One can't hear orientability of surfaces

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Abstract

The main result of this paper is that one cannot hear orientability of a surface with boundary. More precisely, we construct two isospectral flat surfaces with boundary with the same Neumann spectrum, one orientable, the other non-orientable. For this purpose, we apply Sunada's and Buser's methods in the framework of orbifolds. Choosing a symmetric tile in our construction, and adapting a folklore argument of Fefferman, we also show that the surfaces have different Dirichlet spectra. These results were announced in the *C. R. Acad. Sci. Paris Sér. I Math.*, volume 320 in 1995, but the full proofs so far have only circulated in preprint form.

Keywords: Spectrum, Laplacian, Isospectral surfaces, Orientability

MSC 2010: 58J50, 58J32

Contents

1 Introduction

Let M be a compact Riemannian manifold with boundary. The *spectrum* of M is the sequence of eigenvalues of the Laplace-Beltrami operator $\Delta f = -\operatorname{div}(\operatorname{grad} f)$ acting on smooth functions on M; when $\partial M \neq \varnothing$, one can impose either Dirichlet boundary conditions on the

function f (i.e., $f|_{\partial M} = 0$) or Neumann boundary conditions (the normal derivative $\partial f/\partial n$ vanishes on ∂M). Mark Kac's classic paper [27] has stimulated a great deal of interest in the question of what geometric or topological properties of M are determined by its spectrum. In [23], it was shown that, in Kac's terminology, one cannot hear the shape of a drum, or of a bell: that is, there exist pairs of nonisometric planar domains that have the same spectra, for either Dirichlet (in the case of a drum) or Neumann (in the case of a bell) boundary conditions. The construction uses an adaptation to orbifolds of Sunada's technique for constructing isospectral manifolds. For many other examples, see [12].

This paper uses orbifold techniques to exhibit examples of pairs of Neumann isospectral flat surfaces with boundary, one of which is orientable and the other nonorientable. Performing the same construction using a tile with an additional symmetry, we exhibit Neumann isospectral bordered surfaces that are not Dirichlet isospectral by adapting an argument of C. Fefferman.

To our knowledge, these are the first confirmed examples in which Neumann isospectrality holds but Dirichlet isospectrality fails, although examples of Neumann isospectral surfaces that are not believed to be Dirichlet isospectral were constructed in [12]; we are grateful to Peter Doyle for drawing these examples to our attention.

This paper is organized as follows. Section 2 briefly reviews the Sunada construction of isospectral manifolds; this is used to construct the Neumann isospectral surfaces M_1 and M_2 in section 3. Section 4 summarizes some representation-theoretic calculations that furnish a computation of the most general "transplantation" map, which transplants a Neumann eigenfunction on M_1 to a Neumann eigenfunction on M_2 with the same eigenvalue; as in [12] and [4], transplantation of eigenfunctions affords an elementary visual proof of the isospectrality. In section 5, we give an unpublished argument of C. Fefferman; while in section 6, by modifying our construction slightly, we show that there are Neumann isospectral flat surfaces with boundary M_1 and M_2 that are not Dirichlet isospectral by adapting Fefferman's argument. Section 7 contains some concluding observations.

Since the constructions and proofs are quite elementary, the exposition is aimed at a general reader and is essentially self-contained, although reference to [23] may be helpful. These results were announced in [6] and circulated in an MSRI preprint [7]; this paper is a revision of [7] and contains the details of the results announced in [6] (along with some improvements), after a long delay.

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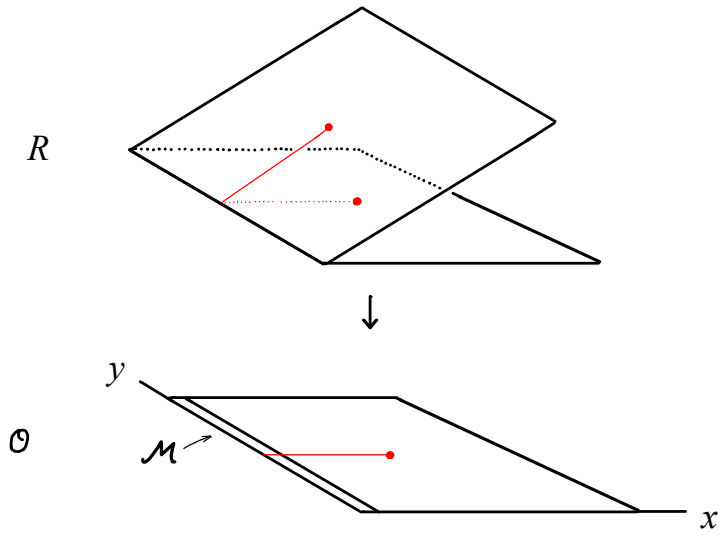
2 Isospectral manifolds

Let G be a finite group, and let Γ be a subgroup of G. Then the left action of G on the coset space G/Γ determines a linear representation $\mathbb{C}[G/\Gamma]$ of G, where $\mathbb{C}[G/\Gamma]$ denotes a complex vector space with basis G/Γ; the G-module $\mathbb{C}[G/\Gamma]$ can be viewed as the representation $(\mathbb{1}_\Gamma)\uparrow_\Gamma^G = \mathbb{C}[G] \otimes_{\mathbb{C}[\Gamma]} \mathbb{C}$ of G induced from the one-dimensional trivial representation $\mathbb{1}_\Gamma$ of the

subgroup Γ.
Now let Γ_1 and Γ_2 be subgroups of G. Then (G, Γ_1, Γ_2) is called a *Gassmann-Sunada triple* if $\mathbb{C}[G/\Gamma_1]$ and $\mathbb{C}[G/\Gamma_2]$ are isomorphic representations of G. The formula for the character of an induced representation (see, e.g., [34], section 7.2 or [26], 21.19) shows that isomorphism of the induced representations $(\mathbb{1}_{\Gamma_1})\uparrow_{\Gamma_1}^G$ and $(\mathbb{1}_{\Gamma_2})\uparrow_{\Gamma_2}^G$ is equivalent to the assertion that Γ_1 and Γ_2 are *elementwise conjugate* or *almost conjugate* subgroups of G — that is, there exists a bijection $\Gamma_1 \to \Gamma_2$ carrying each element $\gamma \in \Gamma_1$ to a conjugate element $g\gamma g^{-1} \in \Gamma_2$, where the conjugating element $g \in G$ may depend upon γ. See R. Perlis [31] or the papers of R. Guralnick (e.g., [22]) for many examples.
Elementwise conjugate subgroups that are not conjugate as subgroups were used by Gassmann [19] to exhibit pairs of nonisomorphic number fields having the same zeta function and hence the same arithmetic. The analogy of Galois theory with covering space theory led T. Sunada to apply Gassmann-Sunada triples to develop a very powerful technique for constructing isospectral Riemannian manifolds that are not isometric, as follows. (Conjugate subgroups are trivially almost conjugate; however, in that case the two Riemannian manifolds arising from Sunada's Theorem are isometric. Thus we seek nonconjugate pairs of almost-conjugate subgroups.)

Theorem 2.1 (Sunada [35])**.** *Let M be a compact Riemannian manifold with boundary, and let G be a finite group acting on M by isometries. Suppose that (G, Γ_1, Γ_2) is a Gassmann-Sunada triple, with Γ_1 and Γ_2 acting freely on M. Then the quotient manifolds $M_1 = \Gamma_1 \backslash M$ and $M_2 = \Gamma_2 \backslash M$ are isospectral. (If $\partial M \neq \varnothing$, then either Dirichlet or Neumann boundary conditions can be imposed.)*

For surveys of some of the extensions and applications of Sunada's technique, see [21], [8], [9], and [5].
P. Bérard [3] gave a representation-theoretic proof of Sunada's theorem, relaxing the requirement that the subgroups Γ_1 and Γ_2 act freely; the conclusion is then that the orbit spaces $\mathcal{O}_1 = \Gamma_1 \backslash M$ and $\mathcal{O}_2 = \Gamma_2 \backslash M$ are isospectral as orbifolds. Recall that an n-dimensional *orbifold* is a space whose local models are orbit spaces of $\mathbb{R}^n$ under action by finite groups G; an orbifold with boundary is similarly modeled locally on quotients of a half-space by finite group actions. See [36], [33], or [1] for more details. The *singular set* of the orbifold consists of all points where the isotropy is nontrivial.
For our purposes, an understanding of one of the simplest examples of an orbifold with boundary will suffice. Consider the rectangle $R = [-1, 1] \times [0, 1]$ in $\mathbb{R}^2$. The group $\Gamma = \mathbb{Z}/2\mathbb{Z}$ acts via the reflection $(x, y) \mapsto (-x, y)$ about the vertical axis, and the quotient orbifold $\mathcal{O} = \Gamma \backslash R$ is, as a point set, the square $[0, 1] \times [0, 1]$. However, $\mathcal{O}$ has a singular set consisting of a distinguished "mirror edge" $\mathcal{M} = \{0\} \times [0, 1]$, the image of the fixed-point set of Γ. At points not in $\mathcal{M}$, the local structure is that of $\mathbb{R}^2$ or of the half-plane, and the projection $R \xrightarrow{\pi} \mathcal{O}$ is locally a double cover; however, at points of the mirror edge $\mathcal{M}$, the local structure is that of $\mathbb{R}^2$ modulo a reflection, and the orbit map π is a covering map only in the sense of orbifolds. To distinguish $[0, 1] \times [0, 1]$ as a point set from $\mathcal{O} = \Gamma \backslash R$ viewed as an orbifold with boundary, we write $|\mathcal{O}|$ for the underlying space $[0, 1] \times [0, 1]$ of $\mathcal{O}$ when the orbifold structure is disregarded. It is important to note that $\partial\mathcal{O}$ (the orbifold boundary) differs from $\partial|\mathcal{O}|$ in that $\partial\mathcal{O}$ does *not* contain the mirror edge $\mathcal{M}$. In the orbifold sense, the fundamental group of $\mathcal{O}$ is $\mathbb{Z}/2\mathbb{Z}$, and R is its universal cover. Indeed, the loop in $\mathcal{O}$ consisting of the straight-line path from $(1/2, 1/2)$ to $(0, 1/2)$ and back again lifts to a non-closed path, as shown below.

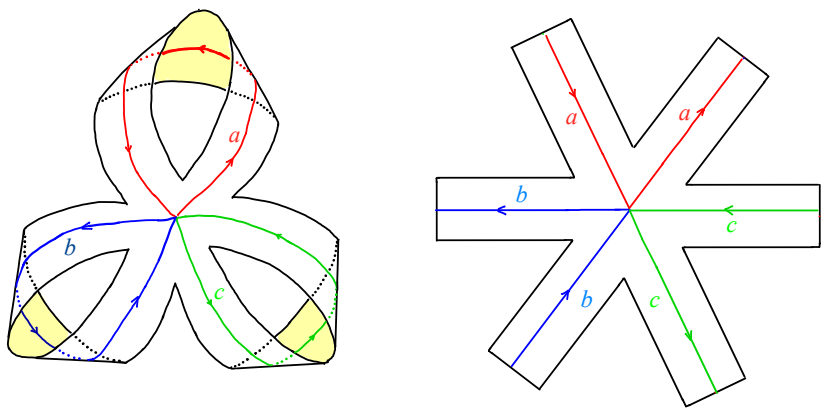


Mirror edges will be denoted in our drawings by doubled lines.
The smooth functions on $\mathcal{O} = \Gamma\backslash R$ are precisely the smooth functions on R that are Γ-invariant, and the *spectrum* of $\mathcal{O}$ is the spectrum of the Laplacian acting on Γ-invariant functions on R. Note that a Γ-invariant smooth function on R restricts to a function on $|\mathcal{O}|$ with zero normal derivative on the mirror edge $\{0\} \times [0,1]$; thus the Dirichlet spectrum of the orbifold $\mathcal{O}$ is the spectrum of the domain $|\mathcal{O}|$ with mixed boundary conditions: Dirichlet conditions on the three edges $[0,1] \times \{0\}$, $[0,1] \times \{1\}$, and $\{1\} \times [0,1]$ forming the orbifold boundary $\partial\mathcal{O}$, but Neumann conditions on the mirror edge $\mathcal{M}$.

Remark 2.2. In general, for a 2-orbifold $\mathcal{O}$ with boundary whose singular set consists of disjoint mirror arcs, the Dirichlet spectrum of $\mathcal{O}$ is simply the spectrum of the underlying space $|\mathcal{O}|$ with mixed boundary conditions: Neumann conditions on the mirror arcs, but Dirichlet conditions on the remainder of the boundary. Similarly, the Neumann spectrum of $\mathcal{O}$ is simply the Neumann spectrum of $|\mathcal{O}|$, since the reflection-invariance forces Neumann conditions on the part of the boundary of $|\mathcal{O}|$ corresponding to mirror arcs of $\mathcal{O}$. This simple observation will be used repeatedly in what follows.

Given a Gassmann-Sunada triple (G, Γ_1, Γ_2), in order to apply Sunada's Theorem one needs a manifold M on which G acts by isometries. Perhaps the easiest way to obtain such an M is to begin with a manifold M_0 whose fundamental group $\widehat{G} = \pi_1(M_0)$ admits a surjective homomorphism $\varphi : \widehat{G} \to G$. Setting $\widehat{\Gamma}_i = \varphi^{-1}(\Gamma_i)$ for $i = 1, 2$ defines two subgroups of $\pi_1(M_0)$; by covering space theory, there are covering spaces M_1 and M_2 corresponding to $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$. Also, there is a common regular covering M of M_1 and M_2 associated to the subgroup $\widehat{K} = \ker \varphi$, so that $M_i = \Gamma_i\backslash M$, $i = 1, 2$, and $M_0 = G\backslash M$.
P. Buser [10] exploited the observation that if $\pi_1(M_0)$ is free, then Schreier graphs furnish a concrete means of constructing M, M_1 and M_2 without explicit reference to the universal cover; when M_0 is a surface with nonempty boundary, he used this construction to exhibit isospectral flat bordered surfaces.
Given a group G, a G-set X, and a generating set S for G, recall that the *Schreier graph* of X relative to the generating set S has vertex set X, with a directed edge labeled by $s \in S$ joining

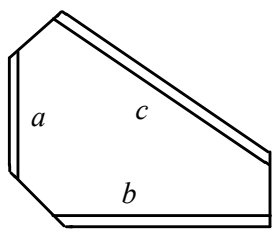
the vertex x to the vertex sx, for each $x \in X$, $s \in S$. Concretely, one chooses the bordered surface M_0 to be a thickened one-point union of circles, with one circle for each generator; the case of a three-element generating set $S = \{a, b, c\}$ is depicted below.



M_0 $\qquad$ T

The construction of M_i for $i = 1, 2$ now goes as follows. Begin with $[G : \Gamma_i]$ copies of the fundamental domain T depicted above, labeled by the cosets G/Γ_i. For each generator $s \in S$, glue the edge of the outgoing leg of the tile labeled $x\Gamma_i$ to the edge of the incoming s-leg of the tile labeled $sx\Gamma_i$. The surface so constructed is a thickened Schreier graph of $G/\Gamma_i \cong \widehat{G}/\widehat{\Gamma}_i$, so it has fundamental group $\widehat{\Gamma}_i$ and covers M_0, and hence must be precisely the manifold M_i defined above.

Suppose now that the group G is generated by a set S of involutions. Then one can construct analogously a pair of isospectral orbifolds with boundary as follows. Let $\mathcal{O}_0$ be a disk with $[G : \Gamma_i]$ nonintersecting mirror arcs (the case of $S = \{a, b, c\}$ is depicted in Figure 1).



$\mathcal{O}_0$

Figure 1

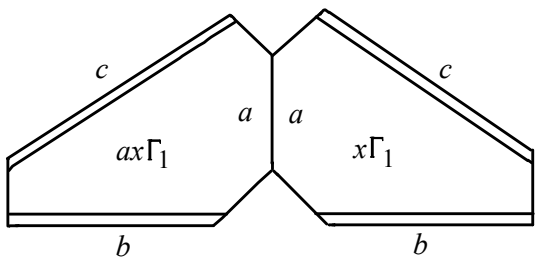


Figure 2

Then the orbifold fundamental group $\widehat{G} = \pi_1(\mathcal{O}_0)$ is the free product of $|S|$ copies of $\mathbb{Z}/2\mathbb{Z}$. Define a surjective homomorphism $\varphi : \widehat{G} \to G$ by sending the obvious "mirror reflection" generators of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}$ to the elements of S. As before, let $\widehat{\Gamma}_i = \varphi^{-1}(\Gamma_i)$, $i = 1, 2$. Then by covering space theory, there is an orbifold cover $\mathcal{O}_i$ of $\mathcal{O}_0$ corresponding to the subgroup $\widehat{\Gamma}_i \subseteq \widehat{G} = \pi_1(\mathcal{O}_0)$, which can be constructed explicitly as follows: begin with $[G : \Gamma_i]$ copies of the fundamental domain $|\mathcal{O}_0|$, labeled by the elements of G/Γ_i. For each $s \in S$, glue the mirror edge labeled s of the tile labeled $x\Gamma_i$ to the mirror edge labeled s of the tile labeled $sx\Gamma_i$ so that reflection in the common edge interchanges the two tiles, as in Figure 2; if s fixes a coset, then no identification is performed on that edge. Then the orbifold

so constructed has fundamental group $\widehat{\Gamma}_i$, so it must be $\mathcal{O}_i$. By Bérard's version of Sunada's Theorem, $\mathcal{O}_1$ and $\mathcal{O}_2$ are isospectral orbifolds.

3 Construction

In this section, we turn to the proof of the following.

Theorem 3.1. *There exists a pair M_1, M_2 of flat surfaces with boundary which are Neumann isospectral, yet M_1 is nonorientable while M_2 is orientable.*

Proof. The surfaces M_1 and M_2 will be constructed as the underlying spaces of Neumann isospectral orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ with boundary whose singular sets consist of disjoint unions of mirror arcs. As noted above in Remark 2.2, this means that Neumann boundary conditions hold on the mirror edges of the underlying surfaces $M_1 = |\mathcal{O}_1|$ and $M_2 = |\mathcal{O}_2|$ as well as on the edges forming the orbifold boundary; since ∂M_i consists of the boundary of $\mathcal{O}_i$ together with the mirror edges, it follows that Neumann conditions hold on the entire boundary of M_i.
The Gassmann-Sunada triple we use was first considered by Gerst [20]; it was also used by Buser [11] to construct isospectral Riemann surfaces, and in [23]. Let G be the semidirect product of a multiplicatively-written cyclic group $\langle s\rangle$ of order 8 by its full automorphism group; the latter is a Klein 4-group, generated by the automorphism t sending $s \mapsto s^7$ and the automorphism u sending $s \mapsto s^3$. Thus G is the semidirect product $\mathbb{Z}_8 \rtimes \mathbb{Z}_8^\times$, with s generating the cyclic subgroup $\mathbb{Z}_8$; a presentation is $G = \left\langle s, t, u \mid s^8 = t^2 = u^2 = [t, u] = 1,\ tst = s^7,\ usu = s^3\right\rangle$. Let $\Gamma_1 = \{1, t, u, tu\}$, $\Gamma_2 = \{1, t, s^4u, s^4tu\}$. Then (G, Γ_1, Γ_2) is a Gassmann-Sunada triple (with Γ_1 and Γ_2 nonconjugate).
Now let $\mathcal{O}_0$ be the orbifold with boundary depicted below, in Figure 3:

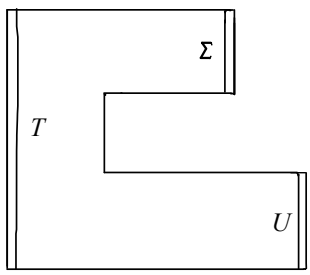
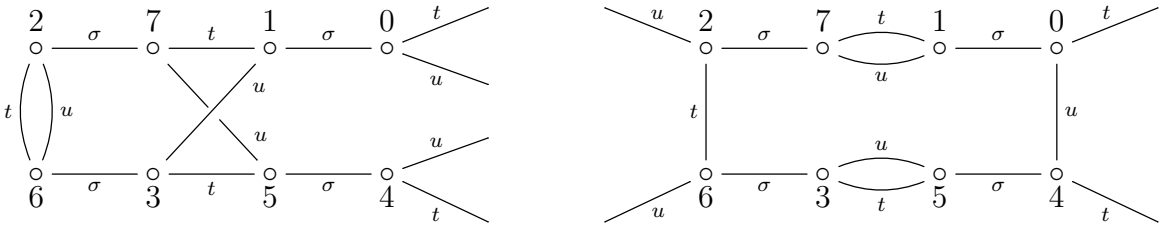


Figure 3

Its fundamental group $\widehat{G}$ is then given by $\widehat{G} = \pi_1(\mathcal{O}_0) = \left\langle \Sigma, T, U \mid \Sigma^2 = T^2 = U^2 = 1\right\rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Let $\sigma = st \in G$. Then σ, t and u are involutions, so we can define $\varphi : \pi_1(\mathcal{O}_0) \to G$ by $\Sigma \mapsto \sigma$, $T \mapsto t$, $U \mapsto u$. Since G is generated by the set $\{\sigma, t, u\}$, it follows that φ is surjective.
As in section 2, let $\widehat{\Gamma}_i = \varphi^{-1}(\Gamma_i)$. The elements $1, s, s^2, \ldots, s^7$ form a set of coset representatives for G/Γ_i and hence for $\widehat{G}/\widehat{\Gamma}_i$ for $i = 1, 2$; we will denote the coset $s^i\Gamma_1$ or $s^i\Gamma_2$ simply by i in the depiction of the Schreier graphs below. The action on G/Γ_1 and G/Γ_2 of the three generators σ, t and u is most easily recorded by the Schreier graphs below; when dealing with generating sets consisting of involutions, we adopt the convention that a single undirected edge labeled by a generator $r \in S$ replaces the two oppositely-directed edges labeled r and r^{-1} joining a pair of vertices; if an edge labeled r leaves a vertex labeled x and does not terminate at

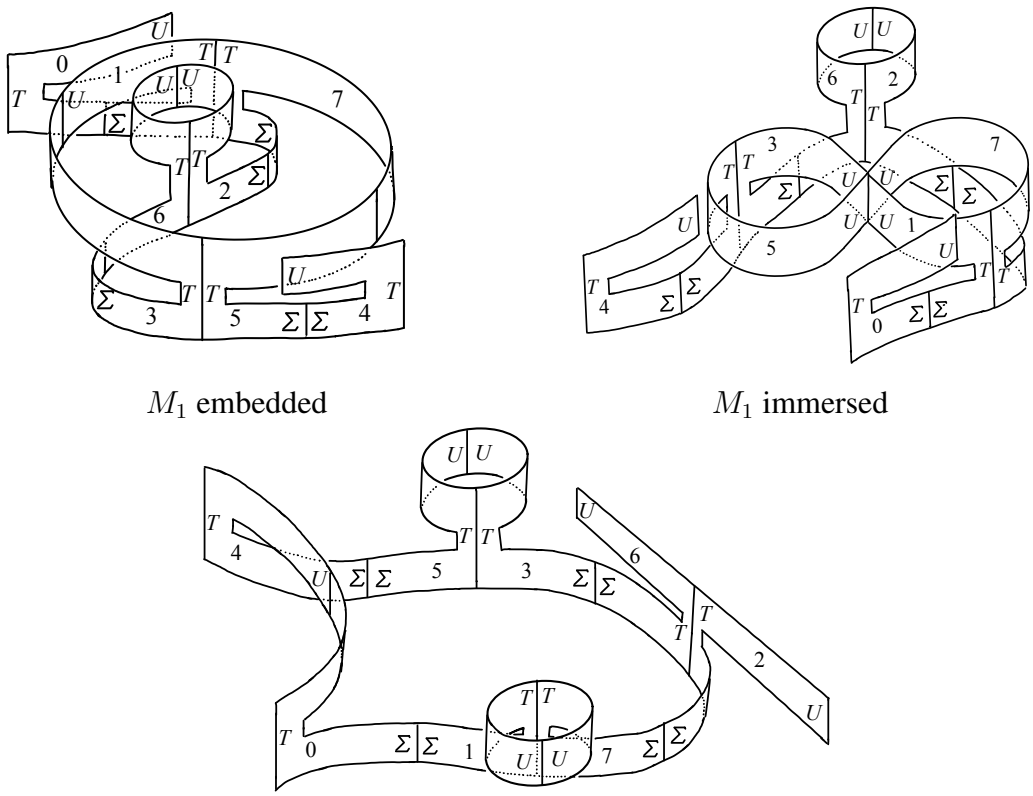
another vertex, this indicates that the generator r fixes the coset x. Thus when dealing with involutive generators, we are replacing a pair of oppositely-directed edges by a single undirected edge, and replacing a loop based at a vertex by a "half-edge" emanating from that vertex.

(3.1)



G/Γ_1 G/Γ_2

Let $\mathcal{O}_i$ be the orbifold covering of $\mathcal{O}_0$ corresponding to the subgroup $\widehat{\Gamma}_i$ of $\widehat{G}$. The bordered surfaces $M_1 = |\mathcal{O}_1|$ and $M_2 = |\mathcal{O}_2|$ are shown below: the first model of M_1 is embedded in $\mathbb{R}^3$, while the second model shows it immersed with an arc of self-intersection; the immersed version enables one to see easily an involutive symmetry that will be exploited in section 6.

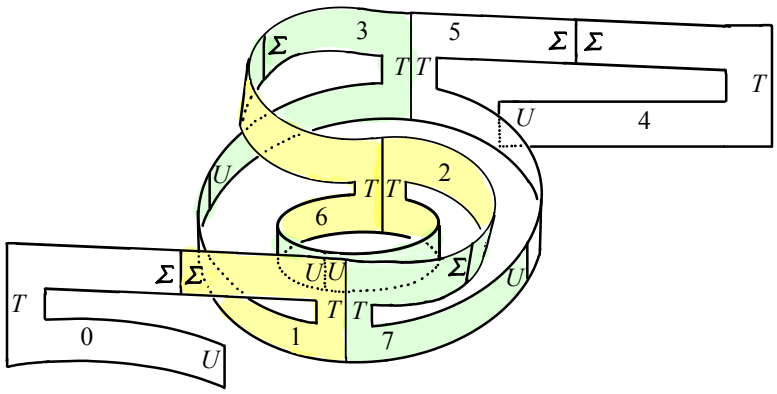


M_1 embedded M_1 immersed

M_2

Figure 4

Clearly, M_1 is nonorientable while M_2 is orientable. Indeed, an embedded Möbius strip in M_1 can be seen in the following picture:



If one colors one side of the fundamental tile yellow and the other side green, then beginning with the yellow side of tile 1, the succession of the U-gluing of tile 1 to tile 3, the Σ-gluing of tile 3 to tile 6, the T-gluing of tile 6 to tile 2, and the Σ-gluing of tile 2 to tile 7 shows that the T-gluing of tile 7 back to tile 1 forces the gluing of the green side of tile 7 to the yellow side of tile 1, exhibiting the union of tiles 1, 3, 6, 2, and 7 as a one-sided surface. Thus one cannot "hear" orientability of surfaces with boundary. □

An immediate consequence of the above result is the following.

Theorem 3.2. *One cannot hear whether a surface with boundary admits a complex structure.*

Proof. Perform the same construction as above, but using a different fundamental tile: rather than the tile in Figure 3, use a right-angled hyperbolic hexagon, three pairwise nonadjacent sides of which are mirror loci, as in Figure 5:

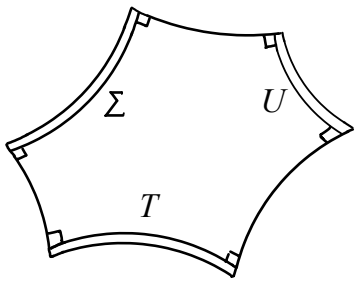


Figure 5

The resulting manifolds M_1 and M_2 are homeomorphic to the manifolds of Figure 4 above. Both are hyperbolic surfaces with piecewise-smooth boundary. An orientable hyperbolic surface has a Riemann surface structure, by the uniformization theorem (see [18], §11.1.1 or [25], Chapter 1). Thus M_2 has a complex structure. However, M_1, being nonorientable, cannot have a complex structure, since any complex manifold is canonically orientable. □

Remarks 3.3. We list a few immediate remarks:

1. The nonorientability of M_1 can be seen algebraically from the presence of cycles of odd length in the Schreier graph of G/Γ_1 in (3.1).

2. A slight modification of the construction in Theorem 3.2 yields a pair of isospectral compact hyperbolic surfaces with smooth boundary whose boundary components are

closed geodesics; the first of these surfaces is nonorientable but the second is orientable. For $i = 1, 2$, perform the same construction, but using as basic tile the pair of pants (in the terminology of [13], section 3.1, a Y-*piece*) obtained by gluing together two copies of the right-angled hyperbolic hexagon along the alternating non-mirror edges of the tile of Figure 5. Since each Y-piece is constructed by "doubling" a right-angled hyperbolic hexagon by gluing along the mirror edges, this amounts to beginning with two copies of the manifold M_i of Theorem 3.2 and gluing the two copies together along the non-mirror edges of their boundaries, thereby effectively doubling the manifold M_i of Theorem 3.2 to obtain a hyperbolic surface N_i with geodesic boundary components (the boundary components correspond to half-edges in the Schreier graphs in (3.1)). For more information on pants decompositions of nonorientable hyperbolic surfaces, see [30].

3. Peter Doyle and Juan Pablo Rossetti have shown [17] that if two *closed* hyperbolic surfaces have the same spectrum, then for every possible length, the two surfaces have the same number of orientation-preserving geodesics and the same number of orientation-reversing geodesics. Thus one *can* hear orientability of closed hyperbolic surfaces.

4. The reader is encouraged to build paper and tape models; this will make it easy to follow the arguments in sections 4 and 6.

4 Transplantation of eigenfunctions

In this section, we present an elementary visual proof of the Neumann isospectrality using the idea of "transplantation" of eigenfunctions from Bérard's proof of Sunada's Theorem. We give an explicit combinatorial recipe for transplanting a Neumann eigenfunction on M_1 to a Neumann eigenfunction for the same eigenvalue on M_2. To do this we first determine the irreducible representations of G, then express the induced representations $\mathbb{C}[G/\Gamma_i] = (\mathbb{1}_{\Gamma_i})\uparrow_{\Gamma_i}^{G}$ in terms of the irreducible representations. The reader who is only interested in the isospectrality proof can skip to Remarks 4.4.

Recall that G is a semidirect product $K \rtimes H$, where the normal subgroup $K = \langle s \rangle$ is cyclic of order 8, and $H = \langle t, u \mid t^2 = u^2 = [t, u] = 1 \rangle$ is a Klein 4-group, with t acting as the automorphism $s \mapsto s^7$ and u acting as the automorphism $s \mapsto s^3$. The rational group algebra $\mathbb{Q}[K \rtimes H]$ of a semidirect product has the structure of a trivial crossed-product or *twisted group algebra* $(\mathbb{Q}K)\sharp H$ (see [15] for general information on crossed products). The group algebra $\mathbb{Q}C_n$ of a cyclic group C_n of order n decomposes as a product of cyclotomic fields: $\mathbb{Q}C_n \cong \prod_{d|n} \mathbb{Q}[\zeta_d]$, where ζ_d is a primitive dth root of unity (see [28]), so $\mathbb{Q}K \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}[i] \times \mathbb{Q}[\zeta]$, where ζ is a primitive eighth root of unity. The H-action on $\mathbb{Q}K$ stabilizes this decomposition, so $\mathbb{Q}G \cong (\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}[i] \times \mathbb{Q}[\zeta])\sharp H \cong \mathbb{Q}H \times \mathbb{Q}H \times (\mathbb{Q}[i]\sharp H) \times (\mathbb{Q}[\zeta]\sharp H)$. Tensoring with $\mathbb{C}$, we obtain

$$\mathbb{C}G \cong \mathbb{C}H \times \mathbb{C}H \times (\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}[i])\sharp H \times (\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta])\sharp H. \tag{4.1}$$

The first two factors of (4.1) yield eight one-dimensional representations, denoted $\mathbb{1}^{abc}$, where each of a, b, c can be +1 or -1; each $\mathbb{1}^{abc}$ is a one-dimensional vector space, where s acts as the scalar a, t acts as b, and u acts as c. We will denote the trivial one-dimensional representation $\mathbb{1}^{+++}$ simply as $\mathbb{1}$, and the "parity" representation $\mathbb{1}^{-++}$ simply as $\mathbb{1}^{-}$. In the third factor of (4.1), H acts on $\mathbb{C} \otimes \mathbb{Q}[i] \cong \mathbb{C} \times \mathbb{C}$ as follows: t and u both act by the involution $(x, y) \mapsto$

(y, x), so $v = tu$ acts trivially; thus $(\mathbb{C} \otimes \mathbb{Q}[i])\sharp H \cong (\mathbb{C} \otimes \mathbb{Q}[i])\langle v\rangle \sharp_c \overline{H}$, a crossed product algebra in which the quotient group $\overline{H} = H/\langle v\rangle$ acts faithfully on the ordinary group ring $(\mathbb{C} \otimes \mathbb{Q}[i])\langle v\rangle$ of the cyclic group $\langle v\rangle$ over the coefficient ring $\mathbb{C} \otimes \mathbb{Q}[i]$, and c is a 2-cocycle defining the extension $1 \to \langle v\rangle \to H \to H/\langle v\rangle \to 1$; but this extension is split, so the cocycle c can be taken to be trivial, and hence $(\mathbb{C} \otimes \mathbb{Q}[i])\sharp H$ is an ordinary twisted group ring $(\mathbb{C} \otimes \mathbb{Q}[i])\langle v\rangle \sharp \overline{H}$. Moreover, $\overline{H}$ acts trivially on $\langle v\rangle$, as the extension is central, so the above reduces to $\mathbb{C}\langle v\rangle \otimes_{\mathbb{Q}} (\mathbb{Q}[i]\sharp\overline{H})$. Now $\overline{H}$ acts faithfully on $\mathbb{Q}[i]$, so by Galois theory (e.g., [16], Chapter III, Proposition 1.2), $\mathbb{Q}[i] \sharp \overline{H} \cong \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}[i]) \cong \mathrm{M}_2(\mathbb{Q})$, while $\mathbb{C}\langle v\rangle \cong \mathbb{C} \times \mathbb{C}$; thus the third factor of (4.1) decomposes as $\mathrm{M}_2(\mathbb{C}) \times \mathrm{M}_2(\mathbb{C})$ and contributes two irreducible representations W^+ and W^-. Tracing through the isomorphisms, one sees that the actions of the generators of G in these representations are as follows:

- On W^+, s acts as $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, while t and u act as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$;
- On W^-, s acts as $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, t acts as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and u acts as $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

Finally, consider the fourth factor $(\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta])\sharp H \cong \mathbb{C} \otimes_{\mathbb{Q}} (\mathbb{Q}[\zeta]\sharp H)$ in (4.1). By Galois theory, $\mathbb{Q}[\zeta]\sharp H \cong \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}[\zeta]) \cong \mathrm{M}_4(\mathbb{Q})$, since $H = \mathrm{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$. Thus the fourth factor of the decomposition (4.1) is $\mathrm{M}_4(\mathbb{C})$, and it contributes a 4-dimensional irreducible representation X. Using the basis $\{1, \zeta, \zeta^2, \zeta^3\}$ for $\mathbb{Q}[\zeta]$, one sees that:

$$s \text{ acts by } \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad t \text{ by } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \text{and } u \text{ by } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

This completes the determination of all the irreducible representations of G; the character table is given in Table 1. As usual, the rows are indexed by the irreducible representations (up to isomorphism) of G and the columns are indexed by (representatives of) the conjugacy classes of G. The additional row at the bottom of the table records the character values of the induced representations $(\mathbb{1}_{\Gamma_i})\uparrow_{\Gamma_i}^G$, which are easily computed via the formula for the character of an induced representation, or directly from the Schreier graphs (3.1).

	1	s^4	s^2	v	s^2v	s	t	u	st	su	sv
$\mathbb{1}$	1	1	1	1	1	1	1	1	1	1	1
$\mathbb{1}^{+-+}$	1	1	1	-1	-1	1	-1	1	-1	1	-1
$\mathbb{1}^{++-}$	1	1	1	-1	-1	1	1	-1	1	-1	-1
$\mathbb{1}^{+--}$	1	1	1	1	1	1	-1	-1	-1	-1	1
$\mathbb{1}^{-}$	1	1	1	1	1	-1	1	1	-1	-1	-1
$\mathbb{1}^{--+}$	1	1	1	-1	-1	-1	-1	1	1	-1	1
$\mathbb{1}^{-+-}$	1	1	1	-1	-1	-1	1	-1	-1	1	1
$\mathbb{1}^{---}$	1	1	1	1	1	-1	-1	-1	1	1	-1
W^+	2	2	-2	2	-2	0	0	0	0	0	0
W^-	2	2	-2	-2	2	0	0	0	0	0	0
X	4	-4	0	0	0	0	0	0	0	0	0
$(\mathbb{1}_{\Gamma_i})\uparrow_{\Gamma_i}^G$	8	0	0	4	0	0	2	2	0	0	0

Table 1

It is easy to compute the character of the induced representation $(\mathbb{1}_{\Gamma_i})\uparrow_{\Gamma_i}^G$ in terms of the irreducible characters by orthonormal expansion, since the latter form an orthonormal basis for the space of class functions relative to the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi}(g).$$

One finds:

Proposition 4.1. $\mathbb{C}[G/\Gamma_i] \cong \mathbb{1} \oplus \mathbb{1}^- \oplus W^+ \oplus X$.

Using the Fourier inversion formula [34]

$$\varepsilon_V = \frac{\dim(V)}{|G|} \sum_{g \in G} \chi_V(g^{-1}) g$$

for the primitive central idempotent ε_V associated to an irreducible representation V of G one can easily determine bases for the irreducible constituents of the representation $\mathbb{C}[G/\Gamma_i]$ in terms of the bases of cosets. Consider first $\mathbb{C}[G/\Gamma_1]$. Let u_i denote the coset $s^i\Gamma_1$, $0 \leq i \leq 7$. Let $e_1 = (\varepsilon_{1^+}) \cdot u_0 = \frac{1}{8}\sum_{i=0}^7 u_i$, $e_2 = (\varepsilon_{1^-}) \cdot u_0 = \frac{1}{8}\sum_{i=0}^7 (-1)^i u_i$, $e_3 = (\varepsilon_{W^+}) \cdot u_0 = \frac{1}{4}(u_0 - u_2 + u_4 - u_6)$, $e_4 = (\varepsilon_{W^+}) \cdot u_1 = \frac{1}{4}(u_1 - u_3 + u_5 - u_7)$, $e_5 = \varepsilon_X \cdot u_0 = \frac{1}{2}(u_0 - u_4)$, $e_6 = \varepsilon_X \cdot u_1 = \frac{1}{2}(u_1 - u_5)$, $e_7 = \varepsilon_X \cdot u_2 = \frac{1}{2}(u_2 - u_6)$, and $e_8 = \varepsilon_X \cdot u_3 = \frac{1}{2}(u_3 - u_7)$. Then the following is immediate:

Proposition 4.2. *Let $e_1, \ldots, e_8$ be as defined above. Then:*

- $\{e_1\}$ *is a basis of the* $\mathbb{1}^+$ *summand of* $\mathbb{C}[G/\Gamma_1]$.
- $\{e_2\}$ *is a basis of the* $\mathbb{1}^-$ *summand of* $\mathbb{C}[G/\Gamma_1]$.
- $\{e_3, e_4\}$ *is a basis of the* W^+ *summand of* $\mathbb{C}[G/\Gamma_1]$.
- $\{e_5, e_6, e_7, e_8\}$ *is a basis of the* X *summand of* $\mathbb{C}[G/\Gamma_1]$.

Similarly, turning to G/Γ_2, let v_i denote the coset $s^i\Gamma_2$, $0 \leq i \leq 7$.

Proposition 4.3. *Let $f_1, f_2 \ldots f_7$ be defined by the same formulas defining the e_i, but with each u_i replaced by v_i. Then:*

- $\{f_1\}$ *is a basis of the* $\mathbb{1}^+$ *summand of* $\mathbb{C}[G/\Gamma_2]$.
- $\{f_2\}$ *is a basis of the* $\mathbb{1}^-$ *summand of* $\mathbb{C}[G/\Gamma_2]$.
- $\{f_3, f_4\}$ *is a basis of the* W^+ *summand of* $\mathbb{C}[G/\Gamma_2]$.
- $\{f_5, f_6, f_7, f_8\}$ *is a basis of the* X *summand of* $\mathbb{C}[G/\Gamma_2]$.

We now make explicit the most general equivalence of the permutation representations $\mathbb{C}[G/\Gamma_1]$ and $\mathbb{C}[G/\Gamma_2]$; since these representations are both equivalent to $\mathbb{1} \oplus \mathbb{1}^- \oplus W^+ \oplus X$, this reduces to determining all the intertwining isomorphisms of the latter.

Remarks 4.4. From the above, the following assertions are immediate:

1. For any nonzero scalar a, the map sending $e_1 \mapsto af_1$ is a G-isomorphism of the $\mathbb{1}^+$ summand of $\mathbb{C}[G/\Gamma_1]$ onto the $\mathbb{1}^+$ summand of $\mathbb{C}[G/\Gamma_2]$.

2. For any nonzero scalar b, the map sending $e_2 \mapsto bf_2$ is a G-isomorphism of the $\mathbb{1}^-$ summand of $\mathbb{C}[G/\Gamma_1]$ onto the $\mathbb{1}^-$ summand of $\mathbb{C}[G/\Gamma_2]$.

3. The actions of s, t and u on e_3 and e_4 coincide with their actions on f_3 and f_4; thus for any nonzero c, the map sending $e_3 \mapsto cf_3$ and $e_4 \mapsto cf_4$ is a G-isomorphism of the W^+ summand of $\mathbb{C}[G/\Gamma_1]$ onto the W^+ summand of $\mathbb{C}[G/\Gamma_2]$. Moreover, by Schur's Lemma, this is the most general such isomorphism.

4. One easily computes that the most general G-isomorphism of $\mathrm{span}\{e_5, e_6, e_7, e_8\}$ onto $\mathrm{span}\{f_5, f_6, f_7, f_8\}$ is given by

$$\begin{bmatrix} 0 & d & 0 & -d \\ d & 0 & d & 0 \\ 0 & d & 0 & d \\ -d & 0 & d & 0 \end{bmatrix},$$

   where $d \neq 0$. Thus, letting $h_5 = f_6 - f_8$, $h_6 = f_5 + f_7$, $h_7 = f_6 + f_8$, $h_8 = -f_5 + f_7$, we see that, relative to the bases $\{e_5, e_6, e_7, e_8\}$ and $\{h_5, h_6, h_7, h_8\}$, the most general isomorphism of the X summand of $\mathbb{C}[G/\Gamma_1]$ onto the X summand of $\mathbb{C}[G/\Gamma_2]$ is given by $e_i \mapsto dh_i$, $i = 5, 6, 7, 8$.

5. Finally, let $h_i = f_i$ for $1 \leq i \leq 4$. Then the most general G-isomorphism $\Phi : \mathbb{C}[G/\Gamma_1] \to \mathbb{C}[G/\Gamma_2]$ is given, relative to the bases $\{e_1, e_2, \ldots, e_8\}$ and $\{h_1, h_2, \ldots, h_8\}$, by the matrix $\mathrm{diag}(a, b, c, c, d, d, d, d)$, where $abcd \neq 0$. Changing bases, we see that relative to the natural coset bases $\{u_0, u_1 \ldots u_7\}$ and $\{v_0, v_1 \ldots v_7\}$, the general intertwining isomorphism Φ is given by the matrix

$$A = \begin{bmatrix} \alpha & \beta & \gamma & \delta & \alpha & \delta & \gamma & \beta \\ \beta & \alpha & \beta & \gamma & \delta & \alpha & \delta & \gamma \\ \gamma & \beta & \alpha & \beta & \gamma & \delta & \alpha & \delta \\ \delta & \gamma & \beta & \alpha & \beta & \gamma & \delta & \alpha \\ \alpha & \delta & \gamma & \beta & \alpha & \beta & \gamma & \delta \\ \delta & \alpha & \delta & \gamma & \beta & \alpha & \beta & \gamma \\ \gamma & \delta & \alpha & \delta & \gamma & \beta & \alpha & \beta \\ \beta & \gamma & \delta & \alpha & \delta & \gamma & \beta & \alpha \end{bmatrix}, \tag{4.2}$$

   where $\alpha = \frac{1}{8}(a + b + 2c)$, $\beta = \frac{1}{8}(a - b + 4d)$, $\gamma = \frac{1}{8}(a + b - 2c)$, $\delta = \frac{1}{8}(a - b - 4d)$ and α, β, δ are chosen so that $abcd \neq 0$.

Remarks 4.5. The above conclusions can also be reached by more elementary computations:

1. The transplantation matrix A relative to the bases given by the cosets can be computed in a naïve way simply by determining the conditions on the entries of A that are forced by the requirement that A intertwine the two permutation representations.

2. The decomposition of the representations $\mathbb{C}[G/\Gamma_i]$ into irreducible representations can also be determined by considering the equation $s^8 - 1 = 0$ and looking at the possible degrees of irreducible constituents. From these irreducible decompositions, one can recover Buser's transplantation of eigenfunctions on the isospectral Riemann surfaces described in [11].

3. For a systematic study of transplantation with many interesting examples, see [24].

A simple choice of A for our purposes is given by setting $c = d = 2$, $a = 6$, $b = -2$, so $\alpha = 1$, $\beta = 2$, $\gamma = \delta = 0$. We can now describe explicitly how to transplant a Neumann eigenfunction from M_1 to M_2. Recall that M_i is constructed by gluing together copies of a fundamental tile $\mathscr{T}$, the copies labeled by the cosets G/Γ_i. Let F be a λ-eigenfunction on M_1; let F_i denote its restriction to the tile labeled by the coset $s^i\Gamma_1$. Then F can be represented by the vector of functions on $\mathscr{T}$ given by $[F_0\ F_1\ F_2 \cdots F_7]$. (Note that there is no ambiguity about what it means to regard an F_i as a function on $\mathscr{T}$; there is a unique isometry of $\mathscr{T}$ with each tile of M_1 or M_2 that preserves the labeling of the boundary edges.) Now let H denote the function on M_2 whose restriction H_i to the tile of M_2 labeled $s^i\Gamma_2$ is given by the matrix product

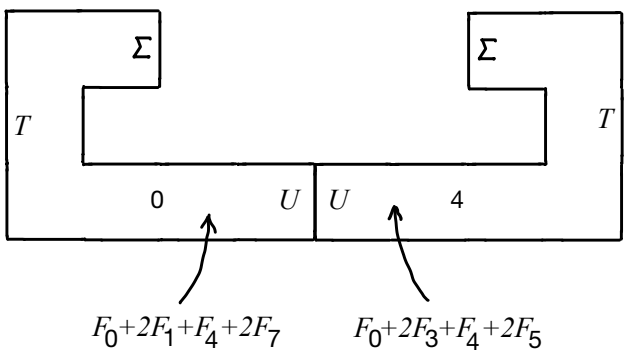
$$\begin{bmatrix} H_0 & H_1 & H_2 & \dots & H_7 \end{bmatrix} = \begin{bmatrix} F_0 & F_1 & F_2 & \dots & F_7 \end{bmatrix} A,$$

where A is the intertwining matrix defined above. If H is smooth, then it is certainly an eigenfunction, since this is a local condition; thus checking Neumann isospectrality reduces to checking that:

1. The functions H_i fit together smoothly across the interfaces between tiles;

2. The function H satisfies Neumann boundary conditions.

These assertions are easily checked by inspection of the paper models or by looking at the graphs (3.1); we illustrate briefly. Consider the interface between tiles 0 and 4 of M_2; it is depicted in the figure below.



We will show that H_0 and H_4 fit together smoothly across this U-edge. It suffices to show that the following condition is satisfied:

(4.3) H_0 and H_4 coincide along their common U-edge, and their inner normal derivatives there are negatives of each other.

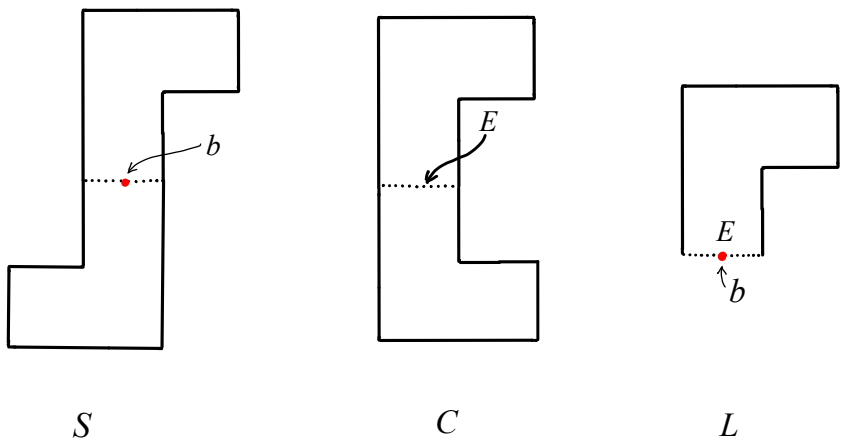
Indeed, if (4.3) holds, then by Green's formula the function H (which equals H_0 on tile 0 and H_4 on tile 4) is a local weak solution of $\Delta u = \lambda u$, so by regularity of local weak solutions, H is smooth along the U-edge. We now verify that (4.3) holds. The functions H_0 and H_4 are given by $H_0 = F_0 + 2F_1 + F_4 + 2F_7$, $H_4 = F_0 + 2F_3 + F_4 + 2F_5$. Inspection of Figure 4 or of the graphs (3.1) shows that in M_1, tiles 1 and 3 meet along a U-edge, so $2F_1$ and $2F_3$ and their derivatives fit together smoothly along their common U-edge; similarly, $2F_7$ and $2F_5$ fit together. As for F_0 and F_4, they have vanishing normal derivatives on the edge. Thus H_0 and H_4 fit together smoothly across the U-interface. To see that H satisfies Neumann boundary conditions on M_2, consider for example the tile 0; we will show that H_0 satisfies the Neumann condition on the T-edge. First, the T-edges of tiles 0 and 4 of M_1 are boundary edges, so F_0 and F_4 have vanishing normal derivative on the T-edge. Figure 4 and the graphs (3.1) also show that tiles 1 and 7 of M_1 share a T-edge; since the (unique) isometry used to identify tiles 1 and 7 of M_1 is a reflection in their common T-edge, it is clear that on the T-edge of tile 0 in M_2, the normal derivatives of F_1 and F_7 are negatives of each other. Thus the normal derivative of $H_0 = F_0 + 2F_1 + F_4 + 2F_7$ vanishes on the T-edge, as desired. One performs a similar verification on each tile of M_2.

Thus any Neumann eigenfunction on M_1 can be transplanted to a Neumann eigenfunction on M_2; using the inverse of the matrix A, one can transplant eigenfunctions from M_2 to M_1. Thus M_1 and M_2 are Neumann isospectral.

5 A folklore argument of Fefferman

In [23], planar domains that are isospectral for either Neumann or Dirichlet boundary conditions were exhibited. The domains are underlying spaces of orbifolds (with boundary) $\mathcal{O}_1$ and $\mathcal{O}_2$ whose singular sets consist of disjoint mirror arcs, and the Neumann isospectrality was established by showing that $\mathcal{O}_1$ and $\mathcal{O}_2$ are isospectral as orbifolds. The Dirichlet isospectrality follows because there are surfaces (with boundary) S_1 and S_2 that can be viewed as the orientation double covers of the orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ (obtained by "doubling" the orbifolds along the mirror edges); the surfaces S_1 and S_2 are themselves isospectral by Sunada's Theorem; these examples were discovered by Buser [10]. In fact, for each of S_1 and S_2, the reflection symmetry decomposes the space of smooth functions on S_i as the direct sum of a $(+1)$-eigenspace (the reflection-invariant functions) and a (-1)-eigenspace (the reflection-anti-invariant functions); the latter are the functions satisfying the Dirichlet boundary condition, and using this decomposition, one easily deduces Dirichlet isospectrality from the isospectrality of S_1 and S_2 and of $\mathcal{O}_1$ and $\mathcal{O}_2$. Thus $|\mathcal{O}_1|$ and $|\mathcal{O}_2|$ are Dirichlet isospectral domains. One can also make explicit a Dirichlet transplantation matrix (see [4]).

As was observed by Peter Doyle in connection with some of the examples in [12], if the orientation double covers are not isospectral, then the above argument fails, so there is no reason to expect M_1 and M_2 to be Dirichlet isospectral. We will show in fact that our M_1 and M_2 have a different lowest Dirichlet eigenvalue, at least when M_1 and M_2 are constructed using a more symmetrical fundamental tile than that used in section 3. The extra symmetry will permit an adaptation of an argument due to C. Fefferman showing that the two planar domains S and C shown below are not Dirichlet isospectral.



The tile L on the right above is the upper half of both the domains S and C.
Fefferman's argument exploits the fact that the domain C has a reflection symmetry (reflection through the dotted line E), while the domain S has an involutive rotation symmetry (rotation by π around the barycenter b). We sketch Fefferman's argument below.
For any Riemannian manifold (with boundary) M and p-forms α, β on M, write $(\cdot,\cdot)_M$ for the L^2 inner product: $(\alpha,\beta)_M = \int_M \langle \alpha,\beta\rangle \, dvol$, where $\langle\cdot,\cdot\rangle$ is the pointwise inner product on p-covectors and $dvol$ is the Riemannian measure. We recall (see [2] or [14]) that, for a Riemannian manifold M with boundary, the lowest nonzero Neumann eigenvalue μ is given by the infimum of the Rayleigh quotients:

$$\mu = \inf_{f\in H^1,\, f\perp 1} R_M(f) = \inf_{f\in H^1,\, f\perp 1} \frac{(df,df)_M}{(f,f)_M}, \tag{5.1}$$

where f ranges over the space $C^\infty(M)$ of smooth functions on M (or equivalently, over its H^1-completion, the Sobolev space H^1 of functions having one distributional derivative in L^2) that are L^2-orthogonal to the constant functions. The lowest Dirichlet eigenvalue λ is given by

$$\lambda = \inf_{f\in H^1_0} R_M(f) = \inf_{f\in H^1_0} \frac{(df,df)_M}{(f,f)_M},$$

the infimum of the Rayleigh quotients with f ranging over the space of smooth functions with compact support in the interior of M (or equivalently, over its H^1-completion H^1_0). Finally, for mixed boundary conditions (i.e., Neumann boundary conditions on an open submanifold N of ∂M, but Dirichlet conditions on an open submanifold $D \subseteq \partial M$ such that $\partial M = \overline{N \cup D}$), one allows f to range over the space of smooth functions on M supported away from D (or over its Sobolev completion H^1_{mixed}). In each case, a function f in the pertinent Sobolev completion whose Rayleigh quotient realizes the infimum is an eigenfunction for the lowest nonzero eigenvalue.

Theorem 5.1. *(Fefferman) The domains S and C depicted above are not Dirichlet isospectral. In fact, if λ_Ω denotes the lowest Dirichlet eigenvalue of a domain $\Omega \subseteq \mathbb{R}^2$, then $\lambda_C < \lambda_S$.*

Proof. Fefferman's argument runs as follows. Let f_C be a normalized (i.e., of unit L^2-norm) eigenfunction on C for the lowest Dirichlet eigenvalue λ_C of C. By Courant's nodal domain theorem (see [14]), f_C is never zero on the interior C° of C, so without loss of generality,

assume that $f_C > 0$ on C°. The eigenspace associated with the lowest eigenvalue is one-dimensional, since no two functions that are strictly positive on C° can be L^2-orthogonal. Let ϱ be the obvious reflection symmetry of C (reflection through the line E in the illustration above). Then $f_C \circ \varrho$ is another normalized λ_C-eigenfunction on C, so $f_C \circ \varrho = \pm f_C$. But $f_C \geq 0$, so $f_C \circ \varrho \geq 0$, and thus $f_C \circ \varrho = f_C$. Thus f_C is invariant under the reflection in the segment E, so restricting f_C to the upper half L of C yields a function $f_L := (f_C)|_L$ that is zero on the three edges of L other than E, but has zero normal derivative on E. Also, f_L is strictly positive on the interior of L. Thus f_L is an eigenfunction on L corresponding to the lowest eigenvalue for the following mixed problem: Neumann boundary conditions on the edge E and Dirichlet boundary conditions on the other three edges of L. Thus by the variational characterization of eigenvalues (see [2]), $(f_C)|_L$ realizes the infimum of the Raleigh quotients

$$\inf_{f \in H^1_{\text{mixed}}} R_L(f) = \inf_{f \in H^1_{\text{mixed}}} \frac{(df, df)_L}{(f, f)_L},$$

where H^1_{mixed} is the Sobolev space for the mixed problem, the H^1-completion of the space of smooth functions supported away from the three edges of L other than E. Because of the reflection invariance of f_C,

$$\frac{(df_C, df_C)_C}{(f_C, f_C)_C} = 2\frac{(df_C, df_C)_L}{(f_C, f_C)_L}.$$

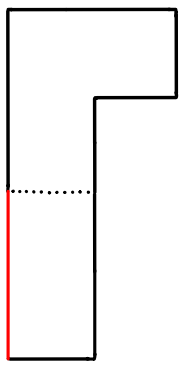
Now consider a normalized eigenfunction f_S for the lowest Dirichlet eigenvalue λ_S of S. By an analogous argument to that above, f_S satisfies $f_S = f_S \circ \sigma$, where σ is the involutive rotation symmetry of S. It follows that

$$\frac{(df_S, df_S)_S}{(f_S, f_S)_S} = 2\frac{(df_S, df_S)_L}{(f_S, f_S)_L}.$$

The restriction $(f_S)|_L$ to L vanishes on the three edges of L other than E, i.e., $(f_S)|_L \in H^1_{\text{mixed}}$. Since $(f_C)|_L$ realizes the infimum and $f_S \in H^1_{\text{mixed}}$, it follows that

$$\frac{(df_C, df_C)_L}{(f_C, f_C)_L} \leq \frac{(df_S, df_S)_L}{(f_S, f_S)_L}. \tag{5.2}$$

If equality holds, then $(f_S)|_L$ is also a nonnegative normalized eigenfunction for the mixed problem, so $(f_S)|_L = (f_C)|_L$. Since eigenfunctions are real analytic, it follows that f_C and f_S must agree upon their common domain of definition when the top halves of S and C are superimposed, i.e., on the domain Γ depicted below:



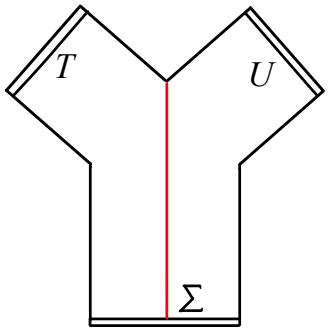
But f_C, as a Dirichlet eigenfunction on C, must vanish on the red edge of Γ, while f_S cannot vanish there, since it is strictly positive on the interior of S. This contradiction implies that the inequality (5.2) is strict. Thus

$$\lambda_C = \frac{(df_C, df_C)_C}{(f_C, f_C)_C} = \frac{2(df_C, df_C)_L}{2(f_C, f_C)_L} < \frac{2(df_S, df_S)_L}{2(f_S, f_S)_L} = \frac{(df_S, df_S)_S}{(f_S, f_S)_S} = \lambda_S,$$

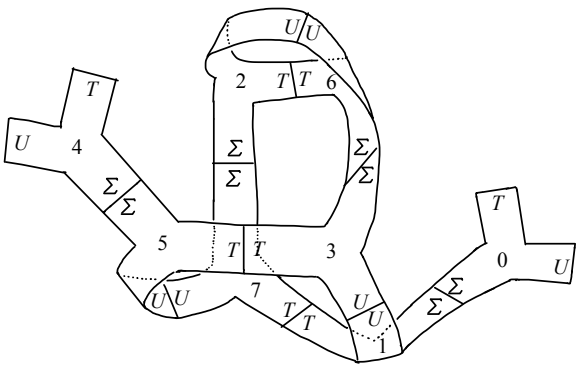
so the lowest eigenvalue of C is smaller than the lowest eigenvalue of S. □

6 Neumann isospectral but not Dirichlet isospectral surfaces with boundary

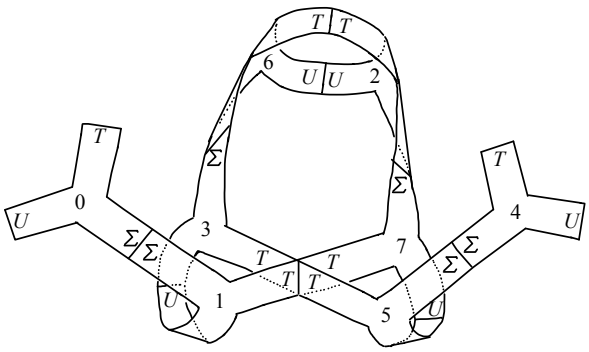
We turn now to an adaptation of Fefferman's argument to our setting. The bordered surfaces M_1 and M_2 are the underlying spaces of orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ constructed exactly as in section 3, but using a Y-shaped fundamental tile: the underlying space $Y := |\mathcal{O}_0|$ of the orbifold $\mathcal{O}_0$ depicted below rather than the tile of Figure 5.



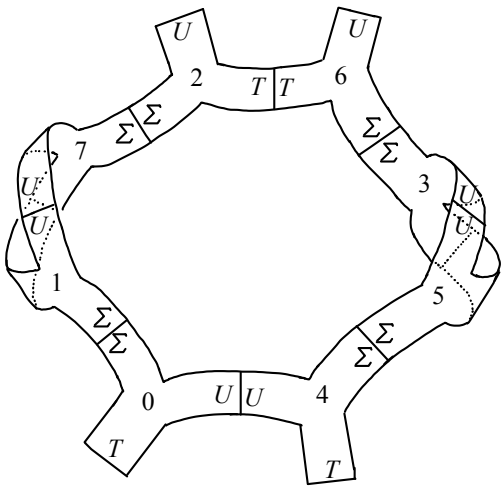
This tile has a symmetry that will be exploited in what follows.
The manifolds M_1 and M_2 thus obtained are depicted in Figure 6; as before, both an embedded and an immersed version of M_1 are shown.



$M_1 = |\mathcal{O}_1|$



$M_1 = |\mathcal{O}_1|$

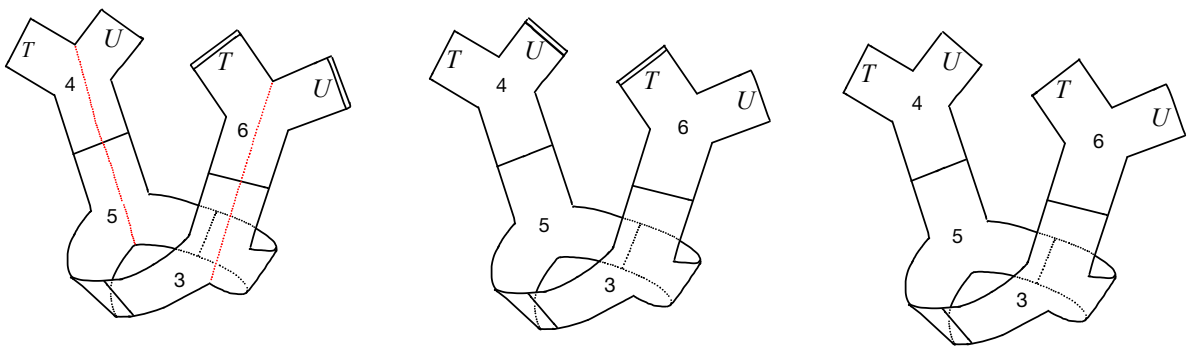


$M_2 = |\mathcal{O}_2|$

Figure 6

Theorem 6.1. *The bordered surfaces M_1 and M_2 depicted in Figure 6 are isospectral with Neumann boundary conditions, but not with Dirichlet boundary conditions.*

Proof. The Neumann isospectrality was established in section 3; it remains only to show that Dirichlet isospectrality fails. Suppose then that M_1 and M_2 are Dirichlet isospectral.
Note that M_1 has an involutive isometry ρ_1 whose effect is to interchange tiles i and $i-4$ (mod 8) (this involutive symmetry is best visualized in the immersed picture in Figure 6). The fixed-point set of ρ_1 consists of the common T-edges and U-edges of tiles 2 and 6. (Although the common T-edge of tiles 1 and 7 and the common T-edge of tiles 3 and 5 coincide in the picture of M_1 immersed, they are of course distinct and are interchanged by ρ_1. Thus ρ_1 is not quite a "reflection in a plane perpendicular to the paper" as the picture would suggest.) The quotient orbifold $Q_1 = M_1/\langle\rho_1\rangle$ is shown in Figure 7. Also, M_2 has a reflection symmetry ρ_2 whose effect is to interchange tiles i and $i-4$ (mod 8); the fixed point set consists of the common T-edge of the tiles 2 and 6 and the common U-edge of tiles 0 and 4. The quotient orbifold $Q_2 = M_2/\langle\rho_2\rangle$ is also shown in Figure 7. Note that Q_1 and Q_2 have the same underlying space $S = |Q_1| = |Q_2|$.



Q_1 Q_2 S

Figure 7

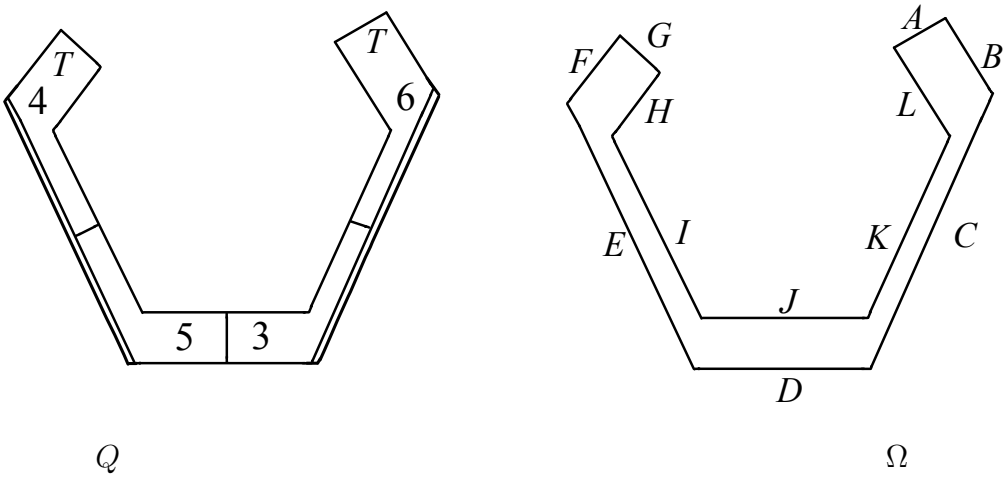
Consider a Dirichlet eigenfunction φ for the lowest Dirichlet eigenvalue λ of M_1. By the Courant nodal domain theorem [14], by replacing φ by $-\varphi$ if necessary, we can assume that φ assumes only positive values on the interior of M_1. Thus the lowest eigenvalue λ has multiplicity one, since no two everywhere-positive functions could be orthogonal. But ρ_1 is an isometry, so $\varphi \circ \rho_1$ is also a λ-eigenfunction; hence $\varphi \circ \rho_1 = \varphi$, i.e., φ is ρ_1-invariant. Thus φ is a Dirichlet eigenfunction on the orbifold Q_1, that is, an eigenfunction on the underlying surface $S = |Q_1|$ in Figure 7 for the following mixed boundary conditions:

$$\text{(6.1)} \qquad \begin{cases} \text{Neumann conditions on the boundary edges } 6T \text{ and } 6U, \\ \text{Dirichlet boundary conditions on all other boundary edges.} \end{cases}$$

Moreover, φ must be a lowest eigenfunction on S for this mixed boundary value problem, since it is everywhere positive on the interior. Thus λ is the lowest eigenvalue on S for the mixed problem (6.1).
Since M_1 and M_2 are assumed Dirichlet isospectral, λ is also the lowest Dirichlet eigenvalue of M_2. Let ψ be a λ-eigenfunction on M_2. By the same argument, ψ is ρ_2-invariant, so is a Dirichlet eigenfunction on the orbifold Q_2, i.e., an eigenfunction on the underlying space S for the following mixed boundary conditions:

$$\text{(6.2)} \qquad \begin{cases} \text{Neumann conditions on the boundary edges } 6T \text{ and } 4U, \\ \text{Dirichlet boundary conditions on all other boundary edges.} \end{cases}$$

Thus λ is also the lowest eigenvalue on S for the problem (6.2). We conclude that the mixed eigenvalue problems (6.1) and (6.2) on S have the same lowest eigenvalue λ.
Now the orbifold Q_1 itself has an involutive symmetry τ (reflection in the red line in the drawing of Q_1 in Figure 7), and the quotient orbifold $Q = Q_1/\langle\tau\rangle$ is shown in Figure 8 together with its underlying space $\Omega = |Q|$. The surface S can be recovered by doubling the planar domain Ω along the boundary edges C and E.



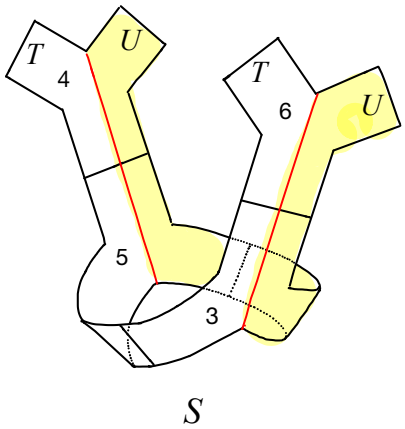


Figure 8

As before, the lowest eigenfunction φ on Q_1 must be τ-invariant, so it must be a lowest eigenfunction on the quotient orbifold Q, i.e., a lowest eigenfunction on $\Omega = |Q|$ for the following boundary conditions:

$$\text{(6.3)} \qquad \begin{cases} \text{Neumann boundary conditions on edges } A, C \text{ and } E, \\ \text{Dirichlet conditions on all other boundary edges.} \end{cases}$$

Thus the lowest eigenvalue of Ω for the boundary condition (6.3) is also λ.
Now the lowest eigenfunction φ on Ω for the mixed conditions (6.3) achieves the infimum of the Rayleigh quotients

$$\inf_{f \in H^1_{\text{mixed}}} \frac{(df, df)_\Omega}{(f, f)_\Omega},$$

where f ranges over the H^1-completion H^1_{mixed} of the space of smooth functions on Ω supported away from all boundary edges except A, C and E.
Now let ψ be a lowest Dirichlet eigenfunction on Q_2. Consider its restriction to the unshaded copy of Ω in Figure 8 which forms the left half of the surface S. Figure 7 shows that ψ satisfies Dirichlet conditions on all boundary edges except A, C and E, since all other edges lie in the orbifold boundary ∂Q_2; thus $\psi \in H^1_{\text{mixed}}$. Now ψ is an eigenfunction on S for the problem (6.2), so $\frac{(d\psi, d\psi)_S}{(\psi, \psi)_S} = \lambda$. Also, note that the orbifold Q_2 has an involutive "rotation by π" symmetry, and ψ must be invariant under this symmetry. Thus

$$\frac{(d\psi, d\psi)_\Omega}{(\psi, \psi)_\Omega} = \frac{\frac{1}{2}(d\psi, d\psi)_S}{\frac{1}{2}(\psi, \psi)_S} = \frac{(d\psi, d\psi)_S}{(\psi, \psi)_S} = \lambda.$$

Thus ψ realizes the infimum

$$\lambda = \inf_{f \in H^1_{\text{mixed}}} \frac{(df, df)_\Omega}{(f, f)_\Omega},$$

so ψ is a lowest eigenfunction on Ω for the boundary conditions (6.3). Since the multiplicity of the lowest eigenvalue is one, it follows that ψ and φ agree on Ω (after multiplying ψ by a nonzero scalar). Since ψ and φ agree on an open set, $\psi = \varphi$ on all of S by the maximum principle.

Reference to Figure 7 shows that φ satisfies Neumann conditions on edge $6U$ of S, while ψ satisfies Dirichlet conditions on that edge. Now consider a small disk of radius ε centered at a point in the interior of the boundary edge $6U$. Since S is flat, this disk is isometric to the open half-disk $H = \{(x, y) \in \mathbb{R}^2$, so we can view φ as an eigenfunction on H. Since φ satisfies Neumann boundary conditions on the x-axis, we can extend φ by reflection to an eigenfunction φ_1 on the open disk $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \varepsilon\}$ that is invariant under reflection in the x-axis. But φ also satisfies Dirichlet conditions on the x-axis, so we can also continue φ to an eigenfunction φ_2 on B that is anti-invariant under reflection in the x-axis. But φ_1 and φ_2 agree on an open set, so $\varphi_1 = \varphi_2$, a contradiction. Thus M_1 and M_2 are not Dirichlet isospectral, and Theorem 6.1 is proved. □

7 Inaudible singularities; concluding remarks

We conclude with a few remarks, references to related results, and open questions.

Remarks 7.1. The results and methods above lead to some other interesting phenomena, and highlight several natural questions.

1. Some of the examples of isospectral surfaces in [12] show that one cannot hear whether there is a singularity in the interior: some of the isospectral pairs have the property that M_1 has an interior cone singularity, while M_2 does not.

   If one constructs isospectral bordered surfaces M_1 and M_2 using the Gerst Gassmann-Sunada triple (G, Γ_1, Γ_2) of section 3 with generators st, t, and tu but using the triangular fundamental tile T in Figure 9, then M_1 and M_2 exhibit this same phenomenon. The surfaces thus constructed are shown in Figure 9. Both surfaces are flat annuli with polygonal boundary, but M_2 has a cone singularity in the interior while M_1 does not (M_1 is a manifold with corners, while M_2 is more singular). Similarly, if one constructs bordered surfaces M_1 and M_2 using the triangular tile T but the generators σ, t and u as in section 3, then M_1 has a single interior cone singularity, while M_2 has two cone singularities. Thus one cannot hear the nature of singularities.

2. We note that the content of Theorem 3.1 can be expressed as the assertion that one cannot hear the vanishing of the first Stiefel-Whitney class w_1 of a Riemannian manifold with boundary. It is then natural to ask: for an orientable Riemannian manifold, can one hear the vanishing of the second Stiefel-Whitney class w_2? That is, can one infer from the Laplace spectrum whether or not the manifold admits a Spin structure? This question was answered negatively by Roberto Miatello and Ricardo Podestá in [29].

3. Recall that an *orbifold chart* on a space X (see [1]) is given by a connected open subset $\widetilde{U}$ of $\mathbb{R}^n$, a finite group G of diffeomorphisms of U, and a G-invariant map $\varphi : \widetilde{U} \to X$ that induces a homeomorphism of the orbit space $G\backslash\widetilde{U}$ with an open subset of X; an orbifold is then a space equipped with a cover of orbifold charts that satisfy a suitable compatibility condition. An orbifold is *locally orientable* if in each such orbifold chart, the action of the group G on $\widetilde{U}$ is by orientation-preserving diffeomorphisms of $\widetilde{U}$. In contrast with our main result, it was recently shown by Sean Richardson and Elizabeth Stanhope [32] that one *can* hear local orientability of a Riemannian orbifold.

Finally, we note that our main result and the example depicted in Figure 9 call attention to two open problems:

- Can a *closed* orientable Riemannian manifold be isospectral to a *closed* nonorientable Riemannian manifold?
- Can a Riemannian orbifold with nonempty singular set be isospectral to a Riemannian manifold?

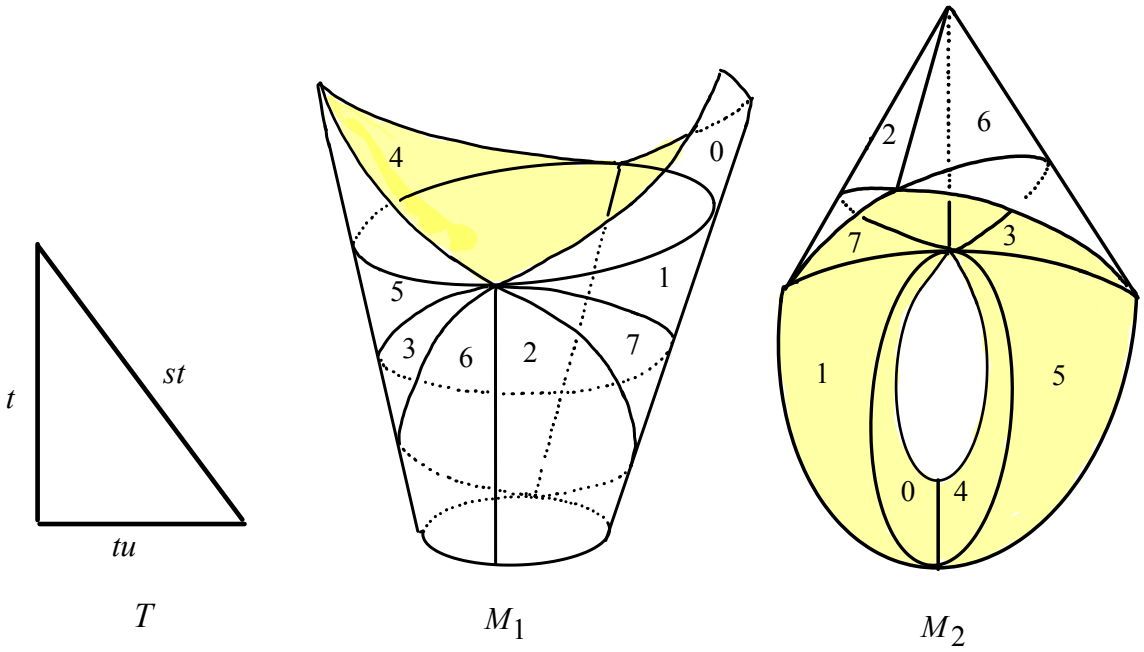


Figure 9

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