



Spontaneous non-Hermiticity in the Nambu–Jona-Lasinio model

Maxim N. Chernodub, Alberto Cortijo, Marco Ruggieri

► To cite this version:

Maxim N. Chernodub, Alberto Cortijo, Marco Ruggieri. Spontaneous non-Hermiticity in the Nambu–Jona-Lasinio model. *Physical Review D*, 2021, 104 (5), pp.056023. 10.1103/PhysRevD.104.056023 . hal-02940002

HAL Id: hal-02940002

<https://hal.science/hal-02940002>

Submitted on 9 Nov 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Spontaneous Non-Hermiticity in Nambu–Jona-Lasinio model

M.N. Chernodub,^{1,2} Alberto Cortijo,³ and Marco Ruggieri⁴

¹*Institut Denis Poisson UMR 7013, Université de Tours, 37200 France*

²*Pacific Quantum Center, Far Eastern Federal University, Sukhanova 8, Vladivostok 690950, Russia*

³*Departamento de Física de la Materia Condensada,*

Universidad Autónoma de Madrid, Madrid E-28049, Spain

⁴*School of Nuclear Science and Technology, Lanzhou University,*

222 South Tianshui Road, Lanzhou 730000, China

(Dated: August 27, 2020)

We explore the physical consequences of a scenario when the standard Hermitian Nambu–Jona-Lasinio (NJL) model spontaneously develops a non-Hermitian $\mathcal{PT}$ -symmetric ground state via dynamical generation of an anti-Hermitian Yukawa coupling. We demonstrate the emergence of a noncompact non-Hermitian (NH) symmetry group which characterizes the NH ground state. We show that the NH group is spontaneously broken both in weak- and strong-coupling regimes. In the chiral limit at strong coupling, the NH ground state develops inhomogeneity, which breaks the translational symmetry. At weak coupling, the NH ground state is a spatially uniform state, which lies at the boundary between the $\mathcal{PT}$ -symmetric and $\mathcal{PT}$ -broken phases. Outside the chiral limit, the minimal NJL model does not possess a stable non-Hermitian ground state.

I. INTRODUCTION

The Nambu–Jona-Lasinio (NJL) model [1] describes the dynamics of interacting relativistic fermions. The model is often employed as a viable low-energy effective theory of quantum chromodynamics (QCD) because the NJL model, similarly to QCD, exhibits both the dynamical mass gap generation and the chiral symmetry breaking. The model naturally describes basic chiral features of the QCD ground state as well as particularities of the mesonic spectrum, and captures effects of finite temperature and baryonic density [2].

Similarly to QCD, the NJL model is described by a Hermitian Lagrangian. The ground states of both theories are naturally assumed to be Hermitian. In our definition, the “Hermiticity” of the ground state means that elementary excitations over this ground state possess a Hermitian dynamics and, therefore, are described by a Hermitian model. In our paper, we attempt to describe a scenario, when a Hermitian model may spontaneously, via a dynamical mechanism, generate a non-Hermitian ground state.

Non-Hermitian terms usually appear in open systems when a physical system interacts with the external environment. Due to the energy exchange with the environment, some of those systems reside in an off-equilibrium but steady regime. A well-balanced steady state may be described by a Hamiltonian with real-valued eigenenergies. This is the case of the class of Parity-Time ($\mathcal{PT}$) symmetric systems, where the Hermiticity condition is traded by the commutation of the Hamiltonian (and the eigenstates) with the combined parity $\mathcal{P}$ and time reversal inversion $\mathcal{T}$ operation [3]. This combined symmetry allows for an unitary dynamical evolution and real spectrum.

Non-Hermitian physics has previously been addressed in the contexts of interacting models of relativistic fermions. For example, the inclusion of non-Hermitian

$\mathcal{PT}$ -symmetric interactions may support real energy spectra in fermionic theories in 1+1 and 3+1 dimensions [4]. Anti-Hermitian Yukawa interactions may lead to an anomalous radiative mass-gap generation in a model of the right-handed sterile neutrinos, while the energetics of the system forbids the emergence of a dynamically generated mass [5]. Inclusion of a particular $\mathcal{PT}$ - and chirally symmetric bilinear non-Hermitian term contributes to the mass gap generation in the NJL model and leads to a rich phase structure [6].

Our proposal differs from the already existing approaches. Instead of incorporation of a non-Hermitian coupling to the original model, we consider the possibility that a perfectly Hermitian model develops, via a dynamical mechanism, a non-Hermitian ground state with a physically meaningful set of features.

Non-Hermitian physics of relativistic fermions may appear in fireballs of quark-gluon plasma (QGP) created in heavy-ion collisions. A fireball of QGP is a relativistically expanding out-of-equilibrium system. Although this system does not reside in a steady regime, the fermionic interactions may generate a non-Hermitian ground state in a steady-state non-equilibrium regime, which is realized in between the early moments of the plasma until the evolution of the QGP approaches the chiral crossover and eventual hadronization. The non-Hermitian description of the QGP in the expanding non-equilibrium regime is supported by the widely acknowledged fact that the equilibrium ground state of a generic non-Hermitian Hamiltonian is often related to an out-of-equilibrium ground state of an appropriate Hermitian system [7]. In many cases, the Hermitian and non-Hermitian Hamiltonians are indeed related to each other by a (non-Unitary) similarity transformations. In a coherence with the above remarks, we notice that non-Hermitian Dirac fermions allow for realization of the chiral magnetic effect [8] in the thermal equilibrium regime [9], while the similar effect is forbidden in the thermal equilibrium for the ordi-

nary Hermitian Dirac fermions [10–15]. The concept of non-Hermitian quantum theory may also be extended to gauge-gravity duality [16].

In Condensed Matter Physics, the situation is similar. The problem of symmetry-broken states in interacting many-body systems with already incorporated non-Hermiticity has been studied in several physical systems in the recent years. The questions of interest include the effect of non-Hermitian terms in topological superconductivity [17], how the phenomenology attributed to Majorana states might appear in the topologically trivial region due to coupling to the environment [18], or the possibility of non-Hermitian superfluidity with a complex-valued interaction constant [19]. Also some problems known to reside in the strongly coupling regime have been investigated when non-Hermitian terms are considered. These are the case, for example, of the Kondo effect [20], the out-of-equilibrium-induced coupling between the Higgs mode and the Leggett modes in driven superconductors [21], or the Kibble-Zurek mechanism in non-Hermitian environments [22]. In all these cases, the non-Hermitian elements are associated to open systems coupled to the environment or when the system are driven out of equilibrium. Only recently, the possibility of a non-Hermitian superconducting ground states out of an interacting Hermitian system has been proposed [23]. The non-Hermitian albeit $\mathcal{PT}$ -symmetric superconductivity displays a different phenomenology than the Hermitian counterpart.

The organization of this article is as follows: In Section II we briefly review the phenomenology of a free model of non-Hermitian Dirac fermions where the non-Hermiticity is explicitly imposed at the Lagrangian level. The NJL model and its standard Hermitian ground state are introduced in Section III. The homogeneous non-Hermitian ground states are discussed in Section IV while their inhomogeneities are address within the Ginzburg-Landau approach in Section V. An outlook on further development and our conclusions are given in the last two sections. The Appendix contains the presentation of the gradient expansion of the effective non-Hermitian action with the nonhomogeneous terms included.

II. FREE NON-HERMITIAN FERMIONS

The $\mathcal{PT}$ -invariant non-Hermitian extension of the Lagrangian for a Dirac fermion [24–28] has the following form:

$$\mathcal{L}_\psi = \bar{\psi} (i\partial\!\!\!/ - m - m_5\gamma^5) \psi, \quad (1)$$

where m is the mass of the fermion field ψ and m_5 is a non-Hermitian mass. Standard notations for the fermionic fields will be employed throughout the article: $\bar{\psi} = \psi^\dagger \gamma^0$, and $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ where γ^μ and γ^5 are the Dirac matrices.

In the momentum space, the Hamiltonian of the sys-

tem (1) is given by the following operator,

$$\hat{H} = \boldsymbol{\alpha} \cdot \mathbf{k} + \beta m + \beta \gamma_5 m_5, \quad (2)$$

where $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$ and $\beta = \gamma^0$ are the original Dirac notations. Due to the presence of the last term, the Hamiltonian (2) is not a Hermitian operator: $\hat{H} \neq \hat{H}^\dagger$ provided $m_5 \neq 0$.

The Hamiltonian (2), however, commutes with the product of parity $\mathcal{P}$ and time $\mathcal{T}$ inversions: $[\mathcal{PT}, \hat{H}] = 0$, thus implying that the theory respects the $\mathcal{PT}$ symmetry. These discrete operations affect the coordinates as follows: $\mathcal{P}(t, \mathbf{x}) = (t, -\mathbf{x})$ and $\mathcal{T}(t, \mathbf{x}) = (-t, \mathbf{x})$. The parity inversion $\mathcal{P}$ acts linearly on the fermionic fields:

$$\mathcal{P}: \begin{cases} \psi(t, \mathbf{x}) \rightarrow \gamma^0 \psi(t, \mathbf{x}), \\ \bar{\psi}(t, \mathbf{x}) \rightarrow \bar{\psi}(t, \mathbf{x}) \gamma^0, \end{cases} \quad (3)$$

while the time inversion $\mathcal{T}$ is represented by an anti-linear operator:

$$\mathcal{T}: \begin{cases} \psi(t, \mathbf{x}) \rightarrow i\gamma^1 \gamma^2 \psi^*(t, \mathbf{x}), \\ \bar{\psi}(t, \mathbf{x}) \rightarrow \bar{\psi}^*(t, \mathbf{x}) i\gamma^1 \gamma^2. \end{cases} \quad (4)$$

Therefore, the Lagrangian (1) describes a $\mathcal{PT}$ symmetric non-Hermitian theory of Dirac fermions.

The Lagrangian (1) gives the following classical equations of motion:

$$(i\partial\!\!\!/ - m - m_5\gamma^5) \psi = 0, \quad (5)$$

$$\bar{\psi} (i\overleftarrow{\partial\!\!\!/} + m + m_5\gamma^5) = 0. \quad (6)$$

The positive-frequency solutions of the Dirac equation,

$$\psi(x) = u(p) e^{-ip \cdot x}, \quad (7)$$

are expressed via the spinor $u(p)$ which satisfies the Dirac equation in the momentum space:

$$(\not{p} - m - m_5\gamma^5) u(p) = 0, \quad (8)$$

$p^\mu = (p^0, \mathbf{p})$ is the the four-momentum and $p \cdot x = p_\mu x^\mu$ is the scalar product in the Minkowski space. The self-consistency of Eq. (8) requires $p^2 = M^2$ and determines the energy spectrum ($p_0 \equiv E_{\mathbf{p}}$):

$$E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + M^2}, \quad (9)$$

where the mass of the fermionic excitation is

$$M = \sqrt{m^2 - m_5^2}. \quad (10)$$

The same statement applies also to the negative-energy solutions. We always take $|m| \geq |m_5|$ to insure that the mass M is a real quantity. This range of parameters correspond to the “ $\mathcal{PT}$ -symmetric” phase. If the Hermitian mass $|m|$ is smaller than the non-Hermitian mass $|m_5|$, then the system resides in the “ $\mathcal{PT}$ -broken” phase which is characterized by paired complex branches of fermionic energies that make the vacuum unstable. The stability

of the non-Hermitian theory in the $\mathcal{PT}$ symmetric region is the direct consequence of $\mathcal{PT}$ symmetry of the Hamiltonian (2).

In the absence of both non-Hermitian m_5 and Hermitian m_5 masses, the free fermionic theory (1) possesses the global vector and global axial symmetries described by the continuous transformations, respectively:

$$U(1)_V : \quad \psi \rightarrow e^{i\omega_V} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{-i\omega_V}, \quad (11a)$$

$$U(1)_A : \quad \psi \rightarrow e^{i\gamma^5 \omega_A} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\gamma^5 \omega_A}, \quad (11b)$$

with the coordinate-independent parameters ω_V and ω_A .

In the limit of the vanishing masses, $m = m_5 = 0$, the continuous symmetries (11) lead to the following vector and pseudo-vector Noether currents, respectively:

$$j_V^\mu = \bar{\psi} \gamma^\mu \psi, \quad (12a)$$

$$j_A^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi, \quad (12b)$$

which are conserved at the classical level.

However, if the masses m and m_5 are nonzero, then the currents (12) are no more conserved:

$$\partial_\mu j_V^\mu = -2im_5 \bar{\psi} \gamma^5 \psi, \quad (13a)$$

$$\partial_\mu j_A^\mu = 2im \bar{\psi} \gamma^5 \psi. \quad (13b)$$

The Hermitian mass m breaks the axial $U(1)_A$ symmetry (11b) while the non-Hermitian mass m_5 breaks the vector $U(1)_V$ symmetry (11a).

However, one can see from the non-conservation pattern (13) that the model admits the following linear combination of the currents (12):

$$J_V^\mu = j_V^\mu + \frac{m_5}{m} j_A^\mu, \quad (14)$$

which is conserved at the classical level:

$$\partial_\mu J_V^\mu = 0. \quad (15)$$

The definition (14) gives us the new conserved vector current.

The new axial current is orthogonal to the new vector current (14):

$$J_A^\mu = j_A^\mu - \frac{m_5}{m} j_V^\mu. \quad (16)$$

This quantity is not conserved at the classical level:

$$\partial_\mu J_A^\mu = 2i \frac{m^2 - m_5^2}{m} \bar{\psi} \gamma^5 \psi, \quad (17)$$

similarly to the ordinary axial current.

A system with a finite density of the conserved charge J_V^0 may be controlled by the thermodynamically conjugated chemical potential. Certain consequences of the finite-density non-Hermitian fermions are discussed in Ref. [9]. Below we proceed to the investigation of the ground state of a Hermitian NJL model in which the non-Hermitian theory (1) may arise as an effective theory in a spontaneously formed non-Hermitian ground state.

III. HERMITIAN GROUND STATE IN THE NJL MODEL

A. Nambu–Jona–Lasinio model

The NJL model provides a simplest description of interacting fermions which features the chiral symmetry breaking and the mass gap generation. We consider the standard Hermitian NJL model described by the following Lagrangian:

$$\mathcal{L}_{\text{NJL}} = \bar{\psi}(i\not{\partial} - m_0)\psi + \frac{G}{2} \left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right], \quad (18)$$

which describes the single fermion species ψ with a bare mass m_0 . The coupling constant G determines the strength of the local four-fermion interaction.

In the massless limit, $m_0 = 0$, the NJL Lagrangian (18) is invariant under vector (11a) and axial (11b) global transformations. The $U(1)_V$ symmetry may be gauged via the coupling of the fermions to the electromagnetic field. Since in the physical world the vector (electric) current is always conserved, the $U(1)_V$ symmetry should never be broken in realistic models. However, the $U(1)_A$ may become broken at the quantum level. In the single-species model that we consider (18), the breaking of the chiral symmetry appears spontaneously due to the four-fermion interaction as we discuss below.¹

In the presence of the fermionic mass, $m_0 \neq 0$, the $U(1)_V$ symmetry (11a) is maintained, while the axial symmetry (11b) gets explicitly broken. In QCD, however, the bare mass (also called “current mass”) m_0 is much smaller than the dynamically generated fermionic mass. Therefore, the effects of the axial mass are small and the axial symmetry (11b) is said to be approximately correct. Below, we review the derivation of the standard Hermitian ground states and the dynamical mass gap generation in the NJL model (18). Afterwards, we will proceed to the investigation of the non-Hermitian mass gap generation in the same model.

In the scenario we are considering here, the non-Hermiticity is broken dynamically within the Hermitian model. For a different version of the NJL model, where the non-Hermiticity is broken explicitly via a coupling of the NJL Lagrangian (18) to a non-Hermitian background, we refer the reader to Ref. [6].

B. Hermitian ground state in the NJL model

In the standard approach, the partition function of the NJL model (18),

$$\mathcal{Z} = \int D\psi D\bar{\psi} \exp \left\{ i \int d^4x \mathcal{L}_{\text{NJL}}(\bar{\psi}, \psi) \right\}, \quad (19)$$

¹ In the single-fermion model, the $U(1)_A$ symmetry (11b) may be called both axial and chiral. This terminological degeneracy is lifted in the multi-species models.

is partially bosonized by inserting the identities²

$$1 = \int D\sigma \exp \left\{ -\frac{i}{2G} \int d^4x (\sigma + G\bar{\psi}\psi)^2 \right\}, \quad (20a)$$

$$1 = \int D\phi \exp \left\{ -\frac{i}{2G} \int d^4x (\phi + G\bar{\psi}i\gamma_5\psi)^2 \right\}, \quad (20b)$$

under the integral over the fermionic fields in Eq. (19). The fields σ and ϕ are real-valued quantities.

The prefactors in the exponents of Eq. (20) are chosen in such a way that, after the substitution of Eqs. (20) to Eq. (19),

$$\mathcal{Z} = \int D\psi D\bar{\psi} D\sigma D\phi \exp \left\{ i \int d^4x \mathcal{L}_{\text{NJL}}(\bar{\psi}, \psi; \sigma, \phi) \right\}, \quad (21)$$

the four-fermion interaction terms of Eq. (18) disappear:

$$\mathcal{L}_{\text{NJL}}(\bar{\psi}, \psi; \sigma, \phi) = \bar{\psi} [i\cancel{\partial} - \sigma - i\gamma_5\phi] \psi - \frac{(\sigma - m_0)^2 + \phi^2}{2G}. \quad (22)$$

The partially-bosonized NJL Lagrangian (21) becomes a bilinear expression in terms the original fermionic fields, coupled, via the Yukawa interactions, with the scalar σ and pseudoscalar ϕ auxiliary fields.³ In the expression (21) we have shifted the σ -field, $\sigma \rightarrow \sigma - m_0$ for the sake of further convenience.

The interpretation of the auxiliary fields σ and ϕ may be deduced from the saddle-point equations of the partition function (21):

$$\langle \sigma \rangle = m_0 + G \langle \bar{\psi}\psi \rangle, \quad \langle \phi \rangle = G \langle \bar{\psi}i\gamma_5\psi \rangle. \quad (23)$$

The vacuum expectation value of the condensate σ plays a role of the mass of the fermion field. At the saddle point, the vacuum expectation values of the scalar and pseudoscalar auxiliary fields acquire contributions proportional, correspondingly, to the scalar and pseudoscalar condensates of fermionic fields (23). The emergence of the scalar fermion condensate (also called ‘‘chiral condensate’’) $\langle \bar{\psi}\psi \rangle$ leads to two dynamical phenomena. First, the chiral condensate is not invariant under the chiral transformations (11b) and therefore it breaks the chiral symmetry. Second, the emergence of the chiral condensate leads to the mass gap generation (23) as the field σ plays a role of the fermion mass (22). The pseudoscalar condensate vanishes in the physical vacuum

$$\langle \bar{\psi}i\gamma_5\psi \rangle = 0, \quad (24)$$

implying, according to Eq. (23), that the mean value of the field ϕ vanishes as well, $\langle \phi \rangle = 0$.

The applicability of the saddle-point approximation becomes justified for the NJL model with many fermion flavors N_f . In the limit $N_f \rightarrow \infty$, the saddle-point calculation becomes exact. The consistency of the physical interpretation (23) is guaranteed by the fact that the bilinears $\bar{\psi}\psi$ and $\bar{\psi}i\gamma_5\psi$, similarly to the auxiliary fields σ and ϕ , are real-valued quantities for any spinor field ψ .

Below, we will treat the auxiliary fields σ and ϕ in the mean-field approximation. For the sake of brevity, we will use the same notations for the fields and their expectation values, $\sigma = \langle \sigma \rangle$ and $\phi = \langle \phi \rangle$. In a leading order, we assume that these fields are independent of the spacetime coordinates. Later, we will consider physical excitations over this uniform background.

The integration over the fermionic fields in the partition function (22) leads to the following purely scalar representation of the NJL model:

$$\mathcal{Z} = \int D\sigma D\phi \exp \left\{ -\frac{i}{2G} \int d^4x [(\sigma - m_0)^2 + \phi^2] + \ln \det [i\cancel{\partial} - (\sigma + i\gamma_5\phi)] \right\}. \quad (25)$$

In the completely bosonized representation of the NJL model (25), the chiral invariance (11b) is translated to a chiral (axial) rotation for the following combination of the scalar and pseudoscalar fields:

$$\sigma + i\gamma_5\phi \rightarrow e^{-i\gamma_5\theta} (\sigma + i\gamma_5\phi) e^{i\gamma_5\theta} \equiv \tilde{\sigma} + i\gamma_5\tilde{\phi}. \quad (26)$$

For coordinate-independent condensates σ and ϕ , the fermionic determinant depends only on the combination $\sigma^2 + \phi^2$ which is invariant under the chiral rotation (26).

It is convenient to use the rotation (26) to turn the combination $\sigma + i\gamma_5\phi$ into the purely scalar field with the absolute value $|\tilde{\sigma}| = \sqrt{\sigma^2 + \phi^2}$. Taking $\tilde{\sigma}$ a real positive number ($\tilde{\sigma} > 0$ and $\tilde{\phi} = 0$) and renaming back $\tilde{\sigma} \rightarrow \sigma$, we find that the fermionic sector of partially bosonized mean-field NJL model corresponds to a theory of massive fermions with an effective mass $m = \sigma$ and vanishing pseudoscalar field $\phi = 0$ after the chiral rotation.

Using then the expression for the functional trace of an operator $\hat{\mathcal{O}}$,

$$\text{tr } \hat{\mathcal{O}} = \int d^4x \int \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \mathcal{O}_{k_0, \mathbf{k}}, \quad (27)$$

we get for the (chirally-rotated) last term in Eq. (25) the following expression:

$$\ln \det (i\cancel{\partial} - \sigma) = -i \int d^4x V_{\text{int}}(\sigma), \quad (28)$$

with the potential for the field σ :

$$V_{\text{int}}(\sigma) = i \int \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \text{tr} \ln \frac{\not{k} - \sigma}{\Lambda}. \quad (29)$$

Here $\not{k} \equiv \gamma^\mu k_\mu$ and the (so far arbitrary) quantity of the dimension of mass Λ is needed to maintain correct

² Hereafter we omit inessential normalization factors in front of the functional integrals.

³ For the model with a single fermion, we reserve the letter ϕ for the pseudoscalar instead of the standard notation π .

dimension. Hereafter we will ignore inessential additive constants in the potentials and actions.

Although it is possible to continue the derivation in Minkowski space-time, it is convenient to perform the Wick rotation to the Euclidean momentum space:

$$k_0 \rightarrow ik_4, \quad \int \frac{dk_0}{2\pi} \rightarrow i \int \frac{dk_4}{2\pi}. \quad (30)$$

We arrive to the potential (29) given by the following expression:

$$V_{\text{int}}(\sigma) = -2 \int \frac{d^4k}{(2\pi)^4} \ln \frac{k^2 + \sigma^2}{\Lambda^2}, \quad (31)$$

where $d^4k = dk_4 d^3k$. Here we used the fact that under the Wick rotation $\not{k} = \gamma^0 k^0 - \boldsymbol{\gamma} \cdot \mathbf{k} \rightarrow -\gamma^4 k^4 - \boldsymbol{\gamma} \cdot \mathbf{k} = -\not{k}$, supplemented by the standard chain of relations:

$$\begin{aligned} & \int \frac{d^4k}{(2\pi)^4} \text{tr} \ln \frac{\not{k} - \sigma}{\Lambda} \\ &= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[\ln \frac{\not{k} + \sigma}{\Lambda} + \ln \frac{-\not{k} + \sigma}{\Lambda} \right] \\ &= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \text{tr} \ln \frac{k^2 + \sigma^2}{\Lambda_0^2} = 2 \int \frac{d^4k}{(2\pi)^4} \ln \frac{k^2 + \sigma^2}{\Lambda^2}. \end{aligned} \quad (32)$$

The effective potential for the scalar field σ is then

$$V(\sigma) = \frac{(\sigma - m_0)^2}{2G} - 2 \int \frac{d^4k}{(2\pi)^4} \ln \frac{k^2 + \sigma^2}{\Lambda^2}. \quad (33)$$

The kinetic term for the σ field is absent in the mean-field approach with the uniform σ background.

The ground state of the model is determined by the minimization of the action with respect to the σ field. We employ the four-momentum regularization scheme with a hard ultraviolet cutoff. The regularized potential (33) is:

$$\begin{aligned} V(\sigma) &= \frac{(\sigma - m_0)^2}{2G} - \frac{1}{4\pi^2} \int_0^\Lambda k^3 dk \ln \frac{k^2 + \sigma^2}{\Lambda^2} \\ &= \frac{\sigma^2}{2G} + \Lambda^4 v \left(\frac{\sigma^2}{\Lambda^2} \right), \end{aligned} \quad (34)$$

where the quantity Λ serves an ultraviolet cutoff which plays a role of a physical parameter in this model and

$$v(x) = -\frac{1}{16\pi^2} [x + x^2 \ln x + (1 - x^2) \ln(x + 1)]. \quad (35)$$

The critical coupling,

$$G_c = 4\pi^2 \Lambda^{-2} \simeq 39.48 \Lambda^{-2}, \quad (36)$$

defines two regimes of the theory. Consider first the case of zero bare mass $m_0 = 0$. For a weak coupling $G < G_c$, the potential $V(\sigma)$ takes its minimum at vanishing field $\sigma = 0$ which defines a chirally symmetric vacuum. At strong coupling $G > G_c$, the minimum of the potential is reached at $\sigma \neq 0$, the chiral symmetry gets broken and the fermions acquire the mass $M = \langle \sigma \rangle \sim \Lambda$ via the dynamical mechanism. A small nonzero bare mass, $m_0 \neq 0$, shifts the minimum σ to a nonzero value $\sigma \sim$

m_0 at weak coupling G . However, the dynamical mass generation overwhelm the bare mass, $\Lambda \gg m_0$, and the in the chirally broken phase, the mass of the fermion is given by the dynamically generated mass.

Alternatively, one can determine the ground state by solving a system of the mass-gap equations that correspond to the extremization of the effective potential with respect to the dynamical field: $\delta V(\sigma, \phi)/\delta \sigma = 0$ and $\delta V(\sigma, \phi)/\delta \phi = 0$. Restoring the pseudoscalar field ϕ in the effective potential (33), we obtain the following system of equations:

$$\sigma - m_0 - 4G \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{\sigma}{k^2 + \sigma^2 + \phi^2} = 0, \quad (37a)$$

$$\phi - 4G \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{\phi}{k^2 + \sigma^2 + \phi^2} = 0, \quad (37b)$$

which has the same solution that we discussed above. While both methods often give identical solutions, the direct minimization of the effective potential allows to verify that the ground state corresponds indeed to a globally stable state given by the global minimum of the effective potential.

IV. NON-HERMITIAN GROUND STATE IN THE NJL MODEL

A. Non-Hermitian bosonization

In what follows we exploit the possibility that the partial bosonization, performed with the help of the standard identities (20), is not the only possible choice that can be employed for the bosonization.

In order to maintain the explicit Hermiticity of the model, we required for the fields σ and ϕ to be real-valued quantities. Below, we leave the condition of the Hermiticity and consider the complex-valued bosonic fields.

To this end, we generalize the identities (19) using four, instead of the two ones (20):

$$1 = \int D\sigma_1 \exp \left\{ -\frac{i}{2G_{\sigma 1}} \int d^4x (\sigma_1 + G_{\sigma 1} \bar{\psi} \psi)^2 \right\}, \quad (38a)$$

$$1 = \int D\sigma_2 \exp \left\{ -\frac{i}{2G_{\sigma 2}} \int d^4x (\sigma_2 + iG_{\sigma 2} \bar{\psi} \psi)^2 \right\}, \quad (38b)$$

$$1 = \int D\phi_1 \exp \left\{ -\frac{i}{2G_{\phi 1}} \int d^4x (\phi_1 + G_{\phi 1} \bar{\psi} i\gamma_5 \psi)^2 \right\}, \quad (38c)$$

$$1 = \int D\phi_2 \exp \left\{ -\frac{i}{2G_{\phi 2}} \int d^4x (\phi_2 - G_{\phi 2} \bar{\psi} \gamma_5 \psi)^2 \right\}. \quad (38d)$$

Inserting these identities into Eq. (19), we find that we might cancel the four-fermion interaction terms of the NJL Lagrangian (18) provided the parameters in Eq. (38) satisfy the following conditions:

$$G_{\sigma 1} - G_{\sigma 2} = G, \quad G_{\phi 1} - G_{\phi 2} = G. \quad (39)$$

Repeating all steps as in the Hermitian counterpart, we get the new partially bosonized Lagrangian:

$$\mathcal{L}_{\text{NJL}} = -\frac{1}{2} \left(\frac{(\sigma_1 - m_0)^2}{G_{\sigma_1}} + \frac{\sigma_2^2}{G_{\sigma_2}} + \frac{\phi_1^2}{G_{\phi_1}} + \frac{\phi_2^2}{G_{\phi_2}} \right) + \bar{\psi} [i\cancel{\partial} - (\sigma_1 + i\sigma_2) - i\gamma^5(\phi_1 + i\phi_2)] \psi, \quad (40)$$

in which we have redefined the field $\sigma_1 \rightarrow \sigma_1 - m_0$ for convenience.

Let us consider the fermionic part of the NJL Lagrangian (40). For the state $\sigma_2 = 0$ and $\phi_2 = 0$, the theory reduces to the standard Hermitian case considered earlier: the axial freedom (26) may be used to remove the field ϕ_1 and we arrive to the bosonized mean-field theory, given by Eqs. (34) and (35), with the condensate $\sigma \equiv \sigma_1$.

In the rest of this section we assume, unless explicitly stated otherwise, that the bosonic fields do not depend on the coordinates. The case of the non-uniform background with space-dependent fields will be considered in the next section.

If the fields σ_2 and ϕ_2 are taken to be nonzero, then the fermionic part of the Lagrangian (40) becomes non-Hermitian. The excitations of a non-Hermitian theory may possess a complex spectrum signaling instabilities. The physical spectrum corresponds to the poles $k_0 = \pm\omega_{\mathbf{p}}$ with the following one-particle energy:

$$\omega_{\mathbf{p}} = \sqrt{\mathbf{k}^2 + (\sigma_1 + i\sigma_2)^2 + (\phi_1 + i\phi_2)^2}. \quad (41)$$

We use the axial (chiral) rotation (26) applied to the combination $\sigma + i\gamma^5\phi$ of now-complex fields $\sigma = \sigma_1 + i\sigma_2$ and $\phi = \phi_1 + i\phi_2$ to reduce an unphysical degree of freedom. Given the fact that the unitarity of the chiral transformation requires the parameter ω_A of Eq. (26) to be real, we may get rid of only one real-valued field. We choose to remove, following the standard approach to the NJL mode, the real component of the complex ϕ field, $\phi_1 = 0$.

Similarly to the case of the standard Hermitian theory, we take $\sigma_1 \neq 0$. In this case, the stability of the ground state of the theory requires $\sigma_2 = 0$. The requirements $\phi_1 = 0$ and $\sigma_2 = 0$ are enforced by taking the corresponding constants to be zero, $G_{\phi_1} \rightarrow 0$ and $G_{\sigma_2} \rightarrow 0$, respectively. According to the algebraic conditions (39), we are left with the two linearly-dependent coupling constants $G_{\sigma_1} = G$ and $G_{\phi_2} = -G$ and two one-component fields $\sigma_1 \equiv \sigma$ and $\phi_2 \equiv -\phi_5$. Here we have redefined the fields again for the sake of convenience.

Using the new non-Hermitian bosonization, we arrive to the following representation of the NJL model:

$$\mathcal{L}_{\text{NJL}} = \frac{\phi_5^2 - (\sigma - m_0)^2}{2G} + \bar{\psi} (i\cancel{\partial} - \sigma - \gamma^5\phi_5) \psi. \quad (42)$$

We immediately recognize that the field ϕ_5 plays the role of the non-Hermitian fermionic mass m_5 already discussed in the context of the free theory (1) while the field σ appears in the standard role of the usual Hermitian mass m .

The coupling between the fermions and the auxiliary field ϕ_5 in the partially bosonized NJL Lagrangian (42) has the form of the anti-Hermitian Yukawa coupling that has been studied recently [5, 29] in the phenomenological context of sterile neutrinos.

The energy spectrum of fermions in the partially bosonized NJL theory (42) is, now:

$$\omega_{\mathbf{p}} = \sqrt{\mathbf{k}^2 + \sigma^2 - \phi_5^2}, \quad (43)$$

which is typical for the fermions which have both Hermitian σ and non-Hermitian ϕ_5 fermionic masses (10).

After integrating out the fermionic fields in the non-Hermitian theory (42), we arrive to the effective potential on the scalar and pseudo-scalar non-Hermitian fields:

$$V(\sigma, \phi_5) = \frac{(\sigma - m_0)^2 - \phi_5^2}{2G} + \Lambda^4 v \left(\frac{\sigma^2 - \phi_5^2}{\Lambda^2} \right), \quad (44)$$

where the function $v(x)$ is given in Eq. (35).

Please notice that the minimal realization of a scalar non-Hermitian model involves two scalar fields, one of them is a true scalar while another one is often chosen as a pseudoscalar [27, 28]. This property is also maintained by the effective bosonized theory (44). The form of the bosonized effective model (44) is qualitatively different from the multi-scalar model considered so far in the literature [30–32].

The non-Hermitian model remains physically meaningful provided the $\mathcal{PT}$ symmetry is unbroken:

$$\sigma^2 \geq \phi_5^2. \quad (45)$$

These considerations could naively (and, incorrectly) imply that there is no obvious effect of the non-Hermiticity on the physical properties of the model. Indeed, we are free to denote the field combination in Eq. (44) as

$$M^2 = \sigma^2 - \phi_5^2 \geq 0, \quad (46)$$

and to come back to the Hermitian model in terms of the new field $\sigma \equiv M$ with the same fermionic spectrum (43): $\omega_{\mathbf{p}} = \sqrt{\mathbf{k}^2 + M^2}$.

Contrary to the Hermitian bosonization of the NJL model, there is no obvious interpretation of the field ϕ_5 in terms of a fermionic condensate. For example, a variation of the action associated with the Lagrangian (42) with respect to the field ϕ_5 would give us the apparent relation

$$\langle \phi_5 \rangle \stackrel{?}{=} G \langle \bar{\psi} \gamma^5 \psi \rangle, \quad (47)$$

which could naively be interpreted – following the Hermitian case – as a relation that determines a saddle-point of the effective theory. We notice, however, that the algebraic self-consistency of Eq. (47) is questionable because the field ϕ_5 at the left-hand side of Eq. (47) is a real-valued quantity while the fermionic condensate at the right-hand side of the same equation takes purely imaginary values.

In an attempt to interpret the field ϕ_5 in line of the fermionic condensate (47), one could consider a possibility that the saddle point (47) is realized for a purely

imaginary field ϕ_5 . In this case, we could redefine the field $\phi_5 = i\phi$ and come back to the Hermitian theory, in which the mean field ϕ may be removed further by the chiral rotation (26). This approach could be sustainable if the minimum of the potential at the purely imaginary ϕ_5 is lower compared to its minimum at the real ϕ_5 . We will explore this possibility below.

Before finishing this section, we would like to comment that we could have arrived to the same non-Hermitian model (42) by employing only the first and the last of the identities of the alternative set (38) instead of the original identities (20). This would be a fully legitimate operation. We could also naively use a non-Unitary chiral rotation (26) with a purely imaginary ω_A and also come to the same conclusion. The latter operation is, however, logically forbidden because the transformation (11b) with a purely imaginary ω_A is not a symmetry of the original theory.

B. Non-Hermitian ground state in the chiral limit

We start this section by noticing that for $m_0 = 0$, the NJL potential in the bosonized mean-field representation (44) possesses the emergent $U(1)$ symmetry group,

$$U(1)_{\text{NH}} : \begin{pmatrix} \sigma \\ \phi_5 \end{pmatrix} \rightarrow \begin{pmatrix} \cosh \omega_{\text{NH}} & \sinh \omega_{\text{NH}} \\ \sinh \omega_{\text{NH}} & \cosh \omega_{\text{NH}} \end{pmatrix} \begin{pmatrix} \sigma \\ \phi_5 \end{pmatrix}, \quad (48)$$

which is parameterized by an arbitrary real-valued angle $\omega_{\text{NH}} \in \mathbb{R}$. The non-Unitary transformation (48) corresponds a noncompact Abelian group which keeps invariant the combination of the scalar fields $\sigma^2 - \phi_5^2$.

In presence of condensates of σ and/or ϕ_5 fields, the non-Hermitian symmetry (48) is broken spontaneously. According to the standard lore, this breaking should lead to new Goldstone bosons in the spectrum. Here we firstly discuss the theory with $m_0 = 0$, and in the next subsection we analyze the effects of a nonzero current mass.

In Fig. 1(a) we show the mean-field potential (44) in the chirally symmetric ($G < G_c$) phase. If we restrict ourselves to the purely Hermitian case, $\phi_5 = 0$, then the minimum of the potential is reached at the symmetric vacuum $\sigma = 0$ (marked by the small red sphere in the figure.) In the full non-Hermitian plane, the minimum is reached exactly at the border of the $\mathcal{PT}$ symmetric region, $\sigma = \pm\phi_5$. This vacuum state breaks the non-Hermitian symmetry group (48).

Figure 1(b) shows the potential (44) in the chirally broken ($G > G_c$) phase. The minimum of the potential, shown by the blue line, respects the non-Hermitian symmetry (48). The green line shows the profile of the potential in the Hermiticity region $\phi_5 = 0$ which is denoted by the greenish plane. Both lines cross at the Hermitian minimum of the potential: $(\sigma, \phi_5)_H = (\sigma_{\text{min}}, 0)$. The Hermitian minimum is related to all other non-Hermitian minima via a non-compact transformation (48).

Thus, we conclude that in the case of $m_0 = 0$, the non-Hermitian ground state with nonvanishing scalar Hermitian field σ and the pseudoscalar non-Hermitian ϕ_5 field can well be realized in the Hermitian NJL model. The non-Hermitian ground state breaks the non-compact non-Hermitian group (48).

C. The instability associated to a non-zero m_0

The presence of the mass m_0 in the potential (44) breaks, mildly and explicitly, the non-Hermitian symmetry (48). However, this small explicit symmetry breaking – which would be non-harmful in the usual Hermitian theory, leads to the instability of the non-Hermitian theory. This fact can be observed both for the weakly coupled phase with $G < G_c$, Fig. 1(c), and for strongly coupled phase ($G > G_c$), Fig. 1(d).

The same conclusions may also be easily reached analytically in the case of a large condensate, $\sigma \gg m_0$. Then the vacuum solution follows two lines $\phi_5 \simeq \pm\sigma$ and the effective potential (44) is a linear function of the condensate:

$$V(\sigma, \phi_5 = \pm\sigma) \simeq -\frac{m_0\sigma}{G} + \text{const}, \quad (\text{at } \sigma \gg m_0) \quad (49)$$

which may be made arbitrarily low. While we do not investigate this issue in detail in the present study, we would like to comment that this instability might be removed by inhomogeneous condensates by bringing further terms to the free energy that depend on space derivatives.

We conclude that the NJL does not possess a stable vacuum in terms of uniform condensates in the presences of a small explicit symmetry breaking given by small (in fact, an infinitely small) mass $m_0 \neq 0$ if the non-Hermitian condensate ϕ_5 is allowed.

V. GINZBURG-LANDAU ANALYSIS

In this section we use a Ginzburg-Landau (GL) effective potential to analyze the ground state of the model. This expansion is known to work well only in proximity of a second-order phase transition: In doing this analysis, we assume that the coupling constant is slightly larger the critical coupling so that the condensate is not large.

We do not specify any concrete temperature or chemical potential here, while we assume that the sixth order coefficients (defined in next paragraphs) are positive to have a bounded-from-below effective potential. The structure of the GL potential, in particular the relations between the coefficients of the polynomials and derivative terms, will be adapted from that of chiral symmetry breaking of the NJL model [33].

The GL potential has to be consistent with the symmetry of the model, in particular it should be invariant under the global transformation in Eq. (48). Up to the

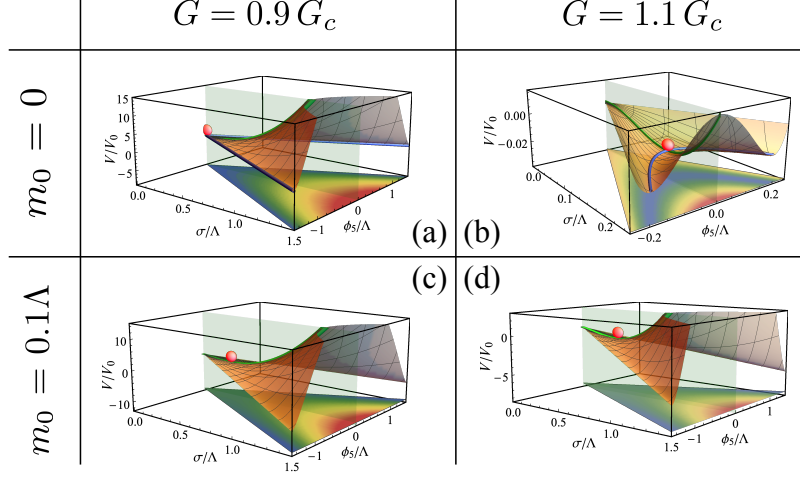


FIG. 1. The mean-field potential (44) as the function of the *uniform, coordinate-independent* σ and ϕ condensates in the chirally symmetric ($G < G_c$) and chirally broken ($G > G_c$) phases at zero ($m_0 = 0$) and non-zero ($m_0 = 0.1\Lambda$) bare masses m_0 . The critical coupling G_c is given in Eq. (36) and $V_0 = (\Lambda/2\pi)^4$. The projection on the (σ, ϕ) plane illustrates the region where the $\mathcal{PT}$ symmetry is unbroken. The density plot at the projection shows the values of the potential ranging from the low (the blue) to the high (the red) values. The greenish cross-section shows the Hermitian region ($\phi_5 = 0$) with the value of the potential highlighted by the green line and the minimum of the effective potential marked by the red spheres. The blue line shows the degenerate minimum of the potential in the full (σ, ϕ) plane.

sixth order in the fields and fourth order in derivatives, the GL potential for the model at hand has to have the form

$$\begin{aligned}
 F_{\text{NH}} = & \frac{\alpha_2}{2} \vec{\chi} \cdot \vec{\chi} + \frac{\alpha_4}{4} (\vec{\chi} \cdot \vec{\chi})^2 + \frac{\alpha_6}{6} (\vec{\chi} \cdot \vec{\chi})^3 \\
 & + \frac{\beta_4}{4} \nabla \vec{\chi} \cdot \nabla \vec{\chi} \\
 & + \frac{\gamma_6}{6} (\vec{\chi} \cdot \vec{\chi}) (\nabla \vec{\chi} \cdot \nabla \vec{\chi})^2 + \frac{\delta_6}{6} (\vec{\chi} \cdot \nabla \vec{\chi})^2 \\
 & + \frac{\varepsilon_6}{6} \Delta \vec{\chi} \cdot \Delta \vec{\chi},
 \end{aligned} \tag{50}$$

where $\vec{\chi} = (\sigma, \phi_5)$ is the vector in the space of fields equipped with the hyperbolic metric: $\vec{\chi} \cdot \vec{\chi} = \sigma^2 - \phi_5^2$ and $\nabla \vec{\chi} \cdot \nabla \vec{\chi} = (\nabla \sigma)^2 - (\nabla \phi_5)^2$. As mentioned earlier, stability requires that the sixth order coefficients are positive: $\alpha_6, \beta_6, \gamma_6, \delta_6, \varepsilon_6 > 0$.

The symmetry alone does not allow to fix the relations between the coefficients of the expansion in Eq. (50). In order to simplify the GL potential we use the known results of the NJL model with chiral condensate: in fact, the GL potential for the non-Hermitian model has to have the same invariance under the chiral rotation of the simpler NJL model, augmented with invariance under rotation Eq. (48) and the latter can be implemented by the replacement $\sigma^2 \rightarrow \sigma^2 - \phi_5^2$ in the potential of the NJL model.

In the NJL model with a chiral condensate the free

energy density in the GL approximation is given by [33]

$$\begin{aligned}
 F_{\text{NJL}}(\sigma(\mathbf{x})) = & \frac{\alpha_2}{2} \sigma^2(\mathbf{x}) + \frac{\alpha_4}{4} [\sigma^4(\mathbf{x}) + (\nabla \sigma(\mathbf{x}))^2] \\
 & + \frac{\alpha_6}{6} \left(\sigma^6(\mathbf{x}) + 5\sigma^2(\mathbf{x}) [\nabla \sigma(\mathbf{x})]^2 + \frac{1}{2} [\Delta \sigma(\mathbf{x})]^2 \right),
 \end{aligned} \tag{51}$$

The effective potential for the non-Hermitian ground state can be obtained from Eq. (51) by the replacements

$$\sigma^2 \rightarrow \vec{\chi} \cdot \vec{\chi}, \tag{52}$$

$$(\nabla \sigma)^2 \rightarrow \nabla \vec{\chi} \cdot \nabla \vec{\chi}, \tag{53}$$

$$\sigma^2 (\nabla \sigma)^2 \rightarrow (\vec{\chi} \cdot \vec{\chi}) \nabla \vec{\chi} \cdot \nabla \vec{\chi} \text{ or } (\vec{\chi} \cdot \nabla \vec{\chi})^2, \tag{54}$$

$$(\Delta \sigma)^2 \rightarrow \Delta \vec{\chi} \cdot \Delta \vec{\chi}, \tag{55}$$

where $\vec{\chi} = (\sigma, \phi_5)$ has been defined above. Notice that the NJL invariant on the left-hand side of Eq. (54) leads to two possible terms in the non-Hermitian model, namely those with coefficients γ_6 and δ_6 in Eq. (50). We analyze the two limiting possibilities here, namely $\gamma_6 = 0$ and $\delta_6 = 0$, noticing that in the former case we get a stable GL potential while in the latter case the free energy is unbounded from below.

Starting with $\delta_6 = 0$, the GL free energy of the non-Hermitian model reads

$$\begin{aligned}
 F_{\text{NH}}(\vec{\chi}(\mathbf{x})) = & \frac{\alpha_2}{2} \vec{\chi} \cdot \vec{\chi} \\
 & + \frac{\alpha_4}{4} [(\vec{\chi} \cdot \vec{\chi})^2 + \nabla \vec{\chi} \cdot \nabla \vec{\chi}] \\
 & + \frac{\alpha_6}{6} [(\vec{\chi} \cdot \vec{\chi})^3 + 5(\vec{\chi} \cdot \vec{\chi}) (\nabla \vec{\chi} \cdot \nabla \vec{\chi})^2 + \frac{1}{2} \Delta \vec{\chi} \cdot \Delta \vec{\chi}].
 \end{aligned} \tag{56}$$

It is now useful to introduce two new fields, ξ and θ , by means of the following transformation

$$\sigma = \xi \cosh \theta, \quad \phi_5 = \xi \sinh \theta. \quad (57)$$

Using the parameterization (57), we rewrite the free energy (56) as follows:

$$\begin{aligned} F_{\text{NH}}(\xi, \theta) = & \frac{\alpha_2}{2} \xi^2 + \frac{\alpha_4}{4} \xi^4 + \frac{\alpha_6}{6} \xi^6 \\ & + \frac{\alpha_4}{4} [(\nabla \xi)^2 - \xi^2 (\nabla \theta)^2] \\ & + \frac{5\alpha_6}{6} [\xi^2 (\nabla \xi)^2 - \xi^4 (\nabla \theta)^2] \\ & + \frac{\alpha_6}{12} [\Delta \xi + \xi (\nabla \theta)^2]^2 \\ & - \frac{\alpha_6}{12} [2(\nabla \xi \cdot \nabla \theta) + \xi \Delta \theta]^2, \end{aligned} \quad (58)$$

where we used the identities:

$$\vec{\chi} \cdot \vec{\chi} = \xi^2, \quad (59)$$

$$\nabla \vec{\chi} \cdot \nabla \vec{\chi} = (\nabla \xi)^2 - \xi^2 (\nabla \theta)^2, \quad (60)$$

$$\Delta \vec{\chi} \cdot \Delta \vec{\chi} = [\Delta \xi + \xi (\nabla \theta)^2]^2 - [2(\nabla \xi \cdot \nabla \theta) + \xi \Delta \theta]^2. \quad (61)$$

The free energy (58) is invariant under rigid shifts of the hyperbolic field $\theta \rightarrow \theta + \theta_0$. This freedom corresponds to the transformations of the global non-Hermitian group (48). The corresponding non-Hermitian Nambu-Goldstone boson is represented by the single "pion" field ϕ_5 , which, in the linear order, is realized as the hyperbolic phase θ with the kinetic-only action. Notice that the kinetic action enters the free energy with the negative sign which is the expected property in the non-Hermitian scalar gauge theories [27]. The negative sign in front of the kinetic term of the θ field is the reason why the presence of the $\xi \neq 0$ condensate breaks the translational symmetry of the strongly-coupled phase. We discuss this question below.

For $\alpha_4 > 0$, the terms proportional to $(\nabla \xi)^2$ and $\xi^2 (\nabla \theta)^2$ in (58) respectively increase and decrease the free energy for an inhomogeneous ground state: therefore, for $\alpha_4 > 0$ it is likely that the lowest free energy is achieved by a ground state of the form

$$\xi = \xi_0, \quad \theta = \theta(x), \quad (62)$$

where ξ_0 is a constant. For simplicity, we limit ourselves here to the following ansatz:

$$\xi = \xi_0, \quad \theta = \theta_0 + \mathbf{k} \cdot \mathbf{x}, \quad (63)$$

where $\mathbf{k}$ is a constant wave vector and θ_0 is an arbitrary constant phase. The vacuum with a nonzero wavevector $\mathbf{k}$ breaks the rotational symmetry of the ground state.

Using the ansatz (63) in Eq. (58) we get the free energy of the ground state:

$$\begin{aligned} F_{\text{NH}} \Big|_{\xi=\xi_0; \theta=\mathbf{k}\mathbf{x}} = & \frac{\alpha_2}{2} \xi_0^2 + \frac{\alpha_4}{4} \xi_0^4 + \frac{\alpha_6}{6} \xi_0^6 \\ & - \left(\frac{\alpha_4}{4} + \frac{5\alpha_6}{6} \xi_0^2 \right) \xi_0^2 \mathbf{k}^2 + \frac{\alpha_6}{12} \xi_0^2 \mathbf{k}^4; \end{aligned} \quad (64)$$

the above equation tells us that an inhomogeneous ground state with a non-zero wavevector $\mathbf{k} \neq 0$ lowers the free energy of the system.

The direction of $\mathbf{k}$ in the ground state will be chosen spontaneously by the system as it does not affect the free energy, while the magnitude of $\mathbf{k}$, namely k_0 , minimizes F_{NH} . Explicitly, the length of the ground-state wavevector is

$$k_0 = \sqrt{\frac{3\alpha_4 + 10\alpha_6 \xi_0^2}{2\alpha_6}}. \quad (65)$$

Using this expression for k_0 in Eq. (64), we get the free energy at the minimum:

$$F_{\text{NH}} \Big|_{\xi=\xi_0; \mathbf{k}=\mathbf{k}_0} = \frac{\tilde{\alpha}_2}{2} \xi_0^2 + \frac{\tilde{\alpha}_4}{4} \xi_0^4 + \frac{\tilde{\alpha}_6}{6} \xi_0^6, \quad (66)$$

with

$$\tilde{\alpha}_2 = \alpha_2 - \frac{3\alpha_4^2}{8\alpha_6}, \quad (67)$$

$$\tilde{\alpha}_4 = \alpha_4 - 5\alpha_4, \quad (68)$$

$$\tilde{\alpha}_6 = \alpha_6 - \frac{25\alpha_6}{2}. \quad (69)$$

The free energy difference between the states with inhomogeneous and homogeneous condensates is

$$\Delta F = -\frac{3\alpha_4^2}{16\alpha_6} \xi_0^2 - \frac{5\alpha_4}{4} \xi_0^4 - \frac{25\alpha_6}{12} \xi_0^6. \quad (70)$$

Notice that the sixth order term in Eq. (70) is negative, which makes the free energy unbounded from below. It is likely that in this case a resummation of all orders in the condensate is necessary. We thus conclude that the GL analysis alone is not enough to determine completely the ground state, although Eq. (70) seems to point in the direction of a free energy is less favorable if the inhomogeneous ground state is taken.

Next we turn to the choice $\gamma_6 = 0$. In this case we replace

$$\xi^2 (\nabla \xi)^2 - \xi^4 (\nabla \theta)^2 \rightarrow \xi^2 (\nabla \xi)^2 \quad (71)$$

in the third line of Eq. (58), and instead of Eq. (64) we would have

$$\begin{aligned} F_{\text{NH}} \Big|_{\xi=\xi_0; \theta=\mathbf{k}\mathbf{x}} = & \frac{\alpha_2}{2} \xi_0^2 + \frac{\alpha_4}{4} \xi_0^4 + \frac{\alpha_6}{6} \xi_0^6 \\ & - \frac{\alpha_4}{4} \xi_0^2 \mathbf{k}^2 + \frac{\alpha_6}{12} \xi_0^2 \mathbf{k}^4; \end{aligned} \quad (72)$$

this would give

$$k_0 = \sqrt{\frac{3\alpha_4}{2\alpha_6}}. \quad (73)$$

instead of Eq. (73), and

$$\tilde{\alpha}_2 = \alpha_2 - \frac{3\alpha_4^2}{8\alpha_6}, \quad (74)$$

$$\tilde{\alpha}_4 = \alpha_4, \quad (75)$$

$$\tilde{\alpha}_6 = \alpha_6. \quad (76)$$

instead of Eqs. (67)-(69). In this case, the sign of the sixth order term coincides with that of α_6 which is assumed to be positive, therefore the free energy is bounded from below and the GL analysis can be completed by estimating the loss in free energy due to the inhomogeneous condensate,

$$\Delta F = -\frac{3\alpha_4^2}{16\alpha_6}\xi_0^2. \quad (77)$$

The numerical value of ΔF depends on the details of the microscopic model; however, regardless of these details we can conclude that the inhomogeneous condensation lowers the free energy of the system.

VI. OUTLOOK

The generalization of our approach to a system with $N_f > 1$ flavors (species) of fermions is rather straightforward. Similarly to the standard Hermitian NJL model [2], the fermions of different flavors form $N_f^2 - 1$ pseudoscalar fields ϕ_5^a which couple with the N_f -multiplet of the fermionic field $\Psi = (\psi_1, \dots, \psi_{N_f})^T$ via the non-Hermitian Yukawa coupling:

$$\mathcal{L}_{\text{NJL}} = \frac{\phi_5^2 - (\sigma - m_0)^2}{2G} + \bar{\Psi} (i\cancel{\partial} - \sigma\mathbb{1} - \gamma^5 \lambda^a \phi_5^a) \Psi. \quad (78)$$

Here λ^a are the generators of the $SU(N_f^2 - 1)$ flavor group and σ is the scalar Hermitian field. The integration of over the fermionic degrees of freedom Ψ produces the one loop action with the pseudoscalar iso-vector field $\phi_5 = (\phi_5^1, \dots, \phi_5^{N_f^2-1})^T$. In the uniform vacuum, it is sufficient to replace $\phi_5^2 \rightarrow \phi_5^2$ in the effective action (44). The effective bosonic action respects a non-compact symmetry group $U(1, N_f^2 - 1)$ which generalizes the single-flavor non-Hermitian group (48) to the multiflavor case. Notice that while for a single-flavor fermion the non-Hermitian symmetry breaking leads to the appearance of the non-Hermitian mass gap given by the last term in Eq. (42), in the multi-flavor case the non-Hermitian ground state may be traced to the emergence of the non-Hermitian Yukawa interaction in the last term of the semi-bosonized Lagrangian (78).

One of the problems that might require future attention concerns the possible mitigation of the destabilizing effect of a finite current mass m_0 on the non-Hermitian ground state of the system as has been noted briefly in Section IV C. To elucidate this problem, let us follow the notation of Section V and parametrize, similarly to Eq. (57), the fields $\sigma + m_0 = \xi \cosh \theta$ and $\phi_5 = \xi \sinh \theta$. The partially bosonized theory gets now the following form:

$$\mathcal{L} = -\frac{1}{2G}(\xi^2 + m_0^2 - 2\xi m_0 \cosh \theta) + \bar{\psi}(i\cancel{\partial} - \xi)\psi. \quad (79)$$

The first term in the effective theory gives us the tree-level potential for the bosonic fields ξ and θ :

$$V_0(\xi, \theta) = \xi^2 + m_0^2 - 2m_0\xi \cosh \theta, \quad (80)$$

which is unbounded from below in the direction of the field θ if $m_0 \neq 0$. The problem is clearly absent in the chiral limit at $m_0 = 0$.

The absence of a finite minimum of the tree-level potential V_0 at $m_0 \neq 0$ cannot be compensated by the fermionic determinant because the determinant may only add the one-loop term $v(\xi/\Lambda)$ to the potential (80). The one-loop correction, computed in Eq.(35), gives a contribution depending on ξ , and, consequently, it cannot lift out the instability of the classical potential (80) in the direction of the field θ . Therefore the quantum corrections can not solve the problem of the absence of the ground state in the minimal NJL model (18) in the presence of the explicit chiral symmetry breaking, $m_0 \neq 0$.

VII. CONCLUSIONS

We elaborated the physical consequences of a scenario in which the standard Hermitian Nambu–Jona-Lasinio model spontaneously develops a non-Hermitian $\mathcal{PT}$ -symmetric ground state. In the semi-bosonized model, the non-Hermitian ground state is catalyzed by the presence of a dynamically generated non-Hermitian Yukawa coupling. The unbroken $\mathcal{PT}$ symmetry of the system guarantees that the spectrum is real. In the exact chiral limit, i.e. in the absence of the explicit chiral symmetry breaking in the NJL model, the uniform non-Hermitian ground state has the same (finite) free energy density as the usual Hermitian ground state.

The known Hermitian and new non-Hermitian ground states are related to each other by a non-Unitary non-compact global symmetry which is spontaneously broken both in weak- and strong-coupling regimes of the model. In the chiral limit at strong coupling, the non-Hermitian ground state develops inhomogeneity, which breaks the translational symmetry of the state. At weak coupling, the ground state is a spatially uniform state, which lies at the boundary between the $\mathcal{PT}$ -symmetric and $\mathcal{PT}$ -broken phases. Outside the chiral limit, at a nonzero current mass $m_0 \neq 0$, the minimal NJL model does not possess a stable ground state because the free energy of the state is unbounded from below. The $m_0 \neq 0$ ground state may perhaps be stabilized in non-minimal NJL models with higher-order fermionic vertices.

ACKNOWLEDGMENTS

The authors are grateful to John Petrucci for inspiration. M.N.C. is partially supported by Grant No. 0657-2020-0015 of the Ministry of Science and Higher Education of Russia. M. R. is supported by the National Science Foundation of China (Grants No.11805087 and No. 11875153) and by the Fundamental Research Funds for the Central Universities (grant number 862946). A.C. acknowledges financial support through MINECO/AEI/FEDER, UE Grant No. FIS2015-73454-

JIN and European Union structural funds, the Comunidad Autónoma de Madrid (CAM) NMT2D-CM Program (S2018-NMT-4511), and the Ramón y Cajal program through the grant RYC2018-023938-I.

Appendix A: Gradient expansion in the effective action

The bosonized NJL model after integrating out fermions gets the following form:

$$\mathcal{L}_{NJL} = -\frac{1}{2G}(\sigma^2(x) - \phi_5^2(x)) + i \text{Tr} \log(G_0^{-1} - (\sigma + \gamma_5 \phi_5)), \quad (\text{A1})$$

where $G_0^{-1} = i\not{\partial}$. The trace operator, “Tr”, includes for the matrix trace “tr” as well as an integral over spatial coordinates.

Factorizing G_0^{-1} and expanding the logarithm in the above expression gives us:

$$\mathcal{L}_{NJL} = -\frac{1}{2G}(\sigma^2(x) - \phi_5^2(x)) + \text{Tr}[G_0^{-1}] - i \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}[G_0(\sigma + \gamma_5 \phi_5)]^n. \quad (\text{A2})$$

Let us consider the case $n = 2$:

$$\begin{aligned} \Gamma_{n=2} &= \frac{i}{2} \int dx dx' \text{tr}(G_0(x, x')(\sigma(x') + \gamma_5 \phi_5(x')) \\ &\quad G_0(x', x)(\sigma(x) + \gamma_5 \phi_5(x))) = \\ &= \frac{i}{2} \int dx dx' \text{tr}[G_0(x, x')G_0(x', x)](\sigma(x')\sigma(x) - \\ &\quad - \phi_5(x')\phi_5(x)). \end{aligned} \quad (\text{A3})$$

In order to perform the derivative expansion, it is mandatory to go to momentum representation, where the functional $\Gamma_{n=2}$ reads as follows:

$$\Gamma_{n=2} = \int \frac{d^D q}{(2\pi)^D} \Pi_2(q)(\sigma(-q)\sigma(q) - \phi_5(-q)\phi_5(q)) \quad (\text{A4})$$

where

$$\Pi_2(q) = \frac{i}{2} \int \frac{d^D k}{(2\pi)^D} \text{tr}[G_0(k)G_0(k-q)], \quad (\text{A5})$$

and $G_0(k) = 1/\gamma^a k_a$. With this, we can do the following:

$$\begin{aligned} \text{tr}[G_0(k)G_0(k-q)] &\simeq \text{tr}[G_0(k)G_0(k)] + \\ &+ \text{tr}[G_0(k)G_0(k) \frac{\partial G_0^{-1}}{\partial q_a} G_0(k)] q_a + \\ &+ \text{tr}[G_0(k)G_0(k) \frac{\partial G_0^{-1}}{\partial k_a} G_0(k) \frac{\partial G_0^{-1}}{\partial k_b} G_0(k)] q_a q_b, \end{aligned} \quad (\text{A6})$$

where we have used the fact that G_0^{-1} is a linear polynomial in the momenta. We plug the above expansion, $G_0 = \gamma^a k_a/k^2$, and $\partial G_0^{-1}/\partial k_a = \gamma^a$ into Eq. (A5). This trick corresponds to the derivative expansion of the effective action $\Gamma_{n=2}$. It is easy to see that the second term in (A6) will be odd in the integration momentum, so it will not contribute to this expansion.

Now we can make the connection with the GL expansion with local terms as all the integrals are formally done at $q = 0$. Then we have just to care of performing divergent integrals in the infrared (connection to non-renormalizable theories). Also, we need the following traces in $D = 4$ ($q^2 = q \cdot q = q^\mu q_\mu$):

$$\text{tr}[\gamma^a \gamma^b] = 4\eta^{ab}, \quad (\text{A7})$$

$$I^{abcd} = \text{tr}[\gamma^a \gamma^b \gamma^c \gamma^d] = 4(\eta^{ab}\eta^{cd} - \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc}), \quad (\text{A8})$$

$$\begin{aligned} F^{abcdef} &= \text{tr}[\gamma^a \gamma^b \gamma^c \gamma^d \gamma^e \gamma^f] = \eta^{ab} I^{cdef} - \eta^{ac} I^{bdef} \\ &+ \eta^{ad} I^{bcef} - \eta^{ae} I^{bcdf} + \eta^{af} I^{bcde}. \end{aligned} \quad (\text{A9})$$

After a bit of algebra, we find

$$\text{tr}[G_0(k)G_0(k-q)] = 4\frac{1}{k^2} + 8\frac{(k \cdot q)^2}{(k^2)^3} - 4\frac{q^2}{(k^2)^2}. \quad (\text{A10})$$

Next, we rotate the integral in (A5) to the Euclidean space (there appears an extra $-i$ factor):

$$\Pi_2(q) = \frac{4}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} + \frac{8}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{(k \cdot q)^2}{k^6} - \frac{4}{2} \frac{q^2}{k^4}. \quad (\text{A11})$$

Performing the angular integration in four dimensions, and putting both UV and IR cutoffs (Λ and l respectively), we finally obtain:

$$\Pi_2(q) = \frac{\Lambda^2}{8\pi^2} - \frac{q^2}{8\pi^2} \log\left(\frac{\Lambda}{l}\right). \quad (\text{A12})$$

From here we can read off α_2 and α_4 coefficients:

$$\begin{aligned} \mathcal{S}_{0+2} &= \int dx \frac{1}{2} \underbrace{\left(-\frac{1}{G} + \frac{\Lambda^2}{4\pi^2}\right)}_{\alpha_2} (\sigma^2 - \phi_5^2) - \\ &- \frac{1}{2} \underbrace{\frac{1}{4\pi^2} \log\left(\frac{\Lambda}{l}\right)}_{\alpha_4} ((\nabla\sigma)^2 - (\nabla\phi_5^2)) \end{aligned} \quad (\text{A13})$$

Let us consider the $n = 4$ contribution to the effective action (this time with the right sign), $\Gamma_{n=4}$:

$$\Gamma_4 = -\frac{i}{4} \int \Pi_{i=1}^4 dx_i \text{tr}[(\sigma(x_1) - \gamma_5 \phi_5(x_1))G_0(x_1 - x_2)(\sigma(x_2) - \gamma_5 \phi_5(x_2))G_0(x_2 - x_3) \cdot (\sigma(x_3) - \gamma_5 \phi_5(x_3))G_0(x_3 - x_4)(\sigma(x_4) - \gamma_5 \phi_5(x_4))G_0(x_4 - x_1)]. \quad (\text{A14})$$

Out of the 16 terms in the expression (A14) only 8 will contribute to the expression, as they have an even number of γ_5 matrices in the product of propagators. Taking into account that the γ_5 matrices anticommute with the Dirac matrices γ_μ , the effective action Γ_4 can be written as

$$\Gamma_4 = -\frac{i}{4} \int \Pi_{i=1}^4 dx_i \text{tr}[G_0(x_1 - x_2)G_0(x_2 - x_3)G_0(x_3 - x_4)G_0(x_4 - x_1)] \cdot (\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) - \sigma(x_1)\sigma(x_2)\phi_5(x_3)\phi_5(x_4) + \sigma(x_1)\phi_5(x_2)\sigma(x_3)\phi_5(x_4) - \sigma(x_1)\phi_5(x_2)\phi_5(x_3)\sigma(x_4) + \phi_5(x_1)\sigma(x_2)\phi_5(x_3)\sigma(x_4) - \phi_5(x_1)\sigma(x_2)\sigma(x_3)\phi_5(x_4) - \phi_5(x_1)\phi_5(x_2)\sigma(x_3)\sigma(x_4) + \phi_5(x_1)\phi_5(x_2)\phi_5(x_3)\phi_5(x_4)). \quad (\text{A15})$$

It will be convenient to write the expression (A15) in momentum space. To avoid too long expressions, we will write up the first term in Eq.(A15) ($(dq) \equiv \frac{d^D q}{(2\pi)^D}$):

$$\Gamma_4^{(1)} = \frac{1}{4} \int (dq)(dp)(dr)\Pi_4(q, p, r)\sigma(q)\sigma(p)\sigma(r)\sigma(-q - p - r), \quad (\text{A16})$$

with

$$\Pi_4(q, p, r) = -i \int (dk) \text{tr}[G_0(k)G_0(k - p)G_0(k - p - r)G_0(k + q)], \quad (\text{A17})$$

being the fourth order susceptibility. The remaining terms in Eq.(A15) are similar, just substituting the corresponding σ fields by ϕ_5 fields.

One can make the connection with the Ginzburg-Landau effective theory by approximating the function $\Pi_4(q, p, r)$ by its value at $q = p = r = 0$, $\Pi_4(0, 0, 0) \equiv \alpha_4$, and then go back to real space. Again, using the term with four σ fields we get:

$$\begin{aligned} \Gamma_4^{(1)} &= \frac{\alpha_4}{4} \int (dq)(dp)(dr)\sigma(q)\sigma(p)\sigma(r)\sigma(-q - p - r) = \\ &= \frac{\alpha_4}{4} \int (dp)(dq)(dr) \int \Pi_{i=1}^4 e^{ix_1 q} e^{ix_2 p} e^{ix_3 r} e^{ix_4 (-p - q - r)} \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) = \\ &= \frac{\alpha_4}{4} \int \Pi_{i=1}^4 dx_i \delta(x_1 - x_4)\delta(x_2 - x_4)\delta(x_3 - x_4)\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) = \\ &= \frac{\alpha_4}{4} \int dx \sigma^4(x). \end{aligned} \quad (\text{A18})$$

Doing the same for the rest of the terms, and collecting all them in (A15) under this approximation, we have

$$\begin{aligned} \Gamma_4 &= \frac{\alpha_4}{4} \int dx (\sigma^4(x) - \sigma^2(x)\phi_5^2(x) + \sigma^2(x)\phi_5^2(x) - \sigma^2(x)\phi_5^2(x) + \sigma^2(x)\phi_5^2(x) - \\ &\quad - \sigma^2(x)\phi_5^2(x) - \sigma^2(x)\phi_5^2(x) + \phi_5^4(x)) = \frac{\alpha_4}{4} \int dx (\sigma^2(x) - \phi_5^2(x))^2 = \\ &= \frac{\alpha_4}{4} \int dx (\vec{\chi} \cdot \vec{\chi})^2. \end{aligned} \quad (\text{A19})$$

This expression coincides with the free energy $F_{\text{NH}}(\vec{\chi})$ of Eq. (50) when the latter is written in terms of $\vec{\chi}$. Also, the expression (A17) is the starting point to start the derivative expansion to get the terms proportional to α_6 .

-
- [1] Y. Nambu and G. Jona-Lasinio, *Phys. Rev.* **122**, 345 (1961).
 - [2] S. P. Klevansky, *Rev. Mod. Phys.* **64**, 649 (1992).
 - [3] C. M. Bender, *Contemporary Physics* **46**, 277 (2005).
 - [4] A. Beygi, S. P. Klevansky, and C. M. Bender, *Phys. Rev. A* **99**, 062117 (2019).
 - [5] J. Alexandre and N. E. Mavromatos, *Physics Letters B* **807**, 135562 (2020).
 - [6] A. Felski, A. Beygi, and S. P. Klevansky, *Phys. Rev. D* **101**, 116001 (2020).
 - [7] I. Rotter and J. P. Bird, *Reports on Progress in Physics* **78**, 114001 (2015).
 - [8] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, *Phys. Rev. D* **78**, 074033 (2008).
 - [9] M. N. Chernodub and A. Cortijo, *Symmetry* **12** (2020), 10.3390/sym12050761.
 - [10] V. A. Rubakov, “On chiral magnetic effect and holography,” (2010), arXiv:1005.1888 [hep-ph].
 - [11] A. Gynther, K. Landsteiner, F. Pena-Benitez, and A. Rebhan, *Journal of High Energy Physics* **2011**, 110 (2011).
 - [12] K. Landsteiner, E. Megías, and F. Peña-Benitez, “Anomalous transport from kubo formulae,” in *Strongly Interacting Matter in Magnetic Fields*, edited by D. Kharzeev, K. Landsteiner, A. Schmitt, and H.-U. Yee (Springer Berlin Heidelberg, Berlin, Heidelberg, 2013) pp. 433–468.
 - [13] M. M. Vazifeh and M. Franz, *Phys. Rev. Lett.* **111**, 027201 (2013).
 - [14] N. Yamamoto, *Phys. Rev. D* **92**, 085011 (2015).
 - [15] M. A. Zubkov, *Phys. Rev. D* **93**, 105036 (2016).
 - [16] D. Aren, K. Landsteiner, and I. Salazar Landea, (2019), arXiv:1912.06647 [hep-th].
 - [17] K. Kawabata, Y. Ashida, H. Katsura, and M. Ueda, *Phys. Rev. B* **98**, 085116 (2018).
 - [18] J. Avila, F. Peñaranda, E. Prada, P. San-Jose, and R. Aguado, *Communications Physics* **2**, 133 (2019).
 - [19] K. Yamamoto, M. Nakagawa, K. Adachi, K. Takasan, M. Ueda, and N. Kawakami, *Phys. Rev. Lett.* **123**, 123601 (2019).
 - [20] J. A. S. Lourenço, R. L. Eneias, and R. G. Pereira, *Phys. Rev. B* **98**, 085126 (2018).
 - [21] H. Krull, N. Bittner, G. S. Uhrig, D. Manske, and A. P. Schnyder, *Nature Communications* **7**, 11921 (2016).
 - [22] B. Dóra, M. Heyl, and R. Moessner, *Nature Communications* **10**, 2254 (2019).
 - [23] A. Ghatak and T. Das, *Phys. Rev. B* **97**, 014512 (2018).
 - [24] C. M. Bender, H. Jones, and R. Rivers, *Phys. Lett. B* **625**, 333 (2005), arXiv:hep-th/0508105.
 - [25] J. Alexandre, C. M. Bender, and P. Millington, *JHEP* **11**, 111 (2015), arXiv:1509.01203 [hep-th].
 - [26] J. Alexandre, C. M. Bender, and P. Millington, *J. Phys. Conf. Ser.* **873**, 012047 (2017), arXiv:1703.05251 [hep-th].
 - [27] J. Alexandre, P. Millington, and D. Seynaeve, *Phys. Rev. D* **96**, 065027 (2017).
 - [28] J. Alexandre, P. Millington, and D. Seynaeve, *J. Phys. Conf. Ser.* **952**, 012012 (2018), arXiv:1710.01076 [hep-th].
 - [29] J. Alexandre, N. E. Mavromatos, and A. Soto, “Dynamical majorana neutrino masses and axions,” (2020), arXiv:2004.04611 [hep-ph].
 - [30] J. Alexandre, J. Ellis, P. Millington, and D. Seynaeve, *Phys. Rev. D* **98**, 045001 (2018), arXiv:1805.06380 [hep-th].
 - [31] P. D. Mannheim, *Phys. Rev. D* **99**, 045006 (2019), arXiv:1808.00437 [hep-th].
 - [32] A. Fring and T. Taira, *Nucl. Phys. B* **950**, 114834 (2020), arXiv:1906.05738 [hep-th].
 - [33] D. Nickel, *Phys. Rev. Lett.* **103**, 072301 (2009).