



A survey of differential operators related to Applied Sciences

Denise Huet

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A survey of differential operators related to Applied Sciences

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Abstract

This survey contains short accounts of publications related to applied sciences. The main topics which are dealt with are the well known models of pattern formation in reaction-diffusion systems, namely Gierer-Meinhardt, Gray-Scott and Klausmeier models, and the linear elasticity in the context of homogenization. These topics are not usually combined in a single document. It aims to be a useful reference for young researchers in mathematics and applied sciences.

AMS Subject Classification: 35K57; 35B32; 35QXX; 35B27; 74B05.

Key Words and Phrases: Patterns formation; Turing/Ginsburg-Landau bifurcation; Klausmeier-Gray-Scott models; Gierer-Meinhardt system; linear elasticity.

INTRODUCTION

The article is divided into several sections as follows

Contents

1	Pattern formation in reaction-diffusion systems	2
1.1	Linear stability analysis: stability of the trivial patterns	2
1.2	Destabilization by the Turing mechanism	3
1.3	The onset of patterns formation: the Turing/Ginsburg-Landau bifurcation . .	4
1.4	The particular case of the 1D-Gray-Scott model and chemistry	4
2	Generalized Klausmeier-Gray-Scott model and echosystems	5
2.1	Initial Klausmeier model	5
2.2	Results by S. van der Stelt et al. [12]	6
2.2.1	The model	6
2.2.2	The background states and the Turing-hopf instability	7
2.2.3	Ginzburg-Landau equation	8
3	Gierer-Meinhardt system and biology	9
3.1	Existence and stability of a single spike solution	9
3.1.1	Existence	9
3.1.2	Linear stability	10

3.2	Stationary Gierer-Meinhardt system	10
3.3	Existence and stability of a cluster of two spikes for the stationnary Gierer-Meinhardt system	10
4	Homogenization and linear elasticity	11
4.1	H and G-convergences	11
4.2	Spectral problems	12
4.2.1	The case of abstract operators defined in different Hilbert spaces . . .	12
4.2.2	Weak * convergence	13
4.2.3	Main result	13
4.3	Periodic homogenization	13
4.3.1	Spaces $L^p_{\sharp}(Y)$, $1 \leq p \leq \infty$	13
4.3.2	Example	13
4.4	Homogenization and linear elasticity	14
4.4.1	The linear elasticity operator	14
4.4.2	Homogenization in perforated domains	14
4.5	Homogenization and Steklov eigenproblems	15

1 Pattern formation in reaction-diffusion systems

Set-up (This section is taken from A. Doelman [4]).

Definition 1. An N -component reaction-diffusion system for $\mathbf{U} = (U_1, U_2, \dots, U_N) \in \mathbb{R}^N$ is a system of the form

$$\mathbf{U}_t = D\Delta\mathbf{U} + \mathbf{F}(\mathbf{U}, \mu), \quad (1)$$

where $\mathbf{U}(x, t)$ depends on $(x, t) \in \Omega \times \mathbb{R}^+$, with $\Omega \subset \mathbb{R}^n$, D is a diffusion matrix i.e. a diagonal $N \times N$ matrix with strictly positives entries, Δ is the Laplace/diffusion operator, $\mu \in \mathbb{R}^m$ represents parameters and the vector field $\mathbf{F}(\mathbf{U}, \mu) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ represents the nonlinear reaction terms.

Definition 2. In [11], A.M. Turing wrote “a system of chemical substances, called **morphogens**, reacting together and diffusing through a tissue, is adequate to account for the main phenomena of morphogenesis. Such a system, although it may originally be quite homogeneous, may later develop a **pattern** or structure due to an **instability** of the homogeneous equilibrium.

Definition 3. The **trivial patterns** of (1) are solutions $\bar{\mathbf{U}} = (\bar{U}_1, \dots, \bar{U}_N)$ of the algebraic equations:

$$\mathbf{F}(\bar{\mathbf{U}}, \mu) = 0 \text{ or } F_1(\bar{\mathbf{U}}, \mu) = 0, \dots, F_N(\bar{\mathbf{U}}, \mu) = 0 \quad (2)$$

From now on, $N=2$, $n=1$, $\Omega = \mathbb{R}$ (Cf. [4], Section 2). Following the notations of [4], set $\bar{U}_1 = \bar{U}$, $\bar{U}_2 = \bar{V}$, $F_1 = G$, $F_2 = H$.

1.1 Linear stability analysis: stability of the trivial patterns

Linearizing (1) around the trivial pattern $(\bar{U} + \alpha e^{ikx + \lambda t}, \bar{V} + \beta e^{ikx + \lambda t})$, we obtain the 2×2 linear eigenvalue problem:

$$\mathcal{A}(k, \mu) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} g_u - k^2 & g_v \\ h_u & h_v - dk^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (3)$$

where

$$g_u(\mu) = \frac{\partial G}{\partial U}(\bar{U}(\mu), \bar{V}(\mu); \mu), \quad g_v(\mu) = \frac{\partial G}{\partial V}(\bar{U}(\mu), \bar{V}(\mu); \mu), \quad (4)$$

and where

$$h_u(\mu) = \frac{\partial H}{\partial U}(\bar{U}(\mu), \bar{V}(\mu); \mu), \quad h_v(\mu) = \frac{\partial H}{\partial V}(\bar{U}(\mu), \bar{V}(\mu); \mu), \quad (5)$$

The characteristic polynomial associated to (3) is

$$\lambda^2 - [(g_u + h_v) - (1 + d)k^2]\lambda + [(g_u - k^2)(h_v - dk^2) - g_v h_u] = 0 \quad (6)$$

It defines 2 functions $k \in \mathbb{R} \rightarrow \lambda_{1,2}(k; \mu) \in \mathbb{C}$ such that (for instance) $\text{Re}(\lambda_2(k)) \leq \text{Re}(\lambda_1(k))$. The trivial state (\bar{U}, \bar{V}) is **spectrally stable** for the values of μ , for which

$$\text{Re}(\lambda_2(k; \mu)) \leq \text{Re}(\lambda_1(k; \mu)) < 0, \quad (7)$$

for all $k \in \mathbb{R}$.

1.2 Destabilization by the Turing mechanism

As $k \rightarrow \infty$, $\lambda_1 \sim -dk^2$ and $\lambda_2 \sim -k^2$ (or vice versa), and condition (7) is satisfied. Let μ_c be the **critical value** beyond which there are values of k for which $\text{Re}(\lambda_1(k, \mu)) > 0$, and k_c such that

$$\text{Re}(\lambda_1(\pm k_c; \mu_c)) = 0 \quad \text{and} \quad \text{Re}(\lambda_2(k, \mu_c)) > 0, \forall k \in \mathbb{R} \quad (8)$$

The planar reaction ODE, associated to (1), with $(U(x, t), V(x, t)) = (u(t), v(t))$ is considered:

$$\dot{u} = G(u, v; \mu), \quad \dot{v} = H(u, v; \mu). \quad (9)$$

The following notations are introduced when μ is taken just beyond μ_c , and ϵ is small

$$\mu = \mu_c + \epsilon^2 \tilde{\mu}, \quad k = k_c + \epsilon \tilde{k} \quad (10)$$

and

$$\lambda_1(k, \mu) = \epsilon^2 \tilde{\lambda}(\tilde{k}, \tilde{\mu}), \quad \lambda_\mu^c = \frac{\partial \lambda_1}{\partial \mu}(k_c, \mu_c) \quad \lambda_{k^2}^c = \frac{\partial^2 \lambda_1}{\partial k^2}(k_c, \mu_c), \quad a(c, \tilde{\mu}, k) = \sqrt{\frac{2\lambda_\mu^c \tilde{\mu}}{|\lambda_{k^2}^c|}} \quad (11)$$

Under the conditions

$$g_u < 0, \quad h_v > 0, \quad g_v h_u < 0, \quad 0 < d < 1 \text{ or } g_u > 0, \quad h_v < 0, \quad g_v h_u > 0, \quad d > 1, \quad (12)$$

it is proved ([4], Lemma 2.1) that the trivial state (\bar{U}, \bar{V}) loses stability as $\tilde{\mu}$ crosses through 0. For $\tilde{\mu} \lambda_\mu^c > 0$, there are two symmetric intervals $k = \pm k_c + \epsilon \tilde{k}$ with $\tilde{k} \in (-a(c, \tilde{\mu}, k) + O(\epsilon), a(c, \tilde{\mu}, k) + O(\epsilon))$, such that there exist (real) perturbations of (\bar{U}, \bar{V}) of the form

$$e^{i(k_c + \epsilon \tilde{k})x + \epsilon^2 \tilde{\lambda}(\tilde{k}, \tilde{\mu})t} \begin{pmatrix} \alpha_1^c + O(\epsilon) \\ \beta_1^c + O(\epsilon) \end{pmatrix} + c.c. \quad (13)$$

where (α_1^c, β_1^c) are the eigenvectors of $\mathcal{A}(k_c, \mu_c)$ corresponding to the eigenvalue $\lambda_1(k_c, \mu_c) = 0$. With

$$\xi = \epsilon x, \quad \tau = \epsilon^2 t, \quad E_c(x) = e^{ik_c x}, \quad (14)$$

the above result can be written

$$\begin{pmatrix} U(x, t) \\ V(x, t) \end{pmatrix} = \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} + \epsilon A(\xi, \tau) \begin{pmatrix} \alpha_1^c \\ \beta_1^c \end{pmatrix} E_c(x) + c.c. + O(\epsilon^2) \quad (15)$$

where $A(\xi, \tau) : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{C}$ satisfies the **Ginzburg-Landau equation**

$$A_\tau = -\frac{1}{2} \lambda_{k^2}^c A_{\xi\xi} + \tilde{\mu} \lambda_\mu^c A + L A |A|^2; \quad (16)$$

Here, $L \in \mathbb{R}$ is the Landau coefficient.

1.3 The onset of patterns formation: the Turing/Ginzburg-Landau bifurcation

It is assumed that $L < 0$. In fact, the author ([4]) shows that no (small amplitude) Turing patterns can exist beyond the destabilization if $L > 0$. Introducing the new variables

$$\hat{\tau} = \tilde{\mu}\lambda_\mu^c \tau, \quad \hat{\xi} = a(c, \tilde{\mu}, k)\xi, \quad \hat{A} = \sqrt{\frac{|L|}{\tilde{\mu}\lambda_\mu^c}} A \quad (17)$$

and dropping the hats, equation (16) becomes

$$A_\tau = A_{\xi\xi} + A - A|A|^2. \quad (18)$$

This equation has a family of stationary spatially periodic solutions $A(\xi, \tau) = Re^{iK\xi}$, $R > 0$ with

$$K^2 + R^2 = 1 \quad \text{and} \quad -1 < K < 1. \quad (19)$$

and by the Eckhaus/Benjamin-Feir-Newell criterion (cf. subsection 2.2.3), this periodic solution is spectrally stable for $-\frac{1}{\sqrt{3}} < K < \frac{1}{\sqrt{3}}$ and unstable for $|K| \in (\frac{1}{\sqrt{3}}, 1)$ ([4], Lemma 2.3).

The main result. Under the above conditions, when ϵ is sufficiently small, a **Turing/Ginzburg-Landau bifurcation** takes place as $\tilde{\mu}$ crosses through 0 i.e. for $\tilde{\mu}\lambda_\mu^c > 0$, there exists a continuous band of asymptotically stable stationary spatially periodic patterns $(U_p(x; k), V_p(x; k))$ of (1), with wave number

$$k(K) = k_c + \epsilon a(c, \tilde{\mu}, k)K \quad (20)$$

and $-\frac{1}{\sqrt{3}} + O(\epsilon) < K < \frac{1}{\sqrt{3}} + O(\epsilon)$. These spatially periodic patterns are $O(\epsilon)$ close to the trivial state (\bar{U}, \bar{V}) and are approximated by

$$\begin{pmatrix} U_p(x; k, \theta) \\ V_p(x; k, \theta) \end{pmatrix} = \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} + \epsilon(\sqrt{1 - K^2} + O(\epsilon))a(c, \tilde{\mu}, k) \begin{pmatrix} \alpha_1^c \cos(k(K)x + \theta) + O(\epsilon) \\ \beta_1^c \cos(k(K)x + \theta) + O(\epsilon) \end{pmatrix} \quad (21)$$

for any phase shift $\theta \in \mathbb{R}$, [4], Theorem 2.4.

Remark 1. In [4], Section 2.5, the author investigates the case of Hopf bifurcation in (9) i.e. the generation of patterns as μ passes through the critical value μ_c .

1.4 The particular case of the 1D-Gray-Scott model and chemistry

(Cf. David S. Morgan et al. [8]). The Gray-Scott model, related to chemical reactions between two species \mathcal{U} and \mathcal{V} , consists of the reaction-diffusion system

$$\begin{aligned} U_t &= D_U \Delta U - UV^2 + A(1 - U) \\ V_t &= D_V \Delta V + UV^2 - BV \end{aligned} \quad (22)$$

Where $U = U(x, t)$, $x \in \mathbb{R}$, $V = V(x, t)$, are the concentrations of \mathcal{U} (inhibitor) and \mathcal{V} (activator), A and B are rate constants, D_U and D_V are diffusivities. Here, for convenience, $D_U = 1$ and $D_V = \delta^{2\sigma}$, $0 < \delta \leq 1$, $\sigma > 0$. The stationary states (trivial patterns) solutions of the system

$$-UV^2 + A(1 - U) = 0 \quad \text{and} \quad UV^2 - BV = 0 \quad (23)$$

are $U \equiv 1, V \equiv 0$ and, when $4B^2 < A$,

$$(U_\pm, V_\pm) = \left(\frac{1}{2} \left[1 \pm \sqrt{1 - \frac{4B^2}{A}} \right], \frac{A}{2B} \left[1 \mp \sqrt{1 - \frac{4B^2}{A}} \right] \right) \quad (24)$$

The state $(U \equiv 1, V \equiv 0)$ is linearly stable for all $A, B > 0$. On the other hand, it is proved ([8], p.117) that the state (U_+, V_+) cannot be marginally stable (cf. Definition 6). Therefore the authors focus only on (U_-, V_-) .

Linear stability of (U_-, V_-) , Turing/Ginsburg-Landau bifurcation. Linearizing (22) around the stationary state (U_-, V_-) leads to

$$\begin{pmatrix} \dot{U} \\ \dot{V} \end{pmatrix} = M \begin{pmatrix} U \\ V \end{pmatrix} \quad (25)$$

where

$$M = \begin{pmatrix} -k^2 - V_-^2 - \delta^\alpha a & -2\delta^\beta b \\ V_-^2 & -\delta^{2\sigma} k^2 + \delta^\beta b \end{pmatrix} \quad (26)$$

with A, B rescaled as $A = \delta^\alpha a, B = \delta^\beta b, \alpha, \beta \geq 0$. The analysis of the eigenvalues of M shows that (U_-, V_-) is linearly stable if and only if $2\alpha \leq 3\beta$ and determines the values a_c and k_c of the parameter a and the wavenumber k such that (U_-, V_-) is marginally stable. In particular,

$$a_c^2 = (3 - 2\sqrt{2})b^3. \quad (27)$$

Remark 2. 1) With the above scaling for A and B , one has, to leading order

$$(U_-, V_-) = (\delta^{2\beta-\alpha} \frac{b^2}{a}, \delta^{\alpha-\beta} \frac{a}{b}) \quad (28)$$

2) For the Gray-Scott model, the Ginsburg-Landau equation (cf.(16)) has the form

$$\mathcal{A}_\tau = \frac{2}{\sqrt{b}} \mathcal{A} + 2\sqrt{2} \mathcal{A}_{\xi\xi} - \frac{2}{9} (10\sqrt{2} - 7) |\mathcal{A}|^2 \mathcal{A} \quad (29)$$

After setting $\mathcal{A}(\xi, \tau) = R e^{i\kappa\xi}$, solutions R and κ satisfy the equation

$$\kappa^2 + \frac{1}{18} (20 - 7\sqrt{2}) R^2 = \frac{1}{\sqrt{2}b} \quad (30)$$

The main result. Finally, the following result is obtained ([8], Theorem 3.2): Let $a = a_c - \gamma^2$ and $3\beta = 2(\sigma + \alpha)$. For $0 < \gamma \ll 1$ small enough, there exists a one parameter family of stationary spatially-periodic solutions of (22) (cf definition 4) that are close to the stationary state (U_-, V_-) :

$$\begin{pmatrix} U(x, \kappa) \\ V(x, \kappa) \end{pmatrix} = \begin{pmatrix} U_- \\ V_- \end{pmatrix} + \gamma R e^{i(k_c + \gamma\kappa)\delta^{\beta-\alpha} x} \begin{pmatrix} 2\delta^{2\beta-\alpha} \\ -(2 - \sqrt{2})\delta^{\alpha-\beta} \end{pmatrix} + c.c. + h.o.t., \quad (31)$$

where R and κ are related by (30).

2 Generalized Klausmeier-Gray-Scott model and echosystems

2.1 Initial Klausmeier model

In [6], C. A. Klausmeier considers the nondimensionalized system:

$$\begin{aligned} \frac{\partial w}{\partial t} &= a - w - wn^2 + \nu \frac{\partial w}{\partial x} \\ \frac{\partial n}{\partial t} &= wn^2 - mn + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) n \end{aligned} \quad (32)$$

for water w and plant biomass n , defined on an infinite two-dimensional domain indexed by x and y . In (32), a controls water input, m measures plant losses and ν controls the rate at which water flows downhill. The corresponding nonspatial model is

$$\begin{aligned} a - w - wn^2 &= 0 \\ wn^2 - mn &= 0 \end{aligned} \quad (33)$$

He shows that this model has a bare stable state $\hat{w} = a$, $\hat{n} = 0$ and an other one vegetated. These stable states correspond to spatially homogeneous equilibria of (32). Then he uses linear stability analysis, for system (32), in two cases: hillsides i.e. $\nu \gg 0$, and flat ground. On hillsides, for given m and ν , there is a critical value of water input a , below which **regular** stripes form. Moreover, this pattern oscillates in time and the strip moves uphill because the eigenvalue that determines the instability of this homogeneous equilibrium is complex with a negative imaginary part. On flat ground, regular pattern formation is impossible when the spatially homogeneous equilibrium is stable. Numerical solutions of (32) show that, in this case, irregular patterns can arise but for parameters which are ecologically unrealistic. His results are illustrated by figures obtained with ecologically realistic parameters.

2.2 Results by S. van der Stelt et al. [12]

2.2.1 The model

On one-dimensional domain, the model introduced by C. A. Klausmeier reads:

$$\begin{aligned} u_t &= k_o u_x + k_1 - k_2 u - k_3 k_5 u v^2 \\ v_t &= d_v v_{xx} - k_4 v + k_5 u v^2 \end{aligned} \quad (34)$$

where $u(x, t), v(x, t) : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$, and $k_i \geq 0, i = 0, \dots, 5, d \geq 0$. The flow of water is denoted by u_t , the slope of the area by $k_o u_x$, the constant precipitation rate by k_1 , an evaporation rate by $-k_2 u$, and an infiltration rate by $-k_3 k_5 u v^2$. The change of biomass is assumed to be controlled by a diffusion term $d_v v_{xx}$. The death rate is denoted by $-k_4 v$ and the infiltration feedback by $k_5 u v^2$. In [12], system (34) is completed by the nonlinear diffusion term $d_u(u^\gamma)_{xx}$:

$$\begin{aligned} u_t &= d_u(u^\gamma)_{xx} + k_o u_x + k_1 - k_2 u - k_3 k_5 u v^2 \\ v_t &= d_v v_{xx} - k_4 v + k_5 u v^2 \end{aligned} \quad (35)$$

where $\gamma \geq 1$ and $0 < d_v \ll d_u$, and rescaled as

$$\begin{aligned} U_t &= U_{xx}^\gamma + C U_x + A(1 - U) - UV^2 \\ V_t &= \delta^{2\sigma} V_{xx} - BV + UV^2 \end{aligned} \quad (36)$$

with $0 < \delta \ll 1$ and

$$\delta^{2\sigma} = \frac{d_v}{d_u} \left(\frac{k_2}{k_1} \right)^{\gamma-1}, \quad \sigma > 0. \quad (37)$$

Remark 3. For ecosystems without a slope $k_0 = 0$ and therefore

$$C = k_0 \frac{k_2^2 k_3}{k_1^2 k_5} \left[d_u \frac{k_3}{k_5} \left(\frac{k_1}{k_2} \right)^{\gamma-3} \right]^{-\frac{1}{2}} = 0 \quad (38)$$

Definition 4. Spatially periodic solutions (patterns) or wave trains are solutions $u(x, t)$ that can be written

$$u(x, t) = u_p(\kappa x + \Omega t) \quad (39)$$

that satisfy $u_p(\xi) = u_p(\xi + 2\pi)$. Here κ is called the **nonlinear wave number**.

Definition 5. A **Busse balloon** for system (36) is a set \mathcal{B} in the (A, κ) -space with the following property: a point (A, κ) lies in \mathcal{B} if system (36) with parameter A allow for at least one stable periodic solution (U_p, V_p) with parameter κ (cf. [12], p.66).

It is pointed out that the ecologically relevant parameter values of γ are $\gamma = 1$ or $\gamma = 2$. For $\gamma = 1$ and $C = 0$, system (36) is the gray-Scott system (22). In [8] Section 6.2, the existence of a Busse balloon, for the Gray-Scott model, is investigated.

2.2.2 The background states and the Turing-hopf instability

The model (36) has the same homogeneous background states as the Gray-Scott model for $A > 4B^2$, namely (cf (23) and (24)).

$$U_0 = 1, V_0 = 0, \text{ and } (U_{\pm}, V_{\pm}) = \left(\frac{1}{2A} [A \mp \sqrt{A^2 - 4AB^2}], \frac{1}{2B} (A \pm \sqrt{A^2 - 4AB^2}) \right). \quad (40)$$

The state $(U_0, V_0) = (1, 0)$ represents the desert since, in this case, $v = \frac{k_1}{k_2 k_3} V_0 = 0$. The state (U_-, V_-) does not represent a homogeneously vegetated state. By linearization of (36) about the state $u_+ = (U_+, V_+)$ we have :

$$u_t = \mathcal{D}u_{xx} + \mathcal{C}u_x + \partial_u F(u_+; A, B)u =: \mathcal{L}(\partial_x)u \quad (41)$$

with $u = (U, V)$, $F(U, V; A, B) = (A(1 - U) - UV^2, -BV + UV^2)$, and suitable matrices \mathcal{C} and \mathcal{D} . Let \mathcal{M} be the matrix defined by

$$\mathcal{M}(a, c, ik) = \begin{pmatrix} -\gamma(U_+)^{\gamma-1}k^2 + ic\delta^\nu k - V_+^2 - \delta^\alpha a & -2b\delta^\beta \\ V_+^2 & -\delta^{2\sigma}k^2 + \delta^\beta b \end{pmatrix} = \mathcal{L}(ik) \quad (42)$$

where $0 < \delta \ll 1$ and

$$A = a\delta^\alpha, \quad B = b\delta^\beta, \quad C = c\delta^\nu; \quad \alpha, \beta > 0, \quad \nu \in \mathbb{R}, \quad (43)$$

with $a, b, c, = O(1)$ with respect to δ . Here k is referred to as **the linear wavenumber**.

Remark 4. For ecosystems without a slope, $C=0$ (cf. remark 3) and therefore $c = 0$.

Remark 5. With the above scaling, (U_+, V_+) can be written out to leading order in δ

$$(U_+, V_+) = \left(\frac{b^2}{a} \delta^{2\beta-\alpha}, \frac{a}{b} \delta^{\alpha-\beta} \right) + \text{h.o.t.} \quad (44)$$

The L^2 -spectrum of (41) is the set of $\lambda \in \mathbb{C}$ such that

$$d(\lambda, ik) = \det[\mathcal{M}(a, c, ik) - \lambda] = 0 \quad (45)$$

Definition 6. $\mathcal{L}(\partial_x)$ is called **marginally stable** with critical Fourier mode $u_0 e^{ik_* x}$ associated to the critical eigenmode $i\omega_*$, up to complex conjugaison, if

$$d(i\omega_*, ik_*) = 0, \quad d(i\omega_*, ik) \neq 0 \text{ for all } k \neq \pm k_* \quad (46)$$

and $d(\lambda, ik) \neq 0$, for all $k \in \mathbb{R}$, all $\lambda \in \mathbb{C}$ with $\lambda \neq i\omega_*$ and $\text{Re}\lambda \geq 0$. In this case, we also say that the background state (U_+, V_+) of (36) is **marginally stable**.

Definition 7. The instability is called a **Turing-Hopf** [resp. **Turing**] instability if $k_* \neq 0, \omega_* \neq 0$ [resp. $k_* \neq 0, \omega_* = 0$].

Main results:1) when $\mathbf{C}=\mathbf{0}$, (U_+, V_+) is marginally stable for $\sigma, a = a_*, k = k_*$ satisfying, to leading order in δ ,

$$(2\gamma + 1)\beta - (\gamma + 1)\alpha = 2\sigma, \quad k_*^2 = \frac{1}{2}(1 - g)b\delta^{-2\gamma\beta + (\gamma+1)\alpha}, \quad a_*^{\gamma+1} = g\gamma b^{2\gamma+1}. \quad (47)$$

where $g = 3 - 2\sqrt{2}$, ([12], Proposition 1),

2) **when $\mathbf{C}=c\delta^{\frac{1}{2}(3-\gamma)\alpha + (\gamma-2)\beta} \neq 0$,** set $\Gamma = \Gamma(\gamma, a) = \gamma(\frac{b^2}{a})^{\gamma-1}$, $\tilde{k} = \delta^{-\frac{1}{2}(\gamma+1)\alpha + \beta\gamma}k$, and drop the tilde on \tilde{k} . Then, the stationary state (U_+, V_+) undergoes a **Turing-Hopf instability** at a uniquely defined critical parameter $a = a_*$ and a critical wavenumber $k = k_*$ that satisfy

$$a_*^{\gamma+1} \geq g\gamma b^{2\gamma+1} \text{ and } k_*^2 < b, \quad (48)$$

moreover, if $c = \frac{2}{3}b\Gamma$,

$$a_*^{\gamma+1} = \frac{1}{3}\gamma b^{2\gamma+1} \text{ and } k_*^2 = \frac{1}{3}b \quad (49)$$

and, if $c \gg 1$

$$a_*^{\gamma+3}(c) = \frac{g}{\gamma}b^{2\gamma+3}c^2 + O(c) \text{ and } k_*^2(c) = \frac{1}{2}(1 - g)b + O(\frac{1}{c}). \quad (50)$$

to leading order in c and δ ([12] Proposition 2).

2.2.3 Ginzburg-Landau equation

If

$$|a - a_*| = r\epsilon^2 \quad (51)$$

and ϵ is small enough, the Ginzburg-Landau equation (16) associated to (36) has the form

$$\mathcal{A}_\tau = (a_1 + ia_2)\mathcal{A}_{\xi,\xi} + (b_1 + ib_2)\mathcal{A} + (L_1 + iL_2)|\mathcal{A}|^2\mathcal{A} \quad (52)$$

whose coefficients are functions of b, c , and γ ([12], Proposition 3). The Turing-Hopf instability of (U_+, V_+) is **supercritical** if $L_1 < 0$ and **subcritical** if $L_1 > 0$.

The Benjamin-Feir-Newell criterion. If the Turing-Hopf bifurcation is supercritical, there exists a band of stable spatially periodic patterns if and only if

$$1 + \frac{a_2 L_2}{a_1 l_1} > 0. \quad (53)$$

By means of the computing system Mathematica, the authors evaluated the coefficients of (52), for $\gamma = (1, 2)$. These evaluations lead them to acute results:

1) For $\gamma = \{1, 2\}$, the coefficients L_1 in (52), is negative for all values of b and c up to $c \sim 10^6$ and $b \sim 10^2$, and condition (53) is satisfied. Therefore, the Turing Hopf bifurcation at $a = a_*$ of the stationary state (U_+, V_+) of (36), with $C > 0$ and $\gamma = \{1, 2\}$ is supercritical ([12] Claim 1, p. 59).

2) For (36) with $C > 0$ and $\gamma = \{1, 2\}$, there exists a stable band of periodic patterns that appears at the Turing-Hopf instability ([12] Claim 2 p. 60).

3) For $c = 0$, all coefficients of (52) are real and $L_1 = L_1(\gamma)$ becomes positive for large γ and equals 0 for $\gamma \approx 13.0446$. Therefore the Turing bifurcation for (35) with $c = 0$ is supercritical for $\gamma < \gamma_{ss}$, and subcritical for $\gamma > \gamma_{ss}$ ([12] proposition 4).

Remark 6. In Section 3 of [12], existence of stable patterns if a is not closed to a_* is studied. Busse Ballons (cf. Definition 5) are numerically constructed for different values of a and fixed values of b, c, γ

3 Gierer-Meinhardt system and biology

The Gierer-Meinhardt system (cf.[5]) is one of the most popular models in biological pattern formation. Let (S, g) be a compact two-dimensional Riemannian manifold without boundary. (For definitions related to Riemannian manifolds, see W. Kühnel [7]). In [10], W. Tse et al. consider the system

$$\begin{cases} A_t = \epsilon^2 \Delta_g A - A + \frac{A^2}{H} \\ \tau H_t = \frac{1}{\beta^2} \Delta_g H - H + A^2 \end{cases} \quad (54)$$

in S . Here, $A(p, t), H(p, t) > 0$ represent the activator and the inhibitor concentrations, respectively, at a point $p \in S$ and at time t ; their corresponding diffusivities are denoted by ϵ^2 and $\frac{1}{\beta^2}$; τ is the time-relaxation constant of the inhibitor and Δ_g is the Laplace-Beltrami operator with respect to the metric g . Their assumptions on the parameters ϵ and β are: ϵ is small enough and $\lim_{\epsilon \rightarrow 0} \frac{\beta^2}{\epsilon^2} = \kappa > 0$. Let G_0 be the Green function defined by

$$\Delta_g G_0(p, q) - \frac{1}{|S|} + \delta_p(q) = 0 \text{ in } S, \quad \text{and} \quad \int_S G_0(p, q) dv_g(q) = 0, \quad (55)$$

$K(p)$ denote the Gauss curvature on S , and w be the solution of the problem

$$\Delta w - w + w^2 = 0, w > 0 \text{ in } \mathbb{R}^2, w(0) = \max_{y \in \mathbb{R}^2} w(y), \quad w(y) \rightarrow 0 \text{ as } |y| \rightarrow \infty, \quad (56)$$

They introduce the function $F(p) = c_1 K(p) + c_2 R(p)$ where $R(p)$ denotes the diagonal of the regular part of the Green function, and

$$c_1 = \frac{\pi}{4} \int_0^\infty \left(\frac{\partial w}{\partial r} \right)^2 r^3 dr \quad c_2 = \frac{|S| \pi \beta^2}{2 \epsilon^2} \int_0^\infty w^2 r dr. \quad (57)$$

3.1 Existence and stability of a single spike solution

3.1.1 Existence

If $p_0 \in S$ is a non-degenerate critical point of $F(p)$, i.e.

$$\nabla F(p_0) = 0, \quad \det(\nabla^2 F(p_0)) \neq 0, \quad (58)$$

it is proved that problem (54) has a positive spiky steady state (A_ϵ, H_ϵ) such that

$$A_\epsilon(x) = \xi_\epsilon \left(w\left(\frac{x - p_\epsilon}{\epsilon}\right) + O(\epsilon^2) \right), \quad (59)$$

uniformly for $x \in S$, where

$$\xi_\epsilon = \frac{|S|}{\epsilon^2 \int_{\mathbb{R}^2} w^2(y) dy} + O\left(\log \frac{1}{\epsilon}\right), \text{ with } p_\epsilon \rightarrow p_0 \text{ as } \epsilon \rightarrow 0, \quad (60)$$

and

$$H_\epsilon(x) = \xi_\epsilon (1 + O(\epsilon^2)) \text{ uniformly in } S \quad (61)$$

([10], Theorem 1.1).

Remark 7. It is assumed that $\xi_{\epsilon, p}$ is the height of the peak and that $p \in \overline{\Lambda_\delta}$ is the location of the peak, where $\Lambda_\delta = S \cap B_g(p_o, \delta)$, with $\delta = \epsilon^\alpha$, for some $0 < \alpha < 1$. ($B_g(p_o, \delta)$ is the ball with center p_o and radius δ , with respect to the metric g).

3.1.2 Linear stability

Linearizing (54) around the equilibrium states $(A_\epsilon + \phi_\epsilon e^{\lambda_\epsilon t}, H_\epsilon + \psi_\epsilon e^{\lambda_\epsilon t})$ the following eigenvalue problem L_ϵ is obtained

$$\begin{cases} \epsilon^2 \Delta_g \phi_\epsilon - \phi_\epsilon + 2 \frac{A_\epsilon}{H_\epsilon} \phi_\epsilon - \frac{A_\epsilon^2}{H_\epsilon^2} \psi_\epsilon = \lambda_\epsilon \phi_\epsilon \\ \frac{1}{\tau} (\frac{1}{\beta^2} \Delta_g \psi_\epsilon - \psi_\epsilon + 2 A_\epsilon \phi_\epsilon) = \lambda_\epsilon \psi_\epsilon \end{cases} \quad (62)$$

where (A_ϵ, H_ϵ) is the above solution of system (54).

Definitions The solution (A_ϵ, H_ϵ) is called **linearly stable** if the spectrum $\sigma(L_\epsilon)$ of L_ϵ lies in the left half plane $\{\lambda \in \mathbb{C} : \text{Re}(\lambda) < 0\}$. It is called **linearly unstable** if there exists an eigenvalue λ_ϵ of L_ϵ , with $\text{Re}(\lambda_\epsilon) > 0$.

Main result (Cf. [10], Theorem 1.2). Let p_o be a non-degenerate **local maximum point** of $F(p)$ i.e.

$$\nabla F(p_o) = 0 \quad \nabla^2 F(p_o) \text{ is negative definite}, \quad (63)$$

let (A_ϵ, H_ϵ) be the above single peaked solution, whose peak approaches p_o , then, there exists a unique $\tau_1 > 0$ such that (A_ϵ, H_ϵ) is linearly stable [resp. unstable], if $\tau < \tau_1$ [resp. $\tau > \tau_1$].

3.2 Stationary Gierer-Meinhardt system

(Cf. [10], Section 3). The proof of the existence theorem, subsection (3.1), is based on the stationary system associated to (54) i.e.

$$\begin{cases} \epsilon^2 \Delta_g A - A + \frac{A^2}{H} = 0, & \text{in } S \\ \frac{1}{\beta^2} \Delta_g H - H + A^2 = 0, & \text{in } S \end{cases} \quad (64)$$

With the rescaled amplitudes

$$a(p) = \frac{1}{\xi_{\epsilon,p}} A(p), \quad h(p) = \frac{1}{\xi_{\epsilon,p}} H(p) \quad (65)$$

an equilibrium solution (a, h) solves the rescaled Gierer-Meinhardt system

$$\begin{cases} \epsilon^2 \Delta_g a - a + \frac{a^2}{h} = 0, & a > 0 \text{ in } S \\ \Delta_g h - \beta^2 h + \beta^2 \xi_{\epsilon,p} a^2 = 0, & h > 0 \text{ in } S \end{cases} \quad (66)$$

with $a = O(1)$ and $h = O(1)$

3.3 Existence and stability of a cluster of two spikes for the stationary Gierer-Meinhardt system

In [2], W. Ao et al. consider the stationary Gierer-Meinhardt system

$$\begin{cases} \epsilon^2 \Delta_g A - A + \frac{A^2}{H} = 0, \\ D \Delta_g H - H + A^2 = 0, \end{cases} \quad (67)$$

on a compact two-dimensional Riemannian manifold (\mathcal{M}, g) without boundary. They assume

$$0 < \epsilon \ll 1, \quad 0 < D \ll 1. \quad (68)$$

It is assumed that there is a non-degenerate local maximum point of the Gaussian curvature function $K(p)$ of \mathcal{M} at $p_o = 0$, i.e. $\nabla K(0) = 0$ and

$$\nabla^2 K(0) = \begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix}, \quad (69)$$

with $K_{11}, K_{22} < 0$. Let p_o be a non-degenerate local maximum point of the gaussian curvature $K(p)$ of \mathcal{M} . they show that, under stronger conditions on ϵ and D , namely

$$0 < \epsilon \ll \sqrt{D} \ll 1, \quad 0 < \sqrt{D} \log \frac{1}{\epsilon^2 D \log \frac{\sqrt{D}}{\epsilon}} \ll 1, \quad (70)$$

the system (67) has, at least, two different 2-spike cluster solutions (A_i, H_i) for $i=1,2$. Moreover, one of the solutions is stable and the other one is unstable ([?] Theorems 1.1 and 1.4)

4 Homogenization and linear elasticity

This section is an introduction to the theory of homogenization. Complements will be found in the references.

4.1 H and G-convergences

(Cf. G. Allaire [1]). Let Ω be a bounded open set in \mathbb{R}^N

Definition 8. Let \mathcal{M}_N be the space of square real matrices of order N and define

$$\mathcal{M}_{\alpha,\beta} = \{M \in \mathcal{M}_N \mid M\xi \cdot \xi \geq \alpha|\xi|^2, M^{-1}\xi \cdot \xi \geq \beta|\xi|^2, \forall \xi \in \mathbb{R}^N\} \quad (71)$$

or

$$\mathcal{M}_{\alpha,\beta} = \{M \in \mathcal{M}_N \mid \alpha|\xi|^2 \leq M\xi \cdot \xi \leq \beta^{-1}|\xi|^2, \forall \xi \in \mathbb{R}^N\}, \quad (72)$$

and also

$$\mathcal{M}_{\alpha,\beta}^s = \{M \in \mathcal{M}_N^s \mid M\xi \cdot \xi \geq \alpha|\xi|^2, M^{-1}\xi \cdot \xi \geq \beta|\xi|^2, \forall \xi \in \mathbb{R}^N\} \quad (73)$$

where $\alpha, \beta > 0$ and $\alpha\beta \leq 1$, and \mathcal{M}_N^s is the space of real symmetric matrices of order N .

Definition 9. A sequence of matrices $A^\epsilon(x) \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ is said to converge in the sense of homogenisation, or **H-converge**, to an homogenized limit or **H-limit** matrix $A^* \in L^\infty(\Omega; \mathbb{M}_{\alpha,\beta})$ if, for any $f \in H^{-1}(\Omega)$, the sequence of solutions u_ϵ of

$$-\operatorname{div} A^\epsilon(x) \nabla u_\epsilon(x) = f(x) \text{ in } \Omega, \quad \text{and } u_\epsilon = 0 \text{ on } \partial\Omega, \quad (74)$$

satisfies

$$u_\epsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega) \text{ and } A^\epsilon \nabla u_\epsilon \rightharpoonup \text{ weakly in } L^2(\Omega)^N, \quad (75)$$

where u is the solution of the homogenized equation

$$-\operatorname{div} A^*(x) \nabla u(x) = f(x) \text{ in } \Omega, \quad \text{and } u = 0 \text{ on } \partial\Omega, \quad (76)$$

(cf.[1], Definition 1.2.15 and Remark 1.2.17). This definition is justified by the following result: for any sequence $A^\epsilon(x)$ of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ there exist a subsequence, still denoted by A^ϵ , and an homogenized matrix $A^*(x) \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ such that A^ϵ **H-converges** to A^* , ([1], Theorem 1. 2. 16)

Definition 10. A sequence of matrices $A^\epsilon(x) \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$ is said to **G-converge**, to an homogenized limit or *G*-limit matrix $A^* \in L^\infty(\Omega; \mathbb{M}_{\alpha,\beta}^s)$ if, for any $f \in H^{-1}(\Omega)$, the sequence of solutions u_ϵ of

$$-\operatorname{div} A^\epsilon(x) \nabla u_\epsilon(x) = f(x) \text{ in } \Omega, \quad \text{and } u_\epsilon = 0 \text{ on } \partial\Omega, \quad (77)$$

satisfies

$$u_\epsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega) \quad (78)$$

where u is the solution of the homogenized equation

$$-\operatorname{div} A^*(x) \nabla u(x) = f(x) \text{ in } \Omega, \quad \text{and } u_\epsilon = 0 \text{ on } \partial\Omega, \quad (79)$$

(cf. [1], Definition 1.3.8). This definition is justified by the following result: for any sequence $A^\epsilon(x)$ of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$ there exist a subsequence, still denoted by A^ϵ , and an homogenized matrix $A^*(x) \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$ such that A^ϵ *H*-converges to A^* , ([1], Theorem 1.3.9)

Remark 8. For symmetric matrices, *H* and *G* convergences are equivalent ([1], Proposition 1.3.11).

4.2 Spectral problems

4.2.1 The case of abstract operators defined in different Hilbert spaces

(Cf. [9], Section 1.2, p.266). Let H_ϵ, H_0 be separable Hilbert spaces, $A_\epsilon : H_\epsilon \rightarrow H_\epsilon, 0 \leq \epsilon \leq 1$, $A_o : H_o \rightarrow H_o$, be continuous linear operators, and V be a subspace of H_o with $\operatorname{Im} A_o \subset V \subset H_o$ satisfying the following conditions:

1-There exist linear continuous operators $R_\epsilon : H_o \rightarrow H_\epsilon$ and a constant γ such that

$$(R_\epsilon f, R_\epsilon f)_{H_\epsilon} \rightarrow \gamma(f, f)_{H_o}, \text{ as } \epsilon \rightarrow 0, \text{ for any } f \in V. \quad (80)$$

2-The operators A_ϵ, A_o are positive, compact, self-adjoint and the norms $\|A_\epsilon\|_{B(H_\epsilon)}$ are bounded by constant independent of ϵ .

3-For any $f \in V$,

$$\|A_\epsilon R_\epsilon f - R_\epsilon A_o f\|_{H_\epsilon} \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (81)$$

4-For each sequence $f^\epsilon \in H_\epsilon$ such that $\sup_\epsilon \|f^\epsilon\|_{H_\epsilon} < \infty$, one can extract a subsequence, still denoted f^ϵ , such that, for some $w \in V$,

$$\|A_\epsilon f^\epsilon - R_\epsilon w\|_{H_\epsilon} \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (82)$$

The following spectral problem, for the operators A_ϵ and A_o , is considered

$$A_\epsilon u_\epsilon^k = \mu_\epsilon^k u_\epsilon^k, \quad k = 1, 2, \dots, \quad u_\epsilon^k \in H_\epsilon \quad (83)$$

$$A_o u_o^k = \mu_o^k u_o^k, \quad k = 1, 2, \dots, \quad u_o^k \in H_o \quad (84)$$

with, for $\epsilon > 0$ or $\epsilon = o$,

$$\mu_\epsilon^1 \geq \mu_\epsilon^2 \geq \dots \geq \mu_\epsilon^k \dots, \quad \mu_\epsilon^k > 0, \quad (u_\epsilon^l, u_\epsilon^m)_{H_\epsilon} = \delta_{lm} \quad (85)$$

It is proved that there is a sequence $\{\beta_\epsilon^k\}$ such that $\beta_\epsilon^k \rightarrow 0$ as $\epsilon \rightarrow 0$, $0 < \beta_\epsilon^k < \mu_o^k$, and

$$|\mu_\epsilon^k - \mu_o^k| \leq \frac{\mu_o^k \gamma^{-1/2}}{\mu_o^k - \beta_\epsilon^k} \sup_{u \in N_o^k, \|u\|_o=1} \|A_\epsilon R_\epsilon u - R_\epsilon A_o u\|_{H_\epsilon}, \quad k = 1, 2, \dots \quad (86)$$

where N_o^k is the eigenspace of operator A_o corresponding to the eigenvalue μ_o^k ([9], Theorem 1.4, p.268)

4.2.2 Weak * convergence

Definition 11. A sequence $(u_\epsilon)_{\epsilon>0}$ is said to converge weakly * in $L^\infty(\Omega)$ to a limit u if, for any $\phi \in L^1(\Omega)$, it satisfies

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} u_\epsilon(x) \phi(x) dx = \int_{\Omega} u(x) \phi(x) dx. \quad (87)$$

This definition is valid for vector valued functions.

4.2.3 Main result

Let $A^\epsilon \in L^\infty(\Omega; \mathcal{M}_{\alpha, \beta}^s)$ be a sequence of symmetric matrices which H -converges to an homogenized matrix A^* . Let ρ_ϵ be a sequence of positive functions, such that

$$0 < \rho_- \leq \rho_\epsilon(x) \leq \rho_+ < \infty \quad (88)$$

which converges weakly * in $L^\infty(\Omega)$ to a limit $\rho(x)$. Let $(\lambda_\epsilon^m)_{m \geq 1}$ be the eigenvalues, labeled by increasing order, and $(u_\epsilon^m)_{m \geq 1}$ be associated normalized, in $L^2(\Omega)$, eigenvectors of the spectral problem

$$-\operatorname{div} A^\epsilon(x) \nabla u_\epsilon^m(x) = \lambda_\epsilon^m \rho_\epsilon(x) u_\epsilon^m(x) \text{ in } \Omega \text{ and } u_\epsilon^m = 0 \text{ on } \partial\Omega. \quad (89)$$

Then, for any fixed $m \geq 1$,

$$\lim_{\epsilon \rightarrow 0} \lambda_\epsilon^m = \lambda^m \quad (90)$$

and, up to a subsequence, u_ϵ^m converges weakly in $H_0^1(\Omega)$, to a normalized eigenvector associated to λ^m , which are solutions of the homogenized eigenvalue problem

$$-\operatorname{div} A^*(x) \nabla u^m(x) = \lambda^m \rho(x) u^m(x) \text{ in } \Omega, \text{ and } u^m = 0 \text{ on } \partial\Omega, \quad (91)$$

and $(\lambda^m)_{m \geq 1}$ is the complete family of eigenvalues of (91), labeled in increasing order ([1], Theorem 1.3.16).

A similar result is obtained in O. A. Oleinik et al. [9], Chapter III Theorem 2.1.

4.3 Periodic homogenization

Cf. [1], Section 1.3.4. Let $Y = (0, 1)^N$ be the unit periodic cell which is identified with the unit N -dimensional torus.

4.3.1 Spaces $L_{\mathbb{H}}^p(Y)$, $1 \leq p \leq \infty$

Definition 12.

$$L_{\mathbb{H}}^p(Y) = \{f \in L_{loc}^p(\mathbb{R}^N) \text{ such that } f \text{ is } Y\text{-periodic}\} \quad (92)$$

equipped with the norm $\|f\|_{L^p(Y)}$.

4.3.2 Example

Let $A(Y)$ be a matrix which belongs to $L_{\mathbb{H}}^\infty(Y; \mathcal{M}_{\alpha, \beta})$ (cf.(92)). Let Ω be a bounded open set in \mathbb{R}^N . For $f \in H^{-1}(\Omega)$, let $u_\epsilon \in H_o^1(\Omega)$ be the unique solution of

$$-\operatorname{div}\left(A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon\right)=f \text { in } \Omega, \quad u_\epsilon=0 \text { on } \partial \Omega \quad (93)$$

Remark 9. Here the conductivity $A(\frac{x}{\epsilon})$ of Ω shows that the domain Ω is highly heterogeneous with periodic heterogeneities of lengthscale ϵ .

For each unit vector e_i , in the canonical basis of \mathbb{R}^N , let w_i be a solution of the following problem in Y :

$$-\operatorname{div} A(y)(e_i + \nabla w_i(y)) = 0 \text{ in } Y, \quad y \rightarrow w_i(y) \text{ is } Y\text{-periodic} \quad (94)$$

Then, the sequence $A_\epsilon(x) = A(\frac{x}{\epsilon})$, H -converges to $A^* \in \mathcal{M}_{\alpha,\beta}$ defined by its entries

$$A_{ij}^* = \int_Y [A(y)(e_i + \nabla w_i) \cdot (e_j + \nabla w_j)] dy \quad (95)$$

The starting point of the proof of this result is to write the solution u_ϵ of equation (93) on the form of a two-scale asymptotic expansion :

$$u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u_i(x, \frac{x}{\epsilon}) \quad (96)$$

where $u_i(x, y)$ is a function of x and y , periodic in y with period Y .

4.4 Homogenization and linear elasticity

4.4.1 The linear elasticity operator

In [9], the usual linear elasticity operator, in a domain $\Omega \subset \mathbb{R}^n$, is defined, with the usual convention of repeated indices, as

$$L(u) \equiv \frac{\partial}{\partial x_h} (A^{hk}(x) \frac{\partial u}{\partial x_k}) \quad (97)$$

where $u = (u_1, \dots, u_n)$ is the displacement vector, and $A^{hk}(x)$ are $(n \times n)$ -matrices whose elements $a_{ij}^{hk}(x)$ are bounded measure functions such that

$$a_{ij}^{hk}(x) = a_{ji}^{kh}(x) = a_{hj}^{ik}(x), \quad x \in \Omega, \quad (98)$$

and

$$\kappa_1 \eta_{ih} \eta_{ih} \leq a_{ij}^{hk}(x) \eta_{ih} \eta_{jk} \leq \kappa_2 \eta_{ih} \eta_{ih}, \quad x \in \Omega \quad (99)$$

for any symmetric matrix with real elements $\{\eta_{ih}\}$, with positive constants κ_1, κ_2 .

Remark:

Are also associated to the above operator (97) the matrix $e(u)$ whose elements are $e_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ which is called the strain tensor, and the Neumann boundary operator

$$\sigma(u) = \nu_h A^{hk}(x) \frac{\partial u}{\partial x_k}(x), \quad x \in \partial\Omega \quad (100)$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the unit outward normal to $\partial\Omega$.

4.4.2 Homogenization in perforated domains

Notations and assumptions (Cf. [9], chapter 1). Let ω be a **smooth** unbounded domain of \mathbb{R}^n invariant under the shifts by any $z = (z_1, z_2, \dots, z_n) \in \mathbb{Z}^n$, $Q = \{x | 0 < x_j < 1, j = 1, 2, \dots, n\}$, $\epsilon G = \{x | \epsilon^{-1}x \in G\}$ satisfying the conditions: $\omega \cap Q$ is a domain with Lipschitz boundary, the sets $Q \setminus \bar{\omega}$ and $V_\delta \cap \partial Q$ consist of a finite number of Lipschitz domains separated from each other and from the edges of Q by a positive distance, where V_δ is a δ -neighborhood ($\delta < \frac{1}{4}$) of ∂Q .

Definition 13. Let Ω be a bounded smooth domain in \mathbb{R}^n . A domain Ω^ϵ is of type I if

$$\Omega^\epsilon = \Omega \cap \epsilon\omega \quad (101)$$

The boundary of Ω_ϵ can be represented as

$$\partial\Omega_\epsilon = \Gamma_\epsilon \cup S_\epsilon \text{ where } \Gamma_\epsilon = \partial\Omega \cap \epsilon\omega, \quad S_\epsilon = (\partial\Omega_\epsilon) \cap \Omega. \quad (102)$$

Example (Cf. [9], p. 119-134). Let Ω_ϵ be a domain of type (101). Consider the following mixed problem:

$$L_\epsilon(u_\epsilon) = \frac{\partial}{\partial x_i} (A^{ij}(\frac{x}{\epsilon}) \frac{\partial u_\epsilon}{\partial x_j}) = f_\epsilon \text{ in } \Omega_\epsilon \quad (103)$$

where the coefficients a_{kl}^{ij} of matrices A^{ij} are sufficiently smooth, satisfont (98)-(99) and are 1-periodic in ξ , with the boundary conditions

$$u_\epsilon = \Phi_\epsilon \text{ on } \Gamma_\epsilon, \quad \sigma_\epsilon(u_\epsilon) = \nu_i A^{ij} \frac{\partial u_\epsilon}{\partial x_j} = 0 \text{ on } S_\epsilon. \quad (104)$$

Here, $f_\epsilon \in L^2(\Omega_\epsilon)$ and $\Phi_\epsilon \in H^1(\Omega_\epsilon)$. The homogenized operator corresponding to L_ϵ has the form

$$\mathcal{L} \equiv \frac{\partial}{\partial x_i} (\mathcal{A}^{ij} \frac{\partial}{\partial x_j}) \quad (105)$$

where the coefficients matrices $\mathcal{A}^{ij}(i, j = 1, \dots, n)$ are given by the formula

$$\mathcal{A}^{ij} = (\text{mes } Q \cap \omega)^{-1} \int_{Q \cap \omega} (A^{ij}(\xi) + A^{ik}(\xi) \frac{\partial N^j(\xi)}{\partial \xi_k}) d\xi \quad (106)$$

and matrices $N^j(\xi)$ are 1-periodic in ξ with $\int_{Q \cap \omega} N^j(\xi) d\xi = 0$ and are solutions of the following boundary problem:

$$\frac{\partial}{\partial \xi_k} (A^{kj}(\xi) \frac{\partial N^q}{\partial \xi_j}) = - \frac{\partial}{\partial \xi_k} A^{kq}(\xi) \text{ in } \omega \quad (107)$$

with

$$\sigma(N^q) = -\nu_k A^{kq}(\xi) \text{ on } \partial\omega. \quad (108)$$

Let $u_o(x)$ be a weak solution of the problem

$$\mathcal{L}(u_o) = f_o \text{ in } \Omega, \quad u_o = \Phi_o \text{ on } \partial\Omega \quad (109)$$

where $f_o \in H^1(\Omega)$, and $\Phi_o \in H^3(\Omega)$. If f_ϵ is the restriction to Ω_ϵ of f_o , and Φ_ϵ is the restriction to Γ_ϵ of Φ_o , then (109) is the homogenized system corresponding to system (103) and (104) and $\|u_\epsilon - u_o\|_{L^2(\Omega_\epsilon)} \rightarrow 0$ as ϵ tends to 0.

4.5 Homogenization and Steklov eigenproblems

In [3], Chechkina, D'Apice and De maio consider a smooth domain Ω in $\mathbb{R}^d, d \geq 2$, whose boundary is $\partial\Omega = \gamma_1 \cup \gamma_\epsilon \cup \Gamma_\epsilon$. Here, ϵ is a small, positive, parameter, Γ_ϵ consists of N_ϵ sets whose diameter is less than or equal to ϵ and the distance between them is greater than or equal to 2ϵ , and $\gamma_\epsilon = \partial\Omega \setminus (\gamma_1 \cup \Gamma_\epsilon)$. **The boundary value problem for the elasticity system (E):**

$$L_k(u_\epsilon) = \frac{\partial}{\partial x_i} (a_{kl}^{ij} \frac{\partial u_\epsilon^l}{\partial x_j}) = 0 \text{ in } \Omega, \quad k = 1, \dots, d \quad (110)$$

and

$$u_\epsilon = 0 \text{ on } \gamma_1 \cup \gamma_\epsilon, \quad (111)$$

$$\sigma(u_\epsilon) = A^{ij}(x) \frac{\partial u_\epsilon}{\partial x_j} \nu_i = g(x) \text{ on } \Gamma_\epsilon \quad (112)$$

is considered. Here, the elements a_{kl}^{ij} of the $d \times d$ -matrices A^{ij} are bounded measurable functions, with

$$a_{kl}^{ij}(x) = a_{lk}^{ji}(x) = a_{il}^{kj}(x), \quad \kappa_1 \xi_{ki} \xi_{ki} \leq a_{kl}^{ij}(x) \xi_{ki} \xi_{lj} \leq \kappa_2 \xi_{ki} \xi_{ki}, \quad x \in \Omega, \quad (113)$$

where κ_1, κ_2 are constants > 0 , $\nu = (\nu_1, \nu_2, \dots, \nu_d)$ is an outward normal vector to the boundary $\partial\Omega$, $g(x) \in (L^2(\partial\Omega))^d$. Let $H^1(\Omega, \gamma_1 \cup \gamma_\epsilon)$ be the completion of the space of functions $v \in C^\infty(\overline{\Omega})$, $v = 0$ on $\gamma_1 \cup \gamma_\epsilon$, with respect to the norm

$$\|v\|_{H^1(\Omega)} = \left(\int_\Omega (v^2 + |\nabla v|^2) dx \right)^{\frac{1}{2}}. \quad (114)$$

The solution u_ϵ of system (E) is in the space $(H^1(\Omega, \gamma_1 \cup \gamma_\epsilon))^d$. The following results are proved:

R1-the solutions u_ϵ of system (E) are uniformly bounded with respect to ϵ , in $H^1(\Omega)$ ([3], Lemma 3) i.e. each u_ϵ^i is bounded in $H^1(\Omega)$

R2-

$$\int_\Omega |u_\epsilon|^2 dx \leq C |\ln \epsilon|^{-\delta}, \quad 0 < \delta < 2 - \frac{2}{d}. \quad (115)$$

([3], Theorem 3). In particular $u_\epsilon \rightarrow 0$, in $L^2(\Omega)$, as $\epsilon \rightarrow 0$.

The Steklov-type problem

$$L(u_\epsilon^n) = \frac{\partial}{\partial x_i} (a_{kl}^{ij} \frac{\partial u_\epsilon^n}{\partial x_j}) = 0 \text{ in } \Omega, \quad k = 1, \dots, d \quad (116)$$

with the boundary conditions

$$u_\epsilon^n = 0 \text{ on } \gamma_1 \cup \gamma_\epsilon, \quad (117)$$

and

$$\sigma(u_\epsilon^n) = A^{ij}(x) \frac{\partial u_\epsilon^n}{\partial x_j} \nu_i = \lambda_\epsilon^n u_\epsilon^n \text{ on } \Gamma_\epsilon \quad (118)$$

is considered. The set $\{\lambda_\epsilon^n\}$, $n = 1, 2, \dots$ is the set of eigenvalues such that $\lambda_\epsilon^1 \leq \lambda_\epsilon^2 \leq \dots \leq \lambda_\epsilon^n \leq \dots$, where the eigenvalues repeat according to their multiplicities.

The following estimate is established, for ϵ sufficiently small, and $N_\epsilon = O(|\ln \epsilon|^{(1-\frac{\delta}{2})d-1})$

$$\lambda_\epsilon^n \geq C |\ln \epsilon|^\delta, \quad (119)$$

where C is a constant independent of ϵ , ([3], Theorem 5). The proof is based on an adaptation of the general abstract result (subsection 4.2.1), thanks to the above results R1 and R2.

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