On the race between primes with an odd versus an even sum of the last k binary digits

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ON THE RACE BETWEEN PRIMES WITH AN ODD VERSUS AN EVEN SUM OF THE LAST k BINARY DIGITS

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ABSTRACT. Motivated by Newman's phenomenon for the Thue-Morse sequence $(-1)^{s(n)}$, where $s(n)$ is the sum of the binary digits of n, we investigate a similar problem for prime numbers. More specifically, for an integer $k \geqslant 2$, we explore the signs of $S_k(x) = \sum_{p \leqslant x} (-1)^{s_k(p)}$, where $s_k(n)$ is the sum of the last k binary digits of n, and p runs over the primes. We prove that $S_k(x)$ changes signs for infinitely many integers x, assuming that all Dirichlet L-functions attached to primitive characters modulo 2^k do not vanish on $(0, 1)$. Our result is unconditional for $k \leqslant 18$. Furthermore, under stronger assumptions on the zeros of Dirichlet L-functions, we show that for $k \geqslant 4$, the sets $\{x > 2 : S_k(x) > 0\}$ and $\{x > 2 : S_k(x) < 0\}$ both have logarithmic density $1/2$.

1. INTRODUCTION

Every integer $n \geqslant 0$ has an unique expansion in base 2:

$$n = \sum_{j \geqslant 0} \varepsilon_j(n) 2^j \quad \text{with } \varepsilon_j(n) \in \{0, 1\} \quad \forall j \geqslant 0,$$

where the $\varepsilon_j(n)$'s are *the binary digits* of n. Let $s(n) = \sum_{j \geqslant 0} \varepsilon_j(n)$ be the sum of digits function. The Thue-Morse sequence on $\{-1, 1\}$ is defined by $\mathbf{t} = (t_n)_{n \geqslant 0}$ where $t_n = (-1)^{s(n)}$. This sequence has been extensively studied and appears in various branches of mathematics, see for example [1]. In a seminal paper [9], Gelfond proved that for every $a \in \mathbb{Z}$ and $q \in \mathbb{N}$,

$$T_{q,a}(x) := \sum_{\substack{n \leqslant x \\ n \equiv a \bmod q}} t_n = o(x) \quad (x \to \infty),$$

and asked whether such an asymptotic holds for other *interesting* subsets of the integers, namely the set of prime numbers, and the set $P(\mathbb{N})$ where P is a polynomial with integral coefficients such that $P(\mathbb{N}) \subset \mathbb{N}$. More generally, one can ask the following question:

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given some specific subset $\mathcal{A}$ of $\mathbb{N}$ with multiplicative, algebraic or analytic structure, do we have

$$\sum_{\substack{n\leqslant x\\ n\in\mathcal{A}}} t_n = o\Big(\sum_{\substack{n\leqslant x\\ n\in\mathcal{A}}} 1\Big) \quad (x\to+\infty)? \tag{1.1}$$

In [9] Gelfond also proved that (1.1) holds when $\mathcal{A}$ is the set of k-free integers. Recently, Mauduit and Rivat made a breakthrough by proving (1.1) when $\mathcal{A}$ is the set of prime numbers [17], answering Gelfond's question, and then when $\mathcal{A}$ is the set of squares [16] (see also [3]).

Hence, for all the aforementioned examples the sequence $\mathbf{t}$ along $\mathcal{A}$ takes the values 1 and -1 with the same frequency. However, Newman [18] discovered that this distribution hides a bias, by showing that $T_{3,0}(x) > 0$ for all $x \geqslant 1$. This surprising fact became known as *Newman's phenomenon*. In fact, Newman established the stronger estimate

$$\frac{3^\alpha}{20} \leqslant T_{3,0}(N)\big\lfloor \tfrac{N-2}{3}\big\rfloor^{-\alpha} \leqslant 5\cdot 3^\alpha \tag{1.2}$$

for all integers $N \geqslant 1$, where $\alpha := \log 3/\log 4$. This shows that $T_{3,0}(N)$ is very large when N grows, and we say that a *drifting phenomenon* occurs in this case.

Whereas the study of the signs of $T_{q,a}(x)$ turns out to be difficult in general, Drmota and Skalba [4] could prove that $T_{q,0}(x) > 0$ for x sufficiently large when $q \equiv 0 \bmod 3$ or $q = 4^k + 1$ with $k \geqslant 1$. We also mention that Drmota and Stoll [5] showed that for every $x \geqslant 1$, we have $T_{3,1}(x) \leqslant -Cx^\alpha$ with some $C > 0$. They also proved that $T_{3,2}(x) \leqslant 0$ with equality holding for infinitely many integers x, which shows that there is a Newman phenomenon but no drift in this case. For more results and references about Newman's phenomenon and its extensions, we refer the reader to the introductions and bibliographies of [5] and [15].

Recently, Mauduit and Moreira [15] provided the first example of a non-linear Newman phenomenon by considering the set $\mathcal{Q}$ of square-free integers. More precisely, they proved that

$$\sum_{\substack{n\leqslant x\\ n\in\mathcal{Q}}} t_n = -\frac{2}{\pi^2}\big(1+o(1)\big)T_{3,0}(x), \text{ as } x\to\infty.$$

In particular, we have $\sum_{\substack{n\leqslant x\\ n\in\mathcal{Q}}} t_n < 0$ for x sufficiently large. They also obtained similar results for the set of k-free integers with $k \geqslant 2$.

In [21] (see also [22]) Shevelev conjectured that a similar phenomenon should hold for the set of prime numbers, namely that for every $x \geqslant 31$, we have

$$S(x) := \sum_{p\leqslant x} (-1)^{s(p)} < 0,$$

where p runs over the primes. He also made the stronger conjecture that

$$\lim_{x\to+\infty} \frac{\log(-S(x))}{\log x} = \alpha. \tag{1.3}$$

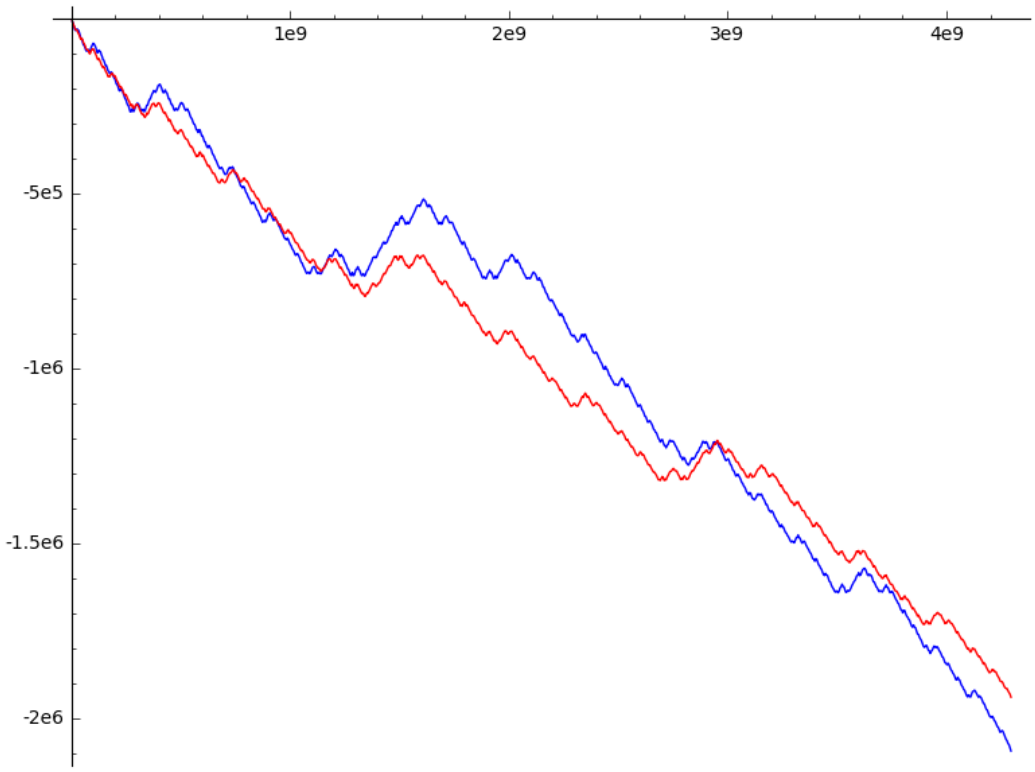

FIGURE 1. graphs of $x \mapsto S(x)$, $x \mapsto -\frac{3}{2}\frac{T_{3,0}(x)}{\log x}$ for $2 \leqslant x \leqslant 2^{32}$

Heuristically, one should expect that for $a \in \{1, 2\}$ we have

$$\sum_{\substack{p \leqslant x \\ p \equiv a \bmod 3}} t_p \sim \frac{3}{\varphi(3) \log x} T_{3,a}(x), \text{ as } x \to \infty.$$

Since $T_{3,0}(x) + T_{3,1}(x) + T_{3,2}(x) = \sum_{n \leqslant x} t_n \in \{-1, 0, +1\}$, this leads to the following conjecture which refines Shevelev's conjectural asymptotic (1.3), in view of Newman's estimate (1.2).

Conjecture 1.1. As $x \to \infty$ we have

$$S(x) \sim -\frac{3}{2} \cdot \frac{T_{3,0}(x)}{\log x}.$$

We have performed numerical computations for x up to $2^{32} \approx 4.3 \cdot 10^9$, which show that the graphs of $S(x)$ and $-3T_{3,0}(x)/(2 \log x)$ are very close in this region (see Figure 1). However, we also noticed that the hypothetical convergence towards 1 of the ratio seems extremely slow.

In this paper we study the analogous question where $s(p)$ is replaced by $s_k(p)$, with $s_k(p)$ being the sum of the last k binary digits of n, and $k \geqslant 2$ is a fixed integer. Note

that the case $k = 1$ is trivial, since all primes $p \geqslant 3$ are odd. We shall investigate the signs of the sum

$$S_k(x) := \sum_{p \leqslant x} (-1)^{s_k(p)}.$$

We note that s_k is a particular case of a weighted sum of digits: let $\gamma = \{\gamma_j\}_{j \geqslant 0}$ be a sequence with $\gamma_j \in \{0, 1\}$ for all $j \geqslant 0$, and introduce

$$s^\gamma(n) = \sum_{j \geqslant 0} \gamma_j \varepsilon_j(n). \tag{1.4}$$

Larcher and Zellinger [14] characterized the sequences γ such that a drifting phenomenon for the sums $\sum_{\substack{n \leqslant x \\ n \equiv a \bmod 3}} (-1)^{s^\gamma(n)}$ occurs. When $s^\gamma = s_k$, it turns out that the sum $\sum_{\substack{n \leqslant x \\ n \equiv a \bmod 3}} (-1)^{s_k(n)}$ vanishes infinitely often for each $a \in \{0, 1, 2\}$.

Our first result shows that $S_k(x)$ changes signs infinitely often, conditionally on a mild assumption on the real zeros of Dirichlet L-functions. This assumption is commonly known as Haselgrove's condition $H(q)$, and states that $L(\sigma, \chi) \neq 0$ for all non-trivial Dirichlet characters $\chi \bmod q$, and all $\sigma \in (0, 1)$. Before stating our result, we recall the Ω notations: if f, g are both real-valued functions with $g > 0$, the notation $f(x) = \Omega_+(g(x))$ (resp. $f(x) = \Omega_-(g(x))$ means that

$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0 \quad \Big(\text{resp. } \liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0\Big),$$

and $f(x) = \Omega_\pm(g(x))$ means that $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$.

Theorem 1.2. *Let $k \geqslant 2$ be an integer and assume $H(2^k)$. Then, we have*

$$S_k(x) = \Omega_\pm\Big(\frac{\sqrt{x}}{\log x}\Big).$$

In particular, the sum $S_k(x)$ changes signs infinitely often.

Haslegrov's condition $H(q)$ is known to hold for all moduli $q \leqslant 400\ 000$ by the work of Platt [19], so our result is unconditional for $k \leqslant 18$. We should also remark that this condition is somewhat necessary to obtain an infinite number of sign changes of $S_k(x)$. Indeed, one can show that if a certain L-function $L(s, \chi_1)$ (where χ_1 is a nonprincipal character mod 2^k) has a real zero $\beta > 1/2$, such that $\beta > \Re(\rho)$ for all non trivial zeros ρ of all L-functions attached to non-principal characters $\chi \neq \chi_1 \bmod 2^k$, then $S_k(x) \gg x^\beta$ for x large.

We have also performed numerical computations to explore the sign changes of $S_k(x)$ (see Figures 2, 3 and 4, which represent the graphs of $S_{15}(x)$, $S_{23}(x)$ and $S_{27}(x)$ respectively, for $2 \leqslant x \leqslant 2^{32}$).

As usual, we denote by $\pi(x)$ the number of primes up to x, and for $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ by $\pi(x; q, a)$ the number of primes $p \leqslant x$ such that $p \equiv a \bmod q$. Introducing for $j \in \{0, 1\}$

$$E_j(k) := \{1 \leqslant a < 2^k \,|\, a \text{ odd},\ s(a) \equiv j \bmod 2\},$$

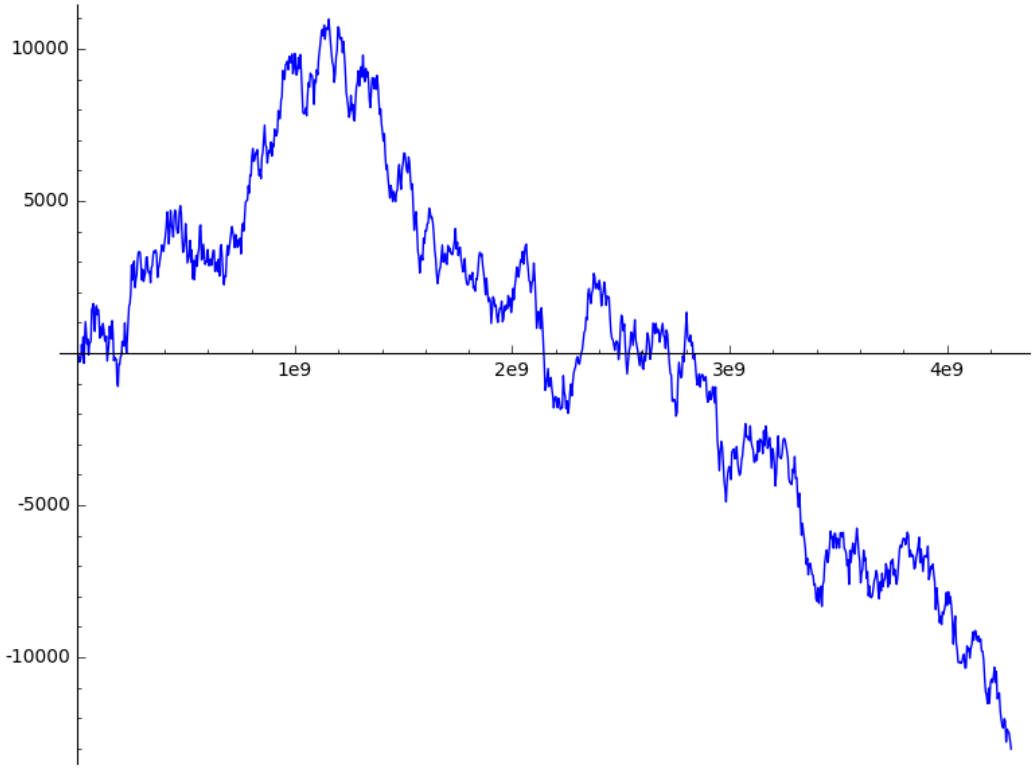

FIGURE 2. graph of $x \mapsto S_{15}(x)$ for $2 \leqslant x \leqslant 2^{32}$

we have for $x \geqslant 3$,

$$S_k(x) = -1 + \sum_{\substack{1 \leqslant a < 2^k \\ a \text{ odd}}} (-1)^{s(a)} \sum_{\substack{p \leqslant x \\ p \equiv a \bmod 2^k}} 1 = -1 + \sum_{a \in E_0(k)} \pi(x; 2^k, a) - \sum_{a \in E_1(k)} \pi(x; 2^k, a).$$

In particular, we have $S_2(x) = -1 + \pi(x; 4, 3) - \pi(x; 4, 1)$ and $S_3(x) = -1 + \pi(x; 8, 3) + \pi(x; 8, 5) - \pi(x; 8, 1) - \pi(x; 8, 7)$. One can easily see that $|E_0(k)| = |E_1(k)|$ for $k \geqslant 2$, and hence we have $S_k(x) = o(\pi(x))\,(x \to \infty)$ by the prime number theorem for arithmetic progressions. In order to study the sign changes of $S_k(x)$, we shall use comparative prime number theory arguments (see [7] for a survey of this area). In particular, the case $k = 2$ of Theorem 1.2 follows from Littlewood's work [10]. When $k = 2, 3$, the odd quadratic residues modulo 2^k are not equally distributed in $E_0(k)$ and $E_1(k)$ (indeed 1 is the only quadratic residue modulo 4 and modulo 8), and this induces a *Chebychev's bias* phenomenon, which can be readily seen in Theorem 1.5 below. However, the case $k \geqslant 4$ is completely different. Indeed, we prove in Proposition 3.1 below that for $k \geqslant 4$, $E_0(k)$ and $E_1(k)$ contain the same number of quadratic residues. In this case, Theorem 1.2 follows from the following result, which is a generalization of the works of Knapowski-Turán [13] and Kátai [12].

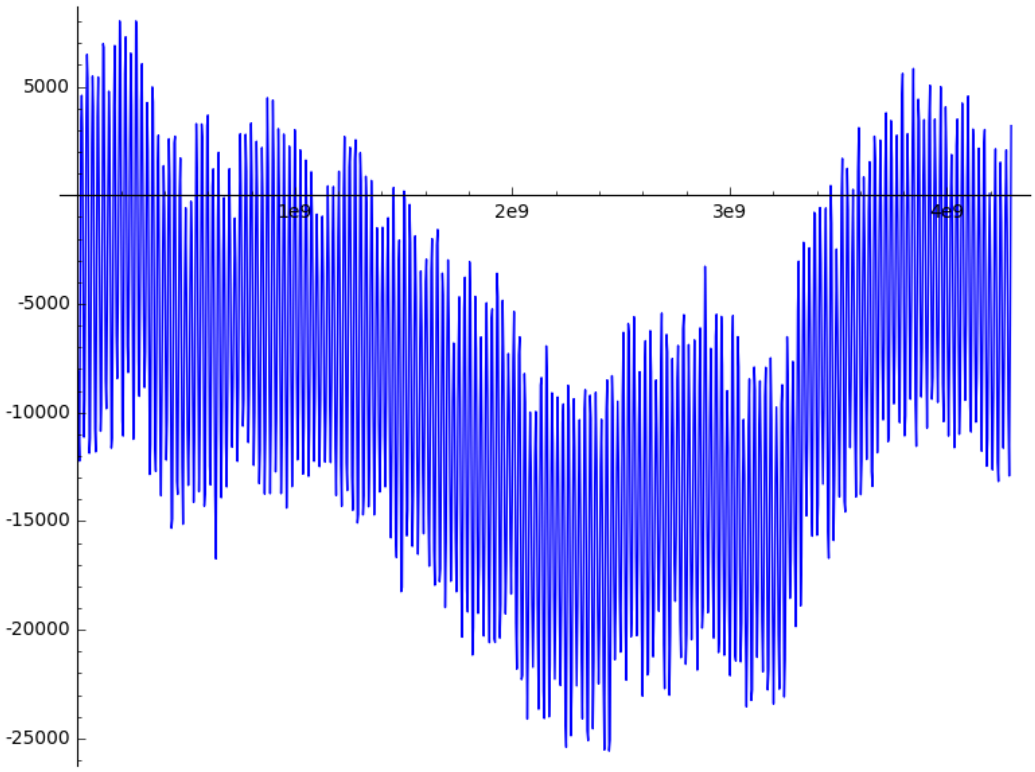

FIGURE 3. graph of $x \mapsto S_{23}(x)$ for $2 \leqslant x \leqslant 2^{32}$

Theorem 1.3. *Let $q \geqslant 2$ be an integer and assume $H(q)$. Let A, B be disjoint subsets of $(\mathbb{Z}/q\mathbb{Z})^*$ such that $|A| = |B|$, and A and B contain the same number of quadratic residues. Then we have*

$$\sum_{a\in A} \pi(x; q, a) - \sum_{b\in B} \pi(x; q, b) = \Omega_{\pm}\left(\frac{\sqrt{x}}{\log x}\right).$$

Let χ_4 denote the nontrivial character modulo 4. Assuming the Riemann hypothesis for $L(s, \chi_4)$, Kaczorowski [11] proved that the set of real numbers $x \geqslant 2$ such that $S_2(x) > 0$ (respectively $S_2(x) < 0$) has a positive *lower logarithmic density*. Recall that for a set $A \subset [0, \infty)$, the upper and lower logarithmic densities of A are defined respectively by

$$\overline{\delta}(A) = \limsup_{x\to\infty} \frac{1}{\log x} \int_{t\in A\cap[2,x]} \frac{dt}{t}, \text{ and } \underline{\delta}(A) = \liminf_{x\to\infty} \frac{1}{\log x} \int_{t\in A\cap[2,x]} \frac{dt}{t}.$$

If $\overline{\delta}(A) = \underline{\delta}(A) = \delta(A)$ we say that $\delta(A)$ is the logarithmic density of A. Using the work of Rubinstein and Sarnak [20] we prove a similar result for $k = 3$. Let $\lambda \geqslant 0$ and define

$$\mathcal{A}_k^+(\lambda) = \left\{x \geqslant 2 \,|\, S_k(x) > \lambda\frac{\sqrt{x}}{\log x}\right\} \quad \text{and} \quad \mathcal{A}_k^-(\lambda) = \left\{x \geqslant 2 \,|\, S_k(x) < -\lambda\frac{\sqrt{x}}{\log x}\right\}.$$

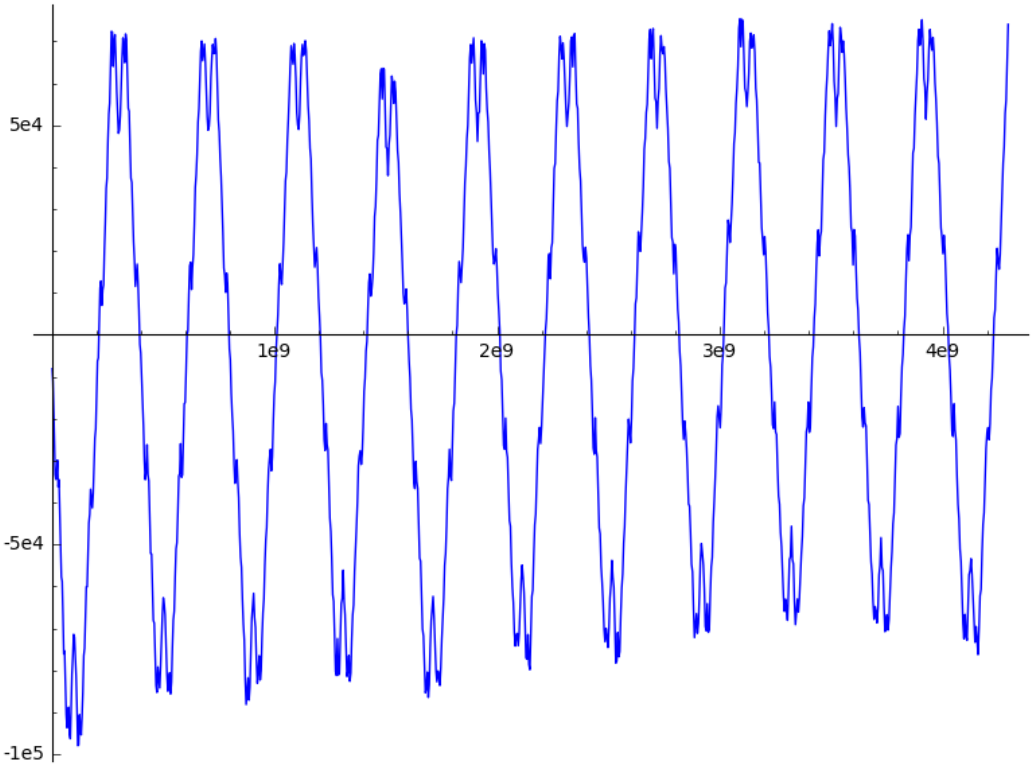

FIGURE 4. graph of $x \mapsto S_{27}(x)$ for $2 \leqslant x \leqslant 2^{32}$

We also let $\chi_8(\cdot) = \left(\frac{8}{\cdot}\right)$ be the Kronecker symbol modulo 8. Then we prove

Theorem 1.4. *Assume the Riemann hypothesis for $L(s, \chi_8)$. There exists a positive constant c such that*

$$\underline{\delta}\left(\mathcal{A}_3^+(\lambda)\right) \gg \exp\big(-\exp(c\lambda)\big), \quad \textit{and} \quad \underline{\delta}\left(\mathcal{A}_3^-(\lambda)\right) \gg \exp\big(-\exp(c\lambda)\big).$$

We should note that the assumption of the Riemann hypothesis in Theorem 1.4 is "necessary" in a certain sense. Indeed, using the work of Ford, Konyagin and Lamzouri [8] one can show that the existence of certain configurations of zeros of $L(s, \chi_8)$ off the critical line implies that $\delta\left(\mathcal{A}_3^+(\lambda)\right) = 0$ (respectively $\delta\left(\mathcal{A}_3^-(\lambda)\right) = 0$).

Rubinstein and Sarnak [20] solved several long standing problems in comparative prime number theory conditionally on the generalized Riemann hypothesis GRH and the Linear Independence hypothesis LI, which is the assumption that the nonnegative imaginary parts of the nontrivial zeros of Dirichlet L-functions attached to primitive characters modulo q are linearly independent over $\mathbb{Q}$. In particular, they showed that $\delta\left(\mathcal{A}_2^+(0)\right) = 1 - \delta\left(\mathcal{A}_2^-(0)\right) = 0.9959...$ under GRH and LI. Using their approach we prove the following result.

Theorem 1.5. *Assume GRH and LI. The logarithmic densities of $\mathcal{A}_k^+(0)$ and $\mathcal{A}_k^-(0)$ exist and are positive. Moreover, for $k \geqslant 4$ we have*

$$\delta\left(\mathcal{A}_k^+(0)\right) = \delta\left(\mathcal{A}_k^-(0)\right) = \frac{1}{2},$$

while for $k = 3$ we have

$$\delta\left(\mathcal{A}_3^+(0)\right) = 0.982201... \quad \text{and } \delta\left(\mathcal{A}_3^-(0)\right) = 1 - \delta\left(\mathcal{A}_3^+(0)\right) = 0.017798...$$

The logarithmic density $\delta\left(\mathcal{A}_3^+(0)\right)$ was computed following the approach of Rubinstein-Sarnak (see Section 4 of [20]). We provide the main steps of the method in §5.

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2. Preliminary results

For an integer $q \geqslant 2$ and $A \subseteq (\mathbb{Z}/q\mathbb{Z})^*$ we define

$$\pi(x, A) := \sum_{a \in A} \pi(x; q, a).$$

As usual we let Λ denote the von Mangoldt function defined by $\Lambda(n) = \log p$ if $n = p^k$, and $\Lambda(n) = 0$ otherwise. We shall use the standard notation $\psi(x; q, a) := \sum_{\substack{n \leqslant x \\ n \equiv a \bmod q}} \Lambda(n)$ and for $A \subseteq (\mathbb{Z}/q\mathbb{Z})^*$ we define

$$\psi(x, A) := \sum_{a \in A} \psi(x; q, a), \quad \text{and} \quad \widetilde{\psi}(x, A) := \frac{\psi(x^+, A) + \psi(x^-, A)}{2}.$$

We also set $K_q := \{a^2 \,|\, a \in (\mathbb{Z}/q\mathbb{Z})^*\}$, and define

$$N_q(a) := |\{b \bmod q \,|\, b^2 \equiv a \bmod q\}| = \begin{cases} N_q(1) & \text{if } a \in K_q, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1. *Let A, B be disjoints subsets of $(\mathbb{Z}/q\mathbb{Z})^*$. Then, we have*

$$\begin{aligned} \pi(x, A) - \pi(x, B) = \frac{\sum_{b \in B} N_q(b) - \sum_{a \in A} N_q(a)}{\varphi(q)} \frac{\sqrt{x}}{\log x} + \frac{\psi(x, A) - \psi(x, B)}{\log x} \\ + \int_2^x \frac{\psi(t, A) - \psi(t, B)}{t(\log t)^2} dt + O_q\left(\frac{\sqrt{x}}{(\log x)^2}\right). \end{aligned} \tag{2.1}$$

Proof. For $a \in (\mathbb{Z}/q\mathbb{Z})^*$ we define

$$\Pi(x; q, a) := \sum_{\substack{n \leqslant x \\ n \equiv a \bmod q}} \frac{\Lambda(n)}{\log n}.$$

On the one hand we have

$$\Pi(x;q,a) = \pi(x;q,a) + \frac{1}{2}\sum_{\substack{p\leqslant\sqrt{x}\\ p^2\equiv a \bmod q}} 1 + O(x^{1/3}) \tag{2.2}$$
$$= \pi(x;q,a) + \frac{1}{\varphi(q)}\frac{\sqrt{x}}{\log x}N_q(a) + O\Big(\frac{\sqrt{x}}{(\log x)^2}\Big),$$

where we use the prime number theorem for arithmetic progressions in the form $\pi(x;a,q) = \frac{x}{\log x} + O\Big(\frac{x}{(\log x)^2}\Big)$. On the other hand, integrating by parts gives

$$\Pi(x;q,a) = \frac{\psi(x;q,a)}{\log x} + \int_2^x \frac{\psi(t;q,a)}{t(\log t)^2}dt. \tag{2.3}$$

Summing (2.2) and (2.3) over $a \in A$ and $b \in B$ completes the proof. □

Let A, B be disjoint subsets of $(\mathbb{Z}/q\mathbb{Z})^*$ such that $|A| = |B|$. Let χ be a non-principal character modulo q and define $\psi(x,\chi) := \sum_{n\leqslant x}\chi(n)\Lambda(n)$. Then, observe that

$$\psi(x,A) - \psi(x,B) = \frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0 \bmod q} c_{A,B}(\chi)\psi(x,\chi), \tag{2.4}$$

where

$$c_{A,B}(\chi) := \Big(\sum_{a\in A}\overline{\chi}(a) - \sum_{b\in B}\overline{\chi}(b)\Big). \tag{2.5}$$

The Mellin transform of $\psi(x,A) - \psi(x,B)$ is defined by

$$G(s,A,B) := \int_0^\infty \big(\psi(x,A) - \psi(x,B)\big)\frac{dx}{x^{s+1}}, \tag{2.6}$$

and is absolutely convergent for $\Re(s) > 1$ since $\psi(x,A) - \psi(x,B) = O_q(x)$. Moreover, in this region we have

$$G(s,A,B) = -\frac{1}{s\varphi(q)}\sum_{\chi\neq\chi_0 \bmod q} c_{A,B}(\chi)\frac{L'}{L}(s,\chi). \tag{2.7}$$

This gives a meromorphic continuation for G to the whole complex plane.

For a non-principal character χ modulo q, we let $\rho_\chi = \beta_\chi + i\gamma_\chi$ runs over the non-trivial zeros of $L(s,\chi)$, and put $\mathcal{Z}_q = \cup_{\chi\neq\chi_0}\{\rho_\chi\}$. We also let $m(\rho_\chi)$ be the multiplicity of ρ_χ, and define

$$\kappa(\rho) := \sum_{\substack{\chi\neq\chi_0 \bmod q\\ \rho_\chi=\rho}} c_{A,B}(\chi)m(\rho_\chi).$$

We have the following explicit formulas for $\psi(x,A) - \psi(x,B)$ in terms of the zeros $\rho \in \mathcal{Z}_q$.

Lemma 2.2. *Let $x \geqslant 2$. Let A, B be disjoint subsets of $(\mathbb{Z}/q\mathbb{Z})^*$ such that $|A| = |B|$. Then, we have*

(2.8)
$$\widetilde{\psi}(x, A) - \widetilde{\psi}(x, B) = -\frac{1}{\varphi(q)} \sum_{\rho \in \mathcal{Z}_q} \kappa(\rho) \frac{x^\rho}{\rho} - \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0 \bmod q} c_{A,B}(\chi) \left(\frac{1 + \chi(-1)}{2} \right) \log x + O_q(1),$$

and

$$\psi(x, A) - \psi(x, B) = -\frac{1}{\varphi(q)} \sum_{\substack{\rho \in \mathcal{Z}_q \\ |\Im(\rho)| \leqslant T}} \kappa(\rho) \frac{x^\rho}{\rho} + O_q\left(\frac{x \log^2(xT)}{T} + \log x \right), \tag{2.9}$$

for $T \geqslant 1$.

Proof. The first estimate (2.8) follows from Eq. (2.4) together with the classical explicit formula (see for example Eq. (3) of Chapter 19 of [2])

$$\widetilde{\psi}(x, \chi) = -\sum_{\rho_\chi} \frac{x^{\rho_\chi}}{\rho_\chi} - \left(\frac{1 + \chi(-1)}{2} \right) \log x + O_q(1). \tag{2.10}$$

where $\widetilde{\psi}(x, \chi) = (\psi(x^+, \chi) + \psi(x^-, \chi))/2$. The second estimate follows from the following truncated form of the explicit formula (see Eq. (7) of Chapter 16 of [2])

$$\psi(x, \chi) = -\sum_{|\Im(\rho_\chi)| \leqslant T} \frac{x^{\rho_\chi}}{\rho_\chi} + O\left(\frac{x \log^2(xT)}{T} + \log x \right), \tag{2.11}$$

which is valid for $T \geqslant 1$. □

By (2.7), the poles of $G(s, A, B)$ in the critical strip $0 < \Re(s) < 1$ are precisely the zeros $\rho \in \mathcal{Z}_q$ for which $\kappa(\rho) \neq 0$. As a consequence of the explicit formula (2.8), we shall deduce that there are infinitely many such poles in the critical strip.

Corollary 2.3. *Let A, B be disjoint subsets of $(\mathbb{Z}/q\mathbb{Z})^*$ such that $|A| = |B|$. Then $G(s, A, B)$ has infinitely many poles in the critical strip.*

Proof. Suppose by contradiction that there are only finitely many $\rho \in \mathcal{Z}_q$ such that $\kappa(\rho) \neq 0$. Then, it follows from the explicit formula (2.8) that $\widetilde{\psi}(x, A) - \widetilde{\psi}(x, B) = f(x) + O_q(1)$, where f is a continuous function of x. However, this contradicts the fact that $\widetilde{\psi}(x, A) - \widetilde{\psi}(x, B)$ has jump discontinuities of order $\log x$ (for example at $x = p$ where p is a prime such that $p \equiv a \bmod q$ with $a \in A$). □

We next show that if all the poles of $G(s, A, B)$ lie on the critical line $\Re(s) = 1/2$, then the third term on the right hand side of (2.1) is small. This will be one of the main ingredients in the proof of Theorem 1.3.

Lemma 2.4. *Let A, B be disjoint subsets of $(\mathbb{Z}/q\mathbb{Z})^*$ such that $|A| = |B|$. Assume that all the poles of $G(s, A, B)$ lie on the critical line $\Re(s) = 1/2$. Then, we have*

$$\int_2^x \frac{\psi(t, A) - \psi(t, B)}{t(\log t)^2} dt \ll_q \frac{\sqrt{x}}{(\log x)^2}.$$

Proof. Let

$$\Psi(x, A, B) = \int_2^x (\psi(t, A) - \psi(t, B)) dt.$$

Integrating the explicit formula (2.9) with respect to t, and letting $T \to \infty$ gives

$$\Psi(x, A, B) = -\frac{1}{\varphi(q)} \sum_{\rho \in \mathcal{Z}_q} \kappa(\rho) \frac{x^{\rho+1}}{\rho(\rho+1)} + O\left(x \log x\right) \ll x^{3/2},$$

since $\Re(\rho) = 1/2$ when $\kappa(\rho) \neq 0$, and $\sum_{\rho \in \mathcal{Z}_q} 1/|\rho|^2 \ll_q 1$. Therefore, by a simple integration by parts we obtain

$$\int_2^x \frac{\psi(t, A) - \psi(t, B)}{t(\log t)^2} dt = \frac{\Psi(x, A, B)}{x(\log x)^2} + \int_2^x \frac{\Psi(t, A, B)}{(t \log t)^2} \left(1 + \frac{2}{\log t}\right) dt \ll \frac{\sqrt{x}}{(\log x)^2},$$

as desired. □

3. Sign changes of $\pi(x, A) - \pi(x, B)$: Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.3. By Corollary 2.3, we know that the Mellin transform $G(s, A, B)$ has infinitely many poles in the critical strip. These correspond to the zeros $\rho \in \mathcal{Z}_q$ with $\kappa(\rho) \neq 0$. Furthermore, none of these poles is real by hypothesis $H(q)$. We shall distinguish two cases:

Case 1: There is a $\rho \in \mathcal{Z}_q$ with $\kappa(\rho) \neq 0$ such that $\Re(\rho) > 1/2$.

In this case, it is more convenient to work with the quantity $\Pi(x, S) := \sum_{a \in S} \Pi(x; q, a)$, and to consider the Mellin transform

(3.1)
$$g(s) = \int_0^\infty (\Pi(x, A) - \Pi(x, B)) \frac{dx}{x^{s+1}} = \frac{1}{s\varphi(q)} \sum_{\chi \neq \chi_0 \bmod q} c_{A,B}(\chi) \log L(s, \chi), \quad \text{for } \Re(s) > 1.$$

Let σ_c be the abscissa of convergence of $g(s)$. The identity (3.1) provides an analytic continuation for g at $s = \sigma_c$. Hence, applying Theorem II.1.11 of [24] (which is a standard consequence of the Phragmen-Landau Theorem), we have for every $\varepsilon > 0$,

$$\Pi(x, A) - \Pi(x, B) = \Omega_\pm(x^{\sigma_c - \varepsilon}).$$

But $\sigma_c \geqslant \sup_{\kappa(\rho) \neq 0} \Re(\rho) > 1/2$, since every ρ with $\kappa(\rho) \neq 0$ induces a logarithmic singularity for $g(s)$. The result follows upon noting that, according to (2.2),

$$\pi(x, A) - \pi(x, B) = \Pi(x, A) - \Pi(x, B) + O_q\left(\frac{\sqrt{x}}{\log^2 x}\right).$$

Case 2: All the zeros $\rho \in \mathcal{Z}_q$ with $\kappa(\rho) \neq 0$ satisfy $\Re(\rho) = 1/2$.

In this case, it follows from Theorem II.1.12 of [24] that

$$\psi(x, A) - \psi(x, B) = \Omega_{\pm}(x^{1/2}).$$

On the other hand, we infer from Lemmas 2.1 and 2.4 that

$$\pi(x, A) - \pi(x, B) = \frac{\psi(x, A) - \psi(x, B)}{\log x} + O_q\left(\frac{\sqrt{x}}{(\log x)^2}\right).$$

Combining these estimates completes the proof. □

We shall deduce Theorem 1.2 from Theorem 1.3 when $k \geqslant 4$, by showing that $E_0(k)$ and $E_1(k)$ have the same number of quadratic residues.

Proposition 3.1. *If $k \geqslant 4$, then there are as many quadratic residues in $E_0(k)$ as in $E_1(k)$.*

Proof. As $k \geqslant 4$, a is an odd quadratic residue modulo 2^k if and only if $a = 8m + 1$ with $m \in \mathbb{Z}$ (see [23] Theorem 16 p. 243). Hence, there are exactly 2^{k-3} odd quadratic residues and they are given by $8m + 1$ with m running from 0 to $2^{k-3} - 1$. Noting that $s(8m + 1) = s(m) + 1$ for $m \geqslant 0$ there are as many odd quadratic residues in $E_0(k)$ as integers m between 0 and $2^{k-3} - 1$ with an odd sum of digits, that is 2^{k-2} since $\sum_{0 \leqslant m < 2^N} (-1)^{s(m)} = 0$ for all $N \geqslant 1$. □

Proof of Theorem 1.2. The case $k \geqslant 4$ follows from Theorem 1.3 and Proposition 3.1, so it only remains to consider the case $k = 3$. Note that in this case, we have

$$\psi(x, E_0(3)) - \psi(x, E_1(3)) = \frac{1}{4} \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod 8}} (\chi(1) + \chi(7) - \chi(3) - \chi(5))\psi(x, \chi) = \psi(x, \chi_8), \tag{3.2}$$

and hence $G(s, E_0(3), E_1(3)) = -\frac{1}{s}\frac{L'}{L}(s, \chi_8)$. It is known that $L(s, \chi_8)$ has no zeroes on $(0, \infty)$. If the Riemann hypothesis is false for $L(s, \chi_8)$, then proceeding as in case 1 of the proof of Theorem 1.3 we obtain $\pi(x, E_0) - \pi(x, E_1) = \Omega_{\pm}(\sqrt{x})$. Finally, if the Riemann hypothesis is true for $L(s, \chi_8)$, then the result follows from Theorem 1.4. □

4. Consequences of GRH and LI: Proof of Theorems 1.4 and 1.5

4.1. **Consequences of GRH: Proof of Theorem 1.4.** Let $1/2 + i\gamma$ runs over the non-trivial zeros of $L(s, \chi_8)$. Then, it follows from (2.11) and (3.2) that

$$\frac{\psi(x, E_0(3)) - \psi(x, E_1(3))}{\sqrt{x}} = \sum_{|\gamma| < x} \frac{x^{i\gamma}}{\frac{1}{2} + i\gamma} + O\left(\frac{\log^2 x}{\sqrt{x}}\right) = 2 \sum_{0 < \gamma < x} \frac{\sin(\gamma \log x)}{\gamma} + O(1),$$

and hence by Lemmas 2.1 and 2.4 we obtain

$$\frac{\log x}{\sqrt{x}} S_3(x) = 2 \sum_{0 < \gamma < x} \frac{\sin(\gamma \log x)}{\gamma} + O(1). \tag{4.1}$$

Let $\lambda > 0$. Then, it follows from the work of Rubinstein and Sarnak (see Section 2.2 of [20]), that

$$\frac{1}{Y}\text{meas}\left\{2 \leqslant y \leqslant Y : \sum_{0<\gamma<e^Y} \frac{\sin(\gamma y)}{\gamma} > \lambda\right\} \gg \exp\big(-\exp(c_0\lambda)\big), \tag{4.2}$$

and

$$\frac{1}{Y}\text{meas}\left\{2 \leqslant y \leqslant Y : \sum_{0<\gamma<e^Y} \frac{\sin(\gamma y)}{\gamma} < -\lambda\right\} \gg \exp\big(-\exp(c_0\lambda)\big), \tag{4.3}$$

for some absolute positive constant c_0, where meas denotes the Lebesgue measure on $\mathbb{R}$. Let X be large and put $Y = \log X$. Then, by making the change of variable $y = \log x$ we deduce that

$$\begin{aligned}\frac{1}{\log X}\int_{\substack{x\in[2,X]\\ S_3(x)>\lambda\sqrt{x}/\log x}} \frac{dx}{x} &= \frac{1}{Y}\text{meas}\left\{\log 2 \leqslant y \leqslant Y : S_3(e^y) > \lambda\frac{e^{y/2}}{y}\right\}\\ &\gg \exp\big(-\exp(c_0\lambda)\big)\end{aligned} \tag{4.4}$$

by (4.1) and (4.2). Finally, the corresponding lower bound for $\underline{\delta}\left(\mathcal{A}_3^-(\lambda)\right)$ follows along the same lines by using (4.1) and (4.3).

4.2. **Consequences of GRH and LI: Proof of Theorem 1.5.** Let $a_1, \dots, a_n$ be distinct reduced residues modulo q, and define

$$E_{q;a_1,\dots,a_n}(x) := \Big(E(x;q,a_1), \dots, E(x;q,a_n)\Big),$$

where

$$E(x;q,a) := \frac{\log x}{\sqrt{x}}\left(\varphi(q)\pi(x;q,a) - \pi(x)\right).$$

As before, we let $\{\gamma_\chi\}$ be the sequence of imaginary parts of the nontrivial zeros of $L(s,\chi)$, and put $\Gamma = \bigcup_{\chi\neq\chi_0 \bmod q}\{\gamma_\chi\}$. If we assume LI then all of the non-negative values of γ_χ are linearly independent over $\mathbb{Q}$, and in particular are distinct. Let $\{U(\gamma_\chi)\}_{\gamma_\chi\in\Gamma,\gamma_\chi>0}$ be a sequence of independent random variables uniformly distributed on the unit circle. Rubinstein and Sarnak [20] proved, under GRH and LI, that for any Lebesgue measurable set $S \subset \mathbb{R}^n$ whose boundary has measure zero, we have

$$\lim_{X\to\infty}\int_{\substack{x\in[2,X]\\ E_{q;a_1,\dots,a_n}(x)\in S}} \frac{dx}{x} = \int_S d\mu_{q;a_1,\dots,a_n}, \tag{4.5}$$

where $\mu_{q;a_1,\dots,a_n}$ is an absolutely continuous measure on $\mathbb{R}^n$, whose density function is real analytic. In fact, it follows from their work that $\mu_{q;a_1,\dots,a_n}$ is the probability measure corresponding to the random vector

$\big(X(q,a_1),\dots,X(q,a_n)\big)$, where

$$X(q,a) := 1 - N_q(a) + \sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \Re\left(2\chi(a)\sum_{\gamma_\chi>0}\frac{U(\gamma_\chi)}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right). \tag{4.6}$$

Note that the inner sum is almost surely convergent by Kolmogorov's three series theorem, since $\mathbb{E}\big(U(\gamma_\chi)\big)=0$ and $\sum_{\gamma_\chi>0}(\frac{1}{4}+\gamma_\chi^2)^{-1}<\infty$. Using (4.5) we shall prove the following result which implies Theorem 1.5.

Theorem 4.1. *Assume GRH and LI. Let A,B be disjoint subsets of $(\mathbb{Z}/q\mathbb{Z})^*$ such that $|A|=|B|$. Let $\mathcal{T}_{A,B}$ be the set of real numbers $x\geqslant 2$ such that*

$$\sum_{a\in A}\pi(x;q,a) > \sum_{b\in B}\pi(x;q,b).$$

Then $\delta(\mathcal{T}_{A,B})$ exists and $\delta(\mathcal{T}_{A,B})=1-\delta(\mathcal{T}_{B,A})>0$. Moreover, we have $\delta(\mathcal{T}_{A,B})=1/2$ if A and B contain the same number of quadratic residues, and $\delta(\mathcal{T}_{A,B})<1/2$ if A contains more quadratic residues than B.

Proof. Let $A=\{a_1,\dots,a_n\}$ and $B=\{b_1,\dots,b_n\}$. Then, note that

$$\pi(x,A)>\pi(x,B) \iff \sum_{j=1}^n E(x;q,a_j) > \sum_{j=1}^n E(x;q,b_j).$$

Therefore, it follows from (4.5) that

$$\begin{aligned}\delta(\mathcal{T}_{A,B}) = \lim_{X\to\infty}\int_{\substack{x\in[2,X]\\ \pi(x,A)>\pi(x,B)}}\frac{dx}{x} &= \int_{\sum_{j=1}^n x_j>\sum_{\ell=n+1}^{2n}x_\ell} d\mu_{q;a_1,\dots,a_n,b_1,\dots,b_n}\\ &= \mathbb{P}\left(\sum_{j=1}^n X(q,a_j) > \sum_{j=1}^n X(q,b_j)\right).\end{aligned} \tag{4.7}$$

Moreover, since $\mu_{q;a_1,\dots,a_n,b_1,\dots,b_n}$ is absolutely continuous and its density function is real analytic, then for any $c\in\mathbb{R}$ and any open interval $I\subset\mathbb{R}$ we have

$$\mathbb{P}\left(\sum_{j=1}^n X(q,a_j)-\sum_{j=1}^n X(q,b_j)=c\right)=0,$$

and

$$\mathbb{P}\left(\sum_{j=1}^n X(q,a_j)-\sum_{j=1}^n X(q,b_j)\in I\right)>0. \tag{4.8}$$

In particular, this shows that $\delta(\mathcal{T}_{A,B})=1-\delta(\mathcal{T}_{B,A})>0$.

Now, since $U(\gamma_\chi)$ is a symmetric random variable ($-U(\gamma_\chi)$ and $U(\gamma_\chi)$ are identically distributed), then

$$\begin{aligned}&\mathbb{P}\left(\sum_{j=1}^{n} X(q,a_j)-\sum_{j=1}^{n} X(q,b_j)>\sum_{j=1}^{n} N_q(b_j)-\sum_{j=1}^{n} N_q(a_j)\right)\\ &=\mathbb{P}\left(\sum_{j=1}^{n} X(q,a_j)-\sum_{j=1}^{n} X(q,b_j)<\sum_{j=1}^{n} N_q(b_j)-\sum_{j=1}^{n} N_q(a_j)\right)=\frac{1}{2}.\end{aligned}$$

The result follows from (4.8) upon noting that $\sum_{j=1}^{n} N_q(a_j)>\sum_{j=1}^{n} N_q(b_j)$ if and only if A contains more quadratic residues than B. □

Proof of Theorem 1.5. The theorem follows from combining Proposition 3.1 and Theorem 4.1. □

5. Computation of $\delta(\mathcal{A}_3^+(0))$

First, it follows from (4.7) that

$$\delta(\mathcal{A}_3^+(0))=\mathbb{P}(X(8,3)+X(8,3)>X(8,7)+X(8,1)),$$

where the random variables $X(q,a)$ are defined in (4.6). Moreover, we have

$$\begin{aligned}&X(8,3)+X(8,3)-X(8,7)-X(8,1)\\ =&4+2\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}}\Re\left(\big(\chi(5)+\chi(3)-\chi(7)-\chi(1)\big)\sum_{\gamma_\chi>0}\frac{U(\gamma_\chi)}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right)\\ =&4-8\sum_{\gamma>0}\frac{\Re(U_\gamma)}{\sqrt{\frac{1}{4}+\gamma^2}},\end{aligned}$$

where γ varies over the imaginary parts of the non-trivial zeros of $L(s,\chi_8)$, and $\{U_\gamma\}_\gamma$ is a sequence of independent random variables uniformly distributed on the unit cercle. Since the U_γ are symmetric about 0, we deduce that

$$\delta(\mathcal{A}_3^+(0))=\mathbb{P}(\mathbb{X}>0)=\int_0^\infty d\mu,$$

where

$$\mathbb{X}=1+2\sum_{\gamma>0}\frac{\Re(U_\gamma)}{\sqrt{\frac{1}{4}+\gamma^2}},$$

and μ is the probability measure of $\mathbb{X}$. The Fourier transform of μ is given by

$$\widehat{\mu}(t)=e^{it}\prod_{\gamma>0}J_0\left(\frac{2t}{\sqrt{\frac{1}{4}+\gamma^2}}\right),$$

where $J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}z)^{2m}}{(m!)^2}$ is the Bessel function of the first kind of order 0. Since $\widehat{\mu}(t)$ has exactly the same form as the Fourier transforms corresponding to the densities computed by Rubinstein and Sarnak (see Section 4 of [20]), we can compute $\delta(\mathcal{A}_3^+(0))$ using the same approach. For the sake of completeness, we will include all the main steps of this method. In what follows, $E_1, E_2, \dots$ shall be understood as error terms.

(1) First, one has

$$\delta(\mathcal{A}_3^+(0)) = \mathbb{P}(\mathbb{X} > 0) = \mathbb{P}(\mathbb{Y} > -1) = \int_{-1}^{+\infty} d\omega,$$

where $\mathbb{Y} = \mathbb{X} - 1$ and ω is the probability measure of $\mathbb{Y}$. The Fourier transform of ω is given by

$$\widehat{\omega}(t) = \prod_{\gamma>0} J_0\left(\frac{2t}{\sqrt{\frac{1}{4}+\gamma^2}}\right). \tag{5.1}$$

(2) By the symmetry of ω around zero and the Fourier inversion formula we obtain

$$\delta(\mathcal{A}_3^+(0)) = \frac{1}{2} + \frac{1}{2}\int_{-1}^{1} d\omega = \frac{1}{2} + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\sin(t)}{t}\widehat{\omega}(t)dt.$$

(3) Let $\varepsilon > 0$ be a parameter to be chosen. Applying the Poisson summation formula

$$\varepsilon \sum_{n\in\mathbb{Z}} \phi(\varepsilon n) = \sum_{n\in\mathbb{Z}} \widehat{\phi}(n/\varepsilon)$$

to $\phi(t) = \frac{1}{2\pi}\frac{\sin(t)}{t}\widehat{\omega}(t)$ gives

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\sin(t)}{t}\widehat{\omega}(t)dt = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{\sin(\varepsilon n)}{n}\widehat{\omega}(\varepsilon n) + E_1.$$

(4) We truncate the last sum over n to get

$$\sum_{n\in\mathbb{Z}} \frac{\sin(\varepsilon n)}{n}\widehat{\omega}(\varepsilon n) = \sum_{-C\leqslant n\varepsilon\leqslant C} \frac{\sin(\varepsilon n)}{n}\widehat{\omega}(\varepsilon n) + E_2,$$

for some positive constant C to be chosen.

(5) To evaluate $\widehat{\omega}$, we truncate the product (5.1) over the γ's to obtain

$$\begin{aligned}\widehat{\omega}(t) &= \prod_{0<\gamma\leqslant X} J_0\left(\frac{2t}{\sqrt{\frac{1}{4}+\gamma^2}}\right)\prod_{\gamma>X} J_0\left(\frac{2t}{\sqrt{\frac{1}{4}+\gamma^2}}\right)\\ &= \prod_{0<\gamma\leqslant X} J_0\left(\frac{2t}{\sqrt{\frac{1}{4}+\gamma^2}}\right)\big(1 + b_1(X)t^2\big) + E_3,\end{aligned}$$

where $b_1(X)$ is defined by

$$\prod_{\gamma>X} J_0\Big(\frac{2t}{\sqrt{\frac{1}{4}+\gamma^2}}\Big) = 1 + b_1(X)t^2 + O_X(t^4) \quad (t \to 0),$$

so that

$$b_1(X) = -\sum_{\gamma>X} (\tfrac{1}{4}+\gamma^2)^{-1} = -\sum_{\gamma>0} (\tfrac{1}{4}+\gamma^2)^{-1} + \sum_{0<\gamma\leqslant X} (\tfrac{1}{4}+\gamma^2)^{-1}.$$

Combining these steps leads to approximate $\delta(\mathcal{A}_3^+(0))$ by the quantity

$$\frac{1}{2} + \frac{1}{2\pi} \sum_{-C\leqslant n\varepsilon\leqslant C} \frac{\sin(n\varepsilon)}{n}\big(1 + b_1(X)(n\varepsilon)^2\big) \prod_{0<\gamma\leqslant X} J_0\left(\frac{2n\varepsilon}{\sqrt{\frac{1}{4}+\gamma^2}}\right). \tag{5.2}$$

We use the list of the first 10^5 zeros of the Dirichlet L-function $L(s,\chi_8)$ compiled by Tomás Oliveira e Silva [6] so that we can choose $X = 7675$. We also set $\varepsilon = 1/20$ and $C = 25$ as in [20], and check that with these values, some technical conditions arising in Rubinstein and Sarnak's method are met. We hence obtain the following approximate value for the density

$$\delta(\mathcal{A}_3^+(0)) = 0.982201...$$

We remark that using the approach of Rubinstein-Sarnak, we can rigorously prove that the error made by these approximations is less than $5 \cdot 10^{-6}$. However, our numerical computations show that the first six digits of the density $\delta(\mathcal{A}_3^+(0))$ do not change once the value of X exceeds 2000.

Finally, it remains to explain how to compute $b_1(X)$, or equivalently $\sum_{\gamma>0}(\frac{1}{4}+\gamma^2)^{-1}$. Following [20, p. 191], we have under GRH

$$\sum_{\gamma>0} \frac{1}{\frac{1}{4}+\gamma^2} = \frac{1}{2}\log\Big(\frac{8}{\pi}\Big) - \frac{\gamma_0}{2} - \frac{\log(2)}{2} + \frac{L'(1,\chi_8)}{L(1,\chi_8)},$$

where γ_0 is the Euler constant. According to [2, p. 50] we have

$$L(1,\chi_8) = -\frac{1}{\sqrt{8}}\Big(\log(\sin(\pi/8)) + \log(\sin(7\pi/8)) - \log(\sin(3\pi/8)) - \log(\sin(5\pi/8))\Big).$$

Furthermore, $L'(1,\chi_8)$ can be computed (see [20, p. 192]) by differentiating the formula

$$\Gamma(s)L(s,\chi_8) = \int_0^\infty \Big(\sum_{n\geqslant 1} \chi_8(n)e^{-nx}\Big)x^{s-1}dx \quad (s>0)$$

which eventually gives

$$L'(1,\chi_8) = \gamma_0 L(1,\chi_8) + \int_0^\infty \frac{e^{-x} + e^{-7x} - e^{-3x} - e^{-5x}}{1-e^{-8x}}\log(x)dx.$$

All of the computations (and also the graphs in the introduction) were performed using the computational software `SAGE`.

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