infty} 1$$

Since this result holds for any $\eta > 0$, the approximated solutions computed with the Upwind-GRU scheme at time T for a CFL number β converge towards the exact solution with probability 1 when N tends towards $+\infty$. This ends the proof of theorem 4.1. \square

Finally, we have proved that the approximated solutions computed with the Upwind-GRU scheme converge towards the exact solution with probability 1 for the specific problem involving: a constant and uniform velocity and an initial condition composed of two states separated by a linear front. Obviously, it is possible to extend this result to the three-dimensional case when focusing of the same kind of configurations. For more complex configurations the same limitations are encountered as those shortly described at the end of section 4.1 concerning the one-dimensional case.

4.3 Three-dimensional case

The results of section (4.2) can straightforwardly be extended to the three-dimensional case. The same arguments can be used even if the calculus may be a little more complex. We thus do not develop entirely here the three-dimensional case, and we limit this section to the statement of the counterpart of theorem (4.1) for the convergence of the Upwind-GRU scheme. In the sequel we consider a uniform mesh composed of cubic cells with a length h . The set of equations is now:

$$\begin{cases} \partial_t (\Phi(t, x, y, z)) + U_x \partial_x (\Phi(t, x, y, z)) + U_y \partial_y (\Phi(t, x, y, z)) + U_z \partial_z (\Phi(t, x, y, z)) = 0, \\ \Phi(t = 0, x, y, z) = \Phi^0(x, y, z). \end{cases} \quad (59)$$

where the velocity field (U_x, U_y, U_z) is such that $U_x > 0$, $U_y > 0$ and $U_z > 0$. As in the previous section, we choose a specific the initial condition associated with a planar front:

$$\Phi^0(x, y, z) = \begin{cases} 1 & \text{if } (x - X_f^0)n_x + (y - Y_f^0)n_y + (z - Z_f^0)n_z < 0, \\ 0 & \text{otherwise.} \end{cases} \quad (60)$$

where $n = (n_x, n_y, n_z)$ is the normal vector to the initial front and (X_f^0, Y_f^0, Z_f^0) a reference point of the initial front. As in section (4.2), we assume that $n_x^2 + n_y^2 + n_z^2 = 1$ with $n_x > 0$, $n_y > 0$ and $n_z > 0$. The approximated initial condition $\Phi_{i,j,k}^0$ for the Upwind-GRU scheme is defined as:

$$\Phi_{i,j,k}^0 = \Phi^0(X_i, Y_j, Z_k), \quad (61)$$

where (X_i, Y_j, Z_k) corresponds to the center of gravity of the cell (i, j, k) .

Theorem 4.2 *Let us assume that the final time T and the CFL number $\beta \in]0, (n_x + n_y + n_z)^{-1}[$ are given, so that both h and Δt are known for any number of iterations N . The approximated solutions computed with the Upwind-GRU scheme at time T converge towards the exact solution with probability 1 when N tends towards $+\infty$ in the sense that:*

$$\forall \eta > 0, P \left(d_n \left(D_f(T), D_f^{N,gru} \right) \leq \eta \right) \xrightarrow{N \rightarrow +\infty} 1,$$

where $D_f(T)$ stands for the exact front at time T and $D_f^{N,gru}$ stands for the approximated front obtained after N iterations of the Upwind-GRU scheme. The distance $d_n(.,.)$ denotes the distance between two planes with the same normal vector n .

5 Conclusion

Some results and properties of the Upwind-GRU scheme have been studied in the sequel. In particular, a proof of convergence of the Upwind-GRU scheme has been proposed considering a two-dimensional setting and a particular class of problems. The latter corresponds to the advection of a planar front by a constant and uniform velocity field.

It is an important point to be quoted here that the choice of the same random number w^k for all the cells at iteration k appears as the key point in order to get a convergent scheme. This is clearly highlighted by the different results proposed in this work. Moreover, the proofs of section 4 are based on the properties of the multinomial probability law which arises from the assumption that the velocity field is constant and uniform. If this assumption does not hold, the proof may become more complex. Indeed, expressing a monotony result as the one stated in proposition 4.10 becomes very tricky. The extension to non-uniform Cartesian grids would lead to the same difficulties.

Nevertheless, some extensions of the proof to more general cases with uniform Cartesian grids could be possible. For instance, initial solution involving a non-planar front could also be considered provided that its local normal vector is not orthogonal to the faces of the cells. When the normal vector can be orthogonal to the faces of the mesh, the approximated solutions may undergo the pathological behavior reported in section 3.2. Therefore, this should be accounted for in the proof.

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