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A remark on n - Jordan homomorphisms

M. El Azhari

Abstract. Let A and B be commutative algebras and $n \geq 2$ an integer. Then each n - Jordan homomorphism $h : A \rightarrow B$ is an n - homomorphism.

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1. Preliminaries

Let A, B be two algebras and $n \geq 2$ an integer. A linear map $h : A \rightarrow B$ is called an n - Jordan homomorphism if $h(a^n) = h(a)^n$ for all $a \in A$. Also, a linear map $h : A \rightarrow B$ is called an n - homomorphism if $h(\prod_{i=1}^n a_i) = \prod_{i=1}^n h(a_i)$ for all $a_1, \dots, a_n \in A$. In the usual sense, a 2- Jordan homomorphism is a Jordan homomorphism and a 2- homomorphism is a homomorphism.

It is obvious that n - homomorphisms are n - Jordan homomorphisms. Conversely, under certain conditions, n - Jordan homomorphisms are n - homomorphisms. For example, if A, B are commutative algebras and $h : A \rightarrow B$ is a Jordan homomorphism, then h is a homomorphism: Let $x, y \in A$, since $h((x + y)^2) = h(x + y)^2$, a simple calculation shows that $h(xy) = h(x)h(y)$. Also, it was shown in [2] that if $n \in \{3, 4\}$, A, B are commutative algebras and $h : A \rightarrow B$ is an n - Jordan homomorphism, then h is an n - homomorphism. This result was also proved for $n = 5$ in [3].

2. Result

In [1], Cheshmavar et al claimed the following result:

Theorem 2.3 of [1]. Let A and B be commutative algebras, $n \geq 3$, and let $h : A \rightarrow B$ be an n - Jordan homomorphism. Then h is an n - homomorphism.

The proof of this theorem is essentially based on a formula which is not correct:

Let $h : A \rightarrow B$ be a linear map, $n \geq 4$, and consider the maps

$$\psi : A^n \rightarrow B, \psi(x_1, \dots, x_n) = h(\prod_{i=1}^n x_i) - \prod_{i=1}^n h(x_i)$$

and for $2 \leq k \leq n$,

$$\varphi_{k-1} : A^k \rightarrow B, \varphi_{k-1}(x_1, \dots, x_k) = h((x_1 + \dots + x_k)^n) - h(x_1 + \dots + x_k)^n.$$

In [1], the authors claimed that

$$\begin{aligned} \varphi_{n-1}(x_1, \dots, x_n) &= \sum_{i,j=1, i < j}^n \varphi_1(x_i, x_j) + \sum_{i,j,k=1, i < j < k}^n \varphi_2(x_i, x_j, x_k) \\ &+ \dots + \sum_{i_1, \dots, i_{n-1}=1, i_1 < \dots < i_{n-1}}^n \varphi_{n-2}(x_{i_1}, \dots, x_{i_{n-1}}) + n! \psi(x_1, \dots, x_n). \end{aligned}$$

This formula is false since, for example, the expression containing exactly the coordinates x_1, x_2 in the left part will be appearing many times in the right part (in $\varphi_1(x_1, x_2), \varphi_2(x_1, x_2, x_3), \varphi_2(x_1, x_2, x_4), \dots$).

We give here a correct proof of [1, Theorem 2.3].

Theorem 2.1. Let A, B be two algebras, $n \geq 2$, and let $h : A \rightarrow B$ be an n -Jordan homomorphism. Then $\sum_{\sigma \in S_n} (h(\prod_{i=1}^n x_{\sigma(i)}) - \prod_{i=1}^n h(x_{\sigma(i)})) = 0$ for all $x_1, \dots, x_n \in A$, where S_n is the set of permutations on $\{1, \dots, n\}$. Furthermore, if A and B are commutative, then h is an n -homomorphism.

Proof. Case $n = 2$: Let $h : A \rightarrow B$ be a Jordan homomorphism and $x_1, x_2 \in A$. Since $h((x_1 + x_2)^2) = h(x_1 + x_2)^2$, a simple calculation shows that $h(x_1 x_2) - h(x_1)h(x_2) + h(x_2 x_1) - h(x_2)h(x_1) = 0$, then $\sum_{\sigma \in S_2} (h(x_{\sigma(1)} x_{\sigma(2)}) - h(x_{\sigma(1)})h(x_{\sigma(2)})) = 0$.
If A and B are commutative, we deduce that h is a homomorphism.

Case $n \geq 3$: Let $h : A \rightarrow B$ be a linear map and $x_1, \dots, x_n \in A$. We will write $h((x_1 + \dots + x_n)^n) - h(x_1 + \dots + x_n)^n$ as the sum of expressions containing respectively $1, 2, \dots, n$ coordinates of (x_1, \dots, x_n) . Let $1 \leq k \leq n$ and $i_1, \dots, i_k \in \{1, \dots, n\}$, $i_1 < \dots < i_k$, we denote by $\psi_k^{(n)}(x_{i_1}, \dots, x_{i_k})$ the expression in $h((x_1 + \dots + x_n)^n) - h(x_1 + \dots + x_n)^n$ containing exactly the k coordinates x_{i_1}, \dots, x_{i_k} . Then

$$\begin{aligned} h((x_1 + \dots + x_n)^n) - h(x_1 + \dots + x_n)^n &= \sum_{i=1}^n \psi_1^{(n)}(x_i) + \sum_{i,j=1, i < j}^n \psi_2^{(n)}(x_i, x_j) \\ &+ \dots + \sum_{i_1, \dots, i_{n-1}=1, i_1 < \dots < i_{n-1}}^n \psi_{n-1}^{(n)}(x_{i_1}, \dots, x_{i_{n-1}}) + \psi_n^{(n)}(x_1, \dots, x_n). \end{aligned}$$

$$\text{For } 1 \leq i \leq n, \psi_1^{(n)}(x_i) = h(x_i^n) - h(x_i)^n.$$

For $i, j \in \{1, \dots, n\}$, $i < j$,

$$\begin{aligned} \psi_2^{(n)}(x_i, x_j) &= h((x_i + x_j)^n) - h(x_i + x_j)^n - (h(x_i^n) - h(x_i)^n) - (h(x_j^n) - h(x_j)^n) \\ &= h((x_i + x_j)^n) - h(x_i + x_j)^n - \psi_1^{(n)}(x_i) - \psi_1^{(n)}(x_j). \end{aligned}$$

For $3 \leq k \leq n$, $i_1, \dots, i_k \in \{1, \dots, n\}$, $i_1 < \dots < i_k$,

$$\begin{aligned} \psi_k^{(n)}(x_{i_1}, \dots, x_{i_k}) &= h((x_{i_1} + \dots + x_{i_k})^n) - h(x_{i_1} + \dots + x_{i_k})^n \\ &- \sum_{m=1}^k \psi_1^{(n)}(x_{i_m}) - \sum_{j_1, j_2 \in \{i_1, \dots, i_k\}, j_1 < j_2} \psi_2^{(n)}(x_{j_1}, x_{j_2}) \\ &- \dots - \sum_{j_1, \dots, j_{k-1} \in \{i_1, \dots, i_k\}, j_1 < \dots < j_{k-1}} \psi_{k-1}^{(n)}(x_{j_1}, \dots, x_{j_{k-1}}). \end{aligned}$$

If $h : A \rightarrow B$ is an n -Jordan homomorphism, then $\psi_1^{(n)}(x_i) = 0$ for $1 \leq i \leq n$, also $\psi_2^{(n)}(x_i, x_j) = 0$ for $i, j \in \{1, \dots, n\}$, $i < j$, and by induction $\psi_n^{(n)}(x_1, \dots, x_n) = \sum_{\sigma \in S_n} (h(\prod_{i=1}^n x_{\sigma(i)}) - \prod_{i=1}^n h(x_{\sigma(i)})) = 0$.

If A and B are commutative,

$$\sum_{\sigma \in S_n} (h(\prod_{i=1}^n x_{\sigma(i)}) - \prod_{i=1}^n h(x_{\sigma(i)})) = n! (h(\prod_{i=1}^n x_i) - \prod_{i=1}^n h(x_i)) = 0,$$

hence h is an n -homomorphism.

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